Sequence and Series and Mean Value Theorem.

Mean Value Theorem:

(1) Rolle's theorem

(2) Lagrange's theorem

(3) cauchy's theorem

(4) Taylor's theorem

(5) Muclunn's theorem

(1) Rolle's Theorem:

* verify the Rolle's theorem for the following functions.

(1) $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$

(2) $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ in $[a, b]$

(3) $f(x) = x(x+3)\, e^{-x/2}$ in $[-3, 0]$

(4) $f(x) = |x|$ in $[-1, 1]$

(5) $f(x) = \frac{1}{x^2}$ in $[-1, 1]$

(6) $f(x) = \sin x$ in $[-\pi, \pi]$

(7) $f(x) = \tan x$ in $[0, \pi]$

(8) $f(x) = \sec x$ in $[0, 2\pi]$

(9) $f(x) = e^x . \sin x$ $[0, \pi]$

(10) $f(x) = (x-a)^m . (x-b)^n$ in $[a, b]$

Rolle's Theorem:

Let $f(x)$ be a function of x defined in (a, b)

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is derivable in (a, b)

(iii) $f(a) = f(b)$

then $\exists$ a $c \in (a, b)$ $\ni$ $f'(c) = 0$.

① $f(x) = \frac{\sin x}{e^x}$ $[0,\pi]$

(i) $f(x) = \frac{\sin x}{e^x}$ is continuous for all 'x'.

$f(x)$ is continuous in $[0,\pi]$

$\Rightarrow f'(x) = \frac{e^x.\cos x - \sin x.e^x}{(e^x)^2}$

$= \frac{e^x(\cos x - \sin x)}{(e^x)^{\cancel{2}}}$

$f'(x) = \frac{\cos x - \sin x}{e^x}$ is exist $\forall x$.

$\Rightarrow f'(x)$ is exist in the interval $[0,\pi]$

$\therefore f(x)$ is derivable in $(0,\pi)$.

$\Rightarrow$ We have to show that $f(0) = f(\pi)$

$f(0) = \frac{\cancel{\cos 0 +}\sin 0}{e^0} = \frac{\cancel{1+}0}{1} = 0.$

$f(\pi) = \frac{\sin\pi}{e^\pi} = \frac{0}{e^\pi} = 0.$

Then $\exists$ exist $c \in (a,b) \ni f'(c) = 0.$

$f(x) = \frac{\sin x}{e^x}$

$f'(x) = \frac{\cos x - \sin x}{e^x}$

$f'(c) = \frac{\cos c - \sin c}{e^c} = 0$

$\cos c - \sin c = 0$

$\sin c = \cos c$

$\tan c = 1$

$c = \tan^{-1}(1)$

$\boxed{c = \pi/4} \in [0,\pi]$

② $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ in $[a,b]$ $a>0, b>0$.

$f(x) = \log(x^2+ab) - \log x(a+b)$ is continuous $\forall$ 'x'.
except at $x=0 \notin [a,b]$.

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x) = \log(x^2+ab) - \log x + \log(a+b)$

$= \frac{1}{x^2+ab}(2x) - \frac{1}{x} - 0.$

$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$ is exist ($\forall$ 'x') in (a,b)

$f(x)$ is derivable in (a,b)

$f(a) = \log(a^2+ab) - \log a(a+b)$

$= \log(a^2+ab) - \log(a^2+ab)$

$= 0.$

$f(b) = \log(b^2+ab) - \log b(a+b)$

$= \log(b^2+ab) - \log(ab+b^2)$

$= 0$

$f(a) = f(b).$

$\exists$ a $c \in (a,b) \ni f'(c) = 0$

We have $f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$

$$f'(c) = \frac{2c}{c^2+ab} - \frac{1}{c} = 0$$

$$2c^2 - c^2 - ab = 0$$

$$c^2 = ab$$

$$c = \pm\sqrt{ab}$$

$$c = \sqrt{ab} \text{ (or) } -\sqrt{ab}.$$

$$\boxed{c = \sqrt{ab}} \in (a,b).$$

③ $f(x) = x\cdot(x+3)\, e^{-x/2}$ in $[-3,0]$

Sol: $f(x)$ is continuous $\forall x$.

(i) $f(x)$ is continuous in $[-3,0]$

(ii) $f(x) = (x^2+3x)\, e^{-x/2}$

$$f'(x) = (x^2+3x)\, e^{-x/2}\cdot\frac{-1}{2} + e^{-x/2}(2x+3)$$

$$= -\frac{(x^2+3x)}{2} e^{-x/2} + (2x+3)\, e^{-x/2}$$

$$= e^{-x/2}\left[(2x+3) - \frac{(x^2+3x)}{2}\right]$$

$$= e^{-x/2}\left[\frac{4x+6-x^2-3x}{2}\right]$$

$$= e^{-x/2}\left[\frac{-x^2+x+6}{2}\right]$$

$$= \frac{e^{-x/2}}{2}(-x^2+x+6)$$

$f'(x)$ is exist in $[-3,0]$.

$\Rightarrow f(x)$ is derivable in $(-3,0)$.

We have to show that $f(-3) = f(0)$

$$f(-3) = -3(-3+3)e^{-3/2}$$

$$= -3(0)\, e^{-3/2}$$

$$= 0$$

$$f(0) = 0\,(0+3)e^{-0/2}$$

$$= 0.$$

$$\boxed{\therefore f(-3) = f(0)}$$

Then $\exists$ a $c \in (a,b) \ni f'(c) = 0$.

$$f'(x) = \frac{e^{-x/2}}{2}(-x^2+x+6)$$

$$f'(c) = \frac{e^{-c/2}}{2}(-c^2+c+6) = 0.$$

$$(-c^2+c+6)\, e^{-c/2} = 0.$$

$e^{-c/2} = 0$ and $-c^2+c+6 = 0$. ✓

$$c^2 - c - 6 = 0$$

$$c^2 - 3c + 2c - 6 = 0$$

$$c(c-3) + 2(c-3) = 0$$

$$(c-3)(c+2) = 0$$

$$c = 3, \ \boxed{c = -2} \in (-3,0)$$

(4) $f(x) = |x|$ in $[-1,1]$

<u>Sol:</u> We know that $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x > 0. \end{cases}$

(i) $f(x) = |x|$ is continuous $\forall x$

$\Rightarrow |x|$ is continuous in $[-1,1]$

(ii) The derivative of $|x|$ does not exist.

Because,

<u>L.H.D</u> $\lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0}$

$$= \lim_{x \to 0^-} \left(\frac{-x-0}{x}\right)$$

$$= \lim_{x\to 0^-} \frac{-x}{x}$$

$$= \lim_{x\to 0^-} (-1) = \underline{-1}$$

<u>R.H.D</u> $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to 0^+} \frac{x-0}{x-0}$$

$$= \lim_{x\to 0^+} \frac{x}{x}$$

$$= \lim_{x\to 0^+} (1) = 1$$

$$\boxed{\therefore L.H.D \neq R.H.D}$$

Hence Rolle's theorem is not verified.

(10). $f(x) = (x-a)^m \cdot (x-b)^n$ in $[a,b]$.

<u>sol:</u> $f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[a,b]$

$\Rightarrow f(x) = (x-a)^m \cdot (x-b)^n$

$$f'(x) = (x-a)^m \cdot n(x-b)^{n-1}(1-0) + (x-b)^n m(x-a)^{m-1}(1-0)$$

$$= n \cdot (x-b)^{n-1} \cdot (x-a)^m + m \cdot (x-a)^{m-1} \cdot (x-b)^n$$

$$= n \cdot (x-b)^n \cdot (x-b)^{-1} (x-a)^m + m \cdot (x-a)^m \cdot (x-a)^{-1} (x-b)^n$$

$$= (x-a)^m (x-b)^n [n \cdot (x-b)^{-1} + m \cdot (x-a)^{-1}]$$

$$= (x-a)^m (x-b)^n \left(\frac{n}{x-b} + \frac{m}{x-a}\right)$$

$$= (x-a)^n \cdot (x-b)^n \left(\frac{n(x-a) + m(x-b)}{(x-a)(x-b)}\right)$$

$f(x)$ is exist $\forall x$. except at $x=a$ and $x=b$. $\notin (a,b)$

$\therefore f(x)$ is exist in (a,b).

$\therefore f(x)$ is derivable in (a,b)

$$f(a) = (a-a)^m \cdot (a-b)^n$$

$$= (0)^m \cdot (a-b)^n$$

$$= 0.$$

$f(b) = (b-a)^m (b-b)^n$

$= (b-a)^m (0)$

$= 0.$

$f(a) = f(b)$

Then $\exists$ a $c \in (a,b) \ni f'(c)=0$

$f(x) = (x-a)^m . (x-b)^n$

$f'(x) = (x-a)^m . (x-b)^n \left(\dfrac{n(x-a) + m(x-b)}{(x-a)(x-b)} \right)$

$f'(c) = (c-a)^m . (c-b)^n . \left(\dfrac{n(c-a) + m(c-b)}{(c-a)(c-b)} \right) = 0.$

$= (c-a)^m (c-b)^n \left[\dfrac{nc - na + mc - mb}{(c-a)(c-b)} \right] = 0$

$= (c-a)^m . (c-b)^n \left[\dfrac{(m+n)c - (na+mb)}{(c-a)(c-b)} \right] = 0$

$(c-a)^m = 0$, $(c-b)^n = 0$ and $(m+n)c - na - mb = 0.$

$\Rightarrow (m+n)c = na + mb.$

$c = \dfrac{na+mb}{m+n} \in (a,b)$

(5) $f(x) = \frac{1}{x^2}$ in $[-1, 1]$.

sol: $f(x) = \frac{1}{x^2}$

~~f(x) is does not exist at x.~~

$\Rightarrow f(x)$ is does not continuous in $[-1, 1]$ Except at $x=0$ $\in [-1,1]$

$f'(x) = -2x^{-3} = \frac{-2}{x^3}$

$\rightarrow$ $f(x)$ is does not derivable in $(-1,1)$ Except at $t=0$.

But $t=0 \in (-1,1)$

$\therefore$ Rolles theorem can not applied.

⑥ $f(x) = \sin x$ in $[-\pi, \pi]$

sol:- $f(x) = \sin x$

$\therefore$ $f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[-\pi, \pi]$.

$f'(x) = \cos x$.

$\Rightarrow f'(x)$ is derivable in $(-\pi, \pi)$

We have to show that $f(-\pi) = f(\pi)$

$f(-\pi) = \sin(-\pi) = -\sin\pi = 0$

$f(\pi) = \sin\pi = 0$

$f(-\pi) = f(\pi)$

then $\exists$ a $c \in (-\pi, \pi) \ni f'(c) = 0$.

$f(x) = \sin x \Rightarrow f'(x) = \cos x$

$f'(c) = 0$

$\cos c = 0$

$c = \cos^{-1}(0)$

$c = \cos^{-1}(\cos \pi/2)$

$\boxed{c = \pi/2} \in (-\pi/2, \pi/2)$

⑦ $f(x) = \tan x$ in $[0, \pi]$.

$f(x) = \tan x$

$f(x)$ is exist $\forall x$. except at $x = \pi/2 \in (0, \pi)$.

$\therefore f(x)$ is does not continuous in $[0, \pi]$

$f'(x) = \sec^2 x$

$f'(x)$ ~~is~~ is ~~exist~~ does not exist $\forall x$. Except at $x = 0 \in (0, \pi)$

Rolle's theorem can not be verified.

(8) $f(x) = \sec x$ in $[0, 2\pi]$

$f(x) = \sec x$.

$f(x)$ is exist $\forall x$. except at $x = \pi/2 \in (0, 2\pi)$

$\Rightarrow f(x)$ is continuous in $[0, 2\pi]$ except at $x = \pi/2 \in (0, 2\pi)$

$f'(x) = \sec x . \tan x$.

$f'(x)$ is exist $\forall x$. Except at $x = \pi/2 \in (0, 2\pi)$.

$\Rightarrow f'(x)$ is derivable ~~at~~ in $(0, 2\pi)$ Except at $x = \pi/2$.

$\Rightarrow f(0) = f(2\pi)$ (we have to S.T)

$f(0) = \sec 0° = 1$

$f(2\pi) = \sec 2\pi = 1$

$\boxed{f(0) = f(2\pi)}$

Then $\exists$ a $c \in (0, 2\pi) \ni f'(c) = 0$

$\sec c . \tan c = 0$

$\tan c = 0$ and $\sec c = 0$

$c = \tan^{-1}(0)$ $\quad c = \sec^{-1}(0)$

$c = \tan^{-1}(\tan 0)$ $\quad$ ~~c = sec⁻¹ sec~~

$\boxed{c = 0}$

(9) $f(x) = e^x . \sin x$ in $[0, \pi]$.

$f(x) = e^x \sin x$

$f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[0, \pi]$

$f'(x) = e^x \cos x + \sin x \, e^x$

$= e^x (\cos x + \sin x)$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0, \pi)$.

we have to show that

$\Rightarrow f(0) = f(\pi)$.

$f(0) = e^0 . \sin 0 = 0$

$f(\pi) = e^\pi \sin \pi = 0$

$\boxed{f(0) = f(\pi)}$

Then $\exists$ a $c \in (0,\pi) \ni f'(c) = 0$

$$e^c(\cos c + \sin c) = 0$$
$$\cos c + \sin c = 0$$

$$\sin c = -\cos c$$
$$\frac{\sin c}{\cos c} = -1$$
$$\tan c = -1$$
$$c = \tan^{-1}(-1)$$
$$c = \tan^{-1}(\tan 3\pi/4)$$
$$\boxed{c = 3\pi/4} \in (0,\pi).$$

Saturday 19/10/19

Lagrange's Mean Value Theorem:

Let $f(x)$ be a function of x. if

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is derivable in (a,b)

(iii) Then $\exists$ a $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

① $f(x) = x(x-1)(x-2)$ in $[0,1/2]$

② $f(x) = \log x$ $[1,e]$

③ $f(x) = e^x$ $[0,1]$

④ $f(x) = \frac{1}{x}$ $[1,4]$

⑤ $f(x) = x - x^3$ $[-2,1]$

⑥ if $x > 0$ show that $x > \log(1+x) > x - \frac{x^2}{2}$

By using LMVT, show that $\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}$ and deduce that

⑦ $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}(4/3) < \frac{\pi}{4} + \frac{1}{6}$

By using LMVT,

⑧ $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}(3/5) > \pi/3 - 1/8.$

⑨ $x \le \sin^{-1}x \le \frac{x}{1-x^2}$

① $f(x) = x(x-1)(x-2)$ in $[0, 1/2]$

$f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[0, 1/2]$

$f(x) = (x^2 - x)(x-2)$

$= x^3 - 2x^2 - x^2 + 2x$

$f(x) = x^3 - 3x^2 + 2x$

$f'(x) = 3x^2 - 6x + 2$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0, 1/2)$.

$\Rightarrow$ Then $\exists$ a $c \in (0, 1/2) \ni f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$$3c^2 - 6c + 2 = \frac{3/8 - 0}{1/2 - 0}$$

$f(1/2) = 1/2(1/2 - 1)(1/2 - 2)$

$= \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)$

$= 3/8$

$$3c^2 - 6c + 2 = \frac{3}{\cancel{8}_4} \times \frac{\cancel{2}}{1}$$

$$3c^2 - 6c + 2 - 3/4 = 0$$

$$3c^2 - 6c + 5/4 = 0$$

$$c = \frac{6 \pm \sqrt{36 - 15}}{2(3)}$$

$$= \frac{6 \pm \sqrt{21}}{6}$$

$$= \frac{6}{6} \pm \frac{\sqrt{21}}{6}, \quad \frac{6}{6} - \frac{\sqrt{21}}{6}$$

$$= 1 + \frac{\sqrt{21}}{6}, \quad 1 - \frac{\sqrt{21}}{6}$$

$$\boxed{c = 1 - \frac{\sqrt{21}}{6} \in (0, 1/2)}$$

② $f(x) = \log x$ in $[1, e]$

$f(x)$ is continuous $\forall x$, except at $x = 0 \notin (1, e)$

$\Rightarrow f(x)$ is continuous in $[1, e]$

$f'(x) = \frac{1}{x}$.

$f'(x)$ is exist $\forall x$. except at $x = 0 \notin (1, e)$

$\Rightarrow f'(x)$ is derivable in $(1, e)$.

Then $\exists$ a $c \in (1,e) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{1}{c} = \frac{\log e - \log 1}{e-1}$$

$$\frac{1}{c} = \frac{1-0}{e-1}$$

$$\frac{1}{c} = \frac{1}{e-1}$$

$e = 2.7 \ldots$

$\Rightarrow \boxed{c = e-1} \in (1,e)$. $\quad e-1 = 2.7-1 = 1.7$

③ $f(x) = e^x$ in $[0,1]$

$f(x)$ is contincuous $\forall x$.

$\Rightarrow f(x)$ is contincuous in $[0,1]$

$f'(x) = e^x$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0,1)$.

Then $\exists$ a $c \in (0,1) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$e^c = \frac{e^1 - e^0}{1-0}$$

$$e^c = \frac{e-1}{1}$$

$$e^c = e-1$$

$$\boxed{c = \log(e-1) \in (0,1)}$$

④ $f(x) = \frac{1}{x}$ in $[1,4]$

$f(x)$ is contincuous $\forall x$. except at $x=0 \notin [1,4]$

$\Rightarrow f(x)$ is contincuous in $[1,4]$

$f'(x) = \frac{-1}{x^2}$ is exist $\forall x$. except at $x=0 \notin (1,4)$

$\Rightarrow f(x)$ is derivable in $(1,4)$

Then $\exists$ a $c \in (1,4) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{-1}{c^2} = \frac{1/4 - 1}{4-1}$$

$$\frac{-1}{c^2} = \frac{\frac{1-4}{4}}{3}$$

$$\frac{-1}{c^2} = \frac{-\cancel{3}/4}{\cancel{3}}$$

$$\cancel{-}\frac{1}{c^2} = \cancel{-}\frac{1}{4}$$

$$c^2 = 4$$

$$c = \sqrt{4} \Rightarrow c = \pm 2$$

$$\boxed{c = 2 \in (1,4)}$$

⑤ $f(x) = x - x^3$ in $[-2,1]$

$f(x) = x - x^3$ is continuous $\forall x$.

$\rightarrow f(x)$ is continuous in $[-2,1]$

$f'(x) = 1 - 3x^2$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(-2,1)$

$\exists$ a $c \in (-2,1) \ni f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$$1 - 3c^2 = \frac{[1-(1)^3] - [-2 - (-2)^3]}{1-(-2)}$$

$$1 - 3c^2 = \frac{(1-1) - [-2-(-8)]}{1+2}$$

$$1 - 3c^2 = \frac{0 - (-2+8)}{3}$$

$$3 - 9c^2 = -6$$

$$\cancel{9}c^2 = \cancel{9}$$

$$c^2 = 1$$

$$c = \pm 1$$

$$\boxed{c = -1 \in (-2,1)}$$

⑥ If $x > 0$ show that $x > \log(1+x) > x - \dfrac{x^2}{2}$

<u>Sol:-</u> Let us take $f(x) = \log(1+x)$

Since $f(x) = \log(1+x)$ is continuous $\forall x > 0$.

and $f(x)$ is derivable $\forall x > 0$.

By using L.M.V.T,

$\exists$ a $c \in (0,x) \ni f'(c) = \dfrac{f(x) - f(0)}{x - 0}$

We have $f(x) = \log(1+x)$

$$f'(x) = \frac{1}{1+x}$$

$$\frac{1}{1+c} = \frac{\log(1+x) - \log(1+0)}{x-0}$$

$$\frac{1}{1+c} = \frac{\log(1+x) - 0}{x}$$

$$\frac{1}{1+c} = \frac{\log(1+x)}{x} \rightarrow (1)$$

Given that $0 < c < x$

$$1 < c+1 < x+1$$

$$1 < \frac{1}{c+1} < \frac{1}{x+1}$$

$$1 < \frac{\log(1+x)}{x} < \frac{1}{x+1}$$

$$x < \log(1+x) < \frac{x}{1+x}$$

(7) Let $f(x) = \tan^{-1}x$ in $[a,b]$

Given, ~~$f(x)$ is continuous $\forall x$. except~~
$f(x)$ is continuous in $[a,b]$
and $f(x)$ is derivable in (a,b).

By using L.M.V.T,
then $\exists$ a $c \in (a,b) \ni f'(c) = \frac{f(b) - f(a)}{b-a}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a}$$

Given that, $a < c < b$

$$a^2 < c^2 < b^2$$

$$1+a^2 < 1+c^2 < 1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\frac{1}{1+a^2} > \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} > \frac{1}{1+b^2}$$

$$\frac{b-a}{1+a^2} > \tan^{-1}(b) - \tan^{-1}(a) > \frac{b-a}{1+b^2}$$

Given that $a=1$, $b=4/3$

$$\frac{4/3-1}{1+(1)^2} > \tan^{-1}(4/3)-\tan^{-1}(1) > \frac{4/3-1}{1+(4/3)^2}$$

$$\frac{1/3}{2} > \tan^{-1}(4/3)-\pi/4 > \frac{1/3}{\frac{25}{9}}$$

$$\frac{1}{6} > \tan^{-1}(4/3)-\pi/4 > \frac{3}{25}$$

$$\frac{1}{6}+\frac{\pi}{4} > \tan^{-1}(4/3) > \frac{3}{25}+\frac{\pi}{4}$$

$$\frac{\pi}{4}+\frac{3}{25} > \tan^{-1}(4/3) > \frac{\pi}{4}+\frac{1}{6}.$$

(8) Let $f(x)=\cos^{-1}x$. in $[a,b]$

Given that, $f(x)$ is continuous in $[a,b]$.

and $f(x)$ is derivable in (a,b)

By using L.M.V.T,

$\exists$ a $c\in(a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$f(x)=\cos^{-1}x \Rightarrow f'(x)=\frac{-1}{\sqrt{1-x^2}}$

$$\frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1}(b)-\cos^{-1}(a)}{b-a}$$

We know that, $a<c<b$

$a^2<c^2<b^2$

$-a^2>-c^2>-b^2$

$1-a^2>1-c^2>1-b^2$

$\sqrt{1-a^2}>\sqrt{1-c^2}>\sqrt{1-b^2}$

Wrong

$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$

$$\frac{1}{\sqrt{1-a^2}} < \frac{\cos^{-1}(a)-\cos^{-1}(b)}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{-(b-a)}{\sqrt{1-a^2}} \geq -[\cos^{-1}(a)-\cos^{-1}(b)] \geq \frac{-(b-a)}{\sqrt{1-b^2}}$$

Given that $a=3/5$, $b=$

$$\frac{a-b}{\sqrt{1-a^2}} > \cos^{-1}(b)-\cos^{-1}(a) > \frac{a-b}{\sqrt{1-b^2}}$$

Given that

Monday

21/10 Cauchy's Mean Value Theorem:

Let $f(x), g(x)$ are a functions of 'x'.

(i) $f(x), g(x)$ are continuous in $[a,b]$

(ii) $f(x), g(x)$ are derivable in (a,b)

Then $\exists$ a $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

verify the cauchy's Mean Value Theorem for the following functions.

① $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$ $0<a<b$.

② $f(x) = \sin x$, $g(x) = \cos x$ in $[0, \pi/2]$

③ $f(x) = e^{x}$, $g(x) = e^{-x}$ in $[a,b]$

④ $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in $[a,b]$ if $0<a<b$

⑤ $f(x) = x^2+2$, $g(x) = x^3-1$ in $[1,2]$

⑥ $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $[1,e]$

⑦ $f(x) = x^3$, $g(x) = 2-x$ in $[0,9]$

① $f(x)$ is always continuous $\forall x$.

$g(x)$ is continuous $\forall x$. Except at $x=0 \notin (a,b)$ $[0<a<b]$

$\Rightarrow f(x), g(x)$ are continuous in $[a,b]$.

$f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = \frac{-1}{2} x^{-3/2}$

$f'(x)$ is exist $\forall x$. Except at $x=0 \notin (a,b)$

$f(x)$ is derivable in (a,b).

$g'(x)$ is exist $\forall x$. Except at $x=0 \in (a,b)$

$g(x)$ is derivable in (a,b)

$\longrightarrow f(x), g(x)$ are derivable in (a,b)

Then $\exists$ a $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{\frac{1}{\cancel{2\sqrt{c}}}}{\frac{-1}{\cancel{2\sqrt{c}}\,c}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\therefore C = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}\sqrt{b}}}$$

$$\therefore C = \frac{\cancel{\sqrt{b}-\sqrt{a}}}{-\cancel{(\sqrt{b}-\sqrt{a})}} \sqrt{ab}$$

$$\boxed{C = \sqrt{ab} \in (a,b)}$$

② $f(x) = \sin x$, $g(x) = \cos x$. $[0, \pi/2]$

$\Rightarrow$ $f(x), g(x)$ are always continuous $\forall x$.

$\Rightarrow$ $f(x), g(x)$ are continuous in $[0, \pi/2]$.

$f'(x) = \cos x$, $g'(x) = -\sin x$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow$ $f(x), g(x)$ are derivable in $(0, \pi/2)$

Then $\exists$ a $c \in (0, \pi/2) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{\cos c}{-\sin c} = \frac{\sin \pi/2 - \sin 0}{\cos \pi/2 - \cos 0}$$

$$\frac{\cos c}{-\sin c} = \frac{1-0}{0-1}$$

$$\frac{\cos c}{\cancel{-}\sin c} = \frac{1}{\cancel{-}1}$$

$$\cos c = \sin c$$

$$\frac{\sin c}{\cos c} = 1$$

$$\tan c = 1$$

$$c = \tan^{-1}(1)$$

$$\boxed{c = \pi/4 \in (0, \pi/2).}$$

③ $f(x) = e^x$, $g(x) = e^{-x}$ in $[a,b]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow$ $f(x), g(x)$ are continuous in $[a,b]$

$f(x) = e^x \qquad g(x) = e^{-x}$

$f'(x) = e^x \qquad g'(x) = -e^{-x}$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow f(x), g(x)$ are derivable in (a,b)

Then $\exists$ a $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-x}}$$

$$-e^c \cdot e^c = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$-(e^c)^2 = \frac{e^b - e^a}{\frac{e^a - e^b}{e^a \cdot e^b}}$$

$$\cancel{-} e^{2c} = \frac{\cancel{e^b - e^a}}{\cancel{-}(\cancel{e^b - e^a})} \quad e^a \cdot e^b$$

$$e^{2c} = e^a e^b$$

$$e^{2c} = e^{a+b}$$

$$2c = a+b$$

$$\boxed{c = \frac{a+b}{2} \in (a,b)}$$

(4) $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in $[a,b]$ $(0 < a < b.)$

$f(x)$ is continuous $\forall x$. Except at $x=0 \notin (a,b)$

$g(x)$ is continuous $\forall x$, Except at $x=0 \notin (a,b)$

$\Rightarrow f(x), g(x)$ are continuous in $[a,b]$

$f'(x) = -2x^{-3} \qquad g'(x) = \cancel{\log x} \ \frac{-1}{x^2}$

$f'(x)$ is exist $\forall x$. Except at $x=0$.

$g'(x)$ is exist $\forall x$, Except at $x=0$

$\rightarrow f(x), g(x)$ are derivable in (a,b)

Then $\exists$ a $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{\frac{\cancel{-}2}{c^{\cancel{3}}}}{\cancel{-}1/\cancel{c^2}} = \frac{1/b^2 - 1/a^2}{1/b - 1/a}$$

$$\frac{2}{x} = \frac{\frac{a^2-b^2}{a^2b^2}}{\frac{a-b}{ab}}$$

$$\frac{2}{x} = \frac{(a+b)\cancel{(a-b)}}{(ab)^{\cancel{2}}} \times \frac{\cancel{ab}}{\cancel{a-b}}$$

$$\boxed{x = \frac{2ab}{a+b} \in (a, b)}$$

⑤ $f(x) = x^2+2$, $g(x) = x^3-1$, in $[1,2]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow$ $f(x)$, $g(x)$ are continuous in $[1,2]$

$f'(x) = 2x$, $g'(x) = 3x^2$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow$ $f(x)$, $g(x)$ are derivable in $(1,2)$

Then $\exists$ a $c \in (1,2) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{2\cancel{c}}{3c^{\cancel{2}}} = \frac{[(2)^2+2]-[(1)^2+2]}{[(2)^3-1]-[(1)^3-1]}$$

$$\frac{2}{3c} = \frac{(4+2)-(1+2)}{(8-1)-(1-1)}$$

$$\frac{2}{3c} = \frac{6-3}{7-0}$$

$$\frac{2}{3c} = \frac{3}{7}$$

$$9c = 14$$

$$\boxed{c = \frac{14}{9} \in (1,2)}$$

⑥ $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $[1,e]$

$f(x)$ is continuous $\forall x$. Except at $x=0 \notin (1,e)$

$g(x)$ is continuous $\forall x$. Except at $x=0 \notin (1,e)$

$\Rightarrow$ $f(x)$, $g(x)$ are continuous in $[1,e]$

$f'(x) = \frac{1}{x}$, $g'(x) = \frac{-1}{x^2}$

$f'(x)$ is exist $\forall x$ except at $x=0 \notin (1,e)$

$g'(x)$ is exist $\forall x$. except at $x=0 \notin (1,e)$

$\Rightarrow f(x), g(x)$ is derivable in $(1,e)$.

