Fractal Reptiles with Holes

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Abstract

This work is the study of fractal rep tiles with holes which was done for open elective course. A cube is an 8-rep-tile: it is the union of eight smaller copies of itself. Is there a set with a hole which has this property? The computer found an interesting and complicated solution, which then could be simplified. We discuss some problems of computer-assisted research in geometry. Will computers help us do geometrical research? Can they find something new? How can we direct them to do those things which we are interested in? On the other hand, will computers change our attitudes? We discuss such issues for elementary problems of fractal geometry [1, 2], using the free software package IFStile [3]. A wide variety of fractal gaskets have been designed from self-replicating tiles. In contrast to the most well-known examples, the Sierpinski carpet and Sierpinski triangle, these gaskets generally have fractal outer boundaries, and the holes in them generally have fractal boundaries. Hamiltonian cycles have been explored that trace out some of these fractal gaskets. We will utilize the geometric method of constructing reptiles in Rd, especially reptiles with holes. We construct n-reptile in R2 with holes. This work will present a few examples of rep tiles with holes including variations.

Keywords: Fractals, Rep Tiles, Iterated Function Systems, Integer Matrices, Rep-tiles with a hole

1. Introduction

A closed bounded set A with non-empty interior in plane or space is called an m-reptile if there are sets $A_1, A_2, ..., A_m$ congruent to A, such that different sets A_k, A_j have no common interior points, and the union

$$B = A_1 \cup ... \cup A_m$$

is geometrically like A. The standard example in the plane is a square, or a parallelogram, or a triangle, with m = 4. 'Rep' stands for 'replication', and the sets are called tiles since they can tile the whole plane. Such tiling's can be obtained by observing that B is also a rep-tile and contained in still larger super-rep-tiles C, D, ..., and they all are unions of copies of A [4][5].

The tilings generated by the flag are non-periodic and quite intricate while the 2×2 subdivision of the square provides only the ordinary periodic checkerboard tiling. Rep-tiles were introduced in 1963 as recreational objects by Gardner [6] and Golomb [7]. In the 1980s they became interesting as models of quasicrystals [4][5], as examples of self-similar fractals [8], as a tool for constructing multidimensional wavelets [9], and as unit intervals for exotic number systems [10]. For the plane, plenty of m-rep-tiles are known for every m [11]. In three-dimensional space, a tetrahedral m-rep-tile can exist only for cubic numbers m, not for $m < 8$ [12]. For $m = 8$, the cube is a standard rep-tile, and the notched cube ('chair') in Figure 1 is another well-known example. The regular tetrahedron is not an 8-rep-tile but some other special tetrahedra are, one of them found by M.J.M. Hill already in 1895, and two others found in 1994 [13]. Recent results support the conjecture that there are no further 8- rep-tile tetrahedra [14]. Figure 1 shows two other polyhedral examples found with the IFStile package.

Figure 1 4-rep-tiles in the plane

The Sierpinski triangle and Sierpinski carpet are two well-known classical fractals based, respectively, on an equilateral triangle and a square. Both are created by removing the center tile from a group of tiles. Such a tile that can be divided into smaller copies of itself is known as a self-replicating tile, or "reptile" [15]. These fractals are formed by iteratively replacing the individual tiles in the reptile with a scaled-down version of the remaining group of tiles. For reptiles such as these, in which the smaller tiles are all the same size, the scaling factor from one generation to the next is the reciprocal of the square root of the number of smaller tiles in the base reptile. E.g., the features of the Sierpinski carpet, which is based on a square divided into nine smaller squares, scale by 1/3 from one generation to the next.

Remaining sections of the work are organized as follows: Section 2 presents the related literature work and section 3 presents the methodology adopted in this study and discusses the results of different methodologies for generating rep tiles with the hole. Finally, section 4 discusses the conclusion remark and presents the future work.

2. Related work

In [16], L`evy tiled the plane with rectangles whose interiors are pairwise disjoint. He then used the replacement strategy to generate the classical fractal known as the L`evy dragon represented in Fig. 2c. Knowing that the plane can be tiled by exotic sets like the L`evy dragon is an intriguing result. One would not expect this since the curve has infinitely many holes, and from the picture, it is not obvious that it has interior points at all.

Definition 2.1. Let (X, d) be a complete metric space. By H(X) we denote the space whose points are the compact subsets of X, other than the null set. H(X) is called space of fractals.

