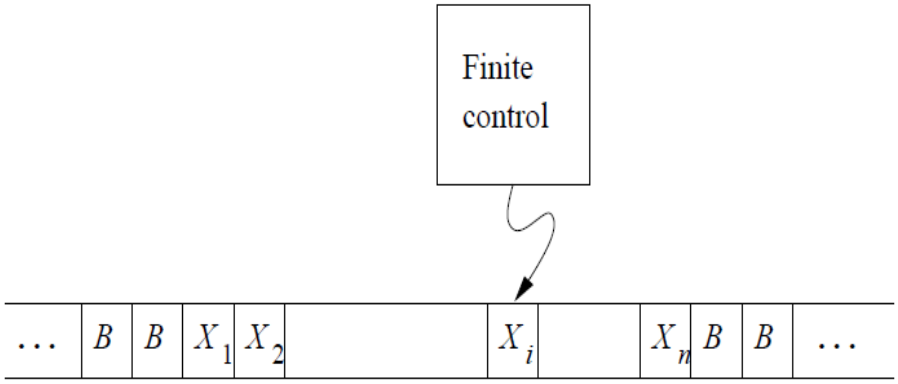
UNIT-V:TURING MACHINES

A Turing machine is an abstract computing machine with the power of real computers. A Turing machine can be visualized as shown in the following figure.



The Turing machine consists of

1. a **finite state control** which can be in any of a finite set of states and

2. an **Infinite tape** divided into cells where each cell holds one of a finite number of tape symbols (including blank symbol)

3. **Read/Write tape head** which can examine one cell at a time.

The Turing machine makes move based on

a) Its current state

b) The tape symbol at the cell scanned by the tape head.

In one move, the Turing machine can

1. Change the state

2. Overwrite the scanned cell with some tape symbol

3. Move the tape head one cell left or right.

Mathematically, a Turing Machine is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$$

where:

- Q is the finite set of *states* of the finite control.
- Σ is the finite set of *input symbols*.
- Γ is the finite set of *tape symbols*; $\Sigma \subset \Gamma$.
- $q_0 \in Q$ is the *start state*.
- $B \in \Gamma$ is the *blank symbol*; $B \notin \Sigma$.
- $F \subseteq Q$ is the set of *final* or *accepting* states.
- δ is a transition function defined as follows.

The arguments of $\delta(q, X)$ are a state q and a tape symbol X. The value of $\delta(q, X)$, if it is defined, is a triple (p, Y, D), where:

1. p is the next state, in Q.
2. Y is the symbol, in Γ, written in the cell being scanned, replacing whatever symbol was there.
3. D is a *direction*, either L or R, standing for "left" or "right," respectively, and telling us the direction in which the head moves.

Representation of a Turing Machine:

A Turing machine can be described by means of a

a) Transition diagram

b) Transition table

c) Instantaneous descriptions using move relations

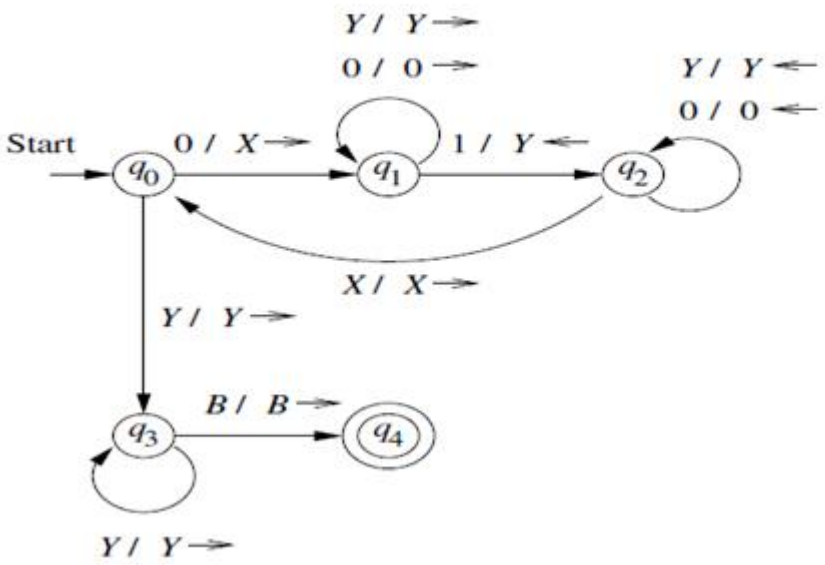
a) Transition diagram:

A transition diagram consists of a set of nodes corresponding to the states of the Turing machine.

An arc from state q to state p is labeled by one or more items of the form X / YD, where X and Y are tape symbols and D is the direction, either L or R. (we can use left arrow ← for L and right arrow → for R)

So whenever $\delta(q,X) = (p,Y,D)$, there is an arc from state q to state p with the label X / YD.

Transition diagram for the Turing machine that accepts the Language L={0^n1^n / n≥1}



b) Transition Table:

The transition function δ of a Turing Machine can also be represented by a table known as transition table.

ex: The transition table of a Turing machine that accepts the Language L={0^n1^n / n≥1}

$$M = (\{q_0, q_1, q_2, q_3, q_4\}, \{0, 1\}, \{0, 1, X, Y, B\}, \delta, q_0, B, \{q_4\})$$

State	Symbol				
	0	1	X	Y	B
q_0	(q_1, X, R)	-	-	(q_3, Y, R)	-
q_1	$(q_1, 0, R)$	(q_2, Y, L)	-	(q_1, Y, R)	-
q_2	$(q_2, 0, L)$	-	(q_0, X, R)	(q_2, Y, L)	-
q_3	-	-	-	(q_3, Y, R)	(q_4, B, R)
q_4	-	-	-	-	-

C) Instantaneous Descriptions:

A Turing machine changes its configuration upon each move.

We use instantaneous descriptions (IDs) for describing such configurations.

An *instantaneous description* is a string of the form

$$X_1X_2\cdots X_{i-1}qX_iX_{i+1}\cdots X_n$$

where

1. q is the state of the Turing machine.
2. The tape head is scanning the ith symbol from the left.
3. $X_1X_2\cdots X_n$ is the portion of the tape between the leftmost and rightmost nonblanks.

We use $\vdash_M$ to designate a move of a Turing machine M from one ID to another.

If $\delta(q, X_i) = (p, Y, L)$ then:

$$X_1X_2\ldots X_{i-1}\ q\ X_iX_{i+1}\ldots X_n \vdash_M X_1X_2\ldots X_{i-2}\ p\ X_{i-1}\ Y\ X_{i+1}\ldots X_n$$

Now, suppose $\delta(q, X_i) = (p, Y, R)$; i.e., the next move is rightward. Then

$$X_1X_2\cdots X_{i-1}qX_iX_{i+1}\cdots X_n \underset{M}{\vdash} X_1X_2\cdots X_{i-1}YpX_{i+1}\cdots X_n$$

Ex:

For the string 0011, initial ID is $q_0$0011

The entire sequence of moves of M is:

$$q_0 0011 \vdash X q_1 011 \vdash X 0 q_1 11 \vdash X q_2 0 Y 1 \vdash q_2 X 0 Y 1 \vdash$$
$$X q_0 0 Y 1 \vdash X X q_1 Y 1 \vdash X X Y q_1 1 \vdash X X q_2 Y Y \vdash X q_2 X Y Y \vdash$$
$$X X q_0 Y Y \vdash X X Y q_3 Y \vdash X X Y Y q_3 B \vdash X X Y Y B q_4 B$$

Ex2: for the string 0010, the sequence of moves of M is given as follows.

$$q_0 0010 \vdash X q_1 010 \vdash X 0 q_1 10 \vdash X q_2 0 Y 0 \vdash q_2 X 0 Y 0 \vdash$$
$$X q_0 0 Y 0 \vdash X X q_1 Y 0 \vdash X X Y q_1 0 \vdash X X Y 0 q_1 B$$

The language of a TM:

Let M = (Q, Σ, Γ, δ, q0, B, F) be a TM. Then L(M) is the set of strings $w \in \Sigma^*$ such that

$$q_0\ w \overset{*}{\vdash} \alpha\ p\ \beta$$

for some state $p \in F$ and any tape string α and β

Note:

1. We always assume that a Turing Machine halts if it accepts
2. The set of languages we can accept using a TM is often called the recursively enumerable languages or RE languages.
3. There is another notion of acceptance that is commonly used for TM: **acceptance by halting**

Acceptance by Halting:

We say a TM halts if it enters a state q scanning a tape symbol X, and there is no move in this situation, i.e., δ(q, X) is undefined.

We can always assume that a TM halts if it accepts. Unfortunately, it is not always possible to require that a TM halts even if it does not accept

Note:

1. Recursive Languages:

Languages for which TM do halt, regardless of whether or not they accept are called as recursive Languages

2. If an algorithm to solve a given problem exists, then we say the problem is decidable.

Problems:

Design a Turing machine that recognizes the language

$$L= \{a^n b^n c^n / n \geq 1\}$$

Sol:

Logic:

First replace 'a' from front by X, then keep moving right till you find a 'b' and replace this 'b' by Y. Again, keep moving right till you find 'c', replace it by Z and move left. Now keep moving left till you find a X. When you find it, move a right, then follow the same procedure as above.

