Option Pricing in Continuous Time

Stochastic Processes

- Describe the way in which a variable such as a stock price, exchange rate or interest rate changes through time
- Incorporates randomness reflecting the uncertainty in the values of these market variables in the future

Example 1

- Each day a stock price
  - increases by $1 with probability 30%
  - stays the same with probability 50%
  - reduces by $1 with probability 20%

Example 2

- Each day a stock price change is drawn from a normal distribution with mean \$0.2 and standard deviation \$1

Outline of lecture

- Basic introduction to stochastic processes (see FE 610)
  - Wiener process, aka Brownian motion
  - Geometric Brownian motion
- Black-Scholes-Merton model
- Option pricing in the BSM model
- Volatility

Markov Processes *(See pages 301-302)*

- Markov property: The past history is irrelevant for the future evolution. Only present state matters
- Future movements in a variable depend only on where we are, not the history of how we got to where we are
- We assume that stock prices follow Markov processes

Weak-Form Market Efficiency

- This asserts that it is impossible to produce consistently superior returns with a trading rule based on the past history of stock prices. In other words technical analysis does not work.
- A Markov process for stock prices is consistent with weak-form market efficiency

Example

- A variable is currently 40
- It follows a Markov process
- The changes of the variable over equal non-overlapping time intervals are i.i.d. (independent identically distributed).
- At the end of 1 year the variable has a normal probability distribution with mean 40 and standard deviation 10

Questions

- What is the probability distribution of the stock price at the end of 2 years?
- ½ years?
- ¼ years?
- Δt years?

Taking the limit $\Delta t \rightarrow 0$ defines a continuous time stochastic process

Variances & Standard Deviations

- In Markov processes changes in successive periods of time are independent
- This means that variances are additive
- Standard deviations are not additive

A Wiener Process *(See pages 302-303)*

- Define $\phi(\mu,\sigma^2)$ as a normal distribution with mean μ and variance σ^2
- A variable z follows a Wiener process (aka Brownian motion) if:
  - The change in z in any interval of time Δt is
    $\Delta z = \varepsilon\sqrt{\Delta t}$ where ε is $\phi(0,1)$
  - The values of Δz for any 2 different (non-overlapping) periods of time are independent

Properties of a Wiener Process

- Mean of $[z(T) - z(0)]$ is 0
- Variance of $[z(T) - z(0)]$ is T
- Standard deviation of $[z(T) - z(0)]$ is $\sqrt{T}$
- The changes $[z(T_3)-z(T_2)]$ and $[z(T_1)-z(T_0)]$ are independent $(T_2 \geq T_1)$

Taking Limits . . .

- The time increments can be taken arbitrarily small in a random walk with normally distributed increments
- In the limit as Δt tends to zero we get a continuous time stochastic process: the Wiener process, or Brownian motion
- The process is “fractal-looking”: it does not have a derivative at any point.

The Example Revisited

- A stock price starts at 40 and has a probability distribution of $\phi(40,100)$ at the end of the year
- If we assume the stochastic process is Markov with no drift, then the process is

$$dS = 10dz$$

- If the stock price were expected to grow by \$8 on average during the year, so that the year-end distribution is $\phi(48,100)$, the process would be

$$dS = 8dt + 10dz$$

Itô Process *(See pages 306)*

- In an Itô process the drift rate and the variance rate are functions of time

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

- The discrete time equivalent

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$$

is true in the limit as Δt tends to zero

Why a Wiener Process Is Not Appropriate for Stocks

- Wiener processes can become negative!
- Empirical evidence shows that expected *percentage* changes of stock prices in a short period of time remains constant (not its expected actual change)
- We can reasonably expect that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price

An Itô Process for Stock Prices

(See pages 306-309)

$$dS = \mu S\,dt + \sigma S\,dz$$

where μ is the expected return and σ is the volatility.

