AE647 Assignment 3 : Single Charged Particle Motion

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Part I

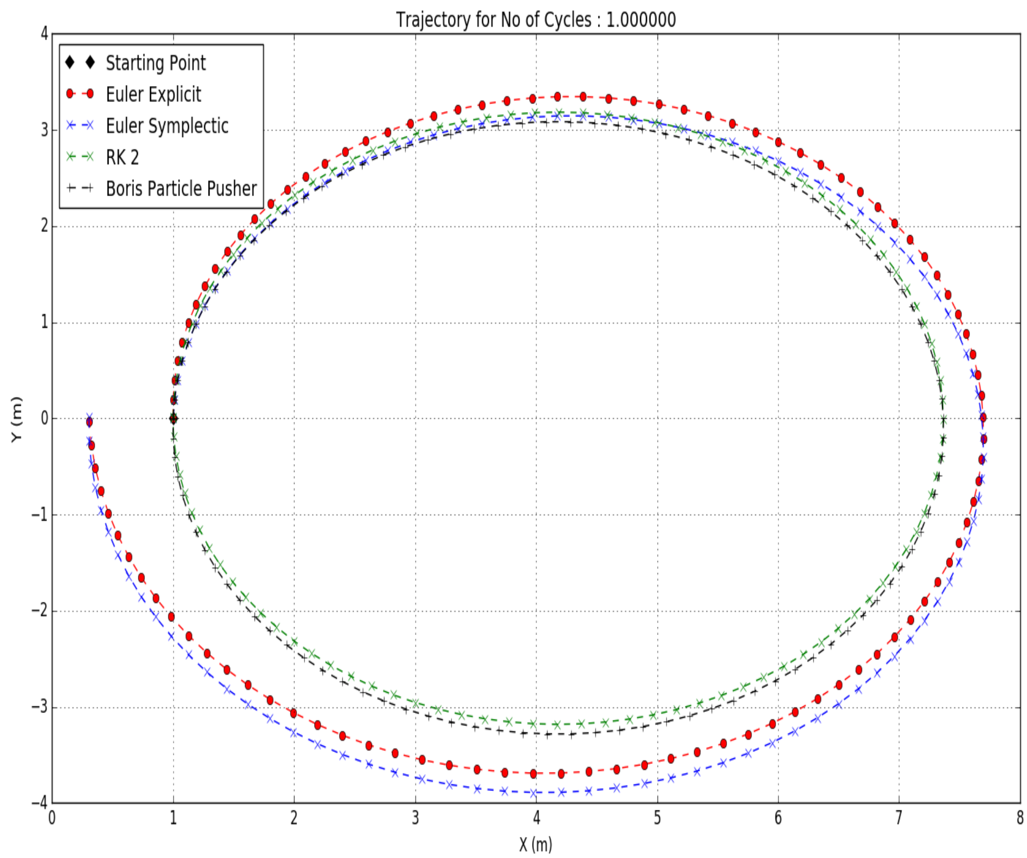
Uniform Out of plane Magnetic Field :

The motion and evolution of energy, of a single charged particle under the influence of a uniform Magnetic Field,:$\bar{B} = B_0\hat{e}_z$ have been plotted and compared for the given Numerical Schemes

