FE 610

Probability Review

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

August 21, 2020

For academic use only.

Sigma Algebras

Definition 1.1.1: Let Ω be a nonempty set, and let $\mathcal{F}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a σ-algebra provided that:

1. The empty set belongs to $\mathcal{F}$
2. Whenever a set A belongs to $\mathcal{F}$, its complement A^c also belongs to $\mathcal{F}$
3. Whenever a sequence of sets $A_1, A_2, \ldots$ belongs to $\mathcal{F}$, their union $(\cup_{i=1}^{\infty} A_i)$ also belongs to $\mathcal{F}$

[1]

Probability Measure

Definition 1.1.2: Let Ω be a nonempty set, and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A probability measure $\mathbb{P}$ is a function that, to every set $A \in \mathcal{F}$, assigns a number in $[0, 1]$, called the probability of A and written $\mathbb{P}(A)$. We require:

1. $\mathbb{P}(\Omega) = 1$
2. Whenever $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.[1]

Definition 1.1.5: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If a set $A \in \mathcal{F}$ satisfies $\mathbb{P}(A) = 1$, we say that the event A occurs almost surely.[1]

Random Variable

Definition 1.2.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable is a real-valued function X defined on Ω with the property that for every Borel subset B of $\mathbb{R}$, the subset of Ω given by

$$\{X \in B\} = \{\omega \in \Omega; X(\omega) \in B\}$$

is in the σ-algebra $\mathcal{F}$.[1]

Expectation

For Discrete:

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega)$$

$$\mathbb{E}[g(X)] = \sum_{\omega \in \Omega} g(X(\omega))\mathbb{P}(\omega)$$

For Continuous:

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)d\mathbb{P}(\omega)$$

$$\mathbb{E}[g(X)] = \int_{\Omega} g(X(\omega))d\mathbb{P}(\omega)$$

Now let Y be another random variable on $(\Omega, \mathcal{F}, \mathbb{P})$

- **Comparison:** If $X \leq Y$ almost surely and X and Y are integrable or almost surely nonnegative, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$. In particular, if $X = Y$ almost surely and one of the random variables is integrable or almost surely nonnegative, then they are both integrable or almost surely nonnegative, respectively, and $\mathbb{E}[X] = \mathbb{E}[Y]$
- **Linearity:** If α and β are real constants and X and Y are integrable or if α and β are nonnegative constants and X and Y are nonnegative, then

$$\mathbb{E}[\alpha X + \beta Y] = \alpha \mathbb{E}[X] + \beta \mathbb{E}[Y]$$

- **Jensen's Inequality:** If φ is a convex, real-valued function defined on $\mathbb{R}$, and if $\mathbb{E}[X] < \infty$, then

$$\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)]$$

[1]

©2020 School of Business

Change of Measure

Let us define the random variable

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$$

but this is better written (to avoid dividing by 0) as

$$Z(\omega)\mathbb{P}(\omega) = \tilde{\mathbb{P}}(\omega)$$

©2020 School of Business

Theorem 1.6.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}[Z] = 1$. For $A \in \mathcal{F}$, define

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

Then $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

$$\tilde{\mathbb{E}}[X] = \mathbb{E}[XZ]$$

If Z is almost surely strictly positive, we also have

$$\mathbb{E}[Y] = \tilde{\mathbb{E}}\left[\frac{Y}{Z}\right]$$

for every nonnegative random variable Y.[1]

Equivalent Measure

Definition 1.6.3: Let Ω be a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of Ω. Two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ are said to be equivalent if they agree which sets in $\mathcal{F}$ have probability zero.[1]

Definition 1.6.5: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\tilde{\mathbb{P}}$ be another probability measure on $(\Omega, \mathcal{F})$ that is equivalent to $\mathbb{P}$, and let Z be an almost surely positive random variable that relates $\mathbb{P}$ and $\tilde{\mathbb{P}}$ via Theorem 1.6.1. Then Z is called the Radon-Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$$

[1]

©2020 School of Business

Filtration

Definition 2.1.1: Let Ω be a nonempty set. Let T be a fixed positive number and assume that for each $t \in [0, T]$ there is a σ-algebra $\mathcal{F}(t)$. Assume further that if $s \leq t$, then every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. Then we call the collection of σ-algebras $\mathcal{F}(t)$, $0 \leq t \leq T$, a filtration

Definition 2.1.3: Let X be a random variable defined on a nonempty sample space Ω. The σ-algebra generate by X, denoted $\sigma(X)$, is the collection of all subsets of Ω of the form $\{X \in B\}$, where B ranges over the Borel subsets of $\mathbb{R}$.[1]

©2020 School of Business

Definition 2.1.5: Let X be a random variable defined on a nonempty sample space Ω. Let $\mathcal{G}$ be a σ-algebra of subsets of Ω. If every set in $\sigma(X)$ is also in $\mathcal{G}$, we say that X is $\mathcal{G}$-measurable.

Definition 2.1.6: Let Ω be a nonempty sample space equipped with a filtration $\mathcal{F}(t), 0 \leq t \leq T$. Let $X(t)$ be a collection of random variable indexed by $t \in [0, T]$. We say that this collection of random variables is an adapted stochastic process if, for each t, the random variable $X(t)$ is $\mathcal{F}(t)$-measurable.[1]

Independence

Definition 2.2.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ and $\mathcal{H}$ be sub-σ-algebras of $\mathcal{F}$. We say these two σ-algebras are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) * \mathbb{P}(B) \text{ for all } A \in \mathcal{G}, B \in \mathcal{H}$$

Let X and Y be random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. We say these two random variables are independent if the σ-algebras they generate, $\sigma(X)$ and $\sigma(Y)$, are independent. We say that the random variable X is independent of the σ-algebra $\mathcal{G}$ if $\sigma(X)$ and $\mathcal{G}$ are independent.[1]

Variance, Covariance, etc.

Definition 2.2.9: Let X be a random variable whose expected value is defined. The variance of X, denoted $Var(X) = \mathbb{V}(X)$, is

$$Var(X) = \mathbb{V}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

Because $(X - \mathbb{E}[X])^2$ is nonnegative, $\mathbb{V}(X)$ is always defined, although it may be infinite. The standard deviation of X is $\sqrt{\mathbb{V}(X)}$. The linearity of expectations shows that

$$\mathbb{V}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Let Y be another random variable and assume that $\mathbb{E}[X], \mathbb{V}(X), \mathbb{E}[Y], \mathbb{V}(Y)$ are all finite. The covariance of X and Y is

$$Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

The linearity of expectations shows that

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

In particular, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ if and only if $Cov(X, Y) = 0$. Assume, in addition to the finiteness of expectations and variances, that $\mathbb{V}(X) > 0$ and $\mathbb{V}(Y) > 0$. The correlation coefficient of X and Y is

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{\mathbb{V}(X)\mathbb{V}(Y)}}$$

If $\rho(X, Y) = 0$, we say that X and Y are uncorrelated.[1]

Definition 2.3.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let X be a random variable that is either nonnegative or integrable. The **conditional expectation of X given $\mathcal{G}$**, denoted $\mathbb{E}[X|\mathcal{G}]$, is any random variable that satisfies

1. **Measurability:** $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable, and
2. **Partial Averaging:**

$$\int_A \mathbb{E}[X|\mathcal{G}] d\mathbb{P}(\omega) = \int_A X(\omega) d\mathbb{P}(\omega), \text{ for all } A \in \mathcal{G}$$

If $\mathcal{G}$ is the σ-algebra generated by some other random variable W (i.e. $\mathcal{G} = \sigma(W)$), we generally write $\mathbb{E}[X|W]$ rather than $\mathbb{E}[X|\sigma(W)]$.[1]

Theorem 2.3.2: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.

1. (Linearity of conditional expectations)
$$\mathbb{E}[c_1 X + c_2 Y|\mathcal{G}] = c_1\mathbb{E}[X|\mathcal{G}] + c_2\mathbb{E}[Y|\mathcal{G}]$$
2. (Taking out what is known) if X is $\mathcal{G}$-measurable
$$\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$$
3. (Iterated conditioning) If $\mathcal{H} \subset \mathcal{G}$
$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$$
4. (Independence) If X is independent of $\mathcal{G}$ then
$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$$
5. (Conditional Jensen's Inequality) If $\varphi(x)$ is a convex function then
$$\mathbb{E}[\varphi(X)|\mathcal{G}] \geq \varphi(\mathbb{E}[X|\mathcal{G}])$$

[1]

Martingale

Definition 2.3.5: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, Let T be a fixed positive number, and let $\mathcal{F}(t), 0 \leq t \leq T$, be a filtration of sub-σ-algebras of $\mathcal{F}$. Consider an adapted stochastic process $M(t), 0 \leq t \leq T$.

1. If
$$\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s), \text{ for all } 0 \leq s \leq t \leq T,$$
we say this process is a martingale.
2. If
$$\mathbb{E}[M(t)|\mathcal{F}(s)] \geq M(s), \text{ for all } 0 \leq s \leq t \leq T,$$
we say this process is a submartingale
3. If
$$\mathbb{E}[M(t)|\mathcal{F}(s)] \leq M(s), \text{ for all } 0 \leq s \leq t \leq T,$$
we say this process is a supermartingale

[1]

©2020 School of Business

Definition 2.3.6: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let T be a fixed positive number, and let $\mathcal{F}(t), 0 \leq t \leq T$, be a filtration of sub-σ-algebras of $\mathcal{F}$. Consider an adapted stochastic process $X(t), 0 \leq t \leq T$. Assume that for all $0 \leq s \leq t \leq T$ and for every nonnegative, Borel-measurable function f, there is another Borel-measurable function g such that

$$\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s))$$

Then we say that the X is a Markov process[1]

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

STEVENS INSTITUTE of TECHNOLOGY
School of Business

FE 610

Random Walks and Brownian Motion

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

August 21, 2020

For academic use only.

Random Walk

For $p = q = \frac{1}{2}$ define the outcomes of a the tosses of a coin as

$$X_j = \begin{cases} 1, & \text{if } \omega_j = H \\ -1, & \text{if } \omega_j = T \end{cases}$$

Define $M_0 = 0$ and

$$M_k = \sum_{j=1}^{k} X_j, k = 1, 2, \dots$$

The process $M_k, k = 0, 1, 2, \dots$ is a symmetric random walk. [1]

Independent Increments

A random walk (both symmetric and asymmetric) has independent increments.
An increment is defined as:

$$M_{k_{i+1}} - M_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j$$

In addition, for a symmetric random walk, each increment has

$$\mathbb{E}[M_{k_{i+1}} - M_{k_i}] = 0$$

$$\mathbb{V}(M_{k_{i+1}} - M_{k_i}) = \sum_{j=k_i+1}^{k_{i+1}} \mathbb{V}(X_j) = \sum_{j=k_i+1}^{k_{i+1}} 1 = k_{i+1} - k_i$$

Martingale Property for Symmetric Random Walk

The argument that a symmetric random walk is as follows.
Given nonnegative integers $k < l$, we have:

$$\begin{aligned}
\mathbb{E}[M_l|\mathcal{F}_k] &= \mathbb{E}[(M_l - M_k) + M_k|\mathcal{F}_k] \\
&= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + \mathbb{E}[M_k|\mathcal{F}_k] \\
&= \mathbb{E}[M_l - M_k|\mathcal{F}_k] + M_k \\
&= \mathbb{E}[M_l - M_k] + M_k \\
&= M_k
\end{aligned}$$

Therefore the symmetric random walk is a martingale.[2]

First Order Variation

The first order variation is determined as:

$$FV_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|$$

If the function $f(t)$ is a continuous differentiable function, then we can express the first order variation as:

$$FV_T(f) = \int_0^T |f'(t)| dt$$

©2020 School of Business

Quadratic Variation

Definition 3.4.1: Let $f(t)$ be a function defined for $0 \leq t \leq T$. The quadratic variation of f up to time T is

$$[f, f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2$$

where $\Pi = \{t_0, t_1, \ldots, t_n\}$ and $0 = t_0 < t_1 < \cdots < t_n = T$
[2]

Quadratic Variation for Random Walk

Let the function $f(t)$ simply be the random walk. This results in the formula for the quadratic variation of a random walk as being:

$$[M, M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2 = k$$

This works for both symmetric and asymmetric random walks.

