Option Pricing in Discrete Time - Binomial Model

Derivatives valuation

We learned so far about:

- Discounted cash flow valuation (e.g. bonds)
- Risky discounted cash flow valuation (e.g. CDS)
- No-arbitrage valuation: Forward contracts

They are all particular cases of risk-neutral valuation – very powerful tool for derivatives valuation

Outline of lecture

- Binomial model for asset prices
- Pricing a European option in the binomial model
- Risk-neutral valuation
- American options in the binomial model

Toy model

- Assume that the market has only two assets:
  - A stock
  - Cash
- The stock price can move only in discrete binary moves (up/down) with known prices
- Assume zero interest rates for simplicity

Binomial Model (Cox, Ross, and Rubinstein)

- The stock price is currently $20
- In 3 months it will be either $22 or $18

Stock Price = $22

Stock price = $20

Stock Price = $18

- What are the probabilities of these future prices?

Binomial Model – naive argument

- The stock price is currently \$20
- In 3 months it will be either \$22 or \$18
- Assume zero interest rate for simplicity
- If probability of \$22 > 1/2:
  - Buy at \$20, in the 3 months expect to have profit
- If probability of \$22 < 1/2:
  - Short at \$20, in 3 months expect to have profit
- The market implied probabilities for \$18 and \$22 appear to be 1/2.

Forward contract

- What if the probability of \$22 is different from 1/2?
- Assume \$22 has probability 0.2
- Recall: a forward contract is a promise to buy the stock at T for price K
- What is the fair forward K?
- Expectation-based price:

$$0.2 * \$22 + 0.8 * \$18 = \$18.8$$

- However, this is incorrect. Go long forward contract to buy stock for \$18.8, sell stock now for \$20 and realize risk-free profit \$1.2. The correct forward price is \$20.

Conclusion

- We cannot price derivatives by expectation of the payoff, with real world probabilities
- This is the so-called “actuarial approach”, commonly used in insurance
- In derivatives one uses no-arbitrage pricing.
- We will add also the replication principle: X replicates Y if they have exactly same payoff in all states of the world
- No-arbitrage + replication can enforce a price

Pricing by replication

- Build a portfolio of stock and cash which replicates the payoff of the forward contract

$$F = S - K \qquad (K = 20)$$

Stock price = \$20
Forward payoff=\$0

Stock Price = \$22
Forward Payoff = \$2

Stock Price = \$18
Forward Payoff = -\$2

Risk-free portfolio

- Another way of looking at it: the forward can be hedged with stock

$$F - S = -K$$

- The resulting hedged position is risk-free so it must grow at the risk-free rate

Call Option (Figure 13.1, page 275)

A 3-month call option on the stock has a strike price of 21.

Stock Price = $22
Option Payoff = $1

Stock price = $20
Option Price=?

Stock Price = $18
Option Payoff = $0

Option pricing by replicating portfolio

- Build a replicating portfolio with same payoff as the option

  x stock + y cash

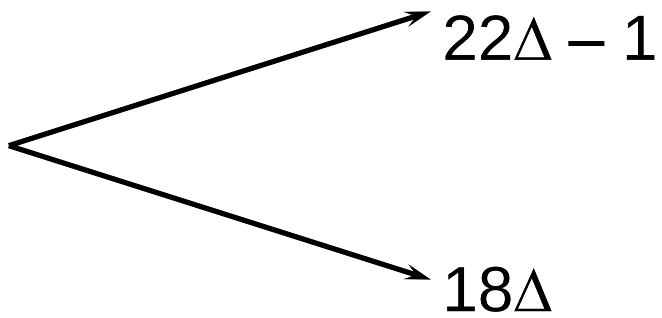
$$22x + y = 1$$

$$18x + y = 0$$

- Solution: x = 0.25, y = -4.5.
- The option must have the same value today as the replicating portfolio = 0.25*$20 - $4.5 = $0.5

Alternative Approach: Riskless portfolio

- Form a portfolio that is long Δ shares and short a call option. The values of this portfolio are



- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio
(Risk-Free Rate is 0%)

- The riskless portfolio is:

  long 0.25 shares

  short 1 call option
- The value of the portfolio in 3 months is

  $22 \times 0.25 - 1 = 4.50$
- The value of the portfolio today is \$20x0.25-C
- Thus, the option price is C=\$0.5

