

Propositional and Predicate Logic

CS 536: Science of Programming, Fall 2022

- Fill in the missing rule names in the proof below of $\neg(p \leftrightarrow q) \Leftrightarrow (q \wedge \neg p) \vee (p \wedge \neg q)$, using the rules from the class notes. (See page 2.)

$$\begin{aligned}
 &\neg(p \leftrightarrow q) \\
 \Leftrightarrow &\neg((p \rightarrow q) \wedge (q \rightarrow p)) && \text{by defn } \leftrightarrow \\
 \Leftrightarrow &\neg(p \rightarrow q) \vee \neg(q \rightarrow p) && \underline{\hspace{2cm}} \\
 \Leftrightarrow &((p \wedge \neg q) \vee (q \wedge \neg p)) && \underline{\hspace{2cm}} \\
 \Leftrightarrow &(q \wedge \neg p) \vee (p \wedge \neg q) && \underline{\hspace{2cm}}
 \end{aligned}$$

- Write a formal proof that shows that $(p \rightarrow p \vee q)$ is a tautology: i.e., prove $(p \rightarrow p \vee q) \Leftrightarrow T$. As a derived rule, this is often called “(left) \vee -introduction”.

$$\begin{aligned}
 &p \rightarrow p \vee q \\
 \Leftrightarrow &\underline{\hspace{2cm}} && \text{by } \underline{\hspace{2cm}} \\
 &\text{etc.}
 \end{aligned}$$

- Which of $\{x = 4, y = 3\}$, $\{x = -2, y = 0\}$, $\{x + y = 6\}$, $\{y = 4\}$, $\{\}$, $\{x = 3, x = 8\}$, and $\{x = 6, y = x\}$ are well-formed? Of the ones that are well-formed, which ones are proper for $2+2 \leq 5$, $x + y^*x$, and x / y ? And, of the proper states, do any of them cause an error when we evaluate?
- For this problem, we'll prove DeMorgan's laws for bounded quantifiers. You'll need to transform the bound quantifiers to unbound quantifiers (and vice versa) and you'll need DeMorgan's laws on unbounded quantifiers. Assume the quantifiers are over variable x .
 - Prove $\neg \forall p.q \Leftrightarrow \exists p.\neg q$.
 - Prove $\neg \exists p.q \Leftrightarrow \forall p.\neg q$. (It's easy to prove this using part (a) and double negation, but for the practice, don't do that.)
- Some logical rules can be derived from others. Use (only) the rules given with each problem (not necessarily in that order). You may need to use a rule more than once.
 - Prove the rule of contraposition, $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p) \Leftrightarrow T$, using (only) Definition of \rightarrow , double negation, commutativity of \vee , and excluded middle.

- b. Prove the rule of transitive contradiction, $(p \rightarrow q) \wedge (q \rightarrow \neg p) \Leftrightarrow \neg p$, using defn \rightarrow , identity, contradiction, and distributivity.
 - c. Prove the rule of left \wedge elimination, $p \wedge q \rightarrow p \Leftrightarrow T$ using defn \rightarrow , distributivity, excluded middle and DeMorgan's law. (There's a similar rule of right \wedge elimination, by the way.)
 - d. Prove a rule that combines left and right \vee -introduction, $(p \rightarrow p \vee q) \wedge (q \rightarrow p \vee q) \Leftrightarrow T$, using excluded middle, identity, domination, and defn \rightarrow .
6. Let $q(x, y) \equiv x < y \rightarrow y < z \wedge f(x) = 2$. Expand $\neg q(x, y)$ to remove \neg signs: Use the rules to find a predicate equivalent to $\neg(x < y \rightarrow y < z \wedge f(x) = 2)$ that doesn't use \neg . Hint: Use DeMorgan's laws to move the negation "inward" to smaller and smaller subexpressions. Show your reasoning as a formal proof. (Don't forget the rule names.)
7. What are the minimal and full parenthesizations for
 - a. $(\forall x. ((\exists y. x > y) \wedge (\exists y. x < y)))$
 - b. $\forall x. \neg(\exists y. p \wedge \forall z. q)$
 - c. $\forall x. \forall y. \exists z. (x \neq y \rightarrow x \leq z \wedge z \leq y \vee x > z \wedge z \geq y)$
8. In general, if $\forall x. \forall y. p(x, y)$ is valid, is $\forall y. \forall x. p(x, y)$ also valid? What about $\exists x. \exists y. p(x, y)$ and $\exists y. \exists x. p(x, y)$?
9. Using propositional and predicate proof rules, find a predicate equivalent to $\neg(\forall x. \exists y. p(x, y))$ that has no negation symbols (i.e., \neg), except possibly in front of $p(x, y)$. Write a formal proof that shows each step needed (don't forget the rule names!). Hint: Use DeMorgan's laws to move the negation inward.
10. Repeat the previous question on $\neg(\exists y. \forall x. p(x, y))$.
11. Write the definition of a predicate function $Repeats(b, m)$ that is true exactly when the first m elements of b match the second m elements of b : I.e., $b[0] = b[m]$, $b[1] = b[m+1]$, ..., $b[m-1] = b[2*m-1]$. (Alternatively, $b[0..m-1]$ and $b[m..2*m-1]$ are pointwise equal.) Example: If b is $[1, 3, 5, 1, 3, 5]$, then $Repeats(b, 3)$ is true but $Repeats(b, 2)$ is false.