The space H(X) itself is a metric space with respect to the Hausdorff metric h defined by: For any A, B ∈ H(X), distance between A and B is given by

$$h(A, B) = max\{d(A, B), d(B,A)\},$$

where $d(A, B) = supx\in A\ infy\in B\{d(x, y)\}$.

To define tiles formally, we need to know the definition of cover of a set.

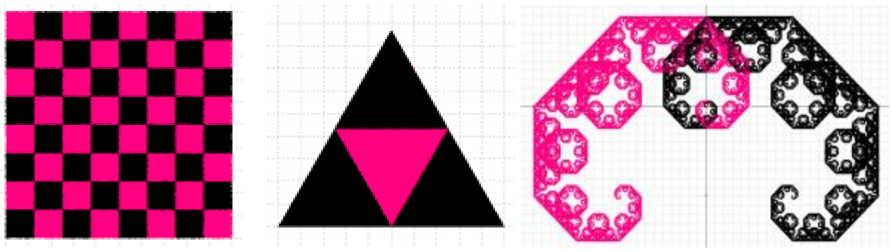
Definition 2.2. A covering of a subset $A \in X$ is a collection of open sets $C = \{C_1, C_2,...,C_n\} \in X$ whose union contains A. i.e.

$$A \subset \bigcup_{i=1}^{n} C_i$$

Definition 2.3. A tiling of the plane is a countable family Ai of compact sets that covers the plane with $int(A_i) \cap int(A_j) \neq \emptyset$ for i $\neq$ j. Here, int(·) denotes the interior of a set.

Thus, the interior of each set does not intersect the interior of any other set, i.e., the tiling will have no overlap.

A chessboard (Fig. 2a) is an elementary example of a self-similar tiling, one that is composed of smaller tiles (-rep tiles) of the same size, each having the same shape. The equilateral triangle in Fig. 2b is another example of a similarity tiling since it is composed of four smaller equilateral triangles.



(a)Chessboard tiling (b)Triangle Tiling (c) The Levy Diagram

Figure 2 Examples of Tiling

Definition 2.4. A transformation $f: X \rightarrow X$ on a complete metric space (X, d) is called contractive or a contraction mapping if there is a constant $0 \leq s < 1$ such that

$$d(f(x), f(y)) \leq s\, d(x, y)\ \forall x, y \in X,$$

where s is called contractivity factor of f.

Definition 2.5. An IFS consists of a complete metric space (X, d) together with a finite set of contraction mappings f_n: X → X, having contractivity factors s_n, for n = 1, 2,...,m. The number

$$s = \max_{1 \le n \le m} s_n$$

is called contractivity factor of the IFS.

Theorem 2.1. Let $\{X, w_n: n = 1, 2,..., w_m\}$ be an IFS with contractivity factor s. Then the transformation W: H(X) → H(X) defined by

$$W(B) = \bigcup_{n=1}^{m} w_n(B)$$

for all $B \in H(X)$ is a contraction mapping on the complete metric space $H(X, h(d))$ with contractivity factor s. That is,

$$h(W(B), W(C)) \le s\, h(B,C)$$

for all $B,C \in H(X)$. Moreover, by contraction mapping theorem, its unique fixed-point $A \in H(X)$ satisfies

$$A = W(A) = \bigcup_{n=1}^{m} w_n(A)$$

and is given by $A = \lim_{n\to\infty} W^{\circ n}(B)$ for any $B \in H(X)$. Here, for $B \in X$ and $k \in N$, $W^{\circ k}(B)$ denotes the k-fold composition of W.

Definition 2.6. The fixed-point A described in Theorem 2.1 is called the attractor of the IFS.

Definition 2.7. A closed set A in R^d(d = 2, 3) with non-empty interior is called an m-rep tile if there are sets A_1,...,A_m congruent to A, such that $intA_i \cap intA_j \neq \emptyset$ for i ≠ j and

$$g(A) = A_1 \cup A_2 \cup ... A_m,$$

where g is a similarity mapping.

An important property of an IFS is the open set condition, briefly denoted as OSC.

Definition 2.8. Let {X, w_i: i = 1, 2,...,m} be an IFS. The IFS is said to satisfy OSC, if there exists a non-empty open set U ⊂ R_n such that

$$\bigcup_{n=1}^{m} w_i(U) \subseteq U, w_i(U) \cap wj\ (U) = \emptyset \text{ for i} \neq \text{ j.}$$

When the OSC is fulfilled, then the attractor A has a nice structure. In particular, when A has non-empty interior and OSC holds, then A is a tile. That is, $R^n = A_1 \cup A_2 \cup A_3 \cup ...$ where each $A_i = h_i(A)$ is a copy of A by some affine map h_j, and the intersection $A_i \cap A_j$ of any two different copies has empty interior.

