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Current to a moving cylindrical electrostatic probe

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The current collection characteristics of a moving cylindrical Langmuir probe are evaluated for a range of probe speeds and potentials which are applicable to earth and planetary measurements. The current expressions derived include the cases of the general accelerated current, sheath area limited current, orbital motion limited current, and retarded current. For the orbital motion limited current, a simple algebraic expression is obtained which includes and generalizes the Mott-Smith and Langmuir expressions for both a stationary probe and a rapidly moving probe. For a rapidly moving probe, a single formula adequately represents both the accelerated and the retarded current.

INTRODUCTION

Cylindrical electrostatic probes have been widely employed on rockets and satellites for charged particle temperature and density measurements.[1,2] In the future these devices may be used on planetary missions. A common feature of the planetary and earth electrostatic probes is the rapid vehicle velocity relative to the particle thermal velocity. The ratio of probe speed to thermal speed is denoted as the speed ratio. Typical speed ratios for 1000 °K ions in the mass range 1—28 lie between 2 and 10.4 for a probe speed of 8 km/sec (earth orbit) and between 12 and 65 for a probe speed of 50 km/sec (Jupiter entry probe). Speed ratios for 1000 °K electrons vary from 0.046 to 0.29 for probe speeds between 8 and 50 km/sec.

Mott-Smith and Langmuir[3] derived two separate cylinder current expressions, one for a stationary probe and the other for a rapidly moving probe. Kanal[4] derived some integral formulas valid for arbitrary probe speed and from them, power series expressions which are valid only for small speed ratios. Bettinger[5] pointed out the need to develop a current expression which fills the void between the formulas for the stationary and the rapidly moving probes. In this paper we derive a cylinder probe current expressions which are valid for a wide range of speed ratios and potentials. The current expressions are presented in a form which allow easy and rapid numerical evaluation

BASIC CURRENT EXPRESSIONS

We review briefly the derivation of the integral expression for the current to a moving cylindrical probe. All current formulas are normalized with respect to the random probe current and are denoted I,

$$I = i/i_{\text{random}}, \tag{1}$$

where i is the actual current, and the random probe current is given by

$$i_{\text{random}} = ANq(kT/2\pi m)^{1/2}, \tag{2}$$

where A is the probe area and N, q, T, and m are, respectively, the particle density, charge, temperature, and mass.

The probe is assumed to behave as an idealized cylindrical Langmuir probe with a coaxial sheath of charged particles which is unperturbed by the relative probe-plasma drift motion. It is interesting to point out that the current expressions derived here are correct in the limit of high drift velocity—this is probably related to the fact that most of the particles collected by the probe came from the undisturbed portion of the sheath in the bow region. The probe is either guarded or is long enough so that end effects can be neglected. We also assume that no collisions take place within the sheath, and that outside the sheath the particles are not influenced by the presence of the probe. Particles which reach the collector surface are assumed to be collected. Under these conditions, the cylinder probe current is given by

$$I = 2\sqrt{\pi}\int_0^{2\pi}(d\alpha/2\pi)(a/r)\int d\mathbf{c}\,\mathbf{c}\cdot\hat{n}f(\mathbf{c}). \tag{3}$$

where a is the sheath radius, r the probe radius, $\hat{n}$ a unit normal vector on the sheath edge, α the azimuthal angle specifying the location of an area element of the cylindrical sheath, and $f(\mathbf{c})$ is the Maxwell-Boltzmann distribution function in a frame fixed on the probe:

$$f(\mathbf{c}) = \pi^{-3/2}\exp[-(\mathbf{c}-\mathbf{s})^2], \tag{4}$$

where $\mathbf{c}$ and $\mathbf{s}$ are, respectively, the particle and drift velocities normalized by the most probable particle velocity:

$$\mathbf{c} = \mathbf{v}(m/2kT)^{1/2}, \quad \mathbf{s} = \mathbf{w}(m/2kT)^{1/2}. \tag{5}$$

The domain of integration in Eq. (3) is determined from the condition that the particles reach the probe surface using the conservation of energy and angular momentum. In terms of the velocity components c_z, c_t, c_n, respectively along the probe axis, tangent to the probe axis and the area element at α, and normal to the area element at α, the velocity limits are for retarded particles

$$-\left(\frac{c_n^2 - V}{a^2/r^2 - 1}\right)^{1/2} < c_t < \left(\frac{c_n^2 - V}{a^2/r^2 - 1}\right)^{1/2},$$

$$-\infty < c_z < \infty, \quad \sqrt{V} < c_n < \infty,$$

for accelerated particles

$$-\left(\frac{c_n^2 + V}{a^2/r^2 - 1}\right)^{1/2} < c_t < \left(\frac{c_n^2 + V}{a^2/r^2 - 1}\right)^{1/2},$$

$$-\infty < c_z < \infty, \quad 0 < c_n < \infty,$$

where $V = |e\phi/kT|$ is the normalized across the sheath potential. The four-dimensional integral is reduced to a single integral using the change of variable $c_n = \xi\cos\psi$, $c_t = \xi\sin\psi$ and integration over α, c_z, and ψ:

$$I_{\text{ret}} = \frac{4}{\sqrt{\pi}}\int_{\sqrt{V}}^{\infty}\xi\,d\xi(\xi^2 - V)^{1/2}\exp[-(\xi^2 + s^2)]I_0(2\xi s) \tag{7}$$

for retarded particles, and

$$I_{\text{acc}} = \frac{4}{\sqrt{\pi}}\int_{[V(a^2/r^2-1)^{-1}]^{1/2}}^{\infty}\xi\,d\xi(\xi^2 + V)^{1/2}\exp[-(\xi^2 + s^2)]I_0(2\xi s).$$

$$+\frac{4}{\sqrt{\pi}}\frac{a}{r}\int_0^{[V(a^2/r^2-1)^{-1}]^{1/2}} \xi^2\, d\xi \exp[-(\xi^2+s^2)]I_0(2\xi s), \tag{8}$$

for accelerated particles. In Eqs. (7) and (8) s is the component of the speed ratio in the plane perpendicular to the probe axis.

Two approximations to the accelerating current are of general interest—the sheath area limited current and the orbital motion limited current. The current is approximated by the sheath area limited current when a/r is close to unity so that all particles entering the sheath are collected. The current is then independent of the across the sheath potential and depends only on s and a/r. To obtain this current, the velocity domain (6b) is modified to allow all values of the tangential velocity, $-\infty < c_t < \infty$:

$$I_{\text{sal}} = (4/\sqrt{\pi})(a/r)\int_0^\infty \xi^2\, d\xi \exp[-(\xi^2+s^2)]I_0(2\xi s). \tag{9}$$

The general accelerated current approaches the orbital motion limited current when the sheath is much larger than the probe radius. In this limit, $a/r \to \infty$, the current depends only on V and s:

$$I_{\text{oml}} = (4/\sqrt{\pi})\int_0^\infty \xi\, d\xi\, (\xi^2+V)^{1/2} \exp[-(\xi^2+s^2)]I_0(2\xi s). \tag{10}$$

The integral current expressions, Eqs. (7)—(10), are the starting point for the derivation of power series representations for the current which are discussed in the next section.

MOVING CYLINDER CURRENT EQUATIONS

The current collection characteristics for the moving cylinder probe are presented in various forms for a range of values of normalized potential V and speed ratio s. The derivations and details are found in the Appendix starting with Eqs. (A11)—(A16).

Sheath area limited current

The sheath area limited current can be expressed as a single confluent hypergeometric function:

$$I_{\text{sal}} = (a/r)\phi(-\tfrac{1}{2}, 1; -s^2), \tag{11a}$$

where ϕ is defined by Eq. (A8). An alternative form has been given by Heatley[6] and Kanal[4]:

$$I_{\text{sal}} = (a/r)\exp(-\tfrac{1}{2}s^2)[(1+s^2)I_0(\tfrac{1}{2}s^2) + s^2 I_1(\tfrac{1}{2}s^2)]. \tag{11b}$$

For small values of the speed ratio s, the power series representation of ϕ may be used to represent I_{sal} [Eq. (A8)]. For large s, the asymptotic form of ϕ [Eq. (A17)] gives the representation

$$I_{\text{sal}} \approx \frac{a}{r}\frac{2}{\sqrt{\pi}} s \sum_{n=0}^{M} \frac{(-\frac{1}{2})_n(-\frac{1}{2})_n}{n!}\left(\frac{1}{s^2}\right)^n$$

$$= \frac{a}{r}\frac{2}{\sqrt{\pi}} s \left\{1 + \frac{1}{4s^2} + \frac{1}{32s^4} + \frac{3}{128s^6} + \cdots\right\}, \quad s \text{ large.} \tag{12}$$

Retarded current

The retarded current is initially given as a double series in V and s:

$$I_{\text{ret}} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-s^2)^n(-V)^m\Gamma(n+m-\frac{1}{2})}{n!\,m!\,\Gamma(m-\frac{1}{2})\Gamma(n+1)}. \tag{13}$$

Expressing the series in V as a confluent hypergeometric function and applying a Kummer transformation [Eq. (A21)], we obtain the retarded current in the form of an exponential in V (the retarded current for a stationary probe is e^{-V}) multiplied by a power series in the speed ratio squared:

$$I_{\text{ret}} = e^{-V}\sum_{n=0}^{\infty}\binom{n-\frac{3}{2}}{n}\frac{(-s^2)^n}{n!}\,\phi\left(-n, -\frac{1}{2}; V\right), \tag{14a}$$

$$I_{\text{ret}} = e^{-V}\sum_{n=0}^{\infty}\frac{(-s^2)^n}{n!}L_n^{(-3/2)}(V) \tag{14b}$$

$$I_{\text{ret}} = e^{-V}\left[1 + \left(\frac{1}{2}+V\right)s^2 + \left(\frac{V^2}{4} - \frac{V}{4} - \frac{1}{16}\right)s^4 + \cdots\right],$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial. Equation (14) is applicable for small values of s and arbitrary values of V. This representation of the retarded current is particularly useful in describing the retarded electron current to satellites and planetary probes.

