The Greeks

Hedging the risks of derivatives

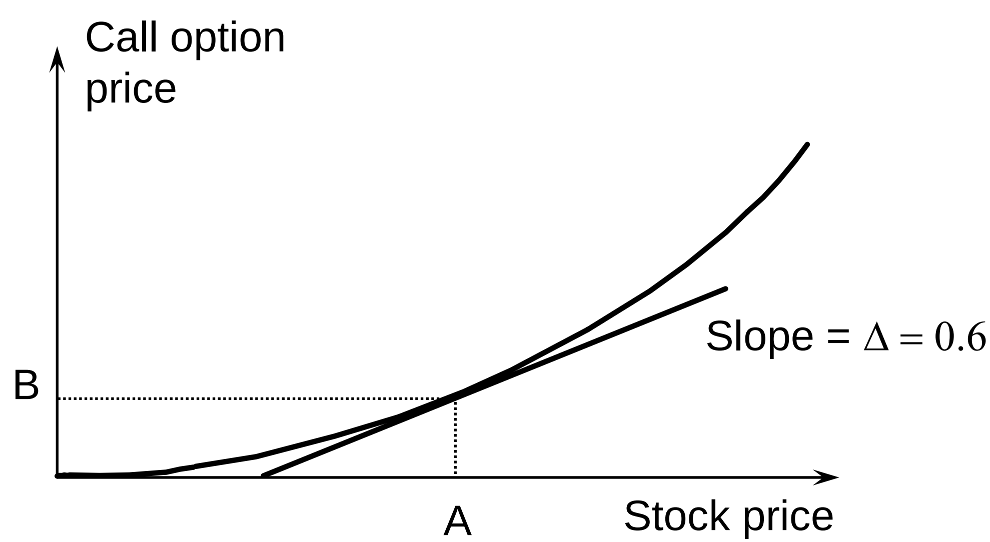
- We learned that an option can be hedged with an appropriate number of shares of stock: riskless portfolio
- This is in general a dynamical strategy: the number of required shares changes in time
- Can we also hedge against changes in volatility, interest rates, etc?
- This requires that we understand better the risks of the derivative: Greeks

Greek Letters

- Greek letters are the partial derivatives with respect to the model parameters that are liable to change
- Usually traders use the Black-Scholes-Merton model when calculating partial derivatives
- The volatility parameter in BSM is set equal to the implied volatility when Greek letters are calculated.

Delta *(See Figure 19.2, page 401)*

- Delta (Δ) is the rate of change of the option price with respect to the underlying asset price



Delta Hedge

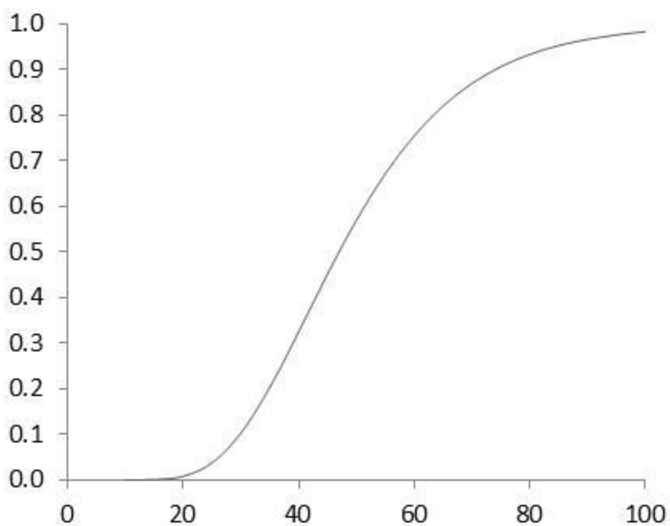
- Trader would be hedged with the position:
  - short 1000 options
  - buy 600 shares
- Gain/loss on the option position is offset by loss/gain on stock position
- Delta changes as stock price changes and time passes
- The hedge position must therefore be rebalanced

European options Delta

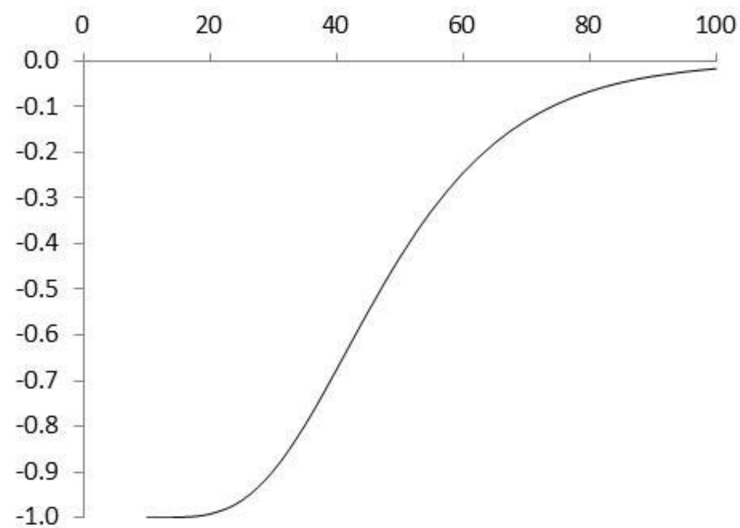
- The Delta of European options in the Black-Scholes model can be computed in closed form
- The delta of a European call on a non-dividend paying stock is $N(d_1)$
- The delta of a European put on the stock is