A condition comes when you find a X immediately followed by a Y. At this point we keep moving right and keep on checking that all b's and c's have been converted to Y and Z. If not then string is not accepted. If we reach Blank 'B' then string is accepted.

The turing machine is given as $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$ where

$Q=\{q0,q1,q2,q3,q4,q5\}$

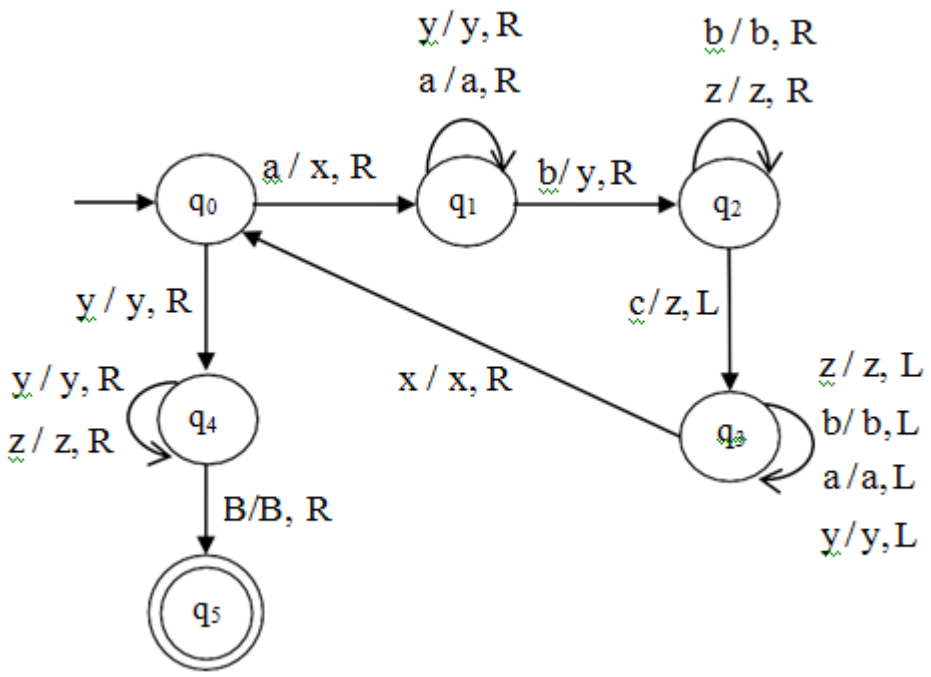
$\Sigma =\{a,b,c\}$

$\Gamma=\{a,b,c,x,y,z,B\}$

$F=\{q_5\}$

and δ is given as follows.

Q \ Γ	a	b	c	x	y	z	B
→ q_0	(q_1, x, R)	-	-	-	(q_4, y, R)	-	-
q_1	(q_1, a, R)	(q_2, y, R)	-	-	(q_1, y, R)	-	-
q_2	-	(q_2, b, R)	(q_3, z, L)	-	-	(q_2, z, R)	-
q_3	(q_3, a, L)	(q_3, b, L)	-	(q_0, x, R)	(q_3, y, L)	(q_3, z, L)	-
q_4	-	-	-	-	(q_4, y, R)	(q_4, z, R)	(q_5, B, R)
* q_5	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ



2. Design a Turing machine that accepts the set of all palindromes over {0,1}

Sol:

The turing machine is given as M = (Q, Σ, Γ, δ, q_0, B, F) where

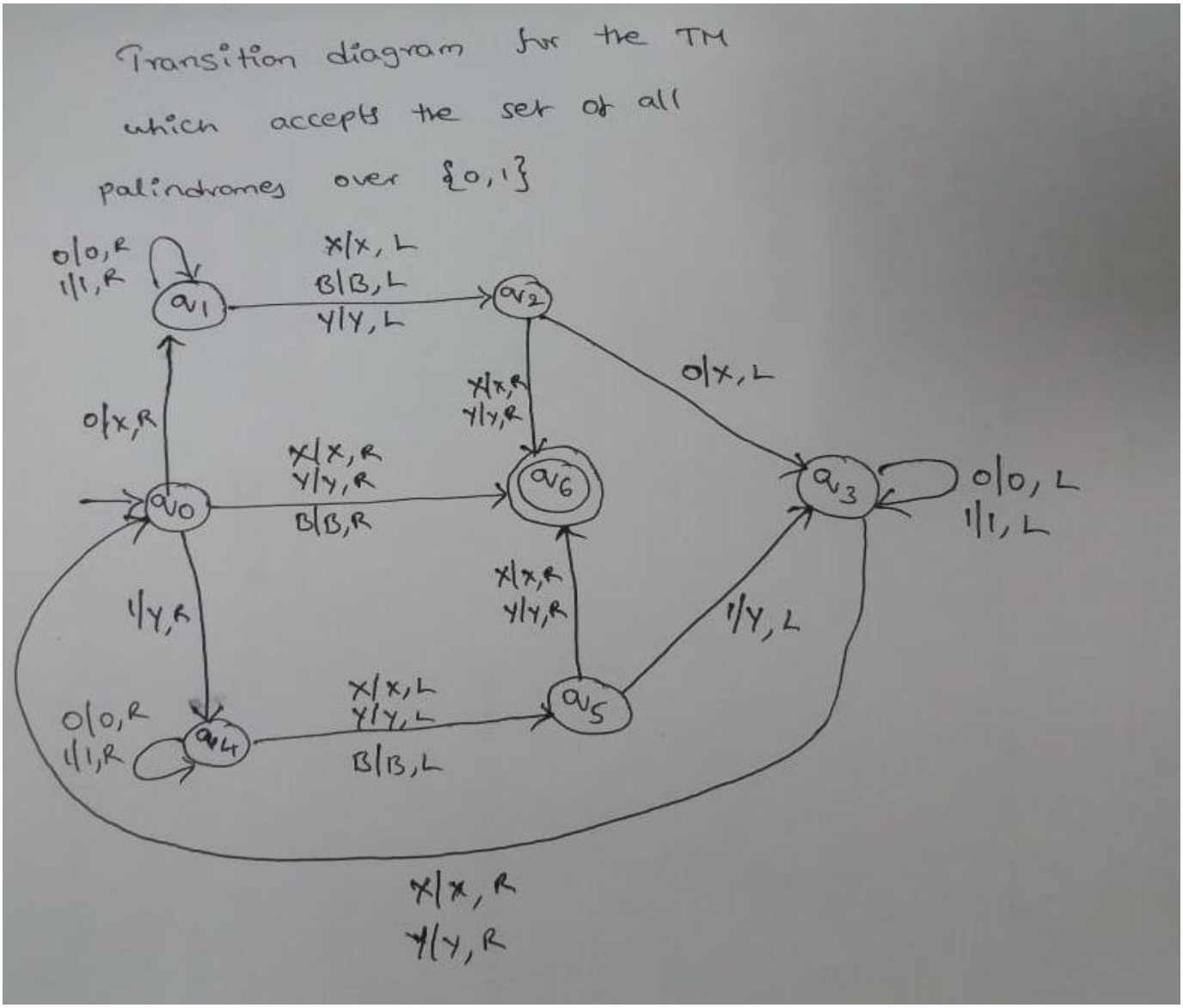
Q={q_0,q_1,q_2,q_3,q_4,q_5,q_6}

Σ ={a,b,c}

Γ={a,b,c,x,y,z,B}

F={q_6}

and δ is given as follows.