0.1 Trajectory:

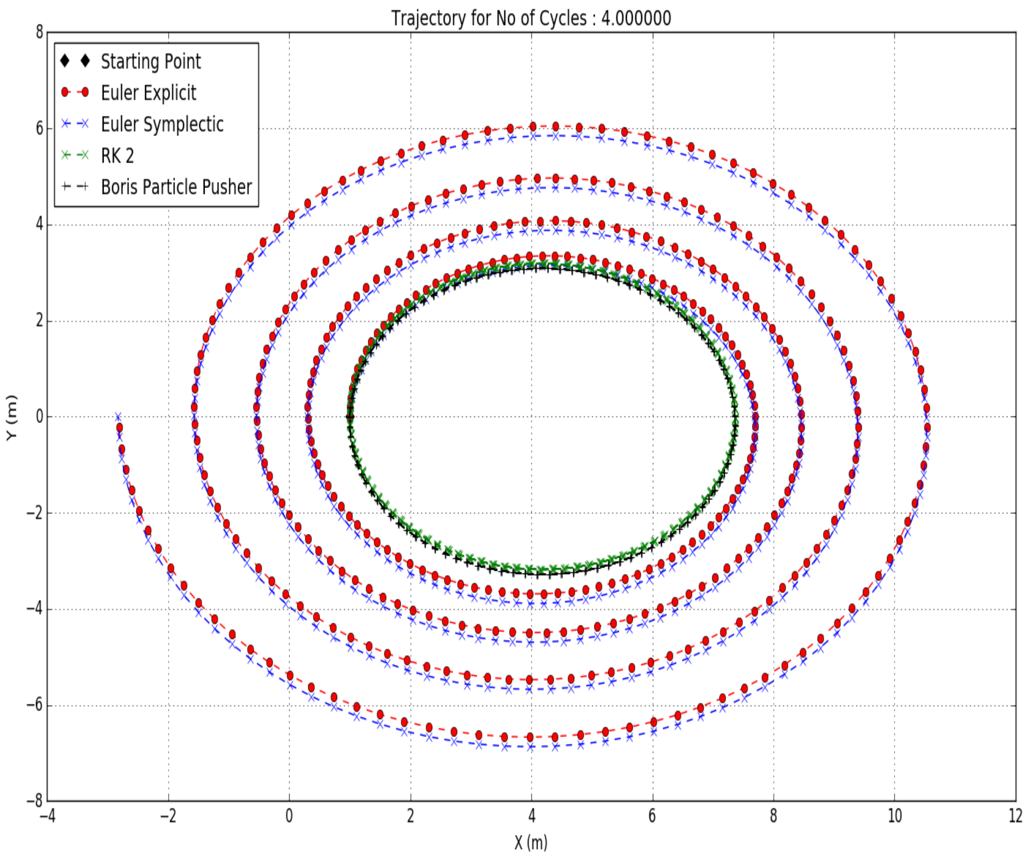
0.1.1 Motion over 1 Cycle :

Figure 1: Trajectory of charged particle



0.1.2 Motion over 4 Cycles :

Figure 2: Trajectory of charged particle over 4 cycles

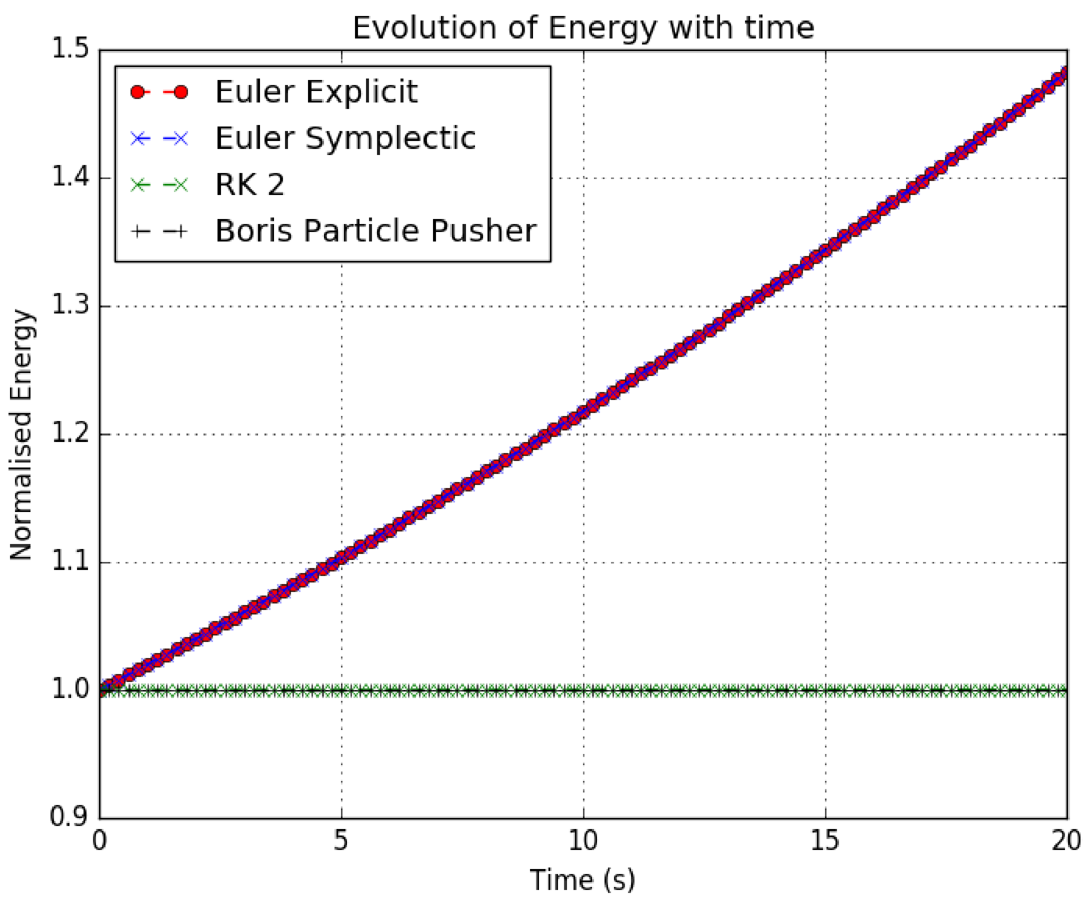


As can be clearly seen, Euler Explicit and Euler Symplectic Numerical Schemes lead to spiralling out of the charged particle while RK-2 and Boris Particle Pusher methods maintain the required circular motion of the charged particle.
It should be noted that the timesteps taken for the above plots are $\frac{1}{100}^{th}$ of the Timeperiod of the circular motion. With larger values of timesteps, say $\frac{1}{10}^{th}$, spiralling out is observed even in RK-2 Method.