Then $\exists$ a $c \in (1,e) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{1/\cancel{c}}{-1/c^{\cancel{2}}} = \frac{\log e - \log 1}{1/e - 1/1}$$

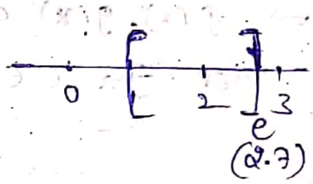
$$-c = \frac{\log_e e - 0}{\frac{1-e}{e}}$$

$$-c = \frac{1-0}{\frac{1-e}{e}}$$

$$-c = \frac{e}{1-e}$$

$$\boxed{c = \frac{e}{e-1} \in (1,e)}$$

$$\boxed{c = 1.58}$$



⑦ $f(x) = x^3$, $g(x) = 2-x$ in $[0,9]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[0,9]$

$f'(x) = 3x^2$

$f'(x)$ is exist $\forall x$.

$f(x)$ is derivable in $(0,9)$

$g'(x) = 0-1 = -1$

$g'(x)$ is exist $\forall x$.

$g(x)$ is derivable in $(0,9)$.

$\Rightarrow f(x), g(x)$ are derivable in $(0,9)$

Then $\exists$ a $c \in (0,9) \ni \dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{3c^2}{-1} = \frac{(9)^3 - (\cancel{2})^3 (2-0)^3}{(2-9)-(2-0)}$$

$$\cancel{+3c^2} \neq \frac{\cancel{81\times 9/3} - 8}{\cancel{+7} \; \cancel{+/2}}$$

$$c^2 \neq \frac{\cancel{81\times 9}}{\cancel{7}}$$

$$-3c^2 = \frac{721}{-7-2}$$

$$\cancel{+}3c^2 = \frac{721}{\cancel{+}9}$$

$$c^2 = \frac{721}{27}$$

$$c = \sqrt{\frac{721}{27}}$$

$$\boxed{c = 5.1675 \in (0,9)}$$

Wednesday
23/10 Taylor's Expansion And Maclurin's:

Taylor's expansion at $x=a$, $x=1$, $x=\pi/2$.

Taylor's expansion about / at $\}$ $x=a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

This is also called as taylor's expansion in powers of $(x-1)$.

Maclurin's:

at $x=0$, $f(x) = f(0) + x.f'(0) + \dfrac{x^2}{2!}f''(0) + \dfrac{x^3}{3!}f'''(0) + \cdots$

① $f(x) = \sin x$

② $f(x) = \log(1+x)$

③ $f(x) = \tan^{-1} x$

④ $f(x) = e^x$ at $x=1$.

⑤ $f(x) = (1-x)^{5/2}$

⑥ $f(x) = \log x$ in powers of $x-1$ and hence evaluate $\log 1.1$ correct to four decimal process.

⑦ $f(x) = 2x^3 - 7x^2 + x + 6$ at $x=2$.

⑥ $f(x) = \log x$

By Taylor's expansion at $x=a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{IV}(a) + \dots$$

$f(x) = \log x \Rightarrow f(1) = \log(1) = 0$

$f'(x) = \frac{1}{x} \rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} \Rightarrow f'''(1) = 2$

$f^{IV}(x) = -6x^{-4} \Rightarrow f^{IV}(1) = -6$

wrong

$\therefore f(x) = \log a + (x-a)$

---

⑥ $f(x) = \log(x)$

By Taylor's explosion at $x=a$

is $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{IV}(a) + \dots$

at $a=1$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \frac{(x-1)^4}{4!}f^{IV}(1) + \dots \rightarrow ①$$

$f(x) = \log x \Rightarrow f(1) = \log(1) = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \rightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} \Rightarrow f'''(1) = 2$

$f^{IV}(x) = -6x^{-4} \Rightarrow f^{IV}(1) = -6$

from ①, $\log x = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}2 + \frac{(x-1)^4}{4!}(-6) + \dots$

$$\log x = (x-1) - \frac{(x-1)^2}{2!} + 2\cdot\frac{(x-1)^3}{3!} - 6\frac{(x-1)^4}{4!} + \dots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{\cancel{6}\,3}(\cancel{2}) - \cancel{6}\cdot\frac{(x-1)^4}{\cancel{24}\,4} + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$\log(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} + \frac{(1.1-1)^4}{4} + \cdots$$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} + \frac{0.00001}{4} + \cdots$$

$$= 0.1 - 0.005 + 0.0003 + 0.000002$$

NO USE

$$= 0.105302$$

$$\log(1.1) = 0.095310179$$

$$\boxed{\log(1.1) \simeq 0.094}$$

⑦ $f(x) = 2x^3 - 7x^2 + x + 6$ at $x = 2$.

By Taylor's expansion at $x = 2$

is $f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2)$

$$+ \frac{(x-2)^4}{4!}f^{IV}(2) + \cdots \rightarrow ①$$

$f(x) = 2x^3 - 7x^2 + x + 6 \Rightarrow f(2) = -4$

$f'(x) = 6x^2 - 14x + 1 \Rightarrow f'(2) = 24 - 28 + 1 = -3$

$f''(x) = 12x - 14 \Rightarrow f''(2) = 24 - 14 = 10$

$f'''(x) = 12 \Rightarrow f'''(2) = 12$

$f^{IV}(x) = 0 \Rightarrow f^{IV}(2) = 0$

from ①,

$$2x^3 - 7x^2 + x + 6 = -4 + (x-2)\overset{(-3)}{\cancel{10}} + \frac{(x-2)^2}{2!}10 + \frac{(x-2)^3}{3!}12$$

$$= -4 + (x-2)\overset{(-3)}{\cancel{10}} + \frac{(x-2)^2}{2!}10 + \frac{(x-2)^3}{3!}$$

$$= -4 - 3(x-2) + 10\frac{(x-2)^2}{2!} + 12\cdot\frac{(x-2)^3}{3!}$$

② $f(x) = \log(1+x)$

Now, the Maclurin's expansion is

$$f(x) = f(0) + x.f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \rightarrow ①$$

$f(x) = \log(1+x) \Rightarrow f(0) = \log(1+0) = 0$

$f'(x) = \frac{1}{1+x} \rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \rightarrow f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$

$f^{IV}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{IV}(0) = -6$

from ①,

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots$$

$$= x - \frac{x^2}{2!} + 2 \cdot \frac{x^3}{3!} - 6\frac{x^4}{4!} + \cdots$$

⑤ $f(x) = (1-x)^{5/2}$

By Maclurin's expansion is

$$f(x) = f(0) + x.f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots \rightarrow ①$$

$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$

$f'(x) = \frac{5}{2}(1-x)^{3/2}(-1) \Rightarrow f'(0) = \frac{-5}{2}$

$f''(x) = \frac{-5}{2} \cdot \frac{3}{2}(1-x)^{1/2}(-1) \Rightarrow f''(0) = \frac{15}{4}$

$f'''(x) = \frac{15}{4} \cdot \frac{1}{2}(1-x)^{-1/2}(-1) \Rightarrow f'''(0) = -\frac{15}{8}$

$\vdots \qquad \vdots$

from ①,

$$(1-x)^{5/2} = 1 + x.\left(\frac{-5}{2}\right) + \frac{x^2}{2!}\left(\frac{15}{4}\right) + \frac{x^3}{3!}\left(\frac{-15}{8}\right) + \cdots$$

$$= 1 - \frac{5}{2}x + \frac{15}{4}\frac{x^2}{2!} - \frac{15}{8} \cdot \frac{x^3}{3!} + \cdots$$

$$= 1 - \frac{5}{2}x + \frac{15}{8}x^2 - \frac{5}{16}x^3 + \cdots$$

① $f(x) = \sin x$.

Now, the Maclurin's expansion is

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots \rightarrow ①$$

$f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = -(-\sin x) \Rightarrow f^{iv}(0) = 0$.

from①,

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

② $f(x) = \tan^{-1} x$.

Now, Maclurin's expansion is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$f(x) = \tan^{-1} x \Rightarrow f(0) = \tan^{-1}(0) = 0$

$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x^2)^2}(2x) \Rightarrow f''(0) = 0$.

$$f'''(x) = \frac{(1+x^2)^2(-2) - (-2x)2(1+x^2)(2x)}{[(1+x^2)^2]^2}$$

$$= \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} \Rightarrow f'''(0) = \frac{-2(1+(0)^2)^2 + 0}{(1+0)^4} = -2$$

from①,

$$\tan^{-1} x = 0 + x.(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \cdots$$

$$\tan^{-1} x = x - 2\frac{x^3}{3!} + \cdots$$

(4) $f(x) = e^x$ at $x=1$

Now, Taylor's expansion is

$$\cancel{f(x) = f(a) + (x-a) f'(x-a) + \frac{x^2}{2!} f''(x-a) + \frac{x^3}{3!} f''' x-a}$$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$\Rightarrow$ at $a=1$

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots \rightarrow (1)$$

$f(x) = e^x \Rightarrow f(1) = e$

$f'(x) = e^x \Rightarrow f'(1) = e$

$f''(x) = e^x \Rightarrow f''(1) = e$

$f'''(x) = e^x \Rightarrow f'''(1) = e$

$f^{IV}(x) = e^x \Rightarrow f^{IV}(1) = e$

from (1),

$$e^x = e + (x-1) e + \frac{(x-1)^2}{2!} e + \frac{(x-1)^3}{3!} e + \cdots$$

$$e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right]$$

Differentiation

Formulae :

- $\frac{d}{dx}(\text{constant}) = 0$
- $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$
- $\frac{d}{dx}(A \cdot x^n) = A \cdot n \cdot x^{n-1}$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = a^x \cdot \log a$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$
- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

$\rightarrow \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

$\rightarrow \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

$\rightarrow \frac{d}{dx}(\log_a x) = \frac{1}{x.\log a}$

$\rightarrow \frac{d}{dx} \cdot \frac{1}{x^n} = \frac{-n}{x^{n-1}}$

$\rightarrow \frac{d}{dx}(\log_e |x|) = \frac{1}{x}$

$\rightarrow \frac{d}{dx}(|x|) = \frac{|x|}{x}$

$\rightarrow \frac{d}{dx}(\sin hx) = \cos hx$

$\rightarrow \frac{d}{dx}(\cos hx) = \sin hx$

$\rightarrow \frac{d}{dx}(\tan hx) = \sec h^2 x$

$\rightarrow \frac{d}{dx}(\cot hx) = -\mathrm{cosec}h^2 x$

$\rightarrow \frac{d}{dx}(\sec hx) = -\sec hx.\tan hx$

$\rightarrow \frac{d}{dx}(\mathrm{cosec}hx) = -\mathrm{cosec}hx.\cot hx$

$\rightarrow \frac{d}{dx}(\sin h^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

$\rightarrow \frac{d}{dx}(\cos h^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

$\rightarrow \frac{d}{dx}(\tan h^{-1}x) = \frac{1}{1-x^2}$

$\rightarrow \frac{d}{dx}(\cot h^{-1}x) = \frac{1}{1-x^2}$

$\rightarrow \frac{d}{dx}(\sec h^{-1}x) = \frac{-1}{|x|\sqrt{1-x^2}}$

$\rightarrow \frac{d}{dx}(\mathrm{cosec}\,h^{-1}x) = \frac{-1}{|x|\sqrt{1+x^2}}$

Integrations

Formulae:

→ $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

→ $\int x \, dx = \frac{x^2}{2} + c$

→ $\int (1) \, dx = x + c$

→ $\int (ax+b)^n \cdot dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

→ $\int \frac{1}{x} \, dx = \log^x_e + c$

→ $\int \frac{1}{ax+b} \, dx = \frac{\log_e (ax+b)}{a} + c$

→ $\int e^x \, dx = e^x + c$

→ $\int e^{ax+b} \, dx = \frac{e^{ax+b}}{a} + c$

→ $\int a^x \cdot dx = \frac{a^x}{\log a} + c$

→ $\int k^{ax+b} \, dx = \frac{k^{ax+b}}{a \cdot \log k} + c$

→ $\int \log x \, dx = x \log x - x$.

→ $\int \sin x \, dx = -\cos x + c$

→ $\int \cos x \cdot dx = +\sin x + c$

→ $\int \sin(ax+b) \cdot dx = \frac{-\cos(ax+b)}{a} + c$

→ $\int \tan x \, dx = \log|\sec x| + c$

→ $\int \tan x \cdot dx = -\log|\cos x| + c$

$\rightarrow \int \tan(ax+b)\, dx = \log \frac{\sec(ax+b)}{a} + C$

$\rightarrow \int \cot x \, dx = \log|\sin x| + C$

$\rightarrow \int \sec x . dx = \log|\sec x + \tan x| + C$

$\rightarrow \int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) + C$

$\rightarrow \int (f(x))^n = \frac{f(x)^{n+1}}{n+1} + C$

$\rightarrow \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$

$\rightarrow \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$

$\rightarrow \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$\rightarrow \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

$\rightarrow \int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$

$\rightarrow \int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

$\rightarrow \int \frac{1}{\sqrt{x^2+a^2}} dx = \sinh^{-1}\left(\frac{x}{a}\right) + c$

(or)

$\log(x + \sqrt{x^2+a^2}) + c$

$\rightarrow \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ (or) $-\cos^{-1}\left(\frac{x}{a}\right) + c$

$\rightarrow \int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}\left(\frac{x}{a}\right) + c$

(or)

$\log(x + \sqrt{x^2-a^2}) + C$

$\rightarrow \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

$\rightarrow \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + c$

$\rightarrow \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$

ILATE

$\rightarrow \int e^{ax}.\sin bx.\,dx = \frac{e^{ax}}{a^2+b^2}\left[a.\sin bx - b.\cos bx\right] + c$

$\rightarrow \int e^{ax}\cos bx.\,dx = \frac{e^{ax}}{a^2+b^2}\left[a.\cos bx + b\sin bx\right] + c$

U,V Formulae:

$\rightarrow d(u \pm v) = d(u) \pm d(v)$

$\rightarrow d(u.v) = \cancel{u.\,dv}\ u.dv + v.du$

$\rightarrow d\left(\frac{u}{v}\right) = \frac{v.du - u.dv}{v^2}$

Solutions of First Order Differential Equations And Applications.

(4) $\cos^2 x \frac{dy}{dx} + y = \tan x$.

Sol:- $\frac{\cos^2 x}{\cos^2 x} \cdot \frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$

$$\frac{dy}{dx} + \frac{1}{\cos^2 x} \cdot y = \tan x \cdot \sec^2 x$$

$$\frac{dy}{dx} + \sec^2 x \cdot y = \tan x \cdot \sec^2 x. \rightarrow (1)$$

Here $P = \sec^2 x \qquad Q = \tan x \cdot \sec^2 x$

Now, I.F $e^{\int P(x) dx} = e^{\int \sec^2 x \cdot dx}$

$$= e^{\tan x}$$

Now the solution of equ (1) is

$$y \cdot e^{\tan x} = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} dx + c$$

Let $\tan x = t$

$\sec^2 x \cdot dx = dt$

$$y \cdot e^{\tan x} = \int t \cdot e^t \cdot dt + c$$

$$= t \cdot e^t - e^t + c$$

$$= e^t (t-1) + c$$

D.	I.
$+t$	e^t
-1	e^t
0	e^t

$$\boxed{y \cdot e^{\tan x} = e^{\tan x}(\tan x - 1) + c.}$$

(2) $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right) \frac{dx}{dy} = 1$

Sol:-
$$\frac{dx}{dy} = \frac{1}{\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}}$$

$$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}.$$

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$\frac{dy}{dx} + \frac{1}{\sqrt{x}} \cdot y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \rightarrow ①$$

where $P = \frac{1}{\sqrt{x}}$, and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Now, $I.F = e^{\int P(x)dx}$

$$= e^{\int \frac{1}{\sqrt{x}} dx}$$

$$= e^{\int x^{-1/2} dx}$$

$$= e^{\frac{x^{1/2}}{1/2}}$$

$$= e^{2\sqrt{x}}$$

Now the solution of equ ① is

$$y \cdot e^{2\sqrt{x}} = \int \frac{\cancel{e^{-2\sqrt{x}}}}{\sqrt{x}} \cdot \cancel{e^{2\sqrt{x}}}\, dx + c$$

$$= \int \frac{1}{\sqrt{x}} dx + c$$

$$= \int x^{-1/2} dx + c$$

$$= \frac{x^{1/2}}{1/2} + c$$

$$\boxed{y \cdot e^{2\sqrt{x}} = 2\sqrt{x} + c}$$

(13) $\frac{dy}{dx} = \frac{y}{2y\log y + y - x}$

Sol:- $\frac{dx}{dy} = \frac{2y\log y + y - x}{y}$

$$\frac{dx}{dy} = \frac{2\cancel{y}\log y}{\cancel{y}} + \frac{y}{y} - \frac{x}{y}$$

$$= 2\log y + 1 - \frac{x}{y}$$

$$\frac{dx}{dy} + \frac{1}{y} \cdot x = 2\log y + 1 \longrightarrow ①$$

where $p = \frac{1}{y}$, $Q = 2\log y + 1$

Now I.F $e^{\int P(y)\,dy}$

$= e^{\int \frac{1}{y} dy}$

$= e^{\log_e y} = y$

Now the solution of equ ① is

$x \cdot e^{\log y} = \int (2\log y + 1)\, e^{\log y}\, dy + c$

$x \cdot y = \int (2\log y + 1)\, y \cdot dy + c$

$xy = \int (2y \cdot \log y + y)\, dy + c$

$xy = 2\int y \cdot \log y \cdot dy + \int y \cdot dy + c$

$= 2\left[\log y \cdot \frac{y^2}{2} - \int \frac{1}{y} \cdot \frac{y^2}{2} dy\right] + \frac{y^2}{2} + c$

$= 2\left[\log y \cdot \frac{y^2}{2} - \frac{1}{2}\int y\, dy\right] + \frac{y^2}{2} + c$

$= 2\left[\log y \cdot \frac{y^2}{2} - \frac{1}{2} \cdot \frac{y^2}{2}\right] + \frac{y^2}{2} + c$

$= \cancel{2} \cdot \log y \cdot y^2 - \cancel{\frac{y^2}{2}} + \cancel{\frac{y^2}{2}} + c$

$x \cdot y = y^2 \log y + c$

H.W

(3) $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$.

sol:-

$\frac{dy}{dx} + \frac{1}{x} \cdot y = x^3 - 3 \longrightarrow ①$

I.F $e^{\int P(x)\,dx} = e^{\int \frac{1}{x} dx}$

$= e^{\log_e x}$

$= x$.

Now, the solution of equ ① is

$y \cdot x = \int (x^3 - 3)\, x \cdot dx + c$

$= \int (x^4 - 3x)\, dx + c$

$xy = \frac{x^5}{5} - 3\frac{x^2}{2} + c$

(5) $x \cdot \log x \frac{dy}{dx} + y = 2 \cdot \log x$

Sol:-

$$\frac{x \cdot \log x}{x \cdot \log x} \cdot \frac{dy}{dx} + \frac{y}{x \cdot \log x} = \frac{2 \log x}{x \cdot \log x}$$

$$\frac{dy}{dx} + \frac{1}{x \cdot \log x} \cdot y = \frac{2}{x} \rightarrow ①$$

where $P = \frac{1}{x \cdot \log x}$, $Q = \frac{2}{x}$

I.F $e^{\int P(x)\,dx} = e^{\int \frac{1}{x \cdot \log x} dx}$ put $\log x = t$

$\frac{1}{x} \cdot dx = dt$

$$= e^{\int \frac{1}{t} dt}$$

$$= e^{\log_e t}$$

$$= t$$

$$= \log x$$

Now the solution of equ ① is

$$y \cdot \log x = \int \frac{2}{x} \cdot \log x \, dx + c$$

$$= 2 \int \frac{1}{x} \cdot \log x \cdot dx + c$$ put $\log x = t$

$\frac{1}{x} \cdot dx = dt$

$$= 2 \int t \cdot dt + c$$

$$= \cancel{2} \cdot \frac{t^2}{\cancel{2}} + c$$

$$= t^2 + c$$

$$\boxed{y \cdot \log x = (\log x)^2 + c}$$

(6) $(1+x^3) \frac{dy}{dx} + 6x^2 y = 1 + x^2$

Sol:-

$$\frac{\cancel{(1+x^3)}}{\cancel{1+x^3}} \frac{dy}{dx} + \frac{6x^2 y}{1+x^3} = \frac{1+x^2}{1+x^3}$$

$$\frac{dy}{dx} + \frac{6x^2}{1+x^3} \cdot y = \frac{1+x^2}{1+x^3} \longrightarrow ①$$

where $P = \frac{6x^2}{1+x^3}$, $Q = \frac{1+x^2}{1+x^3}$

I.F $e^{\int P(x)\,dx} = e^{\int \frac{6x^2}{1+x^3} dx}$

$$= e^{2\int \frac{3x^2}{1+x^3} dx}$$

$$= e^{2 \cdot \log(1+x^3)}$$

$$= e^{\log_e (1+x^3)^2}$$

$$= (1+x^3)^2$$

Now the solution of equ ① is

$$y \cdot (1+x^3)^2 = \int \frac{1+x^2}{\cancel{(1+x^3)}} (1+x^3)^{\cancel{2}} dx + c$$

$$= \int (1+x^3+x^2+x^5)\, dx + c$$

$$y \cdot (1+x^3)^2 = x + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^6}{6} + c$$

Nill

(7) $\frac{dy}{dx} + y \cdot \cot x = \cos x$

Sol:- $\frac{dy}{dx} + \cot x \cdot y = \cos x \longrightarrow ①$

where $p = \cot x$, $Q = \cos x$

I.F $e^{\int p(x)\, dx} = e^{\int \cot x\, dx}$

$= e^{\log_e |\sin x|}$

$= \sin x$

Now, the solution of equ ① is

$$y \cdot \sin x = \int \cos x \cdot \sin x \cdot dx + c$$

put $\sin x = t$

$\cos x \cdot dx = dt$

$$= \int t \cdot dt + c$$

$$= \frac{t^2}{2} + c$$

$$y \cdot \sin x = \frac{(\sin x)^2}{2} + c$$

(8) $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$

Sol:- $\frac{\cancel{1+x^2}}{\cancel{1+x^2}}\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$

$\frac{dy}{dx} + \frac{1}{1+x^2} \cdot y = \frac{e^{\tan^{-1}x}}{1+x^2} \longrightarrow ①$

where $p = \frac{1}{1+x^2}$, $Q = \frac{e^{\tan^{-1}x}}{1+x^2}$

I.F $e^{\int p(x)\, dx} = e^{\int \frac{1}{1+x^2} dx}$

$= e^{\tan^{-1}x}$

Now the solution of equ① is

$$y \cdot e^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{1+x^2} \cdot e^{\tan^{-1}x}\, dx + c$$

put $\tan^{-1}x = t$

$\frac{1}{1+x^2} dx = dt$

$$= \int e^t \cdot e^t\, dt + c$$

$$= \int e^{2t}\, dt + c$$

$$= \frac{e^{2t}}{2} + c$$

$$y \cdot e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + c$$

(10) $e^{-y}\sec^2 y\, dy = dx + x\, dy$

Sol:-

$$e^{-y}\sec^2 y\, dy - x\, dy = dx$$

$$(e^{-y}\sec^2 y - x)\, dy = dx$$

$$e^{-y}\sec^2 y - x = \frac{dx}{dy}$$

$$\frac{dx}{dy} + x = e^{-y}\sec^2 y \rightarrow ①$$

where $P = 1$, $Q = e^{-y}\sec^2 y$

I.F $e^{\int P(y)dy} = e^{\int (1) dy}$

$= e^y$

Now the solution of equ① is

$$x \cdot e^y = \int e^{-y} \cdot \sec^2 y \cdot e^y\, dy + c$$

$$= \int \sec^2 y\, dy + c$$

$$x \cdot e^y = \tan y + c.$$

(11) $y \cdot e^y\, dx = (y^2 - 2x e^y)\, dy$

Sol:-

$$y \cdot e^y\, dx = (y^2 - 2xe^y)\, dy$$

$$\frac{dx}{dy} = \frac{y^2}{y \cdot e^y} - \frac{2x \cdot e^y}{y \cdot e^y}$$

$$\frac{dx}{dy} = \frac{y}{e^y} - \frac{2x}{y}$$

$$\frac{dx}{dy} + \left(\frac{2}{y}\right)x = \frac{y}{e^y} \rightarrow ①$$

where $P = \frac{2}{y}$, $Q = \frac{y}{e^y}$

I.F $e^{\int P(y)dy} = e^{\int \frac{2}{y} dy}$

$= e^{2 \log y}$

$= e^{\log_e y^2}$

$= y^2$

Now, the solution of equ① is

$$x \cdot y^2 = \int \frac{y}{e^y} y^2 \, dy + c$$

$$= \int e^{-y} \cdot y^3 \, dy + c$$

D	I
$+ y^3$	e^{-y}
$- 3y^2$	$-e^{-y}$
$+ 6y$	e^{-y}
$- 6$	$-e^{-y}$
$+ 0$	e^{-y}

$$x \cdot y^2 = -y^3 e^{-y} - 3y^2 e^{-y} - 6y e^{-y} - 6e^{-y} + c$$

$$x \cdot y^2 = -e^{-y}\left[y^3 + 3y^2 + 6y + 6\right] + c.$$

Nill

(12) $(1+y^2) + (x - e^{-\tan^{-1} y}) \frac{dy}{dx} = 0$

Sol:- $(x - e^{-\tan^{-1} y}) \frac{dy}{dx} = -(1+y^2)$

$$\frac{dy}{dx} = \frac{-(1+y^2)}{x - e^{-\tan^{-1} y}}$$

$$\frac{dx}{dy} = \frac{x}{-(1+y^2)} - \frac{e^{-\tan^{-1} y}}{-(1+y^2)}$$

$$\frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = +\frac{e^{-\tan^{-1} y}}{1+y^2} \longrightarrow ①$$

where $P = \frac{1}{1+y^2}$ and $Q = \frac{e^{-\tan^{-1} y}}{1+y^2}$

I.F $e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

Now the solution of equ ① is

$$x.e^{Tan^{-1}y} = \int \frac{e^{-Tan^{-1}y}}{1+y^2} . e^{Tan^{-1}y} dy + c$$

$$= \int \frac{1}{1+y^2} dy + c$$

$$x.e^{Tan^{-1}y} = Tan^{-1}y + c.$$

(14) $\sqrt{1-y^2}\, dx = (\sin^{-1}y - x)\, dy$

Sol:-

$$\frac{dx}{dy} = \frac{\sin^{-1}y - x}{\sqrt{1-y^2}}$$

$$\frac{dx}{dy} = \frac{\sin^{-1}y}{\sqrt{1-y^2}} - \frac{x}{\sqrt{1-y^2}}$$

$$\frac{dx}{dy} + \frac{1}{\sqrt{1-y^2}} . x = \frac{\sin^{-1}y}{\sqrt{1-y^2}} \rightarrow ①$$

Where $p = \frac{1}{\sqrt{1-y^2}}$ and $Q = \frac{\sin^{-1}y}{\sqrt{1-y^2}}$

I.F $e^{\int p(y) dy} = e^{\int \frac{1}{\sqrt{1-y^2}} dy}$

$$= e^{\cancel{Tan}\, \sin^{-1}y}$$

Now the solution of equ ① is

$$x.e^{\sin^{-1}y} = \int \frac{\sin^{-1}y}{\sqrt{1-y^2}} . \cancel{\sin e} e^{\sin^{-1}y} dy + c$$

put $\sin^{-1}y = t$

$$\frac{1}{\sqrt{1-y^2}} dy = dt$$

$$= \int t . e^t dt + c$$

$$= e^t . t - e^t + c$$

$$= e^t (t-1) + c$$

$$x.e^{\sin^{-1}y} = e^{\sin^{-1}y} (\sin^{-1}y - 1) + c.$$

(19) $dr + (2r\cot\theta + \sin 2\theta)\, d\theta = 0$

Sol:- $(2r\cot\theta + \sin 2\theta)\, d\theta = -dr$

$$\frac{d\theta}{dr} = \frac{-1}{(2r\cot\theta + \sin 2\theta)}$$

$$\frac{dr}{d\theta} = -(2r\cot\theta + \sin 2\theta)$$

$$\frac{dr}{d\theta} + 2r\cot\theta = -\sin 2\theta$$

$$\frac{dr}{d\theta} + (2\cot\theta)r = -\sin 2\theta \longrightarrow (1)$$

where $P = 2\cot\theta$ and $Q = -\sin 2\theta$

$$I.F\ e^{\int P(\theta)d\theta} = e^{\int 2\cot\theta\, d\theta}$$

$$= e^{2\int \cot\theta\, d\theta}$$

$$= e^{2\log|\sin\theta|}$$

$$= e^{\log_e (\sin\theta)^2}$$

$$= \underline{\sin^2\theta}$$

Now the solution of equ (1) is

$$r.\sin^2\theta = \int -\sin 2\theta \cdot \sin^2\theta \cdot d\theta + c$$

$$= -\int 2\sin\theta \cdot \cos\theta \cdot \sin^2\theta \cdot d\theta + c$$

$\boxed{\begin{array}{l}\sin\theta = t \\ \cos\theta . d\theta = dt\end{array}}$

$$= -2\int \sin^3\theta \cdot \cos\theta \cdot d\theta + c$$

$$= -2\int t^3 . dt + c$$

$$= -\cancel{2} \cdot \frac{t^4}{\cancel{4}_2} + c$$

$$\boxed{r.\sin^2\theta = -\frac{\sin^4\theta}{2} + c}$$

(20) $\cosh x \frac{dy}{dx} + y \sinh x = 2\cosh^2 x \cdot \sinh x$.

