Basic Numerical Procedures

Approaches to Derivatives Valuation

- Trees
- Monte Carlo simulation
- Finite difference methods

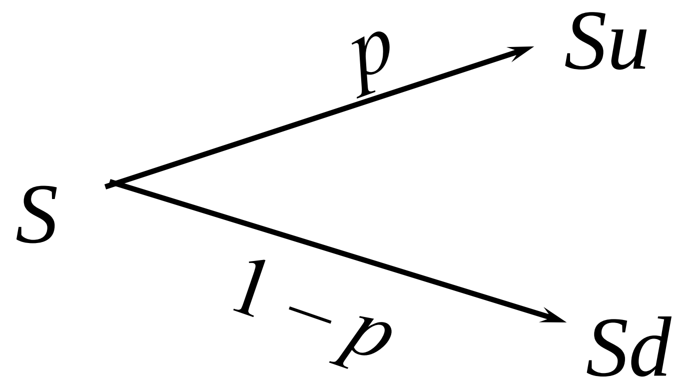
FE 621 – Numerical methods in finance

Binomial Trees

- Binomial trees are frequently used to approximate the movements in the price of a stock or other asset
- In each small interval of time the stock price is assumed to move up by a proportional amount u or to move down by a proportional amount d

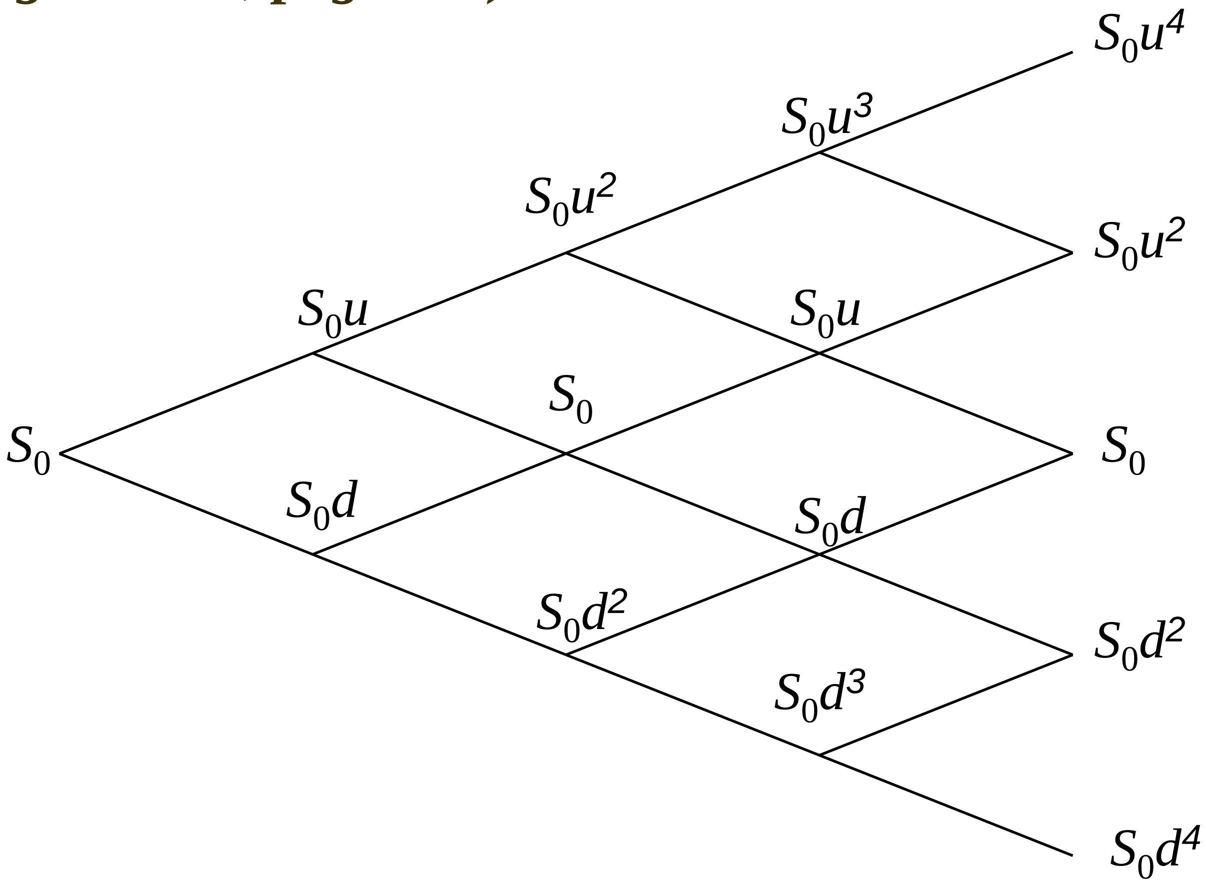
Movements in Time Δt

(Figure 21.1, page 450)