With the speed ratio s kept fixed and small, the retarded current decreases exponentially with increasing potential V; however, when the speed ratio is allowed to increase, with V kept large and fixed, the current increases and loses its exponential character. The behavior is illustrated in the formula of Kanal which may be obtained from Eq. (14a) [see the Appendix, Eqs. (A22)—(A24)]:

$$I_{\text{ret}} = \exp[-(V+s^2)]\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{3}{2})}{n!\,\Gamma(\frac{3}{2})}I_n(2s\sqrt{V})\left(\frac{s}{\sqrt{V}}\right)^n. \tag{15}$$

As s approaches $\sqrt{V}$ with V large, the modified Bessel function behaves asymptotically as

$$\exp(2s\sqrt{V})/(4\pi s\sqrt{V})^{1/2},$$

so that the exponential behavior is $\exp[-(\sqrt{V}-s)^2] \to 1$ as $s \to \sqrt{V}$.

Another representation for the retarded current [see the Appendix, Eqs. (A25)—(A29)] is:

$$I_{\text{ret}} = \frac{\exp[-(s-\sqrt{V})^2]}{(4\pi s\sqrt{V})^{1/2}}\sum_{n=0}^{R}\frac{(1/4s\sqrt{V})^n}{n!}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n$$

$$\times\left(1-\frac{s}{\sqrt{V}}\right)^{-2n-3/2}\psi_n\left(\frac{s}{\sqrt{V}}\right), \quad s\sqrt{V} \text{ large}, \quad s<\sqrt{V} \tag{16}$$

where $\psi_n(x)$ is a polynomial [Eqs. (A28) and (A29)].

Note that because the potential is retarding, the current is small but not exponentially small as $s \to \sqrt{V}$ from below. There is an abrupt change in behavior as the boundary $s = \sqrt{V}$ is crossed; for $s > \sqrt{V}$ the drift motion overwhelms the retarding potential and the current becomes large as s increases. Again the exponential potential dependence is masked by the high drift velocity. The expression for $\sqrt{V} < s$ is obtained by summing Eq. (13) over s:

$$I_{\text{ret}} = \sum_{m=0}^{\infty}\frac{(-V)^m}{m!}\,\phi\left(m-\frac{1}{2}, 1; -s^2\right). \tag{17}$$

The formulas for evaluating ϕ are given in the Appendix in Eqs. (A30). An asymptotic form for $s > \sqrt{V}$, s large, is [see the Appendix, Eqs. (A31)–(A34)]:

$$I_{\rm ret} \approx \frac{2}{\sqrt{\pi}} s \sum_{n=0}^{M} \left(\frac{1}{s^2}\right)^n \frac{(-\frac{1}{2})_n(-\frac{1}{2})_n}{n!} \left(1 - \frac{V}{s^2}\right)^{1/2-2n} P_n\left(\frac{V}{s^2}\right), \quad s > \sqrt{V},\ s \text{ large} \qquad (18)$$

where $P_n(x)$ is a polynomial of order n [Eqs. (A33 and (A34)]. The asymptotic behavior of the retarded current is

$$(2/\sqrt{\pi})(s^2 - V)^{1/2}.$$

Note added in proof: An additional equation is required to evaluate the retarded current between the regions $\sqrt{V} < s$ and $\sqrt{V} > s$ for large s:

$$I_{\rm ret} = \frac{\sqrt{s}}{2} \sum_{n,p=0}^{N,P} \left(\frac{\epsilon/s}{n!}\right)^n \left(\frac{1/s}{p!}\right)^p \frac{\Gamma(\frac{5}{4} + \frac{1}{2}(p-n))}{[\Gamma(\frac{5}{4} - \frac{1}{2}(p+n))]^2},$$

where $\epsilon = s^2 - V$. This formula is obtained from the power series in ϵ and s^2 using the Mellin-Barnes integrals.[8]

Orbital motion limited current

The orbital motion limited current is given by the double series [Eqs. (A11), (A12), (A16)]:

$$I_{\rm oml} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-s^2)^n V^m}{n!\,m!} \left(\frac{\Gamma(n+m-\frac{1}{2})}{\Gamma(m-\frac{1}{2})\Gamma(n+1)} - \frac{V^{3/2}\Gamma(n+m+1)}{\Gamma(m+\frac{5}{2})\Gamma(n+1)} \right). \qquad (19)$$

The formula suitable for small speed ratios is obtained by summing over index m:

$$I_{\rm oml} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (s^2)^n \psi\left(n - \frac{1}{2}, -\frac{1}{2}; V\right), \qquad (20)$$

where ψ is given by Eqs. (A7) and (A35). The first term in the speed ratio expansion [Eq. (20)] is the classical orbital motion limited formula of Mott-Smith and Langmuir.[3] For large values of V, the asymptotic series [Eqs. (A36)–(A41)] is appropriate:

$$I_{\rm oml} \approx \frac{2}{\sqrt{\pi}} (V + s^2)^{1/2} \sum_{m=0}^{R} \left(-\frac{1}{V}\right)^m \left(-\frac{1}{2}\right)_m \left(1 + \frac{s^2}{V}\right)^{1/2-2m} Q_m\left(-\frac{s^2}{V}\right), \quad s^2 < V \qquad (21)$$

where the polyonomial Q_m is given by Eqs. (A40) and (A41).