Sol:-

$$\frac{\cosh x}{\cosh x} \cdot \frac{dy}{dx} + y \cdot \frac{\sinh x}{\cosh x} = \frac{2 \cdot \cosh^2 x \cdot \sinh x}{\cosh x}$$

$$\frac{dy}{dx} + \tanh x \cdot y = 2 \cdot \sinh x \cdot \cosh x \rightarrow (1)$$

Here $p = \tanh x$ and $Q = 2\sinh x \cosh x$

I.F $e^{\int p(x)\,dx} = e^{\int \tanh x\,dx}$

$= e^{\log_e |\operatorname{sech} x|}$

$= \underline{\operatorname{sech} x}$.

Now the solution of equ (1) is

$$y \cdot \operatorname{sech} x = \int 2 \sinh x \cdot \cancel{\cosh x} \cdot \cancel{\operatorname{sech} x}\, dx + C$$

$$= 2 \int \sinh x \cdot dx + C$$

$$\boxed{y \cdot \operatorname{sech} x = 2 \cdot \cosh x + C.}$$

(16) $\frac{dy}{dx} + y \cdot \cot x = 4x \operatorname{cosec} x$ if $y = 0$ when $x = \pi/2$

Sol:-

$$\frac{dy}{dx} + y \cdot \cot x = 4x \cdot \operatorname{cosec} x$$

$$\frac{dy}{dx} + \cot x \cdot y = 4x \cdot \operatorname{cosec} x \rightarrow (1)$$

Here $P = \cot x$ and $Q = 4x \cdot \operatorname{cosec} x$.

I.F $e^{\int P(x)\,dx} = e^{\int \cot x \cdot dx}$

$= e^{\log_e |\sin x|}$

$= \underline{\sin x}$

Now the solution of equ (1) is

$$y \cdot \sin x = \int 4x \cdot \cancel{\operatorname{cosec} x} \cdot \cancel{\sin x} \cdot dx + C$$

$$= 4 \int x \cdot dx + C$$

$$y \cdot \sin x = \cancel{4}^2 \cdot \frac{x^2}{\cancel{2}} + C$$

$$y \cdot \sin x = 2x^2 + C$$

$$(0) \cdot \sin \pi/2 = \cancel{2} \frac{\pi^2}{\cancel{4}_2} + C$$

$$0 = \frac{\pi^2}{2} + c$$

$$\boxed{\therefore c = -\pi^2/2}$$

(17) $\frac{dy}{dx} - y \cdot \tan x = 3 \cdot e^{-\sin x}$ if $y = 4$ when $x = 0$.

Sol:- $\frac{dy}{dx} + (-\tan x) \cdot y = 3 \cdot e^{-\sin x} \longrightarrow ①$

Here $P = -\tan x$ and $Q = 3 \cdot e^{-\sin x}$

I.F $e^{\int P(x)\,dx} = e^{-\int \tan x\, dx}$

$= e^{-(-\log|\cos x|)}$

$= e^{\log_e|\cos x|}$

$= \cos x$

Now the solution of equ ① is

$y \cdot \cos x = \int 3 \cdot e^{-\sin x} \cdot \cos x\, dx + c$

$= 3 \int e^{-\sin x} \cdot \cos x\, dx + c$ put $\sin x = t$, $\cos x\, dx = dt$

$= 3 \cdot \int e^{-t}\, dt + c$

$= 3 \cdot e^{-t} (-1) + c$

$y \cdot \cos x = -3 \cdot e^{-\sin x} + c$

$4 \cdot (\cos 0) = -3\, e^{-\sin 0} + c$

$4 (1) = -3\, e^{0} + c$

$4 = -3(1) + c$

$c = 4 + 3$

$\boxed{c = 7}$

(18) $\frac{dy}{dx} + y \cdot \cot x = 5 \cdot e^{\cos x}$ if $y = -4$ when $x = \pi/2$

Sol:- $\frac{dy}{dx} + \cot x \cdot y = 5 \cdot e^{\cos x} \longrightarrow ①$

Here $P = \cot x$ and $Q = 5 \cdot e^{\cos x}$

I.F $e^{\int P(x)\,dx} = e^{\int \cot x\, dx}$

$= e^{\log_e|\sin x|}$

$= \sin x$

Now the solution of equ① is

$$y \cdot \sin x = \int 5 \cdot e^{\cos x} \cdot \sin x \cdot dx + c$$

$$y \cdot \sin x = 5\int e^{\cos x} \cdot \sin x \, dx + c$$

put $\cos x = t$

$-\sin x \, dx = dt$

$\sin x \cdot dx = -dt$

$$= 5\int e^{t} (-dt) + c$$

$$= -5\int e^{t} dt + c$$

$$= -5e^{t} + c$$

$$y \cdot \sin x = -5 \cdot e^{\cos x} + c$$

$$(-4) \sin \pi/_2 = -5 \cdot e^{\cos \pi/_2} + c$$

$$(-4)(1) = -5 \cdot e^{0} + c$$

$$(-4) = -5(1) + c$$

$$c = -4 + 5$$

$$\boxed{c = 1} \Rightarrow y \cdot \sin x = -5e^{\cos x} + 1$$

(1) $x(1-x^2) \dfrac{dy}{dx} + (2x^2-1) y = x^3.$

Sol:- $x(1-x^2) \dfrac{dy}{dx} + (2x^2-1) y = x^3$

put $x = \cos t$

$dx = -\sin t \cdot dt$

$$\cos t (1-\cos^2 t) \frac{dy}{dx} + (2\cos^2 t - 1) y = \cos^3 t$$

$$\frac{\cancel{\cos t \cdot \sin^2 t}}{\cancel{\cos t \cdot \sin^2 t}} \frac{dy}{dx} + \frac{\cos 2t}{\cos t \cdot \sin^2 t} \cdot y = \frac{\cos^{3} t}{\cos t \cdot \sin^2 t}$$

$$\frac{dy}{dx} + \frac{\cos 2t}{\cos t \cdot \sin^2 t} \cdot y = \cot^2 t.$$

$$\cos t \sin^2 t \cdot \frac{dy}{-\sin t \cdot dt} + \cos 2t \cdot y = \cos^3 t$$

$$\frac{\cancel{-\cos t \cdot \sin t}}{\cancel{\cos t \cdot \sin t}} \frac{dy}{dt} + \frac{\cos 2t}{\cos t \cdot \sin t} \cdot y = \frac{\cos^2 t}{\cos t \cdot \sin t}$$

$$\frac{dy}{dt} + \left(\frac{-\cos 2t}{\cos t \cdot \sin t}\right) y = \frac{-\cos^2 t}{\sin t} \longrightarrow ①$$

Here $P = \dfrac{-\cos 2t}{\cos t \cdot \sin t}$ and $Q = \dfrac{-\cos^2 t}{\sin t}$

$$\text{I.F } e^{\int p(t) dt} = e^{\int \frac{-\cos 2t}{\cos t \cdot \sin t} \cdot dt}$$

$$= e^{-\int \frac{2\cos 2t}{\sin 2t} \cdot dt}$$

$$= e^{-\log |\sin 2t|}$$

$= e^{\log_e (\sin 2t)^{-1}} = (\sin 2t)^{-1} = \frac{1}{\sin 2t}$

Now the solution of equ ① is

$$y \cdot \frac{1}{\sin 2t} = \int \frac{-\cos^2 t}{\sin t} \cdot \frac{1}{\sin 2t} \, dt + c$$

$$\frac{y}{\sin 2t} = -\int \frac{\cos^2 t}{\sin t \cdot 2\sin t \cdot \cos t} \, dt + c$$

$$\frac{y}{\sin 2t} = -\frac{1}{2} \int \operatorname{cosec} t \cdot \cot t \cdot dt + c$$

$$\frac{y}{\sin 2t} = -\frac{1}{2}(-\operatorname{cosec} t) + c$$

$$\frac{y}{\sin 2t} = \frac{\operatorname{cosec} t}{2} + c$$

$$\boxed{\frac{y}{\sin 2(\cos^{-1} x)} = \frac{\operatorname{cosec}(\cos^{-1} x)}{2} + c.}$$

(15) $x\left(\frac{dy}{dx} + y\right) = 1 - y$

Sol :- $x\left(\frac{dy}{dx} + y\right) = 1 - y$

$$\frac{dy}{dx} + y = \frac{1-y}{x}$$

$$\frac{dy}{dx} + y = \frac{1}{x} - \frac{y}{x}$$

$$\frac{dy}{dx} + y + \frac{y}{x} = \frac{1}{x}$$

$$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right) y = \frac{1}{x} \rightarrow ①$$

Here $P = 1 + \frac{1}{x}$ and $Q = \frac{1}{x}$

I.F $e^{\int (1+\frac{1}{x}) dx} = e^{\int (1) dx + \int \frac{1}{x} dx}$

$= e^{x + \log x}$

$= e^x \cdot e^{\log_e x}$

$= x \cdot e^x$

Now the solution of equ ① is

$$y \cdot x \cdot e^x = \int \frac{1}{x} \cdot x \cdot e^x \, dx + c$$

$$= \int e^x \, dx + c$$

$$\boxed{x \cdot y \cdot e^x = e^x + c}$$

(9) $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$

Sol:- $\frac{1-x^2}{1-x^2}\cdot\frac{dy}{dx} + \frac{2xy}{1-x^2} = \frac{x\sqrt{1-x^2}}{1-x^2}$

$$\frac{dy}{dx} + \frac{2x}{1-x^2}\cdot y = \frac{x(1-x^2)^{1/2}}{(\sqrt{1-x^2})^2}$$

$$\frac{dy}{dx} + \frac{2x}{1-x^2}\cdot y = \frac{x}{\sqrt{1-x^2}} \rightarrow ①$$

Equ ① is of linear form $\frac{dy}{dx} + P.y = Q$.

Here $P = \frac{2x}{1-x^2}$ and $Q = \frac{x}{\sqrt{1-x^2}}$

I.F $e^{\int p(x)\,dx}$

$$= e^{\int \frac{2x}{1-x^2}dx}$$

$$= e^{-\int \frac{-2x}{1-x^2}dx}$$

$$= e^{-\log[1-x^2]}$$

$$= e^{\log_e(1-x^2)^{-1}}$$

$$= (1-x^2)^{-1}$$

$$= \frac{1}{1-x^2}$$

Now the solution of equ ① is

$$y\cdot\frac{1}{1-x^2} = \int \frac{x}{\sqrt{1-x^2}}\cdot\frac{1}{1-x^2}dx + c$$

$$= \int \frac{x}{(1-x^2)^{3/2}}dx + c$$

$$= \frac{-1}{2}\int \frac{-2x}{(1-x^2)^{3/2}}dx + c \qquad \text{put } 1-x^2 = t$$

$$-2x.dx = dt$$

$$= \frac{-1}{2}\int \frac{1}{t^{3/2}}dt + c.$$

$$= \frac{-1}{2}\int t^{-3/2}.dt + c$$

$$= \frac{-1}{2}\cdot\frac{t^{-3/2+1}}{-3/2+1} + c$$

$$= \frac{+1}{\cancel{2}}\cdot\frac{t^{-1/2}}{+1/\cancel{2}} + c$$

$$= \frac{1}{t^{1/2}} + c$$

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c.$$

(1) $x(1-x^2)\frac{dy}{dx} + (3x^2-1)y = x^3$

Sol:- $$\frac{dy}{dx} + \frac{3x^2-1}{x(1-x^2)} \cdot y = \frac{x^{3}}{x(1-x^2)}$$

$$\frac{dy}{dx} + \frac{3x^2-1}{x(1-x^2)} \cdot y = \frac{x^2}{1-x^2} \longrightarrow (1)$$

Here $P = \frac{3x^2-1}{x(1-x^2)}$ and $Q = \frac{x^2}{1-x^2}$

$$\text{I.F} \cdot e^{\int \frac{3x^2-1}{x(1-x^2)} dx} = e^{\int \frac{3x^2-1}{x-x^3} \cdot dx}$$

$$= e^{-\int \frac{1-3x^2}{x-x^3} dx} = e^{-\log|x-x^3|}$$

$$= \frac{1}{x(1-x^2)}$$

Now the solution of equ (1) is

$$y \cdot \frac{1}{x(1-x^2)} = \int \frac{x^{2}}{1-x^2} \cdot \frac{1}{x(1-x^2)} dx + C.$$

$$= \frac{-1}{2} \int \frac{-2x}{(1-x^2)^2} dx + C$$

$1-x^2 = t$

$-2x\,dx = dt$

$$= \frac{-1}{2} \int \frac{1}{t^2} \cdot dt + C$$

$$= \frac{-1}{2} \int t^{-2} dt + C$$

$$= \frac{+1}{2} \frac{t^{-1}}{+1} + C$$

$$= \frac{1}{2t} + C$$

$$\frac{y}{x(1-x^2)} = \frac{1}{2(1-x^2)} + C.$$

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Reducible To The Linear Form:

(1) $\frac{dy}{dx} + x.\sin 2y = x^3.\cos^2 y$

Sol:- $\frac{dy}{dx} + x.\sin 2y = x^3.\cos^2 y$

$$\frac{1}{\cos^2 y}.\frac{dy}{dx} + \frac{x.\sin 2y}{\cos^2 y} = \frac{x^3.\cancel{\cos^2 y}}{\cancel{\cos^2 y}}$$

$$\sec^2 y.\frac{dy}{dx} + \frac{x.2\sin y.\cos y}{\cos^2 y} = x^3$$

$$\sec^2 y.\frac{dy}{dx} + 2x.\tan y = x^3$$

$\tan y = t$

$\sec^2 y.dy = dt$

$$\frac{dt}{dx} + 2x.t = x^3 \longrightarrow ①$$

Here $P = 2x$. and $Q = x^3$

I.F $e^{\int 2x.dx} = e^{2\int x.dx}$

$= e^{\cancel{2}.\frac{x^2}{\cancel{2}}}$

$= e^{x^2}$

Now the solution of equ①

$t.e^{x^2} = \int x^2.x e^{x^2}.dx + c$

$= \int v.e^v\frac{dv}{2} + c$

$= \frac{1}{2}\int v.e^v.dv + c$

Let $x^2 = v$

$\cancel{3x^2 dx = dv}$

$2x.dx = dv$

$x.dx = \frac{dv}{2}$

$t.e^{x^2} = \frac{1}{2}e^v(v-1) + c$

$\tan y.e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c.$

(2) $e^y \cdot y' = e^x (e^x - e^y)$

Sol:- $e^y \cdot \frac{dy}{dx} = e^x (e^x - e^y)$

$$e^y \cdot \frac{dy}{dx} = e^x \cdot e^x - e^x \cdot e^y$$

$$e^y \cdot \frac{dy}{dx} = e^{2x} - e^x \cdot e^y$$

$$e^y \frac{dy}{dx} + e^x e^y = e^{2x}$$

$e^y = t$

$$\frac{dt}{dx} + e^x t = e^{2x} \longrightarrow ① \qquad e^y \cdot dy = dt$$

equ① is of the linear form di

$p = e^x$ and $Q = e^{2x}$

I.F $e^{\int p(x) dx} = e^{\int e^x \cdot dx}$

$$= e^{e^x}$$

Now the solution of equ① is

$$t \cdot e^{e^x} = \int e^{2x} e^{e^x} dx + c$$

$$= \int e^x \cdot e^x \cdot e^{e^x} \cdot dx + c \qquad \text{Let } e^x = v$$

$$e^x dx = dv$$

$$= \int v \cdot e^v dv + c$$

$$t \cdot e^{e^x} = e^v (v - 1) + c$$

$$e^y \cdot e^{e^x} = e^{e^x} (e^x - 1) + c$$

(3) $(2x \log x - xy)\, dy = -2y\, dx$

Sol:- $\frac{dy}{dx} = \frac{-2y}{2x \cdot \log x - xy}$

$$\frac{dx}{dy} = \frac{2x \cdot \log x - xy}{-2y}$$

$$\frac{dx}{dy} = \frac{\cancel{2}x \cdot \log x}{-\cancel{2}y} + \frac{x\cancel{y}}{2\cancel{y}}$$

$$\frac{dx}{dy} = \frac{-x \cdot \log x}{y} + \frac{x}{2}$$

$$\cancel{\frac{dx}{dy} + \frac{x}{2} + \frac{-x \cdot \log x}{y}}$$

$$\frac{dx}{dy} + \frac{x.\log x}{y} = \frac{x}{2}$$

$$\frac{1}{x}\cdot\frac{dx}{dy} + \frac{x.\log x}{y}\cdot\frac{1}{x} = \frac{x}{2}\cdot\frac{1}{x}$$

$$\frac{dt}{dy} + \frac{1}{y}\cdot t = \frac{1}{2} \rightarrow (1)$$

put $\log x = t$

$\frac{1}{x}dx = dt$

Here $P = \frac{1}{y}$ and $Q = \frac{1}{2}$

I.F $e^{\int P(y)dy} = e^{\int \frac{1}{y}dy}$

$= e^{\log_e y}$

$= \underline{y}$

Now the solution of equ^n (1) is

$$t\cdot y = \int \frac{1}{2}\cdot y\, dy + C$$

$$t\cdot y = \frac{1}{2}\int y\, dy + c$$

$$= \frac{1}{2}\cdot\frac{y^2}{2} + c$$

$$t\cdot y = \frac{y^2}{4} + c$$

$$\log x\cdot y = \frac{y^2}{4} + c.$$

(4) $\frac{dy}{dx} - \tan x\cdot y = -y^2\cdot\sec^2 x.$

Sol:-

$$\frac{dy}{dx} - \tan x\cdot y = -y^2\sec^2 x$$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{\tan x\cdot y}{y^2} = \frac{-y^2\cdot\sec^2 x}{y^2}$$

$$\frac{1}{y^2}\cdot\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec^2 x.$$

put $\frac{+1}{y} = t$

$\frac{-1}{y^2}dy = dt$

$\frac{1}{y^2}\cdot dy = -dt$

$$\frac{-dt}{dx} + \tan x\cdot t = -\sec^2 x.$$

$$\frac{dt}{dx} - \tan x\cdot t = \sec^2 x \rightarrow (1)$$

Here $P = -\tan x$, $Q = \sec^2 x$.

I.F $e^{\int P(x)\cdot dx} = e^{-\int \tan x\cdot dx}$

$= e^{+\log_e(\cos x)}$

$= \underline{\cos x}$

Now the solution of equ① is

$$t \cdot \cos x = \int \sec^2 x \cdot \cos x \cdot dx + C$$

$$= \int \frac{1}{\cos^{\cancel{2}} x} \cdot \cancel{\cos x \cdot} dx + C$$

$$t \cdot \cos x = \int \sec x \cdot dx + C$$

$$\frac{1}{y} \cdot \cos x = \log|\sec x + \tan x| + C$$

(5) $e^y\left(\frac{dy}{dx} + 1\right) = e^x$

Sol:- $e^y \cdot \frac{dy}{dx} + e^y = e^x$

put $e^y = t$

$e^y \cdot dy = dt$

$$\frac{dt}{dx} + (1)t = e^x \rightarrow ①$$

Here $P = 1$ and $Q = e^x$

I.F $e^{\int P(x)dx} = e^{\int (1) dx}$

$= e^x$

Now the solution of equ① is

$$t \cdot e^x = \int e^x \cdot e^x \cdot dx + C$$

$$= \int e^{2x} \cdot dx + C$$

$$t \cdot e^x = e^{2x} \cdot (2) + C$$

$$e^x \cdot e^y = 2 \cdot e^{2x} + C$$

(6) $(x+1) \cdot \frac{dy}{dx} + 1 = 2e^{-y}$

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(7) $\tan y \cdot \frac{dy}{dx} + \tan x = \cos y \cdot \cos^2 x$.

Sol:-

$$\tan y \cdot \frac{dy}{dx} + \tan x = \cos y \cos^2 x$$

$$\frac{\tan y}{\cos y} \frac{dy}{dx} + \frac{\tan x}{\cos y} = \frac{\cancel{\cos y} \cdot \cos^2 x}{\cancel{\cos y}}$$

$$\sec y \cdot \tan y \cdot \frac{dy}{dx} + \sec y \cdot \tan x = \cos^2 x.$$

$\sec y = t$

$\sec y \cdot \tan y \cdot dy = dt$

$$\frac{dt}{dx} + \tan x \cdot t = \cos^2 x \rightarrow ①$$

Here $P = \tan x$. $Q = \cos^2 x$

I.F $e^{\int P(x)dx} = e^{\int \tan x \cdot dx}$

$= e^{\log_e(\sec x)}$

$= \sec x$

Now the solution of equ① is

$$t \cdot \sec x = \int \cos^2 x \cdot \sec x \cdot dx + c$$

$$= \int \cos^2 x \cdot \frac{1}{\cos x} \cdot dx + c$$

$$= \int \cos x \cdot dx + c$$

$$t \cdot \sec x = \sin x + c$$

$$\sec x \cdot \sec y = \sin x + c$$

(8) $\frac{dz}{dx} + \frac{z}{x} \cdot \log z = \frac{z}{x} (\log z)^2$.

Sol:- $$\frac{dz}{dx} + \frac{z}{x} \cdot \log z = \frac{z}{x} \cdot (\log z)^2$$

$$\frac{1}{z \cdot (\log z)^2} \frac{dz}{dx} + \frac{1}{x} \frac{\cancel{z} \cdot \log z}{\cancel{z} \cdot (\log z)^2} = \frac{1}{x} \cdot \frac{\cancel{z} \cancel{(\log z)^2}}{\cancel{z} \cancel{(\log z)^2}}$$

$$\frac{1}{z \cdot (\log z)^2} \frac{dz}{dx} + \frac{1}{x} \frac{1}{\log z} = \frac{1}{x}$$

$\frac{1}{\log z} = t$

$\frac{-1}{(\log z)^2} \cdot \frac{1}{z} dz = dt$

$\frac{1}{z (\log z)^2} dz = -dt$

$$-\frac{dt}{dx} + \frac{1}{x} \cdot t = \frac{1}{x}$$

$$\frac{dt}{dx} - \frac{1}{x} \cdot t = \frac{-1}{x} \rightarrow ①$$

Here $P = \frac{-1}{x}$ and $Q = \frac{-1}{x}$.

$$I.F = e^{\int P(x) dx} = e^{\int -\frac{1}{x} dx}$$

$$= e^{-\int \frac{1}{x} dx}$$

$$= e^{-\log x} = e^{\log_e x^{-1}}$$

$$\cancel{= \frac{1}{x}} = \frac{1}{x}$$

Now the solution of equ① is

$$t \cdot \frac{1}{x} = \int \frac{-1}{x} \cdot \frac{1}{x} \cdot dx + c$$

$$= -\int \frac{1}{x^2} dx + c$$

$$= -\int x^{-2} dx + c$$

$$= -\frac{x^{-1}}{-1} + c$$

$$t \cdot \frac{1}{x} = \frac{1}{x} + c$$

$$\frac{1}{x \cdot \log z} = \frac{1}{x} + c.$$

(6) $(x+1)\frac{dy}{dx}+1 = 2e^{-y}$.

sol:- $(x+1)\frac{dy}{dx} = 2e^{-y}-1$

$$\frac{dy}{dx} = \frac{2e^{-y}}{x+1} - \frac{1}{x+1}$$

$$\frac{dy}{dx} + \frac{1}{x+1} = \frac{2e^{-y}}{x+1}$$

$$\frac{1}{e^{-y}}\frac{dy}{dx} + \frac{1}{x+1}\cdot\frac{1}{e^{-y}} = \frac{2\cancel{e^{-y}}}{x+1}\times\frac{1}{\cancel{e^{-y}}}$$

$$e^{y}\cdot\frac{dy}{dx} + \frac{1}{x+1}\cdot e^{y} = \frac{2}{x+1}$$ put $e^y = t$

$e^y . dy = dt$

$$\frac{dt}{dx} + \frac{1}{x+1}\cdot t = \frac{2}{x+1} \longrightarrow (1)$$

Here $P = \frac{1}{x+1}$ and $Q = \frac{1}{x+1}$

I.F $e^{\int p(x)dx} = e^{\int\frac{1}{x+1}\cdot dx}$

$= e^{\log_e(x+1)}$

$= x+1.$

Now the solution of equ^n(1) is

$$t\cdot(x+1) = \int\frac{2}{\cancel{x+1}}\cdot\cancel{(x+1)}\,dx + c$$

$$= 2\int(1)\,dx + c$$

$$t\cdot(x+1) = 2x + C$$

$$e^y\cdot(x+1) = 2x + C.$$

(9) $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\,e^{x}\cdot\sec y$

sol:- $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\cdot e^{x}\cdot\sec y$

$$\frac{1}{\sec y}\cdot\frac{dy}{dx} - \frac{\tan y}{1+x}\cdot\frac{1}{\sec y} = \frac{(1+x)\,e^{x}\cdot\cancel{\sec y}}{\cancel{\sec y}}$$

$$\cos y\cdot\frac{dy}{dx} - \frac{1}{1+x}\cdot\frac{\sin y}{\cancel{\cos y}}\cdot\cancel{\cos y} = (1+x)e^{x}$$

$$\cos y\cdot\frac{dy}{dx} - \frac{1}{1+x}\cdot\sin y = (1+x)e^{x}.$$

$\sin y = t$

$$\frac{dt}{dx} - \frac{1}{1+x}\cdot t = (1+x)e^{x} \longrightarrow (2)$$ $\cos y . dy = dt$

~~Eq~~ Equ(1) is of linear form

where $P = \frac{-1}{1+x}$ and $Q = (1+x)\,e^{x}$.

$$I.F \quad e^{\int P(x)dx} = e^{\int -\frac{1}{1+x}\cdot dx}$$

$$= e^{-\int \frac{1}{1+x}\cdot dx}$$

$$= e^{-\log(1+x)}$$

$$= e^{\log_e (1+x)^{-1}}$$

$$= \frac{1}{1+x}$$

Now the solution of equ① is

$$t\cdot\frac{1}{1+x} = \int \cancel{(1+x)}\cdot e^{x}\,\frac{1}{\cancel{1+x}}\cdot dx + C$$

$$= \int e^{x}\,dx + C$$

$$t\cdot\frac{1}{1+x} = e^{x} + C$$

$$\frac{\sin y}{1+x} = e^{x} + C$$

(10) $\frac{dy}{dx} + \frac{y\cdot\log y}{x} = \frac{y(\log y)^2}{x^2}$

Sol:-

$$\frac{dy}{dx} + \frac{y\cdot\log y}{x} = \frac{y\cdot(\log y)^2}{x^2}$$

$$\frac{1}{y\cdot(\log y)^2}\cdot\frac{dy}{dx} + \frac{\cancel{y}\cdot\cancel{\log y}}{x}\cdot\frac{1}{\cancel{y}\cdot(\log y)^{\cancel{2}}} = \frac{\cancel{y}\cdot\cancel{(\log y)^2}}{x^2}\cdot\frac{1}{\cancel{y}\cancel{(\log y)^2}}$$

$$\frac{1}{y(\log y)^2}\cdot\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{\log y} = \frac{1}{x^2}$$

$$\frac{1}{\log y} = t$$

$$\frac{-1}{(\log y)^2}\cdot\frac{1}{y}\cdot dy = dt$$

$$\frac{1}{y\cdot(\log y)^2}\cdot dy = -dt$$

$$\frac{-dt}{dx} + \frac{1}{x}\cdot t = \frac{1}{x^2}$$

$$\frac{dt}{dx} - \frac{1}{x}\cdot t = \frac{-1}{x^2} \longrightarrow ①$$

Here $P = \frac{-1}{x}$ and $Q = \frac{-1}{x^2}$

$$I.F \quad e^{\int -\frac{1}{x}\cdot dx} = e^{-\int \frac{1}{x}\cdot dx}$$

$$= e^{-\log(x)}$$

$$= e^{\log_e (x)^{-1}}$$

$$= \frac{1}{x}$$

Now the solution of equⁿ① is

$$t\cdot\frac{1}{x} = \int -\frac{1}{x^2}\cdot\frac{1}{x}\cdot dx + C$$

$$= \int \frac{-1}{x^3}\cdot dx + C$$

$$= -\int x^{-3}\,dx + c$$

$$= -\frac{x^{-2}}{-2} + c$$

$$t \cdot \frac{1}{x} = \frac{1}{2x^2} + c$$

$$\frac{1}{x \cdot \log y} = \frac{1}{2x^2} + c$$

Monday
16/09

Bernoulli's Equation:

(2) $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$

Sol:- $(xy^2 - e^{1/x^3})\,dx = x^2y\,dy$

$$\frac{dy}{dx} = \frac{xy^2 - e^{1/x^3}}{x^2y}$$

$$\frac{dy}{dx} = \frac{\cancel{x}y^{\cancel{2}}}{x^{\cancel{2}}\cancel{y}} - \frac{e^{1/x^3}}{x^2y}$$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{e^{1/x^3}}{x^2y}$$

$$\frac{dy}{dx} - \frac{1}{x}\cdot y = -\frac{e^{1/x^3}}{x^2}y^{-1} \longrightarrow (1)$$

Equ(1) is of Bernoulli's form $\frac{dy}{dx} + P\cdot y = Q\cdot y^n$.