0.2 Evolution of Energy:

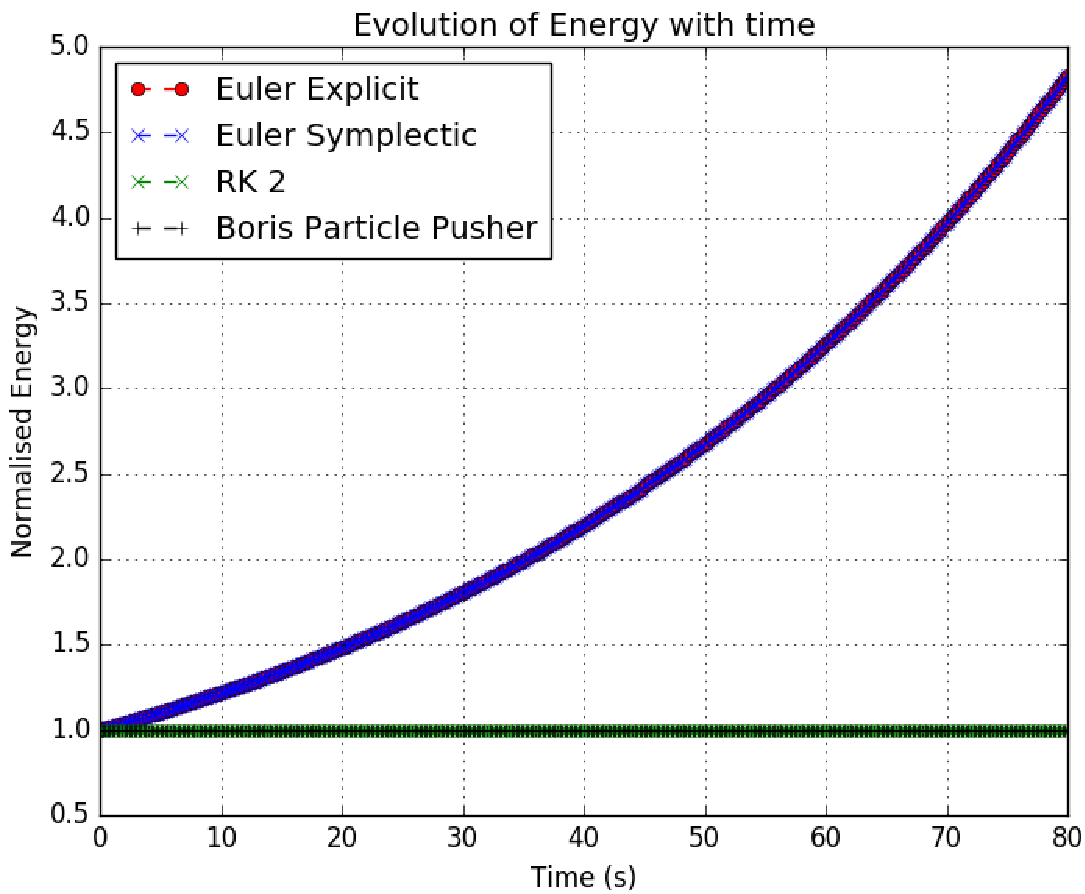
0.2.1 Motion over 1 Cycle :

Figure 3: Evolution of Energy of charged particle



0.2.2 Motion over 4 Cycles :

Figure 4: Evolutionof Energy of charged particle over 4 cycles



As can be clearly seen, Euler Explicit and Euler Symplectic Numerical Schemes lead to increasing energy of the particle while Boris Particle Pusher method conserves the particle energy.For RK-2 scheme, the energy is almost conserved with a barely noticable increase in energy.

