# *Satisfaction, Validity, and State Updates*

## *CS 536: Science of Programming, Fall 2022*

Updated 2022-09-01, -05

## *A. Why*

- A predicate is satisfied or unsatisfied relative to a state.
- A predicate is valid if it is satisfied in all states.
- State updates occur when we introduce new variables or change the values of existing variables.

## *B. Outcomes*

At the end of today, you should

- Know how to check a predicate for satisfaction in a state, how to check a predicate for validity, and know how to update a state.

## *C. Questions*

1. Say u and v stand for variables (possibly the same variable) and α and β are values (possibly equal). When is $\sigma[u \mapsto \alpha][v \mapsto \beta] = \sigma[v \mapsto \beta][u \mapsto \alpha]$? Hint: There are four cases because maybe $u \equiv v$ and maybe $\alpha = \beta$.

2. Let $\sigma(b) = (7, 5, 12, 16)$. Assume out-of-bound indexes cause runtime errors.
   a. Does $\sigma \vDash \exists k .\ 0 \le k \wedge k+1 < size(b) \wedge b[k] < b[k+1]$? If so, what was your witness value for $k$?
   b. Does $\sigma \vDash \exists k .\ 0 \le k\text{-}1 \wedge k+1 < size(b) \wedge b[k\text{-}1] < b[k] < b[k+1]$? If so, what was your witness value for $k$?
   c. Does $\sigma \vDash \forall k .\ 0 \le k < 4 \rightarrow b[k] > 0$? [2022-09-01] Does $\sigma \vDash \forall k .\ 0 \le k < 4 \wedge b[k] > 0$ ?
   d. If $\sigma(k) = -5$, then does $\sigma \vDash \exists k .\ 0 \le k < 4 \wedge b[k] > 0$? Does $\sigma \vDash \exists k .\ 0 \le k < 3 \rightarrow b[k] < 0$ ?

3. For each of the situations below, fill in the blanks to describe when the situation holds.

   Fill in $\_\_\_\_\__1$ with "*some*", "*every*", or "*this*"

   Fill in $\_\_\_\_\__2$ with "*some*" or "*every*"

   Fill in $\_\_\_\_\__3$ with "σ(*x*) must *be undefined*", "σ(*x*) must *be defined and* $\sigma \vDash p$", or "*nothing of* σ(*x*)"

   Fill in $\_\_\_\_\__4$ with "$\vDash p$" or "$\nvDash p$"

   a. $\sigma \vDash (\exists x \in U.\ p)$ iff for $\_\_\_\_\__1$ state σ and $\_\_\_\_\__2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ $\_\_\_\_\__4$

b. $\sigma \vDash (\forall\, x \in U.\ p)$ iff for _____ $_1$ state σ and _____ $_2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ _____ $_4$

c. $\sigma \vDash (\exists\, x \in U.\ p)$ requires _____ $_3$ .

d. $\sigma \vDash (\forall\, x \in U.\ p)$ requires _____ $_3$ .

e. $\sigma \nvDash (\exists\, x \in U.\ p)$ iff for _____ $_1$ state σ for _____ $_2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ _____ $_4$

f. $\sigma \nvDash (\forall\, x \in U.\ p)$ iff for _____ $_1$ state σ for _____ $_2$ $\alpha \in U$, $\sigma[x \mapsto \alpha]$ _____ $_4$

g. $\nvDash (\forall\, x \in U.\ p)$ iff for _____ $_2$ state σ, we have σ _____ $_4$ $(\forall\, x \in U.\ p)$.

h. $\nvDash (\exists\, x \in U.\ p)$ iff for _____ $_2$ state σ, we have σ _____ $_4$ $(\exists\, x \in U.\ p)$.