This can be reduced to linear form.

$$y\frac{dy}{dx} - \frac{1}{x}\cdot y\cdot y = \frac{-e^{1/x^3}}{x^2}\cdot \cancel{y^{-1}}\cdot \cancel{y}$$

$$y\cdot\frac{dy}{dx} - \frac{1}{x}\cdot y^2 = \frac{-e^{1/x^3}}{x^2}$$

$y^2 = t$

$2y\,dy = dt$

$y\,dy = dt\left(\frac{1}{2}\right)$

$$\frac{1}{2}\cdot\frac{dt}{dx} - \frac{1}{x}\cdot t = -\frac{e^{1/x^3}}{x^2}$$

$$\frac{dt}{dx} - \frac{2}{x}\cdot t = \frac{-2\cdot e^{1/x^3}}{x^2} \longrightarrow (2)$$

Equ(2) is in linear form. where $P = \frac{-2}{x}$ and $Q = -\frac{2e^{1/x^3}}{x^2}$.

$$\text{I.F } e^{\int P(x)dx} = e^{-2\int\frac{1}{x}dx}$$

$$= e^{-2\log(x)}$$

$$= e^{\log_e(x)^{-2}}$$

$$= \frac{1}{x^2}$$

Now the solution of equ(2) is

$$t\cdot\frac{1}{x^2} = \int \frac{-2e^{1/x^3}}{x^2}\cdot\frac{1}{x^2}\cdot dx + c$$

$$= -2\int e^{1/x^3}\cdot\frac{1}{x^4}\, dx + c$$

$$= -2\int e^{x^{-3}}\cdot x^{-4}\, dx + c$$

$x^{-3} = v$

$-3\cdot x^{-3-1}dx = dv$

$x^{-4}\cdot dx = \frac{-1}{3}dv$

$$= -2\int e^{v}\cdot\frac{-1}{3}dv + c$$

$$= \frac{2}{3}\int e^{v}\cdot dv + c$$

$$t\cdot\frac{1}{x^2} = \frac{2}{3}e^{v} + c$$

$$y^2\cdot\frac{1}{x^2} = \frac{2}{3}e^{x^{-3}} + c$$

$$\frac{y^2}{x^2} = \frac{2}{3}e^{1/x^3} + c.$$

(1) $x\cdot\frac{dy}{dx} + y = x^3y^6$

Sol:-

$$x\cdot\frac{dy}{dx} + y = x^3y^6$$

$$\frac{x}{y^6}\cdot\frac{dy}{dx} + \frac{\cancel{y}}{y^{\cancel{6}5}} = \frac{x^3\cancel{y^6}}{\cancel{y^6}}$$

$$x\cdot y^{-6}\cdot\frac{dy}{dx} + y^{-5} = x^3$$

$y^{-5} = t$

$-5y^{-6}dy = dt$

$y^{-6}dy = \frac{-1}{5}dt$

$$\cancel{x}\cdot y^{-6}\frac{dy}{dx} + \frac{1}{x}\cdot y^{-5} = x^2$$

$$\frac{-1}{5}\frac{dt}{dx} + \frac{1}{x}\cdot t = x^2$$

$$\frac{dt}{dx} - \frac{5}{x}\cdot t = -5x^2 \longrightarrow (1)$$

Equ(1) is in linear form.

where $P = \frac{-5}{x}$ and $Q = -5x^2$

$$I.F\ e^{\int p(x)dx} = e^{\int\frac{-5}{x}dx}$$

$$= e^{-5\int\frac{1}{x}dx}$$

$$= e^{-5\log x}$$

$$= e^{\log_e (x)^{-5}}$$

$$= x^{-5}$$

$$= \frac{1}{x^5}$$

Now the solution of equ ① is

$$t\cdot\frac{1}{x^5} = \int -5x^{x}\cdot\frac{1}{x^{7}_{3}}\,dx + c$$

$$= -5\int x^{-3}\,dx + c$$

$$= -5\,\frac{x^{-2}}{-2} + c$$

$$t\cdot\frac{1}{x^5} = \frac{5}{2}\cdot\frac{1}{x^2} + c.$$

$$\frac{1}{x^5\cdot y^5} = \frac{5}{2}\cdot\frac{1}{x^2} + c.$$

③ $xy(1+xy^2)\cdot\frac{dy}{dx} = 1$

Sol:- $xy(1+xy^2) = \frac{dx}{dy}.$

$$\frac{dx}{dy} = xy + x^2y^3$$

$$\frac{dx}{dy} - xy = x^2y^3$$

$$\frac{dx}{dy} - y\cdot x = x^2y^3 \rightarrow ①$$

Equ ① is of Bernoulli's form $\frac{dx}{dy} + P\cdot x = Q\cdot x^n$

This can be reduced to linear form.

$$\frac{dx}{dy} - y\cdot x = x^2\cdot y^3$$

$$\frac{1}{x^2}\cdot\frac{dx}{dy} - y\cdot\frac{x}{x^2} = \frac{x^2\cdot y^3}{x^2}$$

$$\frac{1}{x^2}\cdot\frac{dx}{dy} - y\left(\frac{1}{x}\right) = y^3$$

$$+\frac{1}{x} = t$$

$$+\frac{dt}{dy} + y\cdot t = -y^3 \rightarrow ②$$

$$+\left(\frac{-1}{x^2}\right)dx = dt$$

$$\frac{1}{x^2}\,dx = dt$$

Equ② is in linear form.

where $P = y$. and $Q = y^3$.

I.F $e^{\int P(y)\,dy} = e^{\int y\,dy}$

$$= e^{y^2/2}$$

Now the solution of equ ② is

$$t\cdot e^{y^2/2} = \int -y^3\cdot e^{y^2/2}\,dy + c.$$

$= -\int y \cdot y^2 \cdot e^{y^2/2} \cdot dy + C$ $\quad \frac{y^2}{2} = v \rightarrow \boxed{y^2 = 2v}$

$= -\int e^{v} \cdot 2v \cdot dv + C$ $\quad \cancel{\frac{1}{2}} \cdot \cancel{2}y\, dy = dv$

$= -2\int e^{v} \cdot v \cdot dv + C$ $\quad y \cdot dy = dv$

$t \cdot e^{y^2/2} = -2 \cdot e^{v}(v-1) + C$

$\frac{1}{x} \cdot e^{y^2/2} = -2 \cdot e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + C.$

(5) $\frac{dy}{dx} - x^2 y = y^2 \cdot e^{-x^3/3}$

<u>Sol:-</u> $\frac{dy}{dx} - x^2 \cdot y = e^{-x^3/3} \cdot y^2 \rightarrow ①$

Equ^n ① is of Bernoulli's form $\frac{dy}{dx} + P \cdot y = Q \cdot y^n$.

∴ This can be reduced to linear form.

$\frac{1}{y^2}\frac{dy}{dx} - x^2 \cdot \cancel{y} \frac{1}{y^{\cancel{2}}} = e^{-x^3/3} \cdot \cancel{y^2} \cancel{\frac{1}{y^2}}$

$\frac{1}{y^2}\frac{dy}{dx} - x^2 \cdot \frac{1}{y} = e^{-x^3/3}$ $\quad \frac{1}{y} = t$

$-\frac{dt}{dx} - x^2 \cdot t = e^{-x^3/3}$ $\quad -\frac{1}{y^2} \cdot dy = dt$

$\frac{dt}{dx} + x^2 \cdot t = -e^{-x^3/3} \rightarrow ②$ $\quad \frac{1}{y^2} dy = -dt$

Equ^n ② is in linear form.

where $P = x^2$ and $Q = -e^{-x^3/3}$

I.F $e^{\int P(x)dx} = e^{\int x^2 \cdot dx}$

$= e^{x^3/3}$.

Now the solution of equ^n ② is

$t \cdot e^{x^3/3} = \int -\cancel{e^{-x^3/3}} \cdot \cancel{e^{x^3/3}} \cdot dx + C$

$= -1\int (1) dx + C$

$t \cdot e^{x^3/3} = -x + C$

$\frac{1}{y} \cdot e^{x^3/3} = -x + C$

(4) $2.\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$.

Sol:-

$$2.\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$$

$$2.\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2} \rightarrow ①$$

Equ^n ① is of Bernoulli's form $\frac{dy}{dx} + Py = Q.y^n$. This can be reduced to linear form.

$$\frac{2}{y^2}.\frac{dy}{dx} - \frac{1}{x}.\frac{\cancel{y}}{y^{\cancel{2}}} = \frac{1}{x^2}.\frac{\cancel{y^2}}{\cancel{y^2}}$$

$$\frac{2}{y^2}.\frac{dy}{dx} - \frac{1}{x}.\frac{1}{y} = \frac{1}{x^2}$$

put $\frac{1}{y} = t$

$$-\frac{1}{y^2}.dy = dt$$

$$\frac{1}{y^2}\,dy = -dt$$

$$2.\frac{-dt}{dx} - \frac{1}{x}.t = \frac{1}{x^2}$$

$$\frac{dt}{dx} + \frac{1}{2x}.t = \frac{-1}{2x^2} \rightarrow ②$$

~~Equ^{n}① is of Bernoulli's form $\frac{dy}{dx} + P.y = Q.y^n$~~

~~Eq this can be reduced to linear form~~.

Equ^{n}② is in linear form.

Where $P = \frac{1}{2x}$ and $Q = \frac{-1}{2x^2}$

$$\text{I.F } e^{\int P(x)dx} = e^{\frac{1}{2}\int \frac{1}{x}dx}$$

$$= e^{\frac{1}{2}\log x}$$

$$= e^{\log_e (x)^{1/2}}$$

$$= x^{1/2}$$

Now the solution of equ^n ② is

$$t.x^{1/2} = \int \frac{-1}{2x^2}.\frac{1}{2x}\,dx + c$$

$$= \frac{-1}{4}\int \frac{1}{x^3}\,dx + c$$

$$= \frac{-1}{4}\int x^{-3}.dx + c$$

$$= \frac{-1}{4}\frac{x^{-2}}{-2} + c$$

$$t.x^{1/2} = \frac{1}{8x^2} + c$$

$$\frac{1}{y}.x^{1/2} = \frac{1}{8x^2} + c.$$

$$t.x^{1/2} = \int \frac{-1}{2x^2}.x^{1/2}\,dx + c$$

$$= \frac{-1}{2}\int x^{-2}.x^{1/2}\,dx + c$$

$$= \frac{-1}{2}\int x^{-3/2}\,dx + c$$

$$= \frac{-1}{\cancel{2}}\frac{x^{-1/2}}{-\cancel{1/2}} + c$$

$$t.x^{1/2} = \frac{1}{x^{1/2}} + c$$

$$\frac{1}{y}.x^{1/2} = \frac{1}{x^{1/2}} + c.$$

(6) $(x^3y^2 + xy)\,dx = dy$

Sol:- $(x^3y^2 + xy)\,dx = dy$

$$\frac{dy}{dx} = x^3y^2 + xy$$

$$\frac{dy}{dx} - xy = x^3y^2 \rightarrow (1)$$

Equ^n (1) is of Bernoulli's form $\frac{dy}{dx} + P\cdot y = Q y^n$.

This can be reduced to linear form.

$$\frac{1}{y^2}\frac{dy}{dx} - x\cdot \cancel{y}\cdot\frac{1}{y^{\cancel{2}}} = x^3\cdot\frac{\cancel{y^2}}{\cancel{y^2}}$$

$$\frac{1}{y^2}\cdot\frac{dy}{dx} - x\cdot\frac{1}{y} = x^3$$

$$-\frac{dt}{dx} - x\cdot t = x^3 \qquad\qquad \frac{1}{y} = t$$

$$-\frac{1}{y^2}dy = dt$$

$$\frac{dt}{dx} + x\cdot t = -x^3 \rightarrow (2) \qquad\qquad \frac{1}{y^2}dy = dt\,(-1)$$

Equation (2) is in linear form

where $P = x$ and $Q = -x^3$

$$\text{I.F } e^{\int P(x)dx} = e^{\int x\cdot dx}$$

$$= e^{x^2/2}$$

Now the solution of equ^n (2) is

$$t\cdot e^{x^2/2} = \int -x^3\cdot e^{x^2/2}\,dx + c \qquad \Rightarrow \boxed{x^2 = 2v}$$

$$= -\int x^2\cdot x\cdot e^{x^2/2}\,dx + c \qquad \frac{x^2}{2} = v$$

$$\frac{1}{\cancel{2}}\cdot\cancel{2}x\,dx = dv$$

$$= -\int 2v\cdot e^v\cdot dv + c \qquad x\,dx = dv$$

$$= -2\int e^v\, v\, dv + c$$

$$t\cdot e^{x^2/2} = -2\cdot e^v(v-1) + c$$

$$\frac{1}{y}\cdot e^{x^2/2} = -2\cdot e^{x^2/2}\left(\frac{x^2}{2} - 1\right) + c.$$

(7) $\frac{dy}{dx} + y = xy^3$

Sol:- $\frac{dy}{dx} + y = xy^3 \rightarrow ①$

Equⁿ ① is of linear form $\frac{dy}{dx} + P.y = Q y^n$.

This can be reduced to linear form.

$\frac{1}{y^3}\frac{dy}{dx} + \cancel{y}\frac{1}{y^{\cancel{3}2}} = x.\cancel{y^3}\frac{1}{\cancel{y^3}}$

$\frac{1}{y^3}.\frac{dy}{dx} + \frac{1}{y^2} = x.$

$y^{-3}.\frac{dy}{dx} + y^{-2} = x$

Put $y^{-2} = t$

$-2.y^{-3}\,dy = dt$

$y^{-3}.dy = \frac{-1}{2}dt$

$\frac{-1}{2}.\frac{dt}{dx} + t = x.$

$\frac{dt}{dx} - 2t = -2x \rightarrow ②$

Equⁿ ② is in linear form

where $P = -2$ and $Q = -2x$

I.F $e^{\int P(x)dx}$

$= e^{\int -2.dx}$

$= e^{-2\int(1)dx}$

$= e^{-2.x}$ $= e^{-2x}$

Now the solution of equⁿ② is

$t.e^{-2x} = \int -2x.e^{-2x}dx + C$

$= +\int v.e^{v}.\frac{-1}{2}dv + C$

$-2x = v$

$-2(1)dx = dv$

$dx = \frac{-1}{2}.dv$

$= \frac{-1}{2}\int e^{v}.v.dv + C$

$= \frac{-1}{2}.e^{v}(v-1) + C$

$t.e^{-2x} = \frac{-1}{2}.e^{-2x}(-2x-1) + C$

$\frac{1}{y^2}.e^{-2x} = \frac{1}{2}.e^{-2x}(2x+1) + C$

(8) $\frac{dy}{dx} + y \cdot \tan x = y^3 \cdot \cos x$.

Sol:- $\frac{dy}{dx} + y \cdot \tan x = \cos x \cdot y^3 \rightarrow (1)$

Equ^n (1) is of Bernoulli's form $\frac{dy}{dx} + P \cdot y = Q\, y^n$.

This can be reduced to linear form

$$\frac{1}{y^3}\frac{dy}{dx} + \cancel{y}\cdot \tan x \cdot \frac{1}{y^{\cancel{3}2}} = \cos x \cdot \cancel{y^3}\,\cancel{\frac{1}{y^3}}$$

$$y^{-3} \cdot \frac{dy}{dx} + \tan x \cdot y^{-2} = \cos x.$$

put $y^{-2} = t$

$-2y^{-3} dy = dt$

$y^{-3} dy = -\frac{1}{2} dt$

$$-\frac{1}{2}\frac{dt}{dx} + \tan x \cdot t = \cos x$$

$$\frac{dt}{dx} - 2 \cdot \tan x \cdot t = -2 \cdot \cos x$$

Equ^n (2) is in linear form.

where $P = -2\tan x$. and $Q = -2\cos x$

I.F $e^{\int P(x)dx}$

$$= e^{\int -2\tan x \cdot dx}$$

$$= e^{-2\int \tan x\, dx}$$

$$= e^{+2\log(\cos x)}$$

$$= e^{\log_e (\cos x)^2}$$

$$= \underline{\cos^2 x}.$$

Now the solution of equ^n (2) is.

$$t \cdot \cos^2 x = \int -2\cos x \cdot \cos^2 x \cdot dx + c$$

$$= -2\int \cos^3 x \cdot dx + c$$

$$= \cancel{-2 \cdot \frac{\cos^4 x}{4}}$$

$$= \frac{-2}{4}\int (\cos 3x + 3\cos x)\, dx + c$$

$$= \frac{-1}{2}\left[\frac{\sin 3x}{3} + 3\sin x\right] + c.$$

(9) $\frac{dy}{dx} + \frac{x}{1-x^2}\cdot y = x\sqrt{y}$.

Sol:- $\frac{dy}{dx} + \frac{x}{1-x^2}\cdot y = x\sqrt{y}$

$$\frac{dy}{dx} + \frac{x}{1-x^2}\cdot y = x\cdot y^{1/2} \rightarrow ①$$

Equⁿ① is of Bernoulli's form $\frac{dy}{dx} + P\cdot y = Q\cdot y^n$

This can be reduced to linear form.

$$\frac{1}{y^{1/2}}\frac{dy}{dx} + \frac{x}{1-x^2}\cdot y\cdot\frac{1}{y^{1/2}} = x\cdot \cancel{y^{1/2}\cdot\frac{1}{y^{1/2}}}$$

$$\frac{1}{y^{1/2}}\cdot\frac{dy}{dx} + \frac{x}{1-x^2}\cdot y\cdot y^{-1/2} = x.$$

$$\frac{1}{y^{1/2}}\cdot\frac{dy}{dx} + \frac{x}{1-x^2}y^{1/2} = x.$$

$y^{1/2} = t$

$\frac{1}{2}\cdot y^{1/2-1}\, dy = dt$

$\frac{1}{2}\cdot y^{-1/2}\, dy = dt.$

$\frac{1}{y^{1/2}}\cdot dy = 2\, dt$

$$2\frac{dt}{dx} + \frac{x}{1-x^2}\cdot t = x \rightarrow ②$$

$$\frac{dt}{dx} + \frac{\cancel{2}x}{2(1-x^2)}\cdot t = \cancel{2x}. \rightarrow$$

$$\frac{dt}{dx} + \frac{x}{2(1-x^2)}\cdot t = \frac{x}{2} \rightarrow ②$$

Equⁿ ② is in linear form.

where $P = \frac{x}{2(1-x^2)}$ and $Q = \frac{x}{2}$

$$I.F\ e^{\int P(x)\,dx}$$

$$= e^{\int \frac{x}{2(1-x^2)}dx}$$

$$= e^{\frac{1}{2}\int\frac{x}{1-x^2}\cdot dx}$$

$$= e^{\frac{1}{2}\times\frac{-1}{2}\int\frac{-2x}{1-x^2}\cdot dx}$$

$$= e^{-1/4\cdot\log(1-x^2)}$$

$$= e^{\log(1-x^2)^{-1/4}}$$

$$= e^{\log_e(1-x^2)^{-1/4}}$$

$$= (1-x^2)^{-1/4} = \frac{1}{(1-x^2)^{1/4}}$$

Now the solution of equⁿ② is

$$t\cdot\frac{1}{(1-x^2)^{1/4}} = \int\frac{x}{2}\cdot(1-x^2)^{-1/4}\cdot dx + c$$

$$= \frac{1}{2}\int x\cdot(1-x^2)^{-1/4}\, dx + c$$

$$= \frac{1}{2(-2)}\int(-2x)\,(1-x^2)^{-1/4}\, dx + c$$

$$= \frac{-1}{4}\int v^{-1/4}.dv + c \qquad 1-x^2 = v$$
$$-2x.dx = dv$$
$$= \frac{-1}{4} \cdot \frac{v^{-1/4+1}}{-1/4+1} + c$$
$$= \frac{-1}{4} \frac{v^{3/4}}{3/4} + c$$
$$t.(1-x^2)^{-1/4} = \frac{-1}{3} \cdot v^{3/4} + c$$
$$y^{1/2}(1-x^2)^{-1/4} = \frac{-1}{3} \cdot (1-x^2)^{3/4} + c.$$

(10) $y - \cos x \cdot \frac{dy}{dx} = y^2(1-\sin x)\cos x$. G.T $y=2$ when $x=0$.

sol:-
$$y - \cos x \cdot \frac{dy}{dx} = y^2(1-\sin x)\cos x$$
$$-\cos x \frac{dy}{dx} = y^2(1-\sin x)\cos x - y$$
$$\frac{-\cancel{\cos x}}{-\cancel{\cos x}} \frac{dy}{dx} = \frac{y^2(1-\sin x)\cancel{\cos x}}{-\cancel{\cos x}} - \frac{y}{-\cos x}$$
$$\frac{dy}{dx} = -y^2(1-\sin x) + \frac{y}{\cos x}$$
$$\frac{dy}{dx} - \sec x \cdot y = y^2(\sin x - 1) \rightarrow (1)$$

Equ^n(1) is of Bernoulli's form $\frac{dy}{dx} + Py = Q \cdot y^n$

This can be reduced to linear form.

$$\frac{1}{y^2}\frac{dy}{dx} - \sec x \cdot \frac{\cancel{y}}{y^{\cancel{2}}} = \frac{\cancel{y^2}(\sin x - 1)}{\cancel{y^2}}$$
$$\frac{1}{y^2}\frac{dy}{dx} - \sec x \cdot \frac{1}{y} = \sin x - 1 \qquad \frac{1}{y} = t$$
$$\frac{-dt}{dx} - \sec x \cdot t = \sin x - 1 \qquad \frac{-1}{y^2}dy = dt$$
$$\frac{dt}{dx} + \sec x \cdot t = 1 - \sin x \rightarrow (2) \qquad \frac{1}{y^2}dy = -dt$$

Equ^n(2) is in linear form.

where $P = \sec x$ and $Q = 1 - \sin x$

$$\text{I.F } e^{\int P(x)dx} = e^{\int \sec x\, dx}$$
$$= e^{\log_e(\sec x + \tan x)}$$
$$= \sec x + \tan x$$

Now the solution of equ^n ② is

$$t.(\sec x + \tan x) = \int (1-\sin x)(\sec x + \tan x)\,dx + c$$

$$= \int (\cancel{\sec x + \tan x + \sin x.\sec x - \sin x.\tan x})\,dx + c.$$

$$= \int \cancel{\sec x.dx + \int \tan x.dx - \int \tan x.dx}$$

$$= \int (1-\sin x)\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) dx + c$$

$$= \int (1-\sin x)\left(\frac{1+\sin x}{\cos x}\right) dx + c$$

$$= \int \frac{1-\sin^2 x}{\cos x}\,dx + c$$

$$= \int \frac{\cos^2 x}{\cos x}\,dx + c$$

$$t.(\sec x + \tan x) = \sin x + c$$

$$\frac{1}{y}(\sec x + \tan x) = \sin x + c.$$

Given that $y=2$ when $x=0$

$$\frac{1}{2}(\sec 0^\circ + \tan 0^\circ) = \sin 0^\circ + c$$

$$\frac{1}{2}(1+0) = 0 + c$$

$$\frac{1}{2}(1) = c$$

$$\boxed{\therefore c = 1/2}$$

$$\therefore \frac{1}{y}(\sec x + \tan x) = \sin x + 1/2$$

(11) $\frac{dy}{dx} - \tan x.\,y = -y^2.\sec x.$

<u>sol:-</u> $\frac{dy}{dx} - \tan x.\,y = -y^2.\sec x \to ①$ is Bernoulli's.

$$\frac{1}{y^2}.\frac{dy}{dx} - \tan x.\cancel{y}.\frac{1}{y^{\cancel{2}}} = \frac{-\cancel{y^2}\sec x}{\cancel{y^2}}$$

$$\frac{1}{y^2}\frac{dy}{dx} - \tan x.\frac{1}{y} = -\sec x.$$

$\frac{1}{y} = t$

$\frac{-1}{y^2}dy = dt$

$\frac{1}{y^2}dy = -dt$

$$\frac{-dt}{dx} - \tan x.t = -\sec x$$

$$\frac{dt}{dx} + \tan x.t = +\sec x. \to ②$$

Equ^n ② is in linear form.

where $P = \tan x$ and $Q = +\sec x$.

I.F $e^{\int P(x)dx} = e^{\int \tan x\, dx}$

$= e^{\log_e (\sec x)}$

$= \sec x$

Now the solution of equ^n (2) is

$t \cdot \sec x = \int +\sec x \cdot \sec x \cdot dx + c$

$= + \int \sec^2 x \cdot dx + c$

$t \cdot \sec x = +\tan x + c$

$\frac{1}{y} \cdot \sec x = +\tan x + c.$

Tuesday
17/09

Exact Differential Equations:

(2) $[\cos x \tan y + \cos(x+y)]dx + [\sin x \sec^2 y + \cos(x+y)]dy = 0$

Sol:- $[\cos x \tan y + \cos(x+y)]dx + [\sin x \cdot \sec^2 y + \cos(x+y)]dy = 0 \longrightarrow (1)$

Equ^n (1) is of exact form of $M dx + N dy = 0$.

where $M = \cos x \cdot \tan y + \cos(x+y)$

and $N = \sin x \sec^2 y + \cos(x+y)$

$M = \cos x \cdot \tan y + \cos x \cos y - \sin x \sin y$

$\left(\frac{\partial M}{\partial y}\right)_{x = const} = \cos x \cdot \sec^2 y + \cos x(-\sin y) - \sin x(\cos y)$

$\frac{\partial M}{\partial y} = \cos x \cdot \sec^2 y - \cos x \sin y - \sin x \cos y$

$N = \sin x \cdot \sec^2 y + \cos x \cos y - \sin x \sin y$

$\left(\frac{\partial N}{\partial x}\right)_{y = const} = \sec^2 y \cos x + \cos y(-\sin x) - \sin y \cos x$

$\frac{\partial N}{\partial x} = \sec^2 y \cos x - \cos y \sin x - \sin y \cos x$

$= \cos x \cdot \sec^2 y - \cos x \sin y - \sin x \cos y$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Hence Equ^n ① is an exact.

Solution of equ^{n}① is $\int M dx + \int N dy = c$.

$$\int [\cos x \tan y + \cos(x+y)] dx + \int [\sin x \sec^2 y + \cos(x+y)] dy = C$$

$$\int \cos x \tan y \, dx + \int (\cos x \cos y - \sin x \sin y) dx$$

$$+ \int \sin x \sec^2 y \, dy + \int (\cos x \cdot \cos y - \sin x \sin y) dy = C$$

$$\int \cos x \tan y \, dx + \int \cos x \cos y \, dx - \int \sin x \sin y \, dx$$

$$+ \int \sin x \sec^2 y \, dy + \int \cos x \cos y \, dy - \int \sin x \sin y \, dy = C$$

$$\tan y \int \cos x \, dx + \cos y \int \cos x \, dx - \sin y \int \sin x \, dx + 0 + 0 - 0 = C$$

$$\tan y \sin x + \cos y \sin x - \sin y(-\cos x) = C$$

$$\tan y \sin x + \sin x \cos y + \cos x \sin y = C$$

$$\sin x \tan y + \sin(x+y) = C$$

(5) $(1+e^{x/y}) dx + \left(1-\frac{x}{y}\right) e^{x/y} dy = 0$

Sol:- $(1+e^{x/y}) dx + \left(1-\frac{x}{y}\right) e^{x/y} dy = 0 \longrightarrow ①$

$M = 1+e^{x/y}$ and $N = \left(1-\frac{x}{y}\right) e^{x/y}$

$$\frac{\partial M}{\partial y} = 0 + e^{x/y} \cdot \frac{-x}{y^2}$$

$$= -e^{x/y} \cdot \frac{x}{y^2}$$

$$\frac{\partial N}{\partial x} = \left(0-\frac{1}{y}\right) e^{x/y} + e^{x/y} \frac{d}{dx}\left(\frac{x}{y}\right)\left(1-\frac{x}{y}\right)$$

$$= -\frac{1}{y} e^{x/y} + e^{x/y} \frac{1}{y} \cdot \left(1-\frac{x}{y}\right)$$

$$= \cancel{-\frac{1}{y} e^{x/y}} + \cancel{\frac{1}{y} e^{x/y}} - \frac{x}{y^2} e^{x/y}$$

$$= -e^{x/y} \cdot \frac{x}{y^2}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Hence Equ^{n}① is an exact.

