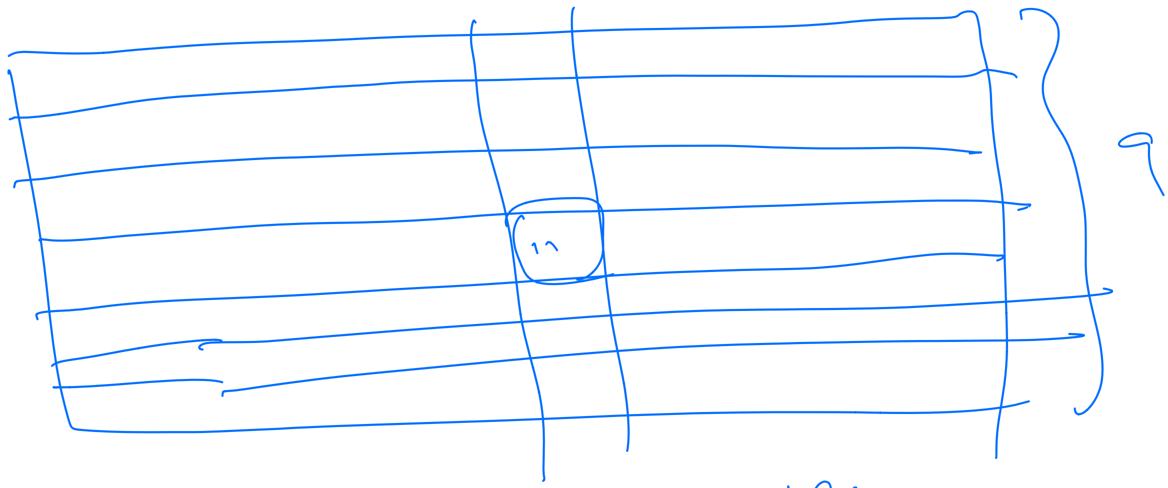
Quiz 1 – Answers.

(1)



Each row has size $n/9$

Half of each row is $n/18$

element less than median of row 5 is $\geq 5n/18$

elements greater than the median of row 5 is

$$\geq 5n/18.$$

Size of each sub problem is at most $\left(1 - \frac{5}{18}\right) n$

$$= \frac{13}{18} \cdot n$$

1 (a) $T(n) = T\left(\frac{13}{18}n\right) + T\left(\frac{n}{9}\right) + O(n)$

$T(1), \dots, T(9) = O(1)$

Lowerbound: $T(n) = \Omega(n)$.

