Denotational Semantics; Runtime Errors

CS 536: Science of Programming, Fall 2022

A. Why

- Our simple programming language is a model for the kind of constructs seen in actual languages.
- Our programs stand for state transformers.
- Runtime errors cause failure of normal program execution.

B. Outcomes

At the end of today, you should be able to:

- Give the denotational semantics of a program in a state.
- Say when and how evaluation of an expression or program fails due to a runtime error.

C. Problems

Denotational Semantics

Problems 1 – 4 are the denotational versions of the similar questions from Practice 5.

1. What is
   a. $M(x := x+1, \{x = 5\})$?
   b. $M(x := x+1, \sigma)$? (Your answer will be symbolic.)
   c. $M(x := x+1;\ y := 2*x, \{x = 5\})$?
2. Let $IF \equiv$ ***if*** $x > 0$ ***then*** $x := x+1$ ***else*** $y := 2*x$ ***fi***.
   a. Let $\sigma(x) = 8$. What is $M(IF, \sigma)$?
   b. Repeat, if $\sigma(x) = 0$.
   c. Repeat, if we don't know what $\sigma(x)$ is. (Your answer will be symbolic.)
3. Let $IF \equiv$ ***if*** $x > 0$ ***then*** $x := x/z$ ***fi***.
   a. What is $M(IF, \sigma)$ if $\sigma = \{x = 8, z = 3\}$? (Don't forget, integer division truncates)
   b. What is $M(IF, \{x = -2, z = 3\})$?
4. Let $W \equiv$ ***while*** $x < 3$ ***do*** S ***od*** where $S \equiv x := x+1;\ y := y*x$.
   a. Evaluate the body S in an arbitrary state τ and give $M(S, \tau)$.
   b. What is $M(W, \sigma)$ if $\sigma \vDash x = 4 \wedge y = 1$?
   c. What is $M(W, \sigma)$ if where $\sigma \vDash x = 1 \wedge y = 1$?

Runtime Errors

5. Let $S \equiv x := y\ /\ b[x]$ and let $\sigma = \{b = (3, 0, -2, 4), x = \alpha, y = 13\}$. Find all σ such that $M(S, \sigma) = \{\perp_e\}$. (Remember, integer division truncates.)
6. Repeat the previous problem on $S \equiv y := y\ /\ sqrt(b[x])$ and $\sigma = \{b = (-1, 9, 12, 0), x = \alpha, y = 8\}$. Treat *sqrt* as returning the truncated integer square root of its argument: $sqrt(3) = 1$, $sqrt(8) = 2$, *sqrt* of $(15) = 3$, etc.

Solution to Practice 6 (Denotational Semantics; Runtime Errors)

Denotational Semantics

1. (Calculate meanings of programs)
   a. $M(x := x+1, \{x = 5\}) = \{\{x = 5\}[x \mapsto \{x = 5\}(x+1)]\} = \{\{x = 6\}\}$
   b. $M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x+1)]\} = \{\sigma[x \mapsto \sigma(x)+1]\}$
   c. $M(x := x+1;\ y := 2*x, \{x = 5\})$
   $= M(y := 2*x,\ M(x := x+1, \{x = 5\})$
   $= M(y := 2*x, \{x = 6\})$, from part (a)
   $= \{\{x = 6\}[y \mapsto \beta]\}$ where $\beta = \{x = 6\}(2*x) = 12$
   $= \{\{x = 6, y = 12\}\}$

2. Let $IF \equiv$ ***if*** $x > 0$ ***then*** $x := x+1$ ***else*** $y := 2*x$ *fi.*
   a. If $\sigma(x) = 8$, then $\sigma(x > 0) = T$, so $M(IF, \sigma) = M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x+1)]\} = \{\sigma[x \mapsto 9]\}$
   b. If $\sigma(x) = 0$, then $\sigma(x > 0) = F$, so $M(IF, \sigma) = M(y := 2*x, \sigma) = \{\sigma[y \mapsto \sigma(2*x)]\} = \{\sigma[y \mapsto 0]\}$
   c. If $\sigma(x) > 0$ then $M(S, \sigma) = M(x := x+1, \sigma) = \{\sigma[x \mapsto \sigma(x)+1]\}$
   If $\sigma(x) \le 0$ then $M(S, \sigma) = M(y := 2*x, \sigma) = \{\sigma[y \mapsto 2 \times \sigma(x)]\}$

3. Let $IF \equiv$ *if* $x > 0$ ***then*** $x := x/z$ ***fi*** $\equiv$ ***if*** $x > 0$ ***then*** $x := x/z$ ***else skip fi***
   a. If $\sigma = \{x = 8, z = 3\}$, then $\sigma(x > 0) = T$, so $M(IF, \sigma) = M(x := x/z, \sigma) = \{\sigma[x \mapsto \beta]\}$ where $\beta = \sigma(x/z)$ $= \sigma[x \mapsto 8 \div 3] = \sigma[x \mapsto 2]$, since integer division truncates.
   b. If $\sigma = \{x = -2, z = 3\}$ then $\sigma(x > 0) = F$, so $M(IF, \sigma) = M($***skip***$, \sigma) = \{\sigma\}$.