Valuing the Portfolio
(Risk-Free Rate is 4%)

- The riskless portfolio is:

  long 0.25 shares

  short 1 call option
- The value of the portfolio in 3 months is

  $22 \times 0.25 – 1 = 4.50$
- The value of the portfolio today is

  $4.5e^{–0.04\times 0.25} = 4.455$

Valuing the Option *(Risk-Free Rate is 4%)*

- The portfolio that is

  long 0.25 shares

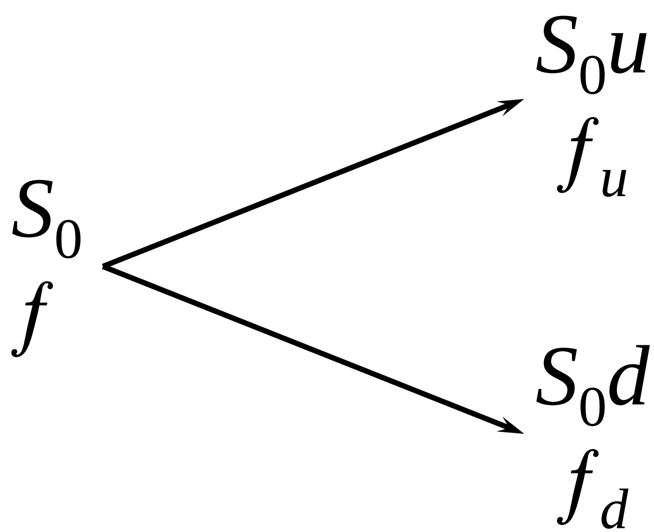
  short 1 option

  is worth 4.455
- The value of the shares is
  5.000 (= 0.25 × 20)
- The value of the option is therefore

  5.000 – 4.455 = 0.545

Generalization *(Figure 13.2, page 276)*

A derivative lasts for time T and is dependent on a stock



Generalization *(continued)*

- Value of a portfolio that is long Δ shares and short 1 derivative:

$$S_0 u\Delta - f_u$$

$$S_0 d\Delta - f_d$$

- The portfolio is riskless when $S_0 u\Delta - f_u = S_0 d\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

Generalization (continued)

- Value of the portfolio at time T is $S_0u\Delta - f_u$
- Value of the portfolio today is $(S_0u\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S_0\Delta - f$
- Hence

$$f = S_0\Delta - (S_0u\Delta - f_u)e^{-rT}$$

Generalization
(continued)

Substituting for Δ we obtain

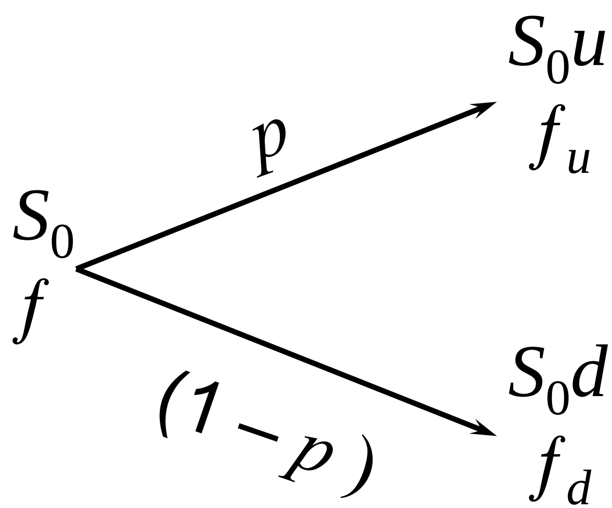
$$f = [\, p f_u + (1 - p) f_d \,] e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

p as a Probability

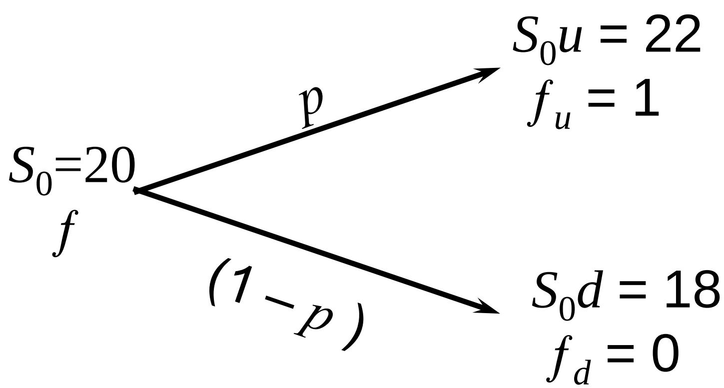
- It is natural to interpret p and 1-p as probabilities of up and down movements
- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate



