Integer Multiplication

5478
2341

Input: $x_1, x_2 \ldots x_n$
$y_1, y_2 \ldots y_n$

$$\begin{array}{r} 5478 \\ 2341 \\ \hline 5478 \\ 20 \\ 00 \\ 00 \\ \hline 78 \end{array}$$

4×5478

$\leq (n-1)$ addition
n multiplica

$\leq n^2$ mul
$\leq n^2$ addition
$\leq n^2$ addition

$c \cdot n^2 \leq$ #operation $\leq n^2 + 2n^2$
$= O(n^2)$

Worst-case

#operation $= \Theta(n^2)$

$(x_1 \ldots x_n)(y_1 \ldots y_n)$

$(100y_{n-2} + 10y_{n-1} + y_n)$

$= (x_1 \ldots x_n) * y_n + 10\, y_{n-1} * x_1 \ldots x_n$

$x_1 \; x_2 \ldots \ldots x_n \leftarrow x$

$y_1 \; y_2 \ldots \ldots y_n \leftarrow y \quad n = 2^k$

$(y_1 y_2 \ldots y_{n/2}) \times 10^{n/2} + y_{n/2+1} \ldots y_n$

$10^{n/2} \cdot c + 10^{n/2} \cdot d$

$(x_1 \ldots x_{n/2}) \times 10^{n/2} + x_{n/2+1}\, x_{n/2+2} \ldots x_n$

$10^{n/2} \cdot a + b$

$(10^{n/2} a + b) * (10^{n/2} c + d)$

$4 = 10^{n} \cdot \underline{ac} + 10^{n/2} (\underline{ad} + \underline{bc}) + \underline{bd}$

① Split as above

② Multiply (a, c)

(ad)

$\underline{(b, c)}$

$\underline{(b, d)}$

(3) Compute $a = 10^n \cdot p + 10^{n/2}(q+r) + s$

$n \quad 2n$

$\leq \underline{\underline{10 \cdot n}}$

$$\begin{cases} T(n) \leq 4T(n/2) + 10 \cdot n \\ T(1) = 1 \end{cases}$$

$T(n) =$

$T(n) \leq O(n)$

$T(n) \leq \cancel{1000} \cdot n$

$$T(n/2) \le \cancel{1000}\, c \,\frac{n}{2}$$

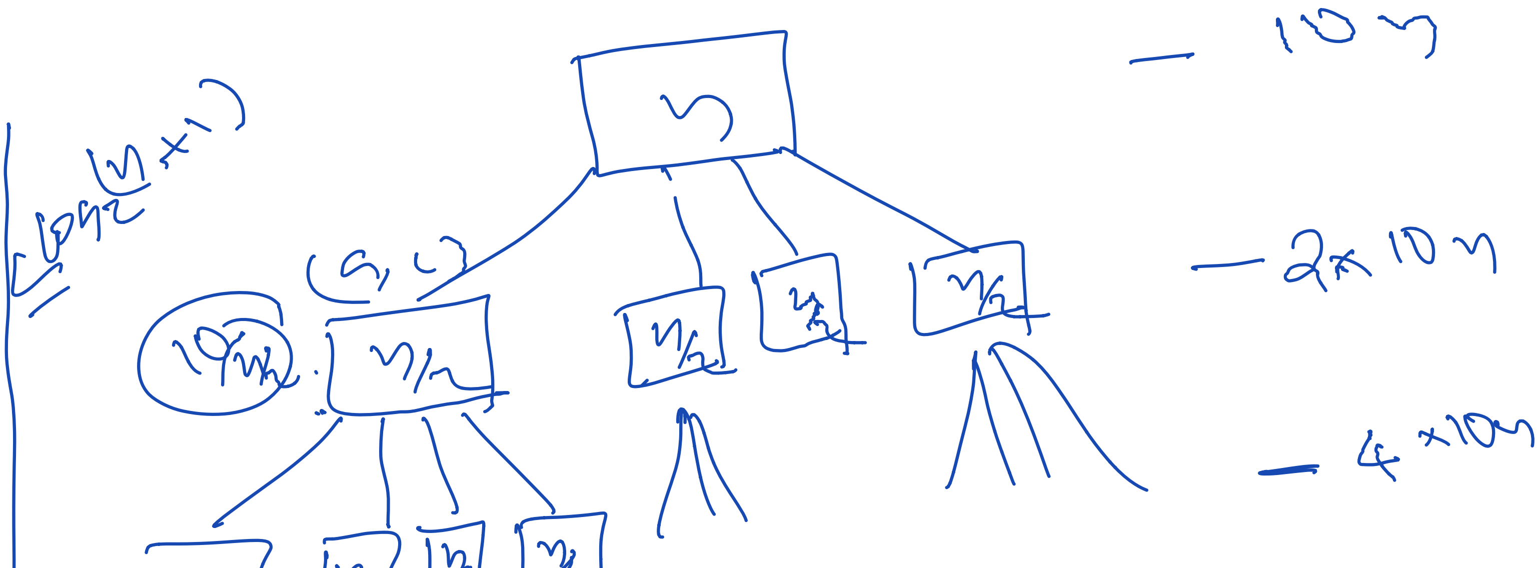
$$T(n) \le \underbrace{\cancel{4} \cdot \cancel{2000}\, c \,\frac{n}{\cancel{2}}}_{2c \cdot n} + 10 \cdot n$$