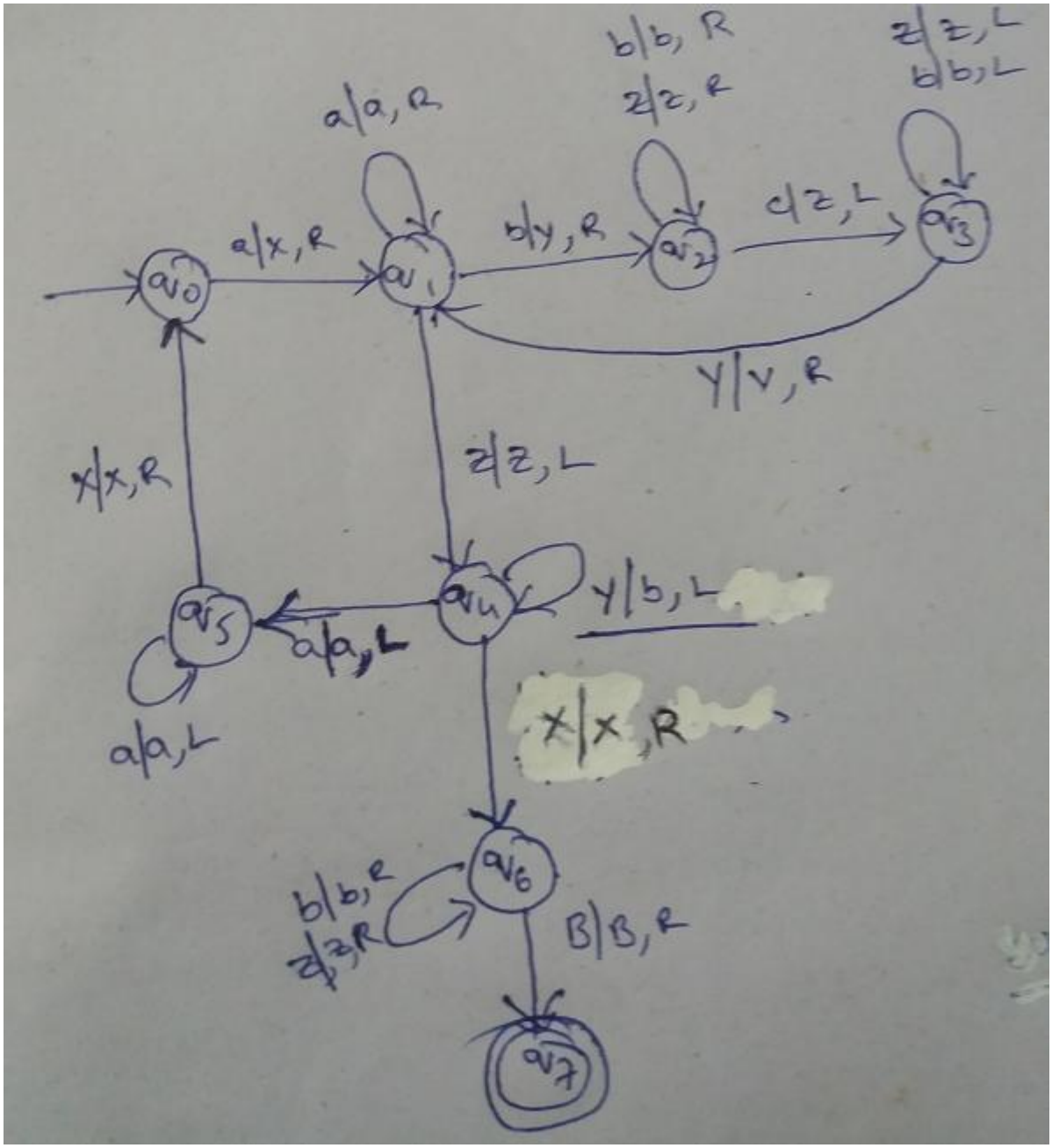
In state q0, when it encounters X, Y Or B then the palindrome is a even length palindrome. In state q2 or q5, when it encounters X or Y then the palindrome is a odd length palindrome.

In state q2, when it encounters 1 it is not a palindrome. Similarly in state q5, when it encounters 0 it is not a palindrome.

Design a turing machine that accepts L={$a^i b^j c^k$ / i*j=k, i,j,k>=1}

Sol:

The transition diagram for the turing machine is given below.



COMPUTABLE LANGUAGES AND FUNCTIONS

Turing machine is capable of performing several operations/functions. It is capable of performing any sort of computations such as

- Addition
- Subtraction
- Multiplication
- Division
- 2's complementation
- 1's complementation
- Comparing two numbers
- Squaring a number
- GCD of numbers
- Finding binary equivalents.

Representation:

Computation of a function „f" is represented as

$$f : \sum_1^* \rightarrow \sum_2^*$$

The function f is Turing computable by the machine,

$$M = (Q, \sum, \Gamma, \delta, q_0, B, F)$$

With the transition function δ of the form,

$$(q_0, \omega) \xrightarrow[M]{*} (q_f, f(\omega))$$

where,

$q_0 \rightarrow$ Initial state ($q_0 \varepsilon Q$)

$\omega \rightarrow$ Input string, where $\omega \varepsilon \sum_1^*$.

$q_f \rightarrow$ Final state ($q_f \varepsilon F$)

$f(\omega) \rightarrow$ Output string after computation of „ω" by f $\left(f(\omega) \varepsilon \sum_2^*\right)$.

Representation of a number

For simplicity, the unary numbers are represented by a single symbol. Let it be "0".

The decimal numbers are represented as,

No input → B

1 → 0

2 → 00

3 → 000

4 → 0000

5 → 00000

.

.

.

n → 'n' number of zeros [0^n]

1. Design a Turing Machine to perform 2's complement of a binary number.

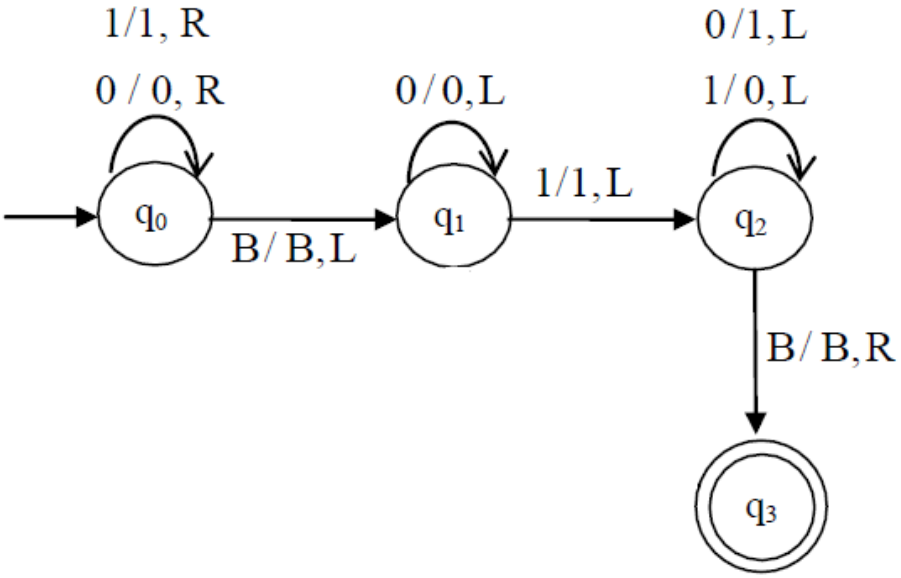
Sol:

2's complement computation:

1. Traverse right and locate the rightmost bit

2. Do not change the bits from the right towards left until the first '1' has been processed

3. Perform complementation to the rest of the bits from right to left (after first 1 is processed)



2. Design a Turing Machine to find 1's complement of a binary number

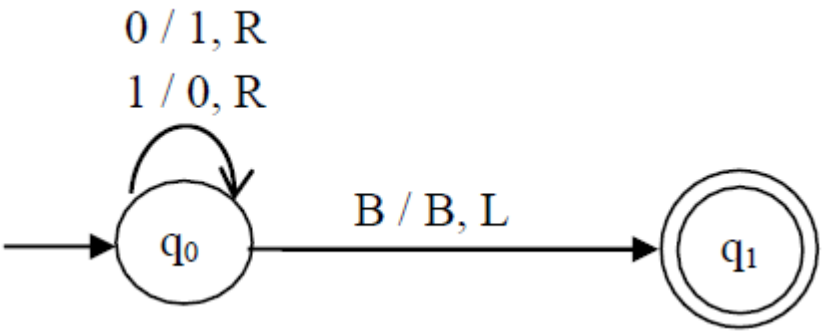
Sol:

1's complement computation:

1. On reading the input, if the symbol is '0', replace it by '1' and move right.

2. If the symbol read is '1' replace it by '0' and move right.

3. Halt the TM if all the string symbols are processed by step 1 and 2



3) Design a Turing Machine that computes addition of two unary numbers.

Sol:

Here the unary number ‘n’ is represented with a symbol ‘0’ n times.

Ex: 2 can be represented as 00

4 can be represented as 0000

6 can be represented as 000000

The separation symbol # is used between two inputs.

Ex: the inputs 5 and 2 can be represented as

00000#00

The input unary numbers m and n can be represented as $0^m\#0^n$

To construct the TM that will add two unary numbers m and n, the Turing Machine M will start with a tape consisting of $0^m\#0^n$ surrounded by blanks. M halts with 0^{m+n} on its tape surrounded with blanks. The required TM is given as

4) Design a Turing Machine that computes the following function

$$f(m,n)=\begin{cases} m-n & if\ m \geq n \\ 0 & otherwise \end{cases}$$

(the above function is called proper subtraction function $m \dot{-} n$.)

Sol: The Turing machine M will start with a tape consisting of $0^m\#0^n$ surrounded by blanks.

M halts with

$$0^{m \dot{-} n}$$

On its tape surrounded by blanks.

The required Turing Machine is given as

Techniques for Turing machine construction:

Describing a Turing machine transitions is not a simple task. To make this task easy, there are some high level conceptual techniques. These techniques can be used in the construction of simple and efficient Turing machines. Some of them are

1. Turing machine with stationary head

2. storage in the state

3. .multiple tracks

4. subroutines

1. Turing machine with stationary head:

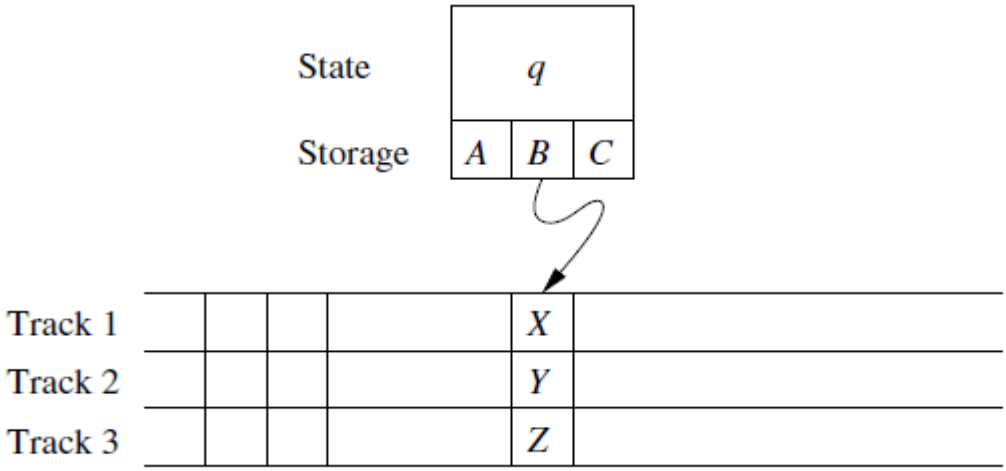
In the definition of TM, the value of $\delta(q,x)$, if it is defined, is given as (p,Y,D) where D= L or R. so the head moves to the left or righ after reading an input symbol.