For small values of the potential, another representation is obtained by summing Eq. (19) over index n:

$$I_{\rm oml} = \sum_{m=0}^{\infty} \frac{V^m}{m!} \left[\phi\left(m - \frac{1}{2}, 1; -s^2\right) - \frac{V^{3/2}\Gamma(m+1)}{\Gamma(m+\frac{5}{2})} \phi(m+1, 1; -s^2) \right]. \qquad (22)$$

The first term is identical to the retarded current expression with V replaced by $-V$. The evaluation of $\phi(m-\frac{1}{2}, 1; -s^2)$ is given by Eq. (A30) and $\phi(m+1, 1; -s^2)$ by Eq. (A42). For small values of the speed ratio s, the retarded current and the orbital motion limited current are not identical under the transform $V \to -V$. However, for high s values, the second term in Eq. (22) decays rapidly as $\exp(-s^2)$ and therefore

$$I_{\rm oml}(V) \approx I_{\rm ret}(-V), \quad s \text{ large}. \qquad (23)$$

Consequently, the asymptotic orbital motion limited current for $s^2 > V$ is obtained from Eq. (18) with $V \to -V$:

$$I_{\rm oml} \approx \frac{2}{\sqrt{\pi}} (s^2 + V)^{1/2} \sum_{n=0}^{R} \left(\frac{1}{s^2}\right)^n \frac{(-\frac{1}{2})_n(-\frac{1}{2})_n}{n!} \left(1 + \frac{V}{s^2}\right)^{-2n} P_n\left(-\frac{V}{s^2}\right), \quad s^2 > V. \qquad (24)$$

In the Appendix it is demonstrated that Eqs. (21) and (24) are term by term identical [Eqs. (A43)–(A45)]. Therefore, these asymptotic expansions may both be written as

$$I_{\rm oml} = \frac{2}{\sqrt{\pi}} (V + s^2)^{1/2} \sum_{k=0}^{R} \frac{A_k}{(V+s^2)^k} \times \sum_{l=0}^{k} B(k,l) \left(\frac{V}{V+s^2}\right)^l \left(\frac{s^2}{V+s^2}\right)^{k-l}, \qquad (25)$$

where

$$A_k = \frac{(-\frac{1}{2})_k(-\frac{1}{2})_k}{k!},$$

$$B(k,l) = (-)^l \frac{(-k)_l(-k)_l}{l!\,(-\frac{1}{2})_l}.$$

Writing out the first few terms of Eq. (25), we have

$$I_{\rm oml} \approx \frac{2}{\sqrt{\pi}} (V + s^2)^{1/2} \left\{ 1 + \frac{\frac{1}{4}}{V+s^2}\left(\frac{s^2}{V+s^2} + \frac{2V}{V+s^2}\right) + \frac{\frac{1}{32}}{(V+s^2)^2}\left[\left(\frac{s^2}{V+s^2}\right)^2 + \frac{8Vs^2}{(V+s^2)^2} - \frac{8V^2}{(V+s^2)^2}\right] + \cdots \right\}. \qquad (26)$$

To first order in $1/(V+s^2)$ this form is identical with the expression

$$I_{\rm oml} \approx \frac{2}{\sqrt{\pi}} \left(V + s^2 + \frac{1}{2}\frac{s^2}{V+s^2} + \frac{V}{V+s^2}\right)^{1/2}, \quad s^2 \text{ large or } V \text{ large}.$$

Formula (27) is the sought-for generalization of the Mott-Smith and Langmuir approximations. For $s = 0$ we obtain the stationary probe current equation

$$I_{\rm oml}\big|_{\rm Eq.\,(27)} = (2/\sqrt{\pi})(1 + V)^{1/2}, \quad s = 0 \qquad (28)$$

and for $s^2 > V$ we obtain the formula

$$I_{\rm oml}\big|_{\rm Eq.\,(27)} = (2/\sqrt{\pi})(V + s^2 + \tfrac{1}{2})^{1/2}, \quad s^2 > V \qquad (29)$$

which generalizes the Mott-Smith and Langmuir formula for high drift velocity by including the random thermal motion in the term $\frac{1}{2}$ [the Mott-Smith and Langmuir formula is $(2/\sqrt{\pi})(V + s^2)^{1/2}$].

General accelerated current

The power series representation for the accelerated cylinder current is obtained from Eqs. (A11)–(A13) but will not be written explicitly. For small values of s, the integral representation is used directly to obtain the representation

$$I_{\rm acc} = \frac{2}{\sqrt{\pi}} \exp(-s^2) \sum_{n=0}^{\infty} \frac{(s^2)^n}{(n!)^2} (f_n + g_n), \qquad (30)$$

where f_n and g_n are given in the Appendix [Eqs. (A46)–(A51)]. No attempt is made to derive an asymptotic form from Eq. (30) for large V since the relation $a/r = a/r(V)$ is unknown. For large values of the speed ratio s the accelerated current reduces to the orbital motion limited current [see the Appendix, Eqs. (A52) and (A53)]:

$$I_{\rm acc} \approx I_{\rm om1}, \quad s \text{ large}. \tag{31}$$

Consequently the current to a rapidly moving cylinder probe is given by a single formula Eq. (24) or Eq. (25) for both accelerated and retarded particles (with V replaced by $-V$ for retarded particles).

