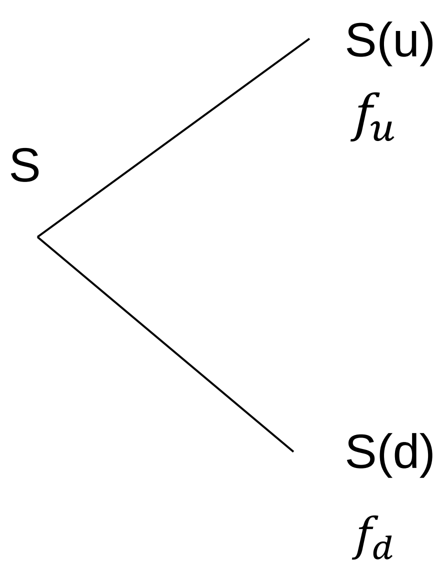
Risk Measurement

Risk neutral pricing - revisit

- Risk neutral pricing assumes market completeness
- Market completeness: the market contains sufficiently many traded instruments such that any claim can be replicated
- Example: the option can be replicated with a portfolio of cash and underlying stock

Binomial tree



In the binomial tree model, any payoff can be replicated with a portfolio of cash and stock

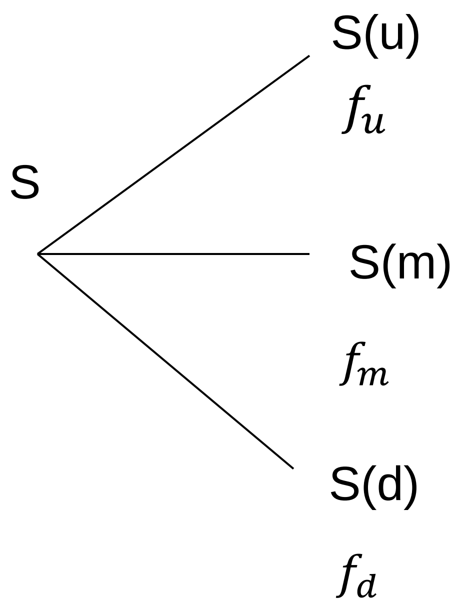
$$\Pi = C + \Delta \cdot S$$

$$C = \frac{u\, f_d - d\, f_u}{u - d} \quad \text{and} \quad \Delta = \frac{f_u - f_d}{S(u-d)}$$

$$\Pi(Su) = C + \Delta \cdot (Su)$$

$$\Pi(Sd) = C + \Delta \cdot (Sd)$$

Trinomial tree



A general payoff cannot be replicated with a portfolio of cash and stock

$$\Pi = C + \Delta \cdot S$$

Since we cannot have simultaneously

$$\Pi(Su) = C + \Delta \cdot (Su)$$

$$\Pi(Sm) = C + \Delta \cdot (Sm)$$

$$\Pi(Sd) = C + \Delta \cdot (Sd)$$

The trinomial tree model describes an incomplete market!

Examples of incomplete markets

- Markets with stochastic volatility. We think of volatility as an independent dynamical asset.
- Such a market will be complete only if we add instruments depending on the (instantaneous) volatility of an asset

Unhedged risk

- We cannot always hedge away all risk by forming riskless portfolios
- It is important to quantify and measure risk
- Current portfolio risk: measured by Greeks
- Future portfolio risk: measured by Value-at-Risk (VaR) and Expected Shortfall