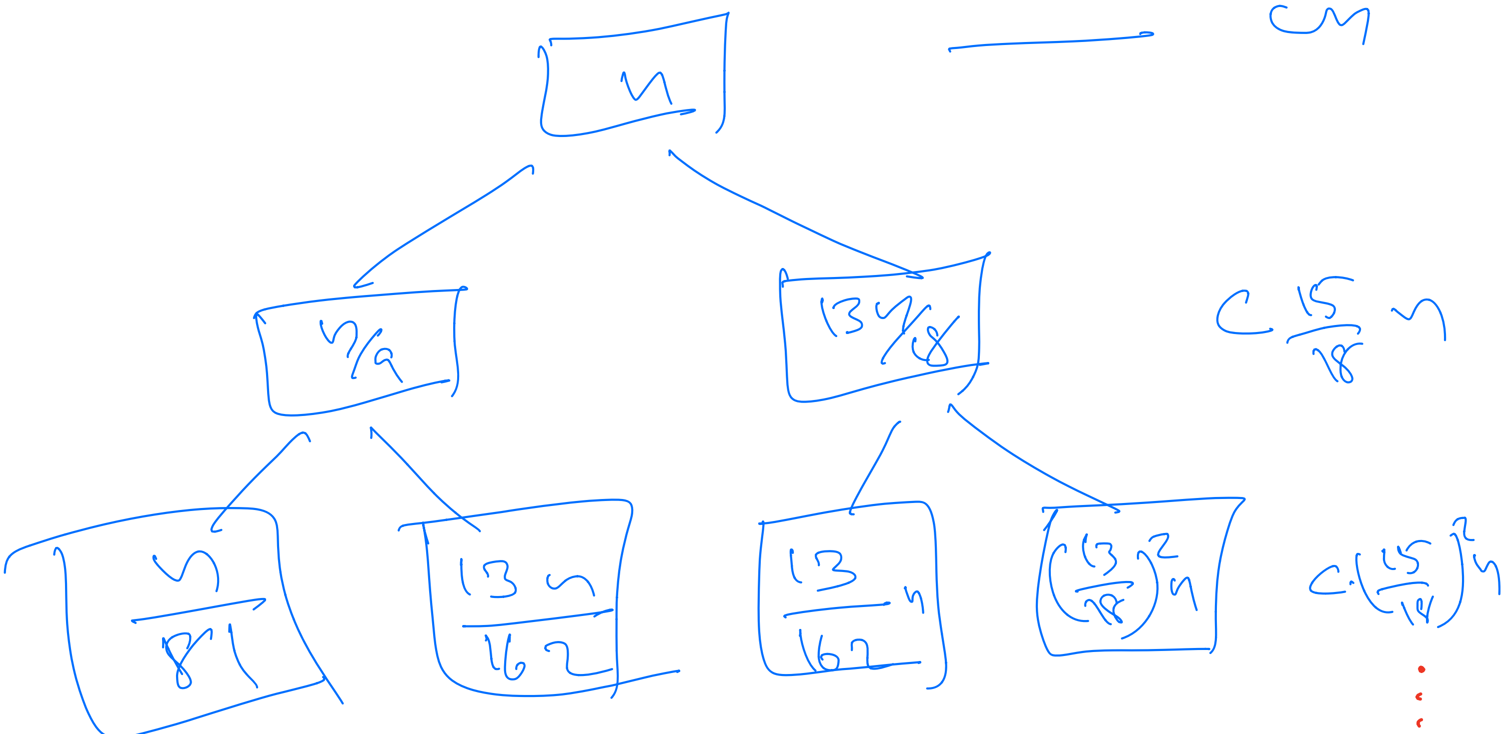
Upperbound: $T(n) = O(n)$

Proof (1 mark)

$$\frac{13}{18} + \frac{1}{9} = \frac{15}{18} < 1$$

$$\Rightarrow T(n) = O(n).$$

Or recurrence tree method



$$\frac{4 + 2\times 13 + 2\times 13 + 13^2}{(18)^2}$$

$$= \frac{(15)^2}{(18)^2}$$

$$T(n) = cn + c \cdot \frac{15}{18} n + \left(\frac{15}{18}\right)^2 n + \cdots$$

$$= cn \cdot \left(\frac{1}{1 - 15/18}\right)$$

$$= \frac{18}{3} \cdot cn = \underline{\underline{6cn}}$$

$$\rightarrow T(n) = \underline{\underline{O(n)}}$$

③ $T(n) = 3T(n-1) + n^2$

$T(0) = 1$

We prove the following statement by induction:

$$T(n) \le 2\times 3^n - n^2$$

<u>Base case:</u>

$$T(0) = 1 \le 2\times 3^0 - 1^2$$

<u>Induction step:</u> $n \ge 1$

By I.H., $T(n-1) = 2\times 3^{n-1} - (n-1)^2$

$$\therefore T(n) \le 3\left(2\times 3^{n-1} - (n-1)^2\right) + n^2$$

$$\le 2\times 3^n - 3(n-1)^2 + n^2$$

$$\le 2\times 3^n - 2n^2 + 6n - 3$$

$$\le 2\times 3^n - 2n^2 \quad (6n-3 \ge 0 \text{ for } n \ge 1)$$

$$\leq 2\times 3^{n} - n^{2}.$$

2(a).

$$T(n) = \sqrt{n}\, T(\sqrt{n}) + 5$$

$$T(1) = T(2) = 5.$$

We use domain and range transformation.

substitute $n = 2^{k}$.

$$T(2^{k}) = 2^{k/2}\, T\left(2^{k/2}\right) + 5$$

Divide by 2^k on both sides

$$\frac{T(2^k)}{2^k} = \frac{T(2^{k/2})}{2^{k/2}} + \frac{5}{2^{k/2}}$$

Let $S(k) = \dfrac{T(2^k)}{2^k}$

(Base case) $S(1) = \dfrac{T(1)}{1} = 5$

Then, $S(k) = S(k/2) + \dfrac{5}{2^{k/2}}$ — (2)

(2) can be solved in two ways.

(i) $S(k) = \dfrac{5}{2^{k/2}} + \dfrac{5}{2^{k/4}} + \dfrac{5}{2^{k/8}} + \cdots$

(iii) $\cancel{k}\ 2^{\ell} = k$

$$S(2^{\ell}) = S(2^{\ell-1}) + \frac{5}{2^{\frac{2^{\ell}}{2}}}$$

$$R(\ell) = R(\ell-1) + \frac{5}{2^{2^{\ell}/2}}$$

$$= \frac{5}{2^{\frac{2^{\ell}}{2}}} + \frac{5}{2^{\frac{2^{\ell-1}}{2}}} + \cdots$$

$$= \frac{5}{2^{k/2}} + \frac{5}{2^{k/4}} + \frac{5}{2^{k/8}} + \cdots$$

Thus:

$$S(k) = \frac{5}{2^{k/2}} + \frac{5}{2^{k/4}} + \frac{5}{2^{k/8}} + \cdots$$

$$S(0) = 5$$

$$S(k) \le 5\left(\cdots + \frac{1}{2^8} + \frac{1}{2^4} + \frac{1}{2^2} + \frac{1}{2}\right)$$

$$\le 5\left(\cdots + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2}\right)$$

$$\le 5 \cdot 2 \le \underline{\underline{10}}.$$

$$T(2^k) = 2^k \cdot S(k)$$

$$\le 10 \cdot 2^k$$

$$T(n) \le 10 \cdot n.$$

$$= O(n).$$

2(b)

