Potential Buildup on an Electron-Emitting Ionospheric Satellite

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An experiment in which a continuous beam of high-energy electrons is emitted from a satellite (e.g. to produce an artificial aurora or to map the geomagnetic field) will be jeopardized if the satellite potential becomes so high that the beam electrons cannot escape. The potential acquired by the satellite is determined by the condition that it collect from the ambient ionosphere a current equal to the emitted current. The geomagnetic field severely limits the current that may be emitted. For example, a spherical satellite of radius 1.5 meters in the *F* layer cannot emit more than about 40 ma of kilovolt electrons.

INTRODUCTION

This paper is concerned with the potential acquired by a satellite from which a continuous beam of fast electrons is emitted. Such a beam might be required, for example, in an attempt to produce an aurora by artificial means (W. N. Hess, personal communication, 1965). If the emitted particles are to escape, the current collected from the ambient ionosphere plasma must be equal to the emitted current when the satellite potential is less than K/e, where K is the kinetic energy of the emitted particles and e is the magnitude of the electron charge. In the presence of the earth's magnetic field, the current tends more strongly to saturate or level off than it would in the absence of this field. As a consequence, the equilibrium potential of the satellite rises with increasing beam current much more rapidly than it would in the absence of the earth's magnetic field.

Beard and Johnson [1961] calculated the potential to which a spherical satellite would become charged if electrons were emitted. The results of Beard and Johnson cannot be used in the situation of interest here since they do not take explicit account of the geomagnetic field as is done in the present paper. On the other hand, Beard and Johnson give more consideration to the plasma polarization effect. Although the present paper does not treat the satellite-plasma-magnetic field interaction in a completely rigorous and self-consistent manner, some new results are obtained that indicate that the potential acquired by a satellite in the ionosphere may be considerably higher than heretofore expected.

The theory of the drift approximation leads to a differential equation that is solved numerically. For large potentials such that the drift approximation is no longer valid, the constants of the motion (nonrelativistic) of an electron are utilized to obtain a rigorous analytic bound on current collection. It is shown that, to collect a given current I, the sphere potential must exceed a certain minimum value, proportional to I^2. This minimum value is independent of the form of the potential distribution. Conversely, for a given potential V, the maximum current that can be collected is proportional to $V^{1/2}$.

ASSUMPTIONS OF THE THEORY

In the ionosphere, electron cyclotron (or gyration) radii are of the order of centimeters, while satellite dimensions are of the order of meters, and collision mean free paths are of the order of kilometers. Therefore, electrons move easily along the magnetic field lines, but they remain essentially 'glued' to the field lines while executing tight spiral motions around them. In the absence of an electric field, the electrons that are collected move within a 'magnetic tube' defined by the bundle of magnetic field lines passing through the satellite (see Figure 1). The cross-sectional area of the tube is slightly larger than the cross-sectional area of the satellite normal to the field lines due to the finite cyclotron radius, but this difference will be neglected here since it is assumed that the cyclotron radius is negligible compared with the satellite radius. The collected electrons may be considered to come from 'infinity' at the ends of the tube. Thus, the electron current that can be collected by a sphere of radius a is the thermal current passing into area $2\pi a^2$, or

$$I_0 = 2\pi a^2 j_0 \tag{1}$$

where j_0 is the component of the ambient mean thermal current density parallel to the magnetic field direction. The current I_0 defined by (1) is the saturation current, i.e. the limiting current for small potentials in the absence of significant mechanisms that cause electrons to move across magnetic field lines.

It is assumed that the magnetic tube that supplies electrons is not depleted. The situation appears to be very complicated, but neglecting depletion effects should give an upper limit to the current that can be collected. The problem is different from that of a passive electrostatic probe in a magnetic field in the laboratory, where depletion is known to cause significant reduction in current [*Chen*, 1965; *Bohm et al.*, 1949]. Under the above assumption, an active electron-emitting satellite collects a current I_0, plus additional current due to mechanisms that cause electrons to move across magnetic field lines. This is in contrast to the case of the passive laboratory probe where these mechanisms constitute the only source of electrons that can be collected.

One mechanism for the transport of electrons across field lines is that of diffusion by stochastic processes. The current contributed by this diffusion is a fraction of I_0, and falls off for strong magnetic fields like B^{-1}, where B is the magnetic field strength [*Bohm et al.*, 1949; *Simon*, 1955; *Dote et al.*, 1964, 1965; *Sugawara and Hatta*, 1965]. The contributions of these stochastic processes will be neglected here, as we are primarily interested in currents much greater than I_0.