Scaled Symmetric Random Walk

For a fixed integer n, the scaled symmetric random walk is defined as

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$$

where nt is an integer. If nt is not an integer use linear interpolation to define the value.

©2020 School of Business

SSRW cont.

For $0 = t_0 < t_1 < \cdots < t_m$ where each nt_j is an integer then the scaled symmetric random walk increments are independent. This can be seen as:

$$
\begin{aligned}
W^{(n)}(t_{j+1}) - W^{(n)}(t_j) &= \frac{1}{\sqrt{n}} M_{nt_{j+1}} - \frac{1}{\sqrt{n}} M_{nt_j} \\
&= \frac{1}{\sqrt{n}} (M_{nt_{j+1}} - M_{nt_j}) \\
&= \frac{1}{\sqrt{n}} \sum_{i=nt_j+1}^{nt_{j+1}} X_i
\end{aligned}
$$

SSRW cont.

In addition, if $0 \leq s < t$ such that ns and nt are integers, we have

$$\begin{aligned}
\mathbb{E}[W^{(n)}(t) - W^{(n)}(s)] &= \mathbb{E}\left[\frac{1}{\sqrt{n}} \sum_{i=ns+1}^{nt} X_i\right] \\
&= \frac{1}{\sqrt{n}} \sum_{i=ns+1}^{nt} \mathbb{E}[X_i] = 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}(W^{(n)}(t) - W^{(n)}(s)) &= \mathbb{V}\left(\frac{1}{\sqrt{n}} \sum_{i=ns+1}^{nt} X_i\right) \\
&= \frac{1}{n} \sum_{i=ns+1}^{nt} \mathbb{V}(X_i) \\
&= t - s
\end{aligned}$$

SSRW cont.

The scaled symmetric random walk is a martingale as can be seen by looking at, for $0 \leq s < t$:

$$\begin{aligned}
\mathbb{E}[W^{(n)}(t)|\mathcal{F}(s)] &= \mathbb{E}[(W^{(n)}(t) - W^{(n)}(s)) + W^{(n)}(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[W^{(n)}(t) - W^{(n)}(s)|\mathcal{F}(s)] + \mathbb{E}[W^{(n)}(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[W^{(n)}(t) - W^{(n)}(s)] + W^{(n)}(s) \\
&= W^{(n)}(s)
\end{aligned}$$

And the quadratic variation is:

$$\begin{aligned}
[W^{(n)}, W^{(n)}](t) &= \sum_{j=1}^{nt} \left[W^{(n)}\left(\frac{j}{n}\right) - W^{(n)}\left(\frac{j-1}{n}\right)\right]^2 \\
&= \sum_{j=1}^{nt} \left[\frac{1}{\sqrt{n}} X_j\right]^2 = t
\end{aligned}$$

[2]

Normal Distribution

A normally distributed random variable is defined by the parameters μ, its mean, and σ^2, its variance. If X is a normally distributed random variable, it is denoted as $X \sim N(\mu, \sigma^2)$. The distribution of this normal random variable is given as:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

In addition it has the properties:

$$\mathbb{E}[X] = \mu$$

$$\mathbb{V}(X) = \sigma^2$$

and

$$\varphi_X(t) = \mathbb{E}[e^{xt}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Central Limit Theorem

Theorem 3.2.1: (Central Limit) Fix $t \geq 0$. As $n \to \infty$, the distribution of the symmetric scaled random walk $W^{(n)}(t)$ evaluated at time t converges to the normal distribution with mean zero and variance t.

Proof done using limits of moment generating functions.[2]

Brownian Motion

Definition 3.3.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous fuction $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. Then $W(t), t \geq 0$, is a Brownian Motion if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0$$

$$\mathbb{V}(W(t_{i+1}) - W(t_i)) = t_{i+1} - t_i$$

[2]

Quadratic Variation and Brownian Motion

Theorem 3.4.3: Let W be a Browninan motion. Then $[W, W](T) = T$ for all $T \geq 0$ almost surely.

The proof of this theorem involves showing convergence in $\mathcal{L}^2$ or mean square convergence. If there is mean square convergence, then there exists a subsequence that converges almost surely.

In addition, a consequence of this theorem is:

$$dW(t) \cdot dW(t) = dt$$

Cross Variation

If we determine the cross variation between Brownian motion and time, we see that

$$[W(t), t](t) = [t, W(t)](t) = 0$$

In addition, because time is a continuous differentiable process, we have the quadratic variation of time

$$[t, t](t) = 0$$

A result of this is that:

$$dW(t) \cdot dt = dt \cdot dW(t) = (dt)^2 = 0$$

Joint Normal Distribution

The joint normal distribution (or multivariate normal) of k different normal random variables $X_1, X_2, \ldots, X_k$ is denoted $N_k(\mu, \Sigma)$, where $\mu = (\mathbb{E}[X_1], \ldots, \mathbb{E}[X_k])$ and Σ is the covariance matrix (positive definite) with entries $a_{i,j} = Cov(X_i, X_j)$. This density is given as:

$$f_{\mathbf{X}}(x_1, \ldots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} e^{-\frac{1}{2}\left((\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\right)}$$

where $|\Sigma|$ is the determinant of the covariance matrix.

Because the increments

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$$

are independent and normally distributed, the variables $W(t_1), W(t_2), \ldots, W(t_m)$ are jointly normally distributed. To determine the covariance matrix, we look at two Brownian motions at times $0 \leq s < t$:

$$\begin{aligned}
Cov(W(t), W(s)) &= \mathbb{E}[W(t)W(s)] - \mathbb{E}[W(t)]\mathbb{E}[W(s)] \\
&= \mathbb{E}[W(s)^2 + W(t)W(s) - W(s)^2] \\
&= \mathbb{E}[W(s)^2] + \mathbb{E}[W(s)(W(t) - W(s))] \\
&= \mathbb{V}(W(s)) + \mathbb{E}[W(s)]\mathbb{E}[W(t) - W(s)] \\
&= s
\end{aligned}$$

Alternate Characterization of Brownian Motion

Theorem 3.3.2: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. The following three properties are equivalent.

1. Definition 3.3.1
2. For all $0 = t_0 < t_1 < \cdots < t_m$, the random variables $W(t_1), W(t_2), \ldots, W(t_m)$ are jointly normally distributed with means equal to 0 and covariance matrix (Σ):

$$\Sigma = \begin{bmatrix} t_1 & t_1 & \ldots & t_1 \\ t_1 & t_2 & \ldots & t_2 \\ \vdots & \vdots & & \vdots \\ t_1 & t_2 & \ldots & t_m \end{bmatrix}$$

3. For all $0 = t_0 < t_1 < \cdots < t_m$, the random variables $W(t_1), W(t_2), \ldots, W(t_m)$ have the joint moment-generating function

$$\varphi_{W(t_1),\ldots,W(t_m)}(u_1, u_2, \ldots, u_m) = \mathbb{E}[e^{u_m W(t_m) + u_{m-1} W(t_{m-1}) + \cdots + u_1 W(t_1)}]$$

[2]

Filtration for Brownian Motion

Definition 3.3.3: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W(t), t \geq 0$. A filtration for the Brownian motion is a collection of σ-algebras $\mathcal{F}(t), t \geq 0$, satisfying:

1. For $0 \leq s < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$.
2. For each $t \geq 0$, the Brownian motion $W(t)$ at time t is $\mathcal{F}(t)$-measurable.
3. For $0 \leq t < u$,the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$.

Let $\Delta(t), t \geq 0$, be a stochastic process. We say that $\Delta(t)$ is adapted to the filtration $\mathcal{F}(t)$ if for each $t \geq 0$ the random variable $\Delta(t)$ is $\mathcal{F}(t)$-measurable.[2]

©2020 School of Business

Martingale Property of Brownian Motion

Theorem 3.3.4: Brownian motion is a martingale
Proof: Let $0 \leq s \leq t$ be given.

$$
\begin{aligned}
\mathbb{E}[W(t)|\mathcal{F}(s)] &= \mathbb{E}[(W(t) - W(s)) + W(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[W(t) - W(s)] + W(s) \\
&= W(s)
\end{aligned}
$$

[2]

Geometric Brownian Motion

For α and $\sigma > 0$ as constants, we define **geometric Brownian motion** as:

$$S(t) = S(0)e^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

For a partition Π as typically defined, the **log returns** of the process is:

$$\log\left(\frac{S(t_{j+1})}{S(t_j)}\right) = \left(\alpha - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j) + \sigma(W(t_{j+1}) - W(t_j))$$

[1] S.E. Shreve. *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*. Number v. 1 in Springer Finance.

[2] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

STEVENS INSTITUTE of TECHNOLOGY
School of Business

FE 610

Markov Processes, Passage Times and Reflection Principle

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

February 25, 2021

For academic use only.

Alternate Definition for Markov Property

For an adapted stochastic process, if the conditional probability distribution for future states depending on the entire history of the process is the same as the conditional probability distribution for future states depending only on the current value of the process, then the process is Markov.

This can be expressed easily for the random walk. Given a random walk M_n as defined previously, we have for some integer x:

$$\mathbb{P}(M_{n+m} = x | M_0, M_1, \ldots, M_n) = \mathbb{P}(M_{n+m} = x | M_n)$$

Theorem 3.5.1: Let $W(t), t \geq 0$, be a Brownian motion and let $\mathcal{F}(t), t \geq 0$, be a filtration for this Brownian motion. Then $W(t), t \geq 0$, is a Markov process.[1]

To prove this, we must show that there exists a function g such that

$$\mathbb{E}[f(W(t))|\mathcal{F}(s)] = g(W(s))$$

for all functions f

Transition Density

Let $\tau = t - s$ and $y = w + x$. This allows the previous formula to be expressed as:

$$g(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\tau}} f(y) e^{-\frac{(y-x)^2}{2\tau}} dy$$

If we let $p(\tau, x, y)$ be the **transition density** for the Brownian Motion and set

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-x)^2}{2\tau}}$$

then

$$g(x) = \int_{-\infty}^{\infty} f(y) p(\tau, x, y) dy$$

Stopping Times

A **stopping time**, τ, is a random variable that takes a value in $[0, \infty)$ and satisfies the condition that if $\mathbb{E}[\tau|\mathcal{F}(t)] = t$, then $\mathbb{E}[\tau|\mathcal{F}(u)] = t$ for all $u > t$.

Theorem A martingale stopped at a stopping time is a martingale.