$$T(n) = 3T(n/4) + T(n/6) + n.$$

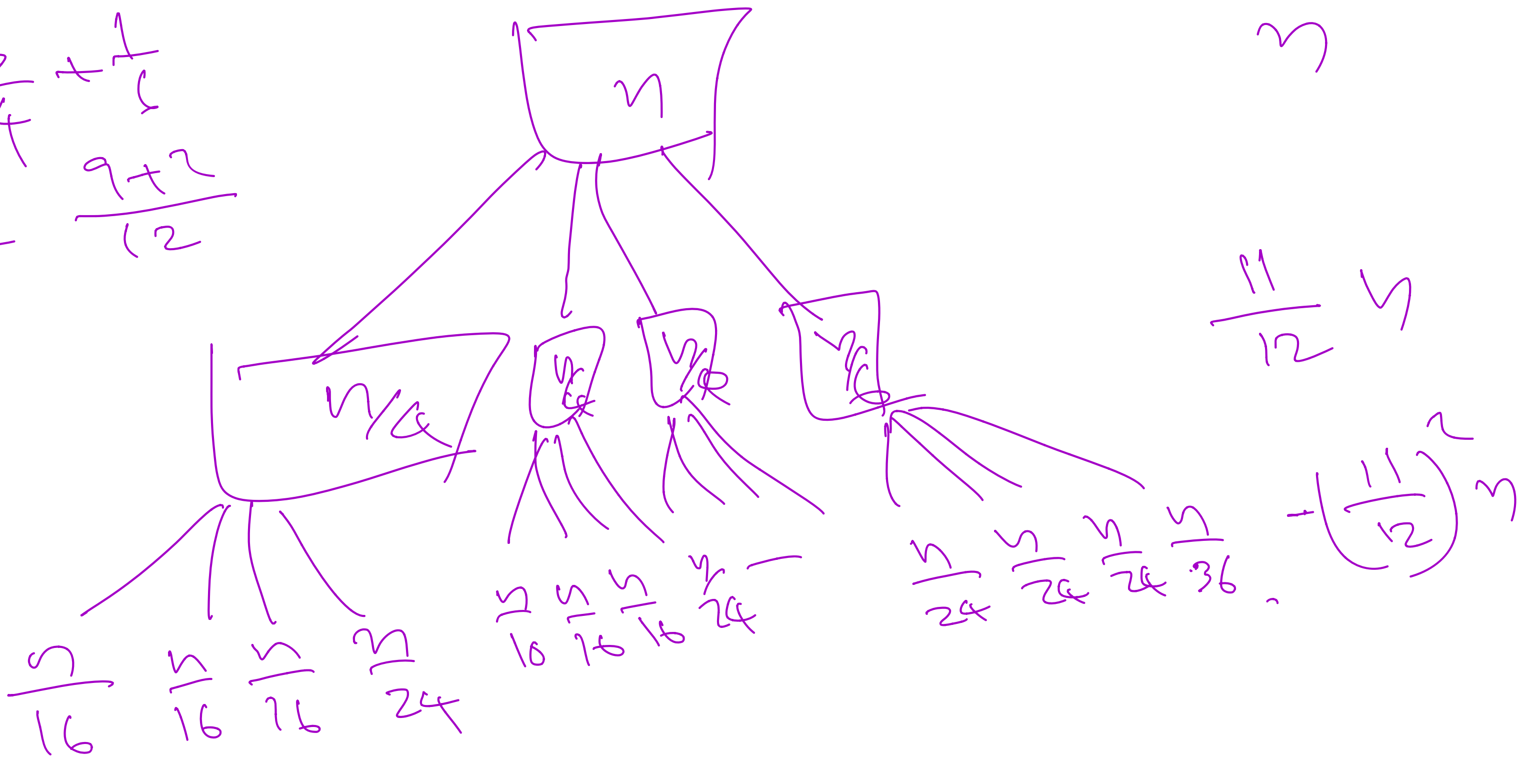
$$T(1) = T(2) = 5$$

Answer $T(n) = O(n)$

Proof: recurrence tree method.

$\frac{3}{4} + \frac{1}{6}$

$= \frac{9+2}{12}$



$$n\left(1 + \frac{11}{12} + \left(\frac{11}{12}\right)^2 + \cdots\right)$$

$$= n\left(\frac{1}{1 - \frac{11}{12}}\right)$$

$$= n(12) = \underline{O(n)}$$

Another method.

Induction

$T(n) \leq 5n$

Base case $T(2) = 5 \leq 10$

$T(1) = 5 \leq 5$

Induction:

$T(n) \leq 3\left(5\frac{n}{4}\right) \quad 5\cdot\frac{n}{6} \quad + n$

$$\leq \frac{15}{4}n + \frac{5n}{6} + n$$

$$\leq \frac{45+10+12}{12}$$

$$\leq \frac{67}{12}n$$

$$\leq \frac{60}{12}n = 5n$$