CS 536: Solution to Practice 2 (Propositional and Predicate Logic)

1. DeMorgan's law; Negation of \rightarrow twice; commutativity of \vee .

$$\begin{aligned}
 2. \quad & p \rightarrow p \vee q \\
 & \Leftrightarrow \neg p \vee (p \vee q) && \text{Defn } \rightarrow \\
 & \Leftrightarrow (\neg p \vee p) \vee q && \text{Associativity of } \vee \\
 & \Leftrightarrow T \vee q && \text{Excluded middle} \\
 & \Leftrightarrow T && \text{Domination}
 \end{aligned}$$

3. $\{x = 4, y = 3\}$ is well-formed, proper for $2+2 \leq 5$, $x + y * x$ and x / y , and all three expressions evaluate successfully (no runtime error).

$\{x = -2, y = 0\}$ is well-formed and proper for the same three expressions, and evaluation is successful for $2+2 \leq 5$ and $x + y * x$, but causes division by zero for x / y .

$\{y = 4\}$ and $\{\}$ are both well-formed, proper for $2+2 \leq 5$, and evaluation is success; they're improper for the other expressions.

$\{x = 6, y = x\}$ is ill-formed (we should bind y to a value, like 6, not to the variable x).

4a. (Negation of bounded \forall)

$$\begin{aligned}
 & \neg \forall x. p \\
 & \Leftrightarrow \neg \forall x. p \rightarrow q && \text{Defn bound } \forall \\
 & \Leftrightarrow \exists x. \neg(p \rightarrow q) && \text{DeMorgan's law } \neg \forall \\
 & \Leftrightarrow \exists x. \neg(\neg p \vee q) && \text{Defn } \rightarrow \\
 & \Leftrightarrow \exists x. \neg \neg p \wedge \neg q && \text{DeMorgan's law } \neg \vee \\
 & \Leftrightarrow \exists x. p \wedge \neg q && \text{Double negation} \\
 & \Leftrightarrow \exists x. p. \neg q && \text{Defn bound } \forall
 \end{aligned}$$

Note Negation of \rightarrow is available as a derived rule (see the notes), then you can go directly from $\exists x. \neg(p \rightarrow q) \Leftrightarrow \exists x. p \wedge \neg q$ in one line.

4b. (Negation of bounded \exists)

$$\begin{aligned}
 & \neg \exists x. p \\
 & \Leftrightarrow \neg \exists x. p \wedge q && \text{Defn bound } \exists \\
 & \Leftrightarrow \neg \exists x. \neg \neg p \wedge q && \text{Double negation} \\
 & \Leftrightarrow \neg \exists x. \neg(\neg p \vee q) && \text{DeMorgan's law } \neg \wedge \\
 & \Leftrightarrow \neg \exists x. \neg(p \rightarrow q) && \text{Defn } \rightarrow \\
 & \Leftrightarrow \forall x. \neg \neg(p \rightarrow q) && \text{DeMorgan's law } \neg \exists \\
 & \Leftrightarrow \forall x. p \rightarrow q && \text{Double negation} \\
 & \Leftrightarrow \forall x. p. q && \text{Defn bound } \forall
 \end{aligned}$$

The version that uses part (a) takes $\neg \exists p.q \Leftrightarrow \neg \exists p.\neg q \Leftrightarrow \neg \neg \forall p.\neg q \Leftrightarrow \forall p.\neg q$. (If you're interested.)

On the off chance that you're interested, the proof that uses part (a) takes $\neg \exists p.q \Leftrightarrow \neg \exists p.\neg q \Leftrightarrow \neg \neg \forall p.\neg q \Leftrightarrow \forall p.\neg q$.