0.3 Error Analysis :

The Phase and 'Dissipation' error for all the Numerical Schemes have been analysed below

0.3.1 Euler Explicit :

The Scheme is given by :

$$\bar{V}^{n+1} = \bar{V}^n + \frac{q}{m}\left(E(\bar{X}^n) + \bar{V}^n x B(\bar{X}^n)\right)\Delta t \tag{1}$$

$$\bar{X}^{n+1} = \bar{X}^n + \bar{V}^n \Delta t \tag{2}$$

Where

1. $mass$ is the mass of the particle
2. q is the charge of the particle
3. $\bar{V}^{n+1}$ is the Velocity at the current time step
4. $\bar{V}^n$ is the Velocity at the previous time step
5. $\bar{X}^{n+1}$ is the Position at the current time step
6. $\bar{X}^n$ is the Position at the previous time step
7. $\bar{E}$ is the Electric Field
8. $\bar{B}$ is the Magnetic Field

Phase Error :

For the present analysis, $\bar{E} = 0$ and $\bar{B} = B_0\hat{e}_z$. By taking Cross Product of Eq(1) with $\bar{V}^n$, we arrive at the following relation

$$\Delta\theta = \arcsin(\Delta\theta_0) \tag{3}$$

Where

1. $\Delta\theta$ is the local phase change over a time step ,obtained numerically
2. $\Delta\theta_0 = \omega_c\Delta t$ is theoretical change in phase due to cyclotron motion.
3. ω_c is the cyclotron frequency

Thus we observe that

$$\Delta\theta = \Delta\theta_0 + O(\Delta t^3) \tag{4}$$

Energy Error :

The error in energy over a time step Δt is given by :

$$\frac{m}{2}\left((\bar{V}^{n+1})^2 - (\bar{V}^n)^2\right) = \frac{qB_0^2|\bar{V}^n|^2(\Delta t)^2}{2m} \tag{5}$$

Thus we observe that error in energy is $O(\Delta t^2)$

0.3.2 Euler Symplectic :

The Scheme is given by :

$$\bar{V}^{n+1} = \bar{V}^n + \frac{q}{m}\left(E(\bar{X}^n) + \bar{V}^n x B(\bar{X}^n)\right)\Delta t \tag{6}$$

$$\bar{X}^{n+1} = \bar{X}^n + \bar{V}^{n+1}\Delta t \tag{7}$$

Where

1. $mass$ is the mass of the particle
2. q is the charge of the particle
3. $\bar{V}^{n+1}$ is the Velocity at the current time step
4. $\bar{V}^n$ is the Velocity at the previous time step
5. $\bar{X}^{n+1}$ is the Position at the current time step
6. $\bar{X}^n$ is the Position at the previous time step
7. $\bar{E}$ is the Electric Field
8. $\bar{B}$ is the Magnetic Field

Phase Error :

For the present analysis, $\bar{E} = 0$ and $\bar{B} = B_0\hat{e}_z$. By taking Cross Product of Eq:(5) with $\bar{V}^n$, we arrive at the following relation

$$\Delta\theta = \arcsin(\Delta\theta_0) \tag{8}$$

Where

1. $\Delta\theta$ is the local phase change over a time step ,obtained numerically
2. $\Delta\theta_0 = \omega_c\Delta t$ is theoretical change in phase due to cyclotron motion.
3. ω_c is the cyclotron frequency

Thus we observe that

$$\Delta\theta = \Delta\theta_0 + O(\Delta t^3) \tag{9}$$

Energy Error :

The error in energy over a time step Δt is given by :

$$\frac{m}{2}\left((\bar{V}^{n+1})^2 - (\bar{V}^n)^2\right) = \frac{qB_0^2|\bar{V}^n|^2(\Delta t)^2}{2m} \tag{10}$$

Thus we observe that error in energy is $O(\Delta t^2)$

0.3.3 RK2 :

The Scheme is given by :

$$\bar{K}_{x1} = \bar{V}^n \Delta t \tag{11}$$

$$\bar{K}_{v1} = \bar{F}^n \Delta t \tag{12}$$

$$\bar{K}_{v2} = \Delta t \bar{F}[\bar{V}^n + \bar{K}_{v1}/2, \bar{X}^n + \bar{K}_{x1}/2] \tag{13}$$

$$\bar{K}_{x2} = [\bar{V}^n + \bar{K}_{v1}/2]\Delta t \tag{14}$$

$$\bar{V}^{n+1} = \bar{V}^n + \bar{K}_{v2} \tag{15}$$

$$\bar{X}^{n+1} = \bar{X}^n + \bar{K}_{x2} \tag{16}$$

Where

1. $mass$ is the mass of the particle
2. q is the charge of the particle
3. $\bar{V}^{n+1}$ is the Velocity at the current time step
4. $\bar{V}^n$ is the Velocity at the previous time step
5. $\bar{X}^{n+1}$ is the Position at the current time step
6. $\bar{X}^n$ is the Position at the previous time step
7. $F(X^{\bar{n}}, V^n) = \frac{q}{m}\left(E(\bar{X}^n) + \bar{V}^n x B(\bar{X}^n)\right)$
8. $\bar{E}$ is the Electric Field
9. $\bar{B}$ is the Magnetic Field

