1.1

1) $E[e^{3N(t)-6W(t)} \mid F(s)]$ for $s<t$

$N(t)$ is a poisson process

$N(t) = N(s) + [N(t) - N(s)]$; $W(t) = W(s) + [W(t) - W(s)]$

$3N(t) - 6W(t) = 3N(s) + 3[N(t) - N(s)] - 6W(s) - 6[W(t) - W(s)]$

$\Rightarrow e^{3N(s)-6W(s)} \cdot e^{3[N(t)-N(s)]} \cdot e^{-6[W(t)-W(s)]}$

$= E[e^{3N(s)-6W(s)} \mid F(s)] = e^{3N(s)-6W(s)} \cdot E[e^{3[N(t)-N(s)]} \cdot e^{-6[W(t)-W(s)]} \mid F(s)]$

increments are indipendent to $F(s)$

$\Rightarrow e^{3N(s)-6W(s)} \cdot E[e^{3[N(t)-N(s)]}] \cdot E[e^{-6[W(t)-W(s)]}]$

for $N(t)$ being poission$(\lambda(t-s))$

$E[e^{3[N(t)-N(s)]}] = e^{\{\lambda(t-s)(e^3-1)\}}$

$E[e^{-6[W(t)-W(s)]}] = \cancel{e^{...}} \; e^{18(t-s)\}}$

$= e^{3N(s)-6W(s)} \cdot e^{(\lambda(t-s)(e^3-1)\}} \cdot e^{18(t-s)}$

1.2

(b) $\operatorname{corr}(N(s), N(t)) = \dfrac{\operatorname{cov}(N(s), N(t))}{\sqrt{\operatorname{var}(N(s)) \cdot \operatorname{var}(N(t))}}$

for poission $N(t)$

mean = variance = $E[N(t)] = \lambda t$

$\operatorname{cov}(N(s), N(t)) = \cancel{\operatorname{var}(N(s))}$

$= E[N(s) \cdot N(t)] - E[N(s)] * E[N(t)]$

$\cancel{N(t) = N(s) + [N(t) - N(s)]}$

$\cancel{= E[N(s)[N(s) + N(t) - N(s)]]}$

$\cancel{= E[N^2(s)]} +$

$E[N(s) \cdot N(t)] =$

$= E[N(s)[N(t) + N(s) - N(s)]]$

$= E[N^2(s)] + E[N(s)(N(t) - N(s))]$

$N(t) - N(s)$ is indipidint

$= E[N^2(s)] + E[N(s)] \cdot E[N(t) - N(s)]$

$=$ 〃 〃 $\cdot \lambda(t-s) \longrightarrow$ 3rd term

$= E[N^2(s)] = \operatorname{var}(N(s)) + E[N(s)]$

$= \lambda s + (\lambda s)^2 \Rightarrow$ 1st term

$= E[N(s)] = \lambda t \rightarrow$ 2nd term

1.3

$E[N(s)\cdot N(t)] =$

~~cov(N(s), N(t))~~ $= \lambda s + \lambda^2 s^2 + \lambda s(\lambda t - \lambda s)$

$\Rightarrow \lambda s + \cancel{\lambda^2 s^2} + \lambda^2 st - \cancel{\lambda^2 s^2} \Rightarrow \lambda s + \lambda^2 st$

$= \lambda s(1+\lambda t)$

$\operatorname{cov}(N(s), N(t)) =$

$\Rightarrow \lambda s + \lambda^2 st - (\lambda s)(\lambda t)$

$\Rightarrow \lambda s + \lambda^2 st - \lambda^2 st$

$\Rightarrow \underline{\underline{\lambda s}}$

$\operatorname{corr}(N(s), N(t)) = \dfrac{\lambda s}{\sqrt{\lambda s \cdot \lambda t}} \Rightarrow \dfrac{\lambda s}{\lambda\sqrt{st}} \Rightarrow \dfrac{s}{\sqrt{st}}$

