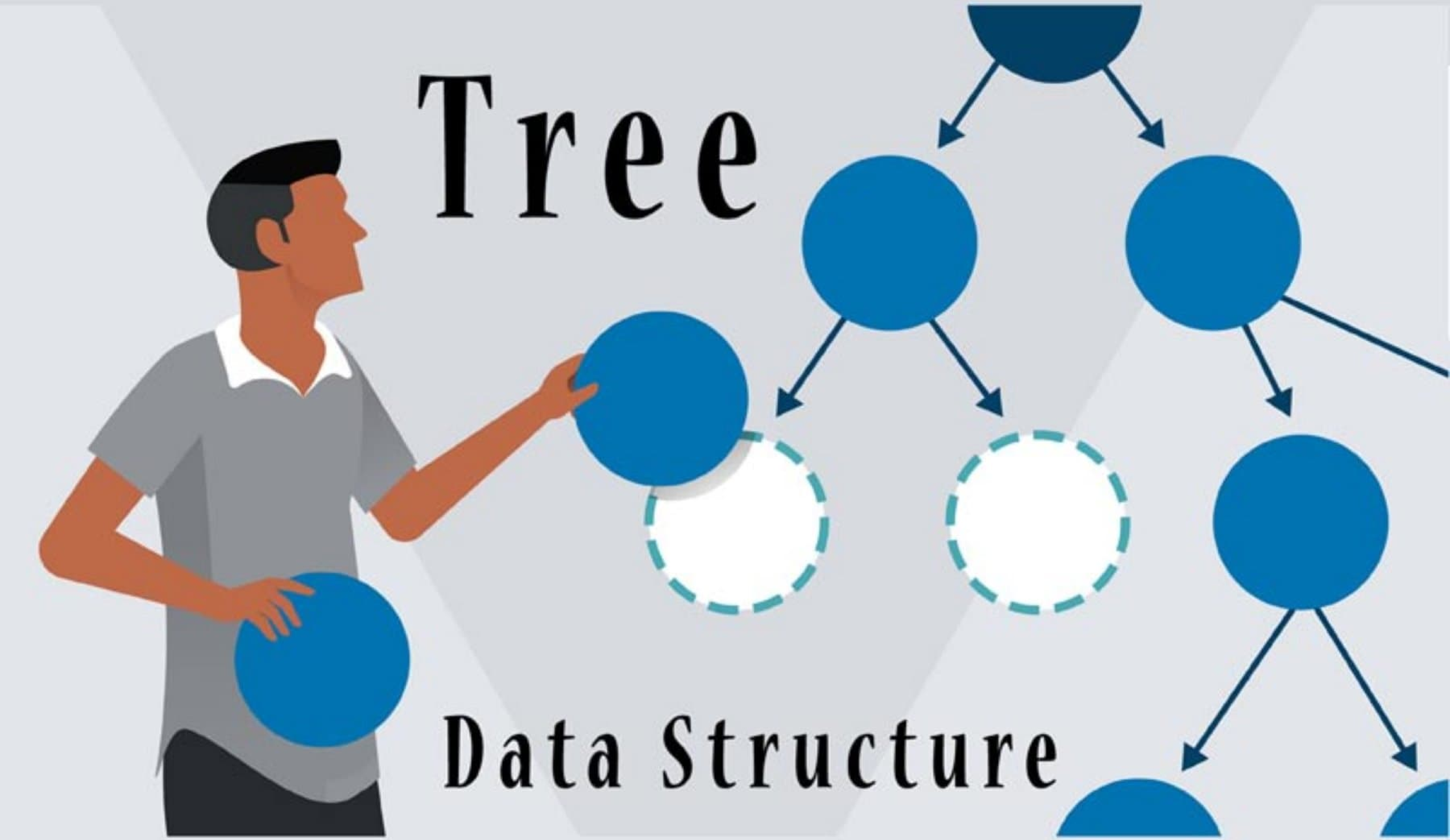
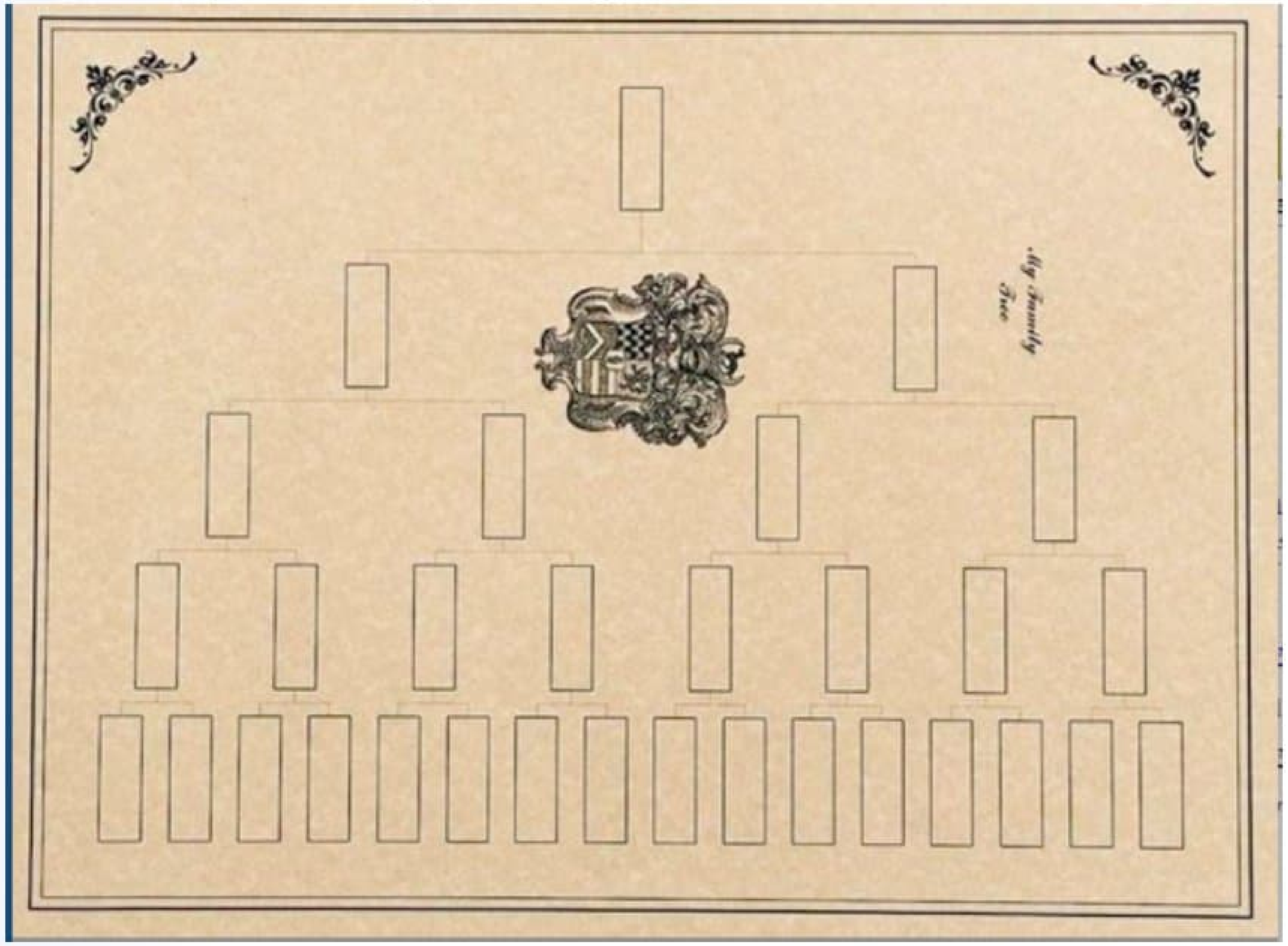

Tree
Data Structure

Summary

- Topics
  - general trees, definitions and properties
  - interface and implementation
  - tree traversal algorithms
    - depth and height
    - pre-order traversal
    - post-order traversal
  - binary trees
    - properties
    - interface
    - implementation
  - binary search trees
    - definition
    - h-n relationship
    - search, insert, delete
    - performance
- READING:
  - GT textbook chapter 7 and 10.1

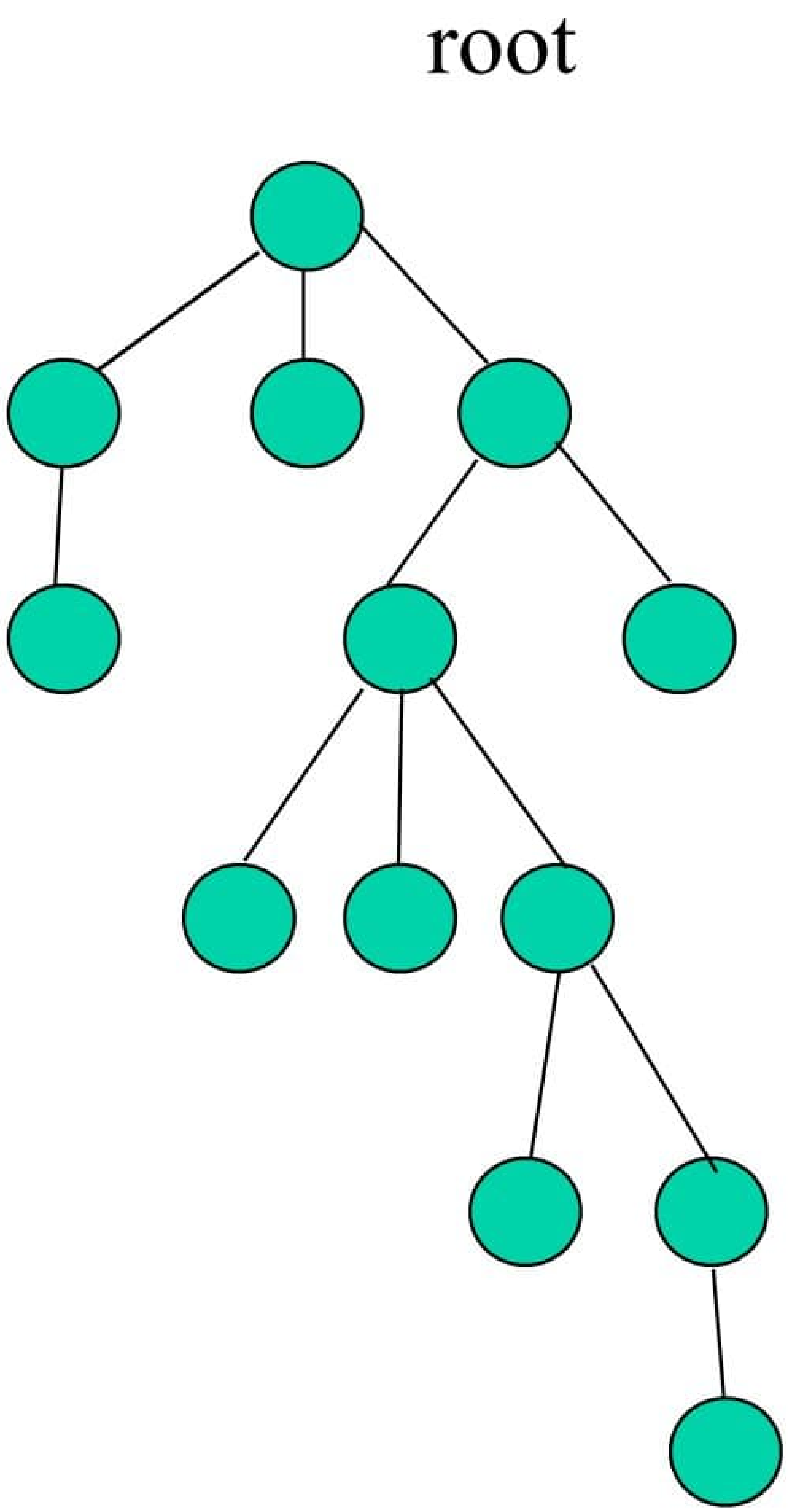
Trees

- So far we have seen linear structures
  - linear: before and after relationship
  - lists, vectors, arrays, stacks, queues, etc
- Non-linear structure: trees
  - probably the most fundamental structure in computing
  - hierarchical structure
  - Terminology: from family trees (genealogy)



Trees

- store elements hierarchically
- the top element: root
- except the root, each element has a parent
- each element has 0 or more children