The discrete time equivalent is

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

The process is known as geometric Brownian motion

Monte Carlo Simulation

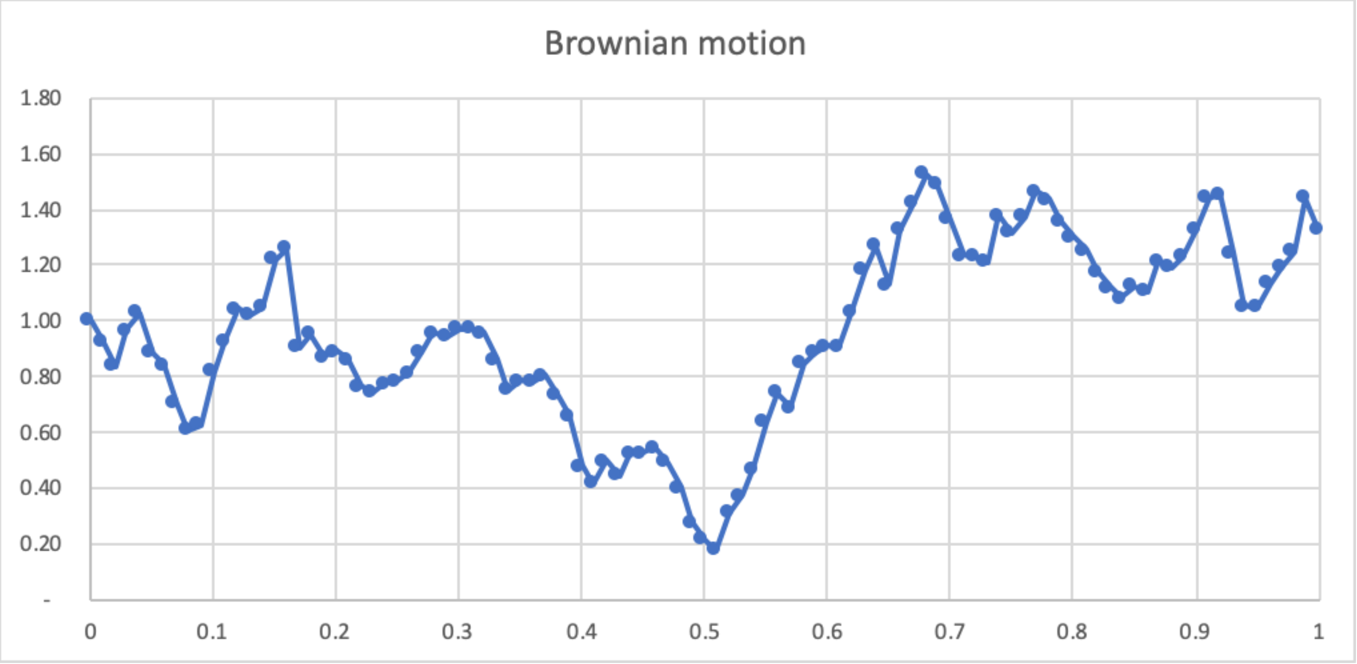
- We can sample random paths for the stock price by sampling values for ε
- Suppose μ= 0.15, σ= 0.30, and Δ*t* = 1 week (=1/52 or 0.192 years), then

$$\Delta S = 0.15 \times 0.0192 S + 0.30 \times \sqrt{0.0192}\varepsilon\, S$$

or

$$\Delta S = 0.00288 S + 0.0416 S\varepsilon$$

Example



Brownian motion
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
-
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Monte Carlo Simulation – Sampling one Path *(See Table 14.1, page 309)*

Week	Stock Price at Start of Period	Random Sample for ε	Change in Stock Price, ΔS
0	100.00	0.52	2.45
1	102.45	1.44	6.43
2	108.88	−0.86	−3.58
3	105.30	1.46	6.70
4	112.00	−0.69	−2.89

Itô's Lemma *(See pages 311-313)*

- If we know the stochastic process followed by x, Itô's lemma tells us the stochastic process followed by some function $G(x, t)$. When $dx = a(x,t)\,dt + b(x,t)\,dz$ then

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\, dz$$

- Since a derivative is a function of the price of the underlying asset and time, Itô's lemma plays an important part in the analysis of derivatives

Intuition for Itô's Lemma

- A Taylor's series expansion of $G(x, t)$ gives

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta x\,\Delta t + ½\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \ldots$$