Phase Error :

For the present analysis, $\bar{E} = 0$ and $\bar{B} = B_0 \hat{e}_z$. By taking Cross Product of Eq:(13) with $\bar{V}^n$, we arrive at the following relation

$$\Delta\theta = \arcsin(\Delta\theta_0) \tag{17}$$

Where

1. $\Delta\theta$ is the local phase change over a time step ,obtained numerically
2. $\Delta\theta_0 = \omega_c \Delta t$ is theoretical change in phase due to cyclotron motion.
3. ω_c is the cyclotron frequency

Thus we observe that

$$\Delta\theta = \Delta\theta_0 + O(\Delta t^3) \tag{18}$$

Energy Error :

The error in energy over a time step Δt is given by :

$$\frac{m}{2}\left((\bar{V}^{n+1})^2 - (\bar{V}^n)^2\right) = \frac{(qB_0)^4 |\bar{V}^n|^2 (\Delta t)^4}{8m^3} \tag{19}$$

Thus we observe that error in energy is $O(\Delta t^4)$

0.3.4 Boris Particle Pusher :

The Scheme is given by :

$$\bar{V}^- = \bar{V}^n + \frac{q\Delta t E(\bar{X}^n)}{2m} \tag{20}$$

$$\frac{\bar{V}^+ - \bar{V}^-}{\Delta t} = \frac{q}{2m}\left(\bar{V}^+ + \bar{V}^-\right) x B(\bar{X}^n) \tag{21}$$

$$\bar{V}^{n+1} = \bar{V}^+ + \frac{q\Delta t E(\bar{X}^n)}{2m} \tag{22}$$

$$\bar{X}^{n+1} = \bar{X}^n + \bar{V}^{n+1}\Delta t \tag{23}$$

Where

1. *mass* is the mass of the particle
2. q is the charge of the particle
3. $\bar{V}^{n+1}$ is the Velocity at the current time step

4. $\bar{V}^n$ is the Velocity at the previous time step
5. $\bar{X}^{n+1}$ is the Position at the current time step
6. $\bar{X}^n$ is the Position at the previous time step
7. $\bar{E}$ is the Electric Field
8. $\bar{B}$ is the Magnetic Field

Phase Error :

For the present analysis, $\bar{E} = 0$ and $\bar{B} = B_0\hat{e}_z$. By taking Cross Product of Eq:(18) with $\bar{V}^n$, we arrive at the following relation

$$\Delta\theta/2 = \arctan(\Delta\theta_0/2) \tag{24}$$

Where

1. $\Delta\theta$ is the local phase change over a time step ,obtained numerically
2. $\Delta\theta_0 = \omega_c\Delta t$ is theoretical change in phase due to cyclotron motion.
3. ω_c is the cyclotron frequency

Thus we observe that

$$\Delta\theta = \Delta\theta_0 + O(\Delta t^3) \tag{25}$$

Energy Error :

The error in energy over a time step Δt is given by :