The Complete Tree

(Figure 21.2, page 452)



Tree Parameters for asset paying a dividend yield of q

Parameters p, u, and d are chosen so that the tree gives correct values for the mean & variance of the stock price changes in a risk-neutral world

Mean: $e^{(r-q)\Delta t} = pu + (1-p)d$

Variance: $\sigma^2 \Delta t = pu^2 + (1-p)d^2 - e^{2(r-q)\Delta t}$

A further condition often imposed is $u = 1/d$

Tree Parameters for asset paying a dividend yield of q *(continued)*

When Δt is small a solution to the equations is

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{a-d}{u-d}$$

$$a = e^{(r-q)\,\Delta t}$$

Backwards Induction

- We know the value of the option at the final nodes
- We work back through the tree using risk-neutral valuation to calculate the value of the option at each node, testing for early exercise when appropriate

Example: Put Option

(Example 21.1, page 452-454)

S_0 = 50; K = 50; r =10%; σ = 40%;

T = 5 months = 0.4167; Δt = 1 month = 0.0833

In this case

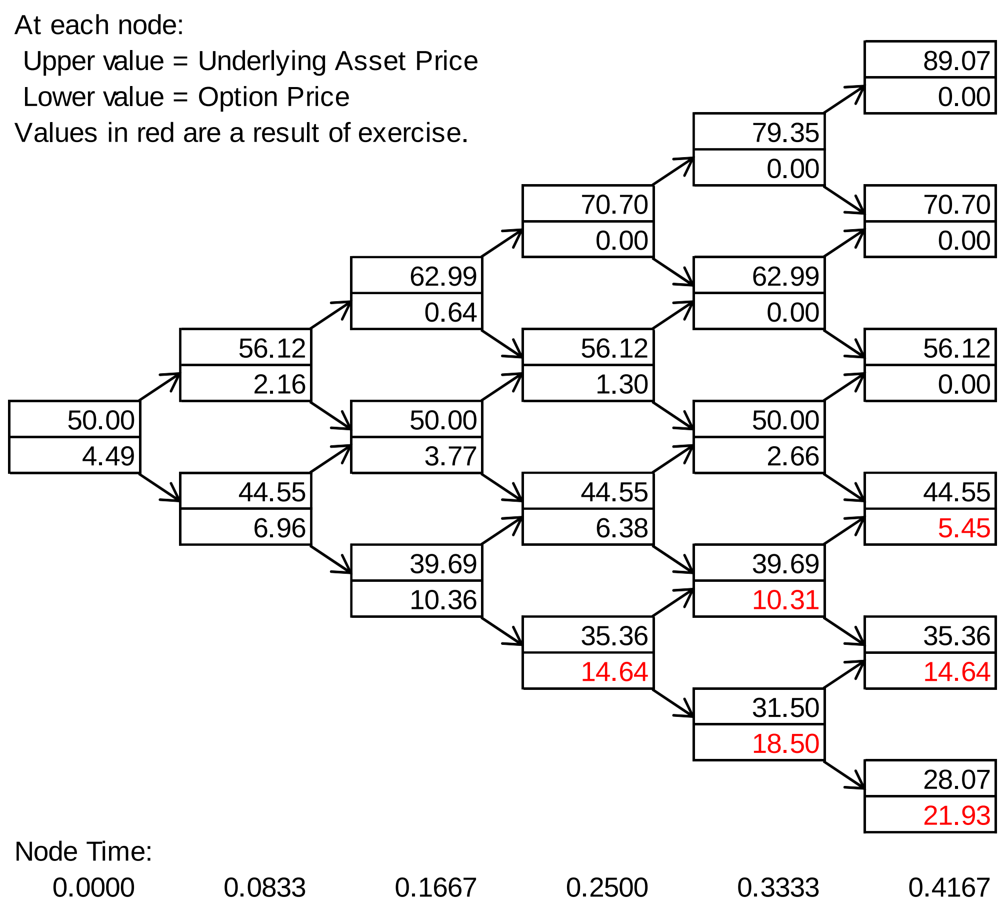
$$a = e^{0.1\times 1/12} = 1.0084$$

$$u = e^{0.4\sqrt{1/12}} = 1.1224$$

$$d = \frac{1}{u} = 0.8909$$

$$p = \frac{1.0084 - 0.8909}{1.1224 - 0.8909} = 0.5073$$

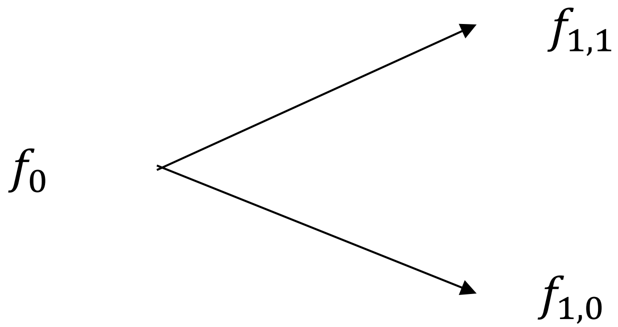
Example (continued; Figure 21.3, page 453)



Calculation of Delta

Delta is calculated from the nodes at time Δt

$$\text{Delta} = \frac{2.16 - 6.96}{56.12 - 44.55} = -0.41$$



