10/26/24

Midterm

① given:-

Hall-white model

$$dR(t) = \alpha(t) - \beta(t) R(t))dt + \sigma(t) dW(t)$$

α, β, σ are non-random adapted processes

$\alpha(t) = 2t^2$

$\beta(t) = 4t^2$

$\sigma(t) = \frac{1}{1+t}$

find:-

① closed form soln of $R(t)$

② $E[R(t)]$

$$M(t) = e^{\int_0^t \beta(u)\,du} R(t)$$

step 1:- ~~calc~~

$\beta(u) = 4u^2$

$\int_0^t \beta(u)\,du = \int_0^t 4u^2\,du = \frac{4}{3}u^3$

$M(t) = e^{\frac{4}{3}t^3} R(t)$

Sum of constant ~~cat~~ ito integral + Rieman integral

$dM(t) = e^{\frac{4}{3}t^3} dR(t)$

Step 2:- we can use ito product rule, we can express $e^{\frac{3}{4}t^3}$ as

$$d(X(t)Y(t)) = X(t)\,dY(t) + Y(t)\,dX(t) + dX(t)\,dY(t)$$

$$dM(t) = e^{\frac{4}{3}t^3} dR(t) + R(t)\, d\left(e^{\frac{4}{3}t^3}\right)$$
$$+ d\left(e^{\frac{4}{3}t^3}\right) dR(t)$$

1st term

$$dR(t) = \left(2t^2 - 4t^2 R(t)\right)dt + \frac{1}{1+t} dw(t)$$

$$e^{\frac{4}{3}t^3} \cdot dR(t) = e^{\frac{4}{3}t^3}\left(2t - 4t^2 R(t)\right)dt$$
$$+ e^{\frac{4}{3}t^3} \cdot \frac{1}{1+t} dw(t)$$

2nd term:-

$$R(t)\, d\left(e^{\frac{4}{3}t^3}\right)$$

$$d\left(e^{\frac{4}{3}t^3}\right) = \text{differentiate} \cdot e^{\frac{4}{3}t^3} \text{ w.r.t } t$$

$$= e^{\frac{3}{4}t^3} \cdot \frac{d}{dt}\left(\frac{4}{3}t^3\right)dt$$

$$= e^{\frac{4}{3}t^3} \cdot 4t^2 dt$$

$$R(t)\, d\left(e^{\frac{4}{3}t^3}\right) = R(t)\, e^{\frac{4}{3}t^3} \cdot 4t^2 dt$$

step 3:-

$$dM(t) = e^{\frac{4}{3}}\left(2t^2 - 4t^2 R(t)\right)dt + e^{\frac{4}{3}t^3} \cdot \frac{1}{1+t} dw(t)$$
$$+ R(t)\, e^{\frac{4}{3}t^3} 4t^2 dt + e^{\frac{4}{3}t^3} 4t^2 dt \cdot \left(\left(2t^2 - 4t^2 R(t)\right) dt + \frac{1}{1+t} dw(t) \right)$$

$dt \cdot dt = 0$

$dt \cdot dw(t) = 0$

$$= e^{\frac{4}{3}t^3} 2t^2 dt + e^{\frac{4}{3}t^3} \frac{1}{1+t} dw(t)$$

$\therefore \int_0^t$ on Both sides

~~$M(t) = M(0)$~~

$$M(t) - M(0) = \int_0^t e^{\frac{4}{3}u^3} 2u^2 du + \int_0^t e^{\frac{4}{3}u^3} \cdot \frac{1}{1+u} dw(u)$$

$$M(\cancel{t}) = e^{\frac{4}{3}t^3} R(t) \underset{\text{where } t=0}{=} e^{\frac{4}{3}0^3} R(0) = R(0)$$

$M(t) = M(0) +$ // $+$ //

$M(t) =$ ~~$R(0) +$ // $+$ //~~

$$R(0) + \int_0^t e^{\frac{4}{3}u^3} 2u^2 du + \int_0^t e^{\frac{4}{3}u^3} \frac{1}{1+u} dw(u)$$

$$M(t) = e^{\frac{4}{3}t^3} R(t)$$

$$R(t) = e^{-\frac{4}{3}t^3} \left(R(0) + \int_0^t e^{\frac{4}{3}u^3} \cdot 2u^2 du + \int_0^t e^{\frac{4}{3}u^3} \cdot \frac{1}{1+u} dw(u) \right)$$

closed form solution of $R(t)$

Step 4:-

$$E[P(t)] = e^{-\frac{4}{3}t^3}\Big(E[P(0)] + E\Big[\int_0^t e^{\frac{4}{3}u^3} \cdot 2u^2\,du\Big] + E\Big[\int_0^t e^{\frac{4}{3}u^3} \cdot \frac{1}{1+u}\,dw(u)\Big]$$

$P(0)$ is constant

2nd term has norandomness ∴ take out

3rd term $dw(u)$ = Brownian moction

$\therefore E[P(0)] = P(0)$

2nd term $= E\Big[\int_0^t e^{\frac{4}{3}u^3} \cdot 2u^2\,du\Big] = \int_0^t e^{\frac{4}{3}u^3} 2u^2\,dt$