Suppose we want to include the option that the **head can continue to be in the same cell for some input symbol.** Then we can define $\delta(q,x)$ as (p,Y,S). This means that the TM, on reading the input symbol x, changes to state p and writes Y in place of x and continues to remain in the same cell.

Thus in this model, the value of $\delta(q,x)$, if it is defined, is given as (p,Y,D) where D=L , R and S.

2. storage in the state:

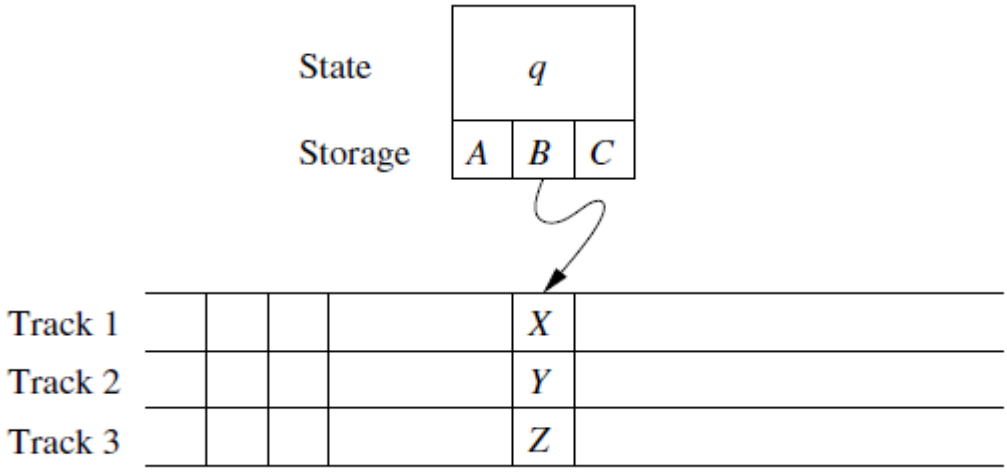
Sometimes it helps to design a TM for a particular language if we imagine that the **state two or more components**. One component is control component and the other components hold data that the TM need to remember. This is illustrated in the following figure.


State
q
Storage
A
B
C
Track 1
X
Track 2
Y
Track 3
Z

Here, the finite control not only consists of a control state q, but also holds three data elements A,B and C. This **technique requires to think state as [q,A,B,C]**. Regarding states this way allows us to describe transitions in a more systematic way.

3.Multiple Tracks:

In a multiple track TM, a single tape is assumed to be divided into several tracks. Each track can hold one symbol, and the tape alphabet of the TM consists of tuples, with one component for each track. For ex, consider the following TM which consists of three tracks.



Here the cell scanned by the tape head contains the symbol [X,Y,Z].

If the number of tracks is equal to k, then the tape alphabet is required to consist of k-tuples of tape symbols. Here the tape symbols are elements of Γ^k .

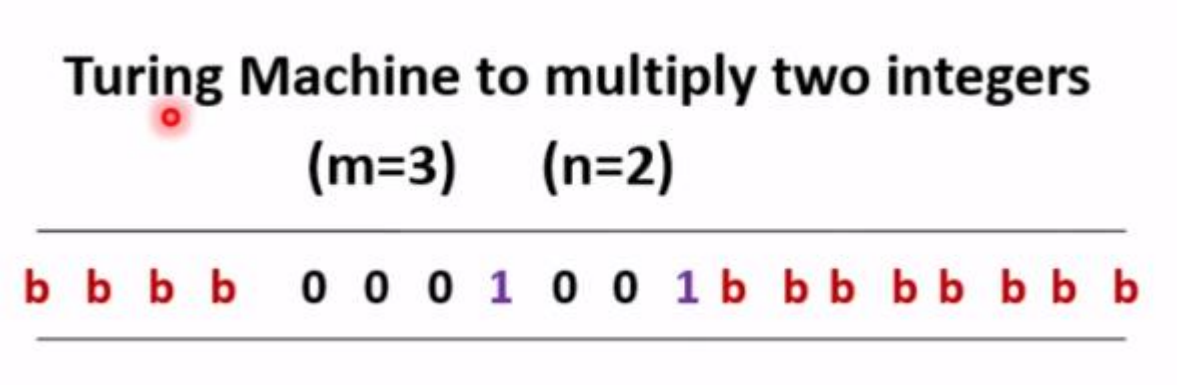
4. subroutines:

Just as a computer program has a main procedure and subroutines, the TM can also be programmed to have a main TM and TMs which serve as subroutines.

A subroutine of a Turing machine is a set of states in the TM such that performs a small useful computation. Usually, a subroutine has single entry state and a single exit state. Many very complicated tasks can be performed by TMs by breaking those tasks into smaller subroutines.

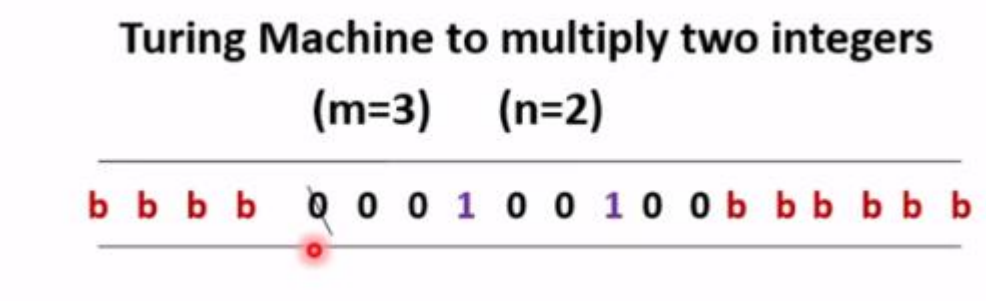
In order that a Turing machine M1 uses another Turing machine M2 as a subroutine, the states of M1 and M2 have to be disjoint. Also when M1 wants to call M2, from a state of M1, the control goes to the initial state of M2. When the subroutine finishes and returns to M1, from the halting state of M2, the machine goes to some state of M1.

Ex: For multiplying two unary numbers m and n, n has to be copied on m times. We can write a sub TM for copying and main TM will call this m times.

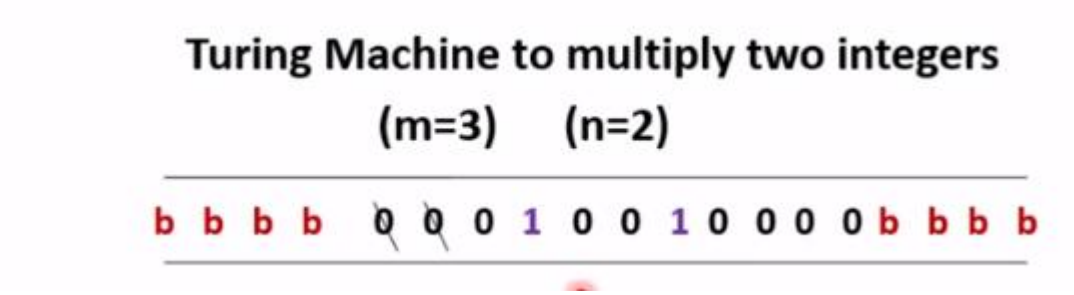


- 0^m 1 0^n 1 is placed on the tape
  (the output will be written after the rightmost 1).

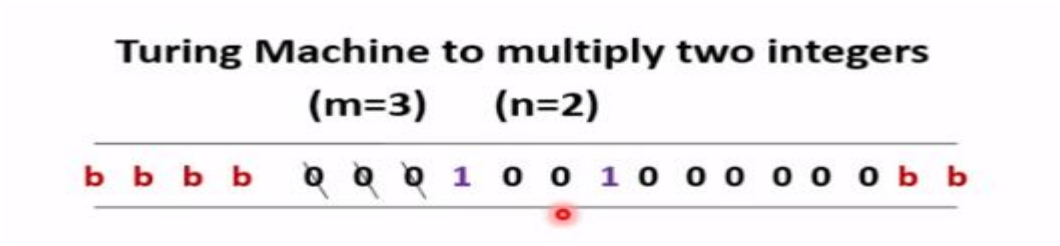
- The leftmost 0 is erased.
- A block of n 0's is copied onto the right end.