$$\Delta = \frac{f_{1,1} - f_{1,0}}{(u-d)S}$$

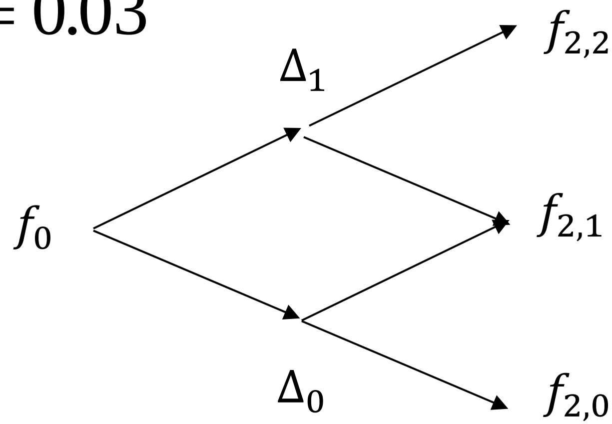
Calculation of Gamma

Gamma is calculated from the nodes at time $2\Delta t$

$$\Delta_1 = \frac{0.64 - 3.77}{62.99 - 50} = -0.24; \;\; \Delta_2 = \frac{3.77 - 10.36}{50 - 39.69} = -0.64$$

$$\text{Gamma} = \frac{\Delta_1 - \Delta_2}{11.65} = 0.03$$

=0.5(62.99-50)+0.5(50-39.69)



$$\Gamma = \frac{\Delta_1 - \Delta_0}{\frac{1}{2}(u^2 - d^2)S}$$

$$\Delta_1 = \frac{f_{2,2} - f_{2,1}}{(u^2 - 1)S}$$

$$\Delta_0 = \frac{f_{2,1} - f_{2,0}}{(1 - d^2)S}$$

Calculation of Theta

Theta is calculated from the central nodes at times 0 and $2\Delta t$

$$\text{Theta} = \frac{3.77 - 4.49}{0.1667} = -4.3 \text{ per year}$$

or -0.012 per calendar day

Calculation of Vega

- We can proceed as follows
- Construct a new tree with a volatility of 41% instead of 40%.
- Value of option is 4.62
- Vega is $4.62 - 4.49 = 0.13$

  per 1% change in volatility

Trees for Options on Indices, Currencies and Futures Contracts

As with Black-Scholes-Merton:

- For options on stock indices, q equals the dividend yield on the index
- For options on a foreign currency, q equals the foreign risk-free rate
- For options on futures contracts $q = r$

Control Variate Technique

- Value American option - f_A
- Value European option using same tree - f_E
- Value European option using Black-Scholes - f_{BS}
- Option price = $f_A + (f_{BS} - f_E)$