Exponential Martingale

Let the process $Z(t)$ be defined as:

$$Z(t) = e^{\sigma W(t) - \frac{1}{2}\sigma^2 t}$$

Theorem 3.6.1: Let $W(t), t \geq 0$, be a Brownian motion with a filtration $\mathcal{F}(t), t \geq 0$, and let σ be a constant. The process $Z(t), t \geq 0$ is a martingale.[1]
To prove this we show that:

$$\mathbb{E}[Z(t)|\mathcal{F}(s)] = Z(s)$$

First Passage Time

For m being a real number, define the **first passage time to level** m

$$\tau_m = \min\{t \geq 0; W(t) = m\}$$

Using this and the property that a stopped martingale is still a martingale we have:

$$1 = Z(0) = \mathbb{E}[Z(t \wedge \tau_m)] = \mathbb{E}[e^{\sigma W(t\wedge\tau_m) - \frac{1}{2}\sigma^2(t\wedge\tau_m)}]$$

Theorem 3.6.2: For $m \in \mathbb{R}$, the first passage time of Brownian motion to level m is finite almost surely, and the Laplace transform of its distribution is given by

$$\mathbb{E}[e^{-\alpha\tau_m}] = e^{-|m|\sqrt{2\alpha}}$$

for all $\alpha > 0$
Note that in a Laplace transform $(F(s))$, for a function $f(t)$ defined on $[0, \infty)$,

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

Reflection Equality:

$$\mathbb{P}\{\tau_m \le t, W(t) \le w\} = \mathbb{P}\{W(t) \ge 2m - w\}, w \le m, m > 0$$

Theorem 3.7.1: For all $m \neq 0$, the random variable τ_m has cumulative distribution function

$$\mathbb{P}\{\tau_m \leq t\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}} dy, t \geq 0$$

and density

$$f_{\tau_m}(t) = \frac{d}{dt}\mathbb{P}\{\tau_m \leq t\} = \frac{|m|}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}}, t \geq 0$$

[1]

Remark 3.7.2: From the previous result we have:

$$\mathbb{E}[e^{-\alpha\tau_m}] = \int_0^\infty e^{-\alpha t} f_{\tau_m}(t)dt$$
$$= \int_0^\infty \frac{|m|}{t\sqrt{2\pi t}} e^{-\alpha t - \frac{m^2}{2t}} dt, \forall \alpha > 0$$

But remember from Theorem 3.6.2, we have

$$\mathbb{E}[e^{-\alpha\tau_m}] = e^{-|m|\sqrt{2\alpha}}$$

[1]

Maximum to Date

Define the **maximum to date** of a Brownian motion to be the process $M(t)$ defined as:

$$M(t) = \max_{0 \leq s \leq t} W(s)$$

Note that $M(t) \geq m$ iff $\tau_m \leq t$, and so we can rewrite the reflection equality as

$$\mathbb{P}\{M(t) \geq m, W(t) \leq w\} = \mathbb{P}\{W(t) \geq 2m - w\}, w \leq m, m > 0$$

Theorem 3.7.3: For $t > 0$, the joint density of $(M(t), W(t))$ is

$$f_{M(t),W(t)}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, w \leq m, m > 0$$

[1]

Corollary 3.7.4: The conditional distribution of $M(t)$ given $W(t) = w$ is

$$f_{M(t)|W(t)}(m|w) = \frac{2(2m - w)}{t} e^{-\frac{2m(m-w)}{t}}, w \leq m, m > 0$$

[1]

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

FE 610

Stochastic Calculus(Integrands)

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

August 21, 2020

For academic use only.

What is a Stochastic Integral

$$\int_0^T \Delta(t)dW(t) = ?$$

If we let $W(t), t \geq 0$ be a Brownian motion with respect to a filtration $\mathcal{F}(t), t \geq 0$ and $\Delta(t)$ be an adapted process such that it is $\mathcal{F}$-measurable, then we can define this expression.

Simple Functions

To do so, we begin with simple functions. A simple function in real analysis is defined to be a function that only takes finite values. We can think of this as:

$$f(x) = \sum_{k \in K} a_k \mathbb{I}_{\{x \in A_k\}}$$

Simple Process

Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ where $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$ be a partition of the interval $[0, T]$

Let $\Delta(t)$ be constant in each interval $[t_j, t_{j+1})$, as such $\Delta(t)$ is a simple process.

Think of $\Delta(t)$ as the position taken in an underlying stock whose price is determined by the process $W(t)$. We can only change our position in the stock on the trading dates $t_0, t_1, \ldots, t_n$. As such the gain (or loss) of our portfolio at time t such that $t_k \leq t \leq t_{k+1}$ is given by the function:

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]$$

As such, this is the same as the integral of the simple process $\Delta(t)$ and as such we now have a representation for:

$$I(t) = \int_0^t \Delta(u) dW(u)$$

This is known as the Ito integral.

©2020 School of Business

Ito as Martingale

Theorem 4.2.1: The Ito integral $I(t)$ is a martingale.[1]
Proof: In order to prove it, let $0 \leq s \leq t \leq T$ be given and show that:

$$\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s)$$

We will need to show it when s and t are in the same partition and when they are not. The case when they are in the same partition is much simpler.

Ito Isometry

Theorem 4.2.2: The Ito integral $I(t)$ satisfies

$$\mathbb{E}[I^2(t)] = \mathbb{E}\left[\int_0^t \Delta^2(u)du\right]$$

[1]
This is the variance of the Ito integral, as because the Ito integral is a martingale and $I(0) = 0$, we have $\mathbb{E}[I(t)] = 0$

Theorem 4.2.3: The quadratic variation accumulated up to time t by the Ito Integral $I(t)$ is

$$[I, I](t) = \int_0^t \Delta^2(u)du$$

[1]
Note that this is not the same as the Isometry

Square Integrability

To expand the Ito integral to non-simple functions we need a couple of conditions. First, let $\Delta(t), t \geq 0$ be adapted to the filtration $\mathcal{F}(t), t \geq 0$. Second, the process $\Delta(t)$ must satisfy:

$$\mathbb{E}\left[\int_0^T \Delta^2(t)dt\right] < \infty$$

This is known as the **square-integrability condition**

Let $\Delta_n(t)$ be a sequence of simple processes, such that $\Delta_n(t) \to \Delta(t)$. By this convergence we mean:

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |\Delta_n(t) - \Delta(t)|^2 dt\right] = 0 \qquad (1)$$

The Ito integral is then defined as

$$\int_0^t \Delta(u)dW(u) = \lim_{n\to\infty} \int_0^t \Delta_n(u)dW(u), 0 \le t \le T \qquad (2)$$

Theorem 4.3.1 Let T be a positive constant and let $\Delta(t), 0 \leq t \leq T$, be an adapted stochastic process that satisfies (1). Then $I(t) = \int_0^t \Delta(u)dW(u)$ defined by (2) has the following properties.

1. **(Continuity)** As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.
2. **(Adaptivity)** For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.
3. **(Linearity)** If $I(t) = \int_0^t \Delta(u)dW(u)$ and $J(t) = \int_0^t \Gamma(u)dW(u)$, then $I(t) \pm J(t) = \int_0^t (\Delta(u) \pm \Gamma(u))dW(u)$; furthermore, for every constant c, $cI(t) = \int_0^t c\Delta(u)dW(u)$
4. **(Martingale)** $I(t)$ is a martingale
5. **(Ito Isometry)** $\mathbb{E}[I^2(t)] = \mathbb{E}[\int_0^t \Delta^2(u)du]$
6. **(Quadratic Variation)** $[I, I](t) = \int_0^t \Delta^2(u)du$

[1]

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

STEVENS INSTITUTE of TECHNOLOGY

School of Business

FE 610

Itô’s Lemma and Applications

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

August 21, 2020

For academic use only.

Differential Form

If $f(x)$ is a differentiable function, we would like to discuss how to find the derivative of $f(W(t))$. If $W(t)$ were a differentiable process, then using the chain rule we would have the result

$$df(W(t)) = f'(W(t))W'(t)dt = f'(W(t))dW(t)$$

But, as we have repeatedly proven and discussed, $W(t)$ has non-zero quadratic variation, and as a result we have the expression

$$df(W(t)) = f'(W(t))dW(t) + \frac{1}{2}f''(W(t))dt$$

Integral Form

If we take the Itô formula in differential form from the previous slide and integrate both sides, we have the **Itô formula in integral form**:

$$\int_0^t df(W(u)) = \int_0^t f'(W(u))dW(u) + \frac{1}{2}\int_0^t f''(W(u))du$$

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(u))dW(u) + \frac{1}{2}\int_0^t f''(W(u))du$$

Notice that the terms on the right hand of the equation are just an Itô integral discussed last week and a Lebesgue integral.

©2020 School of Business

Itô Formula for Brownian Motion

Theorem 4.4.1: (Itô formula for Brownian motion) Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,

$$f(T, W(T)) = f(0, W(0)) + \int_0^T f_t(t, W(t))dt + \int_0^T f_x(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt$$

[1]

Itô Processes

Definition 4.4.3: Let $W(t), t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t), t \geq 0$, be an associated filtration. An Itô process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes. [1]

Lemma 4.4.4: The quadratic variation of the Itô process from the previous definition is

$$[X, X](t) = \int_0^t \Delta^2(u)du$$

[1]

Integral With Respect to an Itô Process

Definition 4.4.5: Let $X(t), t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $\Gamma(t), t \geq 0$, be an adapted process. We define the integral with respect to an Itô process

$$\int_0^t \Gamma(u) dX(u) = \int_0^t \Gamma(u)\Delta(u) dW(u) + \int_0^t \Gamma(u)\Theta(u) du$$

[1]

Theorem 4.4.6: (Itô formula for an Itô process) Let $X(t), t \geq 0$, be an Itô process as described in Definition 4.4.3, and let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous. Then, for every $T \geq 0$,

$$f(T, X(T)) = f(0, X(0)) + \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))dX(t) + \frac{1}{2}\int_0^T f_{xx}(t, X(t))d[X,X](t)$$

[1]

We can express the results of the previous theorem in a different way that can make utilizing the results easier.

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2} f_{xx}(t, X(t))dX(t)dX(t)$$

Generalized Geometric Brownian Motion

Example 4.4.8 Let $W(t), t \geq 0$ be a Brownian motion, let $\mathcal{F}(t), t \geq 0$, be an associated filtration, and let $\alpha(t)$ and $\sigma(t)$ be adapted processes. Define the Itô process

$$X(t) = \int_0^t \sigma(s)dW(s) + \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s))ds$$

Then

$$dX(t) = \sigma(t)dW(t) + (\alpha(t) - \frac{1}{2}\sigma^2(t))dt$$

and

$$dX(t)dX(t) = \sigma^2(t)dW(t)dW(t) = \sigma^2(t)dt$$

Let

$$S(t) = S(0)e^{X(t)}$$

Show that

$$\frac{dS(t)}{S(t)} = \alpha(t)dt + \sigma(t)dW(t)$$

Theorem 4.4.9: (Itô integral of a deterministic integrand) Let $W(s), s \geq 0$, be a Brownian motion, and let $\Delta(s)$ be a nonrandom function of time. Define $I(t) = \int_0^t \Delta(s)dW(s)$. For each $t \geq 0$, the random variable $I(t)$ is normally distributed with expected value zero and variance $\int_0^t \Delta^2(s)ds$.[1]

Vasicek

Let $W(t), t \geq 0$, be a Brownian motion. The Vasicek interest rate model is:

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$$

for α, β, and σ as positive constants. Verify the closed form solution of this stochastic differential equation is:

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dW(s)$$

©2020 School of Business

Cox-Ingersoll-Ross

Let $W(t), t \geq 0$, be a Brownian motion. The CIR interest rate model is:

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t)$$

for α, β, and σ positive constants. This model does not have a closed form solution to verify. However, we can determine the mean and variance of $R(t)$ in order to better understand its distribution.