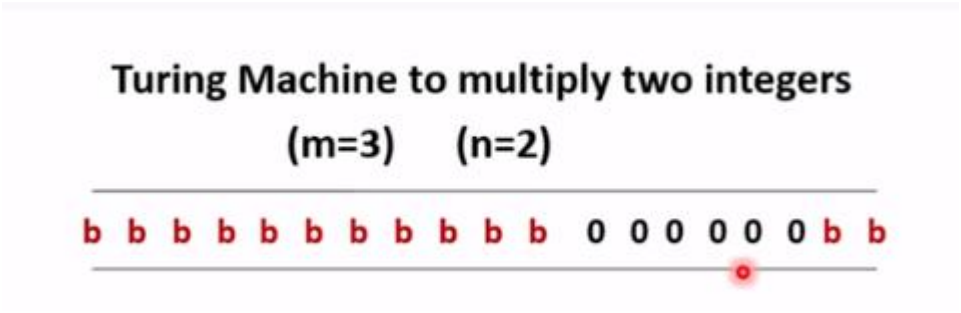
The next 0 is erased and once again block of n 0's is copied onto the right end.

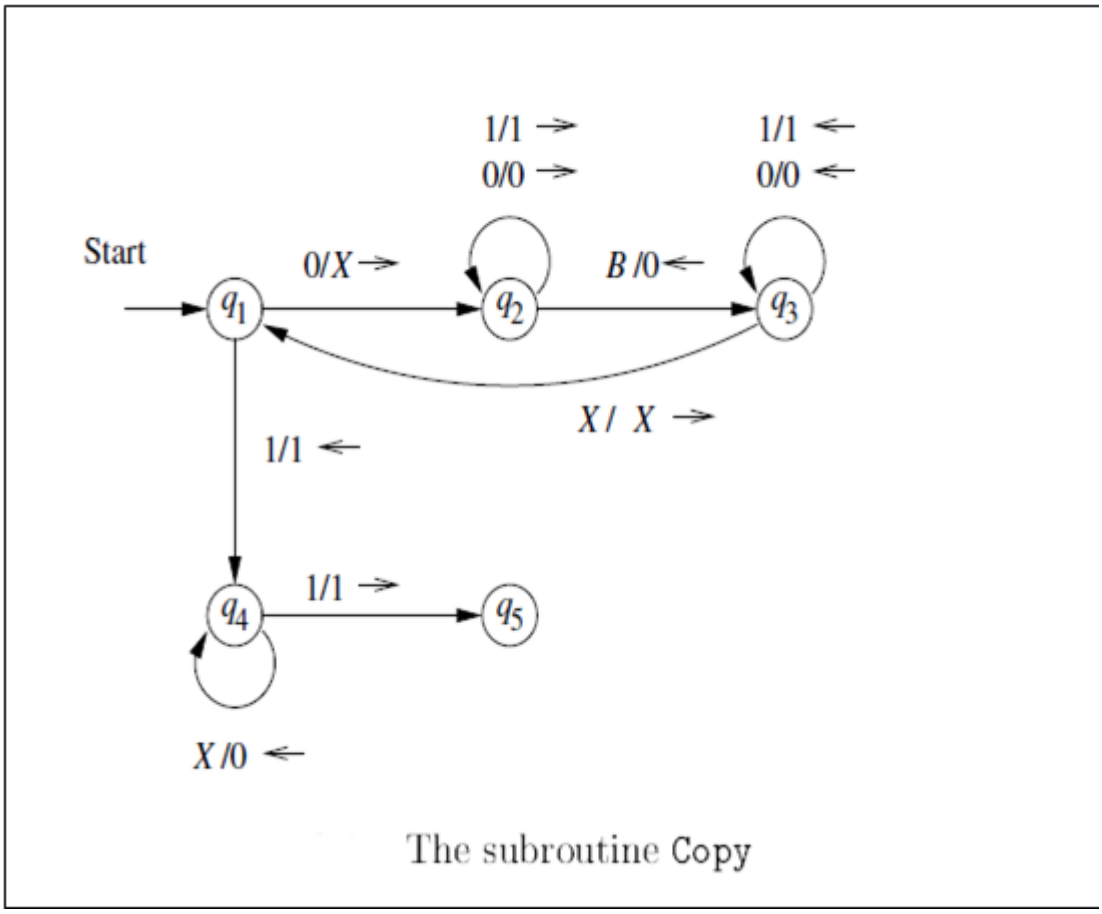


The next leftmost 0 is erased and once again block of n0's is copied on to the right end.

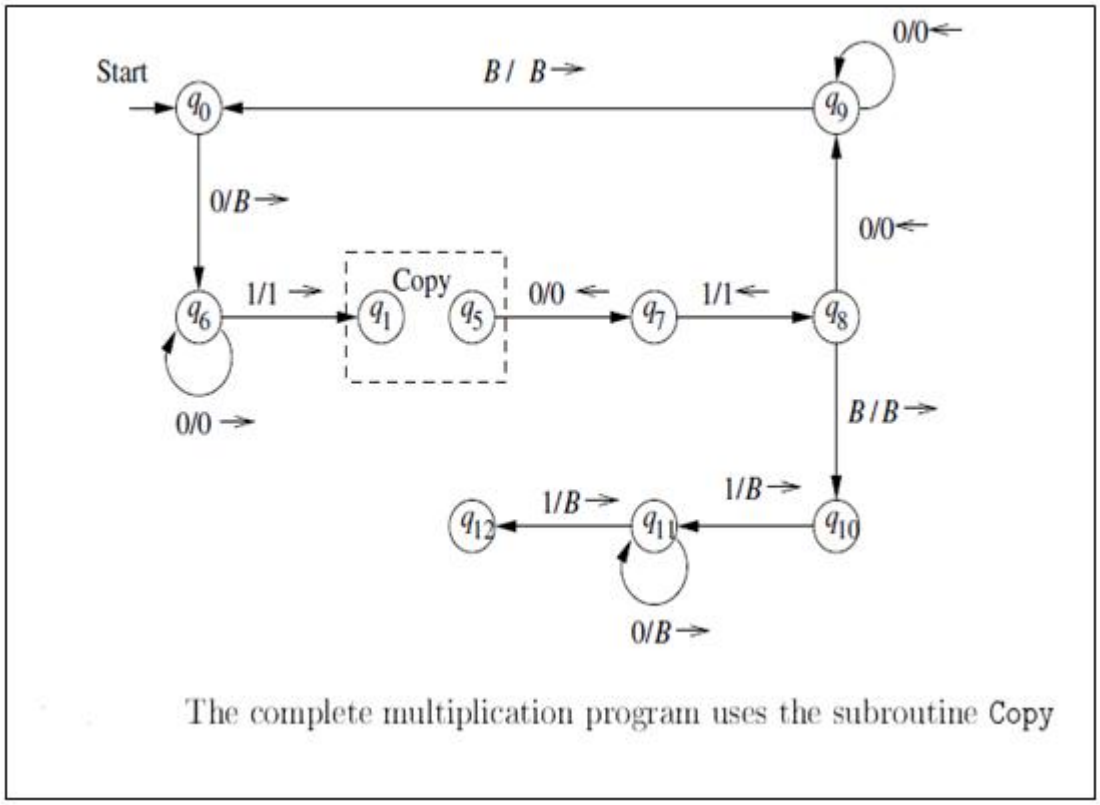


The above process is repeated m times and 10^n10^{mn} is obtained on the tape. Change the leading 10^n1 to blanks, leaving the product 0^{mn} as output.





The subroutine Copy



The complete multiplication program uses the subroutine Copy

Variants of Turing machines:

The following are the variants of turing machines

1. multitape turing machines

2. Nondeterministic Turing Machines

3. Restricted Turing Machines

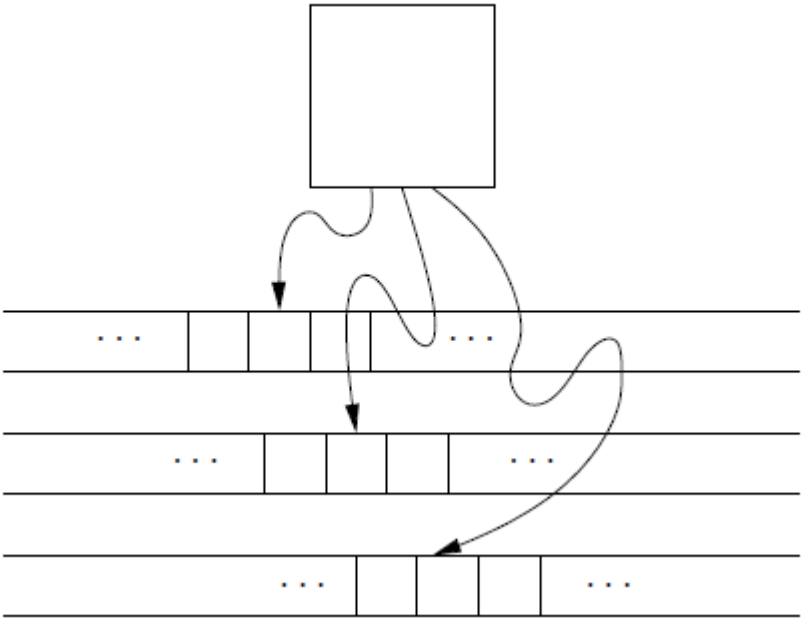
    i) semi-Infinite tape Turing Machines

    ii) multistack machines

    iii)Counter machines

1. Multitape Turing Machines:

A multitape TuringMachine is illustrated in the following figure.



It has a finite control and some finite number of tapes. Each tape is divided into cells, and each cell can hold any symbol of the finite tape alphabet. The set of states includes an initial state and some final states.Initially

1. the Input is placed on the first tape.

