Logic Review

CS 536: Science of Programming, Fall 2022

Due Tue Sep 6, 11:59 pm

A. Why?

- We use propositions and predicates to write program specifications.
- Propositions and predicates can be related or manipulated syntactically or semantically.

B. Objectives

At the end of this homework, you should be able to

- Perform various syntactic operations and checks n propositions and predicates.
- Describe the difference between syntactic and semantic equivalence.
- Form proofs of propositions using some standard proof rules.
- Design predicate functions for simple properties on values and arrays.

C. Problems [60 points total]

Quantified variables range over $\mathbb{Z}$ unless otherwise specified.

1. [8 = 4 * 2 points] Which of the following mean(s) $p \rightarrow q$ and which mean $q \rightarrow p$?
   a. p is sufficient for q
   b. p only if q
   c. p if q
   d. p is necessary for q

2. [4 = 2 * 2 points] Let e_1 and e_2 be expressions.
   a. In general, does $e_1 \neq e_2$ imply $e_1 \not\equiv e_2$? If yes, briefly justify (a sentence or two is fine); if no, give a counterexample (specific values for e_1 and e_2 that show that this implication does not always hold).
   b. In general, does $e_1 = e_2$ imply $e_1 \equiv e_2$,? Again give a brief justification or counterexample.

3. [6 = 3 * 2 points] For each pair below, characterize the state as well- or ill-formed; if well-formed, is it proper? If proper, does the given expression evaluate successfully or cause a runtime error (and if so, how?)
   a. $\{v = 5, w = 6\}$ and $v + 0 * z$
   b. $\{v = -4, w = 6\}$ and $sqrt(v) * sqrt(w)$
   c. $\{y = 2, z = -4\}$ and $y*y / (z+4)$

4. [6 points] The goal is to show that $p \wedge \neg(q \wedge r) \rightarrow q \wedge r \rightarrow \neg p$ is a tautology by proving it is $\Leftrightarrow$ T. To do this, complete the proof of equivalence below using (only) the propositional logic rules (from Lecture 2). Be sure to include the names of the rules. There's more than one correct answer [just give one of them].

   $p \wedge \neg(q \wedge r) \rightarrow q \wedge r \rightarrow \neg p$
   [you fill in] Defn $\rightarrow$
   [you fill in] Defn $\rightarrow$
   [and so on]

5. [6 points] Simplify $\neg(\forall x . (\exists y . x \le y) \vee \exists z . x \ge z)$ to a predicate that has no uses of $\neg$. Present a proof of equivalence. You'll need DeMorgan's Laws. Also use rules like "$\neg(e_1 \le e_2) \Leftrightarrow e_1 > e_2$ by negation of comparison".

6. [4 = 2 * 2 points] What is the full parenthesization of
   a. $p \wedge \neg r \wedge s \rightarrow \neg q \vee r \rightarrow \neg p \leftrightarrow \neg s \rightarrow t$?
   b. $\exists m . 0 \le m < n \wedge \forall j . 0 \le j < m \rightarrow b[0] \le b[j] \le b[m]$ *

7. [4 = 2 * 2 points] Give the minimal parenthesization of each of the following by showing what remains after removing all redundant parentheses. Hint: To avoid getting confused about which parentheses match each other, try rewriting the given parentheses with subscripts: $(_1 \ldots)_1$ versus $(_2$ and $)_2$ and so on.
   a. $((\neg(p \vee q) \vee r) \rightarrow (((\neg q) \vee r) \rightarrow ((p \vee (\neg r)) \vee (q \wedge s))))$
   b. $(\exists i . (((0 \le i) \wedge (i < m)) \wedge (\forall j . (((m \le j) \wedge (j < n)) \rightarrow (b[i] = b[j])))))$. (This predicate asks “Is there a value in $b[0..m-1]$ > every value in $b[m..n-1]$?)
   c. $(\forall x.((\exists y.(p \rightarrow q)) \rightarrow (\forall z.(q \vee (r \wedge s)))))$

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* Leave $(0 \le j < m)$ as is; don't expand it to $((0 \le j) \wedge (j < m))$. Don't forget to parenthesize $(b[0])$, e.g.

8. [10 points total] Say whether the given propositions or predicates are $\equiv$ or $\not\equiv$. Briefly justify your answer.
   a. [2 points] Is $p \wedge q \vee \neg r \to \neg p \to q \equiv ((p \wedge q) \vee ((\neg r \to ((\neg p) \to q))))$?
   b. [2 points] Is $\forall x . p \to \exists y . q \to r \equiv ((\forall x . p) \to (\exists y . q)) \to r$?
   c. [3 points] Is $\exists x . p \wedge \exists y . (q \to r) \vee \exists z . r \to s \equiv \exists x . p \wedge (\exists y . q \to r) \vee (\exists z . r \to s)$?
   d. [3 points] Is $(\forall x . p \vee \forall y . q) \vee (\forall z . r) \to s \equiv \forall x . p \vee (\forall y . q) \vee \forall z . r \to s$?

9. [6 = 2 * 3 points] Say whether each of the following is a tautology, contradiction, or contingency. If it's a contingency, show an instance when the proposition is true and show an instance where it's false.
   a. $(p \to (q \to r)) \leftrightarrow ((p \to q) \to r)$
   b. $(\forall x \in \mathbb{Z}. \forall y \in \mathbb{Z}. f(x, y) > 0) \to (\exists x \in \mathbb{Z}.\ \exists y \in \mathbb{Z}. f(x, y) > 0)$. Rely on the idea that for $(\forall u.\ \varphi)$ to be false, we need some value for u for which φ is false. I.e., we need $(\exists u . \neg\varphi)$. Similarly, for $(\exists v . \psi)$ to be false, we ψ to be false for every value of v. I.e., we need $(\forall v.\ \neg\psi)$.

10. [6 points] Write the definition of a predicate function $GT(b, x, m, k)$ that yields true iff $x > b[m]$, … $b[m+k-1]$. E.g., in the state $\{b = (1, 3, -2, 8, 5)\}$, $GT(b, 4, 0, 3)$ is true; $GT(b, 0, 1, 2)$ is false. You can assume without testing that the indexes m, … $m+k-1$ are all in range. If $k \leq 0$, the sequence $b[m]$, $b[m+1]$, …, $b[m+k-1]$ is empty and $GT(b, x, m, k)$ is true. (It's straightforward to write GT so that this is not a special case.)

    Remember, this has to be a ***predicate function***, not a program that calculates a boolean value. Hint: Check the discussion in the Class 2 notes about trying to translate programs to predicates.

