Types, Expressions, and Arrays

CS 536: Science of Programming, Fall 2022

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A. Why?

- Expressions represent values relative to a state.
- Types describe common properties of sets of values.
- The value of an array is a function value from index values to array values.

B. Outcomes

At the end of this class, you should

- Know what expressions and their values we'll be using in our language
- Know how states are expanded to include values of arrays

C. Types and Expressions

- Let's start looking at programming language we'll be using.
- The ***datatypes*** will be pretty simple (no records or function types, for example).
  - Primitive types: *int* (integers) and *bool* (boolean). We can add other types like characters, strings, and floating-point numbers, but for what we're doing, integers and booleans are enough.
  - Composite types: Multi-dimensional arrays of primitive types of values, with integer indexes.
- ***Expressions*** are built from
  - ***Constants***: Integers (*0*, *1*, *-1*, …) and boolean constants (*T*, *F*).
  - ***Simple variables*** of primitive types.
  - ***Operations***
    - On integers: Binary +, -, *, /, *min*, *max*, %, =, ≠, <, ≤, >, ≥, *divides*, Unary –, *sqrt*.
      - / and *sqrt* truncate toward zero, to an integer. E.g., $13 / 3 = 4$, $13 / -3 = -4$, and $sqrt(17) = 4$. Division and mod (%) by zero and *sqrt* of negative values generate runtime errors.
    - On booleans: ¬, ∧, ∨, →, ↔, =, ≠ (note = and ↔ mean the same thing).
    - On arrays: *size* and array element selection.
  - ***Conditional expressions***
    - *if B then e_1 else e_2 fi.* Semantically, if *B* evaluates to true, then evaluate e_1; if *B* evaluates to false, then evaluate e_2. The C / Java syntax ($B\ ?\ e_1 : e_2$) is also okay.

        - Restrictions: To ensure that the entire conditional expression has a consistent type, e_1 and e_2 must have the same type. (This is sometimes called "balancing".) The type must also be simple (not an array type or function type.

    - ***Arrays***
        - As usual, *b[e]* is array element selection. *size*(*b*) gives the length of *b*. For multi-dimensional arrays, we have $b[e_1][e_2]...[e_n]$ and *size1*(*b*), *size2*(*b*), etc. Arrays are zero-origin and fixed-size.
        - You can have array parameters with functions and predicates (as in *size*(*b*)) .
        - ***Restrictions:*** No array assignments, no expressions of type array; this includes array slices ($b[e_1]$ of a two-dimensional array, for example). To support these, we'd need identifiers to map to memory locations, with a separate function mapping locations to values. (This is also why we don't have pointers.)
    - ***General restrictions***
        - No expressions with functional or array values. (So they all have primitive types.)
            - ***Example***: ***if*** *B* ***then*** *f(x)* ***else*** *g(x)* ***fi*** is legal; ***if*** *B* ***then*** *f* ***else*** *g* ***fi*** *(x)* is not.
        - We don't have assignment expressions (we'll see later how to simulate them).
        - We don't have records (adding them isn't that hard, but they don't really add much. theoretically speaking).
    - We won't explicitly declare variables; we will assume we can infer the types. The default type is integer.
- [2022-08-30] ***Notation***: *a* and *c* and *d* are constants; *e* and *s* are general expressions; *B* and *C* are boolean expressions, *a* and *b* are array names, and *u*, *v*, etc. are variables. Greek letters like α and β stand for semantic values.