3. Methodology

An n-reptile (or n-rep tile) A in R^d is a compact set with non-empty interior that can be tiled by n congruent tiles, each similar to A [16][17]. We assume, as in the literature, that a reptile is the closure of its interior. If the number of pieces n is irrelevant to the discussion, we will simply call an n-reptile a reptile. Reptiles form a special class of self-similar sets. Let $\{f_i\}_{i=1}^{n}$ be an iterated function system (IFS) of contractive similitudes on Rd defined as

$$f_i(x) = \frac{1}{\sqrt[d]{n}} R_i x + d_i, \qquad i = 1, \dots \dots, n,$$

where R_i is an orthogonal transformation, $d_i \in R_d$, and the factor $1/\sqrt[d]{n}$ is the contraction ratio of f_i. Then there exists a unique compact set T satisfying.

$$A = \bigcup_{i=1}^{n} f_i(A) \qquad (1.1)$$

in [18][19] T is called the self-similar set (or n-repset, or attractor) defined by $\{f_i\}_{i=1}^{n}$. It follows from (1.1) and the uniqueness of A that A is the closure of its interior. If the interior of A is non-empty, then it follows from (1.1) that A is an n-reptile. Note that the similarity dimension of $\{f_i\}_{i=1}^{n}$ is d. Thus, the assumption that T has a non-empty interior is equivalent to the requirement that $\{f_i\}_{i=1}^{n}$ satisfies the open set condition [20]. Therefore, T is an n-reptile if and only if T is a self-similar set defined by an iterated function system consisting of n similitudes having the same contraction ratio $1/\sqrt[d]{n}$ and satisfying the open set condition. Identity (1.1) is equivalent to

$$\sqrt[d]{n} A = \bigcup_{i=1}^{n} (R_i A + \sqrt[d]{n} d_i)$$

If A is a reptile, then it follows from the above equality and a standard blow-up argument that R^d can be tiled by essentially disjoint congruent copies of A. We say that a reptile $A \subseteq R^d$ has a hole if the complement of the closure of some component of the interior of A has a bounded component. Answering a question posed by John Conway, Gr¨unbaum gave an example of a 36-reptile in R^2 which has a hole. In [16], Croft et al. asked the following question: what is the least n for which an n-reptile in the plane has any sort of hole? It has been proved recently by Bandt and Wang [21] and Luo et al. [22] that if the interior of an n-reptile in R^2 is connected, then the reptile is a topological disc. We will therefore be interested in reptiles whose interiors are disconnected. In [16], we construct, for every m ≥ 2, a 2*m*-reptile in R^2 that has holes. Define an IFS on R^2 by

$$f_i(x) = \begin{cases} \frac{1}{\sqrt{n}} R\left(\frac{\pi}{2}\right)(x) + \left(i + \frac{1}{2}, 0\right), & -m \le i \le -1 \\ \frac{1}{\sqrt{n}} R\left(-\frac{\pi}{2}\right)(x) + \left(i + \frac{1}{2}, 0\right) & 0 \le i \le m - 2, \end{cases}$$

$$g(x) = \frac{1}{\sqrt{n}} R\left(\frac{\pi}{2}\right) \sigma_y(x) + \left(-m + \frac{1}{2}, 0\right)$$

where R(θ) is the rotation through the angle θ and σ_y denotes the reflection about the y-axis. Let A be the *n*-repset defined by the

$$\text{IFS } F = \{g\} \cup \{f_i\}_{i=-m}^{m=-2}.$$

3.1 Generating a 9-Reptile with a hole

Let $A_0 = [-3,3] \times \left[\frac{-3}{2}, \frac{3}{2}\right]$ and $\rho = \frac{1}{3}$. Now, the IFS generated from F=$\{f_i\}_{i=1}^{9}$ will be:

For,

$$f_i(x) = \begin{cases} \rho R\left(\frac{1}{2}\pi\right)(x) + d_i, & i = 1 \\ \rho R\left(\frac{1}{2}\pi\right)(x)\sigma_y(x) + d_i, & i = 2 \\ \rho x + d_i, & i = 3,4 \\ \rho\sigma_y(x) + d_i, & i = 5,6 \\ \rho\sigma_y(x) + d_i, & i = 7 \\ \rho R(\pi)(x) + d_i, & i = 8,9 \end{cases}$$

Where,

$d_1 = d_2 = \left(\frac{-5}{2}, \frac{1}{2}\right)$, $d_3 = d_5 = (-2, -1)$, $d_4 = d_6 = (-1, 1)$,

$d_7 = (0, -1)$, $d_8 = (-1, 0)$, $d_9 = (1, 0)$.