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

School of Business

FE 610

Black-Scholes-Merton Model

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

August 21, 2020

For academic use only.

The Wealth Equation

Let X_0 be the initial wealth that an individual holds at time 0. If the individual purchases Δ_0 shares of stock with initial price S_0 (possible borrowing money at the risk-free interest rate r to do so) and then investing the surplus in a money market earning the risk-free interest rate. The value at time T of the portfolio (the position in the stock and the money market) is given by the equation:

$$X_T = \Delta_0 S_T + (X_0 - \Delta_0 S_0)(1 + r)$$

If we discretize the time interval, and allow the investor to change the position in the underlying by either investing or borrowing from the money market, we have the evolution of the wealth process X_n as:

$$X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n + \Delta_n S_n)$$

Note that the variable Δ_n is an adapted process.

To look at how we can extend this idea to a continuous, rather than discrete, portfolio valuation, we consider the following. Let the stock be modeled by geometric Brownian motion, and so we have the evolution of the stock price governed by

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

If we have a money market with interest rate r and an adapted process such that at time t, the investor holds $\Delta(t)$ shares of stock and invests the remainder in the money market. The change in the portfolio will result in the change in the stock and the interest earnings.

$$dX(t) = \Delta(t)dS(t) + r\left(X(t) - \Delta(t)S(t)\right)dt$$

In the equation

$$dX(t) = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)$$

the dynamics are

- average underlying rate of return
- risk premium for investing in the stock
- volatility term proportional to the size of the investment

The value of a European call option at expiration T is given as $c(T, S(T)) = (S(T) - K)_+$ where K is the value of the strike and $S(T)$ is the value of the stock at time T. This valuation depends on constants r, σ, and K. This leads, eventually, to the Black-Scholes-Merton differential equation:

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x), \forall t \in [0, T), x \geq 0$$

with terminal condition

$$c(T,x) = (x - K)_+$$

Because of the domain $t, x \in [0, \infty), \mathbb{R}$ we need to observe the terminal condition at these extremes as well.

$$c(t, 0) = 0$$
$$\lim_{x \to \infty} c(t, x) = x$$

We then take all of this information and determine the solution to the differential equation with boundary conditions in the past slides.

Black-Scholes-Merton Formula

We have the solution as:

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)), 0 \leq t < T, x > 0$$

where

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$$

and N is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}dz$$

[1]

Forward Contract

The value of a forward contract is

$$f(t,x) = x - e^{-r(T-t)}K$$

If an agent sets up a static hedge (only takes a single position in the stock) at time 0, then we have:

$$f(0,S(0)) = S(0) - e^{-rT}K$$

The value of this strategy at time T will be:

$$f(T,S(T)) = S(T) - K$$

The value of a call option at expiration is

$$c(T, S(T)) = (S(T) - K)_+$$

The value of a put option at expiration is

$$p(T, S(T)) = (K - S(T))_+$$

If we form a portfolio by buying a call and shorting a put, the value of the portfolio will be

$$\begin{aligned} c(T, S(T)) - p(T, S(T)) &= (S(T) - K)_+ - (K - S(T))_+ \\ &= S(T) - K \\ &= f(T, S(T)) \end{aligned}$$

Because the two portfolios are equal at the end, they must be equal throughout. This lead us to the **put-call parity**

$$f(t,x) = x - e^{-r(T-t)}K = c(t,x) - p(t,x)$$

This allows us to determine the value of a put option without having to derive it.

$$\begin{aligned}
p(t,x) &= c(t,x) - x + e^{-r(T-t)}K \\
&= xN(d_+) - Ke^{-r(T-t)}N(d_-) - x + e^{-r(T-t)}K \\
&= -x(1 - N(d_+)) + Ke^{-r(T-t)}(1 - N(d_-)) \\
&= Ke^{-r(T-t)}N(-d_-) - xN(-d_+)
\end{aligned}$$

©2020 School of Business

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

STEVENS INSTITUTE of TECHNOLOGY
School of Business

FE 610

Multi-variable Stochastic Calculus

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

August 21, 2020

For academic use only.

©2020 School of Business

Definition 4.6.1: A d-dimensional Brownian motion is a process

$$W(t) = (W_1(t), \ldots, W_d(t))$$

with the following properties.

1. Each $W_i(t)$ is a one-dimensional Brownian motion
2. If $i \neq j$, then the processes $W_i(t)$ and $W_j(t)$ are independent
3. **(Information Accumulates)** For $0 \leq s < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$
4. **(Adaptivity)** For each $t \geq 0$, the random vector $W(t)$ is $\mathcal{F}(t)$-measurable
5. **(Independence of future increments)** For $0 \leq t < u$, the vector of increments $W(u) - W(t)$ is independent of $\mathcal{F}(t)$

[1]

Because of the nature of Brownian motion, we have

$$[W_i, W_i](t) = t$$

which can be expressed as

$$dW_i(t)dW_i(t) = dt$$

But, as we shall see, if $i \neq j$ then by the independence of the Brownian motions

$$dW_i(t)dW_j(t) = 0$$

2-dimensional Itô Process

Let $W(t)$ be a 2-dimensional Brownian motion. Recall that we have the form for an Itô process as

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du$$

If we rewrite this process letting $W(t) = (W_1(t), W_2(t))$ and $\Delta(t) = (\sigma_1(t), \sigma_2(t))$ where both functions are adapted processes, we have

$$X(t) = X(0) + \int_0^t \sigma_1(u)dW_1(u) + \int_0^t \sigma_2(u)dW_2(u) + \int_0^t \Theta(u)du$$

Let $X(t)$ and $Y(t)$ be Itô processes. This means we now have

$$X(t) = X(0)+\int_0^t \sigma_{1,1}(u)dW_1(u)+\int_0^t \sigma_{1,2}(u)dW_2(u)+\int_0^t \Theta_1(u)du$$

$$Y(t) = Y(0)+\int_0^t \sigma_{2,1}(u)dW_1(u)+\int_0^t \sigma_{2,2}(u)dW_2(u)+\int_0^t \Theta_2(u)du$$

where Θ_i and $\sigma_{i,j}$ are adapted process for all i and j. We can express this in differential notations as

$$dX(t) = \Theta_1(t)dt + \sigma_{1,1}dW_1(t) + \sigma_{1,2}dW_2(t)$$

$$dY(t) = \Theta_2(t)dt + \sigma_{2,1}dW_1(t) + \sigma_{2,2}dW_2(t)$$

Looking at the quadratic variation for these Itô processes, we have

$$[X, X](t) = \int_0^t (\sigma_{1,1}^2(u) + \sigma_{1,2}^2(u))du$$

or in differential form

$$dX(t)dX(t) = (\sigma_{1,1}^2(t) + \sigma_{1,2}^2(t))dt$$

Similarly we have

$$dY(t)dY(t) = (\sigma_{2,1}^2(t) + \sigma_{2,2}^2(t))dt$$

$$dX(t)dY(t) = \left(\sigma_{1,1}(t)\sigma_{2,1}(t) + \sigma_{1,2}(t)\sigma_{2,2}(t)\right) dt$$

Two-dimensional Itô formula

Theorem 4.6.2: Let $f(t,x,y)$ be a function whose partial derivatives f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, and f_{yy} are defined and are continuous. Let $X(t)$ and $Y(t)$ be Itô processes as discussed. The two-dimensional Itô formula in differential form is

$$\begin{aligned} df\left(t,X(t),Y(t)\right) &= f_t(t,X(t),Y(t))dt \\ &+ f_x(t,X(t),Y(t))dX(t) + f_y(t,X(t),Y(t))dY(t) \\ &+ \frac{1}{2}f_{xx}(t,X(t),Y(t))dX(t)dX(t) \\ &+ f_{xy}(t,X(t),Y(t))dX(t)dY(t) \\ &+ \frac{1}{2}f_{yy}(t,X(t),Y(t))dY(t)dY(t) \end{aligned}$$

[1]

This form can be written much more succinctly. If we suppress the arguments, we have

$$df(t, X(t), Y(t)) = f_t dt + f_x dX + f_y dY + \frac{1}{2} f_{xx} dX dX + f_{xy} dX dY + \frac{1}{2} f_{yy} dY dY$$

If you compare this form to equation (4.6.10) from your book, you can see how much simpler the differential form is to express compared to the integral form.

Corollary 4.6.3: Let $X(t)$ and $Y(t)$ be Itô processes. Then

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t)$$

[1] Proof: Use the differential form from the previous slide, letting $f(t, x, y) = xy$, so $f_t = 0$, $f_x = y$, $f_y = x$ and all second order partial derivatives are 0 except for $f_{xy} = 1$

©2020 School of Business

Lévy, one dimension

Theorem 4.6.4: Let $M(t), t \geq 0$, be a martingale relative to a filtration, $\mathcal{F}(t), t \geq 0$. Assume that $M(0) = 0$, $M(t)$ has continuous paths, and $[M, M](t) = t$ for all $t \geq 0$. Then $M(t)$ is a Brownian motion.[1]
Note that this theorem doesn't mention normality. The key to the proof is to use

$$df(t, M(t)) = f_t(t, M(t)dt + f_x(t, M(t))dM(t) + \frac{1}{2}f_{xx}(t, M(t))dt$$

which holds from the condition $[M, M](t) = t$

Theorem 4.6.5: Let $M_1(t)$ and $M_2(t), t \geq 0$, be martingales relative to a filtration $\mathcal{F}(t), t \geq 0$. Assume that for $i = 1, 2$ we have $M_i(0) = 0$, $M_i(t)$ has continuous paths, and $[M_i, M_i](t) = t$ for all $t \geq 0$. If, in addition, $[M_1, M_2](t) = 0$ for all $t \geq 0$, then $M_1(t)$ and $M_2(t)$ are independent Brownian motions.[1]
To prove this theorem, use the one-dimensional Lévy theorem and an approach similar to its proof as well as the two-dimensional Itô formula.

©2020 School of Business

Correlated Stock Prices

Example 4.6.6: Suppose

$$\frac{dS_1(t)}{S_1(t)} = \alpha_1 dt + \sigma_1 dW_1(t)$$

$$\frac{dS_2(t)}{S_2(t)} = \alpha_2 dt + \sigma_2[\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t)]$$

where $W_1(t)$ and $W_2(t)$ are independent Brownian motions, $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 < \rho < 1$ are constants.

Can we express this instead in terms of correlated Brownian Motions?

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

FE 610

Risk-Neutral Measure and Girsanov

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

August 21, 2020

For academic use only.