Now the solution of equ^n ① is $\int M dx + \int N dy = C$

$$\int (1+e^{x/y}) dx + \int (1 - x/y) e^{x/y} dy = C$$

$$\int (1) dx + \int e^{x/y} dx + \int e^{x/y} dy - \int \frac{x}{y} \cdot e^{x/y} dy = C$$

$$x + \frac{e^{x/y}}{1/y} + 0 - 0 = c$$

$$x + y \cdot e^{x/y} = c.$$

⑥ $(\sec x \tan x \tan y - e^x)dx + \sec x \cdot \sec^2 y \, dy = 0 \longrightarrow ①$

Sol:- Equ^n ① is of exact ~~[illegible]~~ differential equation.

$$M dx + N dy = 0.$$

where $M = \sec x \tan x \tan y - e^x$

$$\frac{\partial M}{\partial y} = \sec x \cdot \tan x \cdot \sec^2 y - 0.$$

$(x = \text{const})$ $= \sec x \cdot \tan x \cdot \sec^2 y$

and $N = \sec x \cdot \sec^2 y.$

$$\frac{\partial N}{\partial x} = \sec^2 y \cdot \sec x \cdot \tan x$$

$(y\ \text{const})$ $= \sec x \cdot \tan x \sec^2 y.$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Hence Equ^n ① is ~~of~~ an exact form.

Now the solution of equ^n ① is $\int M dx + \int N dy = c$

$$\int (\sec x \tan x \tan y - e^x) dx + \int \sec x \sec^2 y \, dy = c$$

$$\tan y \int \sec x \tan x \cdot dx - \int e^x dx + 0 = c$$

$$\tan y \sec x - e^x = c.$$

⑦ $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0. \longrightarrow ①$

Sol:- Equ^n ① is of exact differential equation

of $M dx + N dy = 0.$

where $M = 5x^4 + 3x^2y^2 - 2xy^3$ and $N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y}_{(x=\text{con})} = 0 + 3x^2(2y) - 2x \cdot 3y^2 \qquad \frac{\partial N}{\partial x}_{(y=\text{const})} = 2y(3x^2) - 3y^2(2x) - 0$$

$$= 6x^2y - 6xy^2 \qquad\qquad = 6x^2y - 6xy^2$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Hence Equ^n ① is an exact.

solution of equ^n ① is $\int M dx + \int N dy = c$

$$\int(5x^4+3x^2y^2-2xy^3)\,dx + \int(2x^3y-3x^2y^2-5y^4)\,dy = c$$

$$5\int x^4\,dx + 3y^2\int x^2\,dx - 2y^3\int x\,dx + \int 2x^3y\,dy - \int 3x^2y^2\,dy - \int 5y^4\,dy = c$$

$$\cancel{5}\left(\frac{x^5}{\cancel{5}}\right) + \cancel{3}y^2\left(\frac{x^3}{\cancel{3}}\right) - \cancel{2}y^3\left(\frac{x^2}{\cancel{2}}\right) + 0 - 0 - \cancel{5}\cdot\frac{y^5}{\cancel{5}} = c$$

$$x^5 + x^3y^2 - x^2y^3 - y^5 = c.$$

$$x^5 - y^5 + x^3y^2 - x^2y^3 = c.$$

(3) $$\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$$

Sol:- $$\frac{dy}{dx} = -\frac{y\cos x + \sin y + y}{\sin x + x\cdot\cos y + x}$$

$$(\sin x + x\cos y + x)\,dy = -(y\cos x + \sin y + y)\,dx$$

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0 \longrightarrow (1)$$

Equ^n (1) is of exact differential equation of $M\,dx + N\,dy = 0$

where $M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y + x$.

$$\frac{\partial M}{\partial y} = \cos x\cdot(1) + \cos y + 1 \qquad \frac{\partial N}{\partial x} = \cos x + \cos y(1) + 1$$

$$= \cos x + \cos y + 1 \qquad = \cos x + \cos y + 1$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Hence Equ^n (1) is an exact.

Now the solution of equ^n (1) is $\int M\,dx + \int N\,dy = c$.

$$\int(y\cos x + \sin y + y)\,dx + \int(\sin x + x\cos y + x)\,dy = c$$

$$y\int\cos x\,dx + \sin y\int dx + y\int dx + \int\sin x\,dy + \int x\cos y\,dy + \int x\,dy = c$$

$$y\sin x + \sin y\cdot x + y\cdot x + 0 + 0 + 0 = c$$

$$\sin x\cdot y + x\cdot\sin y + xy = c$$

(4) $(2x^3 - xy^2 - 2y + 3)dx - (x^2y + 2x)\,dy = 0 \rightarrow ①$

Sol:- Equⁿ ① is of an exact differential equation of $Mdx + Ndy = 0$

where $M = 2x^3 - xy^2 - 2y + 3$ and $N = -x^2y - 2x$.

$\frac{\partial M}{\partial y} = 0 - x\,2y - 2 + 0$

$= -2xy - 2$.

$\frac{\partial N}{\partial x} = -y\cdot(2x) - 2$

$= -2xy - 2$.

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Hence Equⁿ① is an exact.

Now the solution of equⁿ① is $\int M dx + \int N dy = C$

$\int(2x^3 - xy^2 - 2y + 3)\,dx + \int(-x^2y^2 - 2x)\,dy = C$.

$2\int x^3 dx - y^2\int x\,dx - \int 2y\,dx + 3\int(1)dx - \int x^2y^2 dy - \int 2x\,dy = C$

$\cancel{2}\cdot\frac{x^4}{4_2} - y^2\cdot\frac{x^2}{2} - 2yx + 3x - 0 - 0 = C$

$\frac{x^4}{2} - \frac{x^2}{2}y^2 - 2xy + 3x = C$

$\frac{x^2}{2}\left[x^2 - y^2\right] - 2xy + 3x = C$

(7) $\left(\cos x\cdot\log(2y-8) + \frac{1}{x}\right)dx + \frac{\sin x}{y-4}\,dy = 0 \rightarrow ①$

Sol:- Equⁿ① is of an exact differential form $Mdx + Ndy = 0$

where $M = \cos x\cdot\log(2y-8) + \frac{1}{x}$. and $N = \frac{\sin x}{y-4}$

$\frac{\partial M}{\partial y} = \cos x\cdot\frac{1}{2y-8}(2-0) + 0$

$= \frac{\cos x}{y-4}$

$\frac{\partial N}{\partial x} = \frac{1}{y-4}(\cos x)$

$= \frac{\cos x}{y-4}$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Hence Equⁿ ① is an exact.

Now the solution of equⁿ① is $\int M dx + \int N dy = C$.

$\int\left(\cos x\cdot\log(2y-8) + \frac{1}{x}\right)dx + \int\frac{\sin x}{y-4}\,dy = C$.

$\log(2y-8)\int\cos x\,dx + \int\frac{1}{x}dx + 0 = C$

$\log(2y-8)\cdot\sin x + \log x = C$

(8) $(2xy\cos x^2 - 2xy + 1)dx + (\sin x^2 - x^2)dy = 0 \rightarrow (1)$

Sol: Equ^n (1) is of an exact differential equation

$$Mdx + Ndy = 0$$

where $M = 2xy\cos x^2 - 2xy + 1$ and $N = \sin x^2 - x^2$

$$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x + 0 \qquad \frac{\partial N}{\partial x} = \cos x^2(2x) - 2x$$

$$= 2x(\cos x^2 - 1) \qquad = 2x(\cos x^2 - 1)$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Hence Equ^n (1) is an exact.

Now the solution of equ^n (1) is $\int M dx + \int N dy = C$

$$\int(2xy\cos x^2 - 2xy + 1)dx + \int(\sin x^2 - x^2)dy = C$$

$$y \int 2x\cdot\cos x^2 dx - 2y\int x\,dx + \int(1)dx + \int \sin x^2 dy - \int x^2\cdot dy = C$$

$$y\int \cos t\cdot dt - \cancel{2}y\frac{x^2}{\cancel{2}} + x + 0 - 0 = C \qquad \text{put } x^2 = t,\ 2xdx = dt.$$

$$y\cdot \sin t - x^2y + x = C$$

$$\sin x^2 \cdot y - x^2y + x = C$$

$$y(\sin x^2 - x^2) + x = C$$

(9) $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0 \rightarrow (1)$

Sol:- Equ^n (1) is of an exact differential equation

$$Mdx + Ndy = 0$$

where

$M = y^2e^{xy^2} + 4x^3$ and $N = 2xye^{xy^2} - 3y^2$

$$\frac{\partial M}{\partial y} = y^2 e^{xy^2}(2y)(x) + e^{xy^2}\cdot 2y$$

$$= 2y\,[xy^2\cdot e^{xy^2} + e^{xy^2}]$$

$$\frac{\partial N}{\partial x} = 2y\,[x\cdot e^{xy^2}(y^2)(1) + e^{xy^2}(1)] - 0$$

$$= 2y\,[xy^2e^{xy^2} + e^{xy^2}]$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Hence Equ^n (1) is an exact.

Now the solution of equ^n (1) is $\int M dx + \int N dy = C$.

$$\int(y^2e^{xy^2} + 4x^3)dx + \int(2xy\cdot e^{xy^2} - 3y^2)dy = C$$

$$\int y^2 e^{xy^2}dx + \int 4x^3 dx + \int 2xye^{xy^2}dy - \int 3y^2 dy = C.$$

$$y^2\int e^{xy^2}dx + 4\int x^3 dx + 0 - 3\int y^2 dy = c$$

$$\cancel{y^2}\cdot\frac{e^{xy^2}}{\cancel{y^2}} + \cancel{4}\,\frac{x^4}{\cancel{4}} - \cancel{3}\,\frac{y^3}{\cancel{3}} = c$$

$$e^{xy^2} + x^4 - y^3 = c.$$

(10) $[y\cdot(1+\frac{1}{x}) + \cos y]\,dx + (x + \log x - x\sin y)\,dy = 0$ $\rightarrow$ (1)

Sol:- Equ^n (1) is of an exact differential equation $Mdx + Ndy = 0$

where $M = y(1+\frac{1}{x}) + \cos y$ and $N = x + \log x - x\sin y$

$$\frac{\partial M}{\partial y} = (1+\frac{1}{x}) + (-\sin y)$$
$$= 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y(1)$$
$$= 1 + \frac{1}{x} - \sin y$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Hence Equ^n (1) is an exact.

Now the solution of equ^n (1) is $\int M dx + \int N dy = c$

$$\int [y(1+\frac{1}{x}) + \cos y]\,dx + \int (x + \log x - x\sin y)\,dy = c$$

$$y\int[(1+\frac{1}{x}) + \cos y]\,dx + \int x\,dy + \int \log x\,dy - \int x\sin y\,dy = c$$

$$y\int(1)dx + \int\frac{1}{x}dx + \cos y\int(1)dx + 0 + 0 - 0 = c$$

$$y\cdot x + \log x + \cos y\cdot x = c$$

$$xy + x\cdot\cos y + \log x = c$$

Non-Exact! 19-09 (Thursday)

(Method - I)

(A)

(4) $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0 \longrightarrow (1)$

Sol:- Equ^n (1) is of exact form $M\,dx + N\,dy = 0$.

where $M = 3xy^2 - y^3$ and $N = -2x^2y + xy^2$.

$$\frac{\partial M}{\partial y} = 3x(2y) - 3y^2 \qquad \frac{\partial N}{\partial x} = -2(2x) + y^2$$

$$= 6xy - 3y^2 \qquad = -4x + y^2$$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equ^n (1) is non-exact.

Equ^n (1) can be reduced to exact by multiplying an integrating factor.

→ clearly equ^n (1) is homogeneous degree "3".

→ $Mx + Ny = (3xy^2 - y^3)x + (-2x^2y + xy^2)y$

$= 3x^2y^2 - xy^3 - 2x^2y^2 + xy^3$

$= x^2y^2 \neq 0$.

$$\boxed{\therefore Mx + Ny \neq 0}$$

$\therefore$ I.F $= \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

from (1),

$(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$

$\frac{1}{x^2y^2}(3xy^2 - y^3)\,dx - \frac{(2x^2y - xy^2)\,dy}{x^2y^2} = 0 \times \frac{1}{x^2y^2}$

$\frac{y^2(3x-y)}{x^2y^2}\,dx - \frac{xy(2x-y)}{x^2y^2}\,dy = 0$.

$\frac{3x-y}{x^2}\,dx - \frac{2x-y}{xy}\,dy = 0$

$\left(\frac{3x}{x^2} - \frac{y}{x^2}\right)dx - \left(\frac{2x}{xy} - \frac{y}{xy}\right)dy = 0$

$\left(\frac{3}{x} - \frac{y}{x^2}\right)dx - \left(\frac{2}{y} - \frac{1}{x}\right)dy = 0 \longrightarrow (2)$

Equ^n (2) is of an exact form $M\,dx + N\,dy = 0$

where $M = \frac{3}{x} - \frac{y}{x^2}$ and $N = -\left(\frac{2}{y} - \frac{1}{x}\right)$

$M = \frac{3}{x} - \frac{y}{x^2}$ $\quad\quad$ $N = -\frac{2}{y} + \frac{1}{x}$

$\frac{\partial M}{\partial y} = 0 - \frac{1}{x^2}(1)$ $\quad\quad$ $\frac{\partial N}{\partial x} = -0 + \left(\frac{-1}{x^2}\right)$

$= \frac{-1}{x^2}$ $\quad\quad$ $= \frac{-1}{x^2}$.

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Equⁿ ② is an exact form.

Now the solution of equⁿ ② is $\int M dx + \int N dy = C$.

$$\int \left(\frac{3}{x} - \frac{y}{x^2}\right) dx + \int \left(-\frac{2}{y} + \frac{1}{x}\right) dy = C$$

$$3\int \frac{1}{x} dx - y \int \frac{1}{x^2} dx - 2\int \frac{1}{y} dy + \int \frac{1}{x} dy = C$$

$$3 \log x - y \int x^{-2} dx - 2 \log y + 0 = C$$

$$3 \log x - y \cdot \frac{x^{-1}}{-1} - 2 \log y = C$$

$$3 \log x + \frac{y}{x} - 2 \log y = C.$$

$$\log x^3 + \frac{y}{x} - \log y^2 = C$$

$$\log \left(\frac{x^3}{y^2}\right) + \frac{y}{x} = C$$

(5) $(x^2 - 3xy + 2y^2)\, dx + x(3x - 2y)\, dy = 0 \rightarrow ①$

Sol: Equⁿ① is of an exact form $M dx + N dy = 0$

Where $M = x^2 - 3xy + 2y^2$ and $N = 3x^2 - 2xy$.

$\frac{\partial M}{\partial y} = 0 - 3x(1) + 2(2y)$ $\quad\quad$ $\frac{\partial N}{\partial x} = 3(2x) - 2y(1)$

$= 4y - 3x.$ $\quad\quad$ $= 6x - 2y$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equⁿ① is non-exact.

Equⁿ① can be reduced to exact form by multiplying an Integrating factor.

→ clearly Equⁿ① is a homogeneous degree "2".

$$Mx + Ny = (x^2 - 3yx + 2y^2)x + (3x^2 - 2xy)y$$

$$= x^3 - \cancel{3x^2y} + \cancel{2xy^2} + \cancel{3x^2y} - \cancel{2xy^2}$$

$$= x^3 \neq 0.$$

$\boxed{Mx+Ny \neq 0}$

$\therefore I.F = \frac{1}{Mx+Ny} = \frac{1}{x^3}$.

from ①,

$$(x^2-3xy+2y^2)\,dx + (3x^2-2xy)\,dy = 0$$

$$\frac{x^2-3xy+2y^2}{x^3}\,dx + \frac{3x^2-2xy}{x^3}\,dy = 0$$

$$\left(\frac{x^2}{x^3} - \frac{3xy}{x^3} + \frac{2y^2}{x^3}\right)dx + \left(\frac{3x^2}{x^3} - \frac{2xy}{x^3}\right)dy = 0$$

$$\left(\frac{1}{x} - \frac{3y}{x^2} + \frac{2y^2}{x^3}\right)dx + \left(\frac{3}{x} - \frac{2y}{x^2}\right)dy = 0. \rightarrow ②$$

Equⁿ ② is an exact. form of $Mdx + Ndy = 0$.

where $M = \frac{1}{x} - \frac{3y}{x^2} + \frac{2y^2}{x^3}$ and $N = \frac{3}{x} - \frac{2y}{x^2}$

$$\frac{\partial M}{\partial y} = 0 - \frac{3}{x^2}(1) + \frac{2}{x^3}(2y) \qquad \frac{\partial N}{\partial x} = 3\left(\frac{-1}{x^2}\right) - 2y(-2)x^{-3}$$

$$= \frac{-3}{x^2} + \frac{4y}{x^3} \qquad\qquad = \frac{-3}{x^2} + 4\frac{y}{x^3}$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Clearly equⁿ ② is an exact.

Now the solution of equⁿ ② is $\int M dx + \int N dy = c$

$$\int\left(\frac{1}{x} - \frac{3y}{x^2} + \frac{2y^2}{x^3}\right)dx + \int\left(\frac{3}{x} - \frac{2y}{x^2}\right)dy = C$$

$$\int \frac{1}{x}dx - 3y\int x^{-2}dx + 2y^2\int x^{-3}dx + 0 = C$$

$$\log x - 3y\frac{x^{-1}}{-1} + \cancel{2}y^2\frac{x^{-2}}{-\cancel{2}} = C$$

$$\log x + 3\frac{y}{x} - \frac{y^2}{x^2} = C$$

(1) $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0. \rightarrow ①$

Sol:- Equⁿ ① is an exact form of $Mdx + Ndy = 0$

Where $M = x^2y - 2xy^2$ and $N = -x^3 + 3x^2y$

$$\frac{\partial M}{\partial y} = x^2(1) - 2x(2y) \qquad \frac{\partial N}{\partial x} = -3x^2 + 3y(2x)$$

$$= x^2 - 4xy \qquad\qquad = -3x^2 + 6xy$$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equⁿ ① is non-exact.

Equⁿ ① can be reduced to exact form by multiplying an integrating factor.

→ clearly equⁿ① is a homogeneous degree '3'

$$Mx+Ny = (x^2y-2xy^2)x - (x^3-3x^2y)y$$

$$= x^3\cancel{y} - 2x^2y^2 - x^3\cancel{y} + 3x^2y^2$$

$$= x^2y^2 \neq 0$$

$$\boxed{Mx+Ny \neq 0}$$

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$$

$$\frac{(x^2y-2xy^2)}{x^2y^2}\cdot dx - \frac{(x^3-3x^2y)}{x^2y^2} dy = 0$$

$$\left(\frac{x^2y}{x^2y^2} - \frac{2xy^2}{x^2y^2}\right) dx - \left(\frac{x^3}{x^2y^2} - \frac{3x^2y}{x^2y^2}\right) dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0 \rightarrow ②$$

clea Equⁿ ② is an exact form $Mdx + N dy = 0$.

where $M = \frac{1}{y} - \frac{2}{x}$ and $N = -\frac{x}{y^2} + \frac{3}{y}$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} - 0 \qquad \frac{\partial N}{\partial x} = -\frac{1}{y^2}(1) + 0$$

$$= -\frac{1}{y^2} \qquad = -\frac{1}{y^2}$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

clearly equⁿ② is an exact.

Now the solution of equⁿ② is $\int M dx + \int N dy = c$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \left(\frac{-x}{y^2} + \frac{3}{y}\right) dy = c$$

$$\frac{1}{y}\int (1)\, dx - 2\int \frac{1}{x} dx - \int \frac{x}{y^2} dy + 3\int \frac{1}{y} dy = c$$

$$\frac{1}{y}\cdot x - 2\cdot \log x - 0 + 3\log y = c$$

$$\frac{x}{y} - \log x^2 + \log y^3 = c$$

$$\log \frac{y^3}{x^2} + \frac{x}{y} = c.$$

(2) $(xy-2y^2)\,dx - (x^2-3xy)\,dy = 0 \rightarrow ①$

Sol:- Equ^n ① is ~~of~~ of an exact form $M\,dx + N\,dy = 0$.

where $M = xy - 2y^2$ and $N = -(x^2 - 3xy)$

$$\frac{\partial M}{\partial y} = x(1) - 2(2y) \qquad \frac{\partial N}{\partial x} = -[2x - 3y(1)]$$

$$= x - 4y \qquad = -2x + 3y$$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Equ^n ① is non-exact.

Equ^n ① can be reduced to exact by multiplying an integrating factor.

→ clearly equ^n ① is a homogeneous degree '2'.

$$Mx + Ny = (xy - 2y^2)x - (x^2 - 3xy)y$$

$$= \cancel{x^2y} - 2xy^2 - \cancel{x^2y} + 3xy^2$$

$$= xy^2 \neq 0.$$

$$\boxed{Mx + Ny \neq 0}$$

$$I.F. = \frac{1}{Mx+Ny} = \frac{1}{xy^2}$$

from ①,

$$\frac{(xy-2y^2)}{xy^2}\,dx - \left(\frac{x^2-3xy}{xy^2}\right) dy = 0.$$

$$\left(\frac{\cancel{xy}}{xy^{\cancel{2}}} - \frac{2\cancel{y^2}}{x\cancel{y^2}}\right) dx - \left(\frac{x^{\cancel{2}}}{\cancel{x}y^2} - \frac{3\cancel{xy}}{\cancel{x}y^{\cancel{2}}}\right) dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0 \rightarrow ②$$

Equ^n ② is an exact form of $M\,dx + N\,dy = 0$

where $M = \frac{1}{y} - \frac{2}{x}$ and $N = -\frac{x}{y^2} + \frac{3}{y}$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2} - 0 \qquad \frac{\partial N}{\partial x} = \frac{-1}{y^2}(1) + 0$$

$$= \frac{-1}{y^2} \qquad = \frac{-1}{y^2}$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

~~So~~ clearly equ^n ② is an exact.

Now the solution of equⁿ ② is $\int M dx + \int N dy = C$

$$\int\left(\frac{1}{y} - \frac{2}{x}\right) dx + \int\left(-\frac{x}{y^2} + \frac{3}{y}\right) dy = C$$

$$\frac{1}{y}\int(1) dx - 2\int\frac{1}{x} dx + \int -\frac{x}{y^2} dy + 3\int\frac{1}{y} dy = C$$

$$\frac{1}{y}(x) - 2\log x + 0 + 3.\log y = C$$

$$\frac{x}{y} - \log x^2 + \log y^3 = C$$

$$\log\left(\frac{y^3}{x^2}\right) + \frac{x}{y} = C.$$

(3). $x^2y\,dx - (x^3+y^3)\,dy = 0 \rightarrow ①$

Sol:-

Equⁿ① is of an exact form $M dx + N dy = 0$

Where $M = x^2y$ and $N = -(x^3+y^3)$

$$\frac{\partial M}{\partial y} = x^2(1) = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2 + 0 = -3x^2$$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equⁿ① is non-exact.

Equⁿ① can be reduced to exact by multiplying Integrating factor.

Clearly equⁿ① is homogeneous degree '3'.

$Mx + Ny = (x^2y)x + [-(x^3+y^3)]y$

$$= x^2y.x - x^3y - y^4$$

$$= \cancel{x^3y} - \cancel{x^3y} - y^4$$

$$= \cancel{x^3(y-1) - y^3} \neq 0 \quad y^4 \neq 0$$

$$\boxed{Mx + Ny \neq 0}$$

$$I.F = \frac{1}{Mx+Ny} = \cancel{\frac{1}{x^3(y-1)-y^3}} \quad \frac{1}{y^4}$$

from①,

$$x^2y.\,dx - (x^3+y^3)\,dy = 0.$$

$$\cancel{\frac{x^2y}{x^3(y-1)-y^3}} \quad \frac{x^2y}{y^{4}_{3}}dx - \frac{(x^3+y^3)}{y^4}dy = 0$$

$$\frac{x^2}{y^3}dx - \left(\frac{x^3}{y^4} + \frac{y^3}{y^4}\right)dy = 0$$

$$\frac{x^2}{y^3}.dx - \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0 \rightarrow ②$$

Equⁿ② is an exact.

Where $M = \frac{x^2}{y^3}$ and $N = -\frac{x^3}{y^4} - \frac{1}{y}$.

$\frac{\partial M}{\partial y} = x^2 (-3) y^{-4}$ $\qquad$ $\frac{\partial N}{\partial x} = \frac{-1}{y^4} 3x^2 - 0$

$= \frac{-3x^2}{y^4}$ $\qquad$ $= \frac{-3x^2}{y^4}$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ $\therefore$ Equn (2) is an exact.

Now the solution of equn (2) is $\int M dx + \int N dy = C$.

$\int \frac{x^2}{y^3} dx + \int \left(\frac{-x^3}{y^4} - \frac{1}{y}\right) dy = C$

$\frac{1}{y^3} \int x^2 dx - \int \frac{x^3}{y^4} dy - \int \frac{1}{y} dy = C$

$\frac{1}{y^3} \frac{x^3}{3} - 0 - \log y = C$

$\frac{x^3}{3y^3} - \log y = C$

Saturday
21/09/2019

Method – II.

(4) $(xy^2 + 2x^2y^3) dx + (x^2y - x^3y^2) dy = 0$. $\rightarrow (1)$

Sol:- Equn (1) is of an exact form $M dx + N dy = 0$

where $M = xy^2 + 2x^2y^3$ and $N = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = x \cdot 2y + 2x^2 (3y^2)$ $\qquad$ $\frac{\partial N}{\partial x} = y(2x) - y^2 \cdot 3x^2$

$= 2xy + 6x^2y^2$ $\qquad$ $= 2xy - 3x^2y^2$

$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$

Hence equn (1) is non-exact.

Equn (1) can be reduced to exact by multiplying Integrating factor.

→ clearly

from (1), $(xy^2 + 2x^2y^3) dx + (x^2y - x^3y^2) dy = 0$

$y(xy + 2x^2y^2) dx + x(xy - x^2y^2) dy = 0$

$Mx - Ny = xy(xy + 2x^2y^2) - xy \cdot (xy - x^2y^2)$

$= \cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3$

$= 3x^3y^3 \neq 0$

$I.F = \frac{1}{Mx - Ny}$

$= \frac{1}{3x^3y^3}$

from ①

$$(xy^2+2x^2y^3)\,dx+(x^2y-x^3y^2)\,dy=0$$

$$\left(\frac{xy^2+2x^2y^3}{3x^3y^3}\right)dx+\left(\frac{x^2y-x^3y^2}{3x^3y^3}\right)dy=0$$

$$\left(\frac{xy^2}{3x^3y^3}+\frac{2x^2y^3}{3x^3y^3}\right)dx+\left(\frac{x^2y}{3x^3y^3}-\frac{x^3y^2}{3x^3y^3}\right)dy=0$$

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy=0 \rightarrow ②$$

Equ^n ② is an exact.

where $M=\frac{1}{3x^2y}+\frac{2}{3x}$ and $N=\frac{1}{3xy^2}-\frac{1}{3y}$

$$\frac{\partial M}{\partial y}=\frac{1}{3x^2}\left(\frac{-1}{y^2}\right)+0 \qquad \frac{\partial N}{\partial x}=\frac{1}{3y^2}\left(\frac{-1}{x^2}\right)-0$$

$$=\frac{-1}{3x^2y^2} \qquad\qquad =\frac{-1}{3x^2y^2}$$

$$\boxed{\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

Clearly Equ^n ② is an exact.