The IFS is:

a	b	c	d	e	f	p
0	-0.3330	0.3333	0	-2.500	0.5	0.1120
0	-0.3330	-0.3333	0	-2.500	0.5	0.1110
0.3333	0	0	0.3330	-2	-1	0.1110
0.3333	0	0	0.3330	-1	1	0.1110
-0.3330	0	0	0.3330	-2	-1	0.1110
-0.3330	0	0	0.3330	-1	1	0.1110
-0.3330	0	0	0.3330	0	-1	0.1110
-0.3330	0	0	-0.3330	-1	0	0.1110
-0.3330	0	0	-0.3330	1	0	0.1110

Then the attractor A of Fractal is a connected 9-reptile whose interior consists of infinitely many components, with the closure of each component consisting of infinitely many holes.

Proof: It is clear that A_0 satisfies the conditions in [23, (1.4)]. Let G = {id, σ_y}. Then Condition in [23, (1.2)] holds. One can check directly that

$f_1\sigma_y = f_2$, $f_2\sigma_y = f_1$, $f_3\sigma_y = f_5$,

$f_5\sigma_y = f_3$, $f_4\sigma_y = f_6$, $f_6\sigma_y = f_4$,

$f_7\sigma_y = \sigma_y f_7$, $f_8\sigma_y = \sigma_y f_9$. $f_9\sigma_y = \sigma_y f_8$.

It follows that $F_1 = \{f_i\}_{i=1}^{6}$, $F_2 = \{f_i\}_{i=7}^{9}$, and Condition in [23, (1.3)] holds. Thus, by Theorem in [23, (1.5(a)]. A is a 9-reptile. It is easy to see that, for all f ∈ F, $V_0 \cap f(T_0) \neq \emptyset$, where V_0 is defined as in [23, (1.8)]. Thus, by Theorem in [23, (1.5(b)], A is connected. The rest of the assertions can be proved as in [23, (3)].

To construct n-reptiles in R^2 whose interior consist of finitely many components, and with the closure of some component having a hole, the smallest n we can obtain is 16. Note that reflections are not involved in the construction of the following reptile.

The final output generated by the ifs is illustrated in Fig. 3.

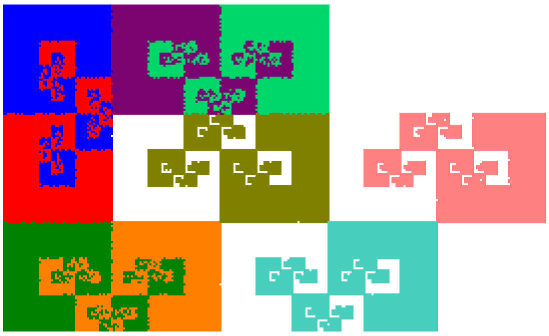


Figure 3 9-Rep Tile with a hole

3.2 Generating a 16-Reptile with a hole

Let $A_0 = [-4,4] \times [-2,2]$ and $\rho = \frac{1}{4}$. Now, the IFS generated from $F=\{f_i\}_{i=1}^{16}$ will be:

For,

$$f_i(x) = \begin{cases} \rho R\left(\frac{1}{2}\pi\right)(x) + d_i, & 1 \leq i \leq 5, \\ \rho R\left(\frac{1}{2}\pi\right)\sigma_y(x) + d_i, & 6 \leq i \leq 10, \\ \rho x + d_i, & i = 11,12, \\ \rho \sigma_y(x) + d_i, & i = 13,14, \\ \rho R\left(\frac{1}{2}\pi\right)(x) + d_i, & i = 15, \\ \rho R\left(-\frac{1}{2}\pi\right)(x) + d_i, & i = 16, \end{cases}$$

Where,

$d_1 = d_6 = \left(\frac{-3}{2}, -1\right)$, $d_2 = d_7 = \left(\frac{-5}{2}, -1\right)$, $d_3 = d_8 = \left(\frac{-7}{2}, -1\right)$,