$= \dfrac{\sqrt{s}}{\sqrt{t}} = \underline{\underline{\sqrt{\dfrac{s}{t}}}}$

(2) $dR(t) = \gamma dt + \delta d\tilde{W}(t)$

$R(0) = 0.01$
$\gamma = 0.3$
$\delta = 0.1$

(a) zero coupon bond

$P(0,T) = \tilde{E}\left[e^{-\int_0^T R(u)\,du}\right]$

$dR(t) = \gamma dt + \delta d\tilde{W}(t)$

$R(t) = R(0) + \gamma t + \delta \tilde{W}(t)$

$\int_0^T R(u)\,du = \int_0^T [R(0) + \gamma u + \delta \tilde{W}(u)]\,du$

$= R(0)T + \gamma \frac{T^2}{2} + \gamma \int_0^T \tilde{W}(u)\,du$

$X = \int_0^T \tilde{W}(u)\,du$

$\mathrm{var}(X) = \int_0^T \int_0^T \tilde{E}[\tilde{W}(u)\tilde{W}(v)]\,du\,dv$

$= \int_0^T \int_0^T \min(u,v)\,du\,dv = \frac{T^3}{3}$

$X \sim N\left(0, \frac{T^3}{3}\right)$

2.2

$$P(0,T) = E^{Q}\left[e^{\left(-R(0)T - \frac{\gamma T^2}{2} - \delta X\right)}\right]$$

$$= E\left[e^{\left(-R(0)T - \frac{\gamma T^2}{2}\right)}\right] \times E^{Q}\left[e^{-\delta X}\right]$$

$$= E^{Q}\left[e^{-\delta X}\right] = \exp\left(\tfrac{1}{2}(-\delta)^2 \frac{T^3}{3}\right)$$

$$= \exp\left(\frac{\delta^2 T^3}{6}\right)$$

$$P(0,T) = \exp\left[-R(0)T - \frac{\gamma T^2}{2} + \frac{\delta^2 T^3}{6}\right]$$

(b) \$5 at $t = \frac{1}{4}$

\$10 at $t = \frac{1}{2}$

\$100 at $t = 1$

at $t = 0$

$\Rightarrow 5 \cdot P(0, 1/4) + 10\, P(0, 1/2) + 100(P(0,1))$

$= 5 \cdot \exp\left(-0.01T - 0.3\frac{T^3}{2} + 0.1\frac{T^3}{6}\right]$

~~$+ 10 \cdot \exp\left(-0.01\frac{1}{2} - 0.3\right.$~~

2.3

$$+ 10 \cdot \exp\left(\frac{-r(0)}{2} \cdot - \frac{\gamma \left(\frac{1}{2}\right)^2}{2} + \frac{\delta^2 \left(\frac{1}{2}\right)^3}{6} \right.$$

$$+ 100 \exp\left(-r(0) \cdot 1 - \frac{\gamma (1)^2}{2} + \frac{\delta^2 (1)^3}{6} \right.$$

$$= 5 \cdot \exp(-0.011848)$$
$$+ 10 \cdot \exp(-0.032292)$$
$$+ 100 \cdot \exp(-0.158334)$$

$$B(0) \simeq 4.94 + 9.59 \cdot 85.40$$

$$\Rightarrow \underline{\underline{99.93}}$$

Bond process $B(0) = \underline{\underline{99.93}}$

3)

$$V(T) = C \cdot \frac{S(T) - S(0)}{S(0)}$$

$$S(T) = S(0)\, e^{(r - \frac{\sigma^2}{2})T + \sigma \tilde{W}(T)}$$

$$V(T) = e^{-r(T-t)} \tilde{E}[V(T) \mid F(t)]$$

$$= e^{-r(T-t)} \tilde{E}\left[C \cdot \frac{S(T) - S(0)}{S(0)} \mid F(t)\right]$$

$$= e^{-r(T-t)} \cdot C\, \tilde{E}\left[\frac{S(T)}{S(0)} - 1 \mid F(t)\right]$$

$$= -e^{r(T-t)} \cdot C\, \tilde{E}\left[\frac{S(T)}{S(0)} \mid F(t)\right]$$

$$= -e^{r(T-t)} \cdot C \cdot e^{(r - \frac{\sigma^2}{2})T} \tilde{E}\left[e^{\sigma \tilde{W}(T)} \mid F(t)\right]$$

$$= \quad '' \quad E\left[e^{\sigma \tilde{W}T + \sigma \tilde{W}t - \sigma Wt} \mid F(t)\right]$$

$$= e^{(r - \cancel{\frac{\sigma^2}{2}})T} \cdot e^{\sigma W^2 t} \cdot e^{\cancel{\frac{1}{2}\sigma^2(T-t)}}$$

$$= e^{rT - \frac{\sigma^2 t}{2} + \sigma \tilde{W} t)}$$

$$V(t) = e^{-r(T-t)} \cdot C \cdot e^{(rT - \frac{\sigma^2 t}{2} + \sigma \tilde{W}^2(t))}$$

$$- e^{-r(T-t)} \cdot C$$

3.2

$$v(t) = C\left[\left(e^{tr - \frac{\sigma^2}{2}t + \sigma\tilde{W}(t)}\right) - e^{-r(T-t)}\right]$$