©2020 School of Business

Theorem 1.6.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}[Z] = 1$. For $A \in \mathcal{F}$, define

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

Then $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

$$\tilde{\mathbb{E}}[X] = \mathbb{E}[XZ]$$

If Z is almost surely strictly positive, we also have

$$\mathbb{E}[Y] = \tilde{\mathbb{E}}\left[\frac{Y}{Z}\right]$$

for every nonnegative random variable Y.[1]

Equivalent Measure

Definition 1.6.3: Let Ω be a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of Ω. Two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ are said to be equivalent if they agree which sets in $\mathcal{F}$ have probability zero.[1]

Definition 1.6.5: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\tilde{\mathbb{P}}$ be another probability measure on $(\Omega, \mathcal{F})$ that is equivalent to $\mathbb{P}$, and let Z be an almost surely positive random variable that relates $\mathbb{P}$ and $\tilde{\mathbb{P}}$ via Theorem 1.6.1. Then Z is called the Radon-Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$$

[1]

Radon-Nikodým Derivative Process

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F}(t)$ for $0 \leq t \leq T$ where T is fixed. Further suppose that Z is an almost surely positive random variable with $\mathbb{E}[Z] = 1$. Define $\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$. This leads us to the definition of the **Radon-Nikodým derivative process:**

$$Z(t) = \mathbb{E}[Z|\mathcal{F}(t)], 0 \leq t \leq T$$

This process is a martingale as can be seen using iterated conditioning:

$$\mathbb{E}[Z(t)|\mathcal{F}(s)] = \mathbb{E}[\mathbb{E}[Z|\mathcal{F}(t)]|\mathcal{F}(s)] = \mathbb{E}[Z|\mathcal{F}(s)] = Z(s)$$

Lemma 5.2.1: Let t satisfying $0 \leq t \leq T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[YZ(t)]$$

[1]

Lemma 5.2.2: Let s and t satisfying $0 \leq s \leq t \leq T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then

$$\tilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]$$

[1]

The proof of this lemma depends on using part of definition 2.3.1 which states

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega)d\mathbb{P}(\omega) = \int_A X(\omega)d\mathbb{P}(\omega), \text{ for all } A \in \mathcal{G}$$

or in our case

$$\int_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)]d\tilde{\mathbb{P}} = \int_A Yd\tilde{\mathbb{P}}, \text{ for all } A \in \mathcal{F}(s)$$

Girsanov

Theorem 5.2.3:(Girsanov, one dimension) Let $W(t), 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \leq t \leq T$, be a filtration for this Brownian motion. Let $\Theta(t), 0 \leq t \leq T$, be an adapted process. Define

$$Z(t) = e^{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du}$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du$$

and assume that

$$\mathbb{E}\left[\int_0^T \Theta^2(u)du\right] < \infty$$

Set $Z = Z(T)$. Then $\mathbb{E}[Z] = 1$ and under the probability measure $\tilde{\mathbb{P}}$, the process $\tilde{W}(t), 0 \leq t \leq T$ is a Brownian motion.[1]

Discount Process

Consider a stock process $(S(t))$ that follows the Generalized Geometric Brownian Motion given as

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), 0 \leq t \leq T$$

Which has the solution

$$S(t) = S_0 e^{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds}$$

In addition, we have an adapted interest rate process $R(t)$. The **discount process** is given as

$$D(t) = e^{-\int_0^t R(s)ds}$$

$$dD(t) = -R(t)D(t)dt$$

Discounted Stock Process

The discounted stock process is

$$D(t)S(t) = S_0 e^{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right)ds}$$

and so its differential is

$$\begin{aligned} d(D(t)S(t)) &= (\alpha(t) - R(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \\ &= \sigma(t)D(t)S(t)[\Theta(t)dt + dW(t)] \end{aligned}$$

where we define

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$$

which is the market price of risk

Risk-Neutral Measure

If we introduce the measure $\tilde{\mathbb{P}}$ and let $\Theta(t)$ be our adapted process in the Girsanov Theorem, we have

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\tilde{W}(t)$$

Note that under this measure, the discounted stock process is now a martingale. As such we call this measure $\tilde{\mathbb{P}}$ the **risk-neutral measure.** To see this just integrate both sides to get

$$D(t)S(t) = S(0) + \int_0^t \sigma(u)D(u)S(u)d\tilde{W}(u)$$

and then take the expectation of each side

Portfolio Process Value under Risk-Neutral

Start with initial capital X_0 and adjust your portfolio at each time $0 \leq t \leq T$. The change of value of your portfolio at each time is given as

$$\begin{aligned}
dX(t) &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\
&= R(t)X(t)dt + \Delta(t)(\alpha(t) - R(t))S(t)dt + \Delta(t)\sigma(t)S(t)dW(t) \\
&= R(t)X(t)dt + \Delta(t)\sigma(t)S(t)[\Theta(t)dt + dW(t)]
\end{aligned}$$

Which leads us to

$$\begin{aligned}
d(D(t)X(t)) &= \Delta(t)\sigma(t)D(t)S(t)[\Theta(t)dt + dW(t)] \\
&= \Delta(t)d(D(t)S(t)) \\
&= \Delta(t)\sigma(t)D(t)S(t)d\tilde{W}(t)
\end{aligned}$$

Pricing Under the Measure

Let $V(T)$ be an $\mathcal{F}(T)$-measurable random variable. We need to know the value of $X(0)$ and the process $\Delta(t)$ in order to have $X(T) = V(T)$ almost surely. Once we have this property satisfied, because $D(t)X(t)$ is a martingale under $\tilde{\mathbb{P}}$ we also have

$$D(t)X(t) = \tilde{\mathbb{E}}[D(T)X(T)|\mathcal{F}(t)] = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]$$

This leads us to the **risk-neutral pricing formula** for the continuous time model

$$V(t) = \tilde{\mathbb{E}}[e^{-\int_t^T R(u)du} V(T)|\mathcal{F}(t)], 0 \leq t \leq T$$

©2020 School of Business

Theorem 5.3.1: (Martingale representation, one dimension) Let $W(t), 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \leq t \leq T$, be the filtration generated by this Brownian motion. Let $M(t), 0 \leq t \leq T$, be a martingale with respect to this filtration. Then there is an adapted process $\Gamma(u), 0 \leq u \leq T$, such that

$$M(t) = M(0) + \int_0^t \Gamma(u) dW(u), 0 \leq t \leq T$$

Corollary 5.3.2: Let $W(t), 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \leq t \leq T$, be a filtration for this Brownian motion. Let $\Theta(t), 0 \leq t \leq T$, be an adapted process. Define

$$Z(t) = e^{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du}$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du$$

and assume that

$$\mathbb{E}\left[\int_0^T \Theta^2(u)du\right] < \infty$$

Set $Z = Z(T)$. Then $\mathbb{E}[Z] = 1$ and under the probability measure $\tilde{\mathbb{P}}$, the process $\tilde{W}(t), 0 \leq t \leq T$ is a Brownian motion.

Now let $\tilde{M}(t), 0 \leq t \leq T$, be a martingale under $\tilde{\mathbb{P}}$. Then there is an adapted process $\tilde{\Gamma}(u), 0 \leq u \leq T$ such that

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u), 0 \leq t \leq T$$

[1]

From previously, we have

$$D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]$$

This discounted option price process is a martingale under the measure $\tilde{\mathbb{P}}$ as can be seen

$$\begin{aligned}
\tilde{\mathbb{E}}[D(t)V(t)|\mathcal{F}(s)] &= \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(t)]|\mathcal{F}(s)] \\
&= \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}(s)] \\
&= D(s)V(s)
\end{aligned}$$

So there is a representation

$$D(t)V(t) = V(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u), 0 \leq t \leq T$$

©2020 School of Business

But we also have

$$D(t)X(t) = X(0) + \int_0^t \Delta(u)\sigma(u)D(u)S(u)d\tilde{W}(u), 0 \leq t \leq T$$

So in order to have $X(t) = V(t)$ for all $0 \leq t \leq T$ we set $X(0) = V(0)$ and

$$\Delta(t)\sigma(t)D(t)S(t) = \tilde{\Gamma}(t)$$

or equivalently

$$\Delta(t) = \frac{\tilde{\Gamma}(t)}{\sigma(t)D(t)S(t)}, 0 \leq t \leq T$$

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

STEVENS INSTITUTE of TECHNOLOGY

School of Business

FE 610

Fundamental Theorems, Dividend Paying Stocks, Futures and Forwards

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

August 21, 2020

For academic use only.

Theorem 5.4.1: Let T be a fixed positive time, and let $\Theta(t) = (\Theta_1(t), \ldots, \Theta_d(t))$ be a d-dimensional adapted process. Define

$$Z(t) = e^{-\int_0^t \Theta(u)\cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 du}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u) du$$

and assume that

$$\mathbb{E}\left[\int_0^T \| \Theta(u) \|^2 Z^2(u) du\right] < \infty$$

Set $Z = Z(T)$. Then $\mathbb{E}[Z] = 1$, and under the probability measure $\widetilde{\mathbb{P}}$ given by

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F},$$

the process $\widetilde{W}(t)$ is a d-dimensional Brownian motion.[1]

Theorem 5.4.2: Let T be a fixed positive time, and assume that $\mathcal{F}(t), 0 \leq t \leq T$, is the filtration generated by the d-dimensional Brownian motion $W(t), 0 \leq t \leq T$. Let $M(t), 0 \leq t \leq T$, be a martingale with respect to this filtration under $\mathbb{P}$. Then there is an adapted, d-dimensional process $\Gamma(u) = (\Gamma_1(u), \ldots, \Gamma_d(u)), 0 \leq t \leq T$, such that

$$M(t) = M(0) + \int_0^t \Gamma(u) \cdot dW(u), 0 \leq t \leq T$$

If, in addition, we assume the notation and assumptions of Theorem 5.4.1 and if $\widetilde{M}(t), 0 \leq t \leq T$, is a $\widetilde{\mathbb{P}}$-martingale, then there is an adapted, d-dimensional process $\widetilde{\Gamma}(u) = (\widetilde{\Gamma_1}(u), \ldots, \widetilde{\Gamma_d}(u))$ such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u) \cdot d\widetilde{W}(u), 0 \leq t \leq T$$

[1]

Multidimensional Market Model

Assume the existence of m stocks, each with a stochastic differential

$$dS_i(t) = \alpha_i(t) S_i(t) dt + S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t), i = 1, \ldots, m$$

In this representation, the σ_{ij}'s denote entries in the covariance matrix of a d-dimensional Brownian motion. Note that if $d = m$ and $\sigma_{ij} = 0$ for $i \neq j$ then each stock price is uncorrelated.