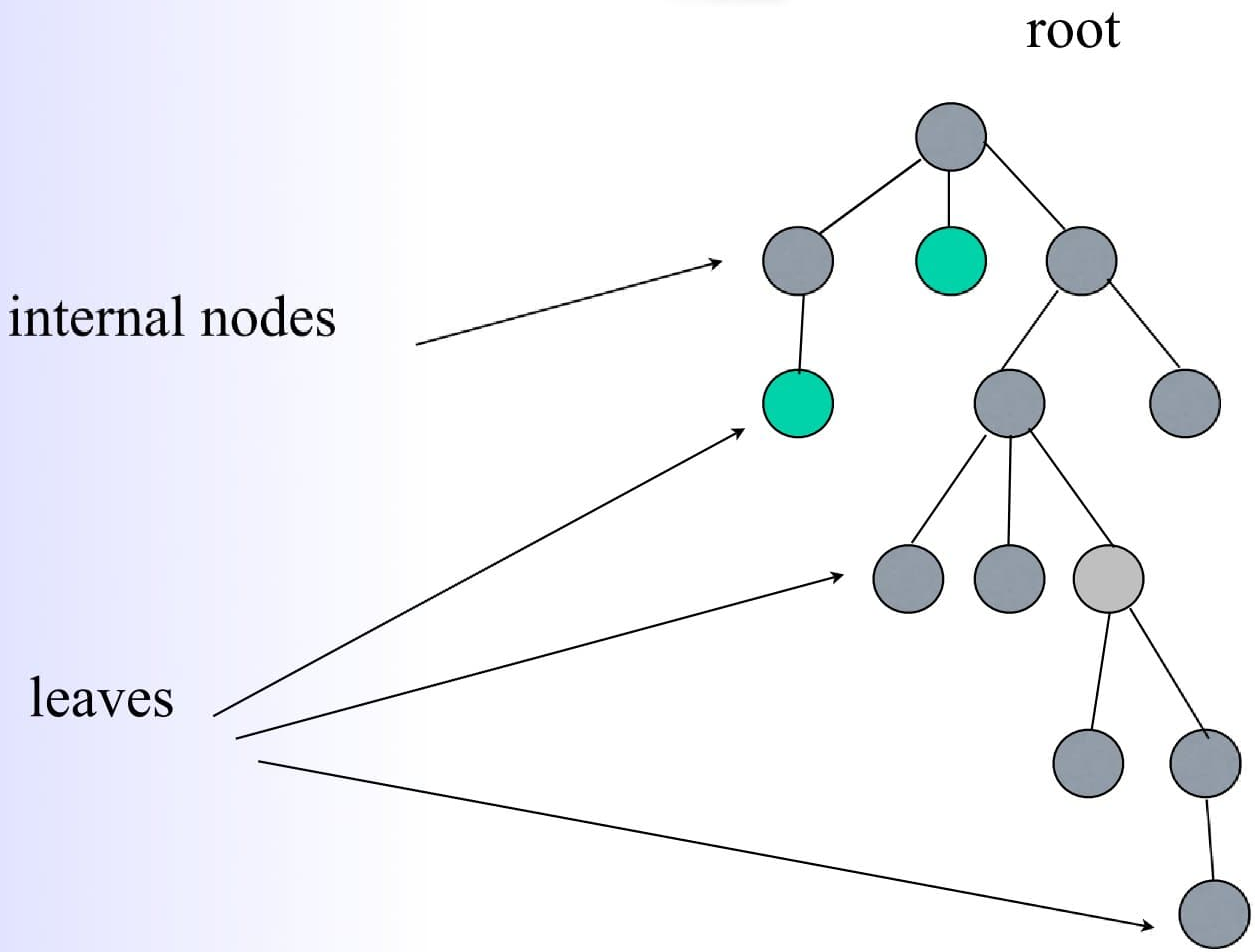
Trees

- Definition
  - A tree T is a set of nodes storing elements such that the nodes have a parent-child relationship that satisfies the following
    - if T is not empty, T has a special tree called the root that has no parent
    - each node v of T different than the root has a unique parent node w; each node with parent w is a child of w

- Recursive definition
  - T is either empty
  - or consists of a node r (the root) and a possibly empty set of trees whose roots are the children of r

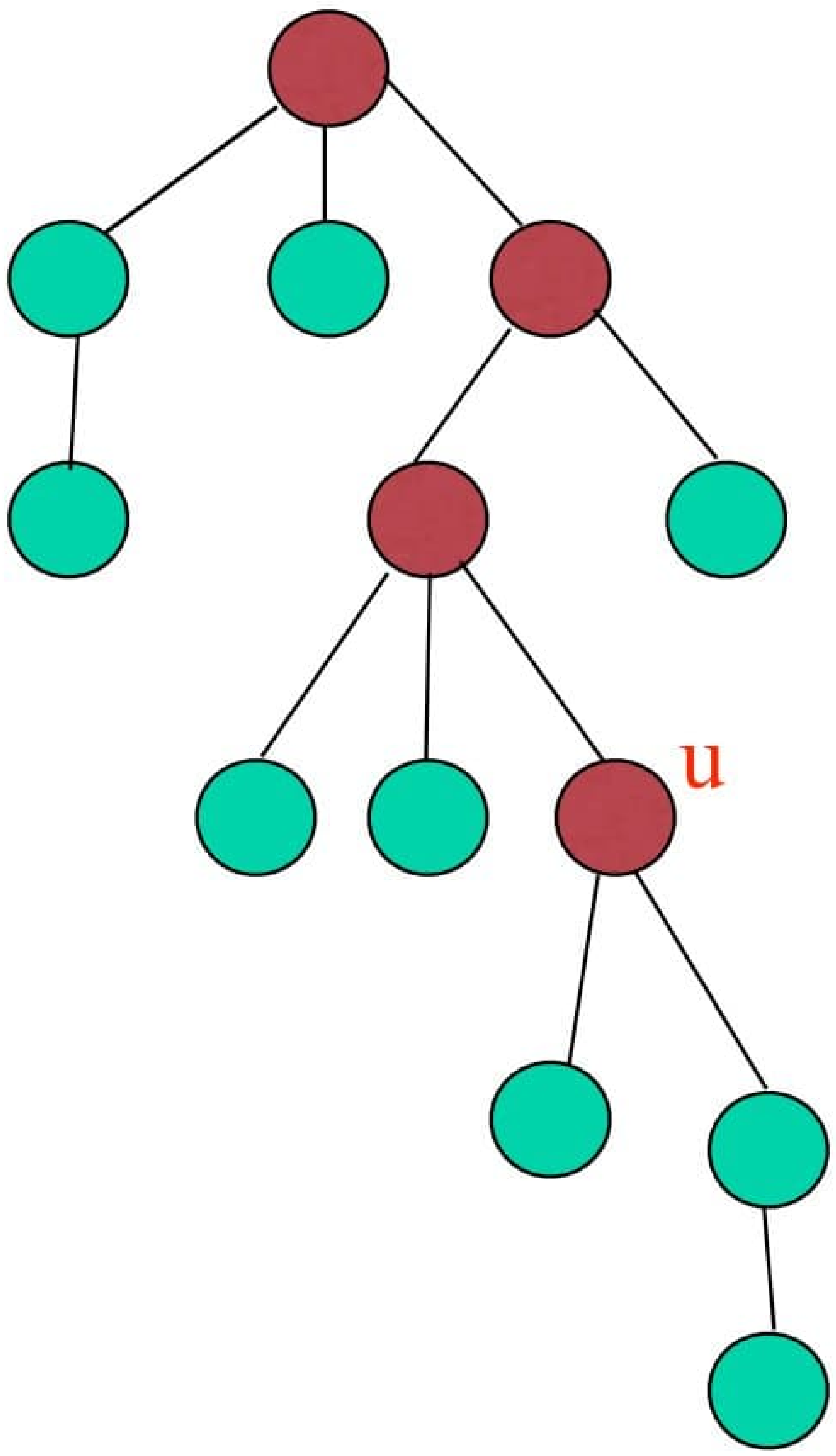
- Terminology
  - siblings: two nodes that have the same parent are called siblings
  - internal nodes
    - nodes that have children
  - external nodes or leaves
    - nodes that don't have children
  - ancestors
  - descendants

Trees



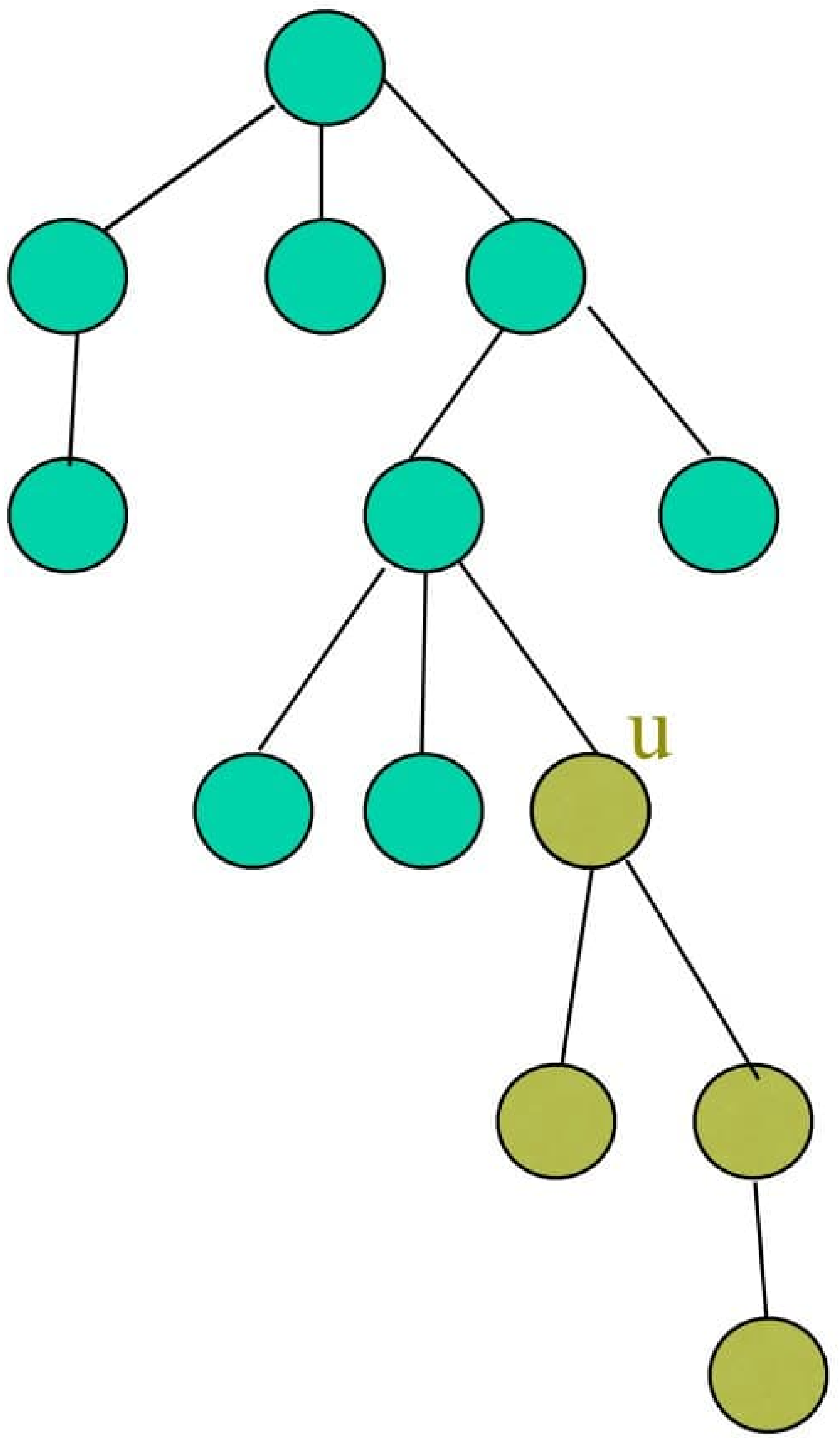
Trees

ancestors of u



Trees

descendants of u



Application of trees

- Applications of trees
  - class hierarchy in Java
  - file system
  - storing hierarchies in organizations

Tree ADT

- Whatever the implementation of a tree is, its interface is the following
  - root()
  - size()
  - isEmpty()
  - parent(v)
  - children(v)
  - isInternal(v)
  - isExternal(v)
  - isRoot()

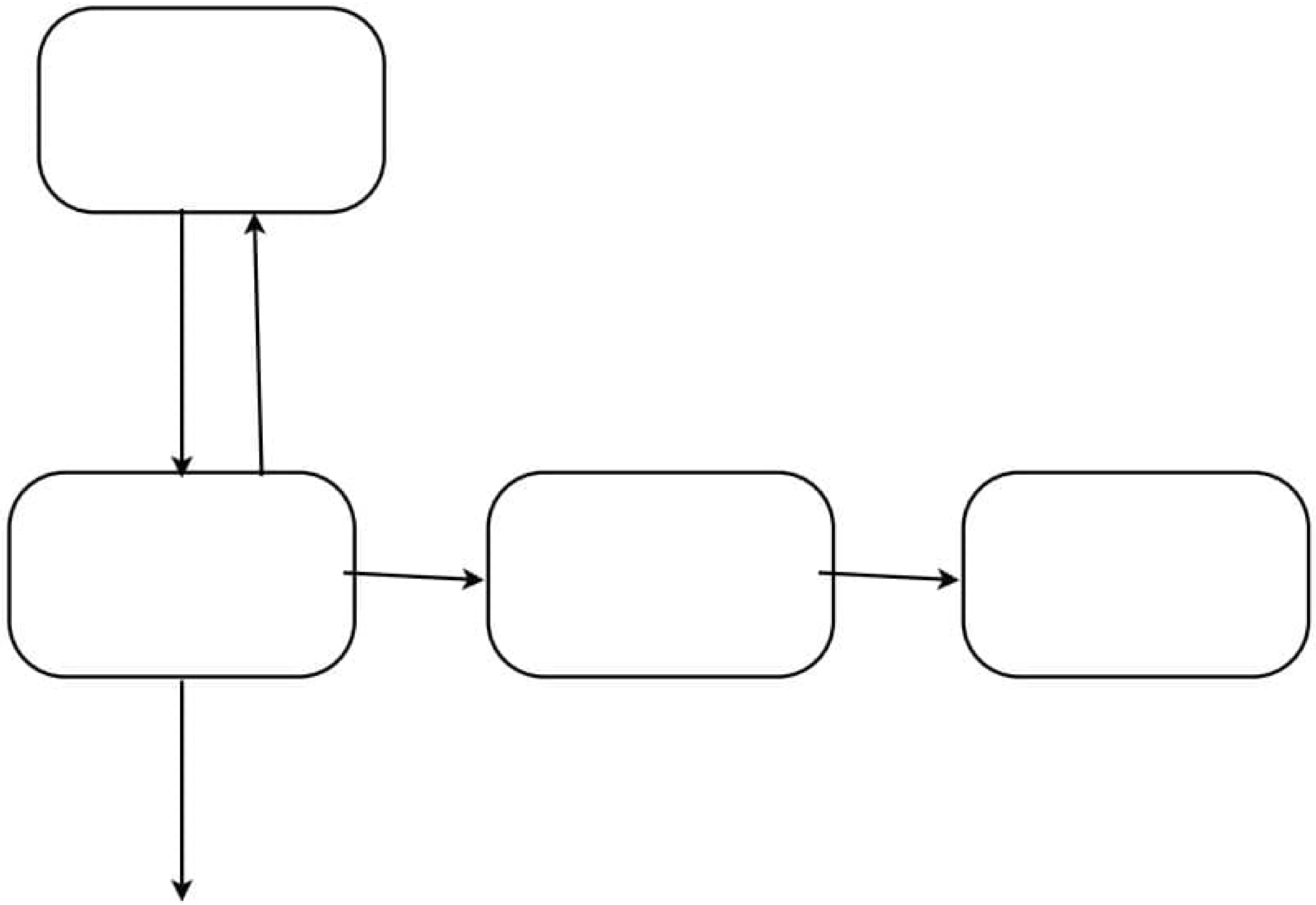
Tree Implementation

```
class Tree {
    TreeNode root;

    //tree ADT methods..
}


class TreeNode<Type> {
    Type data;
    int size;
    TreeNode parent;
    TreeNode firstChild;
    TreeNode nextSibling;


    getParent();
    getChild();
    getNextSibling();
}
```