D. Examples of Expressions

- ***Example 1***: ***if*** *x < 0* ***then*** *0* ***else*** *sqrt(x)* ***fi*** yields 0 if *i* is negative, otherwise it yields the square root of *x*.
- [2022-08-30] (Fix paren) ***Example 2: if*** *x < 0* ***then*** *x+y* ***else*** *x*y+z* ***fi*** means "If *x < 0* evaluates to true, then we evaluate *x+y* and add the result to *z*, otherwise evaluate *x*y* and add the result to *z*." (*x*, *y*, and *z* must all be integers.)
- [2022-08-30] ***Example 3***: ***if*** *i < 0* ***then*** *b[0]* ***else*** ***if*** *i ≥ size*(*b*) ***then*** *b[size*(*b*)*-1]* ***else*** *b[i]* ***fi*** yields *b[i]* if *i* is in range; if *i* is negative, it yields *b[0]*; if *i* is too large, it yields the last element of *b*.
- ***Example 4***: *b[* ***if*** *i < 0* ***then*** *0* ***else*** *i ≥ size*(*b*) ***then*** *size*(*b*)*-1* ***else*** *i* ***fi*** *]* yields the same value as Example 3, but it does this by calculating the index first.
- ***Example 5:*** A (conditional) expression can't yield a function, so ***if*** *B* ***then*** *f(x)* ***else*** *g(x)* ***fi*** is legal; ***if*** *B* ***then*** *f* ***else*** *g* ***fi*** *(x)* is not.
- [2022-08-30] (Fix paren) ***Example 6***: We can't have array-valued expressions, so (assuming *a* and *b* are 1-dimensional arrays), ***if*** *x* ***then*** *a[0]* ***else*** *b[0]* ***fi*** is legal, ***if*** *x* ***then*** *a* ***else*** *b* ***fi****[0]* is not.

E. Syntactic Values and Semantic Values

- When we discuss the meanings of programs, some of the items are syntactic (like expressions) and some items are semantic (values, states). So there's a problem with symbols like “2” or “+”. Sometimes we use them in our programs; this is a syntactic use. But sometimes we mean a mathematical value, the thing denoted by “2” or “two” or "plus" or so on.
- In general, the context tells you whether something is syntactic or semantic. E.g.,
  - ***Example 7:*** In “Does x occur in the predicate p?” since p is a predicate, it is syntactic, so for x to occur in it, x must be syntactic also.
  - ***Example 8:*** In $z \equiv 2+2$, the $\equiv$ symbol is for syntactic equality, so both z and 2+2 are syntactic.
  - ***Example 9:*** In “$\sigma(2+2) = 2+2 = 4$, the σ is semantic (a state) and the first 2+2 is syntactic, since we're looking for its value in σ. The second 2+2 is semantic because σ takes expressions and returns semantic values. (Hence the second + sign is semantic). The result 4 is also semantic. Also, the two equal signs are semantic equality.
  - ***Example 10:*** "The value in σ of 2+2 is two plus two, which is four" is the same as Example 9 but it uses English to write out the semantic values and operations.
- ***Notation:*** If I really want to emphasize that something is semantic, I'll underline it.
- ***Example 11:*** Rewriting Examples 9 and 10: $\sigma(2+2) = \underline{2+2} = \underline{4}$ or: the value in σ of 2+2 is $\underline{\text{two plus two}}$, which is $\underline{\text{four}}$. Technically, the equality tests could be underlined, but $\sigma(2+2) \mathrel{\underline{=}} \underline{2+2} \mathrel{\underline{=}} \underline{4}$ really seems like more trouble than it's worth. Furthermore, $\underline{=}$ (underlined equal) looks a lot like $\equiv$ (syntactic equality).
- ***Example 12:*** If σ is the state that maps x to $\underline{5}$, we could rewrite "σ = {x = 5}” as “$\sigma = \{x = \underline{5}\} = \{(x, \underline{5})\}$”.

