Lecture 10

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18th September 2018

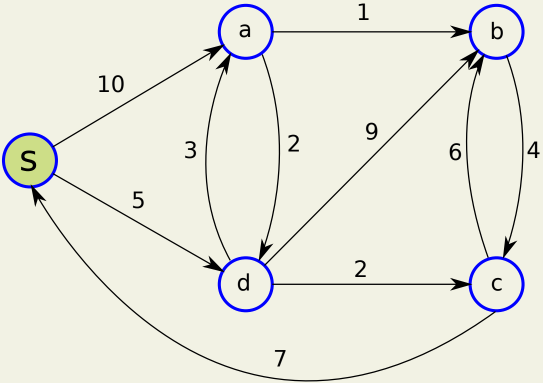
Recap: Shortest paths

Input:

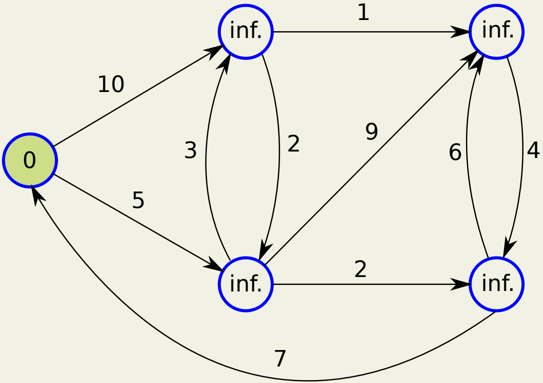
- Graph $G = (V, E)$
- Weight function $w : E \to \mathbb{Z}^+$
- Source vertex $s \in V$

Goal: Compute the shortest path from s to all reachable vertices.

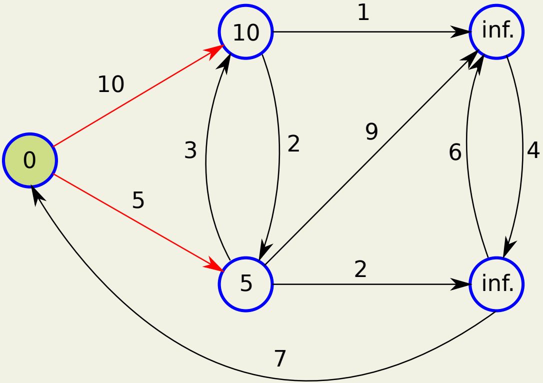
Example graph



Example graph



Example graph



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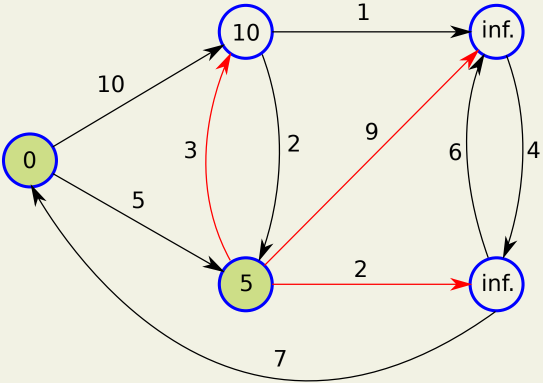
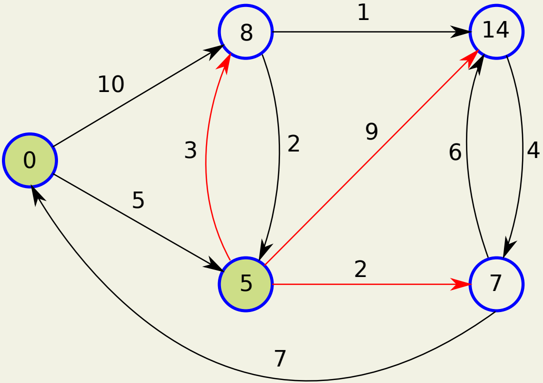
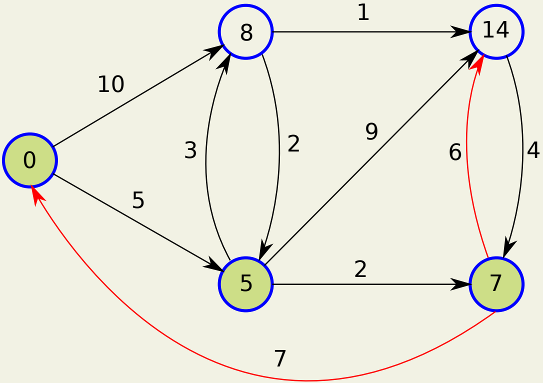