4. Let $W \equiv$ ***while*** $x < 3$ ***do*** S ***od*** where $S \equiv x := x+1;\ y := y*x$.
   a. For arbitrary τ,
   $M(S, \tau) = M(x := x+1;\ y := y*x, \tau)$
   $= M(y := y*x, \tau[x \mapsto \tau(x)+1])$
   $= \{\ \tau[x \mapsto \tau(x)+1][y \mapsto \beta]\ \}$ where $\beta = \tau[x \mapsto \tau(x)+1](y*x) = \tau(y) \times (\tau(x)+1)$
   b. If $\sigma \models x = 4 \wedge y = 1$, then $\sigma(x < 3) = F$ so $M(W, \sigma) = \{\sigma\}$.
   c. If $\sigma \models x = 1 \wedge y = 1$, then $\sigma(x < 3) = T$ so we have at least one iteration to do.
   Let $\sigma_0 = \sigma$, let $\sigma_1 = M(S, \sigma_0) = \sigma_0(y) \times (\sigma_0(x)+1)$, and let $\sigma_2 = M(S, \sigma_1) = \sigma_1(y) \times (\sigma_1(x)+1)$.
   Then,
   $\sigma_0 = \sigma[x \mapsto 1][y \mapsto 1]$
   $\sigma_1 = M(S, \sigma_0) = \sigma_0[x \mapsto \sigma_0(x)+1][y \mapsto \sigma_0(y) \times (\sigma_0(x)+1)] = \sigma[x \mapsto 2][y \mapsto 2]$
   $\sigma_2 = M(S, \sigma_1) = \sigma_1[x \mapsto 2+1][y \mapsto 2 \times (2+1)] = \sigma[x \mapsto 3][y \mapsto 6]$
   Since σ_0 and $\sigma_1 \models x < 3$ but $\sigma_2 \models x \ge 3$, we have $M(W, \sigma) = \{\sigma_2\} = \{\sigma[x \mapsto 3][y \mapsto 6]\}$.

Runtime Errors

5. $M(S, \sigma) = M(x := y/b[x], \sigma) = \{\sigma[x \mapsto \gamma]\}$ *where* $\gamma = \sigma(y/b[x]) = 13/\sigma(b)(\alpha) = \perp_e$
   iff $\sigma(b)(\alpha) = \perp_e$ or $\sigma(b)(\alpha) = 0$
   iff (α is out of range for $\sigma(b)$) or $(\sigma(b)(\alpha) = 0)$ ($b[x]$ fails if x is out of range)
   iff $(\alpha < 0$ *or* $\alpha \geq 4)$ or $(\sigma(b)(\alpha) = 0)$ ($\sigma(b)$ has size 4)
   iff $(\alpha < 0$ *or* $\alpha \geq 4)$ or $(\alpha = 1)$ ($b[1]$ is the only element $= 0$)
   iff $\neg(\alpha = 0, 2,$ or $3)$

6. $M(S, \sigma) = M(y := y/sqrt(b[x]), \sigma) = \{\sigma[y \mapsto \beta]\}$ where $\beta = (\sigma(y)/sqrt(\gamma)) = (8/sqrt(\gamma))$ and $\gamma = \sigma(b)\ (\sigma(x)) = \sigma(b)(\alpha)$.
   So $\beta = \perp_e$ and thus $M(S, \sigma) = \{\sigma[y \mapsto \perp_e]\} = \{\perp_e\}$
   iff $\gamma = \perp_e$ or $\gamma < 0$ *or* $sqrt(\gamma) = 0$ ($b[x]$ fails, $b[x] < 0$, or $sqrt(b[x]]) = 0$)
   iff (α out of range for $\sigma(b)$] or $\gamma < 0$ or $sqrt(\gamma) = 0$ ($\gamma = \perp_e$ iff $b[x]$ has a bad index)
   iff $(\alpha < 0$ or $\alpha \geq 4)$ or $\gamma = \sigma(b)(\alpha) < 0$ or $sqrt(\gamma) = 0$ ($\sigma(b)$ is of size 4)
   iff $(\alpha < 0$ or $\alpha \geq 4)$ or $(\alpha = 0)$ or $sqrt(\gamma) = 0$ (only $b[0] < 0$)
   iff $(\alpha < 0$ or $\alpha \geq 4)$ or $(\alpha = 0)$ or $(\alpha = 3)$ (only $sqrt(b[3]) = sqrt(0) = 0$)
   iff $(\alpha \leq 0$ or $\geq 3)$ (combining terms)