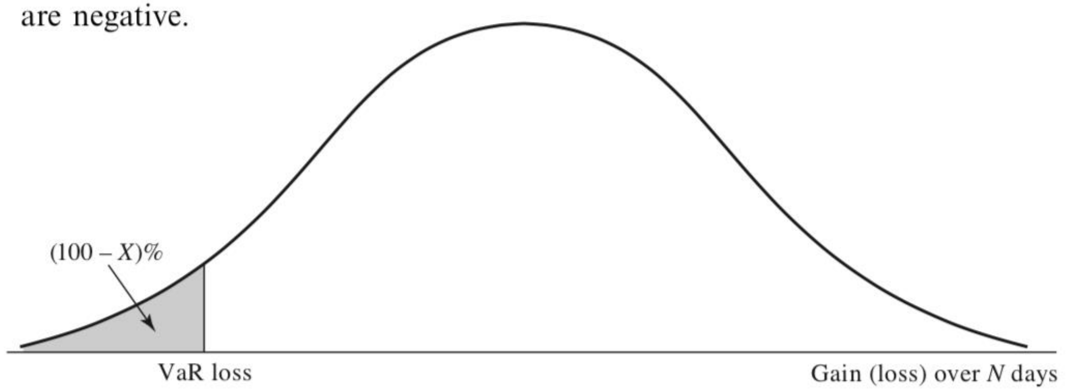
Outline

- Risk metrics: Value-at-Risk (VaR) and Expected Shortfall (ES)
- Regulatory aspects of VaR
- Methods for computing VaR/ES:
  - Historical simulation
  - Model-building approach
- VaR for portfolios of correlated assets

The Question Being Asked in VaR

"What loss level is such that we are X% confident it will not be exceeded in N business days?"

Figure 22.1 Calculation of VaR from the probability distribution of the change in the portfolio value; confidence level is X%. Gains in portfolio value are positive; losses are negative.



VaR vs. Expected Shortfall

- VaR is the loss level that will not be exceeded with a specified probability
- Expected Shortfall (or C-VaR) is the expected loss given that the loss is greater than the VaR level
- Recently there is a move away from VaR to ES in setting bank capital requirements

VaR and ES

- VaR captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: “How bad can things get?”
- ES answers the question: “If things do get bad, just how bad will they be”

VaR/ES specification

- The risk measures depend on two parameters:
  - Time horizon N: 1 Day, 10 Days, etc.
  - Confidence Level X: 95% VaR, 99% VaR
- Denoting *f(L)* the pdf of the loss *L* and *F(L)* the CDF of the loss, we have

$$VaR(X) = F^{-1}(X) \quad \text{or} \quad 1 - X = \int_{VaR(X)}^{\infty} f(L)dL$$

$$ES(X) = \frac{\int_{VaR(X)}^{\infty} Lf(L)dL}{\int_{VaR(X)}^{\infty} f(L)dL} = \frac{1}{1-X}\int_{VaR(X)}^{\infty} Lf(L)dL$$

Methods for computing VaR

- Two main methods for computing VaR/ES:
  - *Historical simulation*. Use past data to imply the probability distribution of the one day moves in the portfolio price
  - *Model-building approach*. Assume that the market moves have a known probability distribution, and use it to determine the price changes of the portfolio

Historical Simulation to Calculate the One-Day VaR or ES

- Create a database of the previous daily movements ($i=1:n$) in all market variables.
- The first simulation trial assumes that the percentage changes in all market variables are as on the day $i=1$
- The second simulation trial assumes that the percentage changes in all market variables are as on the day $i=2$
- and so on

Historical Simulation continued

- Suppose we use 501 days of historical data (Day 0 to Day 500)
- Let v_i be the value of a variable on day i
- There are 500 simulation trials
- The ith trial assumes that the value of the market variable tomorrow is

$$v_{500} \frac{v_i}{v_{i-1}}$$

Example : Calculation of 1-day, 99% VaR or ES for a Portfolio on Sept 25, 2008 *(Table 22.1, page 497)*

Index	*Value ($000s)*
DJIA	4,000
FTSE 100	3,000
CAC 40	1,000
Nikkei 225	2,000

US dollar equivalent of stock indices *(Table 22.2, page 497)*

Day	*Date*	*DJIA*	*FTSE 100*	*CAC 40*	*Nikkei 225*
0	Aug 7, 2006	11,219.38	11,131.84	6,373.89	131.77
1	Aug 8, 2006	11,173.59	11,096.28	6,378.16	134.38
2	Aug 9, 2006	11,076.18	11,185.35	6,474.04	135.94
3	Aug 10, 2006	11,124.37	11,016.71	6,357.49	135.44
…	……	…..	…..	……	……
499	Sep 24, 2008	10,825.17	9,438.58	6,033.93	114.26
500	Sep 25, 2008	11,022.06	9,599.90	6,200.40	112.82