2. All other cells of all the tapes hold the blank

3. the finite control is in the initial state.

4. The head of the first tape is at the left end of the input.

5. All other tape heads are at some arbitrary cell. Some tapes other than the first tape are completely blank.

A move of the multitape TM depends on the state and the symbol scanned by each of the tape heads. In one move, the multitape TM does the following.

1. The control enters a new state, which could be the same as the previous state.

2. On each tape, a new tape symbol is written on the cell scanned. Any of these symbols may be the same as the symbol previously there.

3. Each of the tape heads makes a move, which can be either left, right or stationary. The heads move independently, so different heads may move in different directions, and some may not move at all.

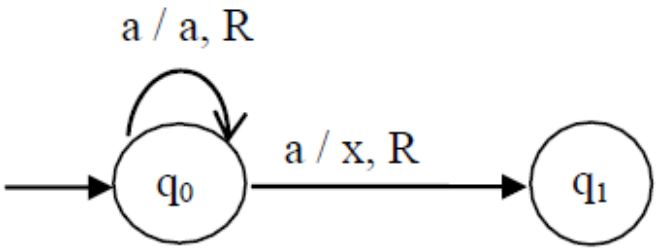
2. Nondeterministic Turing Machine:

For a non deterministic turing machine, the transition function δ is defined such that for each state q and tape symbol x, **δ(q,x) is a set of triplets**

$\{(q_1,y_1,D_1),(q_2,y_2,D_2),\ldots\ldots\ldots\ldots\ldots\ldots,(q_k,y_k,D_k)\}$ where k is any finite integer.

So a NTM has a finite number of choices of next move for each state and symbol scanned. The NTM M accepts an input w if there is any sequence of choices of move that leads from the initial ID with w as input, to an ID with an accepting state.

Ex:



The above transition takes on two paths for the same input 'a'. the transition of 'a' at q_0 is defined as

$$\delta(q_0, a) = \{(q_0, a, R), (q_1, x, R)\}$$

Note:

If M_N is a nondeterministic turing machine, then there exists a deterministic turing machine M_D such that $L(M_N) = L(M_D)$

3. Restricted Turing Machines:

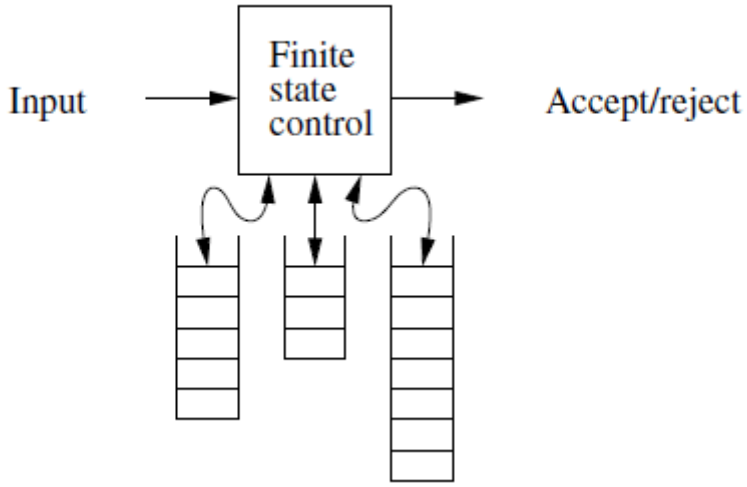
i) Semi Infinite tape Turing Machine:

Restrict a TM to have a tape that is infinite only to the right, with no cells to the left of the initial head position. Here tape head of a Turing machine either move left or right from its initial position but it never moves left of its initial position.

ii) multistack machines:

we can restrict the tapes of a multitape TM to behave like a stacks. A k-stack machine is a deterministic PDA with k stacks. The input is on a separate tape which read once from left-to-right mimicking the input mode for the PDA.

A machine with three stacks is illustrated in the following figure.



It has a finite control which is in one of a finite set of states. It has a finite stack alphabet which it uses for all its stacks. A move of a multistack machine is based on

1. the state of the finite control

2. The Input symbol read which is chosen from the finite input alphabet.

3. the top most stack symbol on each of its stacks/

In one move, the multistack machine can

a) change to a new state

b) replace the top symbol of each stack with a string of zero or more stack symbols. There can be a different replacement string for each stack.

Thus the transition rule for a k-stack machine will be like

$$\delta(q, a, X_1, X_2, \ldots\ldots, X_k) = (p, \alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_k)$$

The interpretation of this rule is that in state q, with X_i on top of i^{th} stack, for i=1,2,3,…..,k , the machine may consume a from its input (either an input symbol or ε), go to state p, and replace X_i on top of i^{th} stack by string α_i for i=1,2,3,….,k. The multistack machine accepts the input by entering into a final state.

iii) counter machines:

The counter machine has the same structure as that of multistack machine, but in place of each stack is a counter. Counters hold any nonnegative integer, but we can distinguish between zero and nonzero counters.

The move of a counter machine depends on its state, input symbol, and which, if any, of the counters are zero. In one move, the counter machine can

a) change state

b) Add or subtract 1 from any of its counters , independently. But a counter is not allowed to become negative, so it cannot subtract 1 from a counter that is currently zero.

Church-Turing Thesis:

By the 1930s the emphasis was on formalising algorithms. **Alan Turing**, at Cambridge, **devised** an abstract machine, now called a **Turing Machine** to define/represent algorithms. **Alonso Church**, at Princeton, **devised the Lambda Calculus** which formalises algorithms as functions. The demonstrated equivalence of their formalisms strengthened both their claims to validity, expressed as the Church-Turing Thesis that shows the equivalence between mathematical concepts of algorithm or computation and Turing machine. The Thesis can be given as follows.

"a problem can be solved by an algorithm iff it can be solved by a Turing Machine"

(or)

"all algorithmically solvable problems can be solved by a Turing Machine"

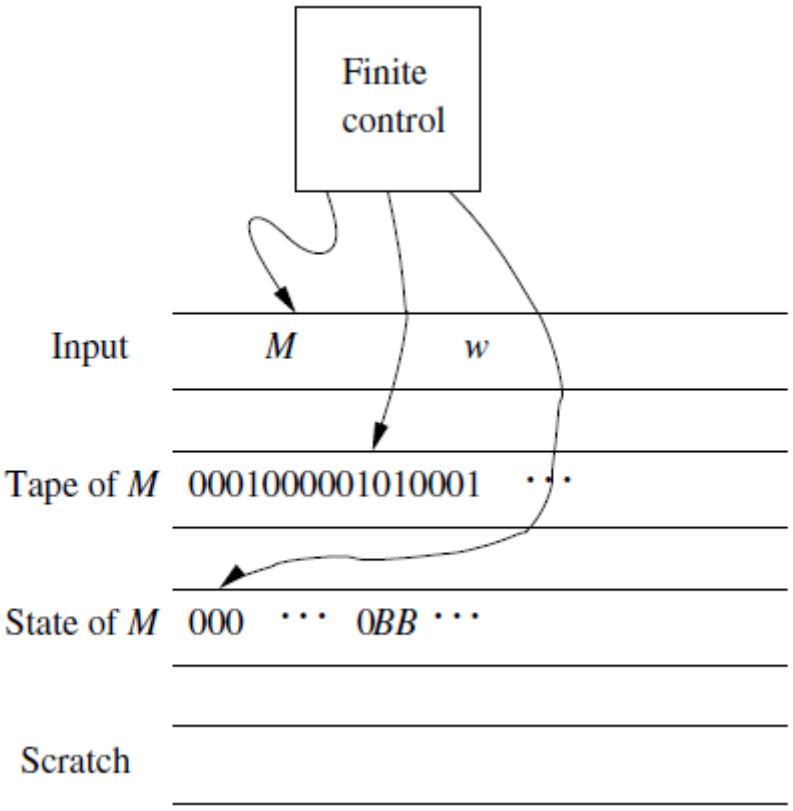
(or)

"a function is computable iff it can be solved by a Turing Machine"

So the church-Turing thesis asserts that if some calculation is effectively carried out by an algorithm, then there exists a Turing Machine which will compute that function.

Universal Turing Machine:

A Universal Turing Machine U is a single Turing Machine that is capable of simulating any other Turing machine. This machine will take coded pair <M ,w> as input, where M is a Turing Machine and string w is an input to M. It will then simulate M's computation on w. It is a multitape Turing machine whose organization is shown in the following figure.



Initially the transitions of M along with the string w are stored on the first tape. A second tape will be used to hold the simulated tape of M where tape symbol x_i of M will be represented by o^i and tape symbols will be separated by single 1's. The third tape of U hold the state of M with state q_i represented as i 0's.

The operation of U can be summarized as follows.

1. Examine the input to make sure that the code for M is a legitimate code for some TM. If not, U halts without accepting.

2. Initialize the second tape to contain the input w in its encoded form. That is, for each 0 of w, place 10 on the second tape, and for each 1 of w , place 100 on the second tape.

3. Place 0, the start state of M, on the third tape, and move the head of U's second tape to the first simulated cell.

4. To simulate a move of M, U searches on its first tape for a transition $0^i10^j10^k10^l10^m$ such that 0^i is the state on tape 3 and 0^j is the tape symbol of M that begins at the position on tape 2 scanned by U. This transition is the one that m would next make. U should:

(a) Change the contents of tape 3 to 0^k; that is, simulate the state change of M. To do so, U first changes all the 0's on tape 3 to blanks, and then copies 0^k from tape 1 to tape 3.