Algorithms on trees: Depth

- Depth:
  - depth(T, v) is the number of ancestors of v, excluding v itself

- Recursive formulation
  - if v == root, then depth(v) = 0
  - else, depth(v) is 1 + depth (parent(v))

- Compute the depth of a node v in tree T: int depth(T, v)

- Algorithm:

```
int depth(T,v) {
    if T.isRoot(v) return 0
    return 1 + depth(T, T.parent(v))
}
```

- Analysis:
  - O(number of ancestors) = O(depth_v)
  - in the worst case the path is a linked-list and v is the leaf
  - ==> O(n), where n is the number of nodes in the tree

Algorithms on trees: Height

- Height:
  - height of a node v in T is the length of the longest path from v to any leaf
- Recursive formulation:
  - if v is leaf, then its height is 0
  - else height(v) = 1 + maximum height of a child of v
- Definition: the height of a tree is the height of its root
- Compute the height of tree T: int height(T,v)
- Height and depth are "symmetrical"
- Proposition: the height of a tree T is the maximum depth of one of its leaves.

Height

- Algorithm:

```
int height(T,v) {
        if T.isExternal(v) return 0;
        int h = 0;
        for each child w of v in T do
                h = max(h, height(T, w))
        return h+1;
}
```

- Analysis:
  - total time: the sum of times spent at all nodes in all recursive calls
  - the recursion:
    - v calls height(w) recursively on all children w of v
    - height() will eventually be called on every descendant of v
    - overall: height() is called on each node precisely once, because each node has one parent
  - aside from recursion
    - for each node v: go through all children of v
      - O(1 + c_v) where c_v is the number of children of v
    - over all nodes: O(n) + SUM (c_v)
      - each node is child of only one node, so its processed once as a child
      - SUM(c_v) = n - 1
  - total: O(n), where n is the number of nodes in the tree

Tree traversals

- A traversal is a systematic way to visit all nodes of T.

- pre-order: root, children
  - parent comes before children; overall root first
- post-order: children, root
  - parent comes after children; overall root last

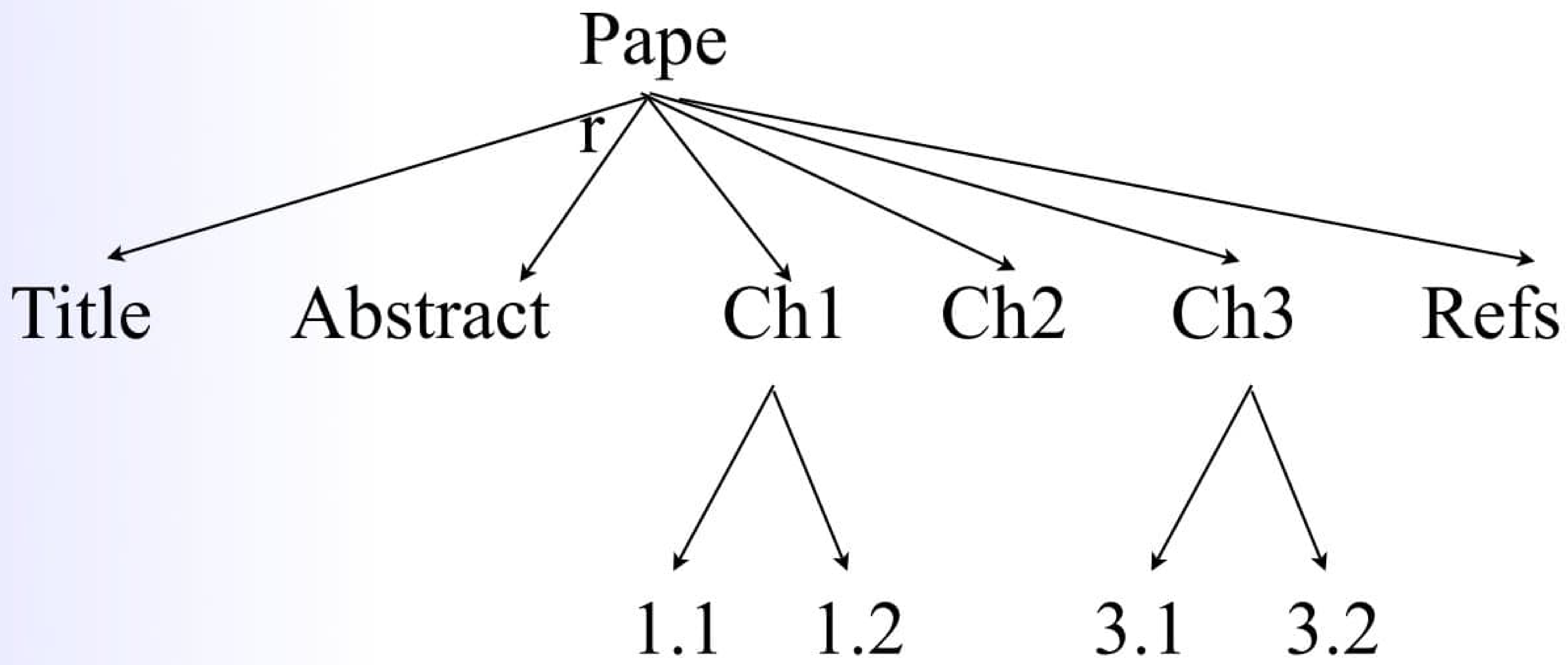
```
void preorder(T, v)
    visit v
    for each child w of v in T do
        preorder(w)


void postorder(T, v)
    for each child w of v in T do
        postorder(w)
    visit v
```

- Analysis: O(n) [same arguments as before]

Examples

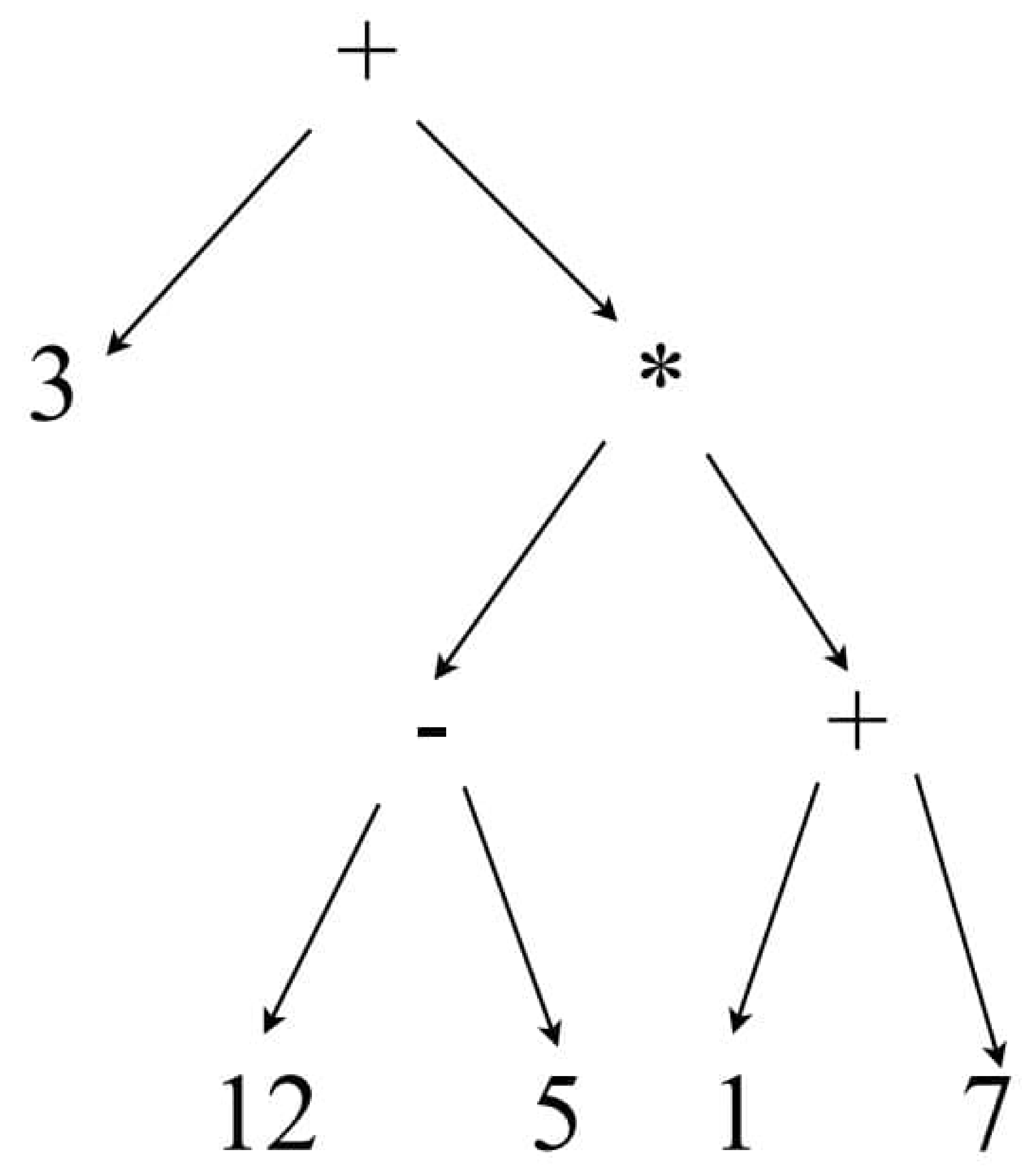
- Tree associated with a document



- In what order do you read the document?

Example

- Tree associated with an arithmetical expression



- Write method that evaluates the expression. In what order do you traverse the tree?

Binary trees

Binary trees

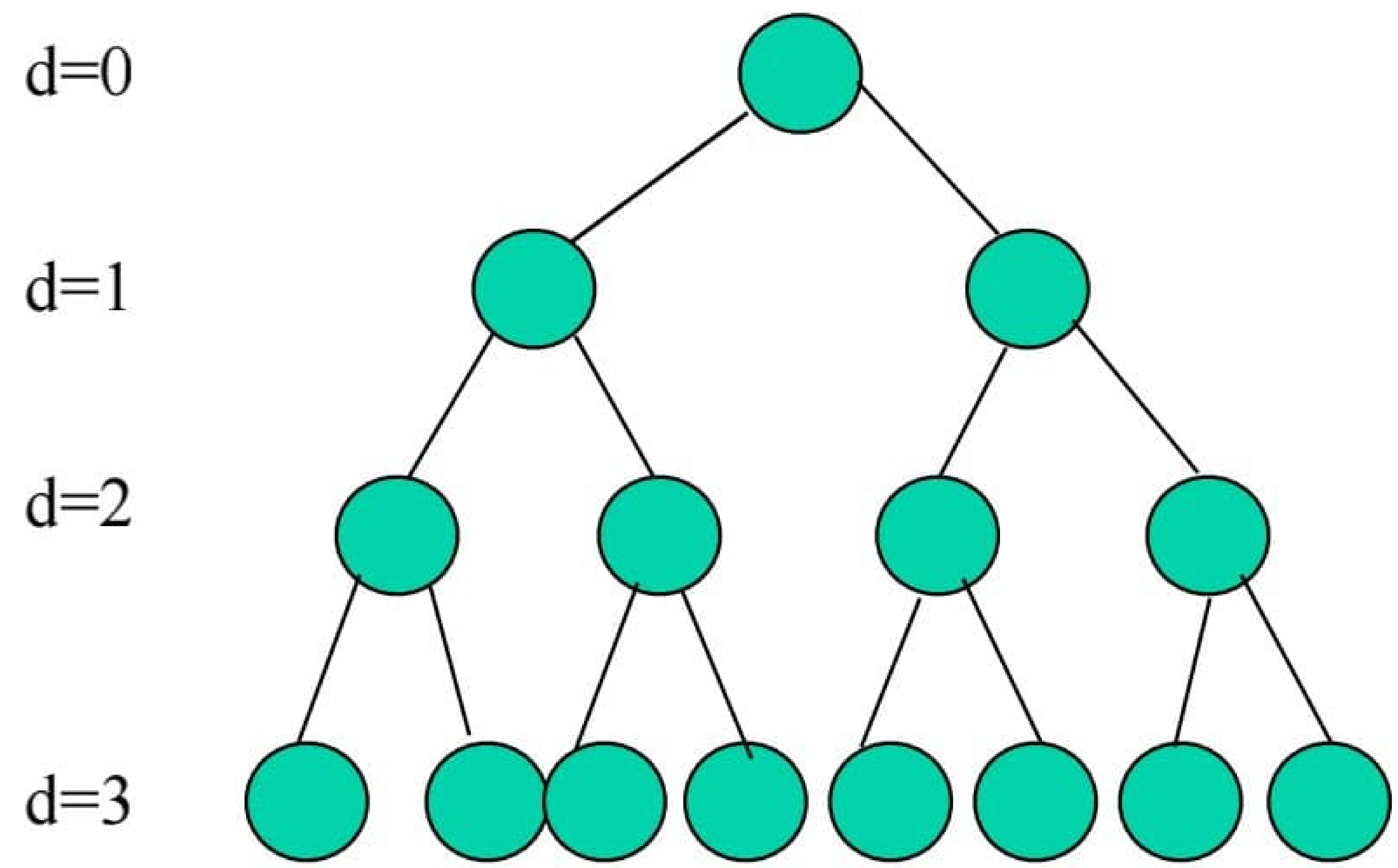
- Definition: A binary tree is a tree such that
  - every node has at most 2 children
  - each node is labeled as being either a left chilld or a right child