Scenarios Generated (Table 22.3, page 498)

$$11{,}022.06 \times \frac{11{,}173.59}{11{,}219.38} = 10{,}977.08$$

Scenario	*DJIA*	*FTSE 100*	*CAC 40*	*Nikkei 225*	*Portfolio Value ($000s)*	*Loss ($000s)*
1	10,977.08	9,569.23	6,204.55	115.05	10,014.334	−14.334
2	10,925.97	9,676.96	6,293.60	114.13	10,027.481	−27.481
3	11,070.01	9,455.16	6,088.77	112.40	9,946.736	53.264
…	…….	…….	…….	……..	…….	……..
499	10,831.43	9,383.49	6,051.94	113.85	9,857.465	142.535
500	11,222.53	9,763.97	6,371.45	111.40	10,126.439	−126.439

Ranked Losses *(Table 22.4, page 499)*

Scenario Number	*Loss ($000s)*
494	477.841
339	345.435
349	282.204
329	277.041
487	253.385
227	217.974
131	205.256

99% one-day VaR = $253,385

99% one-day ES = average of the five worst losses or $327,181

The N-day VaR or ES

- The N-day VaR (ES) for market risk is usually assumed to be $\sqrt{N}$ times the one-day VaR (ES)
- In our example the 10-day VaR would be calculated as $$\sqrt{10} \times 253{,}385 = 801{,}274$$
- This assumption is only perfectly theoretically correct if daily changes are normally distributed with zero mean and independent

Stressed VaR and Stressed ES

- In response to the 2008 financial crisis, regulators added also a requirement for stressed VaR and ES computation
- Stressed VaR and stressed ES calculations are based on historical data for a stressed period in the past (e.g. the year 2008) rather than on data from the most recent past (as in our example)

The Model-Building Approach

- The main alternative to historical simulation is to make assumptions about the probability distributions of the return on the market variables and calculate the probability distribution of the change in the value of the portfolio analytically
- This is known as the model building approach or the variance-covariance approach

Daily Volatilities

- In option pricing we measure volatility “per year”
- In VaR and ES calculations we measure volatility “per day”

$$\sigma_{\text{day}} = \frac{\sigma_{\text{year}}}{\sqrt{252}}$$

- This is the standard deviation of the stock price (portfolio value) percentage change in one day

Microsoft Example *(page 501)*

- We have a position worth $10 million in Microsoft shares
- The volatility of Microsoft is 2% per day (about 32% per year)
- We compute the N-day VaR with N=10 and confidence level X=99%

Microsoft Example continued

- The standard deviation of the change in the portfolio in 1 day is $200,000
- Assume that the expected change is zero (OK for short time periods) and the probability distribution of the change is normal, then:
- The 1-day 99% VaR is

$$200{,}000 \times 2.326 = \$465{,}300$$

- The 10-day 99% VaR is

$$\sqrt{10} \times 465{,}300 = 1{,}471{,}300$$

AT&T Example *(page 502)*

- Consider a position of $5 million in AT&T
- The daily volatility of AT&T is 1% (approx 16% per year)
- The 10-day 99% VaR is

$$\sqrt{10} \times 2.326 \times 50{,}000 = 367{,}800$$

Two stock portfolio

- Now consider a portfolio consisting of both Microsoft and AT&T
- Assume that the returns of AT&T and Microsoft are bivariate normal
- Suppose that the correlation between the returns is 0.3

S.D. of Portfolio

- A standard result in statistics states that

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$$

- In this case σ_X = 200,000 and σ_Y = 50,000 and ρ = 0.3. The standard deviation of the change in the portfolio value in one day is therefore 220,200

VaR for Portfolio

- The 10-day 99% VaR for the portfolio is

$$220{,}200 \times \sqrt{10} \times 2.326 = \$1{,}620{,}100$$

- The benefits of diversification are

  (1,471,300+367,800)–1,620,100=$219,00
- What is the incremental effect of the AT&T holding on VaR?