3rd term = E[of Brownian moction] = 0

$$\therefore E[P(t)] = e^{-\frac{4}{3}t^3}\Big(P(0) + \int_0^t e^{\frac{4}{3}u^3} \cdot 2u^2\,du$$

② $X(t) = \frac{t}{3} + \int_0^t u^2 \, dM(u) - \int_0^t e^{3u} \, du$

$Y(t) = M^3(t)$
$Z(t) = M(t) \, e^t$

$[M,M](t) = 5t$
$M(0) = 4$

① $E[X(t)] = ?$ ② $[X, Z](t) = ?$
$E[Y(t)] = ?$
$E[Z(t)] = ?$

(a) $X(t) = \frac{1}{3} + \int_0^t u^2 \, dM(u) - \int_0^t e^{3u} \, du$

$E[X(t)] = E\left[\frac{1}{3}\right] + E\left[\int_0^t u^2 dM(u)\right] - E\left[\int_0^t e^{3u} du\right]$

$E\left[\frac{1}{3}\right] = \frac{1}{3}$
2nd term $= E[M(t)] = E[\text{Martingale}] = 0$
$E\left[\int_0^t e^{3u} du\right] = \int_0^t e^{3u} du = \frac{1}{3}(e^{3t} - 1)$

$E[X(t)] = \frac{1}{3} + 0 - \frac{1}{3}(e^{3t} - 1)$

$= \frac{1}{3} - \frac{1}{3} e^{3t} + \frac{1}{3} = \frac{2}{3} - \frac{1}{3} e^{3t}$

$E[Y(t)] =$

$Y(t) = M^3(t)$

$E[M^3(t)] \Rightarrow E[M^3(0)]$ (using a hammer as a scalpel)

$= 4^3 = 64$

$E[Z(t)] =$

$Z(t) = M(t)\, e^t$

$= E[M(t)\, e^t] = e^t E[M(t)]$

$= e^t E[M(0)]$

$= 4 \cdot e^t$

(b) $[Y, Z](t)$

$Y(t) = \frac{1}{3} + \int_0^t u^2\, dM(u) - \int_0^t e^{3u}\, du$

$dY(t) = t^2\, dM(t) - e^{3t}\, dt$

$Z(t) = M(t)\, e^t$

$dZ(t) = e^t M_t \cdot dt +$
$e^t\, dM(t)$

$f(t, M(t)) =$
$f(a, b) = b e^a$
$f_a = e^a \cdot b$
$f_b = e^a$
$f_{bb} = 0$

$$[X, Z](t) = \int_0^t u^2 \, d(5u) \, e^u$$

$$= 5\int_0^t u^2 e^u \, du$$

$\cancel{v = e^u} \quad \cancel{dv = e^u du}$

$\cancel{dw} = u^2 du \quad \cancel{w = \frac{u^3}{3} - \frac{2u^2}{3} + \frac{2u}{3} - \frac{2}{3}}$

$u^2 = v$

$e^u = v$, $\cancel{\int u \, dv = v}$ Integrvaction By parts

$$\int uv \, du = u\int v \, dv - \int\left(u' \int v \, du\right) du$$

$$= e^u(u^2 - 2u + 2)$$

$$= 5\left[e^u(u^2 - 2u + 2) - 2\right.$$

$$= 5e^u(u^2 - 2u + 2) - 10$$

(3) we need to prove

$B(t) = (B1(t), B2(t))$ is a 2-D B.M

(1) $B(0) = (0,0)$

(2) $B(t)$ has continus paths

(3) $B(t)$ has independent incvements

(4) the ~~independent~~ incvements are normally distributed with mean 0 and variance t

$\alpha = ?$ $b = ?$ $\rho = ?$

$B1(t)$, $B2(t)$ must be indipident

$$var(B_1(t)) = \gamma^2 t + \alpha^2 t + \frac{1}{2} t$$

$$= \left(\gamma^2 + \alpha^2 + \frac{1}{2}\right) t$$

$$var(B_2(t)) = \frac{4}{9}\left(\frac{1}{2}\right)^2 var(w_1(t)) + \beta^2 var(w_2(t))$$

$$+ \left(\frac{1}{3\sqrt{2}}\right)^2 var(w_3(t))$$

$$= \frac{1}{4} t + \beta^2 t + \frac{1}{18} t$$

$$= \left(\frac{1}{4} + \beta^2 + \frac{1}{18}\right) t$$

$$Var(B_1) = E[B_1^2] - \left(E[B_1]\right)^2$$

$$E[B_1] = 0 \quad \therefore Var(B_1) = E[B_1^2]$$

$$Var(B_2) = E[B_2^2]$$

$$var\, B_1 = E\left[\left(\gamma w_1(t) + \alpha w_2(t) - \frac{1}{\sqrt{2}} w_3(t)\right)^2\right]$$

$$w_1 \cdot w_2 = w_2 \cdot w_3 = w_3 \cdot w_1 = 0$$

$$var\, B_1 = E\left[\gamma^2 w_1^2 t + \alpha^2 w_2^2(t) + \frac{1}{2} w_3^2(t)\right]$$

