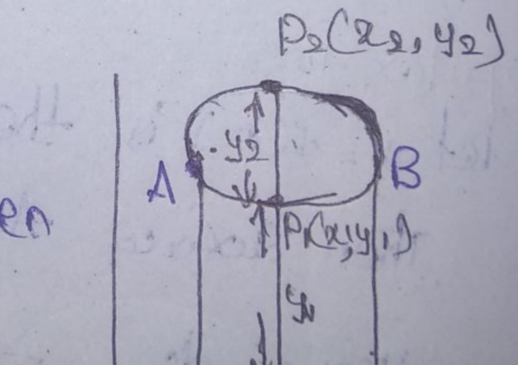
Theorems of Pappus

Volume of Revolution

Statement: If a closed plane curve revolves about a straight line in its plane, (The straight line not intersecting the curve), then the volume of the solid of revolution thus formed is obtained on multiplying area of the region enclosed by the curve with the length of the path described by the centroid of the region.

Proof: Let 'x' axis is the axis of revolution, 'A' is the surface area of the region which is lies b/w $x=a$ & $x=b$

& let $MP_1 = y_1$, $MP_2 = y_2$



∴ The volume 'V' of the solid of revolution is given by

$$V = \int_{x=a}^{b} \pi y_2^2\, dx - \int_{x=a}^{b} \pi y_1^2\, dx$$

(upper curve − lower curve)

$$V = \pi \int_{x=a}^{b} (y_2^2 - y_1^2)\, dx \quad \text{—— (1)}$$

the ordinate $\bar{y}$, of the centroid of the region is given by

$$V = \frac{\int_a^b \frac{y_1+y_2}{2}(y_2-y_1)\,dx}{A}$$

$$\bar{y} = \frac{\int_a^b \frac{y_1+y_2}{2}(y_2-y_1)\,dx}{A}$$

$$\bar{y} = \frac{1}{2A}\int_a^b (y_2^2-y_1^2)\,dx$$

$$\bar{y}\cdot 2A = \int_a^b (y_2^2-y_1^2)\,dx$$

$$\cancel{\bar{y}\,2A = \pi}$$

$$\bar{y}\cdot(2A)\,\pi = \pi\int_a^b (y_2^2-y_1^2)\,dx \quad \text{—— (2)}$$

from (1) & (2)

$$\boxed{V = 2\pi\bar{y}A}$$

Here A is the area of the region

∴ It shows, that, 2π is the length of the path described by the centroid.

Surface of Revolution

If a closed plane curve ~~in its plane~~ in its plane revolves about ~~it~~ a st. line in its plane (The st. line not intersecting the curve) Then the surface of the solid of revolution, thus formed is obtained on multiplying the length of the arc (curve) with that of the path described by the centroid of the curve

Proof: let x axis the axis of revolution l' is the length of the curve which lies between $x=a$ & $x=b$

let the surface S' of the solid of revolution is given by

$$S = \int_0^l 2\pi y \, ds \quad \text{--- (1)}$$

let $\bar{y}$ is the centroid of the curve is given by

$$\bar{y} = \frac{\int_0^l y \, ds}{l}$$

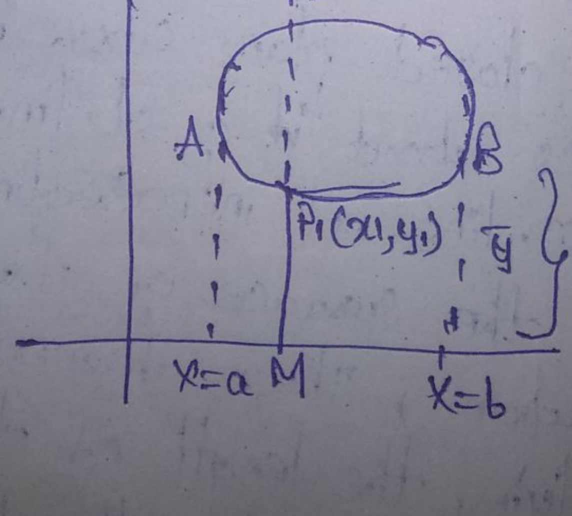
$$l\bar{y} = \int_0^l y \, ds$$

$$2\pi \bar{y} l = \int_0^l 2\pi y \, ds \quad \text{--- (2)}$$

From (1) & (2)

$$\boxed{S = 2\pi \bar{y} l}$$

Here $2\pi \bar{y}$ is the length of the path described by the centroid.



