Red Black Trees Q&A

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Abstract Data Type

Set

maintains a set of elements from the universe

A set has the following functions:

- Insert(x) – Insert x into the set.
- Search(x) – Return True if x is an element of the set.
- Succ(x) – Returns the smallest value larger than x.
- Pred(x) – Returns the largest value smaller than x.
- GetMax() – Returns the largest value in the set.
- GetMin() – Returns the smallest value in the set.
- IsEmpty() – Returns True if and only if the set is empty.
- Delete(x) – Remove x from the set.

Data Structures for Set

Many choices of data structure to implement set:

- Array
- Sorted Array
- Heap
- Binary Search Tree
- Balanced Binary Search Trees

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We now study a balanced Binary Search Tree called Red-Black Trees.

Red-Black Trees

RBTs have the following properties:

1. All nodes are colored either Red or Black.
2. The root node and the leaf nodes (NIL) are black.
3. Both children of a red node are black.
   No double red.
4. For any node x, all paths from x to the descendant leaves have the same number of black nodes. = Black height(x)

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Black height of a red black tree is the black height of its root.

Example

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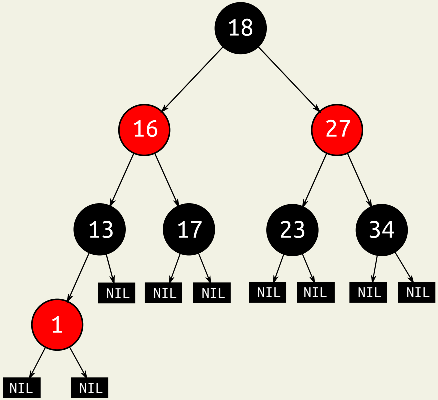

A Red-Black Tree supports all procedures of a BST:

- INSERT(*val*) – Inserts *val* into the RBT rooted at *node*.
- SEARCH(*val*) – Returns True of *val* exists in the BST rooted at *node*. False otherwise.
- SUCC(*val*) – Returns the smallest element greater than *val* in the RBT.
- PRED(*val*) – Returns the largest element lesser than *val* in the RBT.
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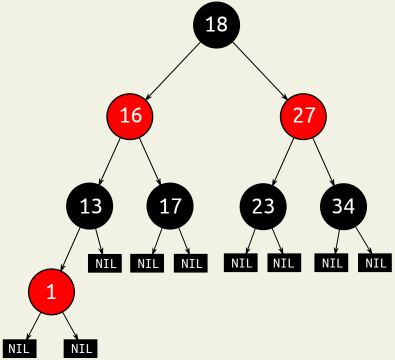
The procedures in green are implemented exactly like in a BST.

Black-Height

The black-height of a node X is the number of black colored nodes encountered on a path starting from X to any leaf (excluding X itself).



The black-height of the node with value 13 is 1.
The black-height of the root node is 2.
The black height of an red-black tree is the black height of its root.

Observations

Claim

A red-black tree with black-height β has height at most 2β.

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Proof sketch:

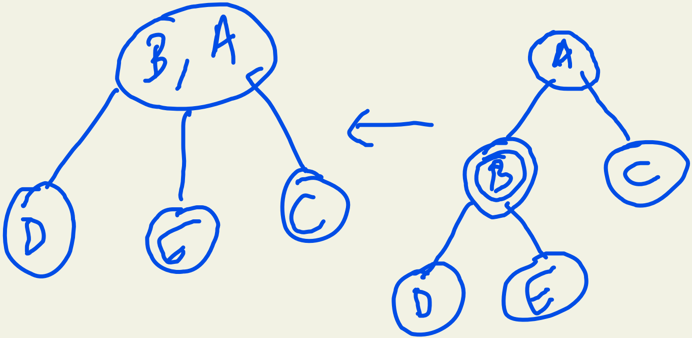
- Try to construct the longest possible path with at most β many black nodes.
- Property 4 will force you to color every alternate node black.

Observations

Theorem

If a red-black tree with n *internal* nodes and black height β, then

$$2^{\beta} \leq n + 1 \leq 4^{\beta}.$$



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If a red-black tree with *n internal* nodes and black height β, then

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Proof sketch

- Merge each red node with its parent.
- Now each node has 1, 2 or 3 values with 2, 3 or 4 children.
  This is a 2-3-4 tree!

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Proof sketch

- Merge each red node with its parent.
- Now each node has 1, 2 or 3 values with 2, 3 or 4 children.
  This is a 2-3-4 tree!
- The above 2-3-4 tree has height β.
- Thus $2^{\beta} - 1 \leq n \leq 4^{\beta} - 1$.

Observations

Theorem

A red-black tree with n *internal* nodes has height at most $2\log(n+1)$.

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- We have seen that $2^{\beta} \leq n+1 \leq 4^{\beta}$.
- That is, $1/2\log(n+1) \leq \beta \leq \log(n+1)$.

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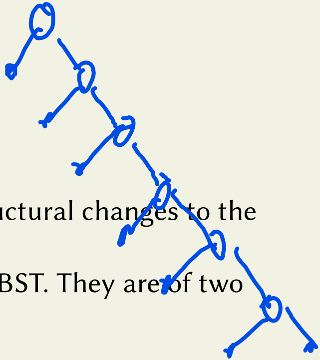
Proof

- We have seen that $2^\beta \leq n+1 \leq 4^\beta$.
- That is, $1/2\log(n+1) \leq \beta \leq \log(n+1)$.
- Use previous claim that height is at most twice the black-height to conclude the Theorem.

Balancing a BST



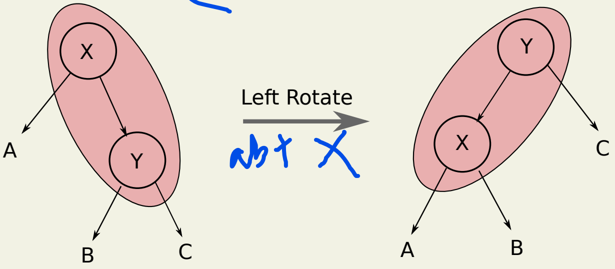
Balancing a BST is done by making structural changes to the underlying tree.