Example graph



Example graph



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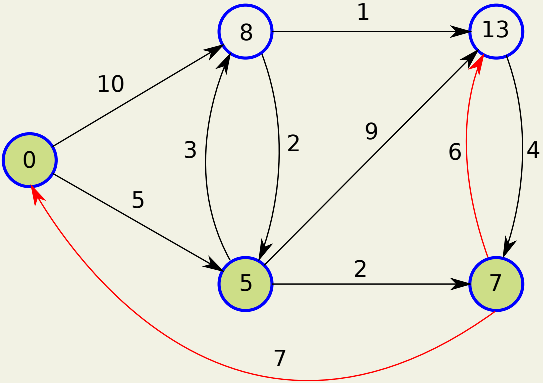

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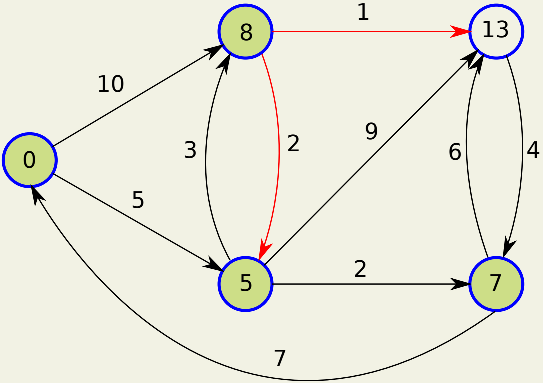
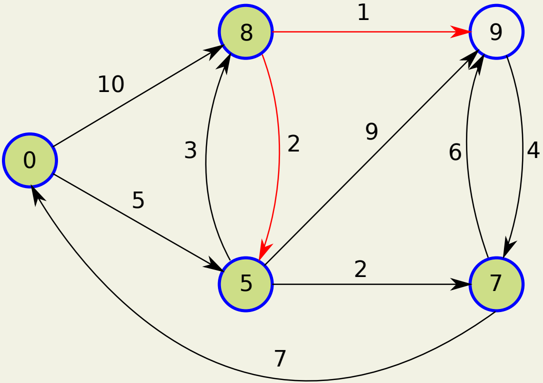
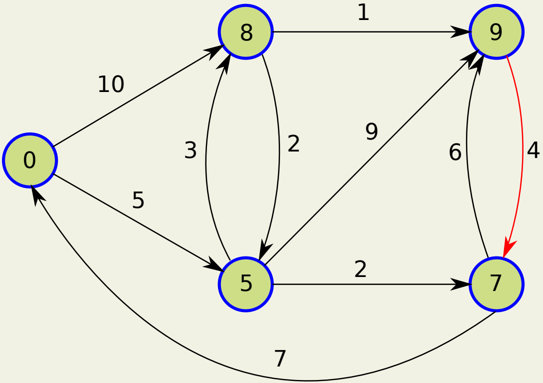


Example graph



Example graph



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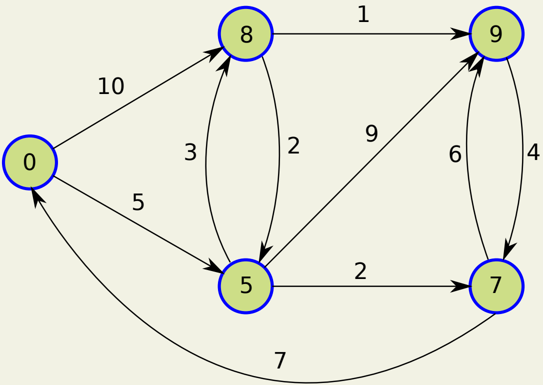


Dijkstra's algorithm

"It is the algorithm for the shortest path, which I designed in about twenty minutes. One morning I was shopping in Amsterdam with my young fiancée, and tired, we sat down on the café terrace to drink a cup of coffee and I was just thinking about whether I could do this, and I then designed the algorithm for the shortest path. As I said, it was a twenty-minute invention."