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1. Find the surface of the solid formed by the revolving co-ordinate $r = a(1+\cos\theta)$ about the initial line.

$S = 2\pi y \frac{ds}{dx}$

Given curve $r = a(1+\cos\theta)$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\left(\frac{dr}{d\theta}\right)^2 = a^2\sin^2\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{1+\cos^2\theta + 2\cos\theta + \sin^2\theta} = \sqrt{2}\,a\sqrt{1+\cos\theta}$$

$$= \sqrt{2}\,a\sqrt{2\cos^2\theta/2}$$

$$= 2a\cos\theta/2$$

Required surface of the

$$\boxed{\text{Solid } S = \int_{\theta_1}^{\theta_2} 2\pi y \frac{ds}{d\theta}\, d\theta}$$

Here $\boxed{y = r\sin\theta}$

$$S = \int_0^{\pi} 2\pi (r\sin\theta)\, 2a\cos\theta/_2\, d\theta$$

$$= 4a\pi \int_0^{\pi} a(1+\cos\theta)\sin\theta\cos\theta/_2\, d\theta$$

$$= 4a^2\pi \int_0^{\pi} (2\cos^2\theta/_2\cos\theta/_2)(2\sin\theta/_2\cos\theta/_2)\, d\theta$$

$$= 16a^2\pi \int_{\theta=0}^{\pi} \cos^4\theta/_2 \sin\theta/_2\, d\theta$$

put $\frac{\theta}{2} = t \Rightarrow \theta = 2t$

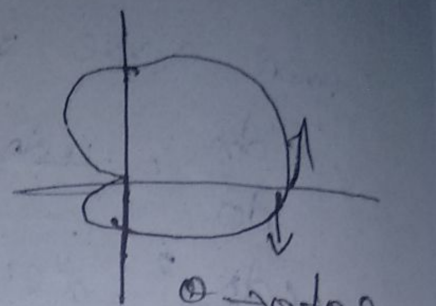
$\Rightarrow d\theta = 2\,dt$

L.L: If $\theta = 0 \Rightarrow 0 = t$

U.L: If $\theta = \pi \Rightarrow t = \pi/2$

$$S = 16a^2\pi \int_{t=0}^{\pi/2} \cos^4 t \sin t \,(2dt)$$

$$= 32a^2\pi \int_{t=0}^{\pi/2} \sin t \cos^4 t \, dt$$



$\theta \to 0$ to 2π

$$= \overset{16}{\cancel{32}}a^2\pi \, \frac{\Gamma(2/2)\,\Gamma(5/2)}{\cancel{2}\,\Gamma(7/2)}$$

$$\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

$$= \frac{16a^2\pi(1)\cancel{\frac{3}{2}}\cancel{\frac{1}{2}}\sqrt{\pi}}{\frac{5}{2}\cancel{\frac{3}{2}}\cancel{\frac{1}{2}}\sqrt{\pi}}$$

$$\frac{2}{5}\,\Gamma(5/2)$$

$$= \frac{32a^2\pi}{5}$$

2. Find the area of the surface of revolution formed by revolving a curve $r = 2a\cos\theta$ about the initial line

Given curve $r = 2a\cos\theta$

$$\frac{dr}{d\theta} = 2a\sin\theta$$

$$\left(\frac{dr}{d\theta}\right)^2 = 4a^2\sin^2\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta}$$

$$= 2a\sqrt{1}$$

$$= 2a$$

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Required

Here

S =

3. Find the ... revolving ... about ...

Given

$\frac{dr}{d\theta}$

$\frac{ds}{d\theta} = \sqrt{\ldots}$

Requir...