Rotations are operations on nodes of a BST. They are of two variants:

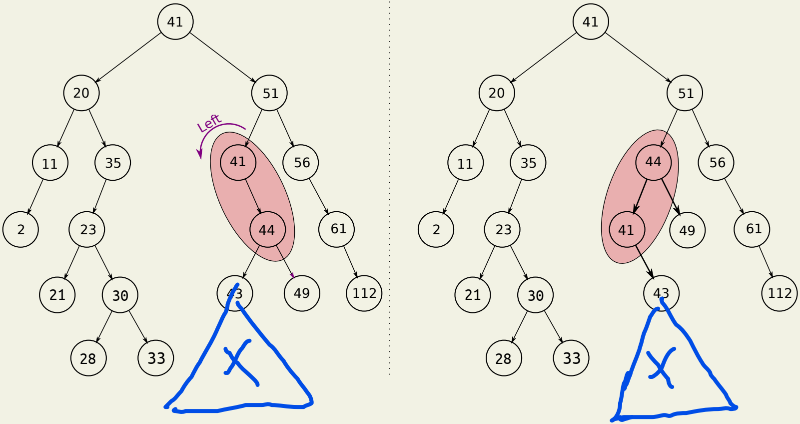
1. Left Rotate.
2. Right Rotate.

Left Rotate

Right rotate abt Y



Example


Left
41
20
51
11
35
41
56
2
23
44
61
21
30
43
49
112
28
33
X
41
20
51
11
35
44
56
2
23
41
49
61
21
30
43
112
28
33
X

INSERT procedure

INSERT(x) – Insert value x into the red-back tree.
High level strategy:

- Create a node X with value x and color red.
- Insert node X just like inserting into a Binary Search Tree.
- Call procedure FIXINSERT at node X.

FixInsert procedure

Which properties might be broken when we insert a new red node?

1. All nodes are colored either Red or Black.
2. The root node is black.
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4. Both children of a red node are black.
5. For any node, all paths from the node to the descendant leaves have the same number of black nodes.

FixInsert procedure

Only properties 2 and 4 could be broken after inserting a red node:

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FixInsert - Fixing property 2

Property 2

The root node is black.

Some invariants when FixInsert is called on a node Z:

- Z is colored Red.
- If Property 2 is violated, then node Z itself is the root.

FixInsert - Fixing property 2

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Some invariants when FixInsert is called on a node Z:

- Z is colored Red.
- If Property 2 is violated, then node Z itself is the root.

Resolution: Simply color Z black.

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Property 4

A red node has black children.

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FixInsert - Fixing Property 4

Property 4

A red node has black children.

Some invariants when FixInsert is called on a node Z:

- Z is colored Red.
- If Property 4 is violated, it is violated only by the node Z and its parent.

There are three cases when FixInsert is called on a node Z.

Cases

- Case 1: Uncle of Z is Red.

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- Case 2: Uncle of Z is black and Z is a right child of a left child.
- Case 3: Uncle of Z is black and Z is a left child of a left child.
- Other cases follow by symmetry.

Case 1

Case 1

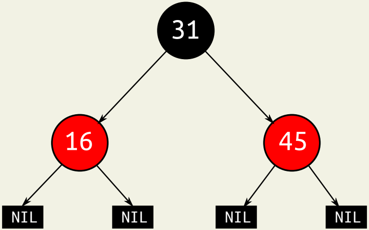
Uncle of Z is Red.

Resolution:

- Recolor parent, uncle and grandparent.
- Call FixInsert(grandparent)

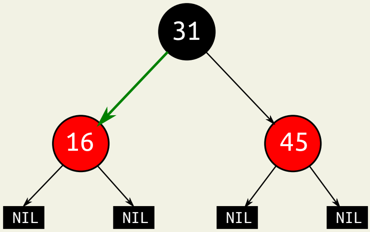
Example 1

Insert 9 to the following RBT:



Example 1

Find the position where 9 should be inserted



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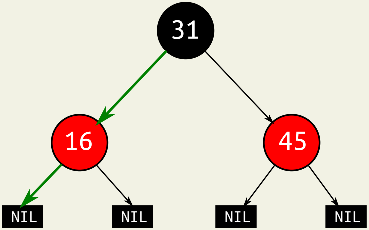
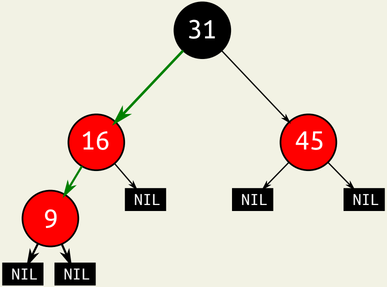


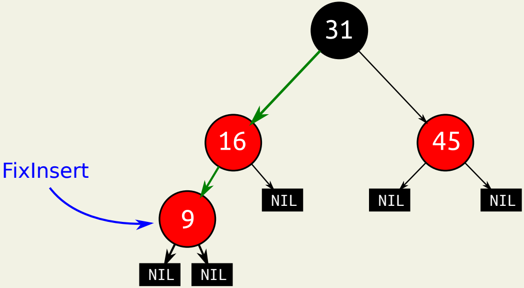
Example 1

Insert 9 as a new node with color red



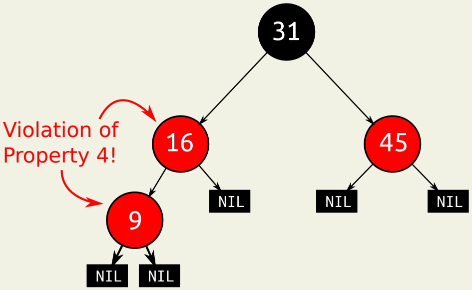
Example 1

Call FixInsert at the inserted location.