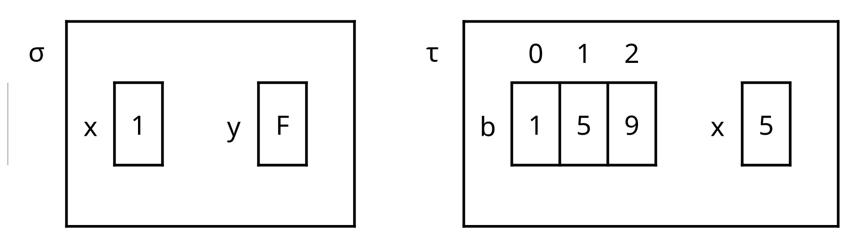
F. Values of Expressions

- In general, expressions have values relative to a state. E.g., relative to $\{x = 1, y = 2\}$, the expression $x+y$ has the value $\underline{3}$. Recall that we write $\sigma(x)$ for the value of the variable x and extend this to $\sigma(e)$ for the value of the expression e.
- The value of $\sigma(e)$ depends on what kind of expression e is, so we use recursion on the structure of e (the base cases are variables and constants and we recursively evaluate subexpressions).
  - $\sigma(x)$ = the value that σ binds variable x to
  - $\sigma(c)$ = the value of the constant c. E.g., $\sigma(2) = \underline{2}$. (Note σ is irrelevant here.)
  - $\sigma(e_1 + e_2) = \sigma(e_1)$ plus $\sigma(e_2)$ [and similar for –, *, etc.]
    [2022-08-30] Added some underlines in next 3 lines
  - $\sigma(e_1 < e_2) = \underline{T}$ iff $\sigma(e_1)$ is less than $\sigma(e_2)$ [similar for ≤, =, etc].
  - $\sigma(e_1 \wedge e_2) = \underline{T}$ iff $\sigma(e_1)$ and $\sigma(e_2)$ are both $= \underline{T}$ [similar for ∨, etc].
  - σ(***if*** B ***then*** e_1 ***else*** e_2 ***fi***) $= \sigma(e_1)$ if $\sigma(B) = \underline{T}$. It $= \sigma(e_2)$ if $\sigma(B) = \underline{F}$.

- We'll put off the σ(*b[e]*) case, the value of the array indexing expression *b[e]*, for just a bit until we look at the value of an array variable.
- ***Example 13***: Let σ = {*x* = 1}, let τ = σ ∪ {*y* = 1}, and let *e* ≡ (*x* = ***if*** *y > 0* ***then*** *17* ***else*** *y* ***fi***).
  - [2022-08-30] To calculate τ(*e*), first we look up τ(*x*) and get $\underline{1}$. (Since τ extends σ with a binding for *y*, τ behaves like σ except on *y*.)
  - Now we need τ(***if*** *y > 0* ***then*** *17* ***else*** *y* ***fi***).
    - τ(*y > 0*) means "Is τ(*y*) greater than zero?" Since τ(y) = $\underline{1}$, the answer is $\underline{T}$.
    - [2022-08-30] τ(*y > 0)* = $\underline{T}$*, so* τ(***if*** *y > 0* ***then*** *17* ***else*** *y* ***fi***) = τ(*17*). I.e., since the test evaluates to T, the value of the conditional is the value of the constant 17.
    - τ(17) = $\underline{17}$, of course.
  - So τ(***if*** *y > 0* ***then*** *17* ***else*** *y* ***fi***) = $\underline{17}$.
  - For the overall expression, we're comparing τ(*x*) and τ(***if*** *y > 0* ***then*** *17* ***else*** *y* ***fi***) for equality. I.e., we test $\underline{1}$ = $\underline{17}$ and we get $\underline{F}$.
  - So τ(*e*) = $\underline{F}$.
- ***The empty state***: Since a state is a set of bindings, the empty set ∅ is a state (the empty state). It's proper for any expression or predicate that doesn't include variables. E.g., In state ∅, the expression *2+2* evaluates to four. (In fact, since we don't care about bindings for variables that don't appear in an expression, we can say that in any state σ, *2+2* evaluates to 4.
- ***Example 14***: Let σ = ∅ (the empty state) then
  - σ(2+2 = 4) = σ(2+2) equals σ(4) = … = $\underline{4}$ equals $\underline{4}$ = $\underline{T}$.
- With operators, you have to distinguish the syntactic symbol from the semantic symbol. So σ(*v*+*w*) = σ(*v*) $\underline{+}$ σ(*w*) is correct: The second plus is the semantic meaning of the syntactic symbol +. You could also write σ(*v*+*w*) = σ(*v*) plus σ(*w*); here, plus has a semantic meaning. (Note σ(*v* plus *w*) only works if "plus" is a binary operator in the language.)

G. Arrays and Their Values

- Compare the usual way we write states on the blackboard. Below, the left state is σ = {*x* = 1, *y* = F} = {(*x*, 1), (*y*, F)}. The right one, τ, defines an array variable *b* and an integer *x*.