$$N(d_1) - 1$$

Delta of a Stock Option *(K=50, r=0, σ = 25%, T=2, Figure 19.3, page 402)*

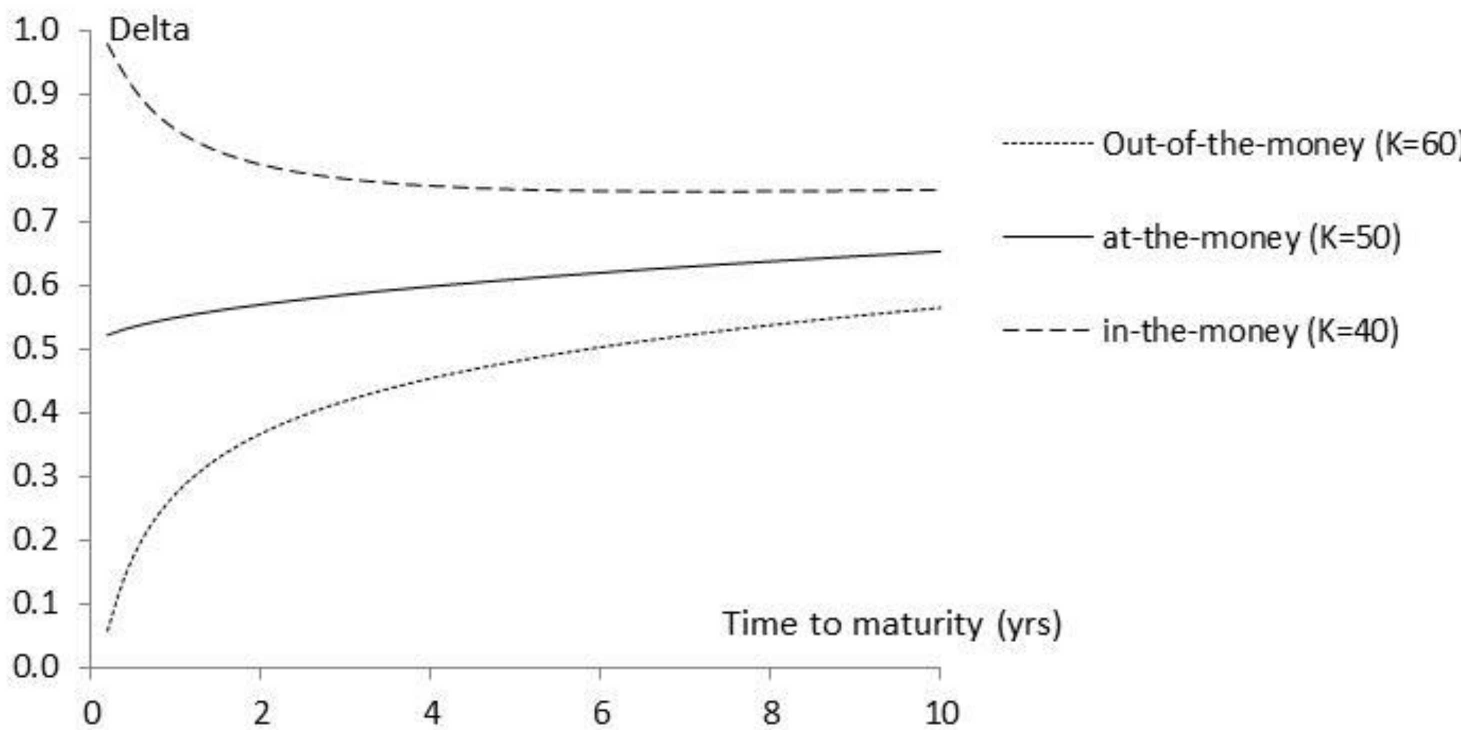


Call



Put

Variation of Delta with Time to Maturity *(S_0=50, r=0, σ=25%, Figure 19.4, page 403)*



The Costs in Delta Hedging continued

- Delta hedging a written option involves a “buy high, sell low” trading rule

First Scenario for the Example:

Table 19.2 page 404

Week	Stock price	Delta	Shares purchased	Cost (‘$000)	Cumulative Cost ($000)	Interest
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
…….	…….	…….	…….	…….	…….	…….
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0	5263.3	

Second Scenario for the Example

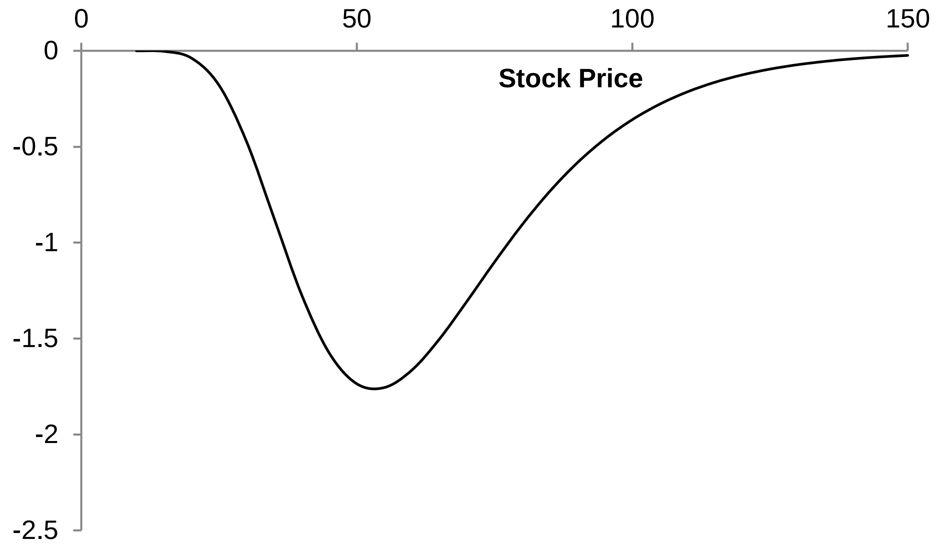
Table 19.3, page 405

Week	Stock price	Delta	Shares purchased	Cost (‘$000)	Cumulative Cost ($000)	Interest
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	49.75	0.568	4,600	228.9	2,789.2	2.7
2	52.00	0.705	13,700	712.4	3,504.3	3.4
....	....	....	....	....	....	....
19	46.63	0.007	(17,600)	(820.7)	290.0	0.3
20	48.12	0.000	(700)	(33.7)	256.6	

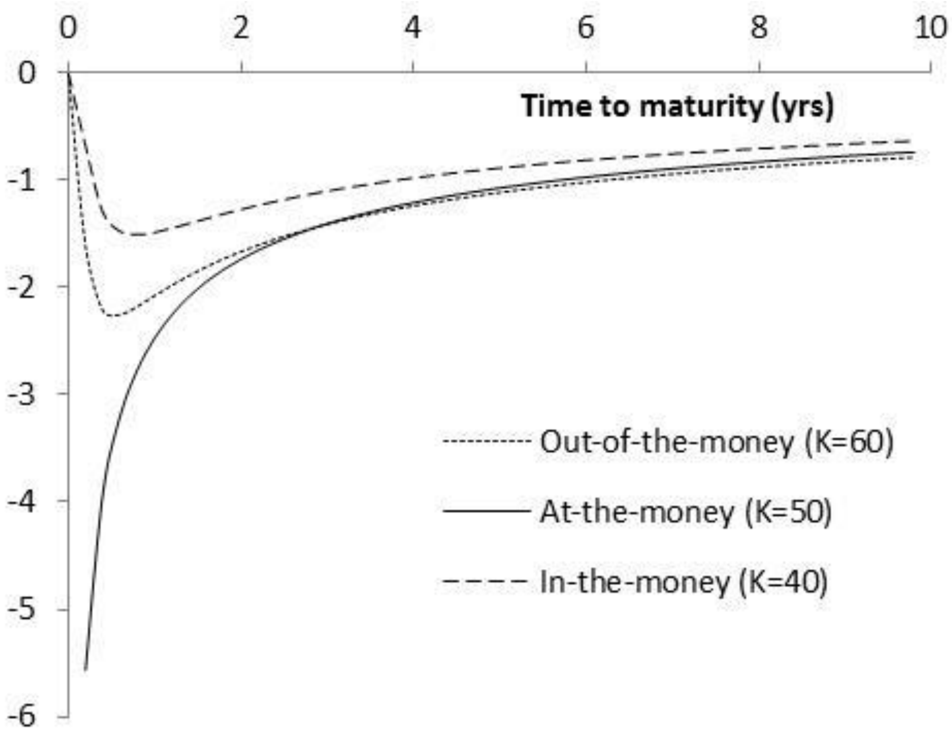
Theta

- Theta (Θ) of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time
- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of a long call or put option declines

Theta for Call Option *(K=50, σ = 25%, r = 0, T = 2, Figure 19.5, page 408)*



Variation of Theta with Time to Maturity *(S_0=50, r=0, σ=25%, Figure 19.6, page 409)*