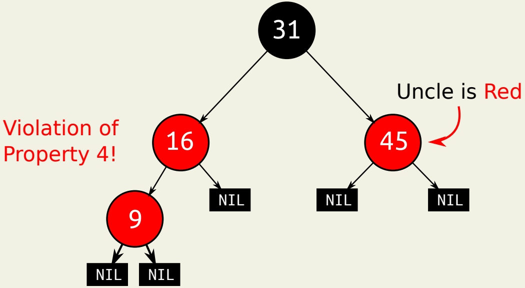
Example 1

Property 4 is violated. Check color of the uncle to determine case.



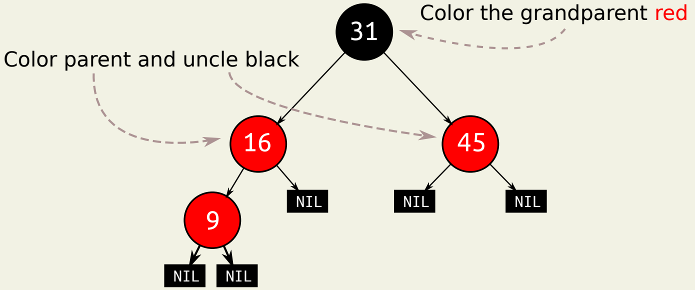
Example 1

We are in Case 1.



Example 1

Case 1 is resolved by recoloring.



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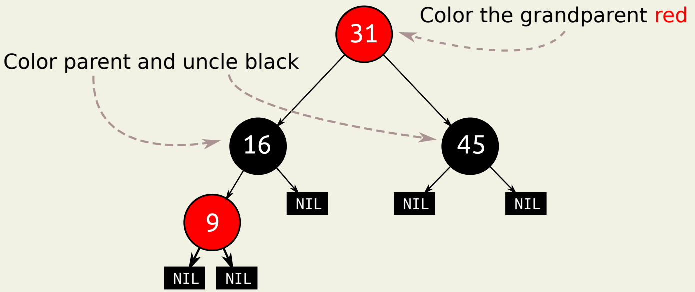
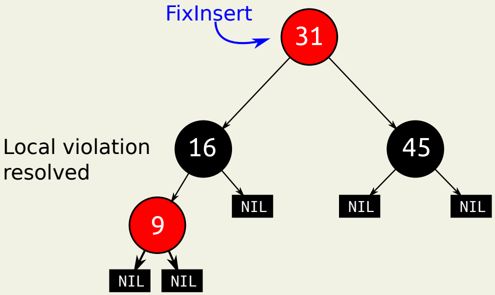


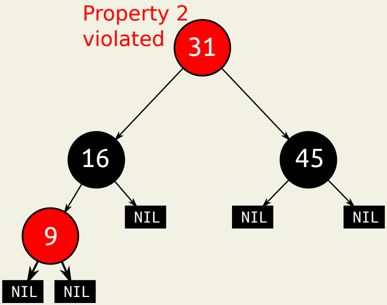
Example 1

Now call FixInsert on the grandparent.



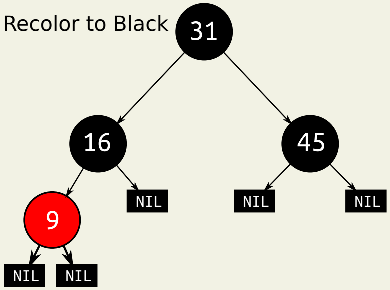
Example 1

Root node is not black.



Example 1

Simply recolor root to black.



Case 3

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Uncle of Z is Black and Z is left child of a left child.

Resolution:

- Recolor parent and grandparent.
- Rotate right at grandparent.

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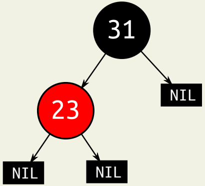
Symmetric case: Z is right child of a right child.

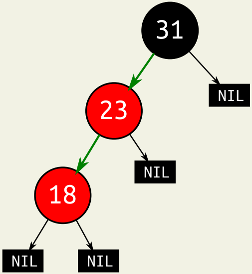
Example 2

Want to insert 18 into this RBT.



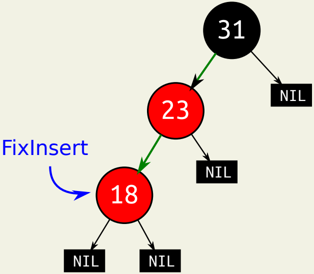
Example 2

Insert 18 as a red node according to BST property.



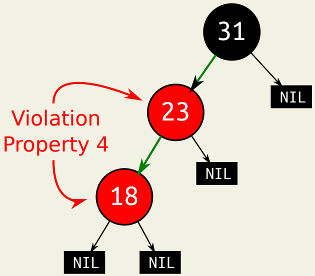
Example 2

Call FixInsert on the new node.



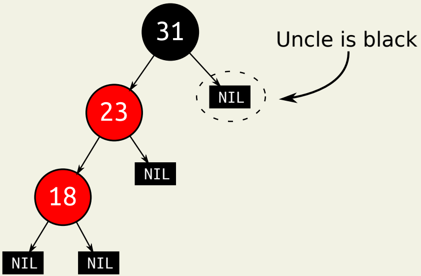
Example 2

FixInsert has to fix property 4.



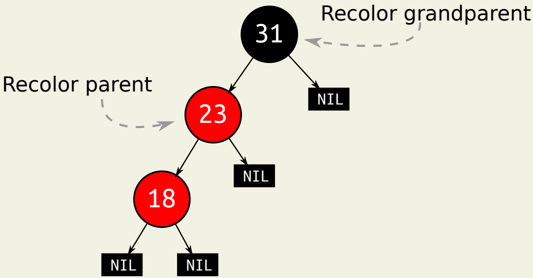
Example 2

Determine the case by looking at uncle.