Required surface of the solid $S = \int_{\theta_1}^{\theta_2} 2\pi y \frac{ds}{d\theta}\, d\theta$

Here $y = r\sin\theta$

$$S = \int_0^{\pi/2} 2\pi (r\sin\theta)\, 2a\, d\theta$$

$$= 4a\pi \int_0^{\pi/2} (2a\cos\theta\sin\theta)\, d\theta$$

$$= 4a^2\pi \int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$= \cancel{4}^{2}a^2\pi \left(\frac{-\cos 2\theta}{\cancel{2}}\right)_0^{\pi/2}$$

$$= -2a^2\pi(-1-1)$$

$$= 4a^2\pi$$

3. Find the surface of the solid generated by revolving the lemniscate $r^2 = a^2\cos 2\theta$ about the initial line.

Given curve $r^2 = a^2\cos 2\theta$

$$r = a\sqrt{\cos 2\theta}$$

$$\frac{dr}{d\theta} = a\frac{1}{\cancel{2}\sqrt{\cos 2\theta}}(-\sin 2\theta)\cancel{2}$$

$$= \frac{-a\sin 2\theta}{\sqrt{\cos 2\theta}}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}}$$

$$= a\sqrt{\frac{1}{\cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}}$$

Required surface of the solid

$$S = \int_{\theta_1}^{\theta_2} 2\pi y \frac{ds}{d\theta}\, d\theta$$

Here $y = r\sin\theta$

$y = a\sqrt{\cos 2\theta}\,(\sin\theta)$

$$S = 2\int_0^{\pi/4} 2\pi(a)\cancel{\sqrt{\cos 2\theta}}\,\sin\theta\,\frac{a}{\cancel{\sqrt{\cos 2\theta}}}\,d\theta$$

$$= 4\pi a^2 \left(-\cos\theta\right]_0^{\pi/4}$$

$$= 4\pi a^2\left(-\cos\pi/4 + \cos 0\right)$$

$$= 4\pi a^2\left(1-\frac{1}{\sqrt2}\right)$$

$$= 4\pi a^2\left(\frac{\sqrt2-1}{\sqrt2}\right)$$

$$= 2\sqrt2\,\pi a^2(\sqrt2-1)$$

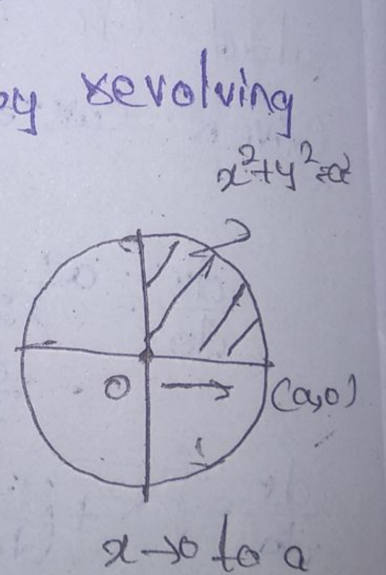
4. Find the surface of a sphere of radius 'a'

Given curve $x^2+y^2=a^2$

let the radius of sphere is 'a'

let the sphere is generated by revolving about x or y axis

$\therefore$ The eqn of circle is



$$x^2+y^2=a^2$$

$$y^2=a^2-x^2$$

$$y=\sqrt{a^2-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}\sqrt{a^2-x^2}}(-\cancel{2}x)$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2-x^2}}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{a^2-x^2}} = \sqrt{\frac{a^2-\cancel{x^2}+\cancel{x^2}}{a^2-x^2}}$$

$$= \frac{a}{\sqrt{a^2-x^2}} = \frac{a}{y}$$

Required Surface of the solid

$$S = \int_{x_1}^{x_2} 2\pi y \frac{ds}{dx}\, dx$$

$$S = 2\int_0^a 2\pi y \frac{a}{y}\, dx$$

$$= 4a\pi\,(x)_0^a$$

$$= 4a^2\pi$$

5. Find the surface of the solid generated by revolving the arc of the parabola by revolving the arc of the parabola its latus rectum $y^2 = 4ax$ bounded by its latus rectum about x-axis