(b) Replace 0^j on tape 2 by 0^l; that is, change the tape symbol of M.

(c) Move the head on tape 2 to the position of the next 1 to the left or right, respectively, depending on whether $m = 1$ (move left) or $m = 2$ (move right). Thus, U simulates the move of M to the left or to the right.

If M has no transition that matches the simulated state and tape symbol then no transition will be found. Thus M halts as well as U halts.

5. If M enters into its accepting state, then U accepts.

In this way, U simulates M on w. U accepts the coded pair <M,w> if and only if M accepts w.

Unit –VI: COMPUTABILITY

Note1:
A Turing machine halts if it enters a state q, scanning a tape symbol X, and there is no move in this situation, i.e., $\delta(q,X)$ is undefined.

Note 2:
We can always assume that a Turing machine halts if it accepts.

Note 3:
Unfortunately, it is not always possible to require that a Turing machine halts even if it does not accept.

Note 4:
Remember that there are three possible outcomes of executing a Turing machine over a given input. The Turing machine may

- Halt and accept the input;
- Halt and reject the input; or
- Never halt.

Recursively Enumeranle Languages and Recursive Languages:

Definition: Recursively Enumerable language

A Language L is said to be recursively enumerable if there exists a Turing Machine M such that L=L(M)

Note:
If L is recursively enumerable language, then L=L(M) for some TM M, and

- If the string w is in L then M halts in a final(or accepting) state
- If the string w is not in L then M may halt in a non-final state or loop forever.(may halt or may not halt)

Definition: Recursive Language:

A language L is said to be recursive language if there exists a Turing Machine M such that L=L(M) and M halts on all inputs.

Note:
If L is recursive language, then L=L(M) for some TM M, and

- If the string w is in L then M halts in a final(or accepting) state
- If the string w is not in L then M may halt in a non-final state or no transition is possible (does not go into infinite loop) (definitely halts).

A Turing Machine of this type corresponds to the notion of "algorithm"

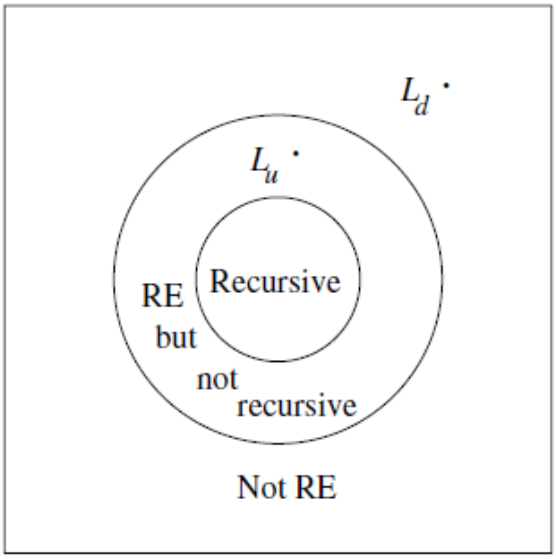
Note 1:

The set of recursive languages is subset of set of recursively enumerable languages

Note 2:

Following are three classes of languages that we can consider now.
a. The recursive languages
b. The languages that are recursively enumerable but not recursive
c. The non –recursively enumerable languages (non RE)

The relation among these languages is given below.

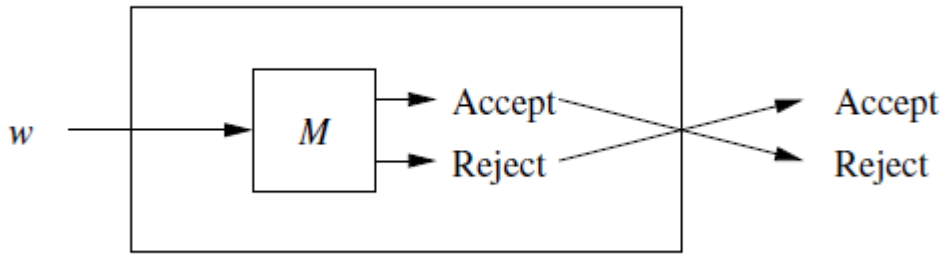


Theorem:

If L is a recursive language, then $\bar{L}$ is also recursive.

Proof:

Let L=L(M) for some TM M that always halts. We construct a TM $\bar{M}$ such that $\bar{L}$ = L($\bar{M}$) that is guaranteed to halt. The construction of $\bar{M}$ is shown in the following figure.



Here M is modified as follows to create $\bar{M}$

1) The accepting states ofM are made nonaccepting states of $\bar{M}$ with no transitions i.e., in these states $\bar{M}$ will halt without accepting.
2) $\bar{M}$ has a new accepting state r ; there are no transitions from r
3) For each combination of a nonaccepting state of M and a tape symbol of M such that M has no transition, add a transition to the accepting state r.

Since M is guaranteed to halt, we know that $\bar{M}$ is also guaranteed to halt. Also $\bar{M}$accepts exactly those strings that M does not accept.Thus $\bar{M}$ accepts $\bar{L}$

Decidable and undecidable problems:

Note: a problem with only two answers yes/no can be considered as a language.

A problem with two answers (Yes/No) is called decidable if the corresponding language is a recursive language (decides on all inputs) and the problem is called undecidable if it is not a decidable.

Examples of decidable languages and problems:

1. Given a DFA does it accept a given word (Acceptance problem for DFA)
   The corresponding language is
   A_{DFA}={<B,w> / B is a DFA that accepts the string w}

   A_{DFA} is decidable

2. Given two DFA's, do they accept the same language(Equivalence problem for DFA)
The corresponding language is

   E_{DFA}={<A,B>/ A,B are DFA's , L(A) = L(B)}

E_{DFA} is decidable

3. A_{CFG}={<G,w> / G is a CFG that accepts the string w} is decidable

Examples of undecidable languages:

1. A_{TM} = {<M,w> / M is a Turing machine and M accepts w} is undecidable

(i.e., Given a TM, does it accept a input string is undecidable)

Proof:

Suppose that A_{TM} is decidable and is decided by a TM H that halts on all inputs.

Then

$$H(\langle M, w\rangle) = \begin{cases} accept & \text{if } M \text{ accepts } w \\ reject & \text{if } M \text{ does not accept } w \end{cases}$$

Using H, now construct another TM D which on input < M >, runs H on input < M, < M > > and outputs opposite of H.(if H exists then definitely D exists)

$$D(\langle M\rangle) = \begin{cases} accept & \text{if } M \text{ does not accept } \langle M\rangle \\ reject & \text{if } M \text{ accepts } \langle M\rangle \end{cases}$$

Finally, run D with its own description <D> as input. According to the construction of D, we must have

$$D(\langle D\rangle) = \begin{cases} accept & \text{if } D \text{ does not accept } \langle D\rangle \\ reject & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

This means that D accepts < D > if D does not accept < D >,which is a contradiction. Hence neither D nor H can exist.

Hence A_{TM} is undecidable.

Halting problem of a Turing Machine:

The halting problem of a Turing machine is given as follows.

"Does a given Turing machine M halts on given input string w".

The corresponding language is given as

$HALT_{TM}$ = { <M,w> / M is a Turing Machine and M halts on input w}

Theorem:

$HALT_{TM}$ = { <M,w> / M is a Turing Machine and M halts on input w} is undecidable.

Proof:

Suppose that $HALT_{TM}$ is decidable.

So by definition of decidability, there exists a TM say R such that L(R) = $HALT_{TM}$ and R halts on all its inputs.

Now construct a TM S as follows.

1. For the TM S, <M,w> is the input
2. Run TM R on input <M,w>
3. If R rejects <M,w> (i.e., TM M does not halt on w), reject

4. If R accepts <M,w> (i.e., TM M halts on w), simulate M on w until it halts.

5. If M accepts w, then S accepts <M,w> , otherwise if M does not accept w, then S does not accept <M,w> .

Now since R halts on all its inputs, S also halts on all inputs. Also, by the above construction, we can show that

$L(S) = \{<M,w> / \text{the TM M accepts w}\}$

So the turing machine S is a decider for A_{TM}. So A_{TM} is decidable which is a contradiction to the fact that A_{TM} is undecidable.

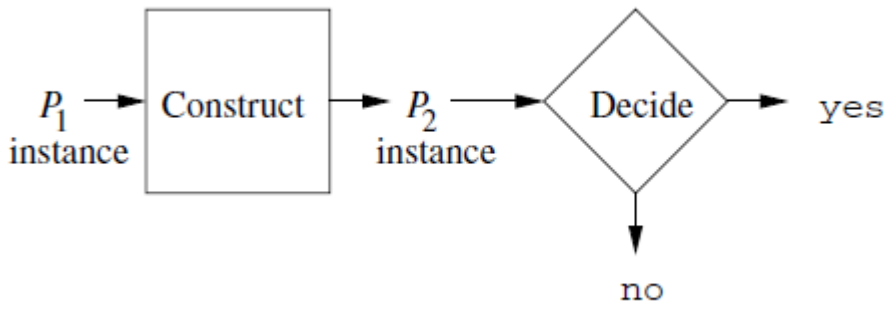
The contradiction arises because of our assumption that $HALT_{TM}$ is decidable.