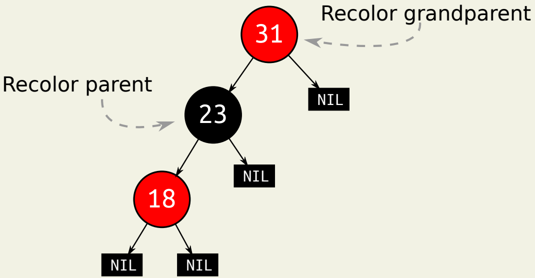
Example 2

Node 18 and its parent 23 are both left children. This is case 3.



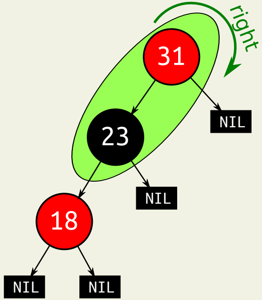
Example 2

Recolor parent to black and grandparent to red.



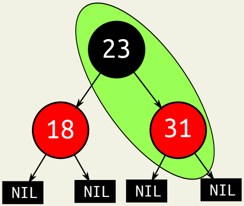
Example 2

Rotate right at grandparent.



Example 2

The resulting tree has no violations.



Case 2

Case 2

Uncle of Z is Black and Z is right child of a left child.

Resolution:

- Assign parent to Z.
- Rotate left at Z.
- Call FixInsert at Z.

The above procedure results in case 3.

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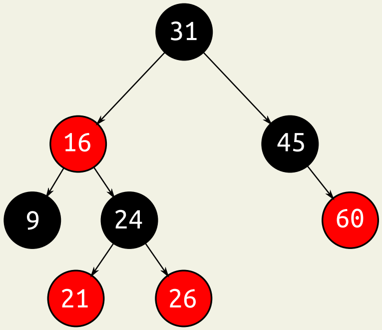
The above procedure results in case 3.
Symmetric case: Z is left child of a right child.

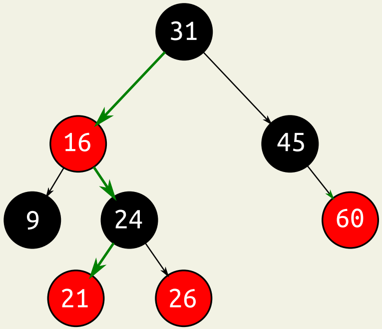
Example 3

Want to insert 18 into the following:



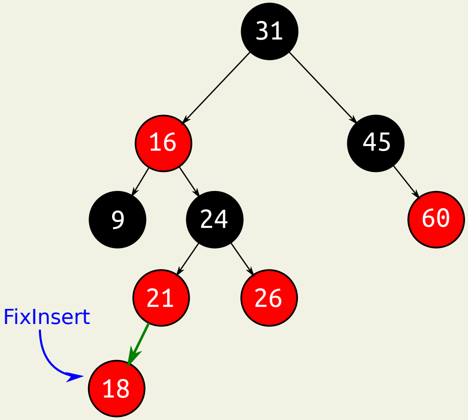
Example 3

Find the position for 18:



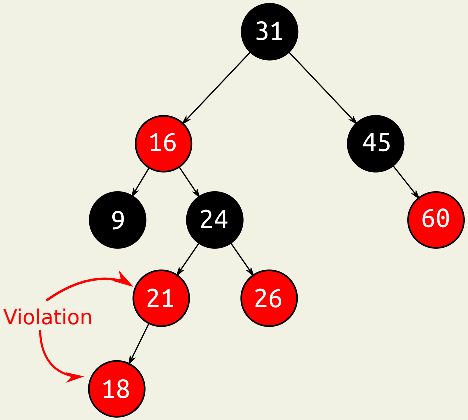
Example 3

Insert a new node with value 18 and color red. Call FixInsert at the inserted location.



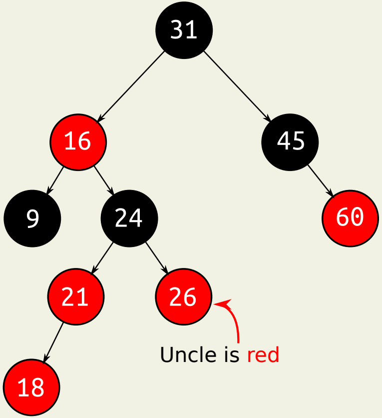
Example 3

Property 4 is violated. Check color of uncle to determine the case.



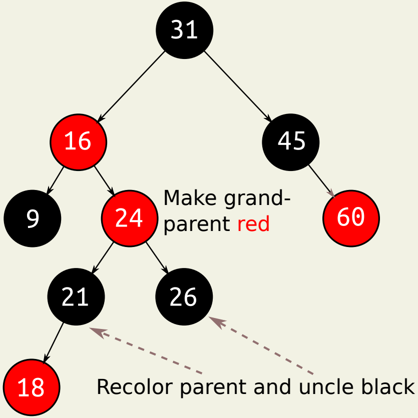
Example 3

Uncle is red. So we are in case 1.



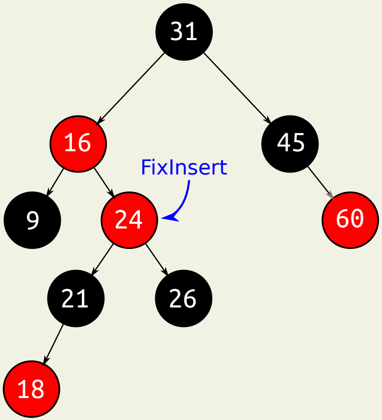
Example 3

Recolor as done earlier.



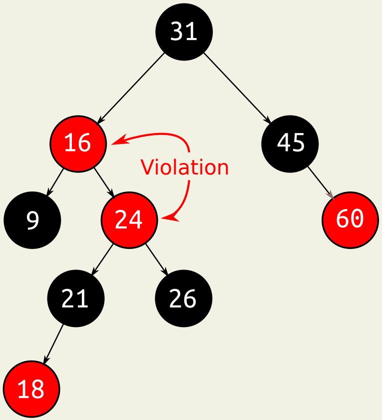
Example 3

Call FixInsert on grandparent.