- Recursive definition:
  - a binary tree is empty;
  - or it consists of
    - a node (the root) that stores an element
    - a binary tree, called the left subtree of T
    - a binary tree, called the right subtree of T

- Binary tree interface
    - left(v)
    - right(v)
    - hasLeft(v)
    - hasRight(v)
    - \+ isInternal(v), is External(v), isRoot(v), size(), isEmpty()

Properties of binary trees

- In a binary tree
  - level 0 has <= 1 node
  - level 1 has <= 2 nodes
  - level 2 has <= 4 nodes
  - ...
  - level i has <= 2^i nodes

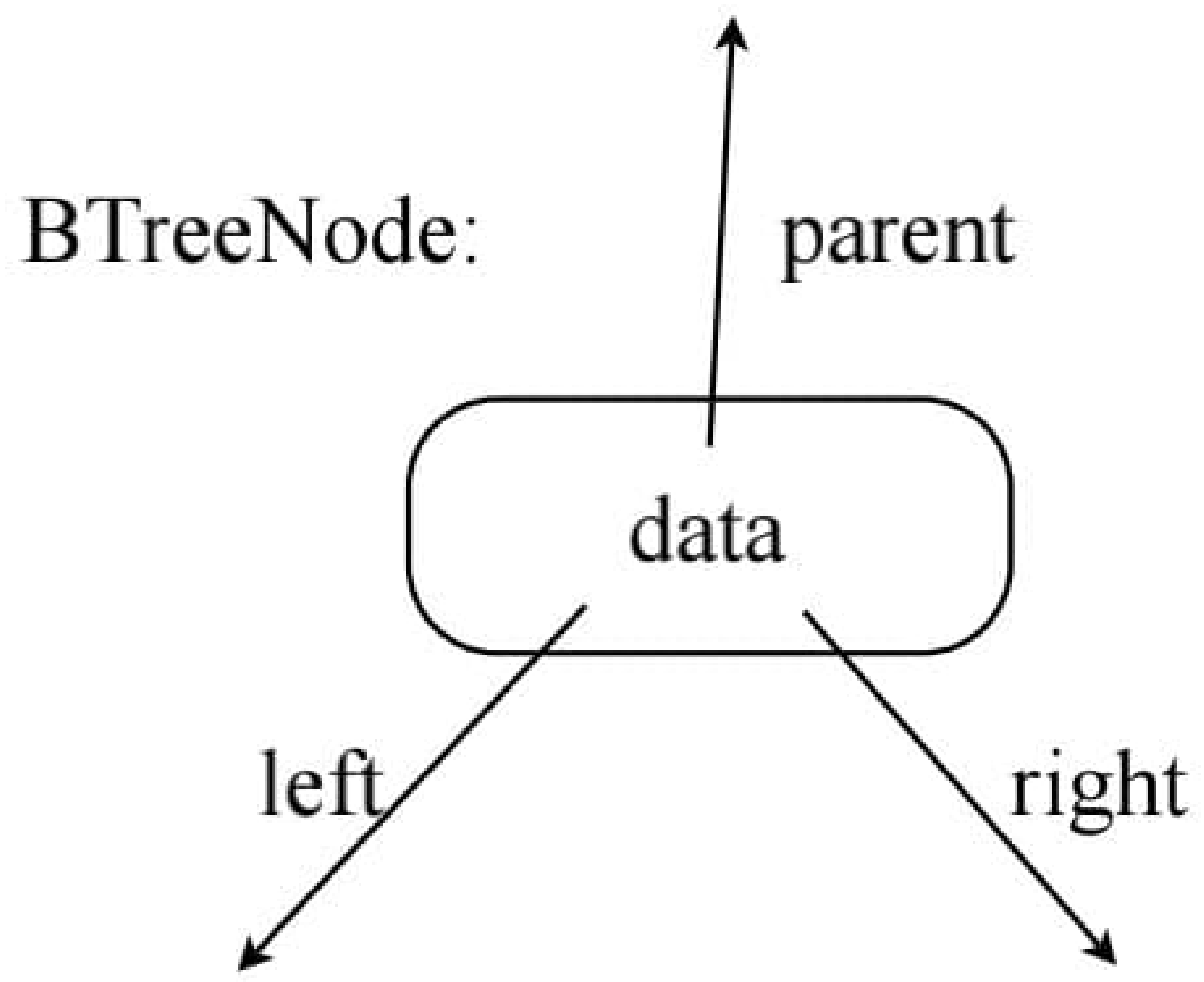


- Proposition: Let T be a binary tree with n nodes and height h. Then
  - h+1 <= n <= 2^{h+1} -1
  - lg(n+1) - 1 <= h <= n-1

Binary tree implementation

- use a linked-list structure; each node points to its left and right children ; the tree class stores the root node and the size of the tree

- implement the following functions:
  - left(v)
  - right(v)
  - hasLeft(v)
  - hasRight(v)
  - isInternal(v)
  - is External(v)
  - isRoot(v)
  - size()
  - isEmpty()
- also
  - insertLeft(v,e)
  - insertRight(v,e)
  - remove(e)
  - addRoot(e)



Binary tree operations

- insertLeft(v,e):
  - create and return a new node w storing element e, add w as the left child of v
  - an error occurs if v already has a left child
- insertRight(v,e)
- remove(v):
  - remove node v, replace it with its child, if any, and return the element stored at v
  - an error occurs if v has 2 children
- addRoot(e):
  - create and return a new node r storing element e and make r the root of the tree;
  - an error occurs if the tree is not empty
- attach(v,T1, T2):
  - attach T1 and T2 respectively as the left and right subtrees of the external node v
  - an error occurs if v is not external

Performance

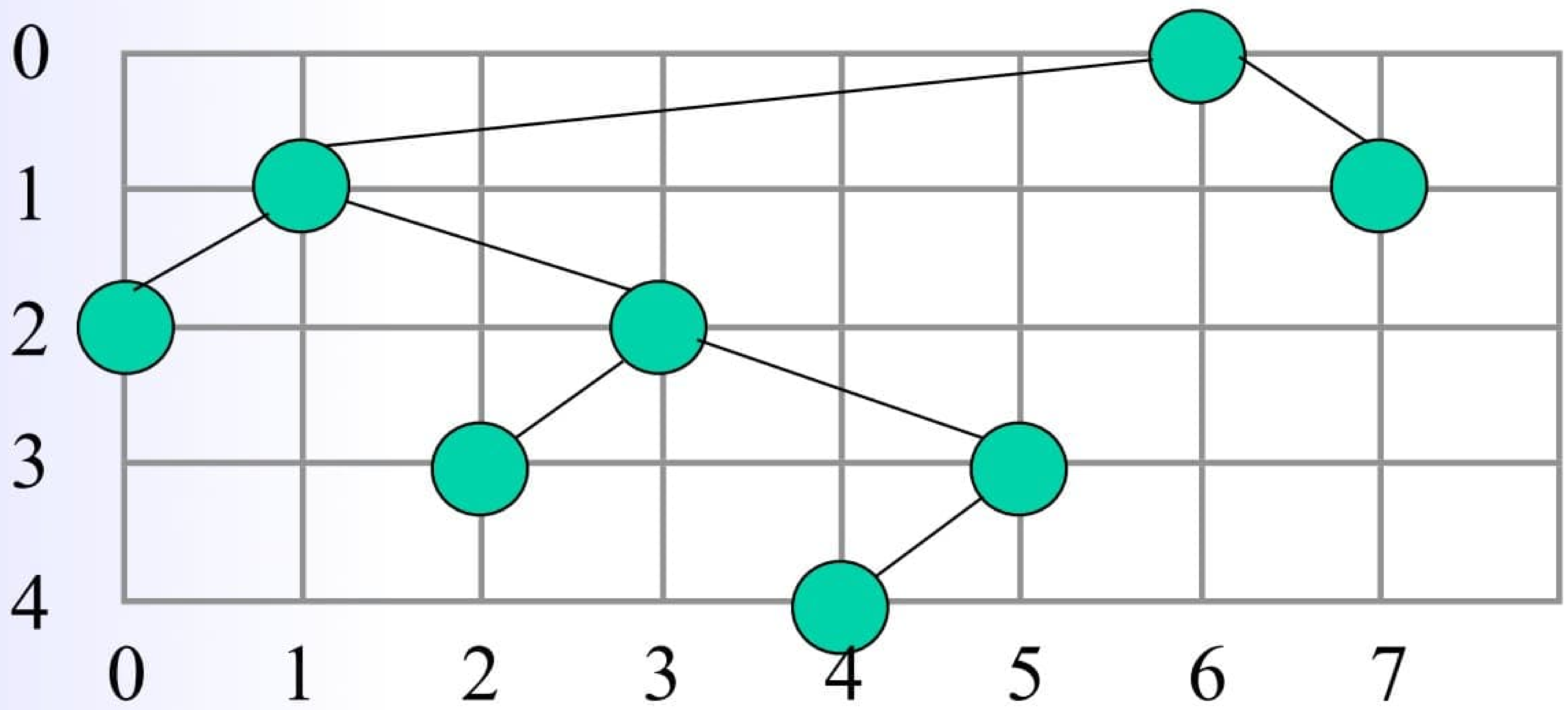
- all O(1)
  - left(v)
  - right(v)
  - hasLeft(v)
  - hasRight(v)
  - isInternal(v)
  - is External(v)
  - isRoot(v)
  - size()
  - isEmpty()
  - addRoot(e)
  - insertLeft(v,e)
  - insertRight(v,e)
  - remove(e)

Binary tree traversals

- Binary tree computations often involve traversals
  - pre-order: root left right
  - post-order: left right root
- additional traversal for binary trees
  - in-order: left root right
    - visit the nodes from left to right
- Exercise:
  - write methods to implement each traversal on binary trees

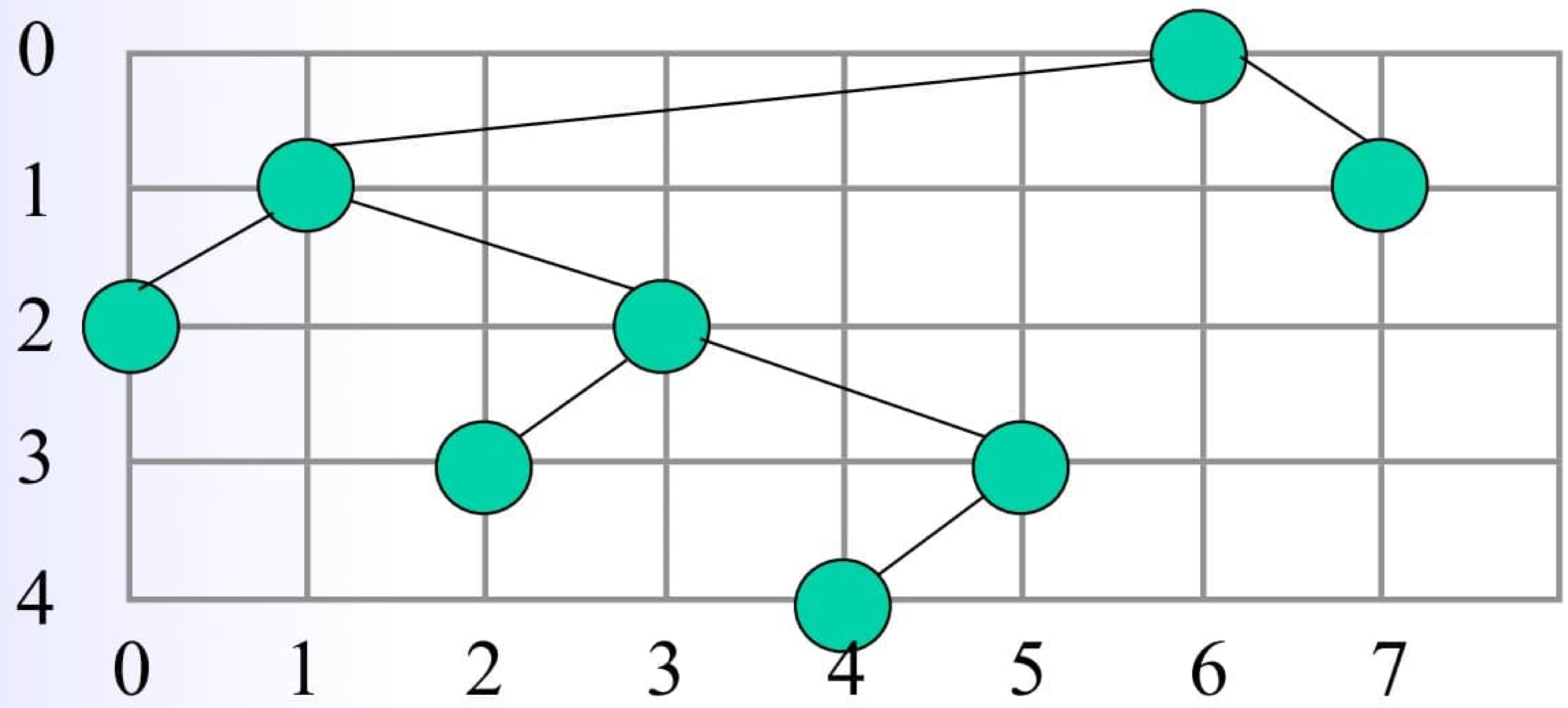
Application: Tree drawing

- Come up with a solution to "draw" a binary tree in the following way. Essentially, we need to assign coordinate x and y to each node.
    - node v in the tree
        - x(v) = ?
        - y(v) = ?