Alternative Binomial Tree

Instead of setting $u = 1/d$ we can set each of the 2 probabilities to 0.5 and

$$u = e^{(r-q-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}}$$

$$d = e^{(r-q-\sigma^2/2)\Delta t-\sigma\sqrt{\Delta t}}$$

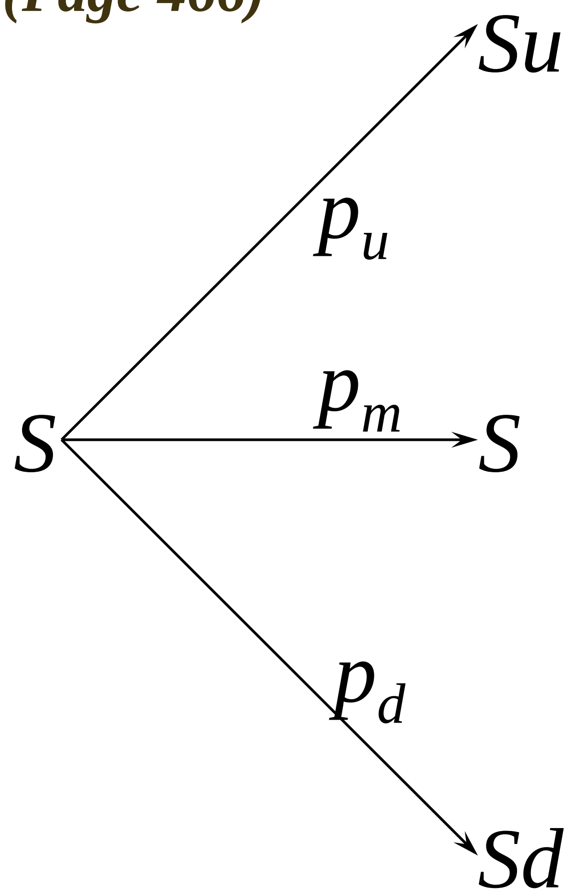
Trinomial Tree *(Page 466)*

$$u = e^{\sigma\sqrt{3\Delta t}} \qquad d = 1/u$$

$$p_u = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

$$p_m = \frac{2}{3}$$

$$p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

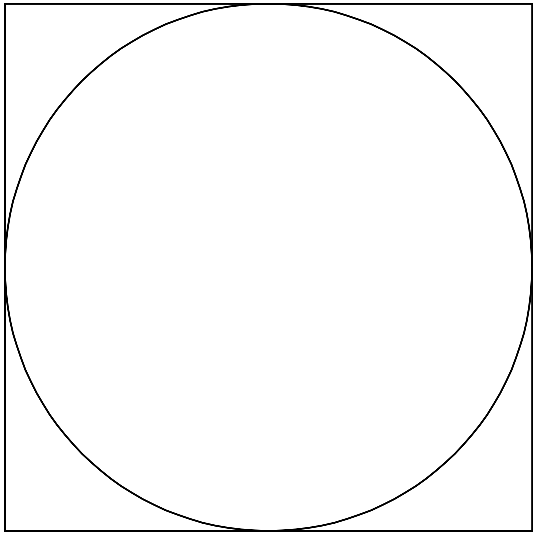


Time Dependent Parameters in a Binomial Tree *(page 467)*

- Making r or q a function of time does not affect the geometry of the tree. The probabilities on the tree become functions of time.
- We can make σ a function of time by making the lengths of the time steps inversely proportional to the variance rate.

Monte Carlo Simulation and π

- How could you calculate π by randomly sampling points in the square?



Monte Carlo Simulation and Options

When used to value European stock options, Monte Carlo simulation involves the following steps:

1. Simulate 1 path for the stock price in a risk neutral world
2. Calculate the payoff from the stock option
3. Repeat steps 1 and 2 many times to get many sample payoffs
4. Calculate mean payoff
5. Discount mean payoff at risk free rate to get an estimate of the value of the option

Sampling Stock Price Movements

- In a risk neutral world, the process for a stock price is

$$dS = \hat{\mu}S\,dt + \sigma S\,dz$$

where $\hat{\mu}$ is the risk-neutral return

- We can simulate a path by choosing time steps of length Δt and using the discrete version of this

$$dS = \hat{\mu}S\,dt + \sigma S\,\varepsilon\sqrt{\Delta t}$$

where ε is a random sample from $\phi(0,1)$

Exact simulation of the gBM

(Equation 21.15, page 470)

Use

$$d\,\ln S = \left(\hat{\mu} - \sigma^2/2\right)dt + \sigma\,dz$$

The discrete version of this is

$$\ln S(t+\Delta t) - \ln S(t) = \left(\hat{\mu} - \sigma^2/2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

or

$$S(t+\Delta t) = S(t)\,e^{\left(\hat{\mu} - \sigma^2/2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}}$$

Extension to multiple assets

- When a derivative depends on several underlying assets, we have to simulate them simultaneously
- Example: basket options on several stocks
- Have to simulate stochastic paths for multiple stocks