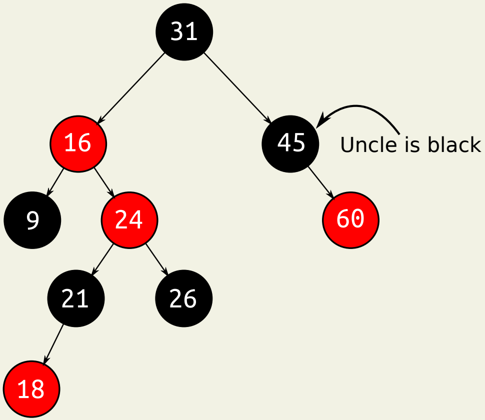
Example 3

Property 4 does not hold. Check color of Uncle to determine case.



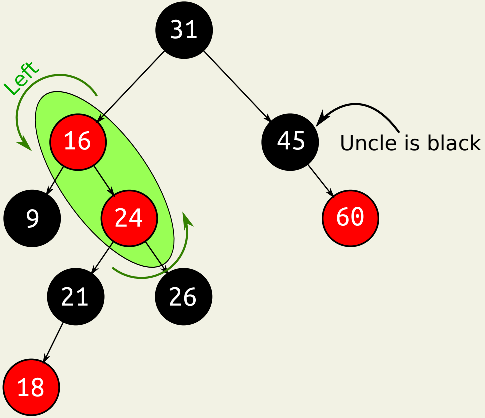
Example 3

Uncle is black and 24 is right child of a left child. So we are in case 2.



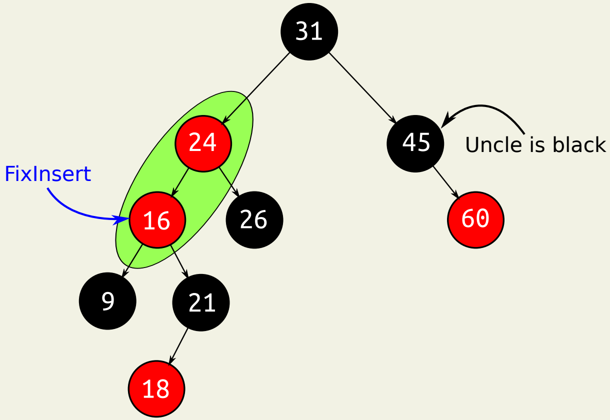
Example 3

Set Z as node with 16. Rotate left at Z and call FixInsert on Z.



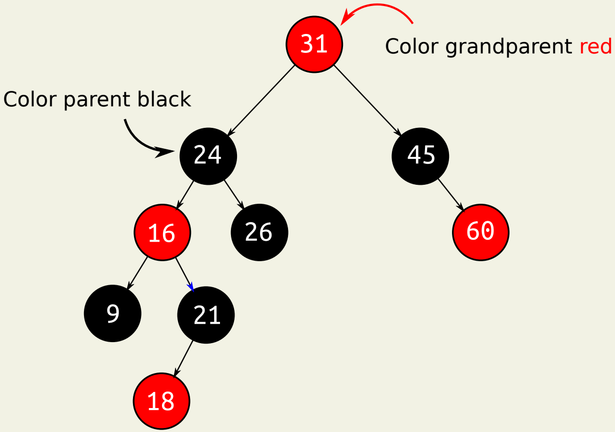
Example 3

Now, Uncle is black and 16 is left child of a left child. This is case 3.



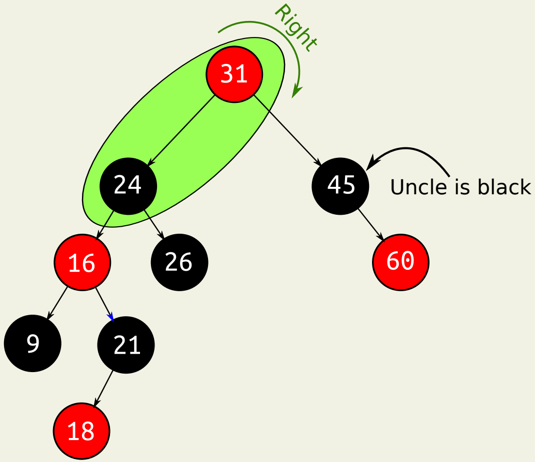
Example 3

Recolor granparent red, parent black.



Example 3

Right rotate at grandparent.



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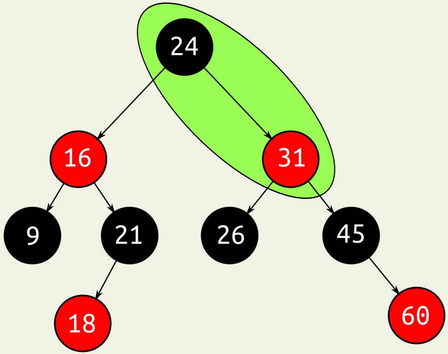
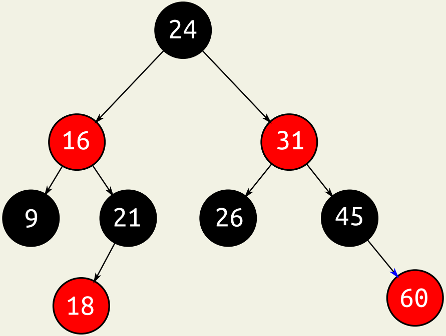


Example 3

Done!