Application: Tree drawing

- We can use an in-order traversal and assign coordinate x and y of each node in the following way:
  - x(v) is the number of nodes visited before v in the in-order traversal of v
  - y(v) is the depth of v



Binary tree searching

- write search(v, k)
  - search for element k in the subtree rooted at v
  - return the node that contains k
  - return null if not found

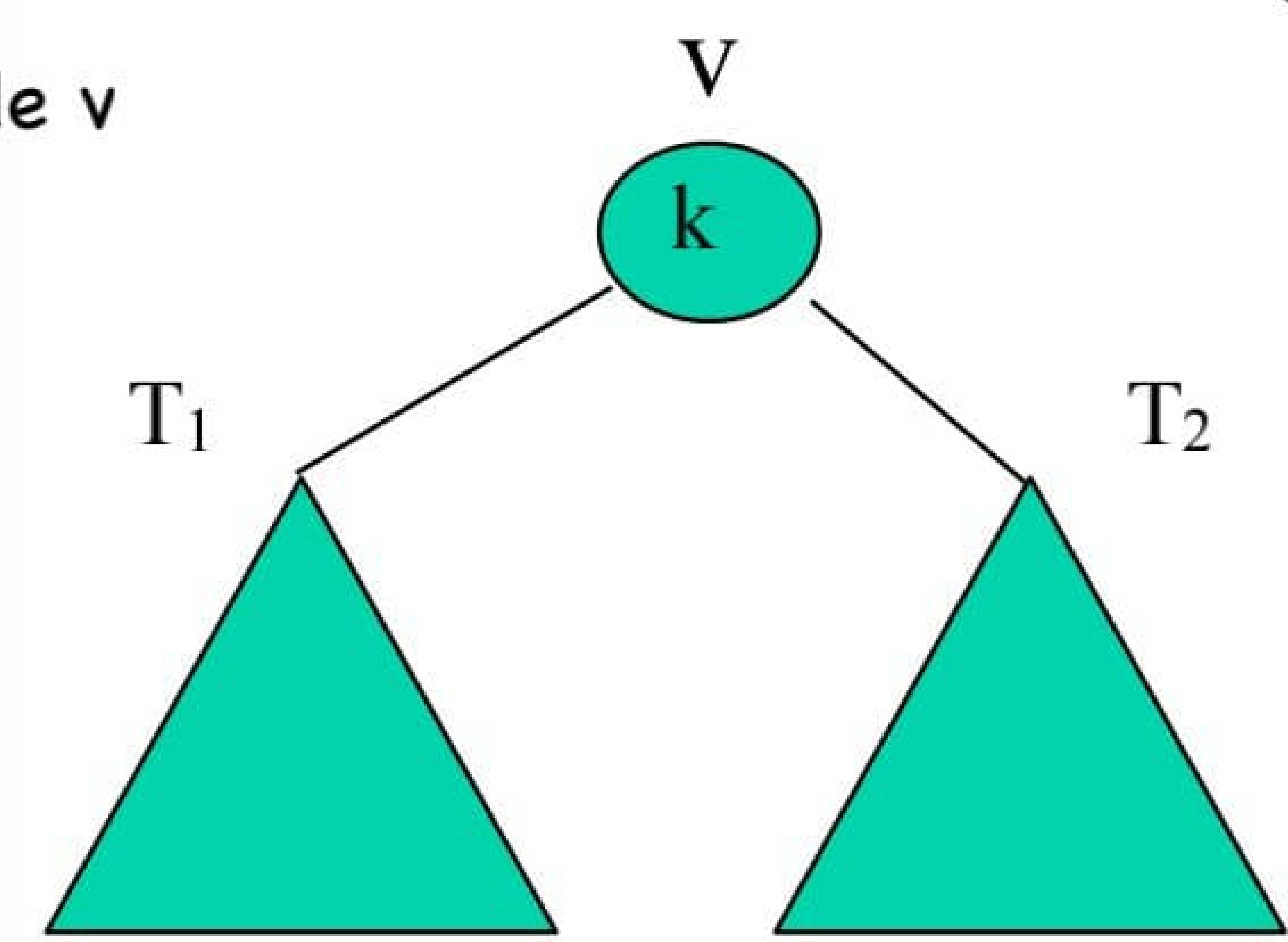
- performance
  - ?

Binary Search Trees (BST)

- Motivation:
  - want a structure that can search fast
  - arrays: search fast, updates slow
  - linked lists: search slow, updates fast
- Intuition:
  - tree combines the advantages of arrays and linked lists
- Definition:
  - a BST is a binary tree with the following "search" property