$$\not\le 1000 \cdot n$$

$$T(n) \le 4 n \log n$$

$$T(n) \le 200 \cdot n^2$$

(Proof by induction,
– substitution.)



$\boxed{n/4}$ $\boxed{n/4}$ $\boxed{n/4}$ $\boxed{n/4}$

$- 8 \times 10 \cdot \frac{n}{8}$

$- 2^{\log_2 n} \cdot 10 \cdot \frac{n}{n}$

$\boxed{1}$

$$10 \cdot n \left(\underline{1 + 2 + 4 + 8 + \cdots 2^{\log_2 n}} \right).$$

$$\leq 2n.$$

$$\leq \underline{20 n^2}.$$

$$T(n) = O(n^2)$$

$$T(n) = \Omega(n^2).$$

$$10^n (ac) + 10^{n/2} (ad + bc) + bd$$

$$(ac + bd) - (a-b)(c-d) = bc + ad$$

a_1 — Multiply (a, c)

a_2 — Multiply (b, d)

a_3 — Multiply $(a-b, c-d)$

— Combine. cn

$a_4 = bc + ad. = (a_1 + a_2) - a_3$

$10^n \cdot a_1 + 10^{n/2} \cdot a_4 + a_2$

$T_2(n) \leq 3T_2(n/2) + \underline{\underline{c \cdot n}}$

$T_1(n) \leq 4T_1(n/2) + \underline{\underline{10 \cdot n}}$

$\rightarrow T_2(n)$?

Solving R.R

$$S(n) = S(n-1) + \frac{c \cdot n}{d(n)}$$

$$S(0) = 0$$

$$S(n) = c \cdot \sum_{i=0}^{n} i \quad d(i)$$

$$= c \cdot \frac{n(n+1)}{2}$$

$$R(0) = 0$$

$$R(n) = aR(n-1) + d(n)$$

$$\frac{R(n)}{a^n} = \frac{R(n-1)}{a^{n-1}} + \frac{d(n)}{a^n}$$

$$S(n) = \frac{R(n)}{a^n}.$$

$\hookrightarrow S(n) = S(n-1) + \frac{d(n)}{a^n}$

$= \sum_{i=0}^{n} \frac{d(i)}{a^i}$

"Range transformation"

$R(n) = a^n \cdot S(n)$.

$T(n) = T(n/b) + d(n)$

$T(1) = 1$

$\underline{n} = b^k$

$S(k) = T(b^k)$

$T(b^k) = T(b^{k-1}) + d(b^k)$

Domain transformation.

$$\boxed{S(k) = S(k-1) + d(k)}$$

$n = 2^k)$

$$T(n) = 3T(n/2) + c \cdot n$$

$$T(2^k) = 3T(2^{k-1}) + c \cdot n$$

$$R(k) = 3R(k-1) + c \cdot n$$

$$\frac{R(k)}{3^k} = \frac{R(k-1)}{3^{k-1}} + \frac{cn}{3^k}$$

$$S(k) = S(k-1) + \frac{c \cdot 2^k}{3^k}$$

$$S(k) = c \sum_{i=0}^{k} \left(2/3\right)^i$$

$$T(n) = T(2^k) =$$

$k = \log_2 n$

$n = 2^k$

$$= c \cdot 3^k \sum_{i=0}^{k} \left(\frac{2}{3}\right)^i$$

$1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots$

$\frac{1}{1 - 2/3}$

$$= c \cdot 3^{\log_2 n} \cdot 3$$

$$= 3 \cdot c \cdot 3^{\log_2 n}$$

$$= 3 \cdot c \cdot \left(2^{\log_2 3}\right)^{\log_2 n}$$

$$= 3 \cdot c \cdot (n)^{\underline{\log_2 3}}$$

$$n^{1.58\ldots}$$

$$= \underline{O\left(n^{1.58\ldots}\right)}$$