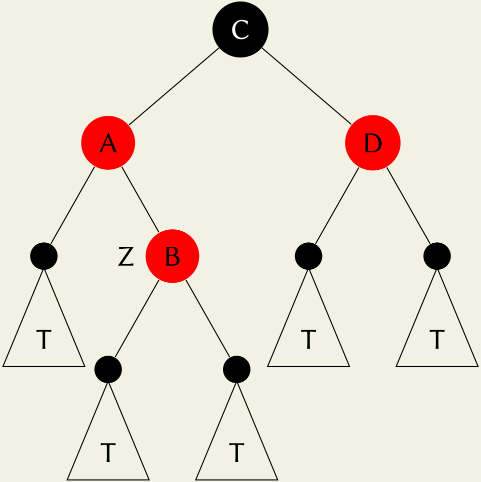


FixInsert pseudocode

Algorithm 1 FixInsert called on node Z

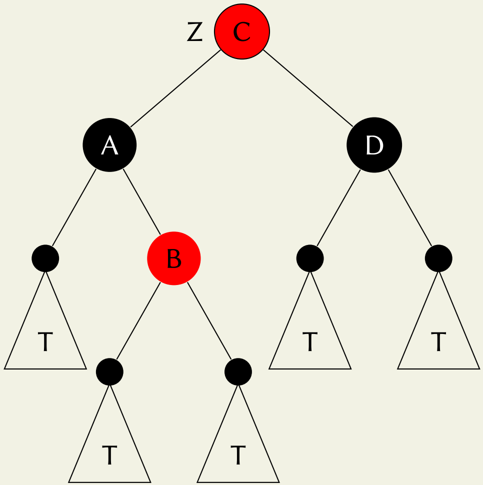
```
while color(parent(Z)) = red do
    U ← Uncle(Z)
    if parent(Z) is the left child of the grandparent then
        if color(U) = red then
            Recolor parent, uncle and grandparent.
            Z ← grandparent(Z).
        else
            if Z is the right child then
                Z ← parent(Z); Left rotate at (Z)
            end if
            Recolor parent and grandparent.
            Right rotate at grandparent(Z).
        end if
    end if
end while
color(root)← black.
```

Case 1



T = subtree of black height k
B could be on either side

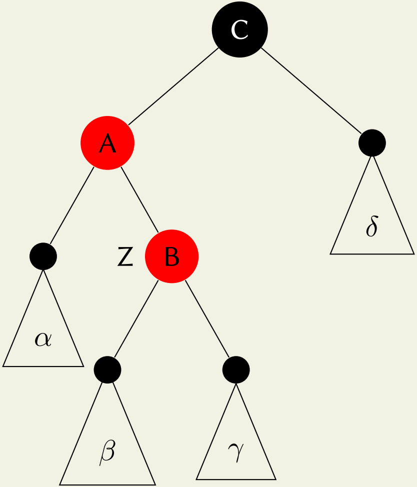
Case 1



T = subtree of black height k

B could be on either side

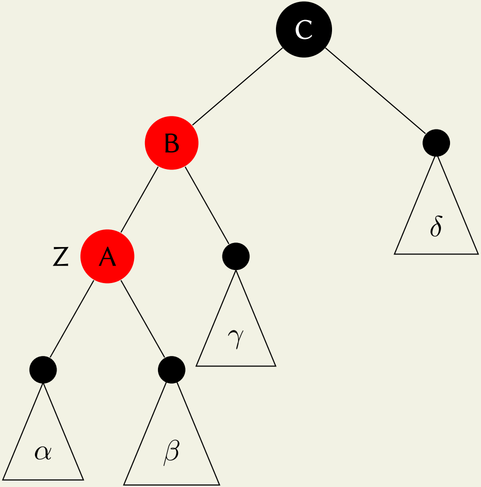
Case 2



$\alpha, \beta, \gamma, \delta =$ subtrees of black height k

We do: Left Rotate at A to get to Case 3

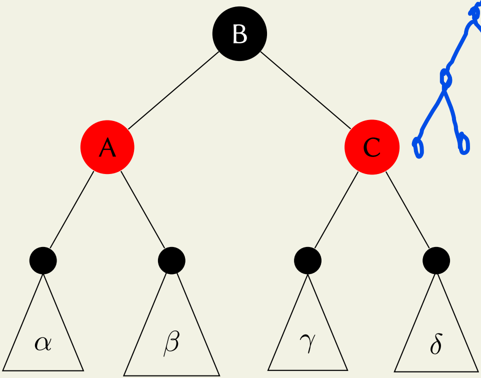
Case 2 –> Case 3

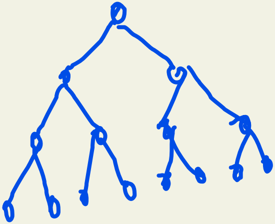


$\alpha, \beta, \gamma, \delta =$ subtrees of black height k

We do right rotate at C

Case 3





$\alpha, \beta, \gamma, \delta =$ subtrees of black height k

Solved!

Summary of Insert

- Case 1: Only recoloring. Pushes up the violation.
- Case 2: Only one rotation. Leads to Case 3.
- Case 3: Only one rotation. No more violations!

Summary of Insert

- Case 1: Only recoloring. Pushes up the violation.
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- Case 3: Only one rotation. No more violations!

- We have $\leq O(\log n)$ recolorings and ≤ 2 rotations
- Total time: $O(\log n)$

How did anyone come up with such an idea?

- 1962: First self-balancing tree invented by Adelson-Velsky and Landis: AVL Trees
- 1972: Bayer invents “symmetric binary B-trees” which became known as 2-3-4 Trees
- 1978: Guibas and Sedgewick studied 2-3-4 trees and introduced the analogous red-black notion
- Improved over the years

Delete procedure

The procedure to delete a node M at a high level:

- Case 1: M has two non-leaf children.
  - Replace (data of) M with the successor.
  - Splice out (delete) successor. This makes it case 2.

Delete procedure

The procedure to delete a node M at a high level:

- Case 1: M has two non-leaf children.
  - Replace (data of) M with the successor.
  - Splice out (delete) successor. This makes it case 2.
- Case 2: M has at most one non-leaf child. Call this C.
  - Trivial case: M is red.
  - Minor case: M is black but C is red.
  - Major case: M and C are both black.

Note: If M has both leaf children (NIL), then C is any one of the NIL nodes.