allows to search efficiently

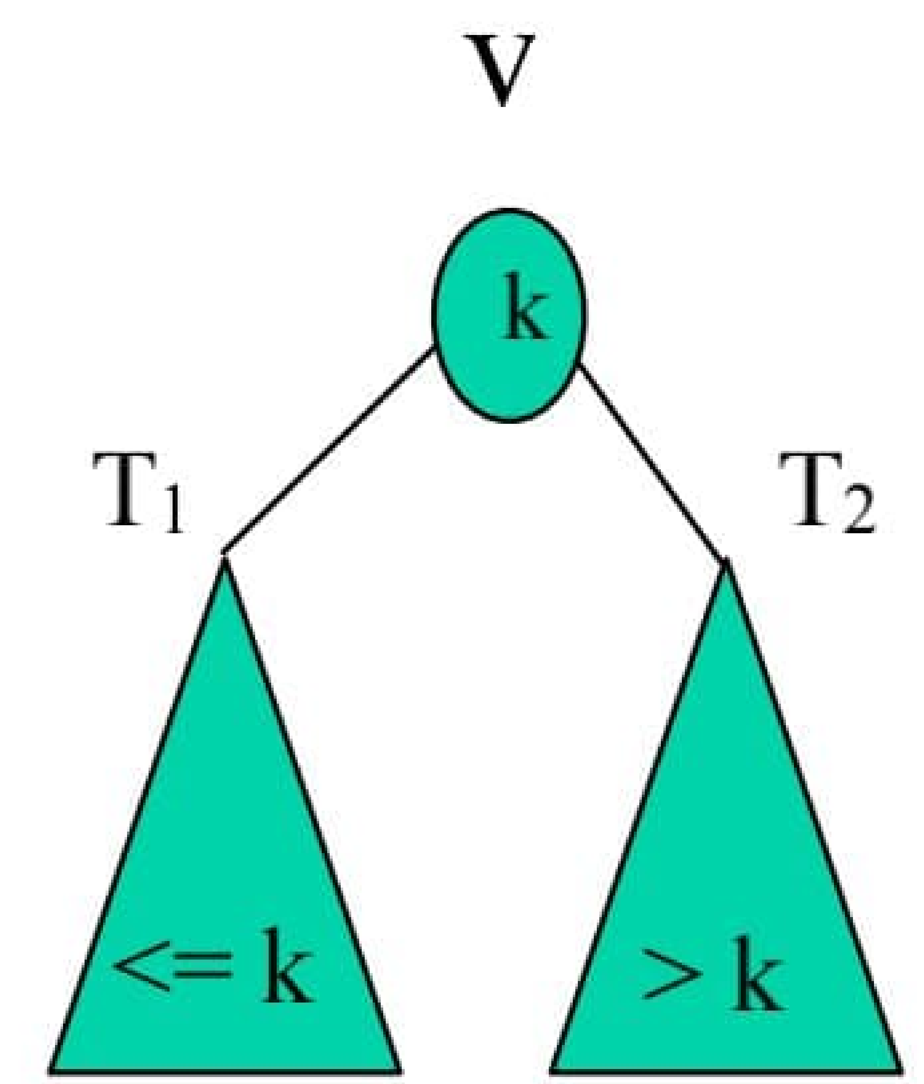
- for any node v

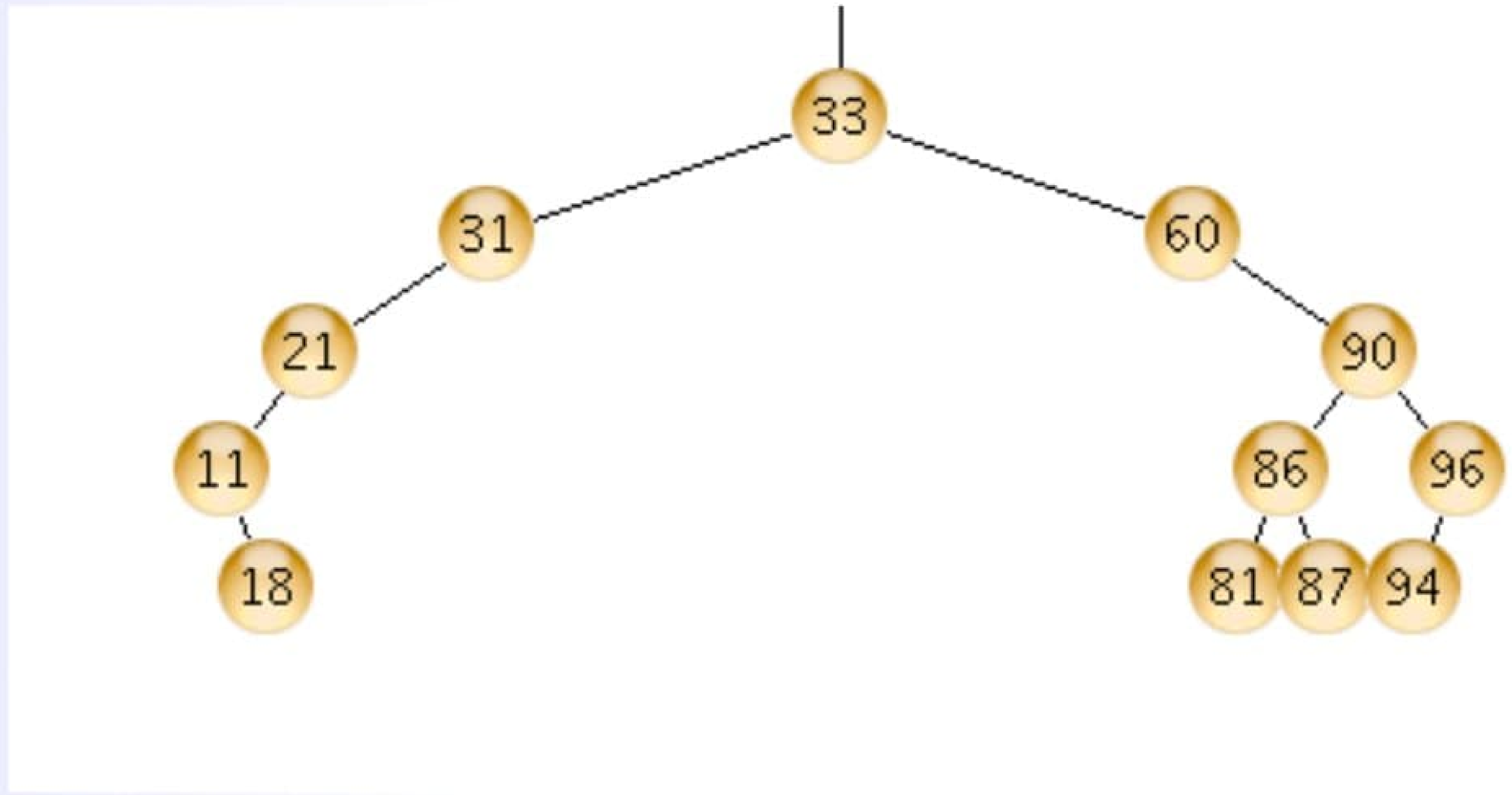


all nodes in T1<= k   all node in T2 > k

BST

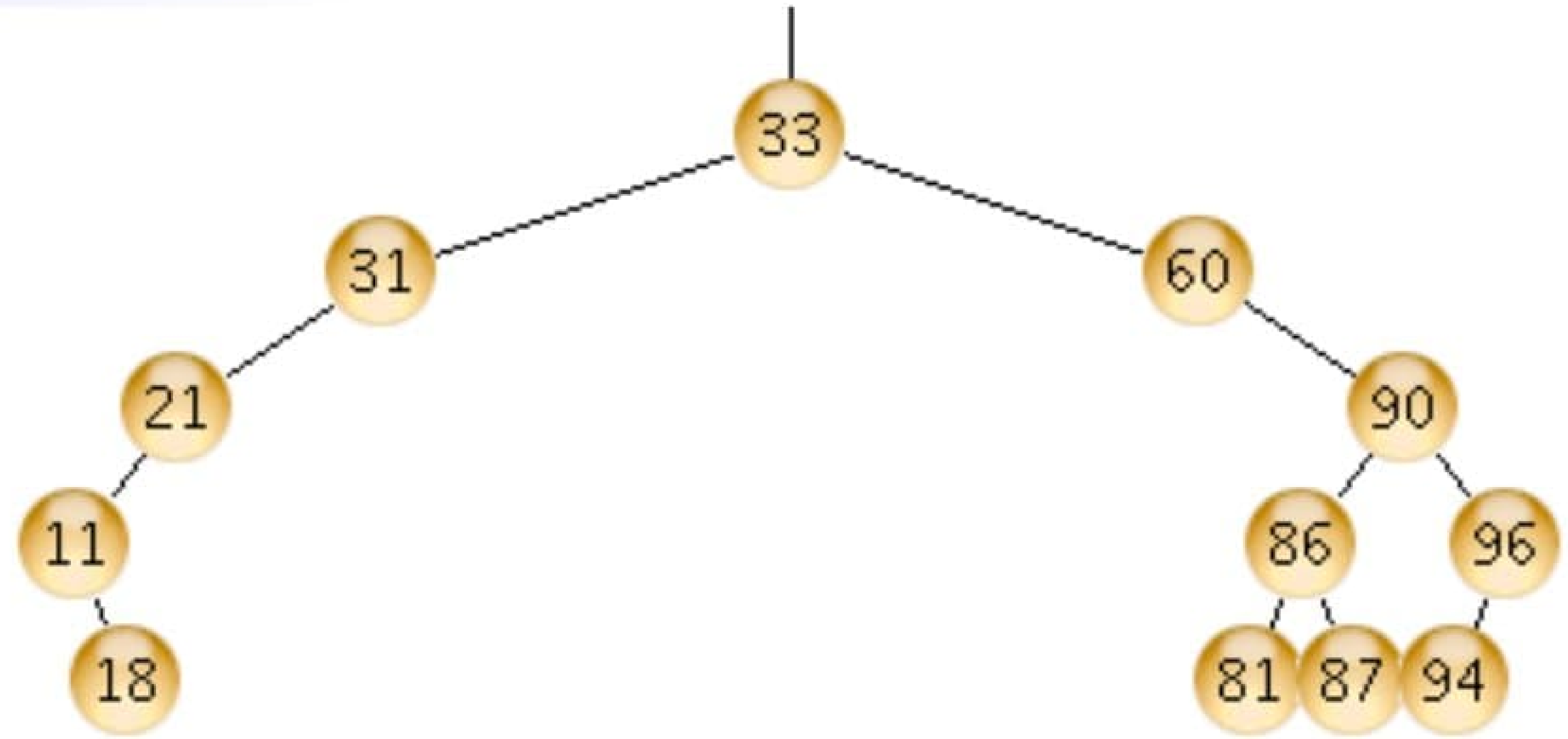
- Example





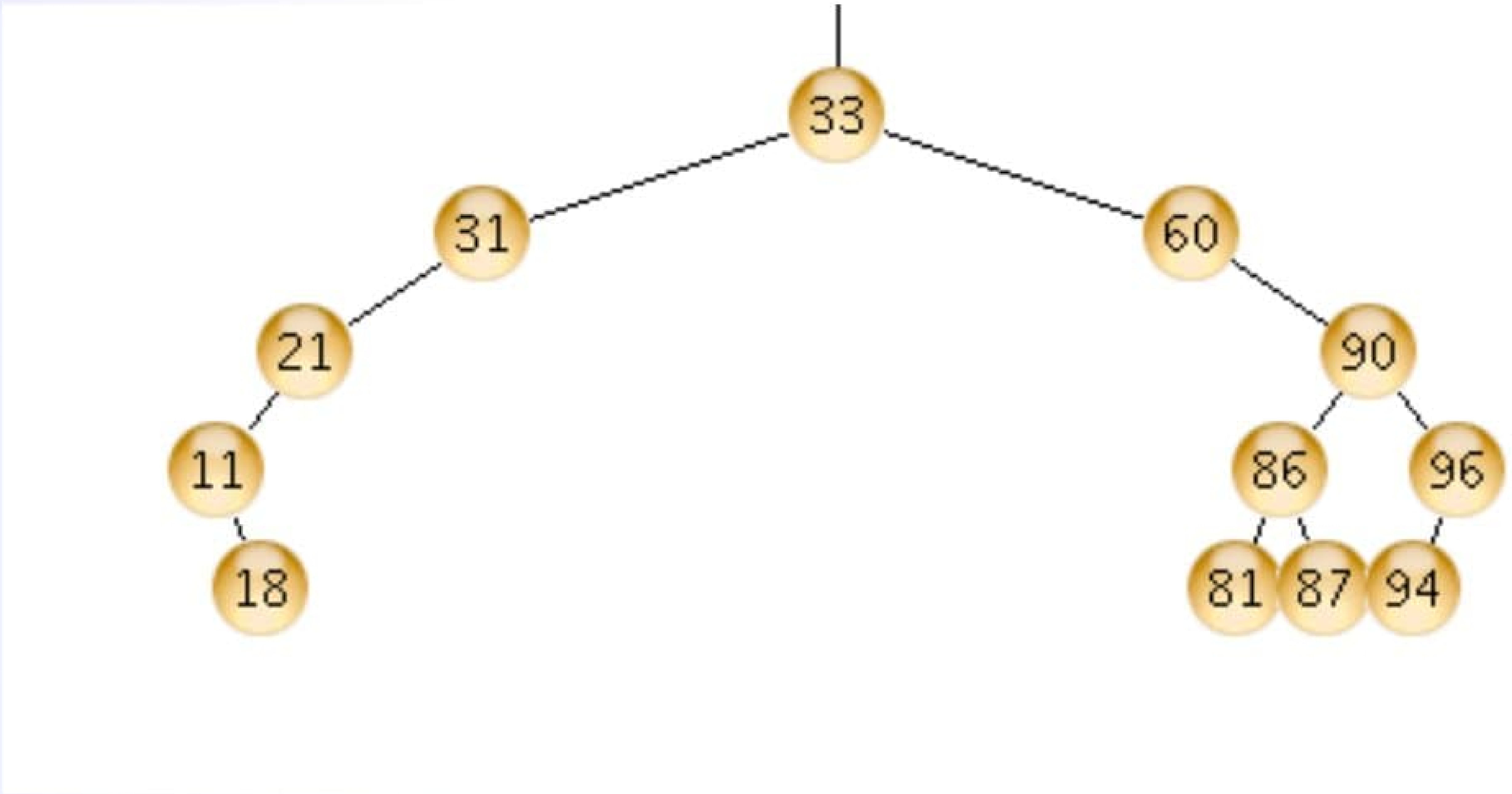
Sorting a BST

- Print the elements in the BST in sorted order



Sorting a BST

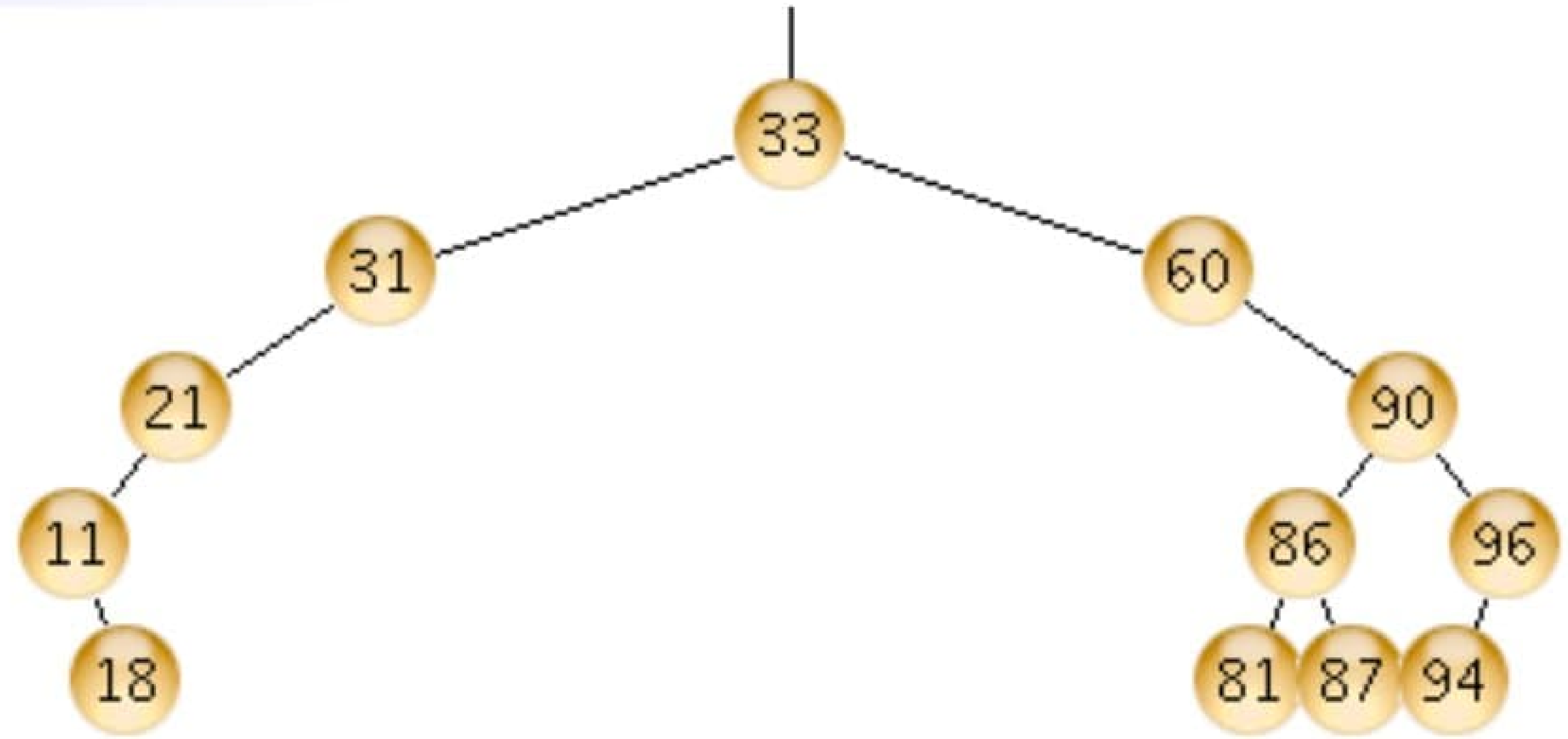
- Print the elements in the BST in sorted order.




```
//print the elements in tree of v in order
sort(BSTNode v)
    if  (v == null) return;
    sort(v.left());
    print v.getData();
    sort(v.right());
```

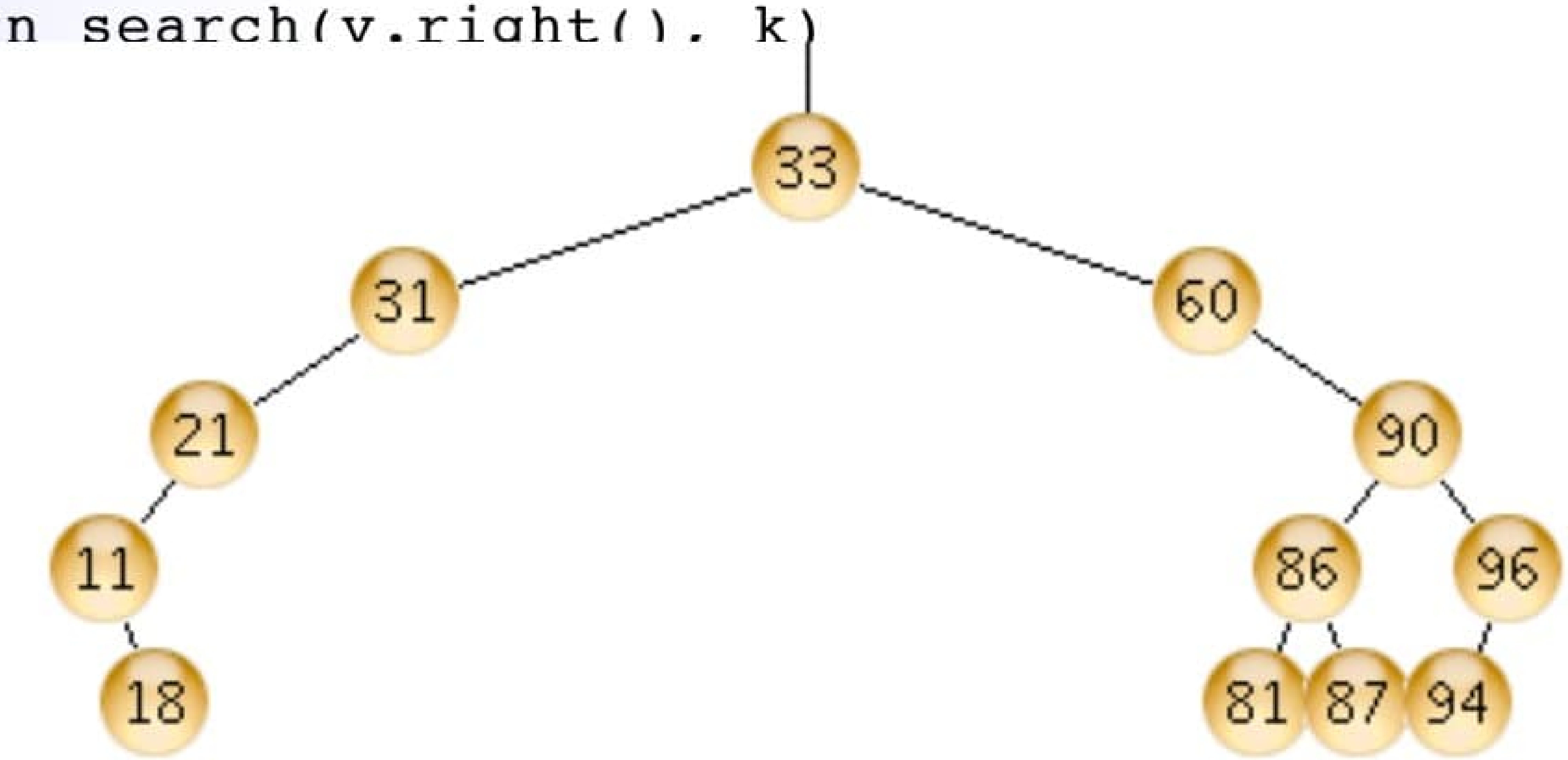
- in-order traversal: left -node-right
- Analysis: O(n)