i. $\nvDash (\forall\, x \in U.\ p)$ iff for _____ $_2$ state σ, and for _____ $_2$ $\alpha \in U$, we have $\sigma[x \mapsto \alpha]$ _____ $_4$

j. $\vDash (\exists\, x \in U\,.\ (\forall\, y \in V\,.\ p))$ iff for _____ $_1$ state σ, for _____ $_2$ $\alpha \in U$, and for _____ $_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ _____ $_4$

k. $\nvDash (\exists\, x \in U\,.\ (\forall\, y \in V\,.\ p))$ iff for _____ $_1$ state σ, for _____ $_2$ $\alpha \in U$, and for _____ $_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ $[\vDash \mid \vDash \neg]\ p$.

l. $\vDash (\forall\, x \in U\,.\ (\exists\, y \in V\,.\ p))$ iff for _____ $_1$ state σ, for _____ $_2$ $\alpha \in U$, and for _____ $_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ $[\vDash \mid \vDash \neg]\ p$.

m. $\nvDash (\forall\, x \in U\,.\ (\exists\, y \in V\,.\ p))$ iff for _____ $_1$ state σ, for _____ $_2$ $\alpha \in U$, and for _____ $_2$ $\beta \in V$, we have $\sigma[x \mapsto \alpha][y \mapsto \beta]$ _____ $_4$

n. [Added 2022-09-05]
$\sigma \nvDash \exists\, x \in U\,.\ (\exists\, y \in V\,.\ p(x, y)) \to (\exists\, z \in W\,.\ q(x, z))$ iff for _____ $_1$ state σ, for _____ $_2$ $\alpha \in U$, if for _____ $_2$ $\beta \in V$, $\sigma[x \mapsto \alpha][y \mapsto \beta]$ _____ $_4$ $p(x, y)$, then for _____ $_2$ $\delta \in W$, $\sigma[x \mapsto \alpha][[z \mapsto \delta]$ _____ $_4$ $q(x, z)$.

4. Let $p \equiv \exists\, y\,.\, \forall\, x\,.\, f(x) > y$, and let $q \equiv \forall\, x\,.\, \exists\, y\,.\, f(x) > y$. (As usual, assume a domain of $\mathbb{Z}$.)

   a. Is it the case that [2022-09-01] for any *f*, if *p* is valid then so is *q*? If so, explain why. If not, give a definition of $f(x)$ and show $\vDash p$ but $\nvDash q$.

   b. (The converse.) Is it the case that [2022-09-01] for any *f*, if *q* is valid then so is *p*? If so, explain why. If not, give a definition of $f(x)$ and show $\vDash q$ but $\nvDash p$.

 

## *CS 536: Solution to Activity 4 (Satisfaction, Validity, and State Updates)*

1. $\sigma[u \mapsto \alpha][v \mapsto \beta] = \sigma[v \mapsto \beta][u \mapsto \alpha]$ iff $u \not\equiv v$ or $\alpha = \beta$, or more precisely, iff $u \not\equiv v$ or ($u \equiv v$ and) $\alpha = \beta$.

2. (Quantified statements over arrays) Let $\sigma(b) = (7, 5, 12, 16)$.
   a. Yes, $\sigma \vDash \exists\, k \,.\, 0 \le k \wedge k+1 < size(b) \wedge b[k] < b[k+1]$ with 1 and 2 as possible witnesses for $k$.
   b. Yes, $\sigma \vDash \exists\, k \,.\, 0 \le k-1 \wedge k+1 < size(b) \wedge b[k-1] < b[k] < b[k+1]$ with 2 as the only witness that works.
   c. Yes, $\sigma \vDash \forall\, k \,.\, 0 \le k < 4 \to b[k] > 0$, since $b[0]$, $b[1]$, $b[2]$, and $b[3]$ are all positive in σ. Recall we're looking for an α such that $\sigma[k \mapsto \alpha] \vDash 0 \le k < 4 \to b[k] > 0$, and for $\sigma[k \mapsto \alpha]$, it doesn't matter whether $\sigma(k)$ has a value or what that value is.