-Edsger Dijkstra

Source: Wikipedia and "An Interview with Edsger W. Dijkstra". Communications of the ACM

Dijkstra's Algorithm Pseudocode

Algorithm 1 Dijkstra's algorithm

For all $u \in V$, $d[u] \leftarrow \infty$, $\pi[u] \leftarrow$ NIL
$d[s] \leftarrow 0$
Initialize min-priority queue $Q \leftarrow V$
$S \leftarrow \emptyset$
while $Q \neq \emptyset$ **do**
    $u \leftarrow \textsc{Extract-Min}(Q)$
    $S \leftarrow S \cup \{u\}$
    for each $v \in \mathcal{N}(u)$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            $d[v] \leftarrow d[u] + w(u, v)$
            DECREASE-KEY$(v, d[v])$.
            $\pi[v] \leftarrow u$
        end if
    end for
end while

Proof of Correctness

Theorem

At the end of Dijkstra's algorithm, we have:

$$\forall u \in U, d[u] = \delta(s, u)$$

Proof

Loop Invariant:
At the start of each iteration, we have $\forall v \in S, d[v] = \delta(v)$.
Init: At the start of the first iteration, $S = \emptyset$.
Maintenance: Look at the start of the iteration in which a vertex $u \in V$ was added to S.
If u is added to S, then it must be reachable.
If $u = s$, then the claim holds. So assume $u \neq s$.

Proof of Correctness

Claim 1: $d[u] \geq \delta(s, u)$

Take a shortest path σ from s to u.

Let y be the first vertex on σ that is outside S.

Let $x \in S$ be the vertex on σ just before y.

So the path σ looks like:

$$s \overset{\sigma_1}{\leadsto} x \to y \overset{\sigma_2}{\leadsto} u$$

Claim 2: $d[y] = \delta(s, y)$.

Proof of Correctness

$$\sigma = s \overset{\sigma_1}{\rightsquigarrow} x \to y \overset{\sigma_2}{\rightsquigarrow} u$$

Claim 1: $d[u] \geq \delta(s, u)$.
Claim 2: $d[y] = \delta(s, y)$.
Since y appears before u in σ, we have $\delta(s, y) \leq \delta(s, u)$.
Hence:

$$d[y] = \delta(s, y) \leq \delta(s, u) \leq d[u]$$

Although y and u were in $V \setminus S$, Extract-Min returned u.
This means $d[u] \leq d[y]$. Hence:
$d[y] = \delta(s, y) = \delta(s, u) = d[u]$

$\square$

Proof of Correctness

Claim 2

$$\sigma = s \overset{\sigma_1}{\rightsquigarrow} x \to y \overset{\sigma_2}{\rightsquigarrow} u$$

We have $d[y] = \delta(s, y)$

Proof

From loop invariant, for all vertices that were added to S before u, we computed the correct shortest distance.
So $d[x] = \delta(s, x)$.
We updated $d[y]$ when we added x to S.
Now we note a *convergence* property:
If y is on a shortest path σ from s to u.
Then, the path formed by σ from s to y is a shortest path from s to y.

$\square$

Proof of Correctness

Claim 1

$d[u] \geq \delta(s, u)$

Proof

Induction on number of times d is updated after initialization.

Base case: Immediately after init, $\forall v$, $d[v] = \infty$ except $d[s] = 0$. So the claim holds.

Step: Assume claim for up to k many updates on d.

The value of $d[u]$ is updated when:

- We visit a vertex v and there exists edge (v, u).
- $d[u] > d[v] + w((v, u))$.

The new $d[u]$ is $d[v] + w((v, u))$.

The hypothesis holds for vertex v: $d[v] \geq \delta(s, v)$. So:

$$d[u] = d[v] + w((u, v)) \geq \delta(s, v) + w((u, v)) \geq \delta(s, u)$$

Spanning Tree

Definition: An undirected graph G is *connected* if every vertex is reachable from every other vertex.

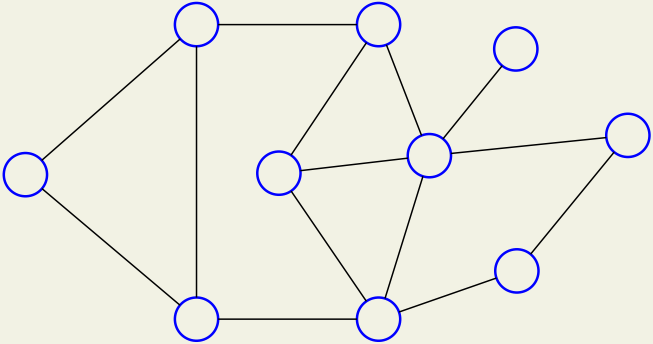
A graph $T = (V, E')$ is a spanning tree of an undirected connected graph $G = (V, E)$ if:

- $E' \subseteq E$.
- T is a *tree*. i.e., there are no cycles in T.
- T is connected.