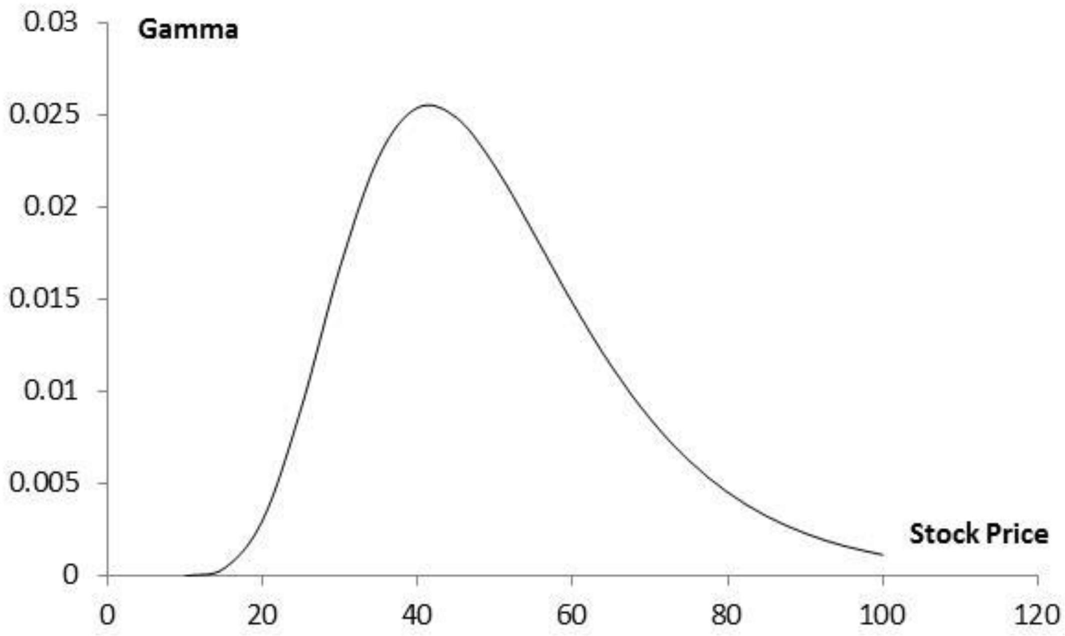


Gamma

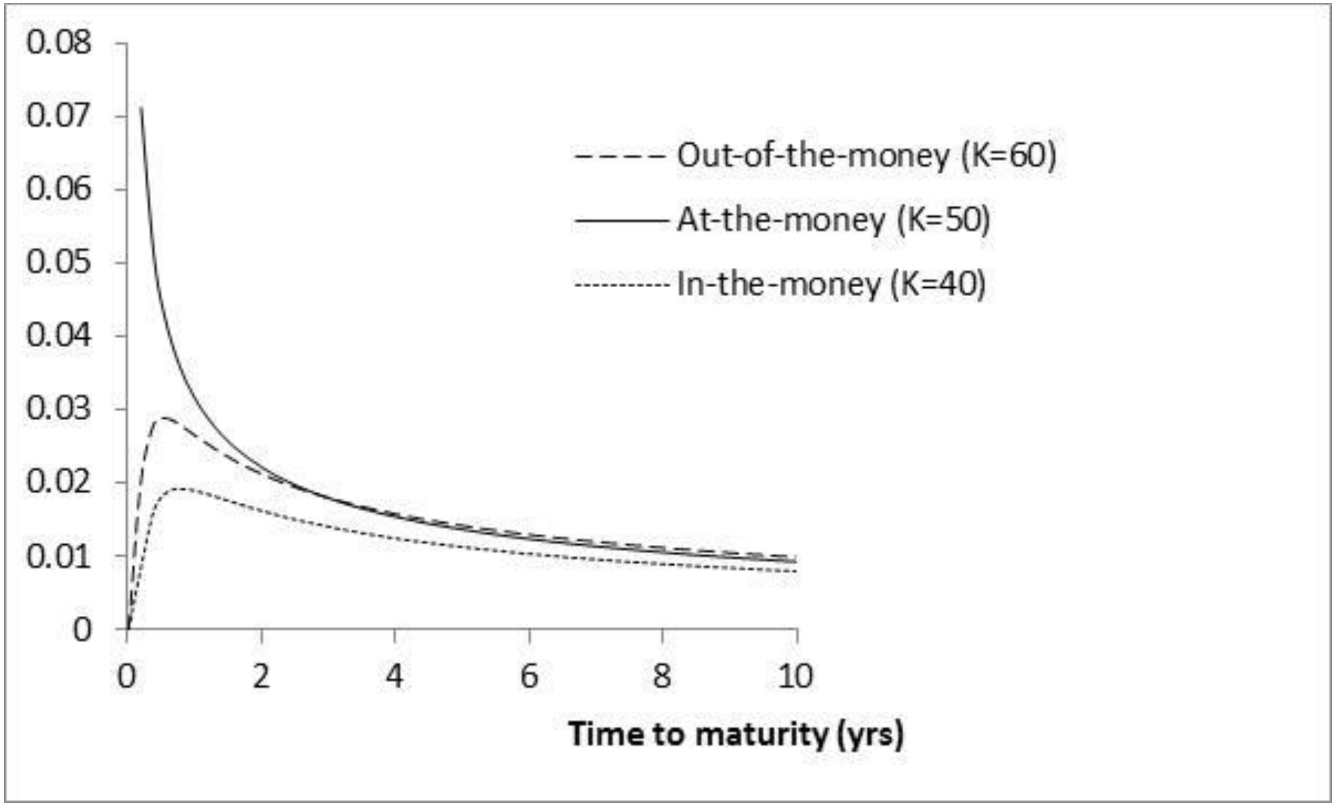
- Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset
- Second order Greek.
- Gamma is greatest for options that are close to the money

Gamma for Call or Put Option:

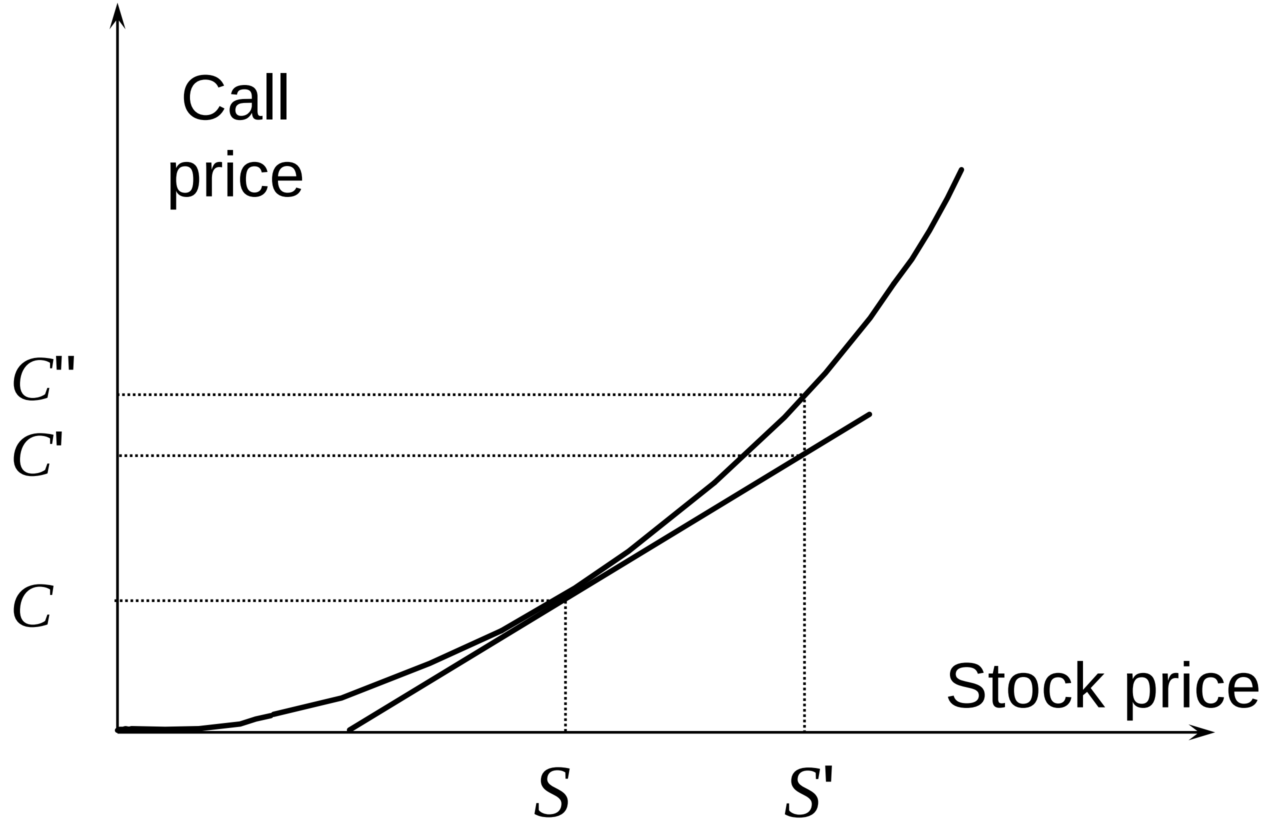
(K=50, σ = 25%, r = 0%, T = 2, Figure 19.9, page 412)