$d_4 = d_9 = \left(\frac{-7}{2}, 1\right)$, $d_5 = d_{10} = \left(\frac{-1}{2}, 1\right)$, $d_{11} = d_{13} = \left(-2, \frac{3}{2}\right)$,

$d_{12} = d_{14} = \left(2, \frac{1}{2}\right)$, $d_{15} = \left(\frac{-1}{2}, -1\right)$, $d_{16} = \left(\frac{1}{2}, -1\right)$, The IFS is:

a	b	c	d	e	f	p
0	-0.25	0.250	0	-1.5	-1	0.63
0	-0.25	0.250	0	-2.5	-1	0.63
0	-0.25	0.250	0	-3.5	-1	0.63
0	-0.25	0.250	0	-3.5	1	0.63
0	-0.25	0.250	0	-0.5	1	0.63
0	-0.25	-0.250	0	-1.5	-1	0.63
0	-0.25	-0.250	0	-2.5	-1	0.63
0	-0.25	-0.250	0	-3.5	-1	0.63
0	-0.25	-0.250	0	-3.5	1	0.63
0	-0.25	-0.250	0	-0.5	1	0.63
0.25	0	0	0.25	-2	1.5	0.63
0.25	0	0	0.25	2	0.5	0.63
-0.25	0	0	0.25	-2	1.5	0.63
-0.25	0	0	0.25	2	0.5	0.63
0	-0.25	0.25	0	-0.5	-1	0.63
0	0.25	-0.25	0	0.5	-1	0.63

Then the attractor A of F is a connected 16-reptile whose interior consists of infinitely many components, with the closure of some of them having holes and the closure of the others being topological discs.

Proof: Let G = {id, σ_y}; thus, Condition in [23, (1.2)] holds. We verify directly that, for $1 \leq i \leq, 5$, $f_i\sigma_y = f_{i+5} \in F$, which implies that $f_{i+5}\sigma_y = f_i$. For i = 11 or 12, we have $f_i\sigma_y = f_{i+5} \in F$, which implies that $f_{i+2}\sigma_y = f_i$. Also, it is easy to check that $f_{15}\sigma_y = \sigma_y f_{16}$, which implies $f_{16}\sigma_y = \sigma_y f_{15}$.

Thus, $F_1 = \{f_i\}_{i=1}^{14}$, $F_2=\{f_i\}_{i=15}^{16}$, and Condition in [23, (1.3)] is satisfied. Therefore, A is a 16-reptile. For all f ∈ F, V1 ∩ f(T0) ≠∅. Hence A is connected. We omit the proofs of the rest of the assertions.

It is not true that if the interior of a reptile consists of infinitely many components, with the closure of some having holes, then the closure of these components must contain infinitely many holes. The following is a counter example.

The final output generated by the ifs is illustrated in Fig. 4.

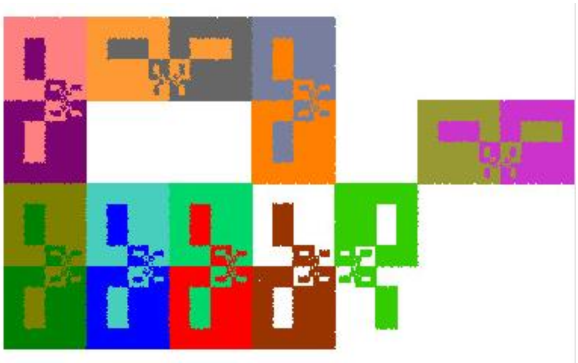


Figure 4 16-Rep Tile with a hole

4. Conclusion

The idea about fractal rep tiles and fractal reptiles with holes is presented here. The fractal reptiles and fractal reptiles with holes are compared. Definitions and few important theorems are also discussed in the report. 9-reptile with holes and 16-reptile with holes are also generated and the corresponding iterative function systems are attached.

The key take away from this project is regrading fractal reptiles with holes. It was understood that We say a reptile has a hole if the complement of the closure of some component of the interior of T has a bounded component. It has been proved that if the interior of an n-reptile in R2 is connected, then the reptile is a topological disc. We mostly focused on in the reptiles whose interiors are disconnected. we constructed a connected reptile in R2 whose interior consists of infinitely many components, with the closure of some of them having holes. Lastly, we construct a connected piecewise polygonal 16-reptile with a hole. All the reptiles constructed in this paper are defined by IFS's that involve reflections, it is worth pointing out that there would be few examples without reflections. The least n-reptile can be built such is 16. So, we described our construction of reptiles in R2 and proved them.

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