M has at most one non-leaf child C

Trivial Case

M is red:

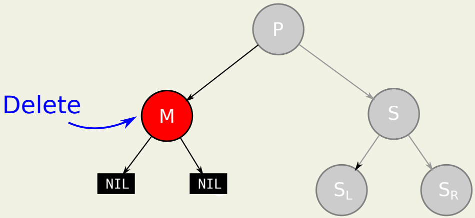
Resolution:

- Then simply replace M with C.

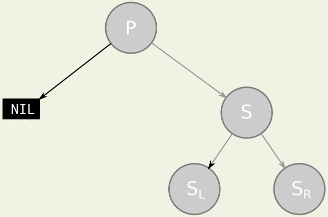
Note:

- M could not have been root.
- For all paths, the black height is not affected.

Trivial Case



Trivial Case



M has at most one non-leaf child C

Minor Case

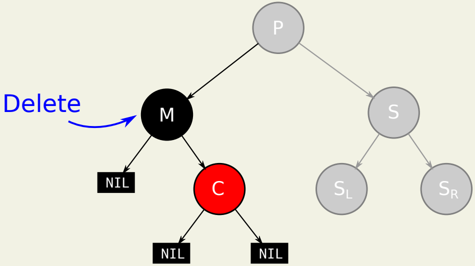
M is black and C is red.

Resolution:

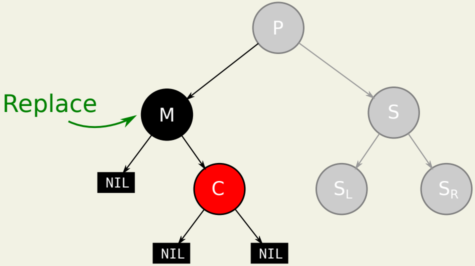
- Replace (splice) M with C.
- Place a “black token” on C.
- Safely discard black token by coloring C black.

The token indicates that the node contributes an extra black to the black count.

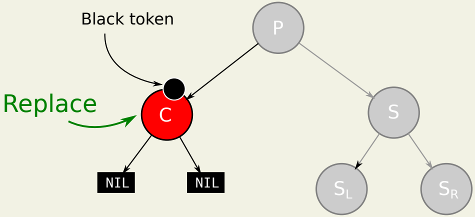
Minor Case



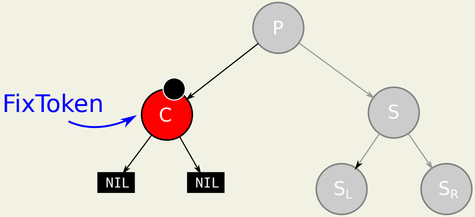
Minor Case



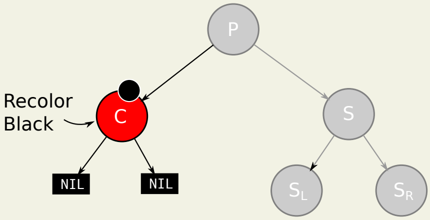
Minor Case



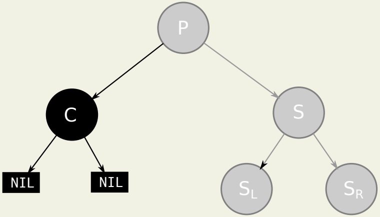
Minor Case



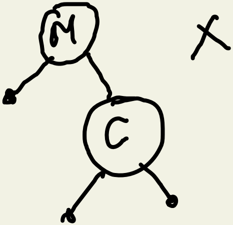
Minor Case



Minor Case



M has at most one non-leaf child C



Major Case

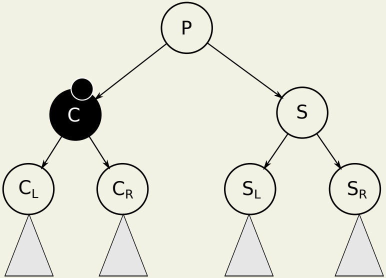
M is black and C is black.

Resolution:

- Replace (splice) M with C.
- Place a black token on C.
- Four cases to safely remove the token placed on C.

Major case - notation

Before replacement of M by C



Removing the token from C

Four cases based on the sibling S of C:

1. S is red.
2. S is black and has both children colored black.
3. S is black and has left child red and right child black.
4. S is black and has right child red.

Case 1

Sibling S of C is red.

Since S is red:

- It must have both black children.
- The parent P of S must be black.

Removing the token from C

Case 1

Sibling S of C is red.

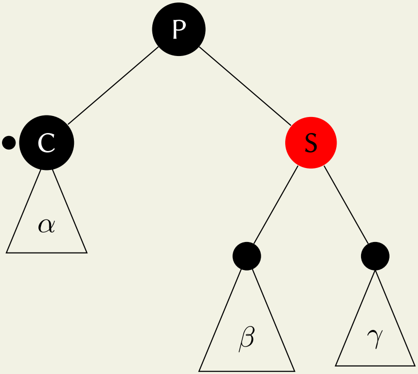
Since S is red:

- It must have both black children.
- The parent P of S must be black.

Resolution:

- Recolor S black and its parent red.
- Rotate at parent. (left rotate if C was left child)
- This is now one of cases 2, 3 or 4.

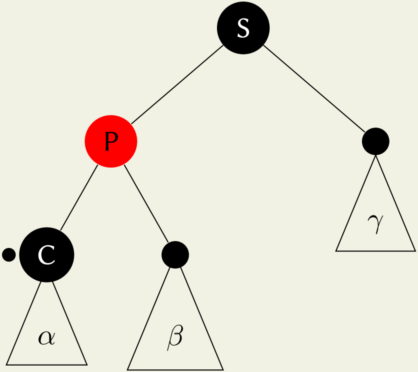
Case 1



$\alpha, \beta, \gamma =$ subtrees of black height k (including the black token)

Left Rotate about P and recolor

Case 1



$\alpha, \beta, \gamma, \delta =$ subtrees of black height *k* (including the black token)

Now in Case 2, 3 or 4

Case 2

Sibling S is black and has both black children.