Sampling from Normal Distribution
(Page 472)

- In Excel: NORMSINV(RAND()) gives a random sample from $\phi(0,1)$
- In R: rnorm(n) gives a sample of n standard normals

To Obtain 2 Correlated Normal Samples

- Obtain independent normal samples x_1 and x_2 and set

$$\varepsilon_1 = x_1$$
$$\varepsilon_2 = \rho x_1 + x_2\sqrt{1-\rho^2}$$

- Use a procedure known as Cholesky's decomposition when samples are required from more than two normal variables (see page 473)

Standard Errors in Monte Carlo Simulation

The standard error of the estimate of the option price is the standard deviation of the discounted payoffs given by the simulation trials divided by the square root of the number of observations.

Application of Monte Carlo Simulation

- Monte Carlo simulation can deal with path dependent options, options dependent on several underlying state variables, and options with complex payoffs
- It cannot easily deal with American-style options

Determining Greek Letters

For Δ:

1. Make a small change to asset price
2. Carry out the simulation again using the same random number streams
3. Estimate Δ as the change in the option price divided by the change in the asset price

Proceed in a similar manner for other Greek letters

Variance Reduction Techniques

- Antithetic variable technique
- Control variate technique
- Importance sampling
- Stratified sampling
- Moment matching
- Using quasi-random sequences

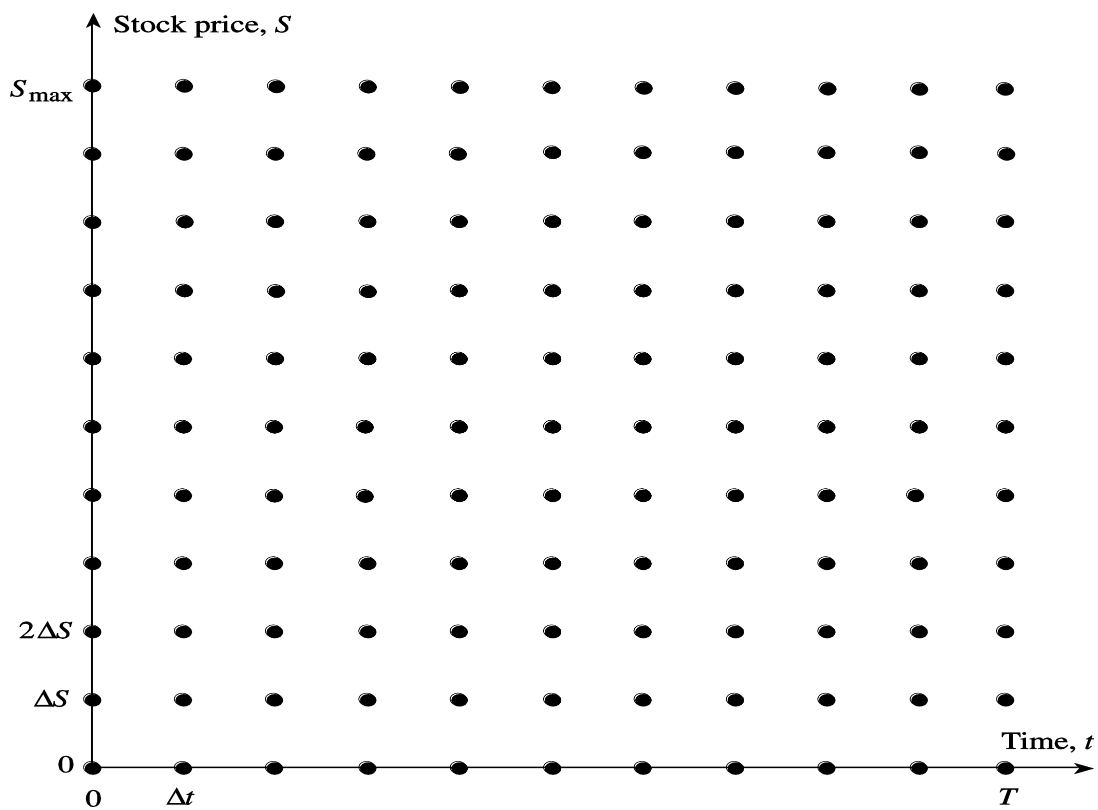
Sampling Through the Tree

- Instead of sampling from the stochastic process we can sample paths randomly through a binomial or trinomial tree to value a derivative
- At each node that is reached we sample a random number between 0 and 1. If it is between 0 and p, we take the up branch; if it is between p and 1, we take the down branch