Searching in a BST



Searching in a BST

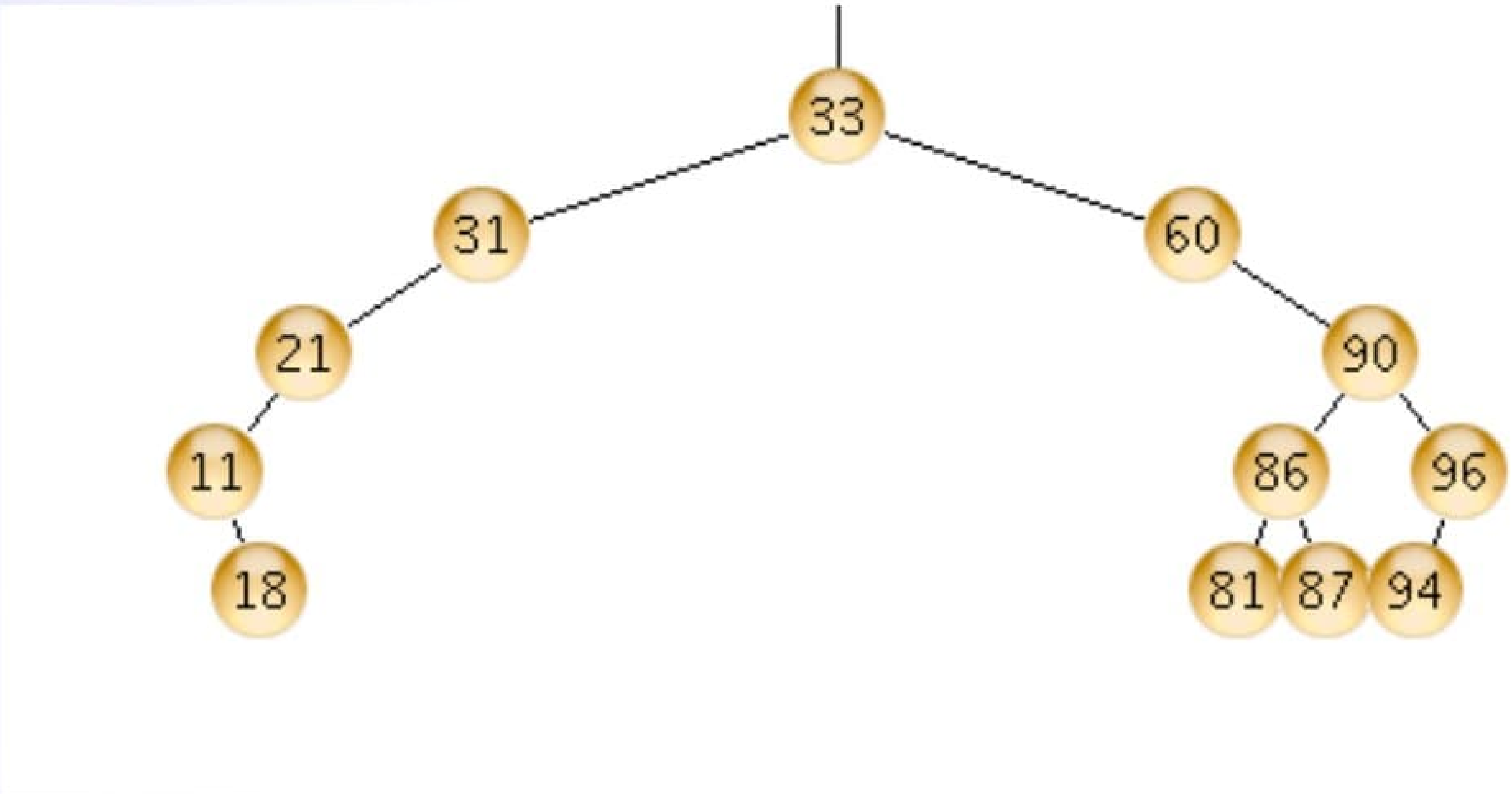
```
//return the node w such that w.getData() == k or null if such a node
//does not exist
BSTNode search (v, k)   {
    if (v == null) return null;
    if (v.getData() == k) return v;
    if (k < v.getData()) return search(v.left(), k);
    else return search(v.right(), k)
}
```



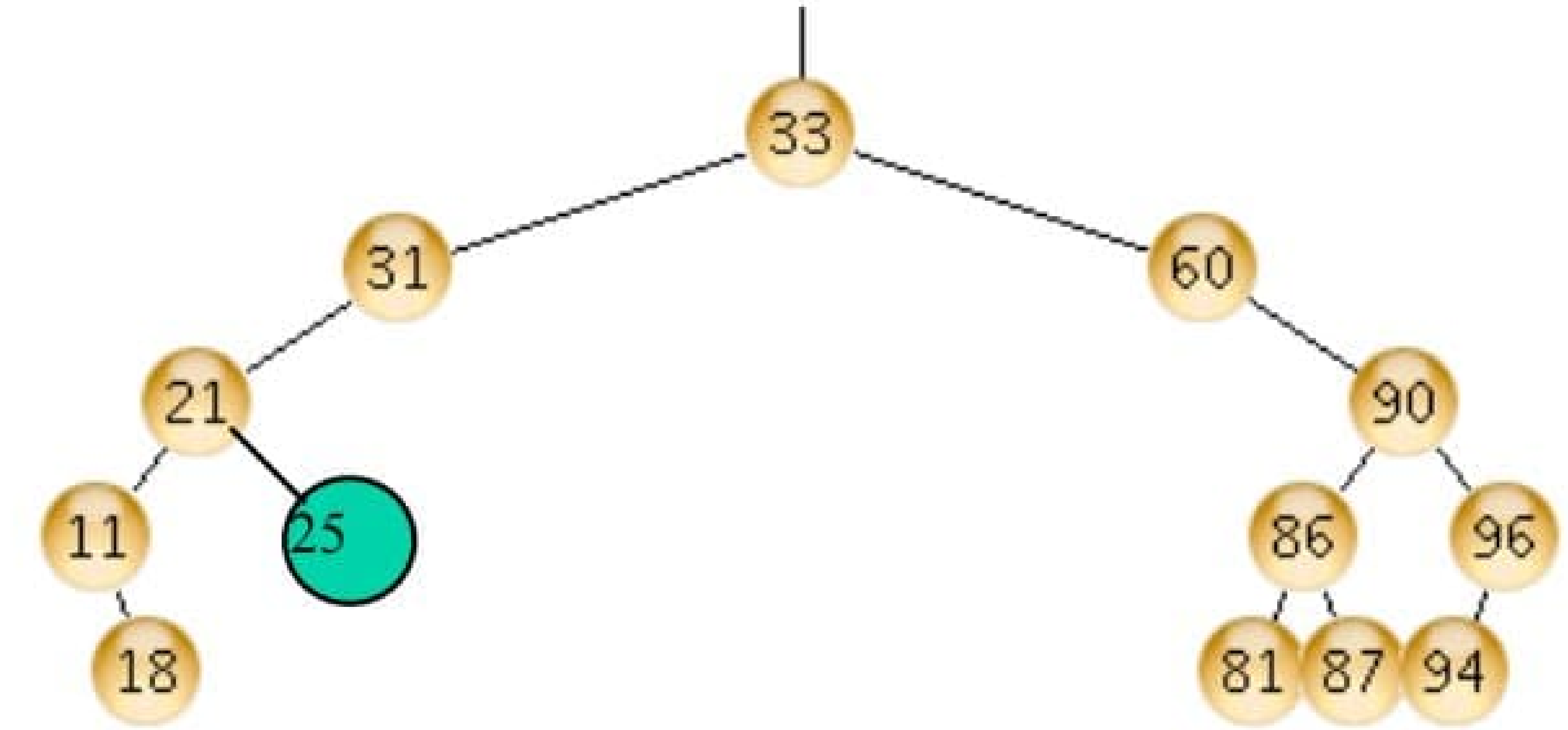
- Analysis:
  - search traverses (only) a path down from the root
  - does NOT traverse the entire tree
  - O(depth of result node) = O(h), where h is the height of the tree

Inserting in a BST

- insert 25



Inserting in a BST



- insert 25
  - There is only one place where 25 can go

- //create and insert node with key k in th
- void insert (v, k) {

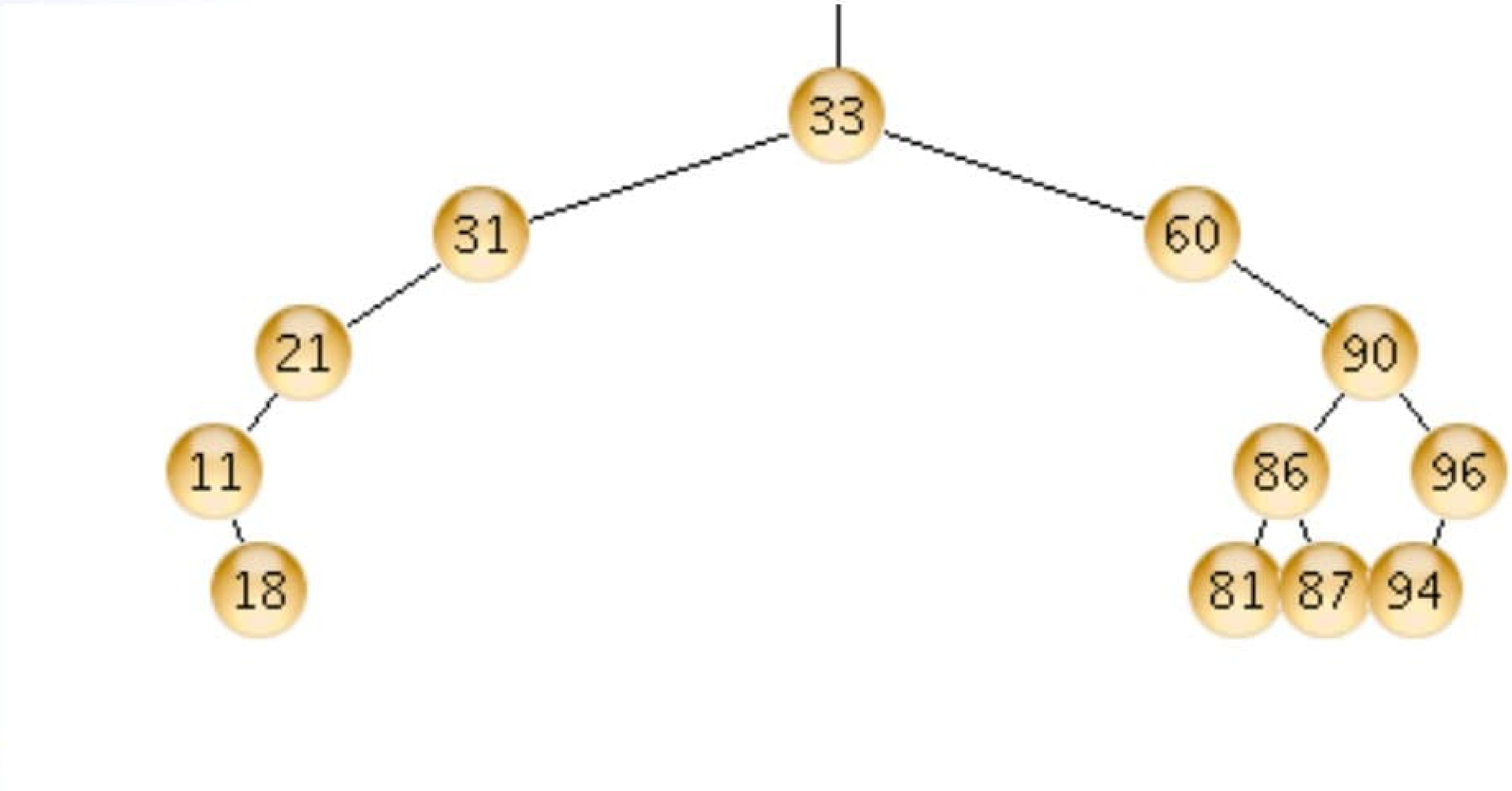
```
void insert (v, k)    {
    //this can only happen if inserting in an empty tree
    if (v == null) return new BSTNode(k);
    if (k <= v.getData()) {
         if (v.left() == null) {
              //insert node as left child of v
              u = new BSTNode(k);
              v.setLeft(u);
        } else {
              return insert(v.left(), k);
        }
    } else //if (v.getData() > k) {
        ...
    }
}
```