$w^2(t) = t$ for a B.M

$$Var\, B_1 = \gamma^2 \cdot t + \alpha^2 \cdot t + \frac{1}{2} t$$

$\text{var } B_2 = \frac{1}{4} t + B^2 t + \frac{1}{18} t$

as $B_2^2(t) = \frac{1}{4} w_1^2(t) + B_2^2 w_2^2(t)$
$+ \frac{1}{18} w_3^2(t)$

$E[B_1 B_2] = 0$

$= E[\gamma \cdot \frac{1}{2} w_1^2(t) + \alpha B w^2(t) - \frac{1}{6} w^3(t)]$

all others $w_1 \cdot w_2 = w_2 \cdot w_3 = w_3 \cdot w_1 = 0$ terms are 0

$E[B_1, B_2] = \frac{\gamma}{2} t + \alpha B t - \frac{1}{6} t$

we know
$\text{var } B_1 = \text{var } B_2 = t$
$E[B_1, B_2] =$

$\gamma^2 + \alpha^2 + \frac{1}{2} = 1 \qquad \gamma^2 + \gamma^2 = \frac{1}{2}$

$\frac{1}{4} + B^2 + \frac{1}{18} = 1$

$B = 5/6$

$$\frac{\gamma}{2} + \alpha\beta - \frac{1}{6} = 0$$

$$\frac{\gamma}{2} + \alpha\left(\frac{5}{6}\right) = \frac{1}{6}$$

$$6\gamma + 10\alpha = 2$$

$$\alpha = \frac{2-6\gamma}{10} \qquad \beta = \frac{5}{6} \qquad \frac{\gamma}{2} + \frac{5\alpha}{6} = \frac{1}{6}$$

$$= \alpha^2 + \gamma^2 = \frac{1}{2} \qquad \alpha = \frac{2-6\gamma}{10}$$

$$= \frac{(2-6\gamma)^2}{100} + \gamma^2 = \frac{1}{2}$$

$$\underline{4 + 36\gamma^2 - 24\gamma} + 100\gamma^2 = 50$$

$$136\gamma^2 - 24\gamma - 46 = 0$$

$$\gamma = \frac{184}{272}$$

$$\alpha = \sqrt{\frac{1}{2} - \left(\frac{184}{272}\right)^2} = \frac{7}{34}$$

$\cancel{\alpha =}$

$\alpha = \frac{2}{34}$ $\beta = \frac{5}{6}$ $\gamma = \frac{184}{272}$

to show 2D B·M
we start at 0
we have shown continus parts
we have exhibiting indipidente parts
possing normal distributed increments

$\therefore$ B(t) is 2-D B·M.

4) given:-

$$Z(t) = \log(\pi) + \int_0^t 3u^2 Z(u)\,du + \int_0^t \sqrt{\cos(u)}\,dW(u)$$

ⓐ $E[Z(t)] = ?$

ⓑ $[Z, Z](\frac{\pi}{2}) = ?$

$$Z(t) = \log(\pi) + \int_0^t 3u^2 Z(u)\,du + \int_0^t \sqrt{\cos(u)}\,dW(u)$$

E on both sides

$$E[Z(t)] = E[\log(\pi)] + E[\int_0^t 3u^2 Z(u)\,du] + E[\int_0^t \sqrt{\cos(u)}\,dW(u)]$$

$= E[\log(\pi)] = \log \pi$

$= E[\int_0^t 3u^2 Z(u)\,du] = \int_0^t 3u^2 E[Z(u)]\,du$

let $m(t) = E[Z(t)]$

$\Rightarrow \int_0^t 3u^2 m(t)\,dt$

$= E[\int_0^t \sqrt{\cos(u)}\,dW(u)]$

ito integral $\therefore E[\;] = 0$

$\therefore m(t) = \log(\pi) \cancel{=} + \int_0^t 3u^2 m(u)\,du$

differentiate on B.S

$$\frac{d}{dt} m(t) = 3t^2 m(t)$$

$$\frac{1}{m(t)} \frac{d\,m(t)}{dt} = 3t^2$$

integrate Both sides with rescept to t

$$\ln(m(t)) = t^3 + \cancel{3}\, C$$

exp on Both sides

$$m(t) = A\, e^{t^3}$$

$A = e^C$ constant

constant = ~~z(0) = log~~

$$m(0) = E[z(0)] = \log \pi$$

$$e^C = \log(\pi)$$

$$\therefore E[z(t)] = \log(\pi)\, e^{t^3}$$

(b) $[z, z](\pi/2)$

$$z(t) = z(0) + \int_0^t a(s)\,ds + \int_0^t b(s)\,dw(s)$$

Q.V is $[z, z](t) = \int_0^t b(s)^2\,ds$

$b(s) = \sqrt{\cos(s)}$ (coefficient of $dW(s)$)

$$[Z, Z](t) = \int_0^t \cos(s)\, ds$$

$$= \int_0^{\pi/2} \cos(s)\, ds$$

$$= [\sin(s)]_0^{\pi/2}$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin(0)$$

$$= 1 - 0 = 1$$