Variation of Gamma with Time to Maturity *(S_0=50, r=0, σ=25%, Figure 19.10, page 413)*

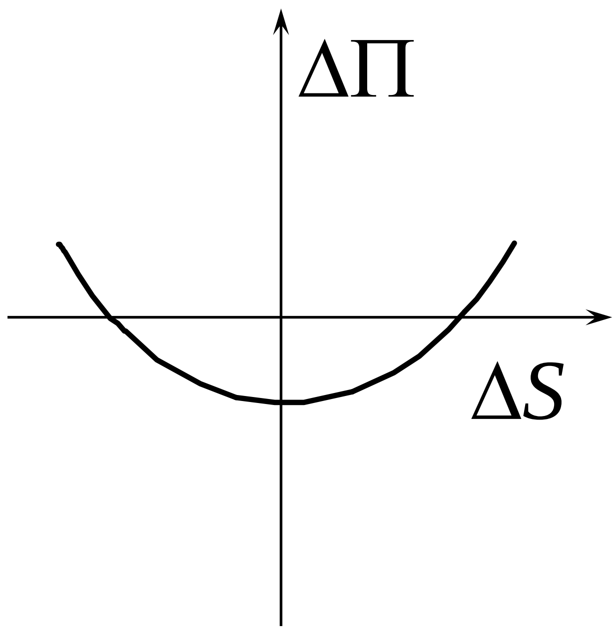


Gamma Addresses Delta Hedging Errors Caused By Curvature *(Figure 19.7, page 411)*

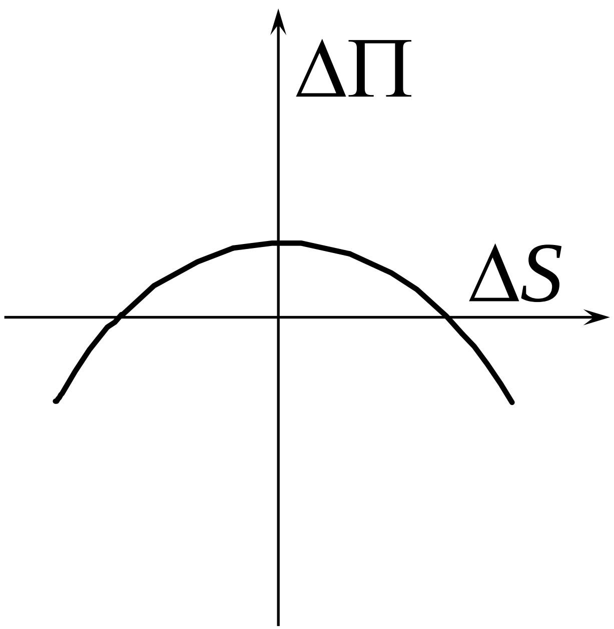


Interpretation of Gamma

For a delta neutral portfolio, $\Delta\Pi \approx \Theta\,\Delta t + ½\Gamma\Delta S^{2}$



Positive Gamma



Negative Gamma

Relationship Between Delta, Gamma, and Theta *(page 415)*

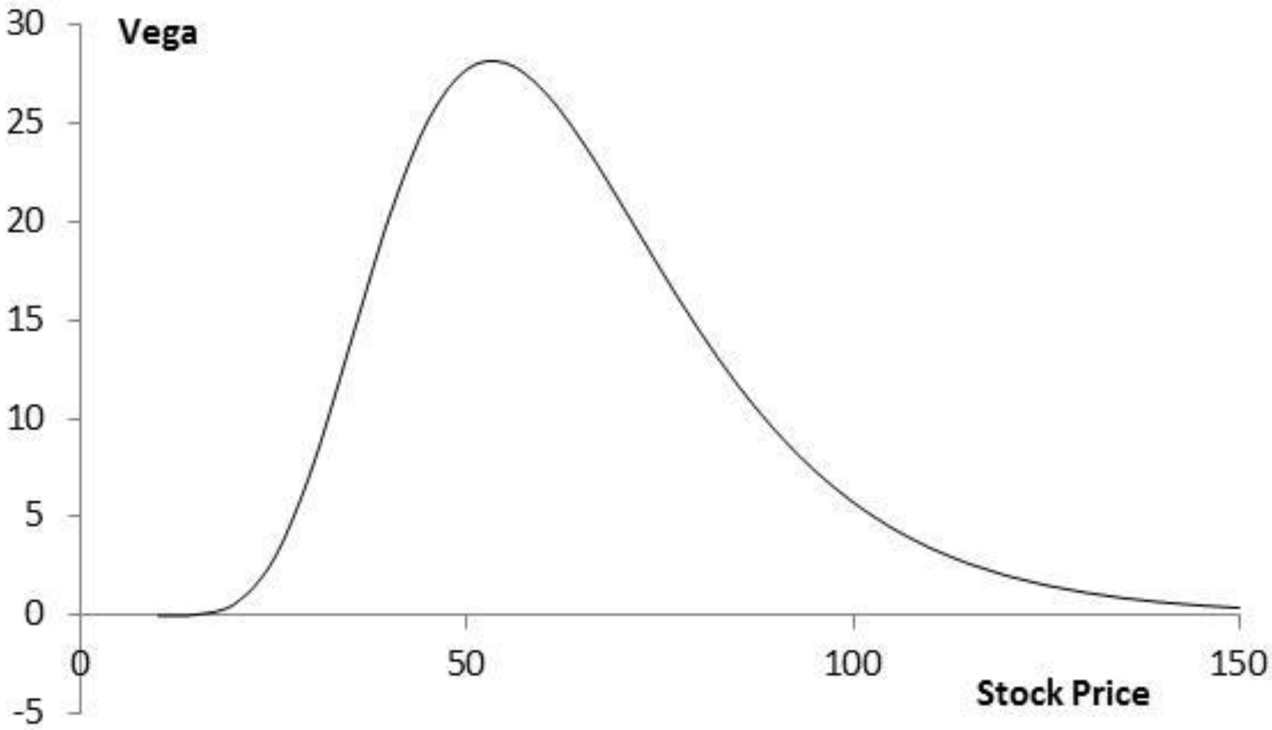
For a portfolio of derivatives on a stock paying a continuous dividend yield at rate q it follows from the Black-Scholes-Merton differential equation that