Now the solution of equ^n ② is $\int M\,dx+\int N\,dy=C$.

$$\int\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\int\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy=c$$

$$\frac{1}{3y}\int\frac{1}{x^2}dx+\frac{2}{3}\int\frac{1}{x}dx+\int\frac{1}{3y^2x}dy-\frac{1}{3}\int\frac{1}{y}dy=c$$

$$\frac{1}{3y}\frac{x^{-1}}{-1}+\frac{2}{3}\log x+0-\frac{1}{3}\log y=c$$

$$\frac{-1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=c$$

$$\frac{-1}{xy}+2\log x-\log y=3c$$

$$\frac{-1}{xy}+\log x^2-\log y=3c$$

$$\frac{-1}{xy}+\log\left(\frac{x^2}{y}\right)=3c$$

$$-\frac{1}{3}\left[\frac{1}{xy}+\log\left(\frac{x^2}{y}\right)\right]=c$$

(6) $(xy\sin xy + \cos xy)\, y\, dx + (xy\sin xy - \cos xy)\, x\, dy = 0$ $\rightarrow ①$

sol:- Equ^{n}① is an exact form $M\,dx + N\,dy = 0$

where $M = xy^2 \sin xy + y\cos xy$

$$\frac{\partial M}{\partial y} = x\left[2y\cdot\sin xy + y^2\cos xy\cdot x\right] + y(\sin xy)(x) + \cos xy\,(1)$$

$$= 2xy\cdot\sin xy + x^2y^2\cos xy - xy\sin xy + \cos xy$$

$$= xy\sin xy + x^2y^2\cos xy + \cos xy$$

and $N = x^2y\sin xy - \cos xy\cdot(x)$

$$\frac{\partial N}{\partial x} = y\left[2x\cdot\sin xy + x^2\cos xy\cdot y\right] - \left[x\cdot(-\sin xy)\,y + \cos xy\,(1)\right]$$

$$= 2xy\sin xy + x^2y^2\cos xy + xy\sin xy - \cos xy$$

$$= 3xy\sin xy + x^2y^2\cos xy - \cos xy$$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equ^{n}① is non-exact.

Equ^{n}① can be reduced to exact by multiplying Integrating factor.

from①,

(1) $y(xy\sin xy + \cos xy)\,dx + x(xy\sin xy - \cos xy)\,dy = 0$

(2) $Mx - Ny = xy(xy\sin xy + \cos xy) - [xy(xy\sin xy - \cos xy)]$

$$= \cancel{x^2y^2\sin xy} + xy\cos xy - \cancel{x^2y^2\sin xy} + xy\cos xy$$

$$= 2xy\cos xy \neq 0$$

$$\boxed{Mx - Ny \neq 0}$$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{2xy\cos xy}$$

from①,

$$\frac{(xy\cdot\sin xy + \cos xy)\,y}{2xy\cos xy}\,dx + \frac{(xy\sin xy - \cos xy)\,x}{2xy\cos xy}\,dy = 0$$

$$\left(\frac{xy^2\sin xy}{2xy\cos xy} + \frac{y\cdot\cos xy}{2xy\cos xy}\right)dx + \left(\frac{x^2y\sin xy}{2xy\cos xy} - \frac{x\cdot\cos xy}{2xy\cos xy}\right)dy = 0$$

$$\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = 0 \rightarrow ②$$

Equ^n ② is an exact.

where $M = \frac{y}{2}\tan xy + \frac{1}{2x}$ and $N = \frac{x}{2}\tan xy - \frac{1}{2y}$

$$\frac{\partial M}{\partial y} = \frac{1}{2}\left[y \cdot \sec^2 xy(x) + \tan xy(1)\right] + 0$$

$$= \frac{1}{2}\left[xy\sec^2 xy + \tan xy\right]$$

$$= \frac{1}{2}xy\sec^2 xy + \frac{1}{2}\tan xy.$$

and $N = \frac{x}{2}\tan xy - \frac{1}{2y}$

$$\frac{\partial N}{\partial x} = \frac{1}{2}\left[x \cdot \sec^2 xy(y) + \tan xy(1)\right] - 0$$

$$= \frac{1}{2}\left[xy\sec^2 xy + \tan xy\right]$$

$$= \frac{1}{2}xy\sec^2 xy + \frac{1}{2}\tan xy$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Clearly equ^n ② is an exact.

Now the solution of equ^n ② is $\int M\,dx + \int N\,dy = c$

$$\int\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \int\left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = c$$

$$\frac{y}{2}\int \tan xy\,dx + \frac{1}{2}\int \frac{1}{x}\,dx + \int \frac{x}{2}\tan xy - \frac{1}{2}\int \frac{1}{y}\,dy = c$$

$$\frac{\cancel{y}}{2}\frac{\log(\sec xy)}{\cancel{y}} + \frac{1}{2}\log x + 0 - \frac{1}{2}\log y = \log c$$

$$\frac{1}{2}\left[\log(\sec xy) + \log x - \log y\right] = \log c$$

$$\log(\sec xy \cdot x) - \log y = 2\log c$$

$$\cancel{\log}\left(\frac{x \cdot \sec xy}{y}\right) = \cancel{\log}\, c^2$$

$$\frac{x}{y}\cdot \sec xy = c.$$

(2) $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$ $\longrightarrow$ ①

Sol:- Equ^n ① is an exact form of $Mdx + Ndy = 0$

where $M = y + xy^2$ and $N = x - x^2y$

$$\frac{\partial M}{\partial y} = 1 + x \cdot 2y = 2xy + 1 \qquad \frac{\partial N}{\partial x} = 1 - y \cdot 2x = 1 - 2xy$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

clearly equⁿ① is non-exact.

Equⁿ① can be reduced to exact by multiplying Integrating factor.

$$Mx - Ny = x(y + xy^2) - (x - x^2y)\cdot y$$

$$= \cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2$$

$$= 2x^2y^2 \neq 0$$

$$\boxed{Mx - Ny \neq 0}$$

$$IF = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$$

from①
$$\frac{(y + xy^2)}{2x^2y^2}dx + \frac{(x - x^2y)}{2x^2y^2}dy = 0$$

$$\left(\frac{y}{2x^2y^2} + \frac{xy^2}{2x^2y^2}\right)dx + \left(\frac{x}{2x^2y^2} - \frac{x^2y}{2x^2y^2}\right)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0 \rightarrow ②$$

Equⁿ② is an exact form of $Mdx + Ndy = 0$

Where $M = \frac{1}{2x^2y} + \frac{1}{2x}$ and $N = \frac{1}{2xy^2} - \frac{1}{2y}$

$$\frac{\partial M}{\partial y} = \frac{1}{2x^2}\left(\frac{-1}{y^2}\right) + 0 \qquad \frac{\partial N}{\partial x} = \frac{1}{2y^2}\left(\frac{-1}{x^2}\right) - 0$$

$$= \frac{-1}{2x^2y^2} \qquad = \frac{-1}{2x^2y^2}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

clearly Equⁿ② is an exact.

Now the solution of equⁿ② is $\int M dx + \int N dy = C$

$$\int\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int\left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = C$$

$$\frac{1}{2y}\int x^{-2}dx + \frac{1}{2}\int\frac{1}{x}dx + \int\frac{1}{2xy^2}dy - \frac{1}{2}\int\frac{1}{y}dy = C$$

$$\frac{1}{2y}\frac{x^{-1}}{-1} + \frac{1}{2}\log x + 0 - \frac{1}{2}\log y = C$$

$$-\frac{1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = C$$

$$+\frac{1}{2}\left[-\log y + \log x - \frac{1}{xy}\right] = C$$

$$\frac{1}{2}\left[\log\left(\frac{x}{y}\right) - \frac{1}{xy}\right] = C$$

(3) $y(2xy+1)\,dx + x(1+2xy-x^3y^3)\,dy = 0$ $\rightarrow ①$

Sol:- Equⁿ ① is an exact form $Mdx+Ndy=0$.

Where $M = y(2xy+1)$ and $N = x(1+2xy-x^3y^3)$

$= 2xy^2+y$ $\qquad = x+2x^2y-x^4y^3$

$\frac{\partial M}{\partial y} = 2x(2y)+1$ $\qquad \frac{\partial N}{\partial x} = 1+2y(2x)-y^3 4x^3$

$= 4xy+1$ $\qquad = 1+4xy-4x^3y^3$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Clearly Equⁿ ① is non-exact.

Equⁿ ① can be converted to exact by multiplying Integrating factor.

$Mx-Ny = (2xy^2+y)x - (x+2x^2y-x^4y^3)y$

$= 2x^2y^2 + xy - xy - 2x^2y^2 + x^4y^4$

$= x^4y^4$

$I.F = \frac{1}{Mx-Ny} = \frac{1}{x^4y^4}$

from ①,

$$\frac{y(2xy+1)}{x^4y^4}dx + \frac{x(1+2xy-x^3y^3)}{x^4y^4}dy = 0$$

$$\left(\frac{2xy^2}{x^4y^4} + \frac{y}{x^4y^4}\right)dx + \left(\frac{x}{x^4y^4} + \frac{2x^2y}{x^4y^4} - \frac{x^4y^3}{x^4y^4}\right)dy = 0$$

$$\left(\frac{2}{x^3y^2} + \frac{1}{x^4y^3}\right)dx + \left(\frac{1}{x^3y^4} + \frac{2}{x^2y^3} - \frac{1}{y}\right)dy = 0$$

$\rightarrow ②$

Equⁿ ① is an exact form of $Mdx+Ndy=0$

Where $M = \frac{2}{x^3y^2} + \frac{1}{x^4y^3}$ and $N = \frac{1}{x^3y^4} + \frac{2}{x^2y^3} - \frac{1}{y}$

$\frac{\partial M}{\partial y} = \frac{2}{x^3}(-2)y^{-3} + \frac{1}{x^4}(-3)y^{-4}$ $\qquad \frac{\partial N}{\partial x} = \frac{1}{y^4}(-3)x^{-4} + \frac{2}{y^3}(-2)x^{-3} + 0$

$= \frac{-4}{x^3y^3} - \frac{3}{x^4y^4}$ $\qquad = \frac{-3}{x^4y^4} - \frac{4}{x^3y^3}$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

clearly Equation is an exact.

Now the solution of equⁿ ② is

$$\int M dx + \int N dy = c$$

$$\int\left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx+\int\left(\frac{1}{x^3y^4}+\frac{2}{x^2y^3}-\frac{1}{y}\right)dy = c$$

$$\frac{2}{y^2}\int(x^{-3})dx+\frac{1}{y^3}\int x^{-4}dx+\int\frac{1}{x^3y^4}dy+\int\frac{2}{x^2y^3}dy-\int\frac{1}{y}dy = c$$

$$\frac{\cancel{2}}{y^2}\left(\frac{x^{-2}}{-\cancel{2}}\right)+\frac{1}{y^3}\left(\frac{x^{-3}}{-3}\right)+0+0-\log y = c$$

$$\frac{-1}{x^2y^2}-\frac{1}{3x^3y^3}-\log y = c$$

$$\cancel{\frac{-1}{x^2y^2}\left[1+\frac{3}{xy}+\log y\right]=c}$$

$$\frac{-1}{x^2y^2}\left[1+\frac{1}{3xy}\right]-\log y = c.$$

Tuesday 24/09

METHOD – III, IV

(1) $(xy^2-e^{1/x^3})\,dx - x^2y\,dy = 0. \to ①$

Sol:- Equⁿ ① is an exact form of $Mdx+Ndy=0$

where $M = xy^2-e^{1/x^3}$ and $N = -x^2y$

$$\frac{\partial M}{\partial y} = x(2y)-0 = 2xy$$

$$\frac{\partial N}{\partial x} = -y(2x) = -2xy.$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equⁿ ① is non-exact.
This can be reduced to exact by multiplying an integrating factor.

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = 2xy-(-2xy) = 4xy$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{4\cancel{xy}}{-x^{\cancel{2}}\cancel{y}} = \frac{-4}{x}$$

Now I.F $e^{\int f(x)dx}$

$$= e^{\int -\frac{4}{x}dx} = e^{-4\int\frac{1}{x}dx} = e^{-4\log x} = e^{\log_e x^{-4}} = \frac{1}{x^4}$$

from ①, $\dfrac{(xy^2 - e^{1/x^3})}{x^4} dx - \dfrac{x^2 y}{x^4} dy = 0$

$$\frac{xy^2}{x^4} - \frac{e^{1/x^3}}{x^4} dx - \frac{x^2 y}{x^4} dy = 0$$

$$\left(\frac{y^2}{x^3} - x^{-4} e^{1/x^3}\right) dx - \frac{y}{x^2} dy = 0 \rightarrow ②$$

Equⁿ ② is an exact form of $Mdx + Ndy = 0$

Where $M = \dfrac{y^2}{x^3} - x^{-4} e^{1/x^3}$ and $N = -\dfrac{y}{x^2}$

$$\frac{\partial M}{\partial y} = \frac{1}{x^3}(2y) - 0 = \frac{2y}{x^3}$$

$$\frac{\partial N}{\partial x} = -y(-2)x^{-3} = \frac{2y}{x^3}$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Clearly equⁿ ② is an exact.

Now the solution of equⁿ ② is $\int M dx + \int N dy = c$

$$\int \left(\frac{y^2}{x^3} - x^{-4} \cdot e^{1/x^3}\right) dx + \int \frac{-y}{x^2} dy = c$$

$$y^2 \int x^{-3} dx - \int x^{-4} \cdot e^{x^{-3}} dx + 0 = c$$

$$y^2 \cdot \frac{x^{-2}}{-2} - \int e^t \left(-\frac{1}{3} dt\right) = c$$

$x^{-3} = t$

$-3 \cdot x^{-4} dx = dt$

$x^{-4} dx = -\frac{1}{3} dt$

$$\frac{-2}{x^2 y^2} + \frac{1}{3} \int e^t dt = c$$

$$\frac{-2}{x^2 y^2} + \frac{1}{3} e^t = c$$

$$\frac{-2}{x^2 y^2} + \frac{1}{3} e^{1/x^3} = c.$$

(2). $(xy^3 + y) dx + 2(x^2 y^2 + x + y^4) dy = 0 \rightarrow ①$

Sol: Equⁿ ① is an exact form of $Mdx + Ndy = 0$

Where $M = xy^3 + y$ and $N = 2x^2 y^2 + 2x + 2y^4$

$$\frac{\partial M}{\partial y} = x \cdot 3y^2 + (1) = 3xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2y^2(2x) + 2(1) + 0 = 4xy^2 + 2$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equⁿ ① is non exact.

This can be reduced to exact by multiplying an Integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3xy^2+1-(4xy^2+2)$$
$$= 3xy^2+1-4xy^2-2$$
$$= -xy^2-1$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-xy^2-1}{xy^3+y}$$

$$= \frac{-xy^2-1}{xy^3+y}$$
$$= \frac{-(xy^2+1)}{y(xy^2+1)}$$
$$= \frac{-1}{y}$$

Now I.F $= e^{-\int g(y)\,dy}$
$$= e^{-\int \frac{-1}{y} dy}$$
$$= e^{\log_e y}$$
$$= y.$$

from ①,

$$\frac{(xy^3+y)}{y} dx + \frac{2(x^2y^2+x+y^4)}{y} dy = 0$$
$$\left(\frac{xy^3}{y}+\frac{y}{y}\right) dx + 2\left[\frac{x^2y^2}{y}+\frac{x}{y}+\frac{y^4}{y}\right] dy = 0$$
$$(xy^2+1)\,dx + 2\left(x^2y+\frac{x}{y}+y^3\right) dy = 0 \longrightarrow ②$$

Eqn ② is an exact form of $Mdx+Ndy=0$

where $M = xy^2+1$ and $N = 2\left[x^2y+\frac{x}{y}+y^3\right]$

$\frac{\partial M}{\partial y} = x(2y)+0$ $\qquad \frac{\partial N}{\partial x} = y(2x)+\frac{1}{y}$

$= 2xy$

from ①,

$$y(xy^3+y)\,dx + 2y(x^2y^2+x+y^4)\,dy = 0$$
$$(xy^4+y^2)\,dx + 2(x^2y^3+xy+y^5)\,dy = 0 \rightarrow ②$$

Eqn ② is an exact form of $Mdx+Ndy=0$

Where $M = xy^4+y^2$ and $N = 2[x^2y^3+xy+y^5]$

$\frac{\partial M}{\partial y} = x\cdot 4y^3+2y$ $\qquad \frac{\partial N}{\partial x} = 2[y^3(2x)+y(1)+0]$

$= 4xy^3+2y$ $\qquad = 4xy^3+2y$

$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ clearly Equⁿ ② is an exact.
Now the solⁿ of Equⁿ ② is $\int M dx + \int N dy = c$

$$\int (x y^4 + y^2) dx + \int 2(x^2y^3 + xy + y^5)\, dy = c$$

$$y^4 \int x\, dx + y^2 \int (1) dx + 2\int x^2y^3 dy + 2\int xy\, dy + 2\int y^5 dy = c$$

$$y^4 \frac{x^2}{2} + y^2.(x) + 0 + 0 + \cancel{2} \frac{y^6}{\cancel{6}_3} = c$$

$$\frac{1}{2}x^2y^4 + xy^2 + \frac{y^6}{3} = c$$

$$\frac{3x^2y^4 + 6xy^2 + 2y^6}{6} = c \Rightarrow 3x^2y^4 + 6xy^2 + 2y^6 = 6c$$

$$\Rightarrow \boxed{3x^2y^4 + 6xy^2 + 2y^6 = c}$$

(7) $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \frac{1}{4}(x + xy^2)\, dy = 0 \longrightarrow ①$

Sol:- Equⁿ ① is an exact form of $M dx + N dy = 0$

where $M = y + \frac{y^3}{3} + \frac{x^2}{2}$ and $N = \frac{1}{4}(x + xy^2)$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{\cancel{3}}\cancel{3}y^2 + 0 = 1 + y^2$$

$$\frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2(1)) = \frac{1}{4}(1 + y^2)$$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equⁿ ① is non-exact.
This can be reduced to exact by multiplying Integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + y^2 - \frac{1}{4}(1 + y^2)$$
$$= 1 + y^2(1 - 1/4)$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\cancel{(1+y^2)}(1 - 1/4)}{1/4\, x\cancel{(1+y^2)}} = \frac{4(1 - 1/4)}{x}$$
$$= \frac{4 - 1}{x}$$
$$= \frac{3}{x}$$

Now I.F $e^{\int f(x) dx}$
$$= e^{\int \frac{3}{x} dx}$$
$$= e^{3\int \frac{1}{x} dx}$$
$$= e^{3 \log x}$$
$$= e^{\log_e x^3}$$
$$= x^3$$

from ①, $\left(y.x^3 + \frac{x^3y^3}{3} + \frac{x^5}{2}\right) dx + \frac{1}{4}(x^4 + x^4y^2)\, dx = 0$
$\longrightarrow ②$

Equ^n (2) is an exact form of $Mdx + Ndy = 0$

where $M = yx^3 + \frac{x^3y^3}{3} + \frac{x^5}{2}$ and $N = \frac{1}{4}(x^4 + x^4y^2)$

$\frac{\partial M}{\partial y} = x^3(1) + \frac{x^3}{\cancel{3}}\cdot\cancel{3}y^2 + 0$

$= x^3(1+y^2)$

$\frac{\partial N}{\partial x} = \frac{1}{4}(4x^3 + 4x^3y^2)$

$= \frac{\cancel{4}x^3}{\cancel{4}}(1+y^2)$

$= x^3(1+y^2)$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

clearly Equ^n (2) is an exact.

Now the solution of equ^n (2) is $\int M dx + \int N dy = c$

$$\int\left(x^3y + \frac{x^3y^3}{3} + \frac{x^5}{2}\right)dx + \int \frac{1}{4}(x^4 + x^4y^2)\,dy = c$$

$$y\int x^3 dx + \frac{y^3}{3}\int x^3 dx + \frac{1}{2}\int x^5 dx + 0 = c$$

$$y\frac{x^4}{4} + \frac{y^3}{3}\frac{x^4}{4} + \frac{1}{2}\cdot\frac{x^6}{6} = c$$

$$\frac{x^4y}{4} + \frac{x^4y^3}{12} + \frac{x^6}{12} = c$$

$$\frac{3x^4y + x^4y^3 + x^6}{12} = c$$

$$3x^4y + x^4y^3 + x^6 = 12c$$

$$3x^4y + x^4y^3 + x^6 = c.$$

(8) $(x\sec^2 y - x^2\cos y)\,dy = (\tan y - 3x^4)\,dx$

Sol: $(\tan y - 3x^4)dx - (x\cdot\sec^2 y - x^2\cos y)\,dy = 0 \rightarrow (1)$

Equ^n (1) is an exact form of $Mdx + Ndy = 0$

Where $M = \tan y - 3x^4$ and $N = +x^2\cos y - x\cdot\sec^2 y$

$\frac{\partial M}{\partial y} = \sec^2 y - 0$

$= (\sec^2 y)$

$\frac{\partial N}{\partial x} = \cos y(2x) - \sec^2 y\cdot(1)$

$= 2x\cdot\cos y - \sec^2 y$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

clearly equ^n (1) is non-exact.

This can be reduced to exact by multiplying Integrating factor.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \sec^2 y - 2x\cos y + \sec^2 y$

$= 2\sec^2 y - 2x\cos y$

$$\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{2\sec^2 y - 2x\cos y}{-(x\sec^2 y - x^2\cos y)}$$

$$= \frac{2(\cancel{\sec^2 y - x\cos y})}{-x(\cancel{\sec^2 y - x\cos y})}$$

$$= \frac{-2}{x}$$

Now I.F $e^{\int f(x)dx}$

$$= e^{\int \frac{-2}{x} dx}$$

$$= e^{-2\log x}$$

$$= \frac{1}{x^2}.$$

from ①, $\frac{(\tan y - 3x^4)}{x^2} dx - \frac{(x\sec^2 y - x^2\cos y)}{x^2} dy = 0$

$$\left(\frac{\tan y}{x^2} - \frac{3x^{\cancel{4}\,2}}{\cancel{x^2}}\right) dx - \left(\frac{\cancel{x}\sec^2 y}{x^{\cancel{2}}} - \frac{\cancel{x^2}\cos y}{\cancel{x^2}}\right) dy = 0$$

$$\left(\frac{\tan y}{x^2} - 3x^2\right) dx - \left(\frac{\sec^2 y}{x} - \cos y\right) dy = 0 \rightarrow ②$$

Equn ② is an exact form of $Mdx + Ndy = 0$

Where $M = \frac{\tan y}{x^2} - 3x^2$ and $N = \cos y - \frac{\sec^2 y}{x}$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2}\sec^2 y - 0 \qquad \frac{\partial N}{\partial x} = 0 - \sec^2 y \cdot \left(\frac{-1}{x^2}\right)$$

$$= \frac{\sec^2 y}{x^2} \qquad = \frac{\sec^2 y}{x^2}$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Clearly equn ② is an exact.

Now the solution of equn ② is $\int M dx + \int N dy = c$

$$\int\left(\frac{\tan y}{x^2} - 3x^2\right) dx + \int\left(\cos y - \frac{\sec^2 y}{x}\right) dy = c$$

$$\tan y \int x^{-2} dx - 3\int x^2 dx + \int \cos y\, dy - \int \frac{\sec^2 y}{x} dy = c$$

$$\tan y\left(\frac{x^{-1}}{-1}\right) - \cancel{3}\frac{x^3}{\cancel{3}} + \sin y - 0 = c$$

$$-\frac{\tan y}{x} - x^3 + \sin y = c$$

$$\frac{1}{x}\tan y + x^3 - \sin y = c.$$

(9) $\left(xy\,e^{x/y}+y^2\right)dx - x^2 e^{x/y}\,dy = 0 \rightarrow (1)$ (I - Method)

Sol: Equⁿ (1) is an exact form of $Mdx + Ndy = 0$

where $M = xy\,e^{x/y}+y^2$

$$\frac{\partial M}{\partial y} = x\left[y\cdot e^{x/y}\left(\frac{-x}{y^2}\right) + e^{x/y}(1)\right] + 2y$$

$$= x\left[e^{x/y}\frac{-x}{y} + e^{x/y}\right] + 2y$$

$$= x\cdot e^{x/y}\left(1-\frac{x}{y}\right) + 2y$$

and $N = -x^2 e^{x/y}$

$$\frac{\partial N}{\partial x} = -\left(x^2\cdot e^{x/y}\cdot\frac{1}{y} + e^{x/y}\cdot 2x\right)$$

$$= -\,x\cdot e^{x/y}\left(\frac{x}{y}+2\right)$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equⁿ (1) is non-exact.

→ This can be reduced to exact by multiplying Integrating factor.

→ clearly equⁿ (1) is a homogeneous ~~of~~ degree '2'.

$$Mx + Ny = \left(xy\,e^{x/y}+y^2\right)x + \left(-x^2 e^{x/y}\right)y$$

$$= \cancel{x^2y\cdot e^{x/y}} + xy^2 - \cancel{x^2y\,e^{x/y}}$$

$$= xy^2 \neq 0$$

$$\boxed{Mx + Ny \neq 0}$$

Now $\quad I.F = \dfrac{1}{Mx+Ny} = \dfrac{1}{xy^2}$

from (1),

$$\frac{\left(xy\,e^{x/y}+y^2\right)}{xy^2}\,dx - \frac{x^2 e^{x/y}}{xy^2}\,dy = 0$$

$$\left(\frac{\cancel{x}\cancel{y}\,e^{x/y}}{\cancel{x}y\cancel{y}} + \frac{\cancel{y^2}}{x\cancel{y^2}}\right)dx - \frac{x\cancel{x}\,e^{x/y}}{\cancel{x}y^2}\,dy = 0$$

$$\left(\frac{e^{x/y}}{y} + \frac{1}{x}\right)dx - \frac{x\,e^{x/y}}{y^2}\cdot dy = 0 \rightarrow (2)$$

Equⁿ (2) is an exact form of $Mdx + Ndy = 0$

where $M = \dfrac{e^{x/y}}{y} + \dfrac{1}{x}$.

$$\frac{\partial M}{\partial y} = \frac{\cancel{y}\cdot e^{x/y}\cdot x\left(\frac{-1}{y^{\cancel{2}}}\right) - e^{x/y}(1)}{y^2} + 0.$$

$$= \frac{-\frac{x}{y} e^{x/y} - e^{x/y}}{y^2}$$

$$= \frac{-\frac{x}{y} \cdot e^{x/y}}{y^2} - \frac{e^{x/y}}{y^2}$$

$$= \frac{-x e^{x/y}}{y^3} - \frac{e^{x/y}}{y^2}$$

$$= -\frac{e^{x/y}}{y^2} \left(\frac{x}{y} + 1\right)$$

and $N = \frac{-x \cdot e^{x/y}}{y^2}$

$$\frac{\partial N}{\partial x} = \frac{-1}{y^2}\left[x \cdot e^{x/y} \frac{1}{y}(1) + e^{x/y} \cdot (1)\right]$$

$$= \frac{-1}{y^2}\left[\frac{x}{y} e^{x/y} + e^{x/y}\right]$$

$$= -\frac{e^{x/y}}{y^2}\left(\frac{x}{y} + 1\right)$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Clearly equⁿ① is an exact.

Now the solution of equⁿ① is $\int M dx + \int N dy = C$

$$\int \left(\frac{e^{x/y}}{y} + \frac{1}{x}\right) dx + \int -\frac{x e^{x/y}}{y^2} dy = c$$

$$\frac{1}{y} \int e^{x/y} dx + \int \frac{1}{x} dx - 0 = c$$

$$\frac{1}{y} \cdot e^{x/y} \frac{1}{y}(1) + \log x = c$$

$$\frac{e^{x/y}}{y^2} + \log x = c$$

(10) $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0 \rightarrow ①$

Sol: Equⁿ① is an exact form of $M dx + N dy = 0$

where $M = 3xy - 2ay^2$ and $N = x^2 - 2axy$

$$\frac{\partial M}{\partial y} = 3x(1) - 2a(2y) \qquad \frac{\partial N}{\partial x} = 2x - 2ay(1)$$

$$= 3x - 4ay \qquad = 2x - 2ay$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equⁿ① is non-exact.

This can be reduced to exact by multiplying Integrating factor.

$\cancel{\text{Here}}\ \frac{dM}{dy} - \frac{dN}{dx} = 3x - 4ay - 2x + 2ay$

$= x - 2ay.$

$$\Rightarrow \frac{\frac{dM}{dy} - \frac{dN}{dx}}{N} = \frac{x - 2ay}{x^2 - 2axy)}$$

$$= \frac{x - 2a\cancel{y}}{x(x - \cancel{2ay})}$$

$$= \frac{1}{x}.$$

Now $I.F = e^{\int f(x)dx} = e^{\int \frac{1}{x} dx}$

$= e^{\log_e x}$

$= \underline{x}.$

from ①,

$$\frac{(3xy - 2ay^2)}{x} dx + \left(\frac{x^2 - 2axy}{x}\right) dy = 0$$

$$\left(\frac{3\cancel{x}y}{\cancel{x}} - \frac{2ay^2}{x}\right) dx + \left(\frac{x^{\cancel{2}}}{\cancel{x}} - \frac{2a\cancel{x}y}{\cancel{x}}\right) dy = 0$$

$$\left(3y - \frac{2ay^2}{x}\right) dx + (x - 2ay)\, dy = 0 \rightarrow ②$$

Equ^n ② is an exact form of $Mdx + Ndy = 0.$

from ①,

$$(3xy - 2ay^2)\, x\, dx + (x^2 - 2axy)\, x\, dy = 0$$

$$(3x^2y - 2axy^2)\, dx + (x^3 - 2ax^2y)\, dy = 0 \longrightarrow ②$$

Equ^n ② is an exact form of $Mdx + Ndy = 0$

where $M = 3x^2y - 2axy^2$ and $N = x^3 - 2ax^2y$

$\frac{dM}{dy} = 3x^2(1) - 2ax(2y)$

$= 3x^2 - 4axy$

$\frac{dN}{dx} = 3x^2 - 2ay(2x)$

$= 3x^2 - 4axy$

$$\boxed{\therefore \frac{dM}{dy} = \frac{dN}{dx}}$$

Clearly equ^n ② is an exact.