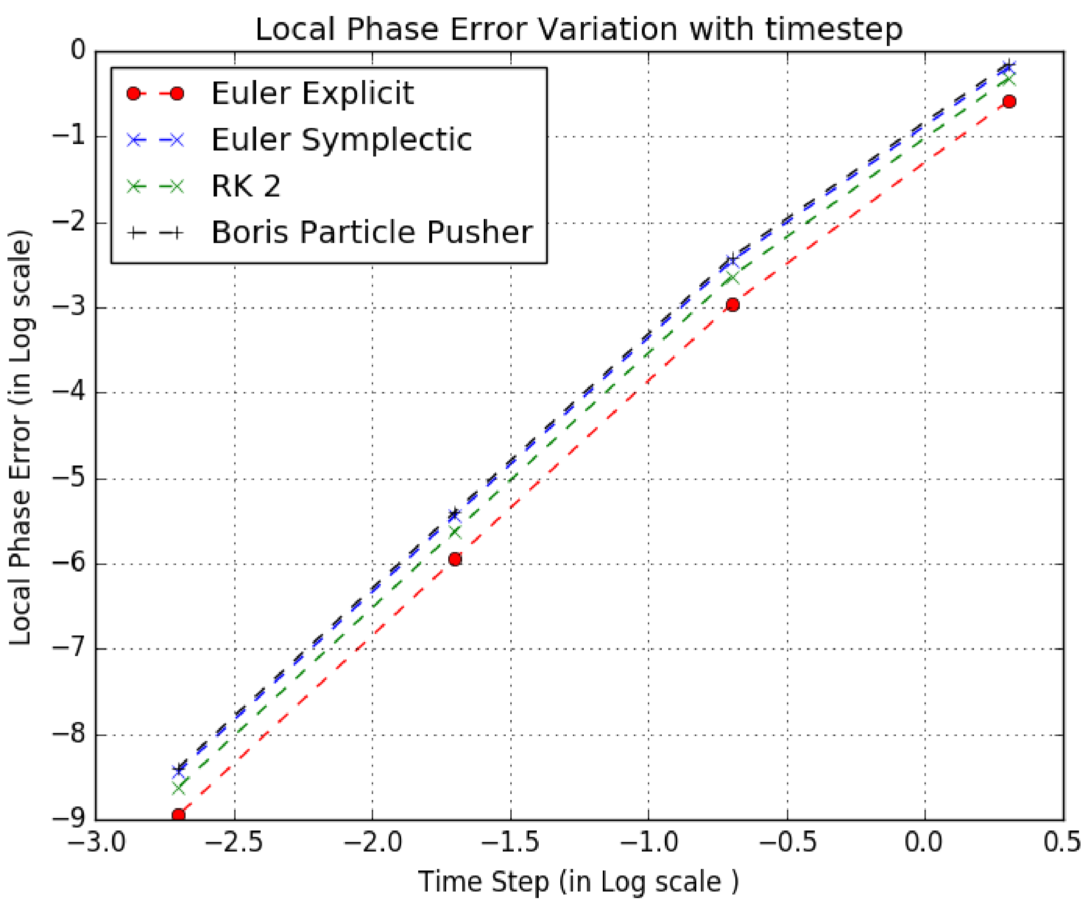
$$\frac{m}{2}\left((\bar{V}^{n+1})^2 - (\bar{V}^n)^2\right) = 0 \tag{26}$$

The above result is directly obtained from Eq :(18) and we observe that error in energy is 0, independent of timestep.

0.3.5 Error Dependency on time step :

Phase Error :

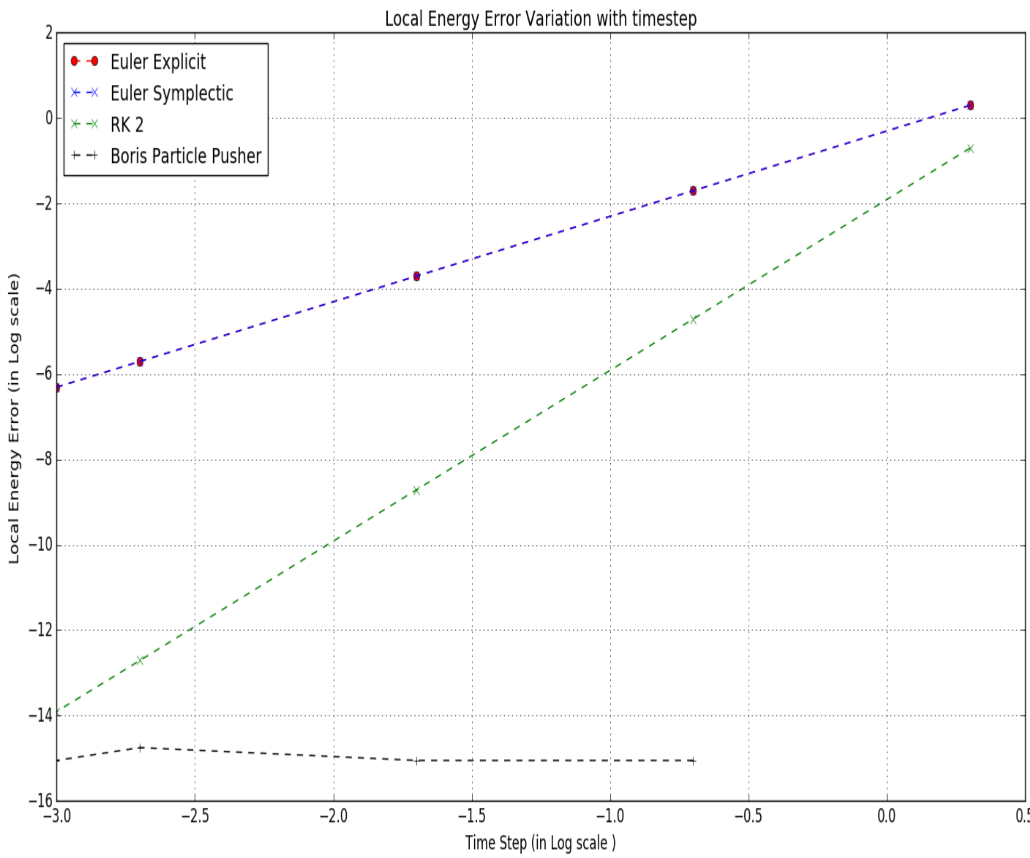
Figure 5: Log plot of Phase Error Vs. Time step



From above figure, we clearly see that the slope of the Phase Error vs. Time step is 3 for all the schemes, thus validating the analytical results arrived at in the previous section.