- We'll take the value of an array to be a function from index values to stored values, so $\tau(b[0]) = 3$, $\tau(b[1]) = 5$, and $\tau(b[2]) = 9$. We could write $\tau = \{b[0] = \underline{3}, b[1] = \underline{5}, b[2] = \underline{9}, x = \underline{5}\} = \{(b[0], 3), (b[1], 5), (b[2], 9), (x, 5)\}$, but a more convenient notation would be nice.
- ***Notation***: Let β be the function with $\beta(\underline{0}) = \underline{3}$, $\beta(\underline{1}) = \underline{5}$, $\beta(\underline{2}) = \underline{9}$, then we can say $\tau = \{b = \beta, x = 5\} = \{(b, \beta), (x, 5)\}$. (I'm using a greek letter β because the function is semantic, taking index values to memory values.). Since a function is a set of ordered pairs, we can also write $\beta = \{(0, 3), (1, 5), (2, 9)\}$. Since β is actually a sequence, let's allow ourselves to abbreviate this to $\beta = (3, 5, 9)$. (Note this last notation looks like the graphical picture of τ.)
- We we have a number of ways to express τ, all valid. Going from shortest to longest we have
  - $\tau = \{b = \beta, x = \underline{5}\}$ where $\beta = (\underline{3}, \underline{5}, \underline{9})$
  - $\tau = \{b[0] = 3, b[1] = \underline{5}, b[2] = \underline{9}, x = \underline{5}\}$
  - $\tau = \{b = \beta, x = \underline{5}\}$ where $\beta = \{(\underline{0}, \underline{3}), (\underline{1}, \underline{5}), (\underline{2}, \underline{9})\}$
  - $\tau = \{b = \beta, x = \underline{5}\}$ where $\beta(\underline{0}) = \underline{3}$, $\beta(\underline{1}) = \underline{5}$, $\beta(2) = \underline{9}$

H. Value of An Array Indexing Expression

- Going back to the definition of the value of an expression in a state, here's the array case:
- $\sigma(b[e]) = \beta(\alpha)$ where $\beta = \sigma(b)$ and $\alpha = \sigma(e)$. The variable b is an array name, so $\sigma(b)$ = a function we're calling β. We call β on the ***value*** of the index expression e, hence $\alpha = \sigma(e)$, and the value $\beta(\alpha)$ is the meaning of $b[e]$.
- You can also write $\sigma(b[e]) = (\sigma(b))(\sigma(e))$ if you don't want to define α and β. Function application is left-associative, so $\sigma(b)(\sigma(e)) = (\sigma(b))(\sigma(e))$. I.e., $\sigma(b)$ is a function we're applying to $\sigma(e)$.
- So another way to write the definition is $\sigma(b[e]) = \sigma(b)(\sigma(e)) = \beta(\alpha)$ where $\beta = \sigma(b)$ and $\alpha = \sigma(e)$.
- With our earlier example then, $\sigma(b[x\text{-}4]) = \sigma(b)(\sigma(x\text{-}4)) = \beta\ (\sigma(x)$ minus four$)$ = β(5 minus four) = $\beta(1) = 5$, where β is as described earlier, $\beta = (\underline{3}, \underline{5}, \underline{9})$.
- ***Example 15***: Let $\sigma = \{x = \underline{1}, b = \alpha\}$ where $\alpha = (\underline{2}, \underline{0}, \underline{4})$. Then
  - $\sigma(x) = \underline{1}$
  - $\sigma(x+1) = \sigma(x) + \sigma(1) = \underline{1+1} = \underline{2}$
  - $\sigma(b) = \alpha$
  - $\sigma(b[x+1]) = (\sigma(b))(\sigma(x+1)) = \alpha(\underline{2}) = \underline{4}$
  - If we don't want to write out the intermediate steps first, we could write
    - $\sigma(b[x+1]) = (\sigma(b))(\ \sigma(x+1)\ ) = \alpha(\underline{\sigma(x)}+1) = \alpha(\underline{1+1}) = \alpha(\underline{2}) = \underline{4}$.
- ***Example 16***: Let $\sigma = \{x = 1, b = \alpha\}$ where $\alpha = (2, 0, 4)$, then
  - $\sigma(b[x+1]\text{-}\ 2) = \sigma(b[x+1]) - \sigma(2) = (\sigma(b))(\sigma(x+1)) - \underline{2}$
    $= (\sigma(b))(\underline{\sigma(x)+\ 1}) - \underline{2}$
    $= \alpha(\underline{1+1}) - \underline{2}$
    $= \underline{\alpha(2)-2} = \underline{4 - 2} = \underline{2}$.