ES for the Model Building Approach

- When the loss over the time horizon has a normal distribution with mean μ and standard deviation σ, the ES is

$$\text{ES} = \mu + \sigma \frac{e^{-Y^2/2}}{\sqrt{2\pi}(1 - X)}$$

  where X is the confidence level and Y is the Xth percentile of a standard normal distribution
- For the Microsoft + AT&T portfolio ES is $1,856,100

The Linear Model

This assumes

- The daily change in the value of a portfolio is linearly related to the daily returns from market variables
- The returns from the market variables are normally distributed

Variance of Portfolio Returns

$$\text{Variance of Portfolio Return} = \sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij} w_i w_j \sigma_i \sigma_j$$

w_i is weight of ith instrument in portfolio

σ_i^2 is variance of return on ith instrument in portfolio

ρ_{ij} is correlation between returns of ith and jth instruments

Covariance Matrix ($var_i = cov_{ii}$)

$$C = \begin{pmatrix} \mathrm{var}_1 & \mathrm{cov}_{12} & \mathrm{cov}_{13} & \cdots & \mathrm{cov}_{1n} \\ \mathrm{cov}_{21} & \mathrm{var}_2 & \mathrm{cov}_{23} & \cdots & \mathrm{cov}_{2n} \\ \mathrm{cov}_{31} & \mathrm{cov}_{32} & \mathrm{var}_3 & \ldots & \mathrm{cov}_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathrm{cov}_{n1} & \mathrm{cov}_{n2} & \mathrm{cov}_{n3} & \ldots & \mathrm{var}_n \end{pmatrix}$$

$\mathrm{cov}_{ij} = \rho_{ij}\,\sigma_i\,\sigma_j$ where σ_i and σ_j are the SDs of the daily returns of variables i and j, and ρ_{ij} is the correlation between them

Handling Interest Rates

- How do we compute VaR for a portfolio of bonds or other interest rate sensitive products?
- Duration approach: Linear relation between ΔP and Δy but assumes parallel shifts)
- Recall that the bond price change ΔP is proportional to the change of the bond yield

$$\Delta P = -D\ P\ \Delta y$$

When a Linear Model Can be Used

We can use a linear model when the price change is linear in the underlying price change

- Portfolio of stocks
- Portfolio of bonds
- Forward contract on foreign currency
- Interest-rate swap

Monte Carlo Simulation *(page 511-512)*

To calculate VaR using MC simulation we

- Value portfolio today
- Sample once from the multivariate distributions of the Δx_i
- Use the Δx_i to determine market variables at end of one day
- Revalue the portfolio at the end of day