Finite Difference Methods

- Finite difference methods aim to represent the differential equation in the form of a difference equation
- We form a grid by considering equally spaced time values and stock price values
- Define $f_{i,j}$ as the value of f at time $i\Delta t$ when the stock price is $j\Delta S$

The Grid



Finite Difference Methods

(continued)

$$\frac{\partial f}{\partial t}+(r-q)S\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}=rf$$

Set $$\frac{\partial f}{\partial S}=\frac{f_{i,j+1}-f_{i,j-1}}{2\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2}=\left(\frac{f_{i,j+1}-f_{i,j}}{\Delta S}-\frac{f_{i,j}-f_{i,j-1}}{\Delta S}\right)\Big/\Delta S$$ or

$$\frac{\partial^2 f}{\partial S^2}=\frac{f_{i,j+1}+f_{i,j-1}-2f_{i,j}}{\Delta S^2}$$

Implicit Finite Difference Method

Set $$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t}$$

to obtain for each node an equation of the form

of the form:

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j}$$

Explicit Finite Difference Method

If $\partial f / \partial S$ and $\partial^2 f / \partial S^2$ are assumed to be the same at the $(i+1, j)$ point as they are at the (i, j) point we obtain the explicit finite difference method

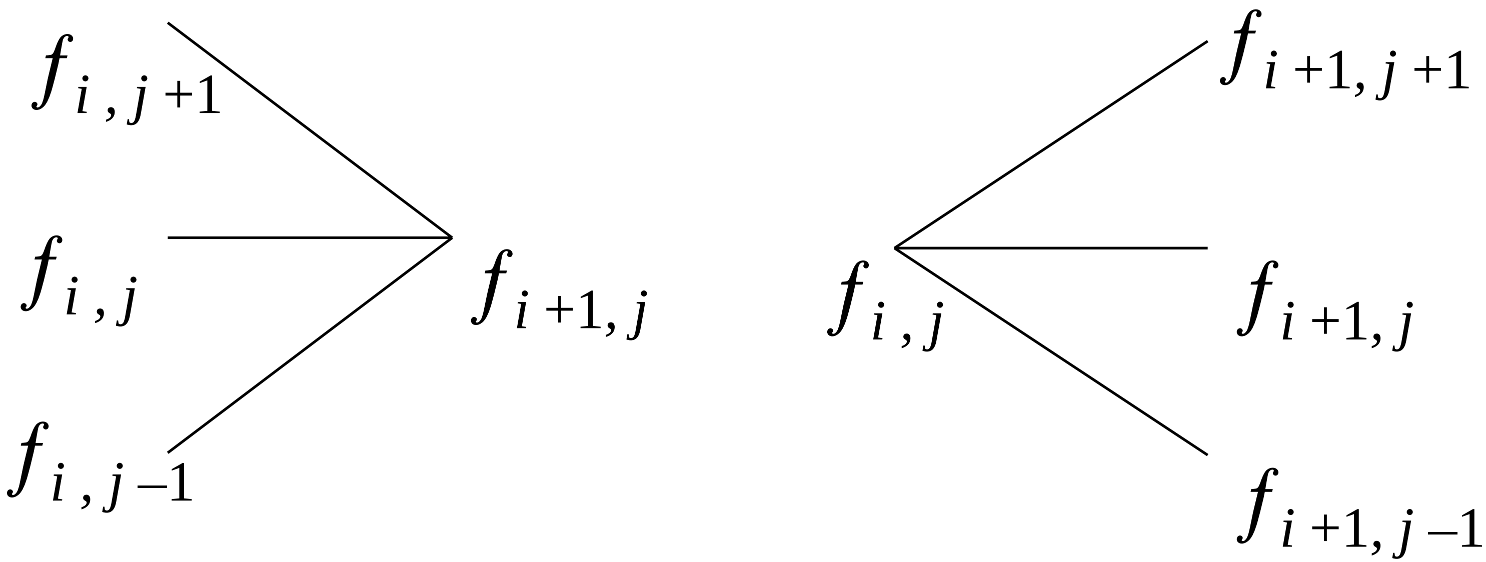
This involves solving equations of the form:

$$f_{i,j} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}$$

Implicit vs Explicit Finite Difference Method

- The explicit finite difference method is equivalent to the trinomial tree approach
- The implicit finite difference method is equivalent to a multinomial tree approach

Implicit vs Explicit Finite Difference Methods *(Figure 21.16, page 483)*



Implicit Method

Explicit Method