      No: $\sigma \nvDash \forall\, k \,.\, 0 \le k < 4 \wedge b[k] > 0$ because there are plenty of values for $k$ that are not in the range 0 through 3. (So whether the body uses → or ∧ is extremely important.)
   d. Yes, $\sigma \vDash \exists\, k \,.\, 0 \le k < 4 \wedge b[k] > 0$, with witnesses k = 0, 1, 2, or 3. (Again, σ(k) is irrelevant.)

      Yes (and perhaps surprisingly), $\sigma \vDash \exists\, k \,.\, 0 \le k < 3 \to b[k] < 0$ with witness $k = 3$: $\sigma[k \mapsto 3]$ satisfies $0 \le k < 3 \to b[k] < 0$ because 3 makes $0 \le k < 3$ false, so the implication is true even though the value of $b[3]$ is positive. (I'm avoiding $k$ outside the range of $b$ because those $b[k]$ cause runtime errors.)

3. (Validity/invalidity of quantified predicates)
   a. this σ, some α, $\vDash p$
   b. this σ, every α, $\vDash p$
   c. nothing of σ($x$)
   d. nothing of σ($x$)
   e. this σ, every α, $\nvDash p$
   f. this σ, some α, $\nvDash p$
   g. some σ, $\nvDash$
   h. some σ, $\nvDash$
   i. some σ, some α, $\nvDash p$
   j. every σ, some α, every β, $\vDash p$
   k. some σ, every α, some β, $\nvDash p$
   l. every σ, every α, some β, $\vDash p$
   m. some σ, some α, every β, $\nvDash p$
   n. this σ, every α, some β, $\vDash q$, every δ, $\nvDash p$ because the negation of $(\exists\, x \,.\, ((\exists\, y \ldots) \to (\exists\, z \ldots)))$ is $(\forall\, x \,.\, ((\exists\, y \ldots) \wedge \neg(\exists\, z \ldots)))$.

4. ($\exists\,\forall$ predicates versus $\forall\,\exists$ predicates, specifically $p \equiv \exists\, y\,.\,\forall\, x\,.\, f(x) > y$, and $q \equiv \forall\, x\,.\,\exists\, y\,.\, f(x) > y$)

   a. The relation does hold: $\vDash p$ implies $\vDash q$. The short explanation is that for satisfaction of $q$, for each value α for $x$, we need to find a value β for $y$ that satisfies the body $f(x) > y$. Now, $p$ says that there's a value that works for every α, so we can use that value for β.

      In more detail, assume $p$ is valid: for every state σ, there is some value β where for every value α, $\sigma[y \mapsto \beta][x \mapsto \alpha] \vDash f(x) > y$.

      To show that $q$ is valid, take an arbitrary state τ with value δ for $x$. We need a witness value for the $\exists\, y$; since $\tau \vDash p$, there's a β for the $\exists\, y$ of $p$, and we'll use that as the witness for the $\exists\, y$ in $q$. To satisfy $q$, we need $\tau[x \mapsto \delta][y \mapsto \beta] \vDash f(x) > y$. Since $x \not\equiv y$, it doesn't matter whether we update using $x$ and then $y$ or vice versa. So it's sufficient to know $\tau[y \mapsto \beta]$ $[x \mapsto \delta] \vDash f(x) > y$, and we know that from $\tau \vDash p$.

   b. The relation does not hold: We can have $\vDash q$ but $\nvDash p$. An easy example is $f(x) = x$, then validity of $p$ would require us to find a value in $\mathbb{Z}$ for $y$ that is > every value of $x$ in $\mathbb{Z}$, but no such value exists.

      As an aside, if use an arbitrary predicate over $x$ and $y$ as the body of the $\exists\,\forall$ and $\forall\,\exists$ predicates, then the relation holds for some predicates and not for others. For example, $\exists\, x.\forall\, y\,.\; x \le y^2$ and $\forall\, y.\exists\, x.\; x \le y^2$ both hold.

 