Sol: Given Curve $y^2 = 4ax \Rightarrow y = 2\sqrt{a}\sqrt{x}$

$$\cancel{2}y\frac{dy}{dx} = \cancel{4}^{2}a\,(1)$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{4a^2}{y^2} = \frac{\cancel{4}a^2}{4ax} = \frac{a}{x}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{a}{x}} = \sqrt{\frac{x+a}{x}}$$

Required surface of the solid

$$S = \int_{x_1}^{x_2} 2\pi y \frac{ds}{dx}\, dx$$

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$$S = \int_{x=0}^{a} 2\pi \, 2\sqrt{a}\,\cancel{\sqrt{x}}\,\frac{\sqrt{x+a}}{\cancel{\sqrt{x}}}\,dx$$

$$= 4\sqrt{a}\,\pi \int_{x=0}^{a} (a+x)^{1/2}\,dx$$

$$= 4\sqrt{a}\,\pi \left[\frac{(a+x)^{3/2}}{3/2}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}\,\pi\left[(2a)^{3/2} - a^{3/2}\right]$$

$$= \frac{8}{3}\sqrt{a}\,\pi\,(a \cdot a^{1/2})\,(2\sqrt{2}-1).$$

$$= 8/_3\,\pi a^2 (2\sqrt{2}-1)$$

$$S = 2\int_0 \ldots dx \ldots \int_0 y \ldots dx = 4\pi a\int_0 dx = 4\pi a\,x\Big]_0 = 4\pi a(a-0) = 4\pi a(a)$$

Surface area of sphere $= 4\pi a^2$.

4) **Find the surfac of the solid generated by revolution of the curve $x^2 + ay^2 = 16$ about $x-axis$.**

Sol. Given curve is $x^2 + 4y^2 = 16$

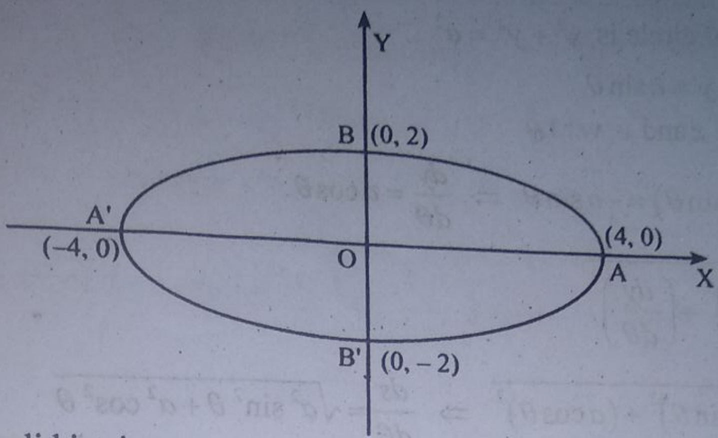
Differentiating equation 1 w.r.t. x

$$\Rightarrow 2x + 4(2y)\frac{dy}{dx} = 0 \Rightarrow 2x + 8y\frac{dy}{dx} = 0 \Rightarrow x + 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{4y}$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1 + \left(\frac{-x}{4y}\right)^2} = \sqrt{1 + \frac{x}{16y^2}^2} = \sqrt{\frac{16y^2 + x^2}{16y^2}}$$

$$=\sqrt{\frac{4\left(4y^2\right)+x^2}{16y^2}} = \frac{1}{4y}\sqrt{4\left(16-x^2\right)+x^2} = \frac{1}{4y}\sqrt{64-4x^2+x^2} = \frac{1}{4y}\sqrt{64-3x^2}$$