Risk-Neutral Valuation

- When the probability of an up and down movements are p and 1-p the expected stock price at time T is S_0e^{rT}
- This shows that the stock price earns the risk-free rate
- Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate
- This is known as using risk-neutral valuation

Original Example Revisited



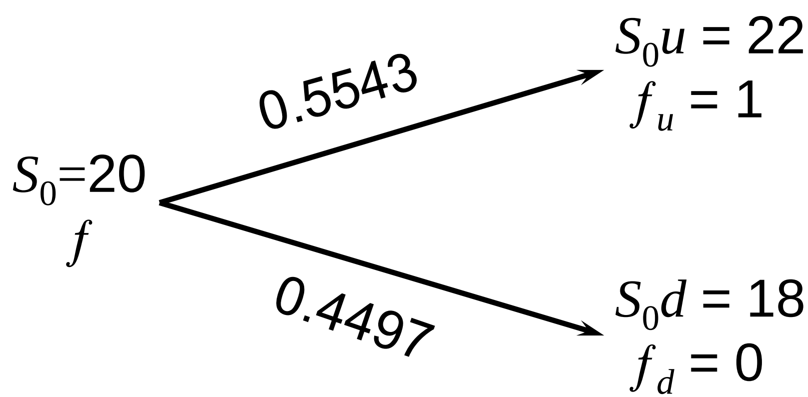
p is the probability that gives a return on the stock equal to the risk-free rate:

$20e^{0.04\times 0.25} = 22p + 18(1-p)$ so that $p = 0.5503$

Alternatively:

$$p = \frac{e^{rT}-d}{u-d} = \frac{e^{0.04\times 0.25}-0.9}{1.1-0.9} = 0.5543$$

Valuing the Option Using Risk-Neutral Valuation



The value of the option is

$e^{-0.04\times0.25}$ (0.5543 ×1 + 0.4497×0)

= 0.545

Three Methods for Pricing

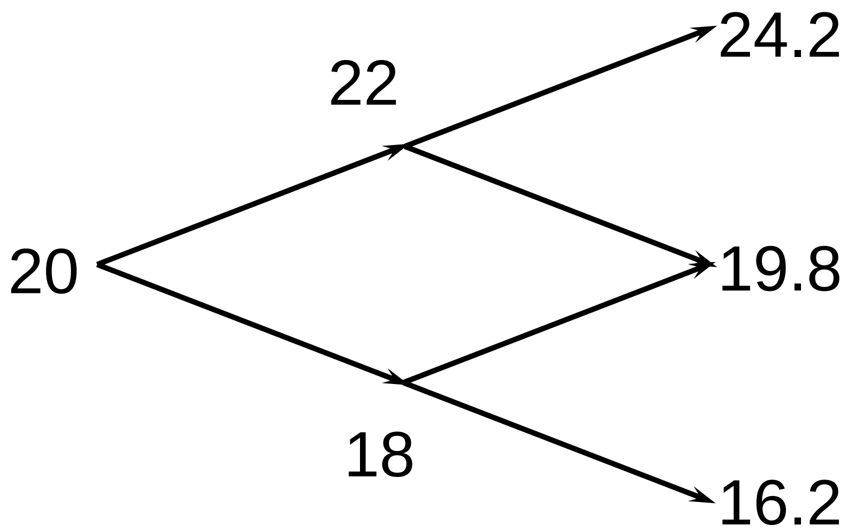
- Replicating portfolio
- Riskless portfolio
- Risk-neutral valuation