SUMMARY

The extensive use of the cylindrical electrostatic probe for *in situ* measurements of charged particles on rockets and earth satellites and its proposed use on planetary probes demonstrate the need to understand the effect of the relative probe to plasma drift velocity on the probe volt-ampere characteristics. To this end we have derived a number of series representations for the current to a moving cylinder probe starting from the four-dimensional integral representations of the current. Four categories of current have been considered: general accelerated current, sheath area limited current, orbital motion limited current, and retarded current. Power series in both s, the speed ratio, and V, the normalized potential, have been given for all four cases. Simplified representations valid for small speed ratio s and arbitrary V as well as representations valid for large s have been presented. These representations fill the gap between the stationary and the rapidly moving probe theories of Mott-Smith and Langmuir.[3] One of the more useful of these is the formula for the orbital motion limited current:

$$I_{\rm om1} = \frac{2}{\sqrt{\pi}}\left(V + s^2 + \frac{V + \frac{1}{2}s^2}{V + s^2}\right)^{1/2}, \tag{32}$$

which is valid for $V + s^2 > 1$. This formula, Eq. (32), can be used to evaluate the accelerated ion current for earth and planetary probes.

Aisenberg has pointed out that the original Mott-Smith and Langmuir work is not completely valid in the ion and electron saturation regions as shown by comparison with experimental probe data. The present work, as well as the Langmuir and Mott-Smith work, does not take into account the effect of plasma depletion in the neighborhood of the probe when the current saturation region is reached; however, at higher speeds this effect is less important. A more rigorous theory of probe current collection entails solving the Boltzmann and Maxwell equations with appropriate probe boundary conditions and is beyond the scope of the present work.

A comparison of the current equations derived here with experimental probe data will be carried out using the series of Atmosphere Explorer satellites[7] which will contain a variety of charged and neutral particle measurements.

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APPENDIX

Basic power series

We derive a power series representation for the cylinder probe current in powers of s^2 and V from which asymptotic formulas valid for high s or V can be obtained. The summation yields series in confluent hypergeometric functions.[8]

The evaluation of the cylinder current as given by Eqs. (7)—(10) reduces to the problem of performing the integral

$$I(q,r,s) = (4/\sqrt{\pi})\int_{q^{1/2}}^{\infty} x\,dx(x^2+r)^{1/2}\exp[-(x^2+s^2)]I_0(2xs). \tag{A1}$$

The dependence of $I(q,r,s)$ on s^2 is simplified by performing a Laplace transform in s^2:

$$\tilde{I}(q,r,p) = \int_0^{\infty} ds^2 \exp(-ps^2)I(q,r,s). \tag{A2}$$

Using the Laplace transform of the modified Bessel function,[9] we obtain the result

$$\tilde{I}(q,r,p) = \frac{2}{\sqrt{\pi}}\frac{1}{1+p}\int_q^{\infty} dz\,(z+r)^{1/2}\exp[-pz/(1+p)]. \tag{A3}$$

This is in the form of a integral representation of the confluent hypergeometric function,[8] $\psi(a,c;z)$. Therefore, using the relation

$$\psi(a,c;z) = z^{1-c}\psi(a-c+1, 2-c; z), \tag{A4}$$

we find

$$\tilde{I}(q,r,p) = \frac{2}{\sqrt{\pi}}\frac{1}{1+p}\exp[-pq/(1+p)]\left(\frac{1+p}{p}\right)^{3/2} \times \psi\left(-\frac{1}{2}, -\frac{1}{2}; \frac{p(q+r)}{1+p}\right). \tag{A5}$$

To facilitate the separation of variables and taking the inverse Laplace transform, we partition I into two parts:

$$I(q,r,s) = I_1(r,s) + I_2(q,r,s), \qquad \tilde{I}(q,r,p) = \tilde{I}_1(r,p) + \tilde{I}_2(q,r,p); \tag{A6}$$

according to the relation between ψ and ϕ,

$$\psi(a,c;z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\phi(a,c;z) + \frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\phi(a-c+1, 2-c; z), \tag{A7}$$

$$\phi(a,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{(c)_n n!}. \tag{A8}$$

The result of the partition is

$$\tilde{I}_1(r,p) = \frac{1}{1+p}\left(\frac{1+p}{p}\right)^{3/2}\exp\left(\frac{pr}{1+p}\right), \tag{A9}$$

$$\tilde{I}_2(q,r,p) = -\frac{1}{1+p}\frac{(r+q)^{3/2}}{\Gamma(\frac{5}{2})}\exp\left(\frac{-pq}{1+p}\right)\phi\left(1, \frac{5}{2}; \frac{p(r+q)}{1+p}\right). \tag{A10}$$

The inverse Laplace transform of $\tilde{I}_1(r,p)$ is obtained by expanding it in a double series in powers of r and $1/p$ and evaluating the residue at $p=0$. The result is a power series in s^2 and r:

$$I_1(r,s)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-s^2)^n r^m\Gamma(n+m-\frac{1}{2})}{n!\,m!\,\Gamma(m-\frac{1}{2})\Gamma(n+1)}. \tag{A11}$$