Removing the token from C

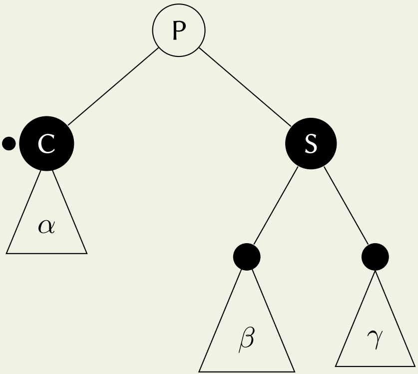
Case 2

Sibling S is black and has both black children.

Resolution:

- Remove a black from both C and S.
- Paste token on the parent P.

Case 2

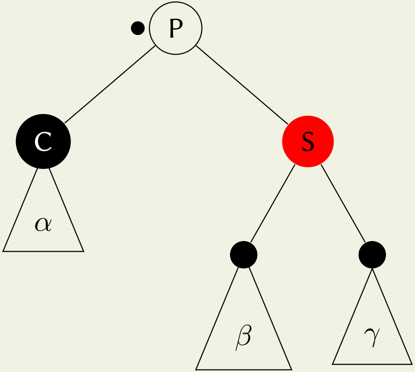


α = subtree of black height $k + 1$ (including the black token)

β, γ = subtrees of black height k

Shift the token to P

Case 2



P can absorb the token if red, else fixup continues

If we came from Case 1, then P is red.

Case 3

Sibling S is black.

Left child S_L is red.

S_R is black.

Case 3

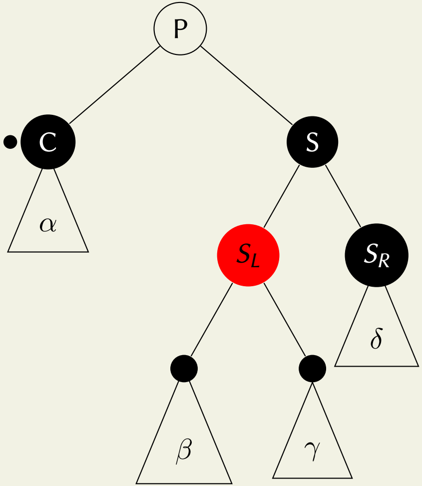
Sibling S is black.
Left child S_L is red.
S_R is black.

Resolution:

- Swap colors of S_L and S.
- Rotate right at S.

This gives us case 4.

Case 3

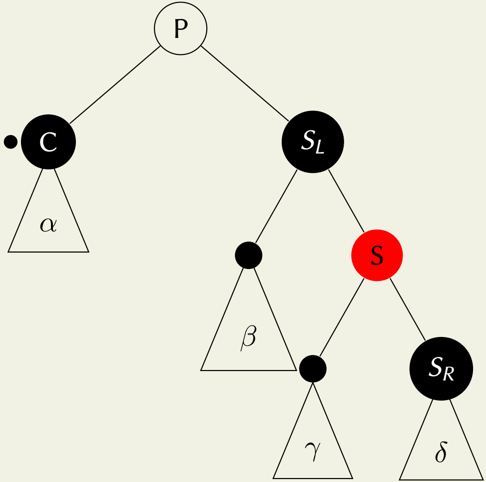


$\alpha =$ subtree of black height $k + 1$ (including the black token)

$\beta, \gamma, \delta =$ subtrees of black height k

Rotate and Recolor

Case 3



Now we are in Case 4

Case 4

Sibling S is black.

Right child S_R is red.

Removing the token from C

Case 4

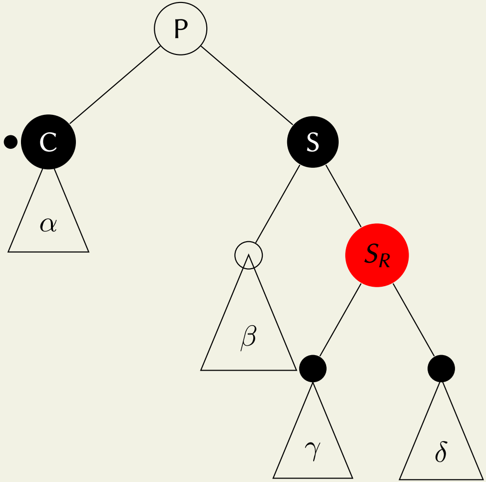
Sibling S is black.

Right child S_R is red.

Resolution:

- Color S_R black.
- S inherits the color of parent P.
- Color P black.
- Rotate left at P.
- Token is removed.

Case 4

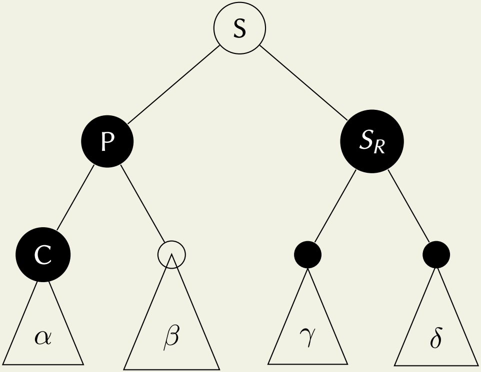


α = subtree of black height $k+1$ (including the black token)

β, γ, δ = subtrees of black height k

Rotate and Recolor

Case 4



$\alpha, \beta, \gamma, \delta =$ subtrees of black height k

Solved!

Case Progression

- Case 1 $\rightarrow$ Case 2 $\rightarrow$ end
- Case 1 $\rightarrow$ Case 3 $\rightarrow$ Case 4 $\rightarrow$ end
- Case 1 $\rightarrow$ Case 4 $\rightarrow$ end
- Case 2 $\rightarrow$ Loop in Case 2 $\rightarrow$ Case 1, 3 or 4
- Case 3 $\rightarrow$ Case 4 $\rightarrow$ end
- Case 4 $\rightarrow$ end