Assignment 15

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Download the latex-tikz codes from

https://github.com/pavanmanesh/EE5609/tree/master/Assignment15

1 Problem

Let n be a positive integer, and let V be the space of polynomials over $\mathbb{R}$ which have degree at most n (throw in the 0-polynomial). Let $\mathbf{D}$ be the differentiation operator on V. What is the minimal polynomial for $\mathbf{D}$?

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left|x\mathbf{I} - \mathbf{A}\right|$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 1: Definitions and theorem used

3 Solution

Given	V is the space of polynomials over $\mathbb{R}$ which have degree at most n.
Matrix Representation	The basis for the space V is $$\mathcal{B} = \left\{1, x, x^2, \ldots, x^n\right\} \tag{3.0.1}$$ Given that $\mathbf{D}$ is the differentiation operator.So, $$\mathbf{D}(1) = 0 \tag{3.0.2}$$ $$\mathbf{D}(x) = 1 \tag{3.0.3}$$ $$\vdots$$ $$\mathbf{D}(x^n) = nx^{n-1} \tag{3.0.4}$$ The vectors of differentiation operator with respect to basis $\mathcal{B}$ $$[\mathbf{D}\,(1)]_{\mathcal{B}} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}_{(n+1)\times 1}, [\mathbf{D}\,(x)]_{\mathcal{B}} = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}_{(n+1)\times 1} \ldots [\mathbf{D}\,(x^n)]_{\mathcal{B}} = \begin{pmatrix}0\\\vdots\\n\\0\end{pmatrix}_{(n+1)\times 1} \tag{3.0.5}$$ The matrix representation can be written as: $$\mathbf{A} = \begin{pmatrix}0&1&0&\ldots&0\\0&0&2&\ldots&0\\\vdots&\vdots&\vdots&\ldots&\vdots\\0&0&0&\ldots&n\\0&0&0&\ldots&0\end{pmatrix} \tag{3.0.6}$$
Characteristic polynomial	$$p\,(x) = \left\|x\mathbf{I} - \mathbf{A}\right\| = \begin{vmatrix}x&-1&0&\ldots&0\\0&x&-2&\ldots&0\\\vdots&\vdots&\vdots&\ldots&\vdots\\0&0&0&\ldots&-n\\0&0&0&\ldots&x\end{vmatrix} \tag{3.0.7}$$ It is equal to the product of diagonal entries. $$p\,(x) = x^{n+1} \tag{3.0.8}$$
Minimal Polynomial	The minimal polynomial of $\mathbf{A}$ can be any of $x, x^2, \ldots, x^{n+1}$ such that, $$m\,(\mathbf{A}) = 0 \tag{3.0.9}$$

Explanation	Let $P(n)$: Minimum polynomial of $\mathbf{D}=x^{n+1}$ i.e $\mathbf{A}^{n+1}=0$ For n=1 $$\mathbf{A}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad (3.0.10)$$ $$\mathbf{A^2}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \quad (3.0.11)$$ So,$P(1)$ is true. Assume $P(k)$ holds for $1 \leq k \leq n$. $$\mathbf{A}_k=\begin{pmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 2 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & k\\ 0 & 0 & 0 & \dots & 0\end{pmatrix}_{(k+1\times k+1)} \implies \mathbf{A}_k^{k+1}=\mathbf{0} \quad (3.0.12)$$ We need to show that $P(k+1)$ is true. $$\mathbf{A}_{k+1}=\begin{pmatrix}0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 2 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & k & 0\\ 0 & 0 & 0 & \dots & 0 & k+1\\ 0 & 0 & 0 & \dots & 0 & 0\end{pmatrix}_{(k+2\times k+2)} \quad (3.0.13)$$ Expressing in terms of block matrices $$\mathbf{A}_{k+1}=\begin{pmatrix}\mathbf{A}_k & \mathbf{x}\\ \mathbf{0}_{1\times k+1} & 0\end{pmatrix}, \mathbf{x}=\begin{pmatrix}0\\ 0\\ \vdots\\ 0\\ k+1\end{pmatrix}_{k+1\times 1} \quad (3.0.14)$$
Finding $\mathbf{A}_{k+1}^{k+2}$	$$\mathbf{A}_{k+1}^2=\begin{pmatrix}\mathbf{A}_k & \mathbf{x}\\ \mathbf{0} & 0\end{pmatrix}\begin{pmatrix}\mathbf{A}_k & \mathbf{x}\\ \mathbf{0} & 0\end{pmatrix}=\begin{pmatrix}\mathbf{A}_k^2 & \mathbf{A}_k\mathbf{x}\\ 0 & 0\end{pmatrix} \quad (3.0.15)$$ $$\mathbf{A}_{k+1}^3=\begin{pmatrix}\mathbf{A}_k^2 & \mathbf{A}_k\mathbf{x}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{A}_k & \mathbf{x}\\ \mathbf{0} & 0\end{pmatrix}=\begin{pmatrix}\mathbf{A}_k^3 & \mathbf{A}_k^2\mathbf{x}\\ 0 & 0\end{pmatrix} \quad (3.0.16)$$ $$\mathbf{A}_{k+1}^{k+2}=\begin{pmatrix}\mathbf{A}_k^{k+2} & \mathbf{A}_k^{k+1}\mathbf{x}\\ 0 & 0\end{pmatrix} \quad (3.0.17)$$ From (3.0.12),We know that $\mathbf{A}_k^{k+1}=\mathbf{0}$ $$\implies \mathbf{A}_{k+1}^{k+2}=\mathbf{0} \quad (3.0.18)$$ So,$P(k+1)$ is true.

Conclusion	From above,by using the principle of induction we can say that the minimal polynomial is $$x^{n+1} \tag{3.0.19}$$

TABLE 2: Finding minimal polynomial