Similarly $\tilde{I}_2(q,r,p)$ is expanded in a triple series in powers of $(q+r)$, q, and $1/p$. Evaluation of the residue yields

$$I_2(q,r,s)=-(r+q)^{3/2}\times\sum_{n,m,l=0}^{\infty}\frac{(-s^2)^n(-q)^m(r+q)^l\,\Gamma(n+m+l+1)}{n!\,m!\,\Gamma(l+\frac{5}{2})\Gamma(m+l+1)\Gamma(n+1)}. \tag{A12}$$

The four basic currents—accelerated, retarded, sheath area limited, and orbital motion limited—are combinations of I_1 and I_2:

$$I_{\rm acc}=I_1(V,s)+I_2\left(\frac{V}{a^2/r^2-1},V,s\right)-\frac{a}{r}I_2\left(\frac{V}{a^2/r^2-1},0,s\right), \tag{A13}$$

$$I_{\rm ret}=I_1(-V,s), \tag{A14}$$

$$I_{\rm sal}=\frac{a}{r}I_1(0,s), \tag{A15}$$

$$I_{\rm oml}=I_1(V,s)+I_2(0,V,s). \tag{A16}$$

Asymptotic formulas

The asymptotic series for the confluent hypergeometric function ϕ are derived from exact series representations:

$$\phi(a,c;-z)=\sum_{n=0}^{\infty}\frac{(a-c+1)_n\Gamma(c)\gamma(a+n,z)}{n!\,\Gamma(c-a)\Gamma(a)z^{a+n}},\quad a-c\neq\text{negative integer} \tag{A17}$$

$$\phi(a,c;z)=\sum_{n=0}^{\infty}\frac{(1-a)_n\Gamma(c)\gamma(c-a+n,z)e^z}{n!\,\Gamma(a)\Gamma(c-a)z^{c-a+n}},\quad a\neq\text{negative integer}. \tag{A18}$$

The incomplete gamma function, γ, may be written as

$$\gamma(a,x)=a^{-1}x^a\phi(a,a+1;-x). \tag{A19}$$

For sufficiently large z, the following holds:

$$\Gamma(a)-\gamma(a,z)\sim z^{a-1}e^{-z}. \tag{A20}$$

The Kummer transform for ϕ is

$$\phi(a,c;z)=e^z\phi(c-a,c;-z). \tag{A21}$$

Kanal formula for retarded current

To obtain the Kanal formula we begin with Eq. (14a), expand ϕ, and sum the series in s:

$$I_{\rm ret}=e^{-V}\sum_{k=0}^{\infty}\frac{(Vs^2)^k}{k!\,k!}\,\phi\left(k-\frac{1}{2},k+1;-s^2\right) \tag{A22}$$

Using the Krummer transform we obtain

$$I_{\rm ret}=\exp[-(V+s^2)]\sum_{k=0}^{\infty}\frac{(Vs^2)^k}{k!\,k!}\,\phi\left(\frac{3}{2},k+1;s^2\right). \tag{A23}$$

Finally we write out the series for ϕ and sum over index k:

$$I_{\rm ret}=\exp[-(s^2+V)]\sum_{m=0}^{\infty}\frac{(\frac{3}{2})_m}{m!}\left(\frac{s}{\sqrt{V}}\right)^m I_m(2s\sqrt{V}). \tag{A24}$$

Using the Bessel function representation

$$I_m(x)=e^x\frac{(x/2)^m}{m!}\,\phi\left(m+\frac{1}{2},2m+1;-2x\right), \tag{A25}$$

we obtain the formula

$$I_{\rm ret}=\exp[-(s-\sqrt{V})^2\times\sum_{m=0}^{\infty}\frac{(\frac{3}{2})_m(s^2)^m}{m!\,m!}\,\phi\left(m+\frac{1}{2},2m+1;-4s\sqrt{V}\right). \tag{A26}$$

Using the asymptotic expansion of ϕ and summing over m, we obtain the result

$$I_{\rm ret}=\frac{1}{(4\pi s\sqrt{V})^{1/2}}\exp[-(s-\sqrt{V})^2]\sum_{n=0}^{R}\frac{(1/4s\sqrt{V})^n}{n!}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\times{}_2F_1(\tfrac{3}{2},\tfrac{1}{2}+n;\tfrac{1}{2}-n;s/\sqrt{V}). \tag{A27}$$

The hypergeometric function may be transformed to the form

$${}_2F_1=(1-s/\sqrt{V})^{-2n-3/2}\,{}_2F_1(-n-1,-2n;\tfrac{1}{2}-n;s/\sqrt{V}), \tag{A28}$$

where ${}_2F_1\equiv\psi_n$ is a polynomial in $s/\sqrt{V}$ of order minimum of $n+1$, $2n$. The first three values are

$$\psi_0(x)=1,\quad \psi_1(x)=1-8x-8x^2,\quad \psi_2(x)=1-8x+48x^2+64x^3. \tag{A29}$$