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Vega

- Vega ($\mathcal{V}$) is the rate of change of the value of a derivatives portfolio with respect to volatility
- It is largest for at-the-money options, and decreases rapidly away for out-of-the-money options
- Vega is always positive, for any derivative

Vega for Call or Put Option
(K=50, σ = 25%, r = 0, T = 2)



Taylor Series Expansion *(Appendix to Chapter 19)*

- The value of a portfolio of derivatives dependent on an asset is a function of of the asset price S, its volatility σ, and time t

$$\Delta\Pi = \frac{\partial\Pi}{\partial S}\Delta S + \frac{\partial\Pi}{\partial\sigma}\Delta\sigma + \frac{\partial\Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}(\Delta S)^2 + \ldots$$

$$= \text{Delta}\times\Delta S + \text{Vega}\times\Delta\sigma + \text{Theta}\times\Delta t + \frac{1}{2}\text{Gamma}\times(\Delta S)^2 + \ldots$$

Managing Delta, Gamma, & Vega

- Delta can be changed by taking a position in the underlying asset
- To adjust gamma and vega it is necessary to take a position in an option or other derivative

Example

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1 and the underlying asset will make the portfolio delta and gamma neutral?
Answer: Long 10,000 options, short 6000 of the asset

What position in option 1 and the underlying asset will make the portfolio delta and vega neutral?
Answer: Long 4000 options, short 2400 of the asset

Example *continued*

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1, option 2, and the asset will make the portfolio delta, gamma, and vega neutral?

We solve

$$-5000+0.5w_1+0.8w_2=0$$
$$-8000+2.0w_1+1.2w_2=0$$

to get $w_1 = 400$ and $w_2 = 6000$. We require long positions of 400 and 6000 in option 1 and option 2. A short position of 3240 in the asset is then required to make the portfolio delta neutral

Rho

- Rho is the rate of change of the value of a derivative with respect to the interest rate

Hedging in Practice

- Traders usually ensure that their portfolios are delta-neutral at least once a day
- Whenever the opportunity arises, they improve gamma and vega
- There are economies of scale
  - As the portfolio becomes larger hedging becomes less expensive per option in the portfolio

Greek Letters for European Options on an Asset that Provides a Yield at Rate q

Greek Letter	*Call Option*	*Put Option*
Delta	$e^{-qT}N(d_1)$	$e^{-qT}\left[N(d_1)-1\right]$
Gamma	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$
Theta	$-S_0N'(d_1)\sigma e^{-qT}/\left(2\sqrt{T}\right)$ $+qS_0N(d_1)e^{-qT}-rKe^{-rT}N(d_2)$	$-S_0N'(d_1)\sigma e^{-qT}/\left(2\sqrt{T}\right)$ $+qS_0N(-d_1)e^{-qT}+rKe^{-rT}N(-d_2)$
Vega	$S_0\sqrt{T}N'(d_1)e^{-qT}$	$S_0\sqrt{T}N'(d_1)e^{-qT}$
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Futures Contract Can Be Used for Hedging

- Delta hedging can be done also using a futures contracts on the underlying (e.g. equity index)
- Small technical detail: The delta of a futures contract on an asset paying a yield at rate q is $e^{(r-q)T}$ times the delta of a spot contract
- The position required in futures for delta hedging is therefore $e^{-(r-q)T}$ times the position required in the corresponding spot contract

Hedging vs Creation of an Option Synthetically

- When we are hedging we take positions that offset delta, gamma, vega, etc
- When we create an option synthetically we take positions that match delta, gamma, vega, etc

Portfolio Insurance

- In October of 1987 many portfolio managers attempted to create a put option on a portfolio synthetically
- This involves initially selling enough of the portfolio (or of index futures) to match the Δ of the put option

Portfolio Insurance

- As the value of the portfolio increases, the Δ of the put becomes less negative and some of the original portfolio is repurchased
- As the value of the portfolio decreases, the Δ of the put becomes more negative and more of the portfolio must be sold

Black Monday

The strategy did not work well on October 19, 1987...


"All the News That's Fit to Print"

The New York Times

Late Edition

VOL. CXXXVII ... No. 47,298

NEW YORK, TUESDAY, OCTOBER 20, 1987

30 CENTS

STOCKS PLUNGE 508 POINTS, A DROP OF 22.6%; 604 MILLION VOLUME NEARLY DOUBLES RECORD

U.S. Ships Shell Iran Installation In Gulf Reprisal

Offshore Target Termed a Base for Gunboats

By STEVEN V. ROBERTS

WORLDWIDE IMPACT

Frenzied Trading Raises Fears of Recession — Tape 2 Hours Late

By LAWRENCE J. De MARIA

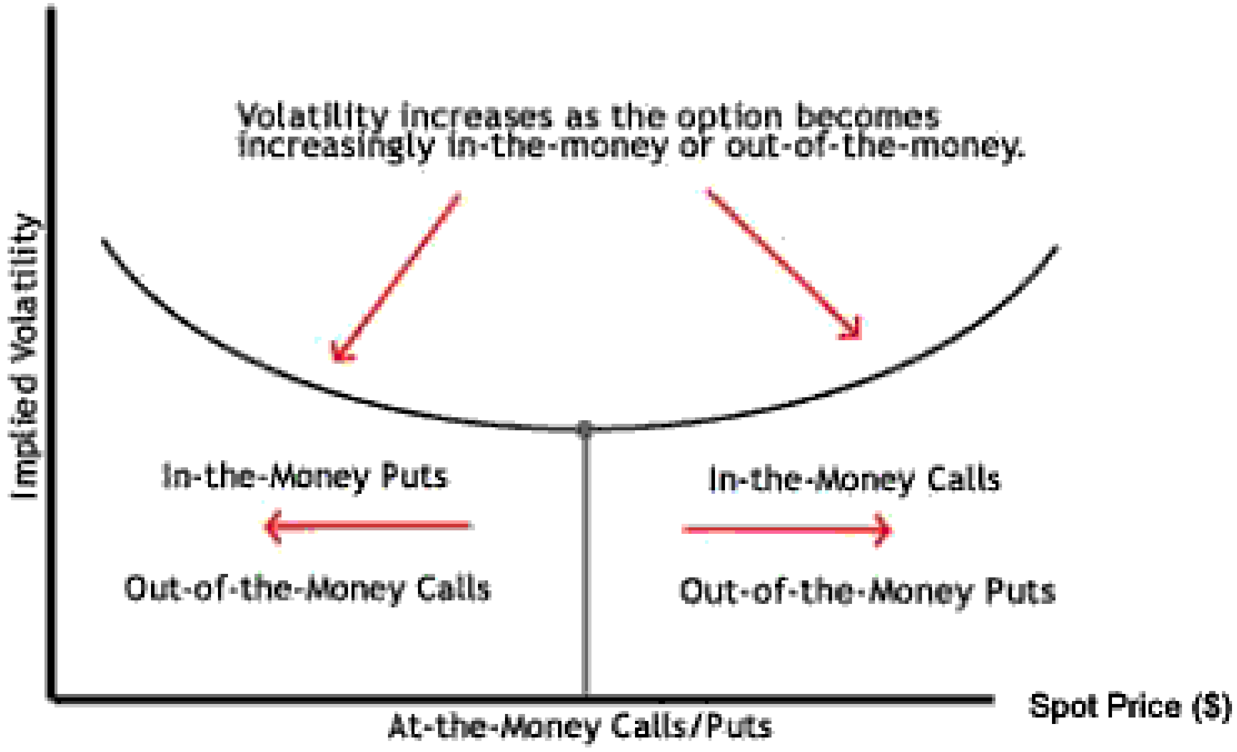
Does 1987 Equal 1929?