Two applications of Ito lemma

1. The forward price of a stock for a contract maturing at time T

$$G = S\,e^{r(T-t)}$$

$$dG = (\mu - r)G\,dt + \sigma G\,dz$$

2. The log of a stock price

$$G = \ln S$$

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dz$$

The Black-Scholes-Merton Model

The Stock Price Assumption

- Consider a stock whose price is S
- In a short period of time of length Δt, the return on the stock is normally distributed:

$$\frac{\Delta S}{S} \approx \phi\left(\mu \Delta t, \sigma^2 \Delta t\right)$$

where μ is expected return and σ is the volatility

- The stock price process is assumed to follow a geometric Brownian motion

The Lognormal Property
(Equations 15.2 and 15.3, page 320)

- It follows from this assumption that

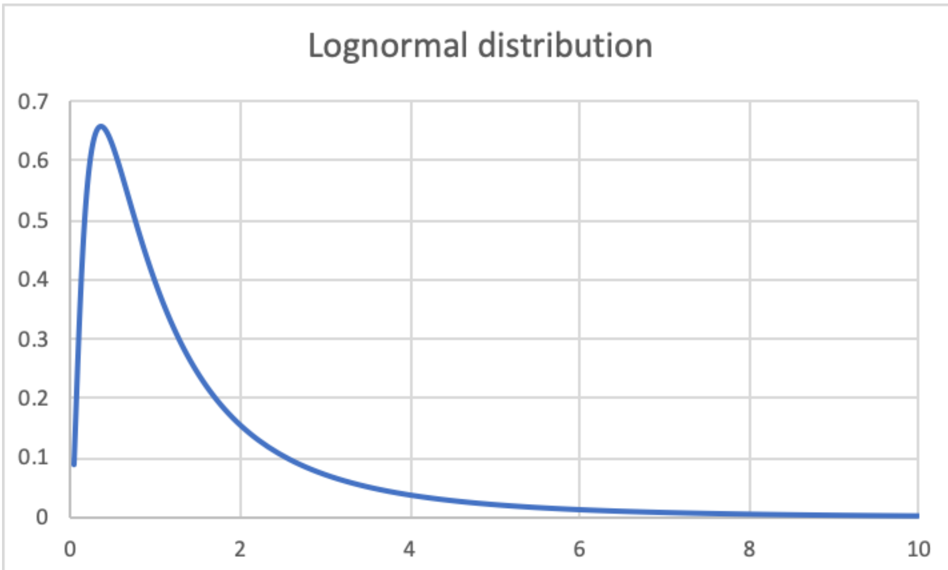
$$\ln S_T - \ln S_0 \approx \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

or

$$\ln S_T \approx \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

- Since the logarithm of S_T is normal, S_T is lognormally distributed

The Lognormal Distribution



$$E(S_T) = S_0 \, e^{\mu T}$$

$$\text{var}(S_T) = S_0^{\,2} e^{2\mu T}(e^{\sigma^2 T} - 1)$$

The Volatility

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year
- The standard deviation of the return in a short time period time Δt is approximately $\sigma\sqrt{\Delta t}$
- If a stock price is $50 and its volatility is 25% per year what is the standard deviation of the price change in one day?

Estimating Volatility from Historical Data *(page 324-326)*

1. Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years (e.g. for weekly data τ = 1/52)
2. Calculate the continuously compounded return in each interval as:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

3. Calculate the standard deviation, s , of the u_i´s
4. The historical volatility estimate is: $\hat{\sigma} = \frac{s}{\sqrt{\tau}}$

Nature of Volatility *(Business Snapshot 15.2, page 327)*

- Volatility is usually much greater when the market is open than when it is closed
- Overnight and over weekend price changes are smaller than intraday price changes
- For this reason, time is usually measured in “trading days” not calendar days when options are valued
- It is assumed that there are 252 trading days in one year for most assets

Black-Scholes-Merton – Main Idea

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes-Merton differential equation

Assumptions for BSM theory

- Market assumptions:
  - There is no arbitrage opportunity
  - It is possible to borrow/lend any amount of cash at the risk free rate
  - It is possible to buy and sell any amount of stock (short selling allowed)
  - There is no transaction cost
- Asset dynamics assumptions:
  - Constant deterministic interest rates
  - The stock price follows a geometric Brownian motion
  - The stock does not pay dividends

The derivation of the BSM equation

$$dS = \mu S\, dt + \sigma S\, dz$$

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dz$$

Set up a portfolio consisting of:

-1 : derivative

$+\dfrac{\partial f}{\partial S}$: shares

This gets rid of the dependence on *dz*

The Derivation of the Black-Scholes-Merton Differential Equation *continued*

The value of the portfolio, Π, is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

The derivation of the BSM differential equation (continued)

The return on the portfolio must be the risk-free rate. Hence

$$d\Pi = r\Pi dt$$

$$-df + \frac{\partial f}{\partial S} dS = r\left(-f + \frac{\partial f}{\partial S} S\right) dt$$

Substituting *dS* and *df* in this equation we get the Black-Scholes differential equation

$$\frac{\partial f}{\partial S} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

The Differential Equation

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In a forward contract the boundary condition is $f = S - K$ when $t = T$
- The solution to the equation is

$$f = S - K e^{-r(T-t)}$$

Risk-Neutral Valuation

- The variable μ does not appear in the Black-Scholes-Merton differential equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation

Applying Risk-Neutral Valuation

1. Assume that the expected return from the stock price is the risk-free rate
2. Calculate the expected payoff from the option
3. Discount at the risk-free rate

Valuing a Forward Contract with Risk-Neutral Valuation

- Payoff is $S_T - K$
- Expected payoff in a risk-neutral world is $S_0 e^{rT} - K$
- Present value of expected payoff is

$$e^{-rT}[S_0 e^{rT} - K] = S_0 - Ke^{-rT}$$

Proving Black-Scholes-Merton Using Risk-Neutral Valuation *(Appendix to Chapter 15)*

$$c = e^{-rT}E[\max(S_T - K, 0)] = e^{-rT}\int_K^{\infty}\max(S_T - K, 0)\, g(S_T)dS_T$$

where $g(S_T)$ is the probability density function for the lognormal distribution of S_T in a risk-neutral world. Recall that ln(S_T /S_0) is normally distributed like $\phi(m, s^2)$ with $m = (r - 1/2\sigma^2)T, s = \sigma\sqrt{T}$

Change the integration variable to $Q = \dfrac{\ln S_T - m}{s}$

so that $$c = e^{-rT}\int_{(\ln K - m)/s}^{\infty}\max(e^{Qs+m} - K,\ 0)h(Q)dQ$$

where h is the probability density function for a standard normal. Evaluating the integral leads to the BS equation.

The Black-Scholes-Merton Formulas for Options *(See pages 333-334)*

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

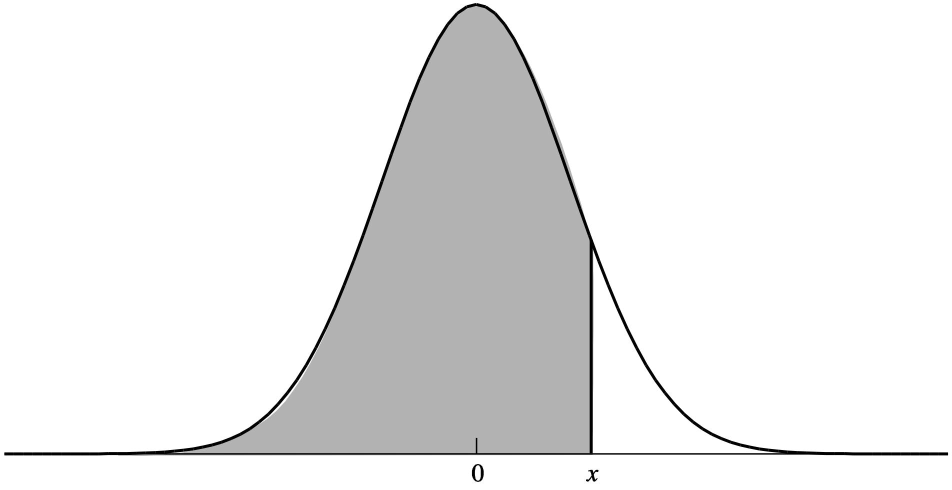
$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The N(x) Function

- $N(x)$ is the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 is less than x



- Can be evaluated in Excel with *NORMDIST(x,0,1,1)*

Properties of Black-Scholes Formula

- As S_0 becomes very large c tends to $S_0 - Ke^{-rT}$ and p tends to zero
- As S_0 becomes very small c tends to zero and p tends to $Ke^{-rT} - S_0$
- What happens as σ becomes very large?
- What happens as T becomes very large?