Another mechanism for the transport of electrons across field lines is that of drift under the action of a transverse electric field [*Northrop*, 1963; *Bertotti*, 1961]. If the electric field is inhomogeneous and sufficiently strong, the drift can increase the collected current appreciably above the value I_0. This is the mechanism with which we will be exclusively concerned here. We will show that the current is quite insensitive to satellite potential and that appreciable increases in collected current can occur only for very large potentials. Only the collection of electrons will be treated. Also, we will neglect distortions of the magnetic field due to induced currents circulating through the satellite and the surrounding ionosphere plasma. Such distortion effects should be small under the conditions of interest.

Current Collection According to Drift Theory

We choose the cylindrical coordinates r, θ, and z to represent the radial, azimuthal, and axial coordinates of the electron, respectively. The sphere is at the origin, and the magnetic field is uniform and parallel to the z axis. An electron experiences a radial drift normal to the axis of the system [*Bertotti*, 1961]. This drift is given by

$$v_r = -(v_z/m\omega^2)\frac{\partial^2\Phi}{\partial r\,\partial z} \tag{2}$$

where v_r and v_z denote the radial and axial components of the drift velocity of the guiding-center, respectively. The electrostatic potential energy of the electron is denoted by $\Phi(r, z)$, and m and ω denote the mass and the cyclotron frequency eB/mc, respectively. If, for example, the electric field could be approximated by an attractive Coulomb field, then an electron (guiding-center) that moves along a magnetic field line toward the source experiences an inward drift as well. That is, in (2) v_z would be negative, but $\partial^2\Phi/\partial r\partial z$ would also be negative, so that v_r is negative. The radial drift given by (2) may also be obtained from a general drift equation given by *Northrop* [1963]. It is a drift term of higher order than the well-known $E \times B$ transverse drift, which causes the guiding-center to describe a helical path around the axis of the system, on the surface of a cylindrical magnetic tube as shown in Figure 1. The radial drift arises from the variation of the $E \times B$ drift as the electron moves toward the sphere. Equation 2 holds for $|v_r/v_z| < 1$, which is the condition under which the drift approximation may be used, since v_r is a drift of higher order than v_z.

Assuming we have a charged sphere of radius a and that $\Phi(r, z)$ is a given axially symmetric function, the radius of the collection area at infinity r_0 may be obtained from the solution of the equation

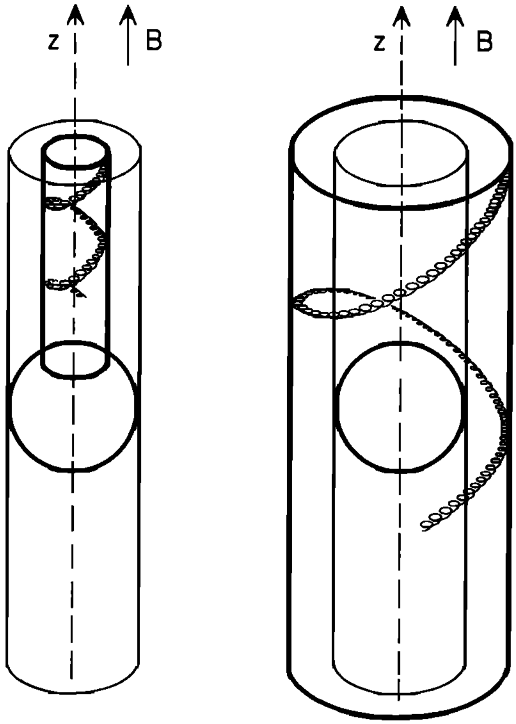

Fig. 1. Electron trajectories for small sphere potentials.

$$\frac{dr}{dz} = -\frac{1}{m\omega^2}\frac{\partial^2\Phi}{\partial r\,\partial z} \quad (3)$$

which follows directly from (2). The desired solution satisfies the condition that $r = a$ for $z = 0$. The solutions have the form depicted in Figure 2. The slope dr/dz vanishes at $z = 0$ and at $z = \infty$. (There is an inflection point that is not evident in Figure 2 since it occurs near $z = 0$.) The solution $r(z)$ increases monotonically from a to r_0 as z goes from zero to infinity. The value r_0 is thus the radius of the collection area at infinity, and the current collected I is given by $(r_0^2/a^2)\,I_0$.