By ERIC GELMAN

What is a Volatility Smile?

- It is the relationship between implied volatility and strike price for options with a certain maturity
- The volatility smile for European call options should be exactly the same as that for European put options
- The same is at least approximately true for American options

Cartoon of a volatility smile

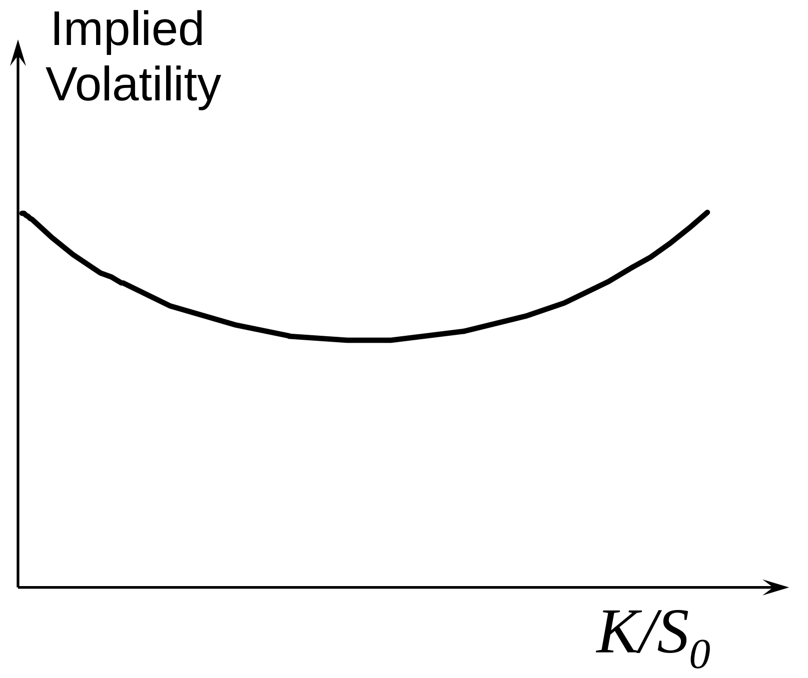


Why the Volatility Smile is the Same for European Calls and Put

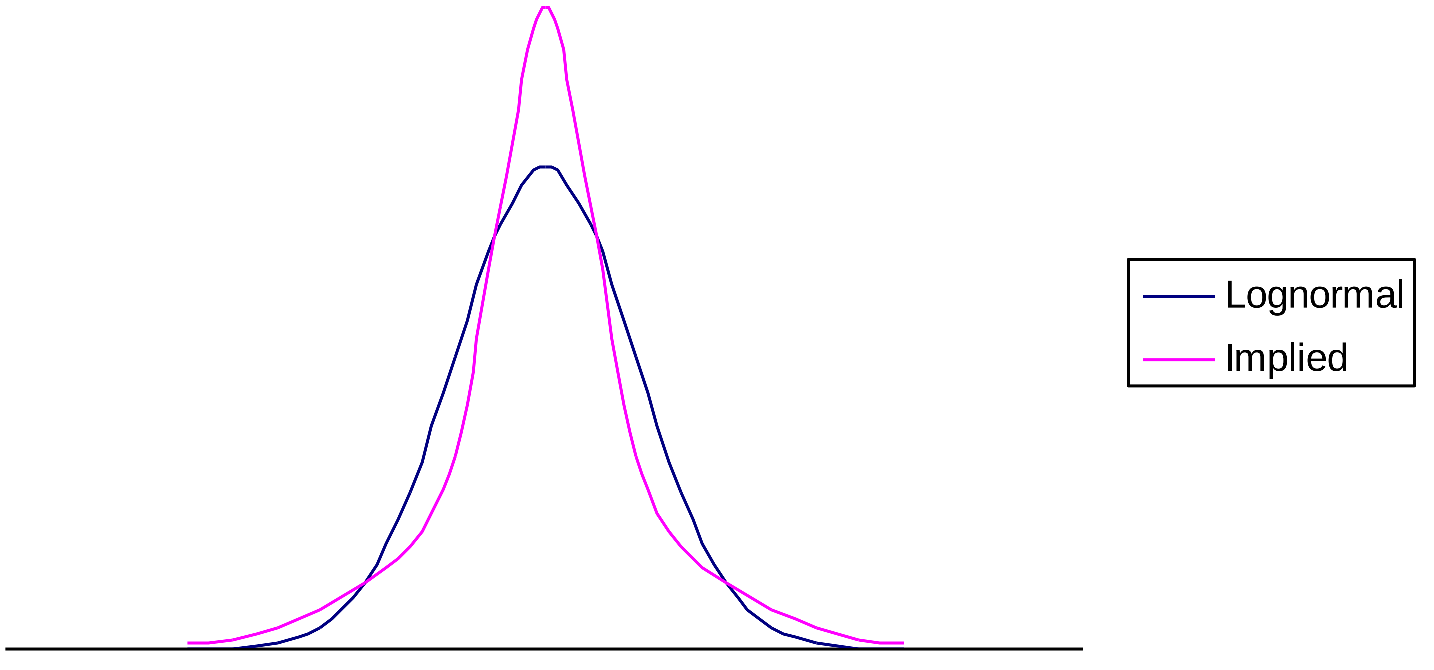
- Put-call parity $p + S_0 e^{-qT} = c + K e^{-rT}$ holds for market prices (p_{mkt} and c_{mkt}) and for Black-Scholes-Merton prices (p_{bs} and c_{bs})
- As a result, $p_{\text{mkt}} - p_{\text{bs}} = c_{\text{mkt}} - c_{\text{bs}}$
- When $p_{\text{bs}} = p_{\text{mkt}}$, it must be true that $c_{\text{bs}} = c_{\text{mkt}}$
- It follows that the implied volatility calculated from a European call option should be the same as that calculated from a European put option when both have the same strike price and maturity

The Volatility Smile for Foreign Currency Options

(Figure 20.1, page 432)



Implied Distribution for Foreign Currency Options



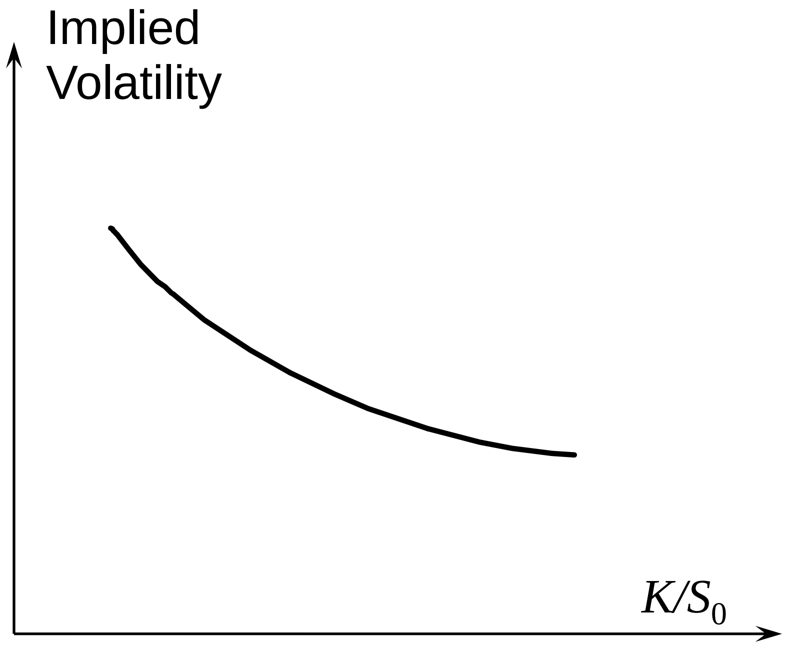
Properties of Implied Distribution for Foreign Currency Options