Irrelevance of Stock's Expected Return

- When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant

A Two-Step Example

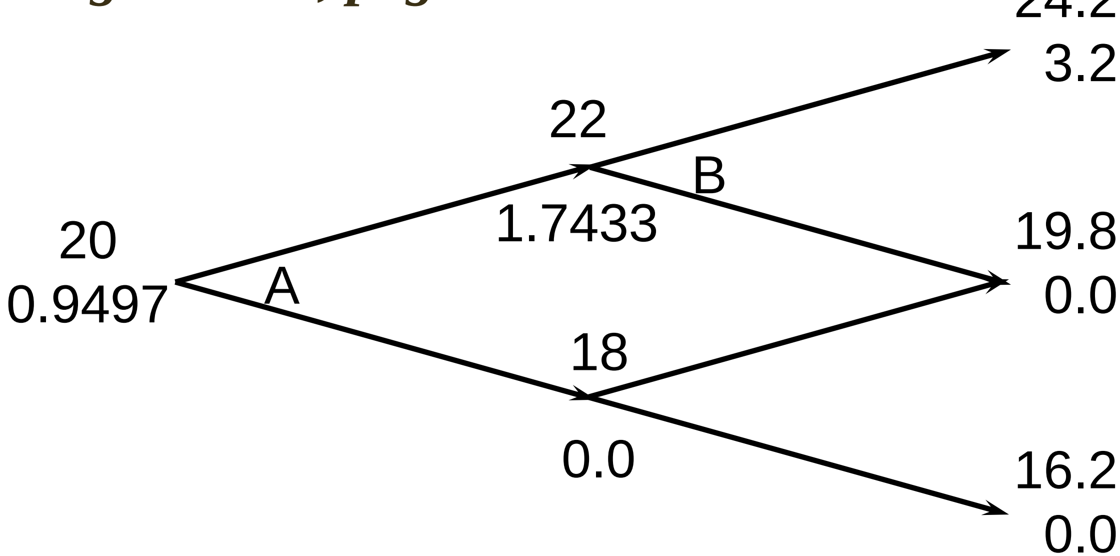
Figure 13.3, page 281



- K = 21, r = 4%
- Each time step is 3 months

Valuing a Call Option

Figure 13.4, page 281



Value at node B

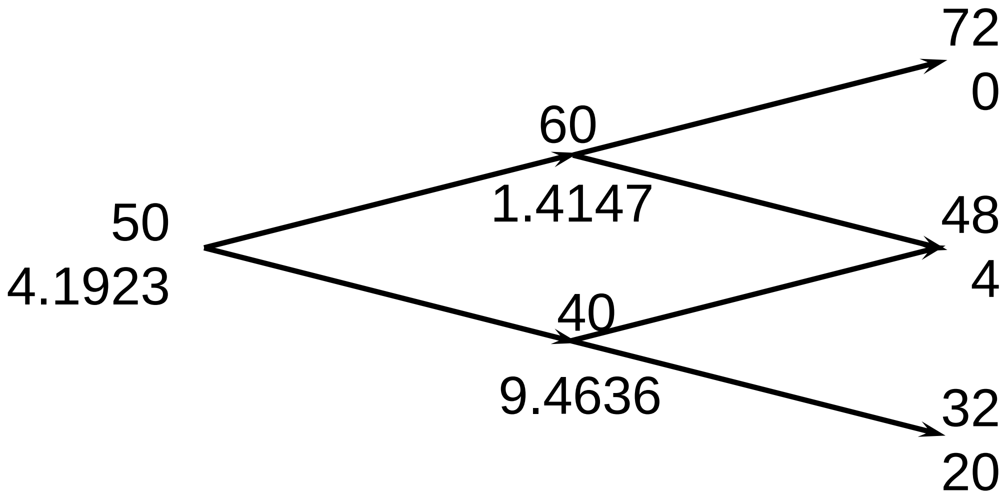
$= e^{-0.04\times0.25}(0.5503\times3.2 + 0.4497\times0) = 1.7433$

