Assignment 17

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Download the latex-tikz codes from

https://github.com/pavanmanesh/EE5609/tree/master/Assignment17

1 Problem

Consider the quadratic forms on $\mathbb{R}^2$

$$Q_1(x,y) = xy, Q_2(x,y) = x^2 + 2xy + y^2, Q_3(x,y) = x^2 + 3xy + 2y^2$$

Choose the correct statements from below

1) Q_1 and Q_2 are equivalent
2) Q_1 and Q_3 are equivalent
3) Q_2 and Q_3 are equivalent
4) All are equivalent

2 Definitions

Matrix representation	The Matrix representation of quadratic forms $$Q(x,y) = ax^2 + 2bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{X}^T\mathbf{A}\mathbf{X} \quad (2.0.1)$$ The symmetric matrix of the quadratic form is $$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.2)$$
Equivalent condition	Two quadratic forms $\mathbf{X}^T\mathbf{A}\mathbf{X}$ and $\mathbf{Y}^T\mathbf{B}\mathbf{Y}$ are called equivalent if their matrices, A and B are congruent. Two real quadratic forms are equivalent over the real field iff they have the same rank and the same index.
Rank	The rank of a quadratic form is the rank of its associated matrix.
Index	The index of the quadratic form is equal to the number of positive eigen values of the matrix of quadratic form.

TABLE 1: Definitions and results used

3 Solution

Given	$Q_1(x,y) = xy, Q_2(x,y) = x^2 + 2xy + y^2, Q_3(x,y) = x^2 + 3xy + 2y^2$
Matrix representation	The corresponding matrices for the quadratic forms are: $$\mathbf{A}_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \mathbf{A}_3 = \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix} \quad (3.0.1)$$
Finding Parameters of $\mathbf{A}_1$	$$\text{rank}(\mathbf{A}_1) = 2$$ $$\left\lvert \mathbf{A}_1 - \lambda\mathbf{I} \right\rvert = \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \quad (3.0.2)$$ $$\implies \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) = 0 \quad (3.0.3)$$ $$\implies \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2} \quad (3.0.4)$$ $$\text{Index of } \mathbf{A}_1 = 1$$
Finding Parameters of $\mathbf{A}_2$	$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.0.5)$$ $$\text{rank}(\mathbf{A}_2) = 1$$ $$\left\lvert \mathbf{A}_2 - \lambda\mathbf{I} \right\rvert = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \quad (3.0.6)$$ $$\implies (\lambda)(\lambda - 2) = 0 \quad (3.0.7)$$ $$\implies \lambda_1 = 0, \lambda_2 = 2 \quad (3.0.8)$$ $$\text{Index of } \mathbf{A}_2 = 2$$
Finding Parameters of $\mathbf{A}_3$	$$\begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & -\frac{1}{4} \end{pmatrix} \xleftrightarrow[R_2 \leftarrow -4R_2]{R_1 \leftarrow R_1 + 6R_2} \quad (3.0.9)$$ $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.10)$$ $$\text{rank}(\mathbf{A}_3) = 2$$ $$\left\lvert \mathbf{A}_3 - \lambda\mathbf{I} \right\rvert = \begin{vmatrix} 1-\lambda & \frac{3}{2} \\ \frac{3}{2} & 2-\lambda \end{vmatrix} = 0 \quad (3.0.11)$$ $$\implies \left(\lambda - \frac{\sqrt{10}+3}{2}\right)\left(\lambda + \frac{\sqrt{10}-3}{2}\right) = 0 \quad (3.0.12)$$

	$$\implies \lambda_1 = \frac{3+\sqrt{10}}{2}, \lambda_2 = \frac{3-\sqrt{10}}{2} \quad (3.0.13)$$ Index of $\mathbf{A}_3 = 1$
Conclusion	From above,we can say that Q_1 and Q_3 are equivalent.

TABLE 2: Finding which quadratic forms are equivalent