- Both tails are heavier than the lognormal distribution
- It is also “more peaked” than the lognormal distribution

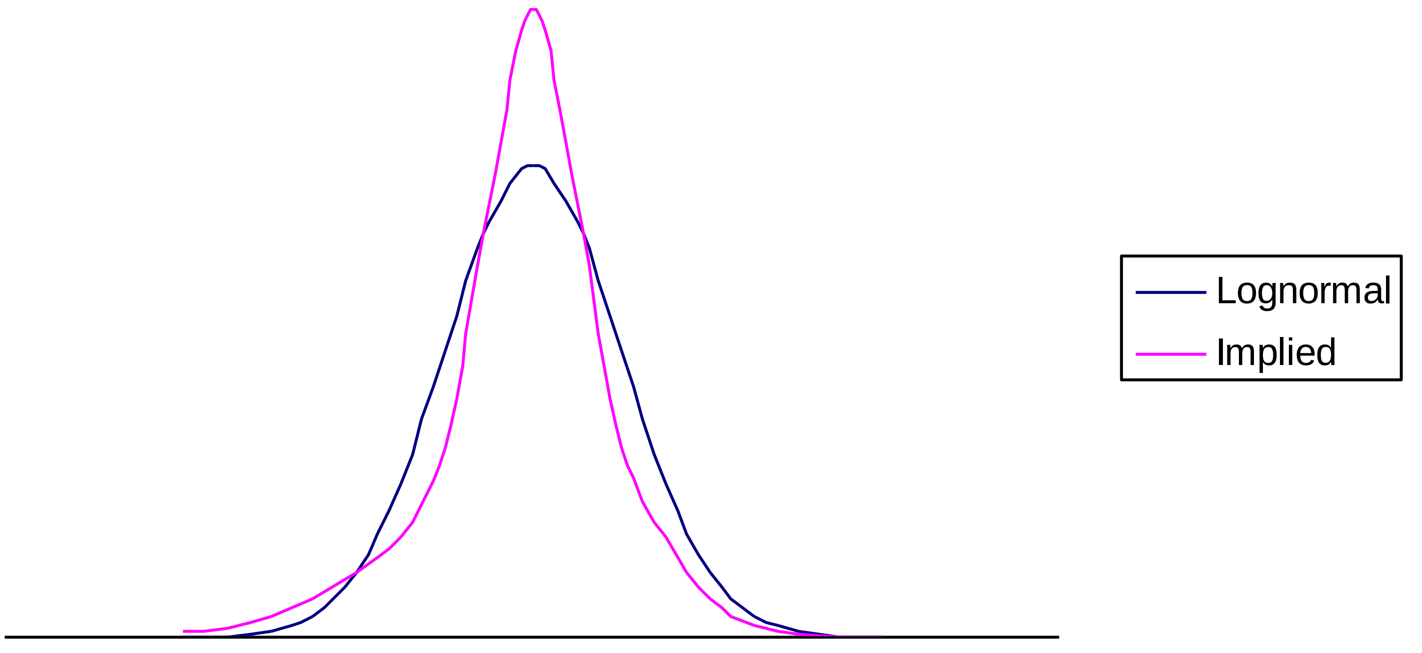
Historical Analysis of Exchange Rate Changes

	Real World (%)	Normal Model (%)
>1 SD	23.32	31.73
>2SD	4.67	4.55
>3SD	1.30	0.27
>4SD	0.49	0.01
>5SD	0.24	0.00
>6SD	0.13	0.00

The Volatility Smile for Equity Options *(Figure 20.3, page 435)*



Implied Distribution for Equity Options



Implied distribution from option prices

- It is possible to imply the risk-neutral distribution of the stock price at some time T in the future, from option prices with maturity T.
- The basic idea: invert the relationship between option prices C(K,T) and the risk-neutral stock price distribution g(S,T).

Determining the Implied Distribution *(Appendix to Chapter 20)*

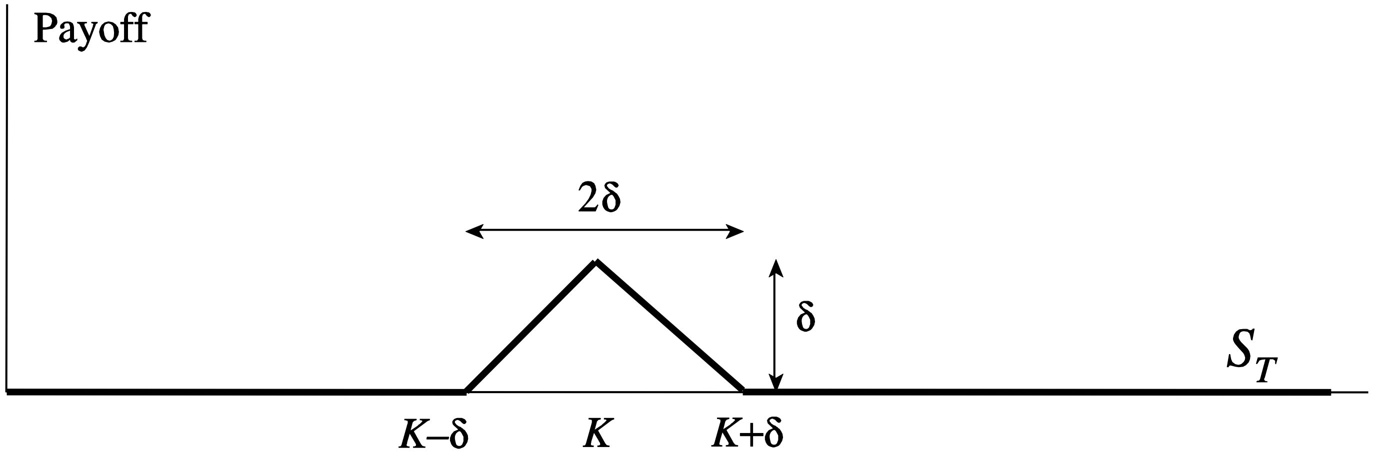
$$c = e^{-rT} \int_{S_T=K}^{\infty} (S_T - K) g(S_T) dS_T$$

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT} g(K)$$

If $c_1, c_2,$ and c_3 are call prices for strikes $K - \delta, K,$ and $K + \delta$ then

$$g(K) = e^{rT} \frac{c_1 + c_3 - 2c_2}{\delta^2}$$

A Geometric Interpretation *(Figure 20A.1, page 447)*



Assuming that density is $g(K)$ from $K-\delta$ to $K+\delta$, $c_1 + c_3 - c_2 = e^{-rT}\,\delta^2\,g(K)$