Value at node A

$= e^{-0.04\times0.25}(0.5503\times1.7433 + 0.4497\times0) = 0.9497$

A Put Option Example

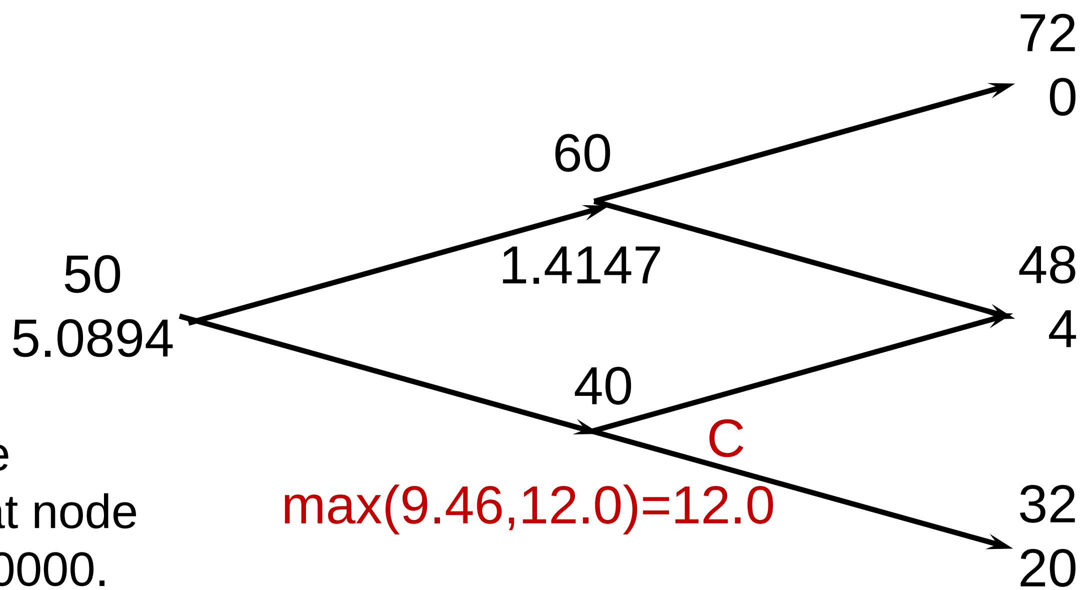
Figure 13.7, page 284



K = 52, time step = 1 year

r = 5%, u =1.2, d = 0.8, p = 0.6282

What Happens When the Put Option is American *(Figure 13.8, page 285)*



The American feature increases the value at node C from 9.4636 to 12.0000.

This increases the value of the option from 4.1923 to 5.0894.

Volatility σ

- The stock price volatility is defined such that the standard deviation of the stock price over time T is $\sigma\sqrt{T}$
- Equivalently, the variance of the stock price is $\sigma^2 T$
- Recall that the variance of a random variable X is

$$var(X) = E[X^2] - (E[X])^2$$

and the standard deviation is

$$stdev(X) = \sqrt{var(X)}$$

Choosing u and d

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

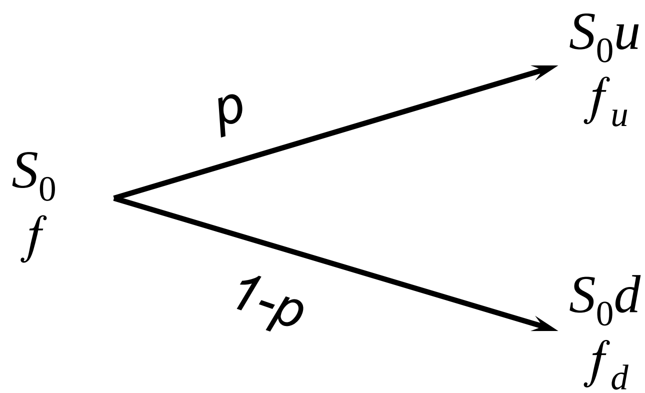
where σ is the volatility and Δt is the length of the time step.

This is the approach used by Cox, Ross and Rubinstein but it is not unique

Risk-Neutral Probabilities

- In summary, we can price any derivative by taking expectations of its payoff with respect to market implied, or risk-neutral probabilities

$$p = \frac{e^{rT} - d}{u - d}$$



Risk-neutral probabilities

- Under the risk-neutral probabilities, non-paying dividend assets grow at the risk-free rate
- Expected stock return

$$E[S(T)] = p(uS_0) + (1-p)(dS_0) = e^{rT}S_0$$

Assets Other than Non-Dividend Paying Stocks

- For options on stock indices, currencies and futures the basic procedure for constructing the tree is the same except for the calculation of p

The Probability of an Up Move

$$p = \frac{a-d}{u-d}$$

$a = e^{r\Delta t}$ for a nondividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index wher e q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk - free rate

$a = 1$ for a futures contract

Growth Rates For Futures Prices

- A futures contract requires no initial investment
- In a risk-neutral world the expected return should be zero
- The expected growth rate of the futures price is therefore zero
- The futures price can therefore be treated like a stock paying a dividend yield of r
- We can use binomial trees to price options on futures, just as on any other asset