Understanding Black-Scholes

$$c = e^{-rT} N(d_2)\left(S_0 e^{rT} N(d_1)/N(d_2) - K\right)$$

e^{-rT} : Present value factor

$N(d_2)$: Probability of exercise

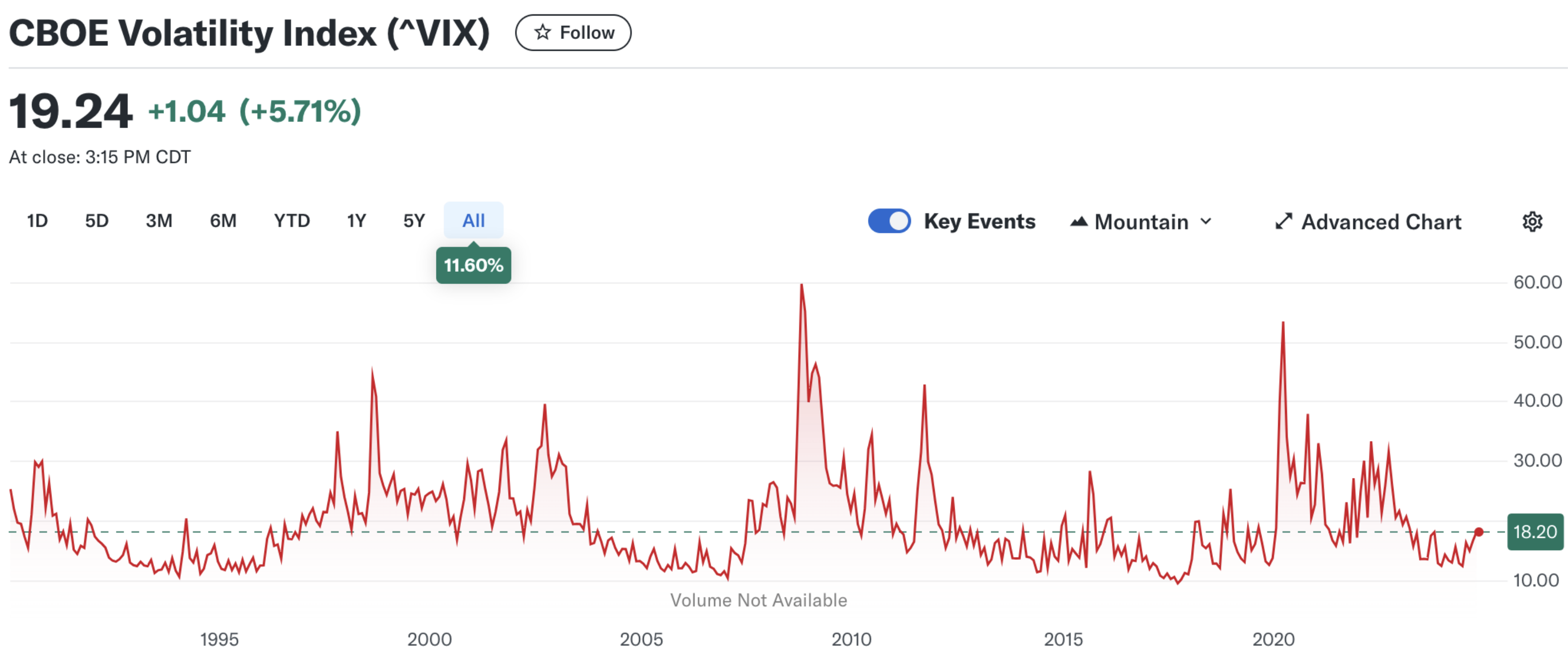
$S_0 e^{rT} N(d_1)/N(d_2)$: Expected stock price in a risk-neutral world if option is exercised

K : Strike price paid if option is exercised

Implied Volatility

- The implied volatility of an option is the volatility for which the Black-Scholes-Merton price equals the market price
- There is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

The VIX S&P500 Volatility Index



Dividends

- European options on dividend-paying stocks are valued by substituting the stock price less the present value of dividends into Black-Scholes
- Only dividends with ex-dividend dates during life of option should be included
- The “dividend” should be the expected reduction in the stock price expected