5a. (Contraposition)

$(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$	
$\Leftrightarrow (\neg p \vee q) \rightarrow (\neg q \rightarrow \neg p)$	Defn \rightarrow
$\Leftrightarrow (\neg p \vee q) \rightarrow (\neg \neg q \vee \neg p)$	Defn \rightarrow
$\Leftrightarrow (\neg p \vee q) \rightarrow (q \vee \neg p)$	$\neg \neg$
$\Leftrightarrow (\neg p \vee q) \rightarrow (\neg p \vee q)$	Comm. of \vee
$\Leftrightarrow \neg(\neg p \vee q) \vee (\neg \neg q \vee \neg p)$	Defn \rightarrow
$\Leftrightarrow T$	Excluded middle (on $(\neg p \vee q)$)

5b. (Transitive contradiction)

$(p \rightarrow q) \wedge (q \rightarrow \neg p)$	
$\Leftrightarrow (\neg p \vee q) \wedge (q \rightarrow \neg p)$	Defn \rightarrow
$\Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee \neg p)$	Defn \rightarrow
$\Leftrightarrow \neg p \vee (q \wedge \neg q)$	Distributivity
$\Leftrightarrow \neg p \vee F$	Contradiction
$\Leftrightarrow \neg p$	identity

5c. (Left \wedge -elimination)

$p \wedge q \rightarrow p$	
$\Leftrightarrow \neg(p \wedge q) \vee p$	Defn \rightarrow
$\Leftrightarrow (\neg p \vee \neg q) \vee p$	DeMorgan
$\Leftrightarrow (p \vee \neg p) \vee (\neg q \vee p)$	Distributivity
$\Leftrightarrow T \vee (q \vee \neg q)$	Excluded middle
$\Leftrightarrow T$	Domination

5d. (Left and right \vee -introduction)

$(p \rightarrow p \vee q) \wedge (q \rightarrow p \vee q)$	
$\Leftrightarrow (\neg p \vee p \vee q) \wedge (\neg q \vee p \vee q)$	Defn \rightarrow (twice)
$\Leftrightarrow (T \vee q) \wedge (T \vee p)$	Excluded middle (twice)
$\Leftrightarrow T \wedge T$	Domination (twice)
$\Leftrightarrow T$	Identity

6. If $q(x, y) \equiv x < y \rightarrow y < z \wedge f(x) = 2$, then

$$\neg q(x, y)$$

$$\Leftrightarrow \neg(x < y \rightarrow y < z \wedge f(x) = 2)$$

Defn of q

$$\Leftrightarrow x < y \wedge \neg(y < z \wedge f(x) = 2)$$

Negation of \rightarrow

$$\Leftrightarrow x < y \wedge (\neg(y < z) \vee \neg(f(x) = 2))$$

DeMorgan's Law

$$\Leftrightarrow x < y \wedge (y \geq z \vee f(x) \neq 2)$$

Negation of comparison, 3 times

7. (Minimal and full parenthesizations)

a. $(\forall x. ((\exists y. x > y) \wedge (\exists y. x < y)))$

Minimal: $\forall x. (\exists y. x > y) \wedge \exists y. x < y$

Full: $(\forall x. ((\exists y. (x > y)) \wedge (\exists y. (x < y))))$ [I showed the outermost parentheses]

b. $\forall x. \neg(\exists y. p \wedge \forall z. q)$ (Is already minimal)

Full: $(\forall x. (\neg(\exists y. (p \wedge (\forall z. q)))))$

c. $\forall x. \forall y. \exists z. (x \neq y \rightarrow x \leq z \wedge z \leq y \vee x > z \wedge z \geq y)$

Minimal: $\forall x. \forall y. \exists z. x \neq y \rightarrow x \leq z \wedge z \leq y \vee x > z \wedge z \geq y$

Full: $(\forall x. (\forall y. (\exists z. ((x \neq y) \rightarrow (((x \leq z) \wedge (z \leq y)) \vee ((x > z) \wedge (z \geq y)))))))$

8. ($Qx.Qy$ versus $Qy.Qx$)

a. Yes: $(\forall x. \forall y. p(x, y))$ is valid if and only if $(\forall y. \forall x. p(x, y))$ is valid

b. Yes: $(\exists x. \exists y. p(x, y))$ is valid if and only if $(\exists y. \exists x. p(x, y))$ is valid

9. $\neg(\forall x. \exists y. p(x, y))$

$$\Leftrightarrow \exists x. \neg \exists y. p(x, y)$$

DeMorgan's Law $\neg \forall$

$$\Leftrightarrow \exists x. \forall y. \neg p(x, y)$$

DeMorgan's Law $\neg \exists$

10. $\neg(\exists y. \forall x. p(x, y))$

$$\Leftrightarrow \forall y. \neg(\forall x. p(x, y))$$

DeMorgan's Law $\neg \exists$

$$\Leftrightarrow \forall y. \exists x. \neg p(x, y)$$

DeMorgan's Law $\neg \forall$

11. First, here's a solution that doesn't check for m being too large:

$$\text{Repeats}(b, m) \equiv \forall j. 0 \leq j < m \rightarrow b[j] = b[m+j]$$

You can also use a bounded quantifier: $\text{Repeats}(b, m) \equiv \forall 0 \leq j < m. b[j] = b[m+j]$.

To check for m being too large, then assuming \wedge is short-circuiting (like $\&\&$ in C), we can use

$$\text{Repeats}(b, m) \equiv 0 \leq 2*m < \text{size}(b) \wedge \forall j. (0 \leq j < m \rightarrow b[j] = b[m+j]).$$