Now the solution of equ^n ② is $\int M dx + \int N dy = c$

$$\int (3x^2y - 2axy^2)\, dx + \int (x^3 - 2ax^2y)\, dy = c$$

$$3y \int x^2 dx - 2ay^2 \int (x)\, dx + 0 = C$$

$$\cancel{3}y \frac{x^3}{\cancel{3}} - \cancel{2}ay^2 \frac{x^2}{\cancel{2}} = C$$

$$x^3y - ax^2y^2 = c$$

$$x^2y(x - ay) = C$$

(11) $(x^4 e^x - 2mxy^2)\,dx + 2mx^2y\,dy = 0 \rightarrow (1)$

Sol: Equn (1) is an exact form of $Mdx + Ndy = 0$

where $M = x^4e^x - 2mxy^2$ and $N = 2mx^2y$

$\frac{\partial M}{\partial y} = 0 - 2mx(2y)$

$= -4mxy$

$\frac{\partial N}{\partial x} = 2my(2x)$

$= 4mxy$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence Equn (1) is non-exact.

This can be reduced to exact by multiplying Integrating factor.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4mxy - 4mxy$

$= -8mxy.$

$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-\cancel{8m}\cancel{x}\cancel{y}\,4}{\cancel{2m}\,x^{\cancel{2}}\cancel{y}}$

$= \frac{-4}{x}.$

Now $I.F = e^{\int f(x)\,dx}$

$= e^{-4\int \frac{1}{x}dx}$

$= e^{-4\log x}$

$= e^{\log_e (x)^{-4}}$

$= x^{-4}$

$= \frac{1}{x^4}.$

from (1),

$\left(\frac{x^4e^x - 2mxy^2}{x^4}\right)dx + \left(\frac{2mx^2y}{x^4}\right)dy = 0$

$\left(\frac{\cancel{x^4}e^x}{\cancel{x^4}} - \frac{2m\cancel{x}y^2}{x^{\cancel{4}3}}\right)dx + \left(\frac{2m\cancel{x^2}y}{x^{\cancel{4}2}}\right)dy = 0$

$\left(e^x - \frac{2my^2}{x^3}\right)dx + \left(\frac{2my}{x^2}\right)dy = 0 \rightarrow (2)$

Equn (2) is an exact form of $Mdx + Ndy = 0$, where $M = e^x - \frac{2my^2}{x^3}$ and $N = \frac{2my}{x^2}$

$\frac{\partial M}{\partial y} = 0 - \frac{2m}{x^3}(2y)$ $\qquad$ $\frac{\partial N}{\partial x} = 2my \cdot (-2) x^{-3}$

$= \frac{-4my}{x^3}$ $\qquad$ $= \frac{-4my}{x^3}$.

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Clearly equⁿ ② is an exact.

Now the solution of equⁿ ① is $\int M dx + \int N dy = c$

$$\int \left(e^x - \frac{2my^2}{x^3}\right) dx + \int \frac{2my}{x^2} dy = c$$

$$\int e^x . dx - 2my^2 \int x^{-3} dx + 0 = c$$

$$e^x - \cancel{2}my^2\left(\frac{x^{-2}}{-\cancel{2}}\right) = c$$

$$e^x + \frac{my^2}{x^2} = c.$$

(12) $y \cdot (2x^2y + e^x) \cdot dx = (e^x + y^3) dy$.

Sol:- $y . (2x^2y + e^x) dx = (e^x + y^3) dy$

$(2x^2y^2 + y . e^x) dx - (e^x + y^3) dy = 0 \rightarrow ①$

Equⁿ ① is an exact form of $Mdx + Ndy = 0$

where $M = 2x^2y^2 + y . e^x$ and $N = -(e^x + y^3)$

$\frac{\partial M}{\partial y} = 2x^2 . (2y) + e^x (1)$ $\qquad$ $\frac{\partial N}{\partial x} = -(e^x + 0)$

$= 4x^2y + e^x$ $\qquad$ $= -e^x$.

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equⁿ ① is non-exact.

This can be reduced to exact by multiplying Integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x^2y + e^x - (-e^x)$$

$$= 4x^2y + e^x + e^x$$

$$= 4x^2y + 2e^x.$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4x^2y + 2e^x}{2x^2y^2 + y \cdot e^x}$$

$$= \frac{2\cancel{(2x^2y + e^x)}}{y\cancel{(2x^2y + e^x)}}$$

$$= \frac{2}{y}.$$

Now $I.F = e^{-\int g(y)\,dy} = e^{-\int \frac{2}{y} dy}$

$$= e^{-2 \cdot \log y}$$

$$= e^{\log_e (y)^{-2}}$$

$$= \frac{1}{y^2}.$$

from ①, $\frac{(2x^2y^2 + y \cdot e^x)}{y^2} dx - \frac{(e^x + y^3)}{y^2} dy = 0$

$$\left(\frac{2x^2\cancel{y^2}}{\cancel{y^2}} + \frac{y \cdot e^x}{y^2}\right) dx - \left(\frac{e^x}{y^2} + \frac{y^3}{y^2}\right) dy = 0$$

$$\left(2x^2 + \frac{e^x}{y}\right) dx - \left(\frac{e^x}{y^2} + y\right) dy = 0 \rightarrow ②$$

Equⁿ ② is an exact form of $Mdx + Ndy = 0$

Where $M = 2x^2 + \frac{e^x}{y}$ and $N = -\left(\frac{e^x}{y^2} + y\right)$

$\frac{\partial M}{\partial y} = 0 + e^x \cdot \frac{-1}{y^2}$ $\qquad$ $\frac{\partial N}{\partial x} = -\left(\frac{1}{y^2} e^x + 0\right)$

$= \frac{-e^x}{y^2}$ $\qquad$ $= \frac{-e^x}{y^2}$

$$\boxed{\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

clearly Equⁿ ② is an exact.

Now the solⁿ of Equⁿ ② is $\int M dx + \int N dy = c$

$$\int \left(2x^2 + \frac{e^x}{y}\right) dx + \int -\left(\frac{e^x}{y^2} + y\right) dy = c$$

$$2\int x^2 dx + \frac{1}{y}\int e^x dx - \int \frac{e^x}{y^2} dy - \int y\, dy = c$$

$$2\frac{x^3}{3} + \frac{1}{y}e^x - 0 - \frac{y^2}{2} = c$$

$$\frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = c.$$

(15) $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0 \rightarrow (1)$

Sol: Eqn (1) is an exact form of $Mdx+Ndy=0$

Where $M=3x^2y^4+2xy$ and $N=2x^3y^3-x^2$

$\frac{\partial M}{\partial y}=3x^2\cdot 4y^3+2x$ $\qquad$ $\frac{\partial N}{\partial x}=2y^3\cdot 3x^2-2x$

$=12x^2y^3+2x$ $\qquad$ $=6x^2y^3-2x.$

$$\boxed{\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}}$$

Hence Eqn (1) is non-exact.

This can be reduced to exact by multiplying an Integrating factor.

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=12x^2y^3+2x-6x^2y^3+2x$$

$$=6x^2y^3+4x.$$

$$=2(3x^2y^3+2x)$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\frac{2(3x^2y^3+2x)}{3x^2y^4+2xy}=\frac{2\cancel{(3x^2y^3+2x)}}{y\cancel{(3x^2y^3+2x)}}$$

$$=\frac{2}{y}$$

I.F $e^{-\int g(y)\,dy}$

$$=e^{-\int\frac{2}{y}dy}$$

$$=e^{-2\cdot\log y}$$

$$=e^{\log(y)^{-2}}$$

$$=\frac{1}{y^2}.$$

from (1),

$$\left(\frac{3x^2y^4+2xy}{y^2}\right)dx+\left(\frac{2x^3y^3-x^2}{y^2}\right)dy=0$$

$$\left(\frac{3x^2y^4}{y^2}+\frac{2xy}{y^2}\right)dx+\left(\frac{2x^3y^3}{y^2}-\frac{x^2}{y^2}\right)dy=0$$

$$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0 \rightarrow (2)$$

Eqn (2) is an exact form of $Mdx+Ndy=0$

Where $M=3x^2y^2+\frac{2x}{y}$ and $N=2x^3y-\frac{x^2}{y^2}$

$\frac{\partial M}{\partial y}=3x^2(2y)+2x\left(\frac{-1}{y^2}\right)$ $\qquad$ $\frac{\partial N}{\partial x}=2y(3x^2)-\frac{1}{y^2}(2x)$

$=6x^2y-\frac{2x}{y^2}$ $\qquad$ $=6x^2y-\frac{2x}{y^2}$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Clearly Equ^n ② is an exact.

Now the sol^n of Equ^n ② is $\int M dx + \int N dy = C$

$$\int \left(3x^2y^2 + \frac{2x}{y}\right) dx + \int \left(2x^3y - \frac{x^2}{y^2}\right) dy = C$$

$$3y^2 \int x^2 dx + \frac{2}{y} \int x dx + 0 = C$$

$$\cancel{3}y^2 \frac{x^3}{\cancel{3}} + \frac{\cancel{2}}{y} \cdot \frac{x^2}{\cancel{2}} = C$$

$$x^3y^2 + x^2 \frac{1}{y} = C.$$

(16) $y \log y \, dx + (x - \log y) \, dy = 0 \rightarrow ①$

Sol:- Equ^n ① is an exact form of $Mdx + Ndy = 0$

where $M = y \log y$ and $N = x - \log y$

$$\frac{\partial M}{\partial y} = \cancel{y} \cdot \frac{1}{\cancel{y}} + \log y (1) \qquad \frac{\partial N}{\partial x} = 1 - 0$$

$$= 1 + \log y \qquad = 1.$$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence Equ^n ① is non-exact.

This can be reduced to exact by multiplying an Integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \cancel{1} + \log y - \cancel{1}$$

$$= \log y.$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{\log y}{y \log y} = \frac{1}{y}.$$

Now I.F $e^{-\int g(y) dy}$

$$= e^{-\int \frac{1}{y} dy}$$

$$= e^{-\log y}$$

$$= e^{\log (y)^{-1}}$$

$$= y^{-1}$$

$$= \frac{1}{y}.$$

from ①, $\frac{\cancel{y}\log y}{\cancel{y}}\,dx + \left(\frac{x-\log y}{y}\right)dy = 0.$

$$\log y\,.dx + \left(\frac{x}{y} - \frac{\log y}{y}\right)dy = 0 \rightarrow ②$$

Equ^n ② is an exact form of $Mdx + Ndy = 0$

where $M = \log y$. and $N = \frac{x}{y} - \frac{\log y}{y}$

$\frac{\partial M}{\partial y} = \frac{1}{y}$.

$\frac{\partial N}{\partial x} = \frac{1}{y}(1) - 0$.

$= \frac{1}{y}$.

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

clearly Equ^n ② is an exact.

Now the sol^n of Equ^n ② is $\int Mdx + \int Ndy = c$.

$$\int \log y\,.\,dx + \int\left(\frac{x}{y} - \frac{\log y}{y}\right)dy = c$$

$$\log y \int(1)dx + \int\frac{x}{y}\,dy - \int\frac{\log y}{y}\,dy = c$$

$$\log y\,.(x) + 0 - \int t\,.\,dt = c.$$

$\log y = t$

$\frac{1}{y}dy = dt.$

$$x\,.\log y - \frac{t^2}{2} = c$$

$$x\,.\log y - \frac{(\log y)^2}{2} = c.$$

③ $(x^2+y^2+x)dx + xy\,dy = 0. \rightarrow ①$

sol:- Equ^n ① is an exact form of $Mdx + Ndy = 0$

where $M = x^2+y^2+x$ and $N = xy$

$\frac{\partial M}{\partial y} = 0 + 2y + 0$

$= 2y$

$\frac{\partial N}{\partial x} = y.(1)$

$= y$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence Equ^n ① is non-exact.

This can be reduced to exact by multiplying an Integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - y = y.$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\cancel{y}}{x\cancel{y}} = \frac{1}{x}.$$

Now I.F $e^{\int f(x)dx} = e^{\int \frac{1}{x}dx}$

$= e^{\log_e x}$

$= x$

from ①,

$\frac{x^2+y^2+x}{x} \cdot dx + \frac{xy}{x} dy = 0$

$\left(\frac{x^4}{x} + \frac{y^2}{x} + \frac{x}{x}\right) dx + y \cdot dy = 0$

$\left(x + \frac{y^2}{x} + 1\right) dx + y \cdot dy = 0 \rightarrow ②$

Equⁿ ② is an exact form of $Mdx + Ndy = 0$

$2 | 2, 4, 3$
$\quad 1, 2, 3$

where $M = x + \frac{y^2}{x} + 1$

$\frac{dM}{dy} = 0 + \frac{1}{x}(2y) + 0$

$= \cancel{\frac{2y}{x}}$

from ①,

$x(x^2+y^2+x) dx + x(xy) dy = 0$

$(x^3 + xy^2 + x^2) dx + x^2y \, dy = 0 \rightarrow ②$

Equⁿ ② is an exact form of $Mdx + Ndy = 0$

where $M = x^3 + xy^2 + x^2$ and $N = x^2y$

$\frac{dM}{dy} = 0 + x(2y) + 0$ $\qquad \frac{dN}{dx} = y(2x)$

$= 2xy$ $\qquad = 2xy$

$$\boxed{\therefore \frac{dM}{dy} = \frac{dN}{dx}}$$

clearly Equⁿ ② is an exact.

Now the solⁿ of Equⁿ ② is $\int Mdx + \int Ndy = c$

$\int (x^3 + xy^2 + x^2) dx + \int (x^2 y) dy = c$

$\int x^3 dx + y^2 \int x \, dx + \int x^2 dx + 0 = c$

$\frac{x^4}{4} + y^2 \cdot \frac{x^2}{2} + \frac{x^3}{3} = c$

$\frac{3x^4 + 6x^2y^2 + 4x^3}{12} = c$

$3x^4 + 6x^2y^2 + 4x^3 = 12c$

$3x^4 + 6x^2y^2 + 4x^3 = c.$

(4) $(x^2+y^2+1)\,dx - 2xy\,dy = 0 \rightarrow ①$

Sol:- Equ^n ① is an exact form of $Mdx+Ndy=0$.

Where $M = x^2+y^2+1$ and $N = -2xy$

$\frac{\partial M}{\partial y} = 0+2y+0$ $\quad$ $\frac{\partial N}{\partial x} = -2y(1)$

$= 2y$ $\quad$ $= -2y$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence Equ^n ① is a non-exact.

This can be reduced to exact by multiplying an Integrating factor.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y-(-2y) = 2y+2y = \underline{4y}$

$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y}{-2xy} = \frac{-2}{x}$

Now I.F $e^{\int f(x)dx} = e^{\int \frac{-2}{x}dx}$

$= e^{-2\log x}$

$= e^{\log_e (x)^{-2}}$

$= \frac{1}{x^2}$

from ①, $\frac{x^2+y^2+1}{x^2}\,dx - \frac{2xy}{x^2}\,dy = 0$

$\left(\frac{x^2}{x^2} + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx - \frac{2y}{x}\,dy = 0$

$\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx - \frac{2y}{x}\,dy = 0 \rightarrow ②$

Equ^n ② is an exact form of $Mdx+Ndy=0$

Where $M = 1+\frac{y^2}{x^2}+\frac{1}{x^2}$ and $N = \frac{-2y}{x}$

$\frac{\partial M}{\partial y} = 0+\frac{1}{x^2}(2y)+0$ $\quad$ $\frac{\partial N}{\partial x} = -2y\left(\frac{-1}{x^2}\right)$

$= \frac{2y}{x^2}$ $\quad$ $= \frac{2y}{x^2}$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Clearly Equ^n ② is an Exact.

Now the soln of Equn ② is $\int M dx + \int N dy = c$

$$\int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx + \int \frac{-2y}{x}\cdot dy = c$$

$$\int (1)\,dx + y^2\int x^{-2}dx + \int x^{-2}dx - 0 = c.$$

$$x + y^2\cdot\frac{x^{-1}}{-1} + \frac{x^{-1}}{-1} = c$$

$$x - \frac{y^2}{x} - \frac{1}{x} = c.$$

$$\frac{x^2-y^2-1}{x} = c$$

$$\frac{1}{x}(x^2-y^2-1) = c.$$

(5) $(x^2+y^2+2x)\,dx + 2y\,dy = 0. \rightarrow ①$

Sol:- Equn ① is an exact form of $Mdx+Ndy=0$

where $M = x^2+y^2+2x$ and $N = 2y$

$$\frac{\partial M}{\partial y} = 0+2y+0 \qquad \frac{\partial N}{\partial x} = 0$$

$$= 2y \qquad\qquad = 0$$

$$\boxed{\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence Equn ① is non-exact.

This can be reduced to exact by multiplying an Integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - 0 = 2y.$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\cancel{2y}}{\cancel{2y}} = \underline{1}$$

Now. I.F $e^{\int f(x)dx} = e^{\int (1)dx}$

$$= \underline{\underline{e^x}}.$$

from ①, $e^x(x^2+y^2+2x)\,dx + 2y\cdot e^x\,dy = 0$

$$(e^x x^2 + e^x y^2 + 2x\cdot e^x)\,dx + 2ye^x\cdot dy = 0 \rightarrow ②$$

Equn ② is an exact form of $Mdx+Ndy=0$

where $M = e^x x^2 + e^x\cdot y^2 + 2x\cdot e^x$

$$\frac{\partial M}{\partial y} = 0 + e^x\cdot(2y) + 0$$

$$= e^x\cdot 2y$$

and $N = 2y \cdot e^x$

$$\frac{\partial N}{\partial x} = 2y(e^x)$$

$$= 2y \cdot e^x$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

clearly Equ^n (2) is an Exact.

Now the solu^n of equ^n (2) is $\int M dx + \int N dy = c$

$$\int (e^x \cdot x^2 + e^x y^2 + 2x \cdot e^x) dx + \int 2y e^x \cdot dy = c.$$

$$\int e^x \cdot x^2 dx + y^2 \int e^x \cdot dx + 2 \int x \cdot e^x dx + 0 = c.$$

D	I
$+ x^2$	e^x
$- 2x$	e^x
$+ 2$	e^x
$- 0$	e^x

$$x^2 e^x - 2x \cdot e^x + 2 \cdot e^x + y^2 \cdot e^x + 2 e^x \cdot (x-1) = c$$

$$x^2 e^x - \cancel{2x \cdot e^x} + \cancel{2e^x} + y^2 \cdot e^x + \cancel{2x \cdot e^x} - \cancel{2e^x} = c$$

$$x^2 \cdot e^x + y^2 \cdot e^x = c$$

$$e^x (x^2 + y^2) = c.$$

(6) $(y^4 + 2y) dx + (xy^3 + 2y^4 - 4x) dy = 0 \longrightarrow (1)$

Sol:- Equ^n (1) is an exact ~~form~~ of $M dx + N dy = 0$

where $M = y^4 + 2y$

$$\frac{\partial M}{\partial y} = 4y^3 + 2$$

$$= 2(2y^3 + 1)$$

and $N = xy^3 + 2y^4 - 4x$

$$\frac{\partial N}{\partial x} = y^3(1) + 0 - 4$$

$$= y^3 - 4$$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equ^n (1) is non-Exact.

This can be reduced to exact by multiplying an Integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4y^3 + 2 - (y^3 - 4)$$

$$= 4y^3 + 2 - y^3 + 4$$

$$= 3y^3 + 6.$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3y^3 + 6}{y^4 + 2y} = \frac{3(\cancel{y^3+2})}{y(\cancel{y^3+2})} = \frac{3}{y}.$$

Now I.F $e^{-\int g(y)\,dy} = e^{-\int \frac{3}{y}\,dy}$

$= e^{-3\log y}$

$= e^{\log_e (y)^{-3}}$

$= \frac{1}{y^3}$

from ①, $\left(\frac{y^4+2y}{y^3}\right)dx + \left(\frac{xy^3+2y^4-4x}{y^3}\right)dy = 0$

$\left(\frac{y^4}{y^3}+\frac{2y}{y^3}\right)dx + \left(\frac{xy^3}{y^3}+\frac{2y^4}{y^3}-\frac{4x}{y^3}\right)dy=0$

$\left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0 \rightarrow ②$

Equⁿ ② is an exact form of $Mdx+Ndy=0$

where $M = y+\frac{2}{y^2}$ and $N = x+2y-\frac{4x}{y^3}$

$\frac{\partial M}{\partial y} = 1+2\cdot(-2)y^{-3}$ $\qquad \frac{\partial N}{\partial x} = 1+0-\frac{4}{y^3}(1)$

$= 1-\frac{4}{y^3}$ $\qquad = 1-\frac{4}{y^3}$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

clearly equⁿ ② is an exact.

Now the soluⁿ of equⁿ ② is $\int M dx + \int N dy = C$

$\int \left(y+\frac{2}{y^2}\right)dx + \int \left(x+2y-\frac{4x}{y^3}\right)dy = C$

$y\int(1)dx + \frac{2}{y^2}\int(1)dx + \int x\cdot dy + 2\int y\,dy - \int \frac{4x}{y^3}\cdot dy = C$

$y(x) + \frac{2}{y^2}\cdot(x) + 0 + 2\cdot\left(\frac{y^2}{2}\right) - 0 = C$

$xy + \frac{2x}{y^2} + y^2 = C.$

(13) $\cdot\, y\,dx - x\,dy + \log x\cdot dx = 0.$

Sol:- $(y+\log x)dx - x\,dy = 0 \rightarrow ①$

Equⁿ ① is an exact form of $Mdx+Ndy=0$

where $M = y+\log x$ and $N = -x$

$\frac{\partial M}{\partial y} = 1+0$ $\qquad \frac{\partial N}{\partial x} = -(1)$

$= 1$ $\qquad = -1$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence Equⁿ ① is non-Exact.

This can be reduced to exact by multiplying an Integrating factor.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1-(-1) = 1+1 = 2$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2}{-x} = \frac{-2}{x}$$

Now I.F $e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx}$

$$= e^{-2\int \frac{1}{x}dx}$$

$$= e^{-2\log x}$$

$$= e^{\log_e (x)^{-2}}$$

$$= \frac{1}{x^2}$$

from ①,

$$\left(\frac{y+\log x}{x^2}\right)dx - \left(\frac{x}{x^2}\right)dy = 0$$

$$\left(\frac{y}{x^2} + \frac{\log x}{x^2}\right)dx - \left(\frac{1}{x}\right)dy = 0 \rightarrow ②$$

Equⁿ ② is an exact form of $Mdx + Ndy = 0$

where $M = \frac{y}{x^2} + \frac{\log x}{x^2}$ and $N = \frac{-1}{x}$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2}(1) + 0 \qquad \frac{\partial N}{\partial x} = (-1)\frac{-1}{x^2}$$

$$= \frac{1}{x^2} \qquad = \frac{1}{x^2}$$

$$\boxed{\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

clearly Equⁿ ② is an Exact.

Now the solⁿ of Equⁿ ② is $\int M dx + \int N dy = c$

$$\int \left(\frac{y}{x^2} + \frac{\log x}{x^2}\right)dx + \int \left(\frac{-1}{x}\right)dy = c.$$

$$y\int x^{-2}dx + \int \log x \cdot \frac{1}{x^2}\, dx - 0 = c.$$

$$y\cdot\left(\frac{x^{-1}}{-1}\right) + \int \log x \cdot x^{-2}\, dx = c.$$

Integration by parts.

D		I
$\log x$	$\rightarrow$	x^{-2}
$\frac{1}{x}$	$\rightarrow$	$\frac{x^{-1}}{-1} = \frac{-1}{x}$

$$\frac{-1}{xy} + \left[\log x \cdot \left(\frac{-1}{x}\right) - \int \left(\frac{1}{x}\right)\left(\frac{-1}{x}\right) dx\right] = C$$

$$\frac{-1}{xy} - \frac{\log x}{x} + \int x^{-2}\, dx = C.$$

$$\frac{-1}{xy} - \frac{\log x}{x} + \frac{x^{-1}}{-1} = C.$$

$$\frac{-1}{xy} - \frac{\log x}{x} - \frac{1}{x} = C$$

$$-\frac{1}{x}\left[\frac{1}{y} + \log x + 1\right] = C.$$

$$\frac{-1}{x}\left[\frac{1+y\cdot\log x + y}{y}\right] = C$$

$$\frac{-1}{xy}\left[1+y+y\cdot\log x\right] = C.$$

(14) $(2x\log x - xy)\, dy + 2y\, dx = 0.$

$2y\cdot dx + (2x\log x - xy)\, dy = 0 \longrightarrow ①$

<u>Sol:-</u> - Equ^n ① is an Exact form of $Mdx + Ndy = 0$

~~where $M = 2x\cdot\log x - xy$ and $N = 2y$~~

~~$\frac{\partial M}{\partial y} = 0 - x\cdot(1) = -x$; $\frac{\partial N}{\partial x} = 0 = 0$~~

~~$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$~~

~~Hence equ^n ① is non-Exact.~~

~~This can be reduced to Exact by multiplying an Integrating factor.~~

~~$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -x - 0 = -x.$~~

~~$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-x}{2y}$~~

where $M = 2y$ and $N = 2x\log x - xy$

$\frac{\partial M}{\partial y} = 2.(1) = 2$

$\frac{\partial N}{\partial x} = 2.\left[x\cdot\frac{1}{x} + \log x(1)\right] - y(1) = 2(1+\log x) - y$

$$\boxed{\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence equ^n ① is non-Exact.

This can be reduced to Exact by multiplying an Integrating factor

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = 2-[2(1+\log x)-y]$$

$$= \cancel{2}-\cancel{2}-2\log x+y$$

$$= y-2\log x$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{y-2\log x}{2x\log x-xy} = \frac{-(\cancel{2\log x-y})}{x(\cancel{2\log x-y})} = \frac{-1}{x}$$

Now. I.F $e^{\int f(x)dx} = e^{\int -\frac{1}{x}dx}$

$$= e^{-\log x}$$

$$= e^{\log_e(x)^{-1}}$$

$$= \frac{1}{x}$$

from ①,

$$\left(\frac{2y}{x}\right)dx+\left(\frac{2x\log x-xy}{x}\right)dy=0$$

$$\left(\frac{2y}{x}\right)dx+\left(\frac{2\cancel{x}\log x}{\cancel{x}}-\frac{\cancel{x}y}{\cancel{x}}\right)dy=0$$

$$\left(\frac{2y}{x}\right)dx+(2\log x-y)\,dy=0 \rightarrow ②$$

Equⁿ ② is an Exact form of $Mdx+Ndy=0$

where $M=\frac{2y}{x}$ and $N=2\log x-y$

$$\frac{\partial M}{\partial y}=\frac{2}{x}(1) = \frac{2}{x} \qquad \frac{\partial N}{\partial x}=2\cdot\left(\frac{1}{x}\right)-0 = \frac{2}{x}$$

$$\boxed{\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

Clearly Equⁿ ② is an Exact.