Evaluation of ϕ $(m-\frac{1}{2},1;-s^2)$

These functions are defined by the first two along with the recursion relation

$$\begin{aligned}\phi(-\tfrac{1}{2},1;-s^2)&=\exp(-\tfrac{1}{2}s^2)[(1+s^2)I_0(\tfrac{1}{2}s^2)+s^2I_1(\tfrac{1}{2}s^2)],\\ \phi(\tfrac{1}{2},1;-s^2)&\equiv\exp(-\tfrac{1}{2}s^2)I_0(\tfrac{1}{2}s^2),\\ \phi(m+\tfrac{3}{2},1;-s^2)&=[1/(m+\tfrac{1}{2})][(2m-s^2)\phi(m+\tfrac{1}{2},1;-s^2)+(\tfrac{1}{2}-m)\phi(m-\tfrac{1}{2},1;-s^2)].\end{aligned} \tag{A30}$$

Asymptotic form of retarded current for $s^2>V$

Substitution of the asymptotic form for ϕ in Eq. (17) yields

$$I_{\rm ret}\approx\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-V/s^2)^m(1/s^2)^{n-1/2}(m-\frac{1}{2})_n(m-\frac{1}{2})_n}{n!\,m!\,\Gamma(\frac{3}{2}-m)} \tag{A31}$$

The sum over index m yields the formula

$$I_{\rm ret}\approx\frac{2}{\sqrt{\pi}}s\sum_{n=0}^{R}\frac{(1/s^2)^n(-\frac{1}{2})_n(-\frac{1}{2})_n}{n!}\times{}_2F_1(n-\tfrac{1}{2};n-\tfrac{1}{2};-\tfrac{1}{2};V/s^2). \tag{A32}$$

A polynomial representation is obtained with the transform

$$_2F_1(n-\tfrac{1}{2},n-\tfrac{1}{2};-\tfrac{1}{2};x)=(1-x)^{1/2-2n}\,{}_2F_1(-n,-n;-\tfrac{1}{2};x), \tag{A33}$$

where $P_n(x)={}_2F_1(-n,-n;-\frac{1}{2};x)$ is a polynomial of order n. The first few polynomials are

$$P_0=1,\quad P_1=1-2x,\quad P_2=1-8x-8x^2. \tag{A34}$$

Orbital motion limited current for small s

The confluent hypergeometric functions, $\psi(n-\frac{1}{2},-\frac{1}{2};V)$ are evaluated from the first two and the recursion relation

$$\begin{aligned}\psi(-\tfrac{1}{2},-\tfrac{1}{2};V)&=\tfrac{1}{2}\sqrt{\pi}\,e^V\operatorname{erfc}(\sqrt{V})+\sqrt{V},\\ \psi(\tfrac{1}{2},-\tfrac{1}{2};V)&=\tfrac{1}{2}\sqrt{\pi}(1-2V)e^V\operatorname{erfc}(\sqrt{V})+\sqrt{V},\\ \psi(n+\tfrac{3}{2},-\tfrac{1}{2};V)&=[1/(n+\tfrac{1}{2})(n+2)][(2n+\tfrac{3}{2}+V)\psi(n+\tfrac{1}{2},-\tfrac{1}{2};V)\\&\quad-\psi(n-\tfrac{1}{2},-\tfrac{1}{2};V)].\end{aligned} \tag{A35}$$

An exact representation for ψ allows us to write the asymptotic form of ψ for large V:

$$\psi(a,c;z)=\sum_{n=0}^{\infty}\frac{(1+a-c)_n}{n!\,\Gamma(a)}\left(\frac{\gamma(a+n,z)}{(-z)^n z^a}+z^{1-c}(-z)^n\Gamma(c-1-n,z)\right). \tag{A36}$$

Consequently, we have the asymptotic form

$$\psi\left(n-\frac{1}{2},-\frac{1}{2};V\right)\approx V^{1/2-n}\sum_{m=0}^{R}\frac{(n+1)_m(n-\frac{1}{2})_m(-)^m}{m!\,V^m}. \tag{A37}$$

Substitution into Eq. (20) yields

$$I_{\text{om1}}\approx\frac{2}{\sqrt{\pi}}\sqrt{V}\sum_{m=0}^{R}\frac{(-1/V)^m}{m!}\sum_{n=0}^{\infty}(s^2/V)\binom{\frac{1}{2}}{n}(n+1)_m(n-\tfrac{1}{2})_m. \tag{A38}$$

When $s^2>V$ the series in n converges and may be written as

$$I_{\text{om1}}\approx\frac{2}{\sqrt{\pi}}\sqrt{V}\sum_{m=0}^{R}\left(-\frac{1}{V}\right)^m\left(-\frac{1}{2}\right)_m{}_2F_1\left(m+1,m-\tfrac{1}{2};1;-\frac{s^2}{V}\right), \tag{A39}$$

where $_2F_1$ has the representation

$$_2F_1=(1+s^2/V)^{1/2-2m}\,{}_2F_1(-m,\tfrac{3}{2}-m;1;-s^2/V), \tag{A40}$$

where $Q_m(x)={}_2F_1(-m,\frac{3}{2}-m;1;x)$ is a polynomial of order m. The first three terms are given as

$$Q_0=1,\quad Q_1=1-\tfrac{1}{2}x,\quad Q_2=1+x-\tfrac{1}{8}x^2. \tag{A41}$$

Evaluation of ϕ $(m+1,1;-s^2)$

Using Eq. (A21) we find

$$\begin{aligned}\phi(m+1,1;-s^2)&=\exp(-s^2)\phi(-m;1;s^2)\\&=\exp(-s^2)L_m^{(0)}(s^2),\end{aligned} \tag{A42}$$

where $L_m^{(0)}(x)$ is the Laguerre polynomial.