Inserting in a BST

- Analysis:
  - similar with searching
  - traverses a path from the root to the inserted node
  - O(depth of inserted node)
  - this is O(h), where h is the height of the tree

Deleting in a BST

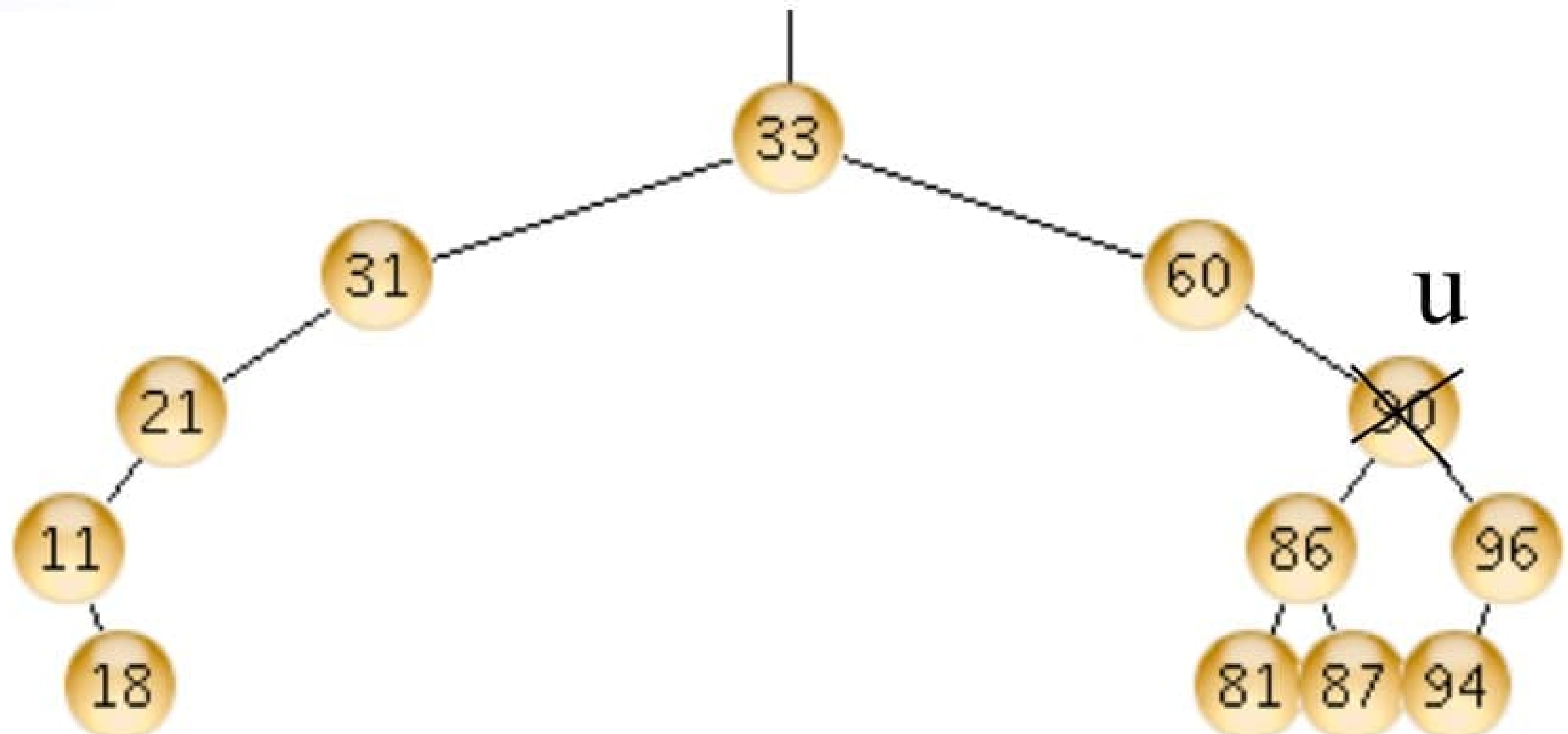
- delete 87
- delete 21
- delete 90



- case 1: delete a
  - if x is left of its parent, set parent(x).left = null
  - else set parent(x).right = null
- case 2: delete a node with one child
  - link parent(x) to the child of x
- case 2: delete a node with 2 children
  - ??

Deleting in a BST

- delete 90



- copy in u 94 and delete 94
  - the left-most child of right(x) ← node has <=1 child
- or
- copy in u 87 and delete 87
  - the right-most child of left(x) ← node has <=1 child

Deleting in a BST

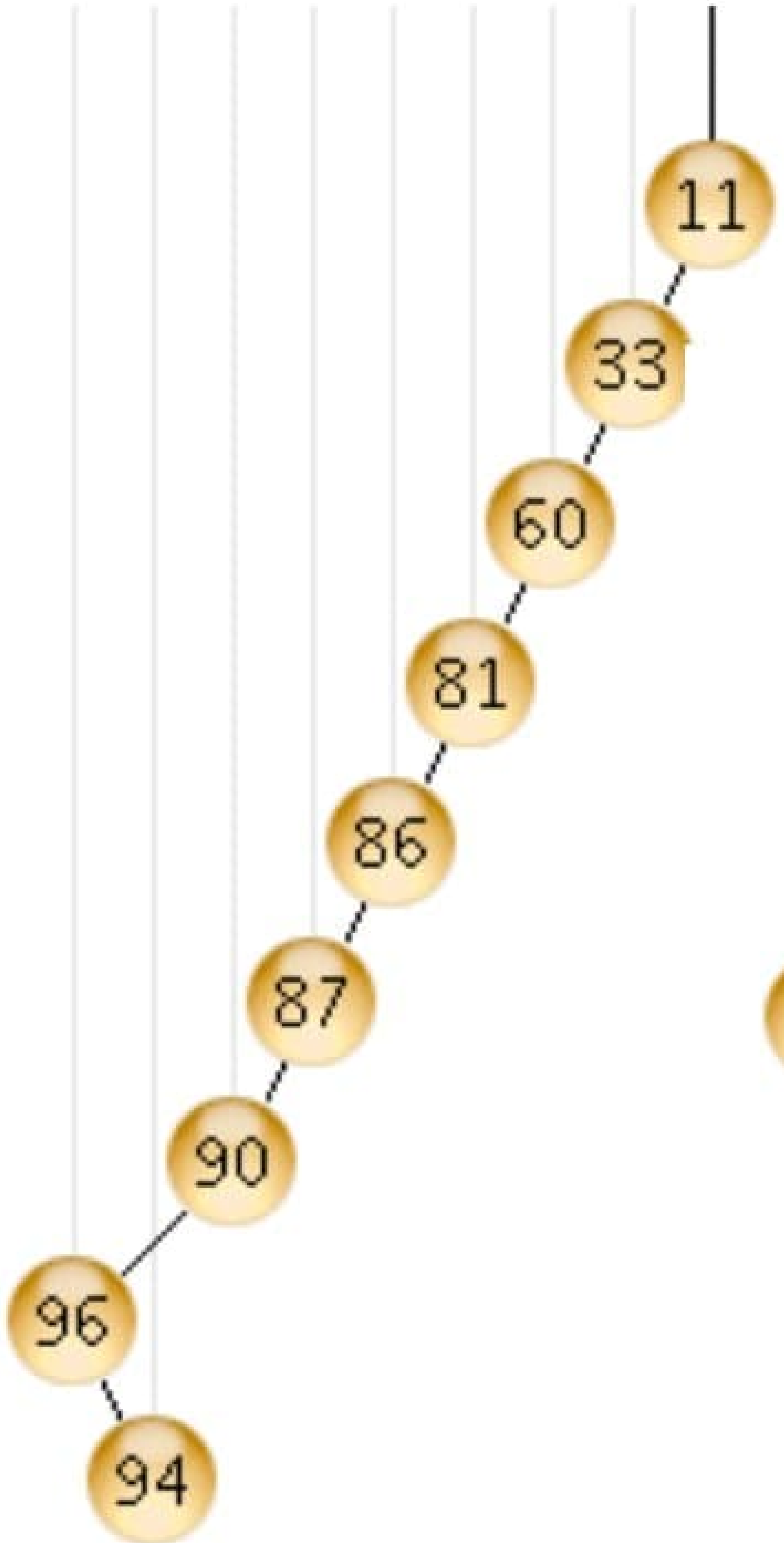
- Analysis:
  - traverses a path from the root to the deleted node
  - and sometimes from the deleted node to its left-most child
  - this is O(h), where h is the height of the tree

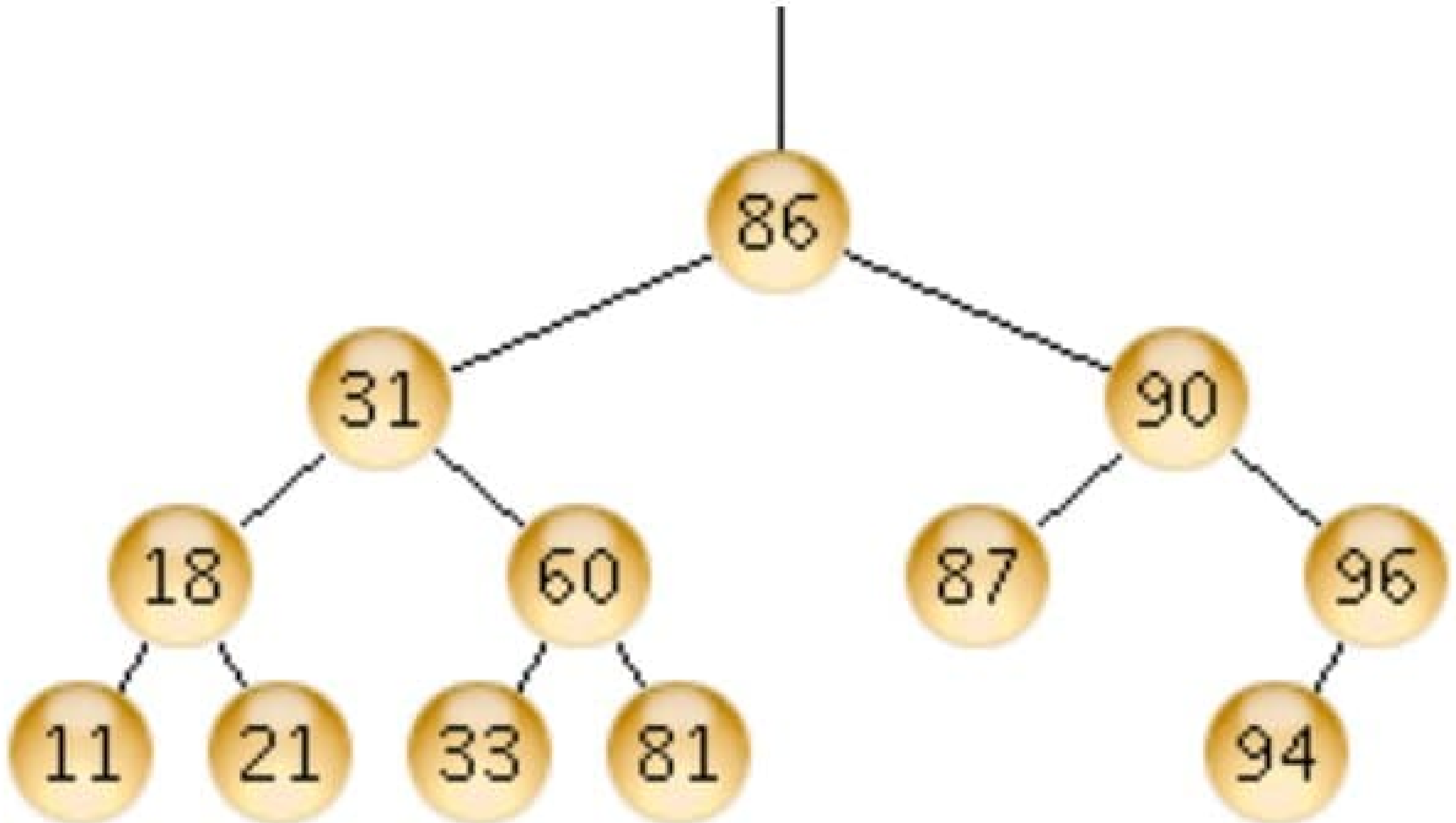
BST performance

- Because of search property, all operations follow one root-leaf path
  - insert: O(h)
  - delete: O(h)
  - search: O(h)

- We know that in a tree of n nodes
  - h >= lg (n+1) - 1
  - h <= n-1

- So in the worst case h is O(n)
  - BST insert, search, delete: O(n)
  - just like linked lists/arrays





BST performance

- worst-case scenario
  - start with an empty tree
  - insert 1
  - insert 2
  - insert 3
  - insert 4
  - ...
  - insert n

- it is possible to maintain that the height of the tree is Theta(lg n) at all times
  - by adding additional constraints
  - perform rotations during insert and delete to maintain these constraints

- Balanced BSTs: h is Theta(lg n)
  - Red-Black trees
  - AVL trees
  - 2-3-4 trees
  - B-trees
- to find out more.... take csci231 (Algorithms)