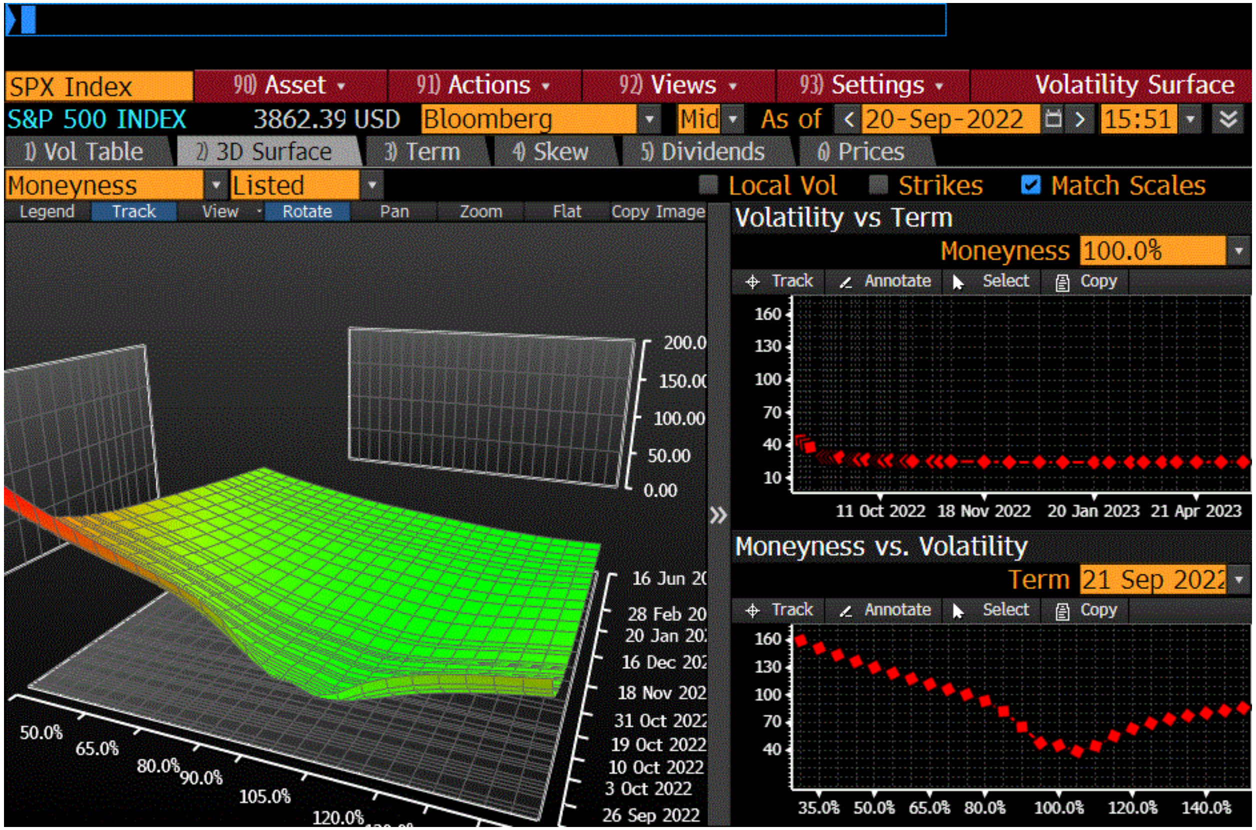
Volatility Term Structure

- In addition to calculating a volatility smile, traders also calculate a volatility term structure
- This shows the variation of implied volatility with the time to maturity of the option
- The volatility term structure tends to be downward sloping when volatility is high and upward sloping when it is low

Volatility Surface

The implied volatility as a function of the strike price and time to maturity is known as the volatility surface

Example: SP500 volatility surface



The vol surface in Bloomberg

Click a volatility point for details

SPX Index | 90) Asset | 91) Actions | 92) Views | 93) Settings | Volatility Surface

S&P 500 INDEX 3862.39 USD Bloomberg Mid As of 20-Sep-2022 15:51

1) Vol Table | 2) 3D Surface | 3) Term | 4) Skew | 5) Dividends | 6) Prices

Moneyness | Listed | 16) Edit | Fwd | Strikes

Exp Date	ImpFwd	60.0%	80.0%	90.0%	95.0%	97.5%	100.0%	102.5%	105.0%	110.0%	120.0%	130.(
		317.4	3089.9	3476.2	3669.3	3765.8	3862.4	3958.9	4055.5	4248.6	4634.9	5021
21 Sep 2022	3862.60	17.77	93.49	65.12	47.54	45.99	44.90	41.71	38.39	44.20	63.14	73.
22 Sep 2022	3862.84	05.94	82.84	54.11	42.85	42.14	41.14	38.67	35.55	35.81	53.36	64.
23 Sep 2022	3863.08	01.49	75.29	49.01	40.57	39.51	38.22	35.55	33.21	34.06	47.44	58.
26 Sep 2022	3863.99	93.07	62.18	38.47	32.77	31.46	29.88	27.38	25.54	25.96	38.04	48.
27 Sep 2022	3864.05	90.69	59.39	38.22	32.79	31.29	29.46	26.96	25.19	26.04	36.14	46.
28 Sep 2022	3864.12	88.66	57.04	37.93	32.81	31.17	29.21	26.62	24.80	25.95	34.59	44.
29 Sep 2022	3864.33	86.81	55.05	37.76	32.88	31.18	29.12	26.50	24.61	25.42	33.31	42.
30 Sep 2022	3864.50	85.11	53.38	37.72	33.02	31.27	29.12	26.43	24.44	25.14	32.15	41.
3 Oct 2022	3864.64	80.68	49.14	34.49	30.30	28.75	26.84	24.46	22.65	22.80	29.38	37.
4 Oct 2022	3864.54	79.40	48.13	34.28	30.24	28.69	26.79	24.53	22.59	23.04	29.31	37.
5 Oct 2022	3864.53	78.19	47.14	34.17	30.14	28.62	26.76	24.53	22.71	22.49	28.46	36.
7 Oct 2022	3865.04	76.01	45.72	33.89	30.17	28.68	26.92	24.81	22.98	22.24	27.33	34.
10 Oct 2022	3864.93	73.15	43.39	32.08	28.66	27.28	25.64	23.70	21.98	21.04	25.92	32.
11 Oct 2022	3865.18	72.29	42.84	32.04	28.67	27.32	25.70	23.79	22.10	20.96	25.12	32.

97) Option Pricing (OVME) 98) Legend Zoom 90%

99) Quick Pricer

Strike	3685.8 Call	Vol	26.69%	Price	312.182	Dividend yield	1.798%
Expiry	22-Dec-2022 Buy	Spot	3862.390	Delta	66.85%	Impl forward	3878.704

Ways of Characterizing the Volatility Smiles

- Plot implied volatility against K/S_0
- Plot implied volatility against K/F_0
  - Note: traders frequently define an option as at-the-money when K equals the forward price, F_0, not when it equals the spot price S_0
- Plot implied volatility against delta of the option
  - Note: traders sometimes define at-the money as a call with a delta of 0.5 or a put with a delta of −0.5. These are referred to as “50-delta options”

Reasons for Smile in Equity Options

There is a negative correlation between equity prices and volatility. Possible reasons:

- Volatility feedback
- Crashophobia

When the price decreases (increases), volatility tends to increase (decrease) making further decreases (increases) more (less) likely