We assume, as an example, that the potential distribution is a Coulomb field due to a sphere of radius a, namely:

$$\Phi(r, z) = -\Phi_0 a/(r^2 + z^2)^{1/2} \quad (4)$$

where Φ_0 is positive and is the magnitude of the potential energy of an electron at the surface of the sphere. (It is not known how well a Coulomb field approximates the true field. The results of the next section, however, are valid for an arbitrary field.) Equation 3 becomes

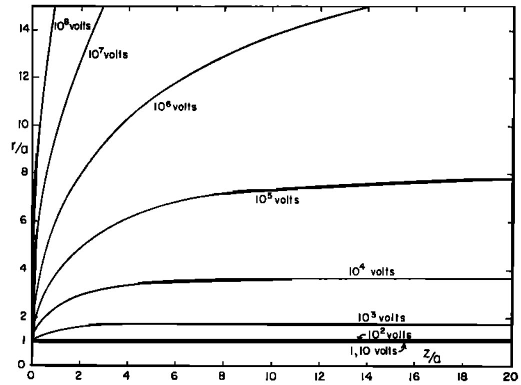

Fig. 2. Solutions of drift equation for a Coulomb field with $r(0) = a = 1.5$ meters. Magnetic field strength $B = 0.45$ gauss. Various sphere potentials.

$$dr/dz = \alpha a^3 rz/(r^2 + z^2)^{5/2} \quad (5)$$

where α is defined by

$$\alpha \equiv 3\Phi_0/(m\omega^2 a^2)$$

$$= 1.71 \times 10^{-3} \times \frac{V(\text{volts})}{[a(\text{meters})B(\text{gauss})]^2} \quad (6)$$

where V(volts) is the potential on the sphere in volts, a(meters) is the sphere radius in meters, and B(gauss) is the strength of the magnetic field in gauss.

Equation 5 has been integrated numerically for several values of sphere potential. The sphere radius and magnetic field strength were chosen to be 1.5 meters and 0.45 gauss, respectively. The results are presented in Table 1 and

TABLE 1. Drift Equation Solutions for Coulomb Potential*

Sphere Potential V, volts	Collection Area Ratio r_0^2/a^2	$r_0/(a\alpha^{1/3})$	$(dr/dz)_{max}$
1	1.0025	6.47	0.0011
10	1.025	3.02	0.011
10^2	1.246	1.55	0.097
10^3	3.107	1.14	0.60
10^4	13.58	1.10	2.2
10^5	62.92	1.10	7.2
10^6	292.0	1.10	22.8
10^7	1355.	1.10	72.1
10^8	6303.	1.10	228.

* $\alpha = 0.00375\ V$, sphere radius = 1.5 meters, magnetic field strength = 0.45 gauss.

in Figure 2. These results show that $r_0/a = (I/I_0)^{1/2}$ becomes proportional to $V^{1/3}$ in the regime of large values of V, i.e. $V > 10^3$ volts. In this regime, $\alpha \gg 1$, and r_0/a may be represented by

$$r_0/a \simeq 1.1\alpha^{1/3} \tag{7}$$

The asymptotic behavior expressed by (7) may be shown to follow from the properties of the differential equation 5. Unfortunately, the convenience afforded by (7) in representing the current ratio $I/I_0 = r_0^2/a^2$ analytically as a function of α, and therefore of V, is mitigated by the fact that the drift approximation breaks down for values of α in excess of a number of the order of unity. This occurs when the particle experiences an appreciable change in the electric field during a gyration period.

When $v_r/v_z > 1$, equation 2, which is based on the drift approximation, becomes invalid since v_r is a drift of higher order than v_z. This corresponds to $dr/dz > 1$ in (5). For large α, it may be shown that (5), together with the boundary condition $r = a$ at $z = 0$, implies that dr/dz achieves its maximum at a value of z small compared with r. This is verified by the computer results and is consistent with Figure 2. For $z/r \ll 1$, the right-hand side of (5) may be written approximately as

$$dr/dz \simeq \alpha a^3 z/r^4 \tag{8}$$

Considering the solution of this equation, we find that the maximum value of dr/dz is $(\alpha/7.2)^{1/2}$. The condition of validity is that α be less than 7.2. Therefore, using (6), we obtain the approximate criterion:

$$V(\text{volts}) < 4200[a(\text{meters})B(\text{gauss})]^2 \tag{9}$$

It may be inferred that the regime of asymptotic behavior (large α) of the solution of (5) is also the regime where the drift approximation breaks down.

For $a = 1.5$ meters and $B = 0.45$ gauss, (9) implies that the critical potential below which the drift approximation is valid is 1900 volts. This is consistent with the computed results. The quantity $(dr/dz)_{max}$ listed in Table 1 denotes, for a given potential, the maximum value of dr/dz in the solution $r(z)$ corresponding to that potential. It is a monotonic function of potential and is less than unity for potentials less than about 2000 volts.