Informally: A spanning tree for G is a tree that can be found inside G which *spans* all vertices of G.

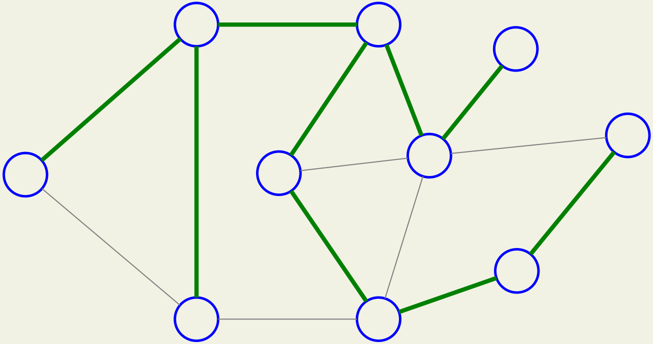
Examples

What are the possible spanning trees for this graph?



Examples

Is this a spanning tree?



Examples

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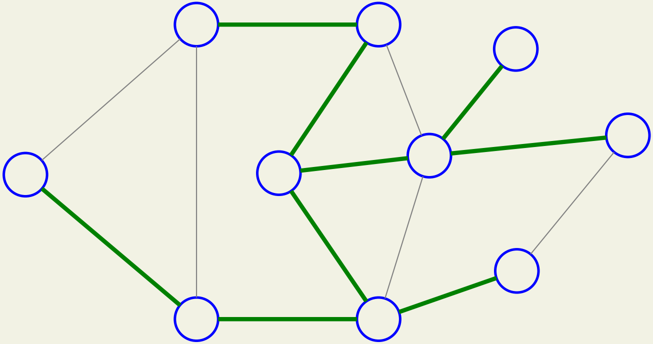

Examples

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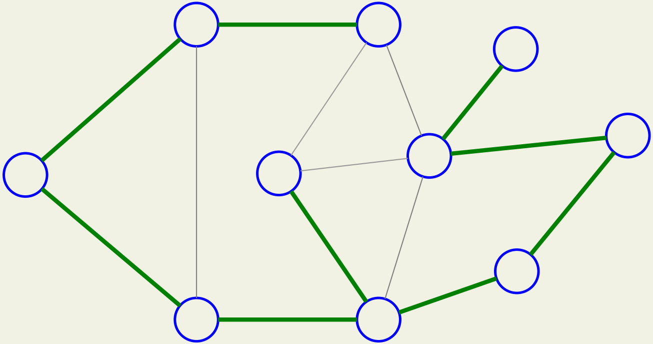

Examples

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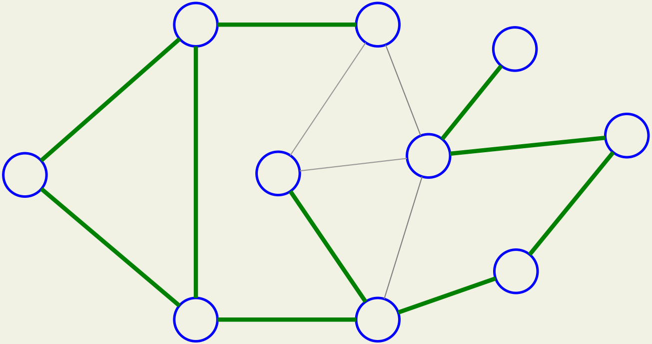
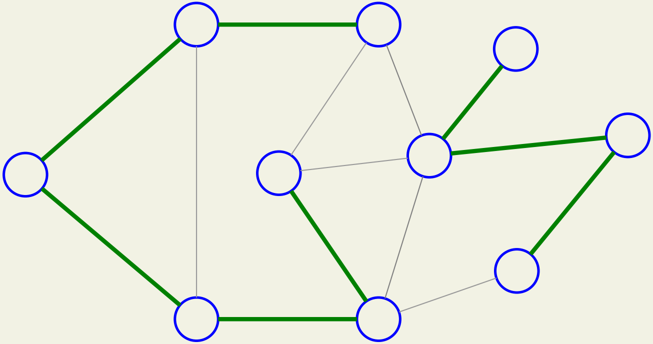


Examples

Is this a spanning tree?



Minimum Spanning Tree Problem

Input

- Undirected connected graph $G = (V, E)$
- Weight function $w : E \to \mathbb{Z}^+$

Goal

Compute a spanning tree for G with minimum total weight.

Kruskal's algorthm example