Now the solⁿ of Equⁿ ② is $\int M dx+\int N dy=c$

$$\int\left(\frac{2y}{x}\right)dx+\int(2\log x-y)\,dy=c$$

$$2y\int\frac{1}{x}dx+\int 2\log x\cdot dy-\int y\,dy=c$$

$$2y\cdot\log x+0-\frac{y^2}{2}=c$$

$$2y\log x-\frac{y^2}{2}=c$$

$$\frac{4y\cdot\log x-y^2}{2}=c$$

$$4y\log x-y^2=2c$$

$$4y\log x-y^2=C.$$

Thursday
26/09/19

Inspection Method

(3) $y(2xy+e^x)\,dx = e^x\,dy$

Sol: $(2xy^2+ye^x)\,dx = e^x dy$

$2xy^2\,dx + ye^x\,dx = e^x\,dy$

$2xy^2\,dx + ye^x\,dx - e^x\,dy = 0$

$\frac{2xy^2}{y^2}dx + \frac{ye^x dx - e^x dy}{y^2} = 0$

$2x\,dx + d\left(\frac{e^x}{y}\right) = 0$

$2\int x\,dx + \int d\left(\frac{e^x}{y}\right) = c$

$\cancel{2}\left(\frac{x^2}{\cancel{2}}\right) + \frac{e^x}{y} = c$

$x^2 + \frac{e^x}{y} = c.$

(4) $(y\log y - 2xy)\,dx + (x+y)\,dy = 0$

Sol:- $y\log y\,dx - 2xy\,dx + x\,dy + y\,dy = 0.$

$y\log y\,dx + y\,dy + x.dy - 2xy\,dx = 0.$

$y\log y\,dx + x.dy - 2xy\,dx + y\,dy = 0$

$\frac{\cancel{y}\log y\,dx}{\cancel{y}} + \frac{x}{y}dy - \frac{2x\cancel{y}}{\cancel{y}}dx + \frac{\cancel{y}}{\cancel{y}}dy = 0$

$\log y.dx + \frac{1}{y}.x\,dy - 2x\,dx + dy = 0$

$d.(\log y.x) - 2x\,dx + dy = 0$

$\int d(\log y.x) - 2\int x\,dx + \int(1)\,dy = c$

$\log y.x - \cancel{2}.\frac{x^2}{\cancel{2}} + y = c$

$x.\log y - x^2 + y = c.$

(7) $x\,dy - y\,dx = (4x^2+y^2)\,dy$

Sol:- $x\,dy - y\,dx = 4x^2\,dy + y^2\,dy$

$x\,dy - y\,dx - 4x^2\,dy + y^2\,dy = 0$

(9) $(x+y)^2 \cdot \left(x\frac{dy}{dx}+y\right) = xy\left(1+\frac{dy}{dx}\right)$

Sol:- $(x+y)^2\left(\frac{x\,dy+y\,dx}{dx}\right) = xy\left(\frac{dx+dy}{dx}\right)$

$$(x+y)^2 \cdot (x\,dy+y\,dx) = xy\,(dx+dy).$$

$$\frac{x\,dy+y\,dx}{xy} = \frac{dx+dy}{(x+y)^2}$$

$$d\left(\log(xy)\right) = -\left(\frac{-1}{(x+y)^2}\right)(dx+dy)$$

$$d\left(\log(x\cdot y)\right) = -d\left(\frac{1}{x+y}\right)$$

$$\int d\left(\log(x.y)\right) + \int d\left(\frac{1}{x+y}\right) = C$$

$$\log(x.y) + \frac{1}{x+y} = c.$$

(7) $x\,dy - y\,dx = (4x^2+y^2)\,dy$

Sol:- $x\,dy - y\,dx = ((2x)^2+y^2)\,dy$

$$\frac{x\,dy-y\,dx}{(2x)^2+y^2} = dy$$

$$\frac{1}{2}\,d\left(\tan^{-1}\left(\frac{y}{2x}\right)\right) = dy$$

$$\frac{1}{2}d\left(\tan^{-1}\left(\frac{y}{2x}\right)\right) - dy = 0$$

$$\frac{1}{2}\int d\left(\tan^{-1}\left(\frac{y}{2x}\right)\right) - \int (1)\,dy = C$$

$$\frac{1}{2}\tan^{-1}\left(\frac{y}{2x}\right) - y = C$$

$\because\ d\left(\tan^{-1}\left(\frac{y}{2x}\right)\right) = \frac{1}{1+\left(\frac{y}{2x}\right)^2}\left[\frac{2x.dy - y.2dx}{(2x)^2}\right]$

$$= \frac{2\,(x\,dy-y\,dx)}{\left(1+\frac{y^2}{(2x)^2}\right)(2x)^2}$$

$$= \frac{2\,(x\,dy-y\,dx)}{\frac{(2x)^2+y^2}{\cancel{(2x)^2}}\,\cancel{(2x)^2}}$$

$$\frac{1}{2}d\left(\tan^{-1}\left(\frac{y}{2x}\right)\right) = \frac{x\,dy-y\,dx}{(2x)^2+y^2}$$

(10) $x\,dy - y\,dx = x\sqrt{x^2-y^2}\,dx$

Sol:- $x\,dy - y\,dx = x\sqrt{x^2\left(1-\frac{y^2}{x^2}\right)}\,dx$

$$x\,dy - y\,dx = x^2\sqrt{1-\left(\frac{y}{x}\right)^2}\,dx$$

$$\frac{x\,dy-y\,dx}{x^2\sqrt{1-\left(\frac{y}{x}\right)^2}} = dx$$

$$d\left(\sin^{-1}\left(\frac{y}{x}\right)\right) - dx = 0.$$

$$\int d\left(\sin^{-1}\left(\frac{y}{x}\right)\right) - \int (1)\,dx = c$$

$$\sin^{-1}\left(\frac{y}{x}\right) - x = c.$$

(5) $x\,dy - y\,dx = xy^2\,dx$

Sol:- $-(y\,dx - x\,dy) = xy^2\,dx$

$$\frac{y\,dx - x\,dy}{y^2} = -x\,dx.$$

$$d\left(\frac{x}{y}\right) + x\,dx = 0$$

$$\int d\left(\frac{x}{y}\right) + \int x\,dx = c$$

$$\frac{x}{y} + \frac{x^2}{2} = c.$$

(6) $x\,dy = (x^2y^2 - y)\,dx.$

Sol:- $x\,dy = x^2y^2\,dx - y\,dx.$

$$x\,dy + y\,dx = x^2y^2\,dx$$

$$x\,dy + y\,dx = (xy)^2\,dx$$

$$\frac{x\,dy + y\,dx}{(xy)^2} = dx$$

$$-d\left(\frac{1}{xy}\right) = dx$$

$$dx + d\left(\frac{1}{xy}\right) = 0$$

$$\int (1)\,dx + \int d\left(\frac{1}{xy}\right) = c$$

$$x + \frac{1}{xy} = c$$

(8) $(y + y^2\cos x)\,dx - (x - y^3)\,dy = 0$

Sol:- $y\,dx + y^2\cos x\,dx - x\,dy + y^3\,dy = 0.$

$$y\,dx - x\,dy + y^3\,dy = -y^2\cos x\,dx.$$

$$\frac{y\,dx - x\,dy}{y^2} + \frac{y^3\,dy}{y^2} = -\cos x\,dx$$

$$d\left(\frac{x}{y}\right) + y\,dy + \cos x\,dx = 0.$$

$$\int d\left(\frac{x}{y}\right) + \int y\,dy + \int \cos x\,dx = c$$

$$\frac{x}{y} + \frac{y^2}{2} + \sin x = c.$$

(11) $x\,dx + y\,dy - a^2 d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0.$

Sol:- $x\,dx + y\,dy - a^2\,d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0$

$$\int x\,dx + \int y\,dy - a^2 \int d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = C$$

$$\frac{x^2}{2} + \frac{y^2}{2} - a^2 \tan^{-1}\left(\frac{y}{x}\right) = C.$$

APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

Monday 30/09/2019

(5) $y^2 = \frac{x^3}{a-x}$ (orthogonal trajectory)

Sol:

$$y^2 = \frac{x^3}{a-x} \rightarrow ①$$

$$y^2(a-x) = x^3 \rightarrow ①$$

Diff. Equn ① w.r.t. 'x'

$$\frac{d}{dx}(y^2(a-x)) = \frac{d}{dx}(x^3)$$

$$y^2(0-1) + (a-x)2y\frac{dy}{dx} = 3x^2$$

$$-y^2 + (a-x)2y\frac{dy}{dx} = 3x^2$$

$$2y \cdot \frac{dy}{dx}(a-x) - y^2 = 3x^2$$

$$2y \cdot \frac{dy}{dx}(a-x) = 3x^2 + y^2$$

$$2yy'(a-x) = 3x^2 + y^2$$

$$\boxed{a-x = \frac{3x^2+y^2}{2yy'}}$$

$$a = \frac{3x^2+y^2}{2yy'} + x$$

from ①,

$$y^2\left(\frac{3x^2+y^2}{2yy'}\right) = x^3$$

$$y^{\cancel{2}}(3x^2+y^2) = 2\cancel{y}y'x^3$$

$$3x^2y + y^3 = 2 \cdot \frac{dy}{dx}x^3 \rightarrow ②$$

Replace $\cancel{\frac{dy}{dx}}$ ~~by~~ $-\frac{dx}{dy}$ by $\frac{dy}{dx}$.

$$3x^2y + y^3 = 2\frac{-dx}{dy}x^3$$

$$3x^2y + y^3 = -2 \cdot \frac{dx}{dy}x^3$$

$$-2x^3\frac{dx}{dy} = 3x^2y + y^3 \rightarrow ③$$

$$-2x^3 \cdot dx = (3x^2y + y^3)dy$$

$$(3x^2y + y^3)dy + 2x^3 \cdot dx = 0$$

$$\frac{dx}{dy} = \frac{-(3x^2y+y^3)}{2x^3}$$

$$\frac{dx}{dy} = \frac{-3x^2y}{2x^3} - \frac{y^3}{2x^3}$$

$$\frac{dx}{dy} = -\frac{3y}{2x} - \frac{y^3}{2x^3}$$

$$\frac{dx}{dy} + \left(\frac{3}{2x}\right)y = \frac{-y^3}{2} \cdot x^{-3}. \quad \text{(Bernoulli's)}$$

put $y = vx \Rightarrow \boxed{v = \frac{y}{x}}$

$\underline{dy} = x\,dv$

$$\frac{dx}{x\,dv} = \frac{-(3x^2(vx)^2 + (vx)^3)}{2x^3}$$

$$\frac{dx}{x.dv} = \frac{-(3x^3v^2 + v^3x^3)}{2x^3}$$

$$\frac{1}{x}\cdot\frac{dx}{dv} = -\frac{\cancel{x^3}(3v^2+v^3)}{2\cancel{x^3}}$$

$$-\frac{2}{x}.dx = (3v^2+v^3)dv$$

$$-2\int\frac{1}{x}dx = 3\int v^2dv + \int v^3 dv$$

$$-2\log x = 3\left(\frac{v^2}{2}\right) + \left(\frac{v^4}{4}\right) + c$$

$$-2\log x = \frac{3}{2}(v^2) + \frac{1}{4}(v)^4 + c$$

$$-2\log x = \frac{3}{2}\left(\frac{y^2}{x^2}\right) + \frac{1}{4}\left(\frac{y}{x}\right)^4 + c.$$

(6) $y = \frac{x^3 - a^3}{3x}$

Sol:- $3xy = x^3 - a^3 \rightarrow ①$

Diff. equ^n ① w.r.to 'x'.

$$3\left[x\cdot\frac{dy}{dx} + y\cdot\frac{dx}{dx}\right] = 3x^2 - 0.$$

$$\cancel{3}\left[x\cdot\frac{dy}{dx} + y\cdot\right] = \cancel{3}x^2$$

$$x\cdot\frac{dy}{dx} + y = x^2 \rightarrow ②$$

Replace $-\frac{dx}{dy}$ by $\frac{dy}{dx}$

$$x \cdot \left(-\frac{dx}{dy}\right) + y = x^2$$

$$y - \frac{dx}{dy}(x) = x^2$$

$$y - \frac{dx}{dy} \cdot x = x^2$$

$$x\frac{dx}{dy} = y - x^2$$

$$x\,dx = (y - x^2)\,dy$$

$$x\,dx - (y - x^2)\,dy = 0.$$

$M = x$. and $N = -(y - x^2)$

$\frac{\partial M}{\partial y} = 0$ $\qquad \frac{\partial N}{\partial x} = -(0 - 2x)$

$= 2x.$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Hence eqn ② is non exact.

This can be reduced to exact by multiplying an integrating factor.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 - 2x = \underline{-2x}$ $\qquad \Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-2\cancel{x}}{\cancel{x}} = -2$

$$I.F\; e^{-\int g(y)\,dy} = \cancel{e^{\int 2y\,dy}} \qquad = e^{+2\int (1)\,dy}$$

$$\cancel{= e^{\frac{2y^2}{2}}}$$

$$= e^{+2y}.$$

$$\cancel{= e^{y^2}}$$

$$e^{2y}\left[x\cdot dx - (y - x^2)\,dy\right] = 0$$

$$e^{2y}\int x\,dx - \int e^{2y}\,y\,dy + \int \frac{x^2}{e^{2y}}\,dy = 0.$$

$$e^{2y}\left(\frac{x^2}{2}\right) - \left[\frac{e^{2y}}{2}\cdot y - \frac{e^{2y}}{4}\right] + 0 = 0$$

$$\frac{1}{2}\cdot x^2 e^{2y} - \left[\frac{e^{2y}}{2}\cdot y - \frac{1}{4}\cdot e^{2y}\right] = 0.$$

$$\frac{1}{2}e^{2y}\left(x^2 - y + \frac{1}{2}\right) = 0.$$

$$\frac{1}{2}e^{2y}\left(x^2 - y + \tfrac{1}{2}\right) = 0.$$

(7) $y^2 = ax^3$ $\rightarrow$ ①

Sol:- diff. eqn ① w.r.to 'x'

$$2y \cdot \frac{dy}{dx} = a \cdot 3x^2$$

$$2y \cdot \frac{dy}{dx} = 3ax^2$$

$$2y \cdot \frac{dy}{dx} = 3ax^2$$

$$a = \frac{2y}{3x^2} \cdot \frac{dy}{dx}$$

$$\boxed{a = \frac{2yy'}{3x^2}}$$

from①,

$$y^{\cancel{2}} = \left(\frac{2\cancel{y}y'}{3\cancel{x^2}}\right)x^{\cancel{3}}$$

$$y = \frac{2y'x}{3}$$

$$3y = 2xy'$$

$$3y = 2x\frac{dy}{dx}$$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$$3y = 2x\left(-\frac{dx}{dy}\right)$$

$$3y + 2x \cdot \frac{dx}{dy} = 0$$

$$2x \cdot dx = 3y\,dy$$

$$\frac{\cancel{2}x^2}{\cancel{2}} = 3\frac{y^2}{2} + c$$

$$x^2 = \frac{y^2}{2} + c.$$

(8) $y = c(\sec x + \tan x)$ $\rightarrow$ ①

Sol:- differentiate with respect to 'x'

$$\frac{dy}{dx} = c(\sec x \cdot \tan x + \sec^2 x)$$

$$y' = c\left(\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} + \frac{1}{\cos^2 x}\right)$$

$$y' = c\left(\frac{\sin x + 1}{\cos^2 x}\right)$$

$$y' = c\left(\frac{\sin x+1}{1-\sin^2 x}\right)$$

$$y' = c\left(\frac{\cancel{\sin x+1}}{(1+\sin x)(1-\sin x)}\right)$$

$$y' = \frac{c}{1-\sin x}.$$

$$\boxed{c = (1-\sin x)\,y'.}$$

from ①,

$$y = (1-\sin x)\,y'(\sec x+\tan x)$$

$$y = y'(1-\sin x)\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right)$$

$$y = y'(1-\sin x)\left(\frac{1+\sin x}{\cos x}\right)$$

$$y = y'\left(\frac{1-\sin^2 x}{\cos x}\right)$$

$$y = y'\left(\frac{\cos^2 x}{\cancel{\cos x}}\right)$$

$$y = y'\cos x.$$

$$y = \frac{dy}{dx}\cdot\cos x. \Rightarrow \text{Replace } \frac{dy}{dx} = -\frac{dx}{dy}$$

$$\frac{1}{\cos x}\cdot dx = \frac{1}{y}\,dy \qquad\qquad y = -\frac{dx}{dy}\cdot\cos x$$

$$\int \sec x\, dx = \int \frac{1}{y}\,dy \qquad\qquad y\,dy = -\cos x\, dx$$

$$\log(\sec x+\tan x) = \log y+\log c \qquad\qquad \int y\,dy = -\int \cos x\,dx$$

$$\log(\sec x+\tan x) = \log(c\cdot y) \qquad\qquad \frac{y^2}{2} = -\sin x + c$$

$$\boxed{cy = \sec x+\tan x} \qquad\qquad \boxed{\frac{y^2}{2}+\sin x = c}$$

③ Find the particular no. of orthogonal trajectories $x^2+cy^2=1$ passing through the point (2,1).

Soln $\quad x^2+cy^2=1 \rightarrow ①$

diff w.r. to 'x'

$$2x + c\cdot 2y\frac{dy}{dx} = 0.$$

$$\cancel{2}x = -\cancel{2}cy\frac{dy}{dx}$$

$$x = -cy\frac{dy}{dx}$$

$x = -c\, y \cdot y'$

$$\boxed{c = \frac{-x}{y y'}}$$

from ①,

$$x^2 + \left(\frac{-x}{\cancel{y} y'}\right) y^{\cancel{2}} = 1$$

$$x^2 - \frac{xy}{y'} = 1$$

$$x^2 = 1 + \frac{xy}{y'}$$

$$x^2 - 1 = xy \cdot \frac{1}{y'}$$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$$x^2 - 1 = xy \cdot \frac{1}{\frac{-dx}{dy}}$$

$$x^2 - 1 = xy\left(-\frac{dy}{dx}\right)$$

$$\frac{x^2-1}{x} \cdot dx = -y \cdot dy$$

$$\left(\frac{x^{\cancel{2}}}{\cancel{x}} - \frac{1}{x}\right) dx = -y \cdot dy$$

$$\int x \cdot dx - \int \frac{1}{x} \cdot dx = -\int y\, dy$$

$$\frac{x^2}{2} - \log x = \frac{y^2}{2} + c$$

$$\frac{x^2}{2} - \log x = -\frac{y^2}{2} + c$$

$$\frac{x^2}{2} + \frac{y^2}{2} - \log x = c.$$

$$\frac{x^2}{2} + \frac{y^2}{2} = \log x + c$$

Given that
the curve passes through the point (2, 1)

$$\frac{(2)^{\cancel{2}}}{\cancel{2}} + \frac{(1)^2}{2} = \log 2 + c$$

$$2 + \frac{1}{2} = \log 2 + c$$

$$\frac{5}{2} = 0.301 + c$$

$2.5 = 0.301 + C$

$C = 2.5 - 0.301$

$\boxed{C = 2.199}$

Approximately $C = 2.2$

(9) $x^2+y^2+2gx+c=0$ where 'g' is the parameter.

Sol:- $x^2+y^2+2gx+c=0 \to ①$

diff. w. r. to 'x'.

$2x+2y.\frac{dy}{dx}+2g+0=0$

$x+y\frac{dy}{dx}+g=0$

$x+y.y'+g=0.$

$g = -(x+yy')$

from①,

$x^2+y^2+2(-(x+yy'))x+c=0 \longrightarrow ②.$

$x^2+y^2-2x^2-2xyy'+c=0$

$-x^2+y^2-2xyy'+c=0$

$c = x^2-y^2+2xyy'$

from②

$\cancel{x^2+y^2+2gx+x^2-y^2+2xyy'=0}$ $-2(x+yy')$

$\cancel{2x+2yy'+2(2x)}$

$x^2+\cancel{y^2}-2x-2yy'+x^2-\cancel{y^2}+2xyy'=0$

$2x^2-2x-2yy'+2xyy'=0$

$2x^2-2x-2y\frac{dy}{dx}+2xy\frac{dy}{dx}=0$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$2x^2-2x+2y\frac{dx}{dy}-2xy\frac{dx}{dy}=0.$

$2x^2-2x+(2y-2xy)\frac{dx}{dy}=0.$

$\cancel{2}(x^2-x) = -\cancel{2}(y-xy)\frac{dx}{dy}$

$x^2-x = -y(1-x)\frac{dx}{dy}.$

$x(x-1) = y(x-1)\frac{dx}{dy}$

$$\frac{1}{y}\cdot dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\log y = \log x + \log c$$

$$\cancel{\log} y = \cancel{\log}(c\cdot x)$$

$$\boxed{y = c\cdot x}$$

(10) $y^2 = 4ax$

Sol: $y^2 - 4ax = 0 \rightarrow (1)$

diff. w. r. to 'x'

$$2yy' - 4a = 0.$$

$$\cancel{4}^{2}a = \cancel{2}y\cdot y'$$

$$a = \frac{yy'}{2}$$

from (1), $y^2 - \cancel{4}^{2}x\left(\frac{yy'}{\cancel{2}}\right) = 0$

$$y^2 - 2xy\cdot y' = 0.$$

$$y^2 - 2xy\frac{dy}{dx} = 0$$

Replace $\frac{dy}{dx} = \frac{-dx}{dy}$.

$$y^2 + 2xy\cdot\frac{dx}{dy} = 0.$$

$$y\cancel{^2} = -2x\cancel{y}\cdot\frac{dx}{dy}$$

$$y\,dy = -2x\,dx.$$

$$\int y\,dy = -2\int x\,dx$$

$$\frac{y^2}{2} = -\cancel{(2)}\frac{x^2}{\cancel{2}} + c.$$

$$\frac{x^2}{1} + \frac{y^2}{2} = c.$$

(11) $xy = c.$

Sol: $xy - c = 0 \rightarrow (1)$

diff. w. r. to 'x'

$$\left[x\cdot\frac{dy}{dx} + y(1)\right] - 0 = 0.$$

$$xy' + y = 0$$

Replace $\frac{dy}{dx} = \frac{-dx}{dy}$

$$x\cdot\left(-\frac{dx}{dy}\right) + y = 0.$$

$$\cancel{-}x \cdot dx = \cancel{-}y \cdot dy.$$

$$\int x \cdot dx = \int y\, dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + C.$$

$$\frac{x^2}{2} - \frac{y^2}{2} = C.$$

(3) $e^x + e^{-y} = C.$

Sol:- $e^x + e^{-y} - C = 0 \rightarrow ①$

diff w. r. to 'x'.

$$e^x + e^{-y}\left(-\frac{dy}{dx}\right) - 0 = 0.$$

$$e^x - e^{-y}\cdot\frac{dy}{dx} = 0$$

Replace $\frac{dy}{dx} = \frac{-dx}{dy}$

$$e^x + e^{-y}\,\frac{dx}{dy} = 0.$$

$$e^x = -e^{-y}\cdot\frac{dx}{dy}.$$

$$\frac{1}{e^{-y}}\, dy = \frac{-1}{e^x}\cdot dx$$

$$e^{-y}\, dy = -e^{-x}\, dx$$

$$\int e^{-y}\, dy = -\int e^{-x} dx$$

$$e^{-y}(-1) = -e^{-x}\cdot(-1) + C$$

$$-e^{-y} = e^{-x} + C.$$

$$e^{-x} + e^{-y} + C = 0.$$

(4) $x^2 + y^2 = c^2.$

Sol:- $x^2 + y^2 - c^2 = 0 \rightarrow ①$

differentiate w. r. to 'x'

$$2x + 2y\frac{dy}{dx} - 0 = 0$$

$$x + y\cdot\frac{dy}{dx} = 0$$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$$x - y \cdot \frac{dx}{dy} = 0$$

$$x = y \frac{dx}{dy}$$

$$\frac{1}{y} dy = \frac{1}{x} dx$$

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$\log y = \log x + \log c$$

$$\log y = \log (c \cdot x)$$

$$\boxed{y = cx}$$

Tuesday
01/10/2019

Polar Form

(1). $r = a(1+\cos\theta)$

Sol:- $r = a + a\cos\theta \longrightarrow (1)$

diff. w. r. to 'θ'.

$$\frac{dr}{d\theta} = 0 + a \cdot (-\sin\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta \Rightarrow a = \frac{-1}{\sin\theta} \cdot \frac{dr}{d\theta}$$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

from (1),

$$r = \frac{-1}{\sin\theta} \frac{dr}{d\theta} (1+\cos\theta)$$

$$r = \frac{-1}{\sin\theta} \frac{dr}{d\theta} - \cot\theta \frac{dr}{d\theta}$$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$r = -\operatorname{cosec}\theta \left(-r^2 \frac{d\theta}{dr}\right) - \cot\theta \left(-r^2 \frac{d\theta}{dr}\right)$$

$$\cancel{r} = r^{\cancel{2}} \left[\operatorname{cosec}\theta \cdot \frac{d\theta}{dr} + \cot\theta \cdot \frac{d\theta}{dr}\right]$$

$$\frac{1}{r} = (\operatorname{cosec}\theta + \cot\theta) \frac{d\theta}{dr}$$

$$\frac{1}{r} dr = (\operatorname{cosec}\theta + \cot\theta) d\theta$$

$$\int \frac{1}{r} dr = \int \operatorname{cosec}\theta d\theta + \int \cot\theta \cdot d\theta.$$

$$\log r = \log(\operatorname{cosec}\theta - \cot\theta) + \log(\sin\theta) + \log c.$$

$$\cancel{\log} r = \cancel{\log}\left[(\operatorname{cosec}\theta - \cot\theta)(\sin\theta) \cdot c\right]$$

$r = (\text{cosec}\theta \cdot \sin\theta - \cot\theta \cdot \sin\theta)c$

$r = \left(\frac{1}{\sin\theta} \cdot \sin\theta - \frac{\cos\theta}{\sin\theta} \sin\theta\right) c$

$r = (1 - \cos\theta)\, c$

(2) $r^n \sin n\theta = a^n$

Sol:- $\log(r^n \cdot \sin n\theta) = \log a^n$

$\log r^n + \log \sin n\theta = n \cdot \log a$

$n \cdot \log r + \log \sin n\theta = n \cdot \log a$

diff. w. r. to 'θ'

$n \cdot \frac{1}{r} \frac{dr}{d\theta} + \frac{1}{\sin n\theta} (\cos n\theta)\, n = 0$

$\frac{\cancel{n}}{r} \frac{dr}{d\theta} = -\cancel{n} \cdot \cot n\theta$

$\frac{1}{r} \frac{dr}{d\theta} = -\cot n\theta$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$\frac{1}{\cancel{r}} (\cancel{-}r^{\cancel{2}}) \frac{d\theta}{dr} = \cancel{-}\cot n\theta$

$r \cdot \frac{d\theta}{dr} = \cot n\theta$

$\frac{1}{\cot n\theta} d\theta = \frac{1}{r} dr$

$\int \tan n\theta \, d\theta = \int \frac{1}{r} dr$

$\frac{\log(\sec n\theta)}{n} = \log r + \log c$

$\frac{1}{n} \cdot \log(\sec n\theta) = \log(r \cdot c)$

$\cancel{\log} (\sec n\theta)^{1/n} = \cancel{\log} (c \cdot r)$

$(\sec n\theta)^{1/n} = c \cdot r$

$\sec n\theta = (cr)^n$

$\sec n\theta = c \cdot r^n$

(3). $r^2 = a^2(\cos 2\theta)$

Sol:- $\log r^2 = \log(a^2 \cos 2\theta)$

$2\log r = \log a^2 + \log \cos 2\theta$
$(2\log a)$

diff. w. r. to 'θ'

$2 \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\cos 2\theta}(-\sin 2\theta)\, 2.$

$\cancel{2} \cdot \frac{1}{r}\frac{dr}{d\theta} = -(\tan 2\theta)\, \cancel{2}$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$\frac{1}{r}\left(\cancel{-}r^{\cancel{2}} \frac{d\theta}{dr}\right) = \cancel{-}\tan 2\theta$

$\frac{1}{\tan 2\theta} \cdot d\theta = \frac{1}{r} \cdot dr$

$\int \cot 2\theta \cdot d\theta = \int \frac{1}{r}\, dr$

$\frac{\log(\sin 2\theta)}{2} = \log r + \log c$

$\frac{1}{2}\log(\sin 2\theta) = \log(c \cdot r)$

$\cancel{\log}(\sin 2\theta)^{1/2} = \cancel{\log}(c \cdot r)$

$\sin 2\theta = (c\, r)^2$

$\sin 2\theta = c \cdot r^2.$

(4) $r^n = a \sin n\theta.$

Sol:- $\log r^n = \log(a \sin n\theta)$

$n \cdot \log r = \log a + \log \sin n\theta.$

diff. w. r. to 'θ'.

$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\sin n\theta}(\cos n\theta)\, n$

$\cancel{n} \cdot \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta \cdot \cancel{n}$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$\frac{1}{r}\left(-r^{\cancel{2}} \frac{d\theta}{dr}\right) = \cot n\theta.$

$-\frac{1}{\cot n\theta}\, d\theta = \frac{1}{r}\, dr$

$-\int \tan n\theta\, d\theta = \int \frac{1}{r}\, dr$

$$-\frac{\log(\sec n\theta)}{n} = \log r + \log c$$

$$-\frac{1}{n}\log(\sec n\theta) = \log(c.r)$$

$$\log(\sec n\theta)^{-1/n} = \log(c.r)$$

$$\sec n\theta = (cr)^{-n}$$

$$\sec n\theta = c^{-n}.r^{-n}$$

$$\sec n\theta = \frac{1}{c^n.r^n}$$

$$c.\sec n\theta = \frac{1}{r^n}.$$

(5) $r = \frac{2a}{1+\cos\theta}$

Sol: $r(1+\cos\theta) = 2a$

$(r + r.\cos\theta)$

diff. w.r. to 'θ'

$$\frac{dr}{d\theta} + \left[r.(-\sin\theta) + \cos\theta\frac{dr}{d\theta}\right] = 0.$$

$$\frac{dr}{d\theta} - r\sin\theta + \cos\theta\frac{dr}{d\theta} = 0.$$

$$(1+\cos\theta)\frac{dr}{d\theta} - r\sin\theta = 0.$$

Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$(1+\cos\theta)\left(