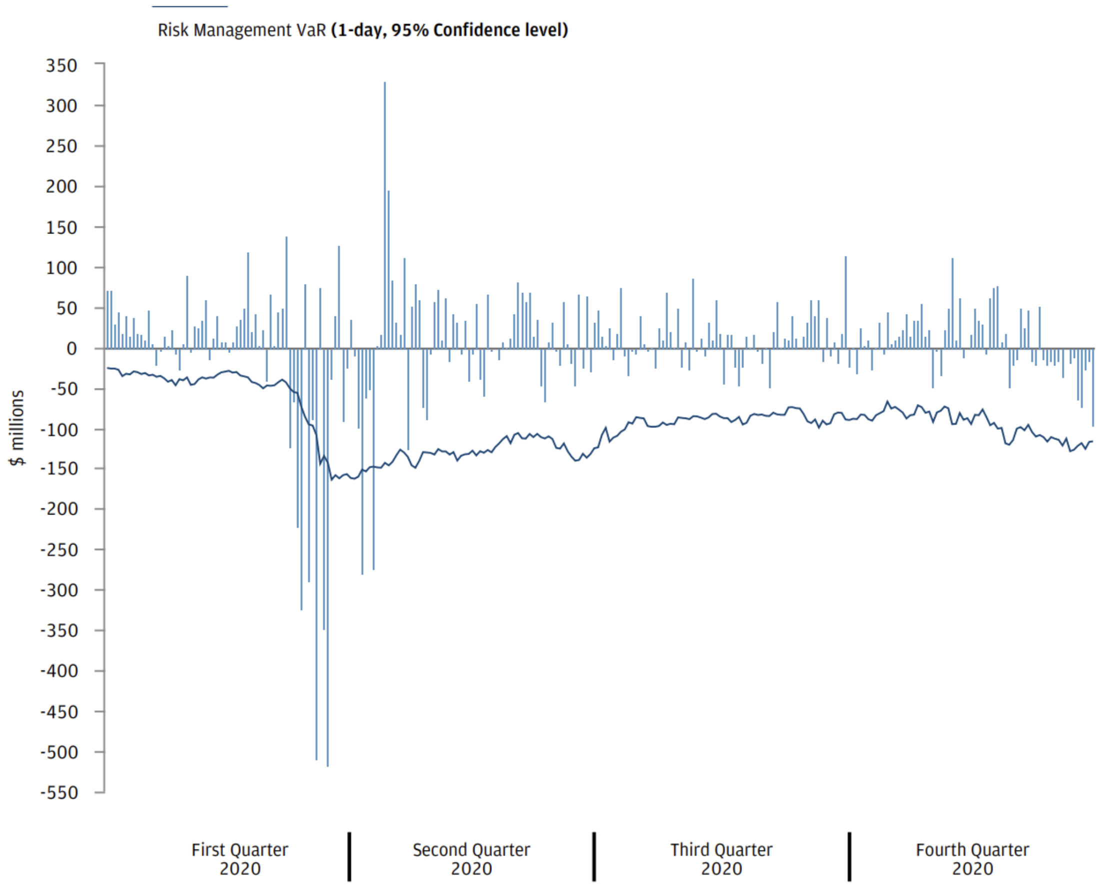
Monte Carlo Simulation *continued*

- Calculate ΔP
- Repeat many times to build up a probability distribution for ΔP
- VaR is the appropriate fractile of the distribution times square root of N
- For example, with 1,000 trial the 1 percentile is the 10th worst case.

Back-Testing

- Tests how well VaR estimates would have performed in the past
- We could ask the question: How often was the actual 1-day loss greater than the 99%/1- day VaR?

Example: JPM 2020 VaR



JPM VaR computation

VaR backtesting

The Firm performs daily VaR model backtesting, which compares the daily Risk Management VaR results with the daily gains and losses that are utilized for VaR backtesting purposes. The gains and losses in the chart below do not reflect the Firm's revenue results as they exclude select components of total net revenue, such as those associated with the execution of new transactions (i.e., intraday client-driven trading and intraday risk management activities), fees, commissions, certain valuation adjustments and net interest income. These excluded components of total net revenue may more than offset backtesting gains and losses on a particular day. The definition of backtesting gains and losses above is consistent with the requirements for backtesting under Basel III capital rules.

The following chart compares Firmwide daily backtesting gains and losses with the Firm's Risk Management VaR for the year ended December 31, 2020. The results in the chart below differ from the results of backtesting disclosed in the Market Risk section of the Firm's Basel III Pillar 3 Regulatory Capital Disclosures reports, which are based on Regulatory VaR applied to the Firm's covered positions.

For the year ended December 31, 2020, the Firm posted backtesting gains on 162 of the 260 days, and observed 10 VaR backtesting exceptions, which were predominantly driven by volatility at the onset of the COVID-19 pandemic that was materially higher than the levels realized in the historical data used for the VaR calculation. Firmwide backtesting loss days can differ from the loss days for which Fixed Income Markets and Equity Markets posted losses, as disclosed in CIB Markets revenue, as the population of positions which compose each metric are different and due to the exclusion of select components of total net revenue in backtesting gains and losses as described above. For more information on CIB Markets revenue, refer to pages 74-75.

Binomial distribution (coin tossing)

- The random variable X is said to be distributed as Bin(p) if it can take only one of two values X={0,1} with probabilities {p,1-p}
- What is the probability that in N draws, we observe n zeros? Denote k the number of zeros.

$$P(k=n) = \frac{N!}{n!\,(N-n)!} p^n (1-p)^{N-n}$$

VaR backtesting

- If the profit/loss of each day is independent, then the event that VaR(p) is breached should follow a binomial distribution Bin(p).
- For example, the probability that 99% VaR should follow a Bin(0.01) distribution: in 252 days, it should be breached 2.52 days on average.
- If this happens more frequently, our VaR model is not conservative enough.