Definition 5.4.3: A probability measure $\tilde{\mathbb{P}}$ is said to be risk-neutral if

1. $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent (i.e. for every $A \in \mathcal{F}, \mathbb{P}(A) = 0$ if and only if $\tilde{\mathbb{P}}(A) = 0$)
2. under $\tilde{\mathbb{P}}$, the discounted stock price $D(t)S_i(t)$ is a martingale for every $i = 1, \ldots, m$

[1]

©2020 School of Business

Lemma 5.4.5: Let $\tilde{\mathbb{P}}$ be a risk-neutral measure, and let $X(t)$ be the value of a portfolio. Under $\tilde{\mathbb{P}}$, the discounted portfolio $D(t)X(t)$ is a martingale.[1]

Definition 5.4.6: An arbitrage is a portfolio value process $X(t)$ satisfying $X(0) = 0$ and also satisfying for some time $T > 0$

$$\mathbb{P}(X(T) \geq 0) = 1, \mathbb{P}(X(T) > 0) > 0$$

[1]

Theorem 5.4.7: (First fundamental theorem of asset pricing) If a market model has a risk-neutral probability measure, then it does not admit arbitrage.[1]

Definition 5.4.8: A market model is complete if every derivative security can be hedged.[1]

©2020 School of Business

Theorem 5.4.9: (Second fundamental theorem of asset pricing) Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.[1]

Continuously Paying Dividends

For a continuously dividend paying process, $A(t)$, we can express as the model of our stock price

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) - A(t)S(t)dt$$

This would result in the portfolio dynamics given by

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + \Delta(t)A(t)S(t)dt + R(t)[X(t) - \Delta(t)S(t)]dt \\ &= R(t)X(t)dt + \Delta(t)S(t)\sigma(t)\left[\Theta(t)dt + dW(t)\right] \end{aligned}$$

Lump Payments of Dividends

For given times $0 < t_1 < t_2 < \cdots < t_n < T$ where a payment is made and the payment at time t_j is given by $a_j S(t_{j-})$. This gives us the price of the stock at time t_j given by

$$S(t_j) = S(t_{j-}) - a_j S(t_{j-}) = (1 - a_j) S(t_{j-})$$

Despite these payments from the stock, the portfolio doesn't see jumps in its value (as the dividend is paying into the portfolio). This means that we still have the dynamic

$$dX(t) = R(t)X(t)dt + \Delta(t)\sigma(t)S(t)\left[\Theta(t)dt + dW(t)\right]$$

Definition 5.6.1: A forward contract is an agreement to pay a specified delivery price K at a delivery time T, where $0 \leq T \leq \bar{T}$, for the asset whose price at time t is $S(t)$. The T-forward price $For_S(t, T)$ of this asset at time t, where $0 \leq t \leq T \leq \bar{T}$, is the value of K that makes the forward contract have no-arbitrage price zero at time t.[1]

Theorem 5.6.2: Assume that zero-coupon bonds of all maturities can be traded. Then

$$For_S(t, T) = \frac{S(t)}{B(t, T)}, 0 \leq t \leq T \leq \bar{T}$$

[1]

©2020 School of Business

Definition 5.6.4: The futures price of an asset whose value at time T is $S(T)$ is given by the formula

$$Fut_S(t,T) = \widetilde{\mathbb{E}}[S(T)|\mathcal{F}(t)], 0 \le t \le T$$

A long position in the futures contract is an agreement to receive as a cash flow the changes in the futures price (which may be negative as well as positive) during the time the position is held. A short position in the futures contract receives the opposite cash flow.[1]

Theorem 5.6.5: The futures price is a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$, it satisfies $Fut_S(T,T) = S(T)$, and the value of a long (or a short) futures position to be held over an interval of time is always zero.[1]

©2020 School of Business

Forward-Futures Spread

The **forward-futures spread** is given by:

$$For_S(0,T) - Fut_S(0,T) = \frac{1}{B(0,T)} \widetilde{Cov}(D(T), S(T))$$

if the interest rate is a constant, r, then the covariance will be 0 (as D(T) will be deterministic) and the forward price and the future price will agree.

©2020 School of Business

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

STEVENS INSTITUTE of TECHNOLOGY
School of Business

FE 610

PDE's and SDE's

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

August 21, 2020

For academic use only.

©2020 School of Business

A stochastic differential equation is of the form

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u)$$

where $\beta(u, X(u))$ is called the drift and $\gamma(u, X(u))$ is called the diffusion of the process. Additionally, because it is a differential equation, a boundary condition is needed. This is given by the initial condition

$$X(t) = x, t \geq 0, x \in \mathcal{R}$$

©2020 School of Business

The goal is then to find the process $X(T)$, $T \geq t$ such that

$$X(t) = x$$

$$X(T) = X(t) + \int_t^T \beta(u, X(u))du + \int_t^T \gamma(u, X(u))dW(u)$$

Note that $X(T)$ is $\mathcal{F}(T)$-measurable.

One type of solvable SDE is the **one-dimensional linear stochastic differential equation**

$$dX(u) = (a(u) + b(u)X(u))du + (\gamma(u) + \sigma(u)X(u))dW(u)$$

where a, b, γ, and σ are nonrandom functions of time. We will look at some examples of this type of SDE and discuss their solutions.

Recall the formula

$$dS(u) = \alpha S(u)du + \sigma S(u)dW(u)$$

In this equation, $a(u) = 0, \gamma(u) = 0, b(u) = \alpha$, and $\sigma(u) = \sigma$. So the solution is found as:

$$S(T) = xe^{\sigma(W(T)-W(t))+\left(\alpha-\frac{1}{2}\sigma\right)(T-t)}$$

$$S(t) = x$$

Hull-White Interest Rate Model

$$dR(u) = (a(u) - b(u)R(u))du + \sigma(u)d\tilde{W}(u)$$

where a, b, and σ are nonrandom positive functions of time. To help with clarity, we use the dummy variable r instead of x when discussing interest rate processes. This can be solved with initial condition $R(t) = r$ as

$$R(T) = re^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv} a(u)du + \int_t^T e^{-\int_t^T b(v)dv} \sigma(u)d\tilde{W}(u)$$

©2020 School of Business

Slightly different from the Hull-White model, we have

$$dR(u) = (a - bR(u))du + \sigma\sqrt{R(u)}d\tilde{W}(u)$$

where a, b, and σ are positive constants. For this model, unlike the previous one, it is impossible for the interest rate to drop below 0. There is no formula for $R(T)$, there is a unique solution given an initial condition.

Given the stochastic differential equation from the beginning of class,

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u)$$

let $0 \leq t \leq T$ be given and let $h(y)$ be a Borel-measurable function. Denote

$$g(t, x) = \mathbb{E}^{t,x} h(X(T))$$

as the expectation of $h(X(T))$ where $X(T)$ is a solution to the SDE with initial condtion $X(t) = x$.

Euler's Method

If we don't have a formula for the distribution of $X(T)$, there are a multiple of ways to approximate the solution. One of which is **Euler's Method** in which, you approximate small steps and use Monte-Carlo simulations to determine the expected values. To do so:

1. Choose a small step size δ such that

$$X(t+\delta) = x + \beta(t,x)\delta + \gamma(t,x)\sqrt{\delta}\epsilon_1$$

2. Work through the process setting

$$X(t+(i+1)\delta) = X(t+i\delta)+\beta(t+i\delta, X(t+i\delta))\delta+\gamma(t+i\delta, X(t+i\delta))\sqrt{\delta}\epsilon_{i-1}$$

3. Repeat first two steps taking the average of results to get $\mathbb{E}[X(T)$ (or $\mathbb{E}[h(X(T))])$

©2020 School of Business

Theorem 6.3.1: Let $X(u), u \geq 0$, be a solution to the stochastic differential equation with initial condition given at time 0. Then, for $0 \leq t \leq T$,

$$\mathbb{E}[h(X(T))|\mathcal{F}(t)] = g(t, X(t))$$

[1]

Corollary 6.3.2: Solutions to stochastic differential equations are Markov processes.[1]

Theorem 6.4.1:(Feynman-Kac) Consider the stochastic differential equation

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u)$$

Let $h(y)$ be a Borel-measurable function. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

$$g(t, x) = \mathbb{E}^{t,x}[h(X(T))]$$

Then $g(t, x)$ satisfies the partial differential equation

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0$$

and the terminal condition

$$g(T, x) = h(x), \forall x$$

[1]

Lemma 6.4.2: Let $X(u)$ be a solution to the stochastic differential equation with initial condition given at time 0. Let $h(y)$ be a Borel-measurable function, fix $T > 0$, and let $g(t, x)$ be given as in the previous theorem. Then the stochastic process

$$g(t, X(t)), 0 \leq t \leq T$$

is a martingale.[1]

General principle behind the proof of the Feynman-Kac is:

1. find the martingale
2. take the differential
3. set the dt term equal to zero

Theorem 6.4.3: (Discounted Feynman-Kac) Consider the stochastic differential equation

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u)$$

Let $h(y)$ be a Borel-measurable function and let r be a constant. Fix $T > 0$, and let $t \in [0, T]$ be given. Define the function

$$f(t, x) = \mathbb{E}^{t,x}[e^{-r(T-t)}h(X(T))]$$

Then $f(t, x)$ satisfies the partial differential equation

$$f_t(t, x) + \beta(t, x)f_x(t, x) + \frac{1}{2}\gamma^2(t, x)f_{xx}(t, x) = rf(t, x)$$

and the terminal condition

$$f(T, x) = h(x), \forall x$$

[1]

Option Pricing Using GGBM

Given motion described as

$$dS(u) = rS(u)du + \sigma(u, S(u))S(u)d\tilde{W}(u)$$

This leads us to the PDE

$$v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2(t,x)x^2 v_{xx}(t,x) = rv(t,x)$$

If we solve this numerically, we determine option prices based on the parameters. We can also calculate the **implied volatility** based on the option price we determine. If the value of an option (for example a European call) based on this model is X then, the implied volatility is the value of the parameter σ^2 such that:

$$c_{BS}(t,x;r,\sigma^2,K) = X$$

For the simplest interest rate models, we will use the one factor equation:

$$dR(t) = \beta(t, R(t))dt + \gamma(t, R(t))d\tilde{W}(t)$$

The discount process is

$$D(t) = e^{-\int_0^t R(s)ds}$$

and the **money market account price process** is

$$\frac{1}{D(t)} = e^{\int_0^t R(s)ds}$$

This leads us to the formulas

$$dD(t) = -R(t)D(t)dt, d\left(\frac{1}{D(t)}\right) = \frac{R(t)}{D(t)}dt$$

Zero-Coupon Bond

A **zero-coupon bond** is a contract promising to pay a certain "face" amount, which we take to be 1, at a fixed maturity date T.[1]

For $0 \leq t \leq T$, the price of the bond ($B(t, T)$) will satisfy

$$D(t)B(t,T) = \tilde{\mathbb{E}}[D(T)|\mathcal{F}(t)]$$

Which gives us the formula

$$B(t,T) = \tilde{\mathbb{E}}[e^{-\int_t^T R(s)ds}|\mathcal{F}(t)]$$

We can now define the **yield** between times t and T as

$$Y(t,T) = -\frac{1}{T-t}\log B(t,T)$$

Because R is given by a stochastic differential equation, it must be Markov, and so

$$B(t, T) = f(t, R(t))$$

for some function $f(t, r)$. To find the partial differential equation for this function, we will need to find a martingale, determine its differential, and set its dt term equal to 0.

$$\begin{aligned} d(D(t)f(t, R(t))) &= f(t, R(t))dD(t) + D(t)df(t, R(t)) \\ &= D(t)\left[-Rfdt + f_t dt + f_r dR + \frac{1}{2} f_{rr} dRdR\right] \\ &= D(t)\left[-Rf + f_t + \beta f_r + \frac{1}{2}\gamma^2 f_{rr}\right] dt + D(t)\gamma f_r d\tilde{W} \end{aligned}$$

Hull-White Interest Rate Model

In this model, the evolution of the interest rate is given by

$$dR(t) = (a(t) - b(t)R(t))dt + \sigma(t)d\tilde{W}(t)$$

The PDE for the zero-coupon bond is then

$$f_t(t,r) + (a(t) - b(t)r)f_r(t,r) + \frac{1}{2}\sigma^2(t)f_{rr}(t,r) = rf(t,r)$$

This has an explicit formula solution as a function of the interest rate

$$B(t,T) = e^{-R(t)C(t,T)-A(t,T)}, 0 \leq t \leq T$$

$$A(t,T) = \int_t^T \left(a(s)C(s,T) - \frac{1}{2}\sigma^2(s)C^2(s,T) \right) ds$$

$$C(t,T) = \int_t^T e^{-\int_t^s b(v)dv} ds$$

Cox-Ingersoll-Ross Interest Rate Model

The evolution for the interest rate in this model is given as

$$dR(t) = (a - bR(t))dt + \sigma\sqrt{R(t)}d\tilde{W}(t)$$

Which has the PDE

$$f_t(t,r) + (a - br)f_r(t,r) + \frac{1}{2}\sigma^2 r f_{rr}(t,r) = rf(t,r)$$

This solution will also be of the form

$$f(t,r) = e^{-rC(t,T)-A(t,T)}$$

but with different $A(t,T)$ and $C(t,T)$ from the previous problem.

