EE5609: Matrix Theory
Assignment-13

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Download all latex-tikz codes from

https://github.com/pavanmanesh/EE5609/tree/master/Assignment13

1 Problem

Let $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and eigen values of} \mathbf{A} \in \mathbb{Q} \right\}$

1) $\mathbf{M}$ is empty
2) $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$
3) If $\mathbf{A} \in \mathbf{M}$ then the eigen values of $\mathbf{A} \in \mathbb{Z}$
4) If $\mathbf{A}, \mathbf{B} \in \mathbf{M}$ such that $\mathbf{AB} = \mathbf{I}$ then $\left|\mathbf{A}\right| \in \{+1, -1\}$

2 Solution

M is empty	Consider **A**=**I**=$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.The elements of **A** $\in \mathbb{Z}$ and it's eigen values $1 \in \mathbb{Q}$. So, **M** is not empty.
$\mathbf{M} = \left\{\begin{pmatrix}a & b\\c & d\end{pmatrix} : a, b, c, d \in \mathbb{Z}\right\}$	Let **A**=$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ where elements of **A** $\in \mathbb{Z}$.The characteristic equation can be written as : $$\lambda^2 + 1 = 0 \implies \lambda = \pm i$$ We see that $\lambda \in \mathbb{C}$ which is contradicting the main definition of **M** .So,this is not correct.
Eigen values of **A** $\in \mathbb{Z}$	Given ,**A** $\in$ **M**.Let λ_1,λ_2 be the eigen values of **A**.The characteristic polynomial can be written as: $$\lambda^2 - tr(\mathbf{A}) + \det \mathbf{A} = 0, \text{where } tr(\mathbf{A}) = \lambda_1 + \lambda_2, \det \mathbf{A} = \lambda_1\lambda_2$$ Given the eigen values λ_1,λ_2 $\in \mathbb{Q}$,For this to be possible the discriminant of above equation should $\in \mathbb{Z}$ $$\sqrt{(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2} \in \mathbb{Z}$$ $$\implies \sqrt{(\lambda_1 - \lambda_2)^2} \in \mathbb{Z}$$ $\implies \lambda_1 - \lambda_2 \in \mathbb{Z}$ This is possible when both λ_1,$\lambda_2 \in \mathbb{Z}$.
If **AB**=**I** then $\lvert\mathbf{A}\rvert \in$ {+1,-1}	As **A**,**B** $\in$ **M**, $\implies$ $\lvert\mathbf{A}\rvert$,$\lvert\mathbf{B}\rvert \in \mathbb{Z}$ Given **AB**=**I** $\implies$ $\lvert\mathbf{A}\rvert\,\lvert\mathbf{B}\rvert$=1 This is possible only when $\lvert\mathbf{A}\rvert$=$\lvert\mathbf{B}\rvert$= $\pm$ 1
Conclusion	options 3) and 4) are correct.