The surface area of solid is given as

$$S=2\int_0^4 2\pi y\frac{ds}{dx}dx = 4\pi\int_0^4 y.\frac{1}{4y}\sqrt{64-3x^2}\,dx = \frac{\cancel{4}\pi}{\cancel{4}}\int_0^4\sqrt{3\left(\frac{64}{3}-x^2\right)}\,dx$$

$$=\pi\int_0^4\sqrt{3\left[\left(\frac{8}{\sqrt{3}}\right)^2-x^2\right]}\,dx = \sqrt{3}\pi\int_0^{\pi}\sqrt{\left(\frac{8}{\sqrt{3}}\right)^2-x^2}\,dx = \sqrt{3}\pi\left|\frac{x}{2}\sqrt{\left(8/\sqrt{3}\right)^2-x^2}+\frac{\left(8/\sqrt{3}\right)^2}{2}\sin^{-1}\left(\frac{x}{\frac{8}{\sqrt{3}}}\right)\right|_0^4$$

$$=\pi\sqrt{3}\left|\frac{x}{2}\sqrt{\frac{64}{3}-16}+\frac{32}{3}\sin^{-1}\left(\frac{4\sqrt{3}}{8}\right)-\frac{0}{2}\sqrt{\frac{64}{3}-0}-\frac{32}{3}\sin^{-1}\left(\frac{\sqrt{3}-0}{8}\right)\right| = \pi\sqrt{3}\left[\frac{x}{2}\sqrt{\frac{64}{3}-x^2}+\frac{64}{3\times 2}\sin^{-1}\left(\frac{\sqrt{3}x}{8}\right)\right]_0^4$$

$$=\pi\sqrt{3}\left[\frac{4}{2}\sqrt{\frac{64}{3}-16}+\frac{32}{3}\sin^{-1}\left(\frac{4\sqrt{3}}{8}\right)-\frac{0}{2}\sqrt{\frac{64}{3}-0}-\frac{32}{3}\sin^{-1}\left(\frac{\sqrt{3}-0}{8}\right)\right]$$

$$=\pi\sqrt{3}\left|\frac{x}{2}\sqrt{\frac{64}{3}-x^2}+\frac{64}{3\times 2}\sin^{-1}\left(\frac{\sqrt{3}x}{8}\right)\right|_0^4$$

$$=\pi\sqrt{3}\left|\frac{4}{2}\sqrt{\frac{64}{3}-16}+\frac{32}{3}\sin^{-1}\left(\frac{4\sqrt{3}}{8}\right)-\frac{0}{2}\sqrt{\frac{64}{3}-0}-\frac{32}{3}\sin^{-1}\left(\frac{\sqrt{3}-0}{8}\right)\right|$$

$$=\pi\sqrt{3}\left|2\left(\frac{4\sqrt{3}}{3}\right)+\frac{32}{3}\sin^{-1}\frac{\sqrt{3}}{2}-0-0\right| = \pi\sqrt{3}\left|\frac{8\sqrt{3}}{8}+\frac{32}{3}\left(\frac{\pi}{3}\right)\right| = \pi\sqrt{3}\left|\frac{8\sqrt{3}}{3}+\frac{32\pi}{9}\right|$$

$$=\frac{8\pi\left(\cancel{3}\right)}{\cancel{3}}+\frac{32\pi^2\sqrt{3}}{9} = 8\pi+\frac{32\sqrt{3}\pi^2}{9}$$

$\therefore$ Surface area of $x^2+4y^2=16$ is $8\pi+\frac{32\sqrt{3}}{9}\pi^2$

5) An arc of a circle of radius a revolves about its chord. If the length of the arc is $2a\alpha\left(\alpha<\frac{\pi}{2}\right)$. Show that the area of the surface generated is $4\pi a^2(\sin\alpha-\alpha\cos\alpha)$.