Multidimensional Feynman-Kac

Let $W(t) = (W_1(t), W_2(t))$ be a two-dimensional Brownian Motion. Consider the two SDE's

$$\begin{aligned} dX_1(u) &= \beta_1(u, X_1(u), X_2(u))du + \gamma_{1,1}(u, X_1(u), X_2(u))dW_1(u) \\ &\quad + \gamma_{1,2}(u, X_1(u), X_2(u))dW_2(u) \\ dX_2(u) &= \beta_2(u, X_1(u), X_2(u))du + \gamma_{2,1}(u, X_1(u), X_2(u))dW_1(u) \\ &\quad + \gamma_{2,2}(u, X_1(u), X_2(u))dW_2(u) \end{aligned}$$

The solution to this pair of SDE's, starting at initial conditions $X_1(t) = x_1$ and $X_2(t) = x_2$, depends on initial time t and positions x_1 and x_2.[1]

Let $h(y_1, y_2)$ be given. Corresponding to the initial conditions t, x_1, x_2 where $0 \le t \le T$, we define

$$g(t, x_1, x_2) = \mathbb{E}^{t,x_1,x_2}[h(X_1(T), X_2(T))]$$

$$f(t, x_1, x_2) = \mathbb{E}^{t,x_1,x_2}\left[e^{-r(T-t)}h(X_1(T), X_2(T))\right]$$

Then

$$g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} + \frac{1}{2}(\gamma_{1,1}^2 + \gamma_{1,2}^2)g_{x_1,x_1}$$
$$+ (\gamma_{1,1}\gamma_{2,1} + \gamma_{1,2}\gamma_{2,2})g_{x_1,x_2} + \frac{1}{2}(\gamma_{2,1}^2 + \gamma_{2,2}^2)g_{x_2,x_2} = 0$$

$$f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} + \frac{1}{2}(\gamma_{1,1}^2 + \gamma_{1,2}^2)f_{x_1,x_1}$$
$$+ (\gamma_{1,1}\gamma_{2,1} + \gamma_{1,2}\gamma_{2,2})f_{x_1,x_2} + \frac{1}{2}(\gamma_{2,1}^2 + \gamma_{2,2}^2)f_{x_2,x_2} = rf$$

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

School of Business

FE 610

Poisson Processes and Jump Diffusion

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

April 14, 2022

For academic use only.

Exponential Random Variables

Let τ be a random variable with pdf

$$f_\tau(t) = \begin{cases} \lambda e^{-\lambda t}, t \geq 0 \\ 0, t < 0 \end{cases}$$

We have $\mathbb{E}[\tau] = \frac{1}{\lambda}$ and cdf

$$F_\tau(t) = \mathbb{P}(\tau \leq t) = 1 - e^{-\lambda t}, t \geq 0$$

This is a memoryless random variable, or

$$\mathbb{P}(\tau > t) = \mathbb{P}(\tau > t + s | \tau > s)$$

Construct a sequence of i.i.d. exponential random variables $\{\tau_i\}$ with parameter λ. The "jump" times can then be defined as

$$S_n = \sum_{i=1}^{n} \tau_i$$

We have a resulting Poisson process that we call $N(t)$ that counts the number of jumps before a given time

$$N(t) = \begin{cases} 0, 0 \leq t < S_1 \\ 1, S_1 \leq t < S_2 \\ \vdots \\ n, S_n \leq t < S_{n+1} \\ \vdots \end{cases}$$

Lemma 11.2.2 The Poisson process $N(t)$ with intensity λ has the distribution

$$\mathbb{P}(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, k = 0, 1, \ldots$$

[1]

Theorem 11.2.3: Let $N(t)$ be a Poisson process with intensity $\lambda > 0$, and let $0 = t_0 < t_1 < \cdots < t_n$ be given. Then the increments

$$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$$

are stationary and independent, and

$$\mathbb{P}(N(t_{j+1}) - N(t_j) = k) = \frac{\lambda^k (t_{j+1} - t_j)^k}{k!} e^{-\lambda(t_{j+1} - t_j)}, k = 0, 1, \ldots$$

[1]

Poisson Increments

As a result of Theorem 11.2.3, we have the following results about our Poisson increments for $0 \leq s < t$

$$\mathbb{P}(N(t) - N(s) = k) = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}, k = 0, 1, \ldots$$

$$\mathbb{E}[N(t) - N(s)] = \lambda(t-s)$$

$$\mathbb{V}[N(t) - N(s)] = \lambda(t-s)$$

Martingale Property

Theorem 11.2.4: Let $N(t)$ be a Poisson process with intensity λ. We define the compensated Poisson process

$$M(t) = N(t) - \lambda t$$

Then $M(t)$ is a martingale.[1] Proof: Let $0 \le s < t$ be given.

$$\begin{aligned}
\mathbb{E}[M(t)|\mathcal{F}(s)] &= \mathbb{E}[M(t) - M(s)|\mathcal{F}(s)] + \mathbb{E}[M(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[N(t) - N(s) - \lambda(t-s)|\mathcal{F}(s)] + M(s) \\
&= \mathbb{E}[N(t) - N(s)] - \lambda(t-s) + M(s) \\
&= M(s)
\end{aligned}$$

Compound Poisson Process

Let $N(t)$ be a Poisson process with intensity λ, and let $\{Y_i\}$ be a sequence of i.i.d. random variables with $\mathbb{E}[Y_i] = \beta$ that are independent of both each other and of the Poisson process. We define the **compound Poisson process** as

$$Q(t) = \sum_{i=1}^{N(t)} Y_i, t \geq 0$$

Using this definition and previous results we have

$$\mathbb{E}[Q(t)] = \beta\lambda t$$

Theorem 11.3.1: Let $Q(t)$ be the compound Poisson process. Then the compensated Poisson process

$$Q(t) - \beta\lambda t$$

is a martingale.[1]

Theorem 11.3.2: Let $Q(t)$ be a compound Poisson process and let $0 = t_0 < t_1 < \cdots < t_n$ be given. The increments

$$Q(t_1) - Q(t_0), Q(t_2) - Q(t_1), \ldots, Q(t_n) - Q(t_{n-1})$$

are independent and stationary. In particular, the distribution of $Q(t_j) - Q(t_{j-1})$ is the same as the distribution of $Q(t_j - t_{j-1})$[1]

Definition 11.4.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{F}(t), t \geq 0$, be a filtration on this space. We say that a Brownian motion $W(t)$ is a Brownian motion relative to this filtration if $W(t)$ is $\mathcal{F}(t)$-measurable for every t and for every $u > t$ the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$. Similarly, we say that a Poisson process N is a Poisson process relative to this filtration if $N(t)$ is $\mathcal{F}(t)$-measurable for every t and for every $u > t$ the increment $N(u) - N(t)$ is independent of $\mathcal{F}(t)$. Finally, we say that a compound Poisson process Q is a compound Poisson process relative to this filtration if $Q(t)$ is $\mathcal{F}(t)$-measurable for every t and for every $u > t$ the increment $Q(u) - Q(t)$ is independent of $\mathcal{F}(t)$.[1]

We need to define the stochastic integral

$$\int_0^t \Phi(s)dX(s)$$

where X can have jumps. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which $\mathcal{F}(t), t \geq 0$, is a filtration. All processes are adapted to this filtration from the previous definition and will be right-continuous and can be expressed as

$$X(t) = X(0) + I(t) + R(t) + J(t)$$

A **pure jump process** $J(t)$ is an adapted, right-continuous process with $J(0) = 0$ and $J(t) = \lim_{s \downarrow t} J(s)$

Definition 11.4.2: A process $X(t)$ of the form in the previous slide, with Itô integral part $I(t)$, Riemann integral part $R(t)$, and pure jump part $J(t)$ will be called a jump process. The continuous part of this process is $X^c(t) = X(0) + I(t) + R(t)$.[1]

Definition 11.4.3: Let $X(t)$ be a jump process of the previous form and let $\Phi(s)$ be an adapted process. The stochastic integral of Φ with respect to X is defined to be

$$\begin{aligned}\int_0^t \Phi(s)dX(s) = \int_0^t \Phi(s)\Gamma(s)dW(s) + \int_0^t \Phi(s)\Theta(s)ds \\ + \sum_{0<s\leq t} \Phi(s)\Delta J(s)\end{aligned}$$

or in differential notation,

$$\begin{aligned}\Phi(t)dX(t) &= \Phi(t)dI(t) + \Phi(t)dR(t) + \Phi(t)dJ(t) \\ &= \Phi(t)dX^c(t) + \Phi(t)dJ(t)\end{aligned}$$

[1]

Theorem 11.4.5: Assume that the jump process $X(s)$ is a martingale, the integrand $\Phi(s)$ is left-continuous and adapted, and

$$\mathbb{E}\left[\int_0^t \Gamma^2(s)\Phi^2(s)ds\right] < \infty, \forall t \geq 0$$

Then the stochastic integral $\int_0^t \Phi(s)dX(s)$ is also a martingale.

Quadratic Variation

For a partition Π, we denote, as usual, the sample quadratic variation of a process as:

$$Q_{\Pi}(X) = \sum_{j=0}^{n-1}(X(t_{j+1}) - X(t_j))^2$$

And then to get the actual quadratic variation of the process $X(t)$, we let $||\Pi|| \to 0$ to get $[X, X](t)$. For the processes $X_1(t)$ and $X_2(t)$ we define the sample cross variation as

$$C_{\Pi}(X) = \sum_{j=0}^{n-1}(X_1(t_{j+1}) - X_1(t_j))(X_2(t_{j+1}) - X_2(t_j))$$

Theorem 11.4.7: Let $X_1(t) = X_1(0) + I_1(t) + R_1(t) + J_1(t)$ be a jump process, where $I_1(t) = \int_0^t \Gamma_1(s)dW(s)$, $R_1(t) = \int_0^t \Theta(s)ds$, and $J_1(t)$ is a right continuous pure jump process. Then $X_1^c(t) = X_1(0) + I_1(t) + R_1(t)$ and

$$
\begin{aligned}
[X_1, X_1](T) &= [X_1^c, X_1^c](T) + [J_1, J_1](T) \\
&= \int_0^T \Gamma_1^2(s)ds + \sum_{0<s\leq T} (\Delta J_1(s))^2
\end{aligned}
$$

Define $X_2(t)$ similarly to $X_1(t)$.

$$
\begin{aligned}
[X_1, X_2](T) &= [X_1^c, X_2^c](T) + [J_1, J_2](T) \\
&= \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\leq T} \Delta J_1(s)\Delta J_2(s)
\end{aligned}
$$

[1]

Ito Formula for One Jump Process

Theorem 11.5.1: Let $X(t)$ be a jump process and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined and continuous. Then

$$f(X(t)) = f(0, X(0)) + \int_0^t f'(X(s))dX^c(s)$$
$$+ \frac{1}{2}\int_0^t f''(X(s))dX^c(s)dX^c(s) + \sum_{0<s\leq T} [f(X(s)) - f(X(s-))]$$

Corollary 11.5.3: Let $W(t)$ be a Brownian motion and let $N(t)$ be a Poisson process with intensity $\lambda > 0$, both defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and relative to same filtration $\mathcal{F}(t), t \geq 0$. Then the processes $W(t)$ and $N(t)$ are independent.