It may be of interest to compare the criterion expressed by (9) with that which would be obtained from the more commonly invoked requirement that the electric field strength be smaller than the magnetic field strength. This requirement is also connected with the validity of the drift approximation, but is less sharply defined than the requirement leading to (9). Taking the maximum electric field to be that at the surface of the sphere, we obtain for a Coulomb field the criterion $V < 3.0 \times 10^4\ aB$, where V, a, and B are expressed in the same units as in (9). Equation 9, which is the appropriate criterion, is also seen to be the more conservative one, assuming aB of the order of 1 meter-gauss.

For large potentials such that the drift approximation breaks down, it is still possible to obtain a rigorous limit for the current that can be collected. This limit is derived in the next section.

Rigorous Dynamical Bounds on Current Collection

The integrals of the motion under the conditions of the previous section may be used to obtain the shape of the allowed domain in r-z space that contains the trajectory of the electron. For a given sphere potential, sphere radius, and magnetic field strength, a necessary condition for the electron to be collected is that the domain boundary must include a portion of the sphere surface. This condition is not sufficient, since the electron may come from infinity, make many passes in the vicinity of the sphere without hitting it, and pass out again to infinity [*Parker*, 1966]. Thus the question of sufficiency is difficult, but the condition of necessity is easily answered and is of interest since it gives an unambiguous lower bound on the sphere potential required to collect current from a given cross-sectional area at infinity. The nonrelativistic equations will be considered, so that sphere potentials are restricted to values somewhat under 10^6 volts.

We assume that at infinity the electron is on a magnetic field line a distance r_0 from the axis and that its kinetic and potential energies are zero. This leads to a simplification that does not change the essential character of the problem. The constants of the motion are

$$C_1 = r^2(\dot{\theta} + \omega/2) \tag{10}$$

and

$$C_2 = \frac{m}{2}(\dot{r}^2 + \dot{z}^2 + r^2\dot{\theta}^2) + \Phi(r, z) \tag{11}$$

corresponding to conservation of canonical angular momentum and energy, respectively. The dots signify time derivatives, and ω denotes eB/mc. Under the stated conditions, $\dot{r} = 0$ whenever:

$$\left[\frac{r}{r_0} - \frac{r_0}{r}\right]^2 = \frac{8}{m\omega^2 r_0^2}\left[-\Phi(r, z) - \frac{m\dot{z}^2(r, z)}{2}\right] \tag{12}$$

The relation between r and z expressed in (12) defines the boundary of the allowed domain in r–z space that contains the trajectory. There is no real solution unless Φ, the potential energy of the electron, is negative. The square bracket on the right-hand side of (12) is the kinetic energy associated with the transverse motion of the electron. Far from the satellite, the right-hand side is small, and the allowed domain consists of a thin cylindrical 'magnetic shell' of radius r_0. At all finite values of z, the right-hand side of (12) is finite, and there are two roots, namely, $r_{out}'(z)$ greater than r_0, and $r_{in}'(z)$ less than r_0. Thus the outer and inner surfaces of the shell are described by the functions $r_{out}'(z)$ and $r_{in}'(z)$, respectively.

The necessary condition for electron collection is that the curve $r'_{in}(z)$ intersect the curve $r^2 + z^2 = a^2$, i.e. the surface of the sphere. Now $\dot{z}^2(r, z)$ is not known analytically. However, if we set $\dot{z} = 0$ in (12), we obtain a shell that is thicker than before, i.e. we obtain new functions $r_{out}(z) > r'_{out}(z)$ and $r_{in}(z) < r'_{in}(z)$. These functions are illustrated in Figure 3, for the case where $r_{in}(0) = a$ and $r_0 = 2a$. Thus setting $r = a$ and $\Phi = -\Phi_0 < 0$ in (12) with $\dot{z} = 0$ yields a lower limit on the potential required such that an electron initially at r_0 is collected. The resulting criterion may be expressed in the form

$$I/I_0 = r_0^2/a^2 < 1 + [8\Phi_0/(m\omega^2 a^2)]^{1/2} \tag{13}$$

Conversely, (13) may be considered as an upper limit on the current that may be collected when the potential is given. The radicand in (13) is identical to $8\alpha/3$, where α is defined by (6). Using (6), we may express (13) as

$$I/I_0 = r_0^2/a^2 < 1 + [4.56 \times 10^{-3} \cdot V(\text{volts})/a^2(\text{meters})B^2(\text{gauss})]^{1/2} \tag{14}$$

Note that (14) holds regardless of the form of the potential $\Phi(r, z)$. In Figure 4, the right-hand side of (14) is plotted, assuming $a = 1.5$ meters and $B = 0.45$ gauss. On the same graph, the computer results given in Table 1 are also plotted. The two curves cross at $V \simeq 3400$ volts and $I/I_0 \simeq 6.8$. For values of V less than 3400 volts, the drift approximation gives the smaller current, so that (14) is satisfied. For values of V greater than 3400 volts, the computed curve lies above the limit given by (14). The correct value of the current collected is given in Figure 4 by the solid curve well to the left of 3400 volts and by an unknown curve that lies below the solid curve well to the right of 3400 volts.

