① Explain about vector spaces and subspaces

sol

Vector space:- A vector space (V) is a non empty set containing objects or vectors (i.e., u, v, w) on which are defined two operations called addition and multiplication by scalar's, that satisfy the following conditions.

i) $u+v \in V \ \forall \ u,v \in V$

ii) $u+v = v+u \ \forall \ u,v \in V$

iii) $(u+v)+w = u+(v+w) \ \forall \ u,w \in V$

(iv) $u+0 = u \ \forall \ u \in V$

(v) $u+(-u) = 0, \ \forall \ u \in V.$

(vi) $c(du) = (cd)u$

$(c+d)u = cu+du$

$c(u+v) = cu+cv$

$1u = u$

Subspace:- A subset H of a vector space V is called subspace of V if it satisfy the following three conditions.

(i) The zero vector of V is in H i.e., $0 \in H$

(ii) H is closed under vector addition i.e., $u, v \in H, u+v \in H$

(iii) H is closed under scalar multiplication i.e., for each $u \in H$, there exists a scalar 'c' such that $cu \in H$.

① Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$ show that. H is subspace of R^3.

sol

Given H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$

clearly $H = \text{span}\{v\}$

The vector set H can be written as

$$H = t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

which shows the span of $\{v\}$

$$v = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore$ H is a subspace of R^3.

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② Let w be the set of all vectors of the form $\begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix}$ where b & c are arbitary. Find vector u & v such that $w = \text{span}\{u, v\}$ Does w is a subspace of R^3.

sol

Given set H is the set of all vectors of the form

$$\begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix}$$

clearly the set $H = \text{span}\{v\}$ the vector set H can be written as $H = b\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ which shows span of v

$$v = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

$$H = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}\right\}$$

$\therefore H$ is a subspace of $R^3(R)$.

③ Prove the set of solutions (x, y, z) of the equation $x+y+2z=0$ is a subspace of the space $R^3(R)$.

sol

Given the set of solutions $(x, y, z) / x+y+2z=0$

Let $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ where $\alpha, \beta \in W$

$\boxed{\text{Let } a, b \in R \text{ and } \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W.}$

$$a\alpha + b\beta = a(x_1+y_1+z_1) + b(x_2+y_2+z_2)$$

$$= (ax_1+ay_1+az_1) + (bx_2+by_2+bz_2)$$

$$= (ax_1+bx_2,\ ay_1+by_2,\ az_1+bz_2)$$

Let us consider

$p = ax_1 + bx_2$

$q = ay_1 + by_2$

$r = az_1 + bz_2$

since $\alpha \in W \Rightarrow x_1 + y_1 + 2z_1 = 0$ — ①

$\beta \in W = x_2 + y_2 + 2z_2 = 0$ — ②

$p+q+2r = ax_1 + bx_2 + ay_1 + by_2 + 2(az_1 + bz_2)$

$= ax_1 + bx_2 + ay_1 + by_2 + 2az_1 + 2az_2$

$= (ax_1 + ay_1 + 2az_1) + (bx_2 + by_2 + 2bz_2)$

$= a(x_1 + y_1 + 2z_1) + b(x_2 + y_2 + 2z_2)$

$= a.0 + b.0$

$p + q + 2r = 0.$

∴ The above condition is satisfied

W is a subspace of $R^3(R)$.

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* Define null space of an $m \times n$ matrix A.

A:- Null space of an $m \times n$ matrix A is the set of all solutions to the homogeneous equation.

∴ $Ax = 0$.

which is represented in set notation form as

Null A = $\{x : x \text{ is in } R^n \ \& \ Ax = 0\}$

which is denoted by Nul A. $\boxed{\text{Nul A}}$

*** Define column space of a $(m \times n)$ matrix A.

sol The column space of a $m \times n$ matrix A is the linear combination of the columns of A

i.e., $A = \{a1, a2, a3 \cdots an\}$ then

column space $A = \text{span}\{a1, a2, \cdots an\}$ which is denoted by col A.

*** Define linear transformation

→ If V and W represente two vector spaces then a mapping $T: V \to W$ is said to be linear transformation if the following conditions are satisfied.

(i) $T(u+v) = Tu + Tv \ \forall u, v \in V$

(ii) $T(cu) = cT(u) \ u \in V$ and all scalars c.

*** Define Co-ordinates

A:- Let $S = \{\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n\}$ be a basis set of a finite dimensional vector space $V(F)$. Let $\beta \in V$

$\beta = a_1\alpha_1 + a_2\alpha_2 + \cdots a_n\alpha_n \ \forall a_1, a_2 \cdots a_n \in F$

then $\exists$ scalars $(a1, a2, \cdots an)$ are called co-ordinates.

*** Linearly Independent

→ An indexed set of vectors $\{v_1, \cdots, v_p\}$ in R^n is said to be linearly independent if the vector eauation

$$\boxed{x_1 v_1 + x_2 v_2 + \cdots x_p v_p = 0}$$

*** Linearly dependent

→ It has only trivial solution. The set $\{v_1, \ldots v_p\}$ is said to be linearly dependent if there exist weights $c_1, \ldots c_p$ not all zero, such that

$$\boxed{c_1 v_1 + c_2 v_2 + \cdots c_p v_p = 0}$$

*** Define kernal and Range of a linear transformation

→ If V and W are two vector spaces and T is a linear transformation from V into W then the kernel or null space of a linear transformation N(T) is defined as the set of all vectors u in V such that, $T(u) = 0$. where, 0 is the zero vector in W

It represented in set notation form as,

$$N(T) = \{v \in V : T(u) = 0 \in W\}$$

Range of a Linear transformation

→ If V and W are two vector spaces and T is a linear transformation from V into W then range of linear transformation R(T) is defined as the set of all vector in W of the form $T(x)$ for some $x \in V$.

It is represented in set notation form as.

$$R(T) = \{T(x) : x \in V\}.$$