Two Dimensional Ito Formula

Theorem 11.5.4: Let $X_1(t)$ and $X_2(t)$ be jump processes, and let $f(t, x_1, x_2)$ be a function whose first and second partial derivatives appearing in the following formula are defined and continuous. Then

$$
\begin{aligned}
f(t, X_1(t), X_2(t)) = {} & f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))ds \\
& + \int_0^t f_{x_1}(s, X_1(s), X_2(s))dX_1^c(s) + \int_0^t f_{x_2}(s, X_1(s), X_2(s))dX_2^c(s) \\
& + \frac{1}{2}\int_0^t f_{x_1,x_1}(s, X_1(s), X_2(s))dX_1^c(s)dX_1^c(s) \\
& + \int_0^t f_{x_1,x_2}(s, X_1(s), X_2(s))dX_1^c(s)dX_2^c(s) \\
& + \frac{1}{2}\int_0^t f_{x_2,x_2}(s, X_1(s), X_2(s))dX_2^c(s)dX_2^c(s) \\
& + \sum_{0<s\leq t} [f(s, X_1(s), X_2(s)) - f(s-, X_1(s-), X_2(s-))]
\end{aligned}
$$

[1]

Ito Product Rule for Jump Processes

Corollary 11.5.5: Let $X_1(t)$ and $X_2(t)$ be jump processes, Then

$$
\begin{aligned}
X_1(t)X_2(t) &= X_1(0)X_2(0) + \int_0^t X_2(s)dX_1^c(s) + \int_0^t X_1(s)dX_2^c(s) \\
&+ [X_1^c, X_2^c](t) + \sum_{0<s\leq t} [X_1(s)X_2(s) - X_1(s-)X_2(s-)] \\
&= X_1(0)X_2(0) + \int_0^t X_2(s-)dX_1(s) + \int_0^t X_1(s-)dX_2(s) \\
&+ [X_1, X_2](t)
\end{aligned}
$$

[1]

Corollary 11.5.6: Let $X(t)$ be a jump process. The **Doleans-Dade exponential** of X is defined to be the process

$$Z^X(t) = e^{X^c(t) - \frac{1}{2}[X^c, X^c](t)} \prod_{0 < s \leq t} (1 + \Delta X(s))$$

This process is the solution to the stochastic differential equation

$$dZ^X(t) = Z^X(t-)dX(t)$$

with initial condition $Z^X(0) = 1$, which in integral form is

$$Z^X(t) = 1 + \int_0^t Z^X(s-)dX(s)$$

[1]

Let $\tilde{\lambda}$ be a positive number and define:

$$Z(t) = e^{(\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda} \right)^{N(t)}$$

Fix a time $T > 0$, and we will use $Z(T)$ to change to measure $\tilde{\mathbb{P}}$ under which $N(t)$ has intensity $\tilde{\lambda}$ instead of λ.

Lemma 11.6.1: The process $Z(t)$ satisfies:

$$dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} Z(t_-) dM(t)$$

In particular, Z(t) is a martingale under $\mathbb{P}$ and $\mathbb{E}[Z(t)] = 1$ for all t.[1]

Theorem 11.6.2 (Change of Poisson Intensity): Under the probability measure $\tilde{\mathbb{P}}$, the process $N(t)$, $0 \leq t \leq T$, is Poisson with intensity $\tilde{\lambda}$.[1]

Define:

$$Z(t) = e^{(\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda} \right)^{N(t)}$$

Theorem 11.6.4: The process $Z(t)$ is a martingale. In particular, $\mathbb{E}[Z(t)] = 1$ for all t.

Theorem 11.6.5: Under $\widetilde{\mathbb{P}}$, $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda}$

Lemma 11.6.6: For $Z(t)$ given by

$$Z(t) = e^{(\lambda-\tilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$$

this process is a martingale. In particular, $\mathbb{E}[Z(t)] = 1$ for all $t \geq 0$.

Define:

$$Z_1(t) = e^{-\int_o^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du}$$

$$Z_2(t) = e^{(\lambda-\tilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$$

$$Z(t) = Z_1(t)Z_2(t)$$

Lemma 11.6.8: The process $Z(t)$ is a martingale. In particular, $\mathbb{E}[Z(t)] = 1$ for all $t \geq 0$.[1]

Theorem 11.6.9: Under the probability measure $\widetilde{\mathbb{P}}$, the process

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(s)ds$$

is a Brownian motion, $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda}$ and independent, identically distributed jump sizes having density $\tilde{f}(y)$, and the processes $\widetilde{W}(t)$ and $Q(t)$ are independent.[1]

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.

FE 610

Exotic Options

Thomas Lonon
Financial Engineering
Stevens Institute of Technology

August 21, 2020

For academic use only.

©2020 School of Business

Define:

$$\widehat{W}(t) = \alpha t + \widetilde{W}(t), 0 \le t \le T$$

$$\widehat{M}(T) = \max_{0 \le t \le T} \widehat{W}(t)$$

Theorem 7.2.1: The joint density under $\widetilde{\mathbb{P}}$ of the pair $(\widehat{M}(T), \widehat{W}(T))$ is

$$\tilde{f}_{\widehat{M}(T),\widehat{W}(T)}(m, w) = \frac{2(2m - w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, w \le m, m \ge 0$$

and is zero for other values of m and w[1]

Corollary 7.2.2: We have

$$\widetilde{\mathbb{P}}\{\widehat{M}(T) \leq m\} = N\left(\frac{m - \alpha T}{\sqrt{T}}\right) - e^{2\alpha m} N\left(\frac{-m - \alpha T}{\sqrt{T}}\right), m \geq 0$$

and the density under $\widetilde{\mathbb{P}}$ of the random variable $\widehat{M}(T)$ is

$$\tilde{f}_{\widehat{M}(T)}(m) = \frac{2}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2\alpha m} N\left(\frac{-m - \alpha T}{\sqrt{T}}\right), m \geq 0$$

and is zero for $m < 0$.[1]

Given a process $\widehat{W}(t)$ and a risky asset given by

$$S(t) = S(0)e^{\sigma\widetilde{W}(t)+(r-\frac{1}{2}\sigma^2)t} = S(0)e^{\sigma\widehat{W}(t)}$$

with $\alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$. Using our process $\widehat{M}(t)$ we then have

$$\max_{0\leq t\leq T} S(t) = S(0)e^{\sigma\widehat{M}(t)}$$

This option will have pay off given by

$$V(T) = \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)_+ \mathbb{I}_{\{S(0)e^{\sigma\widehat{M}(T)}\leq B\}}$$

Theorem 7.3.1: Let $v(t, x)$ denote the price at time t of the up-and-out call under the assumption that the call has not knocked out prior to time t and $S(t) = x$. Then $v(t, x)$ satisfies the Black-Scholes-Merton partial differential equation

$$v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rv(t, x)$$

in the rectangle $\{(t, x); 0 \leq t < T, 0 \leq x \leq B\}$ and satisfies the boundary conditions

$$\begin{aligned} v(t, 0) &= 0, 0 \leq t \leq T, \\ v(t, B) &= 0, 0 \leq t < T, \\ v(T, x) &= (x - K)_+, 0 \leq x \leq B \end{aligned}$$

[1]

Lemma 7.3.2: We have

$$V(t) = v(t, S(t)), 0 \leq t \leq \rho$$

In particular, $e^{-rt}v(t, S(t))$ is a $\widetilde{\mathbb{P}}$-martingale up to time ρ, or, put another way, the stopped process

$$e^{-r(t\wedge\rho)}v(t \wedge \rho, S(t \wedge \rho)), 0 \leq t \leq T$$

is a martingale under $\widetilde{\mathbb{P}}$.[1]

$$
\begin{aligned}
v(t,x) = & x\left[N\left(\delta_+\left(\tau,\frac{x}{K}\right)\right) - N\left(\delta_+\left(\tau,\frac{x}{B}\right)\right)\right] \\
& - e^{-rt}K\left[N\left(\delta_-\left(\tau,\frac{x}{K}\right)\right) - N\left(\delta_-\left(\tau,\frac{x}{B}\right)\right)\right] \\
& - B\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(\tau,\frac{B^2}{Kx}\right)\right) - N\left(\delta_+\left(\tau,\frac{B}{x}\right)\right)\right] \\
& e^{-r\tau}K\left(\frac{x}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(\tau,\frac{B^2}{Kx}\right)\right) - N\left(\delta_-\left(\tau,\frac{B}{x}\right)\right)\right], \\
& \text{for } 0 \le t < T, 0 < x \le B
\end{aligned}
$$

Let

$$Y(t) = \max_{0 \le u \le t} S(u) = S(0)e^{\sigma \widehat{M}(t)}$$

So the lookback option has payoff

$$V(T) = Y(T) - S(T)$$

This leads us to the valuation:

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(Y(T) - S(T))|\mathcal{F}(t)\right]$$

Because the two-dimensional process $(S(t), Y(t))$ is Markov there exists a function:

$$V(t) = \upsilon(t, S(t), Y(t))$$

Theorem 7.4.1: Let $v(t, x, y)$ denote the price at time t of the floating strike lookback option under the assumption that $S(t) = x$ and $Y(t) = y$. Then $v(t, x, y)$ satisfies the Black-Scholes-Merton partial differential equation

$$v_t(t, x, y) + rxv_x(t, x, y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x, y) = rv(t, x, y)$$

in the region $\{(t, x, y); 0 \leq t < T, 0 \leq x \leq y\}$ and satisfies the boundary conditions

$$\begin{aligned} v(t, 0, y) &= e^{-r(T-t)}y, 0 \leq t \leq T, y \geq 0 \\ v_y(t, y, y) &= 0, 0 \leq t \leq T, y > 0 \\ v(T, x, y) &= y - x, 0 \leq x \leq y \end{aligned}$$

[1]

$$
\begin{aligned}
\upsilon(t,x,y) = &\left(1+\frac{\sigma^2}{2r}\right) xN\left(\delta_+\left(\tau,\frac{x}{y}\right)\right) + e^{-r\tau} yN\left(-\delta_-\left(\tau,\frac{x}{y}\right)\right) \\
&- \frac{\sigma^2}{2r} e^{-r\tau}\left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}} xN\left(-\delta_-\left(\tau,\frac{y}{x}\right)\right) - x \\
&\text{for } 0 \le t < T, 0 < x \le y
\end{aligned}
$$

[1] S.E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Number v. 11.