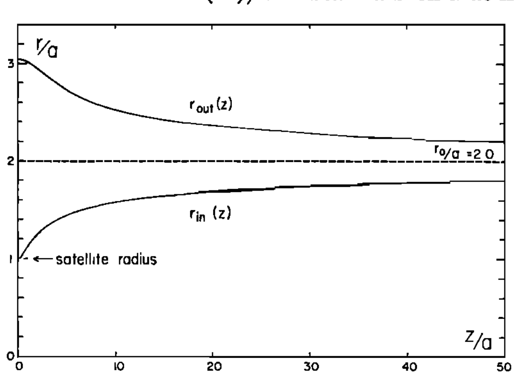

Fig. 3. Allowed domain for electron trajectory in r-z space.

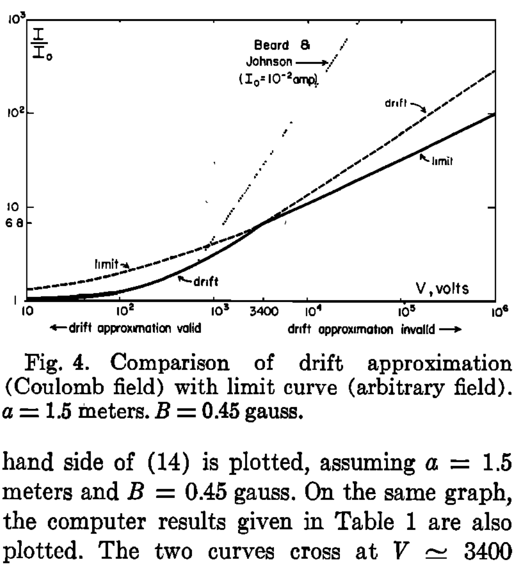

Fig. 4. Comparison of drift approximation (Coulomb field) with limit curve (arbitrary field). $a = 1.5$ meters. $B = 0.45$ gauss.

When the computed values of $(dr/dz)_{max}$ (Table 1) are plotted against potential, $(dr/dz)_{max}$ is found to be between 1.2 and 1.3 at a potential of 3400 volts. Thus, the breakdown of the drift approximation, $(dr/dz)_{max} > 1$, occurs in the neighborhood of the potential at which the drift approximation violates (14). This potential is within a factor of 2 of that given by the approximate analytical formula (9) for the critical potential (1900 volts).

Conclusions

For a satellite in the ionosphere, the current collected is a function of the quantity $V/(a^2B^2)$, where V is the satellite potential, a is the satellite radius, and B is the magnetic field strength. As Figure 4 shows, for a given magnetic field strength the collected current I is bounded at large potentials by a limiting curve that is proportional to $V^{1/2}$, where V is the satellite potential. If we assume a spherical satellite of radius 1.5 meters and a mean thermal electron current density of 0.7×10^{-3} ampere/meter^2 in the F layer, than $I_0 \simeq 10^{-2}$ ampere. According to Figure 4, where $B = 0.45$ gauss, no more than about 40 milliamperes of kilovolt electrons can be emitted. Currents of the order of amperes can be emitted only if the electrons have energies in excess of 10^6 volts.

Figure 4 may be used for an arbitrary sphere radius a in meters and magnetic field strength B in gauss if the sphere potential V in volts is replaced by $(2.19\ a^2B^2)V$. For example, if the product aB is doubled, the current ratio I/I_0 remains unchanged if V is increased by a factor of 4.

It is of interest to compare these results with those expressed by equation 44 of *Beard and Johnson* [1961]. For large potentials, the formula of Beard and Johnson may be written:

$$I \simeq 0.8 \times 10^{-6}(Va)^{3/2} \qquad (15)$$

where I is in amperes, V is in volts, and a is in meters. This formula is independent of B and of I_0, and does not 'saturate' since I increases more rapidly than V. Using (15) for I, with $a = 1.5$ meters, the ratio I/I_0 with $I_0 = 10^{-2}$ ampere is shown in Figure 4 as a dotted line. For potentials greater than about a kilovolt, (15) is seen to underestimate considerably the potential required to collect a given current.

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