Energy Error :

Figure 6: Log plot of Energy Error Vs. Time step



From above figure, we clearly see that the slope of the Energy Error vs. Time step is 2 for Euler Explicit and Euler Symplectic Schemes, while the slope is equal to 4 for RK2 and 0 for Boris Particle Pusher schemes.These results are in excellent agreement with the theoretical results obtained in the previous section.

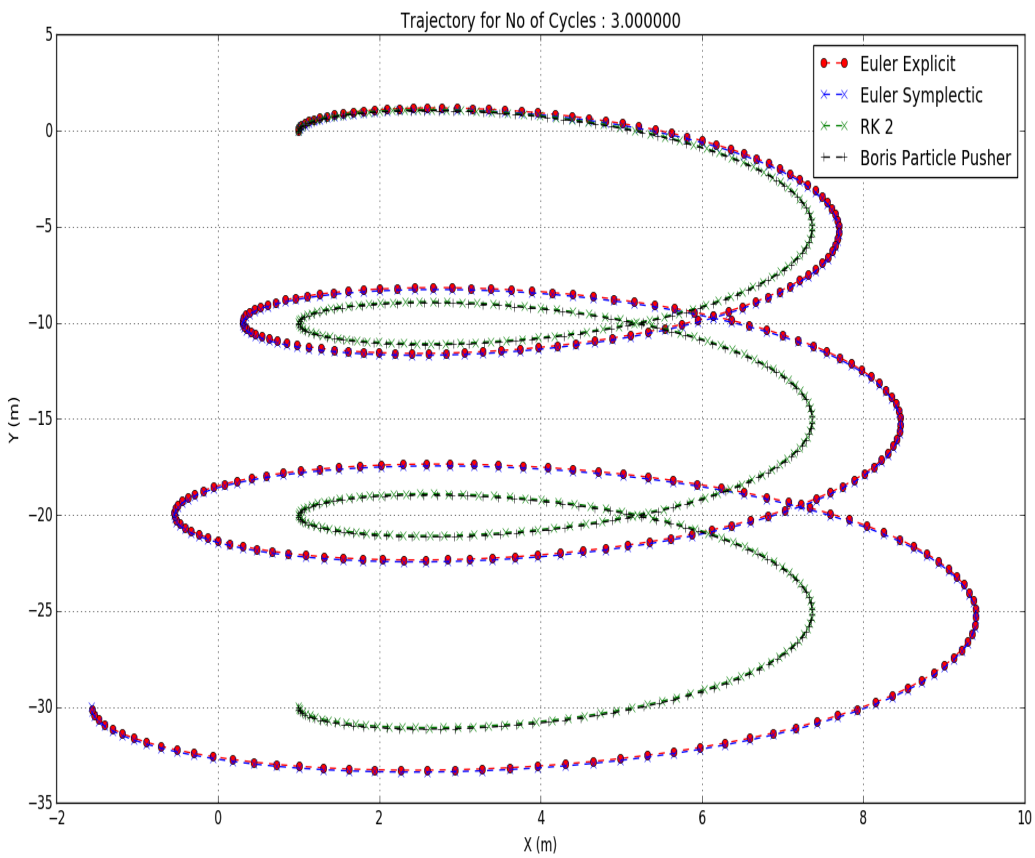
Part II

External Eectric field :

With an additional external electric field, $\bar{E} = E_0\hat{e}_x$, the charged particle along with its cyclotron motion would have a drift velocity given by :

$$\bar{V}_{ExB} = \frac{\bar{E}x\bar{B}}{|B|^2} \tag{27}$$

Figure 7: Particle Trajectory with External Constant E Field



Scheme	V_x	V_y
Euler Explicit	-0.0856	-1.0025
Euler Symplectic	-0.0856	-1.0025
RK2	0	-0.998
Boris	0	-1.00065

Table 1: ExB Drift Velocity obtained from Simulation

For the above simulation , the theoretical Drift Velocity comes out to be : $\bar{V}_{ExB} = -1\hat{e}_y m/s$