Sol. The equation of circle is $x^2+y^2=a^2$

Let $x=a\cos\theta, y=a\sin\theta$

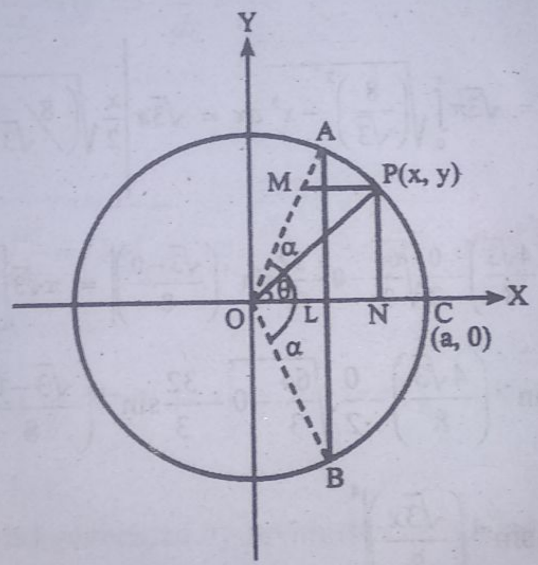
Differentiating x and y w.r.t θ

$$\Rightarrow \frac{dx}{d\theta}=a(-\sin\theta)=-a\sin\theta \Rightarrow \frac{dy}{d\theta}=a\cos\theta$$

$$\therefore \frac{ds}{d\theta}=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}$$

$$\Rightarrow \frac{ds}{d\theta}=\sqrt{(-a\sin\theta)^2+(a\cos\theta)^2} \Rightarrow \frac{ds}{d\theta}=\sqrt{a^2\sin^2\theta+a^2\cos^2\theta}$$

$$\Rightarrow \frac{ds}{d\theta}=\sqrt{a^2(\sin^2\theta+\cos^2\theta)} \Rightarrow \frac{ds}{d\theta}=\sqrt{a^2(1)} \Rightarrow \frac{ds}{d\theta}=a$$



Let $P(x,y)$ be any point on the circle and M be the foot of the perpendicular from P on AB such that $PM=LN$.

$ON=x, OL=a\cos\alpha$

$PM=ON-OL$

$\Rightarrow PM=x-a\cos\alpha$

The curve is symmetrical about $x-axis$ the surface area of solid is given as

$$S=2\int_0^{\alpha}2\pi(PM)\frac{ds}{d\theta}d\theta = 4\pi\int_0^{\alpha}(x-a\cos\alpha)a\,d\theta = 4\pi a\int_0^{\alpha}(a\cos\theta-a\cos\alpha)d\theta$$

$$=4\pi a^2\int_0^{\alpha}(\cos\theta-\cos\alpha)d\theta = 4\pi a^2\left[\sin\theta-\cos\alpha(\theta)\right]_0^{\alpha} = 4\pi a^2\left[(\sin\alpha-\sin 0)-\cos\alpha(\alpha-0)\right]$$

$$= 4\pi a^2\left[\sin\alpha - 0 - \cos\alpha\left(\alpha\right)\right] = 4\pi a^2\left[\sin\alpha - \alpha\cos\alpha\right]$$

$\therefore$ Surface area of solid $= 4\pi a^2\left[\sin\alpha - \alpha\cos\alpha\right]$

SHORT QUESTIONS

1. Find the length an arc of the curve $r = ae^{\theta\cot\alpha}$ taking $s = 0$ when $\theta = 0$.(V.IMP)
2. Find the length of the arc of the curve $y = \log\sec x$ from $x = 0$ to $x = \frac{\pi}{3}$
3. Find the length of the arc of the catenary $y = c\cosh\left(\frac{x}{c}\right)$ measured from the vertex (point (x, y).
4. Find the volume generated by the revolution of an arc of the catenary $y = c\cosh\frac{x}{c}$.