Assignment 15

M Pavan Manesh - EE20MTECH14017

Download the latex-tikz codes from

https://github.com/pavanmanesh/EE5609/tree/master/Assignment15

1 Problem

Let n be a positive integer, and let V be the space of polynomials over $\mathbb{R}$ which have degree at most n (throw in the 0-polynomial). Let $\mathbf{D}$ be the differentiation operator on V. What is the minimal polynomial for $\mathbf{D}$?

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 1: Definitions and theorem used

3 Solution

Given	V is the space of polynomials over $\mathbb{R}$ which have degree at most n.
Matrix Representation	The basis for the space V is $$\mathcal{B} = \left\{1, x, x^2, \dots, x^n\right\} \quad (3.0.1)$$ Given that $\mathbf{D}$ is the differentiation operator.So, $$\mathbf{D}(1) = 0 \quad (3.0.2)$$ $$\mathbf{D}(x) = 1 \quad (3.0.3)$$ $$\vdots$$ $$\mathbf{D}(x^n) = nx^{n-1} \quad (3.0.4)$$ The vectors of differentiation operator with respect to basis $\mathcal{B}$ $$[\mathbf{D}(1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(n+1)\times 1}, [\mathbf{D}(x)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(n+1)\times 1} \dots [\mathbf{D}(x^n)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ n \\ 0 \end{pmatrix}_{(n+1)\times 1} \quad (3.0.5)$$ The matrix representation can be written as: $$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (3.0.6)$$
Characteristic polynomial	$$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert = \begin{vmatrix} x & -1 & 0 & \dots & 0 \\ 0 & x & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -n \\ 0 & 0 & 0 & \dots & x \end{vmatrix} \quad (3.0.7)$$ It is equal to the product of diagonal entries. $$p(x) = x^{n+1} \quad (3.0.8)$$
Minimal Polynomial	The minimal polynomial of $\mathbf{A}$ can be any of $x, x^2, \dots, x^{n+1}$ such that, $$m(\mathbf{A}) = 0 \quad (3.0.9)$$

For n=3	$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.0.10)$$ $$\mathbf{A^2} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.0.11)$$ $$\mathbf{A^3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.0.12)$$ $$\mathbf{A^4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.0.13)$$ For n=3, We can see that $\mathbf{A^4}$=0
Conclusion	From above we can say that the minimal polynomial of **A** is $$x^{n+1} \quad (3.0.14)$$

TABLE 2: Finding minimal polynomial