So $HALT_{TM}$ is undecidable.

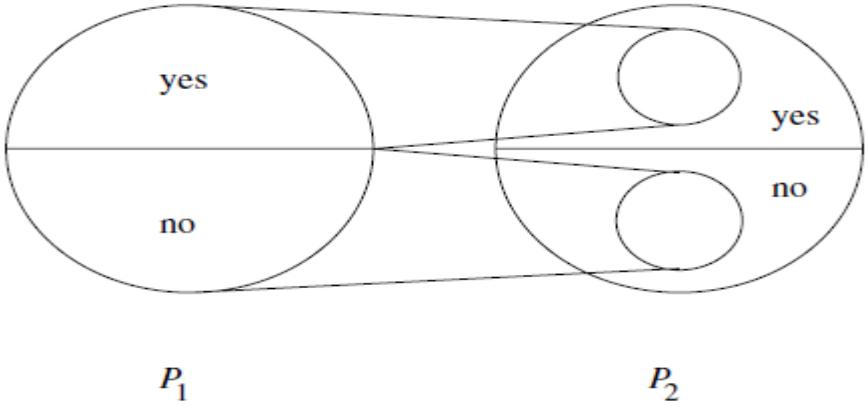
Reductions:

In general, if we have an algorithm to convert instances of a problem P1 to instances of a problem P2 that have the same answer, then we say that P1 reduces to P2.

The reduction strategy is illustrated in the following figure.



A reduction must turn any instance of P1 that has a "yes" answer into an instance of P2 with a "yes" answer, and every instance of P1 with a "no" answer must be turned into an instance of P2 with a "no" answer. This is illustrated in the following figure.



If we could solve problem P_2, then we could use its solution to solve problem P_1. Also if there is a reduction from P_1 to P_2, then following will hold.

a) If P_1 is undecidable, then so is P_2.

b) If P_1 is non-RE, then so is P_2.

Rice's Theorem:

Definition 1:

A property of RE languages is simply a set of RE languages. We say L satisfies the property P if $L \in P$.

Definition 2:

For any property P, define language L_P to consist of Turing Machines which accept a language in P i.e., $L_P = \{<M> \mid L(M) \in P\}$

Also decidability of a property P means that decidability of the language L_P i.e., deciding if a language represented as a TM satisfies the property P.

Ex: 1. {<M> | L(M) is infinite}

2. $L_e = \{<M> \mid L(M) = \Phi\}$

Definition 3: Trivial Properties:

A property is trivial if it is either empty(i.e., satisfied by no language at all) or is all RE languages(i.e., satisfied by all languages) . otherwise it is nontrivial property

Ex:

1. P = { L | L is recognized by a TM with an even number of states}

This is a **trivial property** because for any RE language we have a TM and even if that TM is having odd number of states we can make an equivalent TM having even number of states by adding one extra state. Thus this property satisfied by all the RE languages

2. $P=\{L \mid L \subseteq \Sigma^*\}$

This is a **trivial property** since all languages are subset of Σ* and hence this property satisfied by all recursively enumerable languages.

3. P = { L | L is a recognized by a TM and is finite}

This is a **non trivial property** since it cannot satisified by some Languages and satisfied by some languages.

RICE's theorem:

Statement:

Every nontrivial property of the RE languages is undecidable.

POST CORRESPONDENCE PROBLEM (PCP PROBLEM):

The PCP problem is given as follows.

Given two lists of the strings over some alphabet Σ and of equal length, whether we can pick a sequence of corresponding strings from the two lists and form the same string by concatenation.

An instance of PCP problem is, Given the two lists of strings $A=w_1,w_2,\ldots\ldots\ldots,w_k$ and $B=x_1,x_2,\ldots\ldots\ldots,x_k$ for some integer k. And for each i, the pair (w_i, x_i) is said to be the corresponding pair.

This instance has a solution, if there is a sequence of one or more integers $i_1,i_2,i_3,\ldots\ldots\ldots,i_m$ such that

$$w_{i_1} w_{i_2} \cdots w_{i_m} = x_{i_1} x_{i_2} \cdots x_{i_m}.$$

The sequence $i_1,i_2,i_3,\ldots\ldots\ldots,i_m$ is a solution to this instance of PCP.

Ex:
Let Σ = {0,1} and let A and B are given as follows.

	List A	List B
i	w_i	x_i
1	11	111
2	100	001
3	111	11

The above instance of PCP has a solution. The solution is 1, 2, 3 since

$$w_1 w_2 w_3 = x_1 x_2 x_3$$

Ex:
Let Σ = {0,1} and let A and B are given as follows.

	List A	List B
i	w_i	x_i
1	110	110110
2	0011	00
3	0110	110

The above instance of PCP has a solution. The solution is 2,3,1 since

$$w_2 w_3 w_1 = x_2 x_3 x_1$$

Ex:
Let Σ = {0,1} and let A and B are given as follows.

	List A	List B
i	w_i	x_i
1	01	0101
2	1	10
3	1	11

Here $|w_i| < |x_i|$ for each i=1,2,3 i.e., the list B is having strings of greater lengths than that of list A. so to get the same string after concatenation is difficult. So this instance of PCP has no solution.

Modified PCP (MPCP) problem:

An instance of MPCP is given as follows. Given the two lists of strings $A=w_1,w_2,\ldots\ldots,w_k$ and $B=x_1,x_2,\ldots\ldots,x_k$ for some integer k. And for each i, the pair (w_i, x_i) is said to be the corresponding pair.

This instance has a solution, if there is a sequence of zero or more integers $i_1,i_2,i_3,\ldots\ldots,i_m$ such that

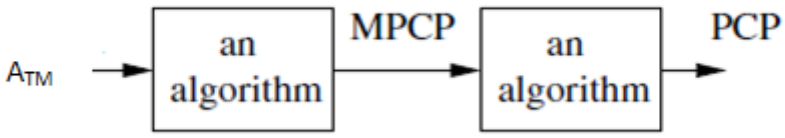
$$w_1 w_{i_1} w_{i_2} \cdots w_{i_m} = x_1 x_{i_1} x_{i_2} \cdots x_{i_m}$$

So in MPCP, the first pair on the two lists A and B must be the first pair in the solution.

Note:

Post's Correspondence Problem is undecidable.

First we can reduce Modified PCP to the original PCP. Then we can reduce A_{TM} to the modified PCP. The chain of reductions is shown in the following figure.



Since A_{TM} is undecidable, we conclude that PCP is undecidable.

Note:

PCP is an important example of an undecidable problem. PCP is a good choice for reducing to other problems and thereby proving them undecidable.

Classes of P and NP:

P consists of all those languages or problems accepted by some deterministic Turing Machine that runs in some polynomial amount of time, as a function of its input length.

So a language L is in class P if there is some polynomial T(n) such that L=L(M) for some deterministic TM M of time complexity T(n)

Ex:

1. Finding the minimum weight spanning tree of a graph.

2. Sorting an array

NP is the class of languages or problems that are accepted by nondeterministic Turing Machine with a polynomial bound on the time taken along any sequence of nondeterministic choices.

So a language L is in class NP, if there exists a nondeterministic TM M and a polynomial time complexity T(n) such that L=L(M) and when M is given an input of length n, there are no sequences of more than T(n) moves of M.

Ex:

1. Travelling Salesman Problem

NP- hard and NP-completeness:

Def: Polynomial Time reduction:

If we can transform instances of one problem P1 in polynomial time into instances of a another problem P2 that has the same answer yes or no, then we say that Problem P1 is polynomial – time reducible to problem P2.

NP-Hard:

A language L is said to be NP-hard if for every language L' in NP there is a polynomial time reduction of L' to L. Here we cannot able to prove that L is in NP.

NP-complete:

A language L is said to be NP-complete if the following statements are true about L

1. L is in NP

2. For every language L' in NP there is a polynomial time reduction of L' to L

Ex: 1.The SAT problem (or) Satisfiability problem- "Given a Boolean expression, is it satisfiable? " is NP-complete.

2. Hamilton Circuit problem

3. Travelling Salesman problem

Boolean satisfiability Problem(SAT Problem):

A *literal* is a either a propositional variable or the negation of a propositional variable.

Ex: Variables, such as x and y, are called *positive literals* and Negated variables, such as

$\neg x$ and $\neg y$, are called *negative literals*.

A *clause* is a disjunction (*or*) of one or more literals.

Ex:

$$(x \vee \neg y \vee z)$$

is a clause.

A *clausal formula*, which we will just call a formula, is a conjunction (*and*) of one or more clauses.

For example,

$$(\neg x \vee y) \wedge$$
$$(\neg y \vee z) \wedge$$
$$(x \vee \neg z \vee y)$$

is a (clausal) formula.

Definition:

A boolean clausal formula φ is satisfiable if there exists an assignment of values to its variables that makes φ true.

For example, formula

$$(\neg x \vee y) \wedge$$
$$(\neg y \vee z) \wedge$$
$$(x \vee \neg z \vee y)$$

is satisfiable since Choosing

$$x = \text{false}$$
$$y = \text{true}$$
$$z = \text{true}$$

The above formula becomes true.

The satisfiability problem, usually called SAT, is the following language.

$$\text{SAT} = \{\varphi \mid \varphi \text{ is a Boolean clausal formula that is satisfiable}\}.$$