Term-by-term equivalence of I_{om1} expansions

The term-by-term equivalence of the two orbital motion limited series for $V>s^2$ and $s^2>V$ follow from the proof of the equality

$$\frac{(-\frac{1}{2})_n}{n!}(s^2)^n P_n\left(-\frac{V}{s^2}\right)=(-)^n V^n Q_n\left(-\frac{s^2}{V}\right). \tag{A43}$$

The proof is accomplished by rearranging the polynominals:

$$\begin{aligned}(-)^m V^m Q_m\left(-\frac{s^2}{V}\right)&=(-)^m V^m\sum_{k=0}^{m}\frac{(-m)_k(\frac{3}{2}-m)_k(-s^2/V)^k}{k!\,k!}\\&=(s^2)^m\sum_{k=0}^{m}\frac{(-m)_k(\frac{3}{2}-m)_k(-V/s^2)^{m-k}}{k!\,k!}.\end{aligned} \tag{A44}$$

Using the relation

$$(\alpha-m)_k=\frac{(1-\alpha)_m(-)^k}{(1-\alpha)_{m-k}}, \tag{A45}$$

then

$$(-)^m V^m Q_m\left(-\frac{s^2}{V}\right)=(s^2)^m\sum_{k=0}^{m}\frac{(1)_m(-\frac{1}{2})_m(-V/s^2)^{m-k}}{k!\,k!\,(1)_{m-k}(-\frac{1}{2})_{m-k}}. \tag{A44$'$}$$

Changing the dummy index $m-k\to k$, and using

$$m!/(m-k)!=(-)^k(-m)_k,$$

we find

$$\begin{aligned}(-)^m V^m Q_m\left(-\frac{s^2}{V}\right)&=(s^2)^m\frac{(-\frac{1}{2})_m}{m!}\sum_{k=0}^{m}\frac{(-m)_k(-m)_k(-V/s^2)^k}{k!\,(-\frac{1}{2})_k}\\&=(s^2)^m\frac{(-\frac{1}{2})_m}{m!}P_m(-V/s^2).\end{aligned}$$

Accelerated current functions

The functions f_n and g_n used in the speed ratio expansion for I_{acc} are given by

$$f_0=[(1+\gamma^2)V]^{1/2}\exp(-\gamma^2 V)+\tfrac{1}{2}\sqrt{\pi}\,e^V\operatorname{erfc}[(1+\gamma^2)V]^{1/2}, \tag{A46}$$

$$\begin{aligned}f_1&=(\tfrac{3}{2}+V\gamma^2)[(1+\gamma^2)V]^{1/2}\exp(-\gamma^2V)\\&\quad+(\tfrac{3}{2}-V)\tfrac{1}{2}\sqrt{\pi}\,e^V\operatorname{erfc}[(1+\gamma^2)V]^{1/2},\end{aligned} \tag{A47}$$

$$\gamma^2=\frac{1}{a^2/r^2-1}, \tag{A48}$$

$$\begin{aligned}f_{n+2}&=(n+\tfrac{5}{2}-V)f_{n+1}+V(n+1)f_n\\&\quad+\gamma^2[(1+\gamma^2)V]^{3/2}\exp(-\gamma^2V)(\gamma^2V)^n,\end{aligned} \tag{A49}$$

$$g_0=(a/r)[\tfrac{1}{2}\sqrt{\pi}\operatorname{erf}(\gamma\sqrt{V})-\gamma\sqrt{V}\exp(-\gamma^2V)], \tag{A50}$$

$$g_{n+1}=(n+\tfrac{3}{2})g_n-(a/r)(\gamma^2V)^{n+3/2}\exp(-\gamma^2V). \tag{A51}$$

Form of I_{acc} for large s

From Eqs. (A13) and A16) we note that

$$I_{\text{acc}}=I_{\text{om1}}+\text{algebraic sum of three } I_2\text{'s}. \tag{A52}$$

The asymptotic behavior of $I_2(q.r.s)$ for s large and fixed q and r is obtained by summing Eq. (A12) over index n:

$$I_2(q\ r,s)=-(r+q)^{3/2}\sum_{m,l\atop 0}^{\infty}\frac{(-q)^m(r+q)^l}{m!\,\Gamma(l+\frac{5}{2})}\phi(m+l+1,1;-s^2), \qquad \text{(A53)}$$

and

$$\phi(m+l+1,1;-s^2)=\exp(-s^2)\phi(-m-l,1;+s^2),$$

which decays exponentially for large s.

The hypergeometric functions in Eqs. (A27), (A32) and (A39) were expressed as polynomials using the theorem

$$F(a,b;c;x)=(1-x)^{c-a-b}F(c-a,c-b;c;x).$$

An alternative method of evaluating these functions is directly through their recursion relations. The recusion relations may be obtained either from the contiguous function relations or by the use of successive raising and/or lowering operators. In the latter method, any two of the hypergeometric functions are expressed as products of raising and lowering operators acting on the third function. Derivatives of order two or more are written as derivatives of order one and zero using the differential equation. Finally, the recursion relation is obtained by eliminating the first derivative in the two resulting equations.

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