(b) $v(T) = c * \log\left(\frac{S(T)}{S(0)}\right)$

$$v(t) = e^{-r(T-t)} \cdot c \cdot \tilde{E}\left[\log\left(\frac{S(T)}{S(0)}\right) \mid F(t)\right]$$

$$= e^{-r(T-t)} \cdot c \cdot \tilde{E}\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma\tilde{W}T \mid F(t)\right]$$

$$= e^{-r(T-t)} \cdot c \cdot \left(r - \frac{\sigma^2}{2}\right)T + 0$$

$$v(t) = e^{-r(T-t)} \cdot c \cdot \left(r - \frac{\sigma^2}{2}\right)T$$

(c) $S(0) = 87$

$\alpha = 0.03$

$\sigma = 0.01$

$T = 0.23$

$r = 0.02$

3.3

$$V(t) = e^{-r(T-t)} \cdot C \cdot \left(r - \frac{\sigma^2}{2}\right) T$$

$$= e^{-0.02(0.25)} \cdot 100\left(0.02 - \frac{0.01}{2}\right) 0.25$$

$$0.375 \times 0.9950$$

$$= \$0.3731 \ \cancel{\text{cents}} \Rightarrow 37.31 \text{ cents.}$$

$$V(0) = 100 \cdot \left[e^{0} - e^{-0.2 \times 0.25}\right]$$

$$= 100 \cdot [1 - 0.9950]$$

$$= 100 \cdot 0.004987$$

$$= 0.4987$$

$$\cong 0.5 \$ = 50 \text{ cents}$$

4) $S(t) = S(0)\, e^{\alpha t^2 + \beta W(t)}$ stock

$P(t) = P(0)\, e^{\gamma t + \delta W(t)}$ Bond

find:
quadratic variation of $S^2(t)$
ito decomposition $Z(t) = S^2(t) \cdot P(t)$
covariance of $S(t)$ and $P(t)$

a) $S(t) = S(0)\, e^{\alpha t^2 + \beta W(t)}$

using ito integral

$$df(S(t)) = \frac{\partial f}{\partial t}\, \partial t + \frac{\partial f}{\partial s}\, dS(t) + \frac{1}{2} \frac{\partial^2 f}{\partial s^2} \frac{(\sigma(t, S(t)))^2}{dt}$$

$= \cancel{S(t)}$

$dS(t) = S(t) \cancel{\cdot (2\alpha t\, dt + \beta\, dW(t))}$
$S(t) (2\alpha t + \frac{1}{2}\beta^2) dt + S(t)\, \beta\, dW(t)$

q.v of $S(t)$

$$[S](t) = \int_0^t \beta^2 S^2(s)\, ds \Rightarrow d[S](t) = \beta^2 S^2(t)\, dt$$

for $S^2(t)$

$[S^2](t) = 4S^2(t)\, [S](t)$

$[S^2](t) \neq 4S^2(t) \cdot \int_0^t \beta^2 S^2(s)\, ds$

$= 4\beta^2 S^4(t)\, t$

$$d[S^2(t)] = 2S(t)\Big(S(t)\big(2\alpha t + \tfrac{1}{2}\beta^2\big)dt + S(t)\beta\, dw(t)\Big) + \beta^2 S^2(t)\,dt$$

$$= d[S^2(t)] = \big(4\alpha + S^2(t) + \beta^2 S^2(t)\big)dt + 2\beta S^2(t)\,dw(t)$$