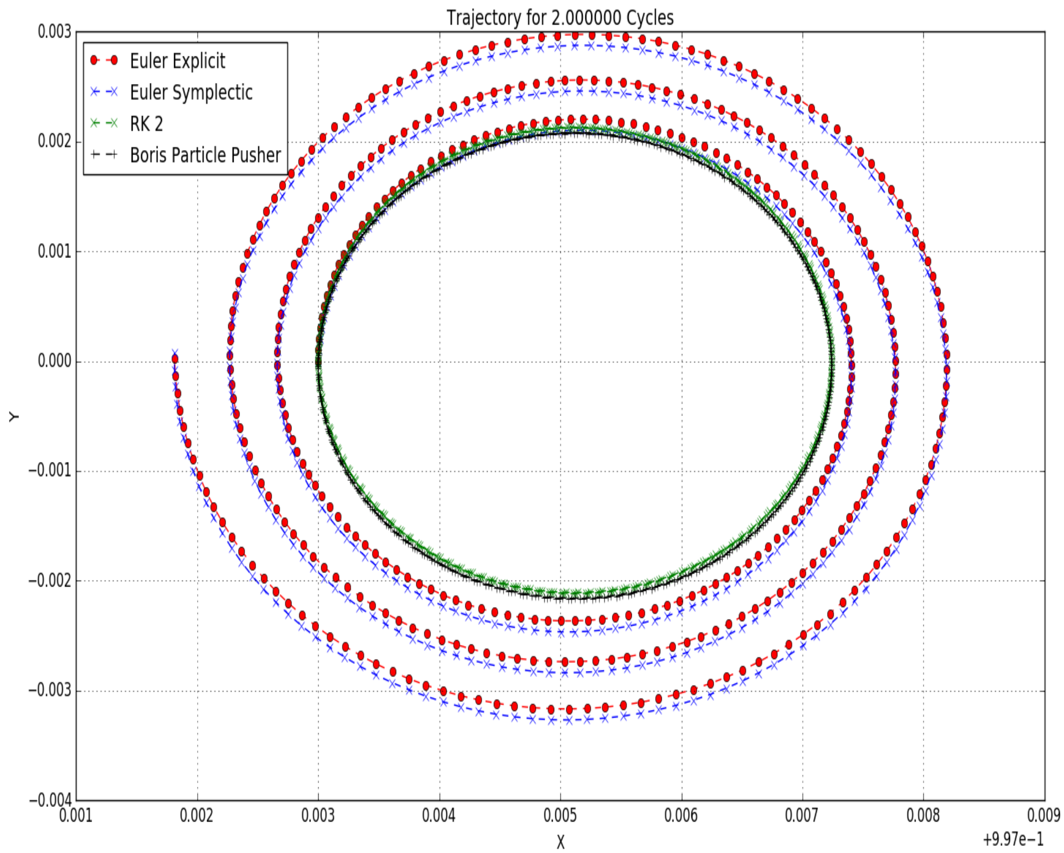
Part III

Gradient of Magnetic field :

With an varying Magnetic field, $B\bar{(x)} = B_0(1 + x/l_b)\hat{e}_z$, the charged particle along with its cyclotron motion would have a drift velocity given by :

$$\bar{V}_{\Delta B} = \frac{mV_0^2(\bar{B}x\bar{\Delta}B}{2q|B|^3} \tag{28}$$

Figure 8: Particle Trajectory with Gradient of Magnetic field Field



Scheme	V_x	V_y
Euler Explicit	-2.96E-5	1.213E-6
Euler Symplectic	-2.97E-5	1.212E-6
RK2	-2.71E-9	1.42E-6
Boris	2.55E-9	8.74E-7

Table 2: Grad B Drift from simulations

For the above simulation , the theoretical Drift Velocity comes out to be : $\bar{V}_{\Delta B} = 7.957 * 10^{-7} \hat{e}_y m/s$

The above results could be obtained only under the following conditions :

1. $\rho_c/l_b = \dfrac{mV}{qB_0} <= 10^{-2}$
2. Time Period $T = \dfrac{2\pi m}{qB_0} >= 200\Delta t$
3. No of Cycles as even, as the simulated motion is out of ohase with the cyclotron motion by a factor of 1.5

Part IV

Conclusion

0.4 Summary :

The trajectory of a charged particle was simulated using the four given Numerical Schemes.Errors in phase and energy for each scheme was analysed and compared with the results from simulation.The dependency of errors on time steps for each numerical scheme was also validated.

The trajectory of a particle under the effect of constant Electric and Magnetic Field was simulated using the given schemes.The corresponding Drift Velocity was also calculated and compared with theoretical results.

The trajectory of a particle under a varying Magnetic Field was simulated using the given schemes.The corresponding Drift Velocity was also calculated and compared with theoretical results.