Assignment 19

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Download the latex-tikz codes from

https://github.com/pavanmanesh/EE5609/tree/master/Assignment19

1 Problem

Let $\mathbf{A}$ be an invertible real $n \times n$ matrix .Define a function F: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by F$(\mathbf{x}, \mathbf{y}) = \langle \mathbf{Ax}, \mathbf{y} \rangle$ where $\langle \mathbf{x}, \mathbf{y} \rangle$ denotes the inner product of $\mathbf{x}$ and $\mathbf{y}$.Let $DF(\mathbf{x}, \mathbf{y})$ denote the derivative of F at $(\mathbf{x}, \mathbf{y})$ which is a linear tranformation from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.Then

1) If $\mathbf{x} \neq 0$,then $DF(\mathbf{x}, 0) \neq 0$
2) If $\mathbf{y} \neq 0$,then $DF(0, \mathbf{y}) \neq 0$
3) If $(\mathbf{x}, \mathbf{y}) \neq 0$,then $DF(\mathbf{x}, \mathbf{y}) \neq 0$
4) If $\mathbf{x} = 0$ or $\mathbf{y} = 0$,then $DF(\mathbf{x}, \mathbf{y}) = 0$

2 Definitions

Invertible	A square matrix is invertible if and only if it does not have a zero eigenvalue. So, from the definition of eigen vector we can write that $$\mathbf{Ax} \neq 0 \quad (2.0.1)$$ The transpose of an invertible matrix is also invertible with inverse $(\mathbf{A}^{-1})^T$. $$\mathbf{AA}^{-1} = \mathbf{I} \implies (\mathbf{A}^{-1})^T \mathbf{A}^T = \mathbf{I}^T = \mathbf{I} \quad (2.0.2)$$ So,similarly we can say that $$\mathbf{A}^T \mathbf{y} \neq 0 \quad (2.0.3)$$
Derivative of F	Suppose F: $\mathbb{R}^n \to \mathbb{R}^m$,the derivative of a function F is given by the Jacobian matrix $$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \quad (2.0.4)$$
Inner product	The inner product of $\mathbf{x}$ and $\mathbf{y}$ is given by $$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} \quad (2.0.5)$$

TABLE 1: Definition and Properties used

3 Solution

Given	$$F(\mathbf{x},\mathbf{y}) = \langle \mathbf{A}\mathbf{x},\mathbf{y}\rangle \quad (3.0.1)$$
using inner product definition	$$F(\mathbf{x},\mathbf{y}) = (\mathbf{A}\mathbf{x})^T\mathbf{y} = \mathbf{x}^T\mathbf{A}^T\mathbf{y} \quad (3.0.2)$$ $$F(\mathbf{x},\mathbf{y}) = \mathbf{y}^T\mathbf{A}\mathbf{x} \quad (3.0.3)$$
Derivative of F	using (2.0.4), We can write that $$DF(\mathbf{x},\mathbf{y}) = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{y}^T\mathbf{A} & \mathbf{x}^T\mathbf{A}^T \end{pmatrix} \quad (3.0.4)$$
If $\mathbf{x} \neq 0$,then $DF(\mathbf{x},0) \neq 0$	using (3.0.4), $$DF(\mathbf{x},0) = \begin{pmatrix} 0 & \mathbf{x}^T\mathbf{A}^T \end{pmatrix} \quad (3.0.5)$$ From (2.0.1),we know that $$\mathbf{A}\mathbf{x} \neq 0 \quad (3.0.6)$$ $$\implies \mathbf{x}^T\mathbf{A}^T \neq 0 \quad (3.0.7)$$ So, We can say that $$DF(\mathbf{x},0) \neq 0 \quad (3.0.8)$$
If $\mathbf{y} \neq 0$,then $DF(0,\mathbf{y}) \neq 0$	using (3.0.4), $$DF(0,\mathbf{y}) = \begin{pmatrix} \mathbf{y}^T\mathbf{A} & 0 \end{pmatrix} \quad (3.0.9)$$ From (2.0.3),we know that $$\mathbf{A}^T\mathbf{y} \neq 0 \quad (3.0.10)$$ $$\implies \mathbf{y}^T\mathbf{A} \neq 0 \quad (3.0.11)$$ So, We can say that $$DF(0,\mathbf{y}) \neq 0 \quad (3.0.12)$$
If $(\mathbf{x},\mathbf{y}) \neq 0$,then $DF(\mathbf{x},\mathbf{y}) \neq 0$	using (3.0.4),

	$$DF(\mathbf{x},\mathbf{y}) = \begin{pmatrix} \mathbf{y}^T\mathbf{A} & \mathbf{x}^T\mathbf{A}^T \end{pmatrix} \tag{3.0.13}$$ As $(\mathbf{x},\mathbf{y}) \neq 0, DF(\mathbf{x},\mathbf{y}) = 0$ iff $\mathbf{A}$=0 From (2.0.1),we know that $$\mathbf{A} \neq 0 \tag{3.0.14}$$ So, We can say that $$DF(\mathbf{x},\mathbf{y}) \neq 0 \tag{3.0.15}$$
If $\mathbf{x} = 0$ or $\mathbf{y} = 0$,then $DF(\mathbf{x},\mathbf{y}) = 0$	From (3.0.12), $$DF(0,\mathbf{y}) \neq 0 \tag{3.0.16}$$ From (3.0.8), $$DF(\mathbf{x},0) \neq 0 \tag{3.0.17}$$ So, if $\mathbf{x} = 0$ or $\mathbf{y} = 0$, $$DF(\mathbf{x},\mathbf{y}) \neq 0 \tag{3.0.18}$$
Conclusion	From above,we can say that options 1),2),3) are correct.

TABLE 2: Finding derivative of linear transformation