q.v of $S^2(t)$

the diffusion coff $= 2\beta S(t)^2 dw(t)$

$$d[S^2] = (2\beta S^2(t))^2 dt$$

$$= 4\beta^2 S^4(t)\,dt$$

$$\therefore \text{ q.v of } S^2(t) = \int_0^t 4\beta^2 S^4(u)\,du$$

b) $Z(t) = S(t)P(t)$

$d[Z(t)] = d[S(t)P(t)]$

$= S(t)\,dP(t) + P(t)\,dS(t) + dS(t)\,dP(t)$

$dP(t) = P(t)\left[(\gamma + \frac{1}{2}\delta^2)dt + \delta\,dW(t)\right]$

$dS(t) = S(t)\left[(2at + \frac{1}{2}\beta^2)dt + \beta\,dW(t)\right)$

$d\langle S_t, P_t\rangle = \beta S(t)\,\delta P(t)\,dt$

$= \beta\delta S(t)P(t)dt$

$d[Z(t)] = S(t)\,P(t)\left[(\gamma + \frac{1}{2}\delta^2)dt + \delta\,dW(t)\right]$

$+ S(t)P(t)\left[2at + \frac{1}{2}\beta^2)dt + \beta\,dW(t)\right]$

$+ \beta\delta S(t)P(t)dt$

$= S(t)P(t)\left[(\gamma + \frac{1}{2}\delta^2) + (2at + \frac{1}{2}\beta^2) + \beta\delta\right]dt$

$+ S(t)P(t)[\delta + \beta]dW(t)$

$d[Z(t)] = Z(t)\left[\gamma + 2at + \frac{1}{2}(\beta^2 + \delta^2) + \beta\delta\right)dt$

$+ (\beta + \delta)\,dW(t)\Big]$

ito decomposition

c) $\text{cov}(S(t), P(t)) = E[S(t)P(t)] - E[S(t)]\,E[P(t)]$

log normal process

$S(t) = S(0)\, e^{\alpha t^2 + \beta W(t)}$

$W(t) \sim N(0,t)$ so $E[e^{\beta W(t)}] = e^{\frac{1}{2}\beta^2 t}$

$E[S(t)] = S(0) e^{\alpha t^2} \cdot e^{\frac{1}{2}\beta^2 t}$

$= S(0)\, e^{\alpha t + \frac{1}{2}\beta^2 t}$

$P(t) = P(0)\, e^{\gamma t + \delta W(t)}$

$W(t) \sim N(0, t)$ so $E[e^{\delta W(t)}] = e^{\frac{1}{2}\delta^2 t}$

$E[P(t)] = P(0)\, e^{\gamma t} \cdot e^{\frac{1}{2}\delta^2 t}$

$= P(0)\, e^{\gamma t + \frac{1}{2}\delta^2 t}$

$S(t)\,P(t) =$

$S(0)\,P(0)\, e^{\alpha t^2 + \beta W(t) + \gamma t + \delta W(t)}$

$= S(0)\,P(0)\, e^{\gamma \alpha t^2 + \delta t + (\beta + \delta)\, W(t)}$

$W(t) \sim N(0,t)$

$E[S(t)P(t)] = S(0)\,P(0)\, e^{\alpha t^2 + \gamma t} \cdot e^{\frac{1}{2}(\beta + \gamma)^2 t}$

4.5

$$\operatorname{cov}(S(t), P(t)) =$$

$$E[S(t)P(t)] = S(0)P(0)e^{\alpha t^2 + \gamma t + \frac{1}{2}(\beta + \delta)^2 t}$$

$$E[S(t)]E[P(t)] = S(0)e^{\alpha t^2 + \frac{1}{2}\beta^2 t} \cdot P(0)e^{\gamma t + \frac{1}{2}\delta^2 t}$$

$$\Rightarrow S(0)P(0)e^{\alpha t^2 + \gamma t + \frac{1}{2}(\beta + \delta)^2 t} - S(0)P(0)e^{\alpha t^2 + \gamma + \frac{1}{2}(\beta^2 + \delta^2)t}$$

$$\Rightarrow S(0)P(0)e^{\alpha t^2 + \gamma t}\left(e^{\frac{1}{2}(\beta + \delta)^2 t} - e^{\frac{1}{2}(\beta^2 + \delta^2)t}\right)$$

$$\frac{1}{2}(\beta + \delta)^2 = \frac{1}{2}\beta^2 + \frac{1}{2}\delta^2 + \beta\delta$$

$$\Rightarrow \operatorname{cov}(S(t), P(t)) = S(0)P(0)e^{\alpha t^2 + \gamma t + \frac{1}{2}(\beta^2 + \delta^2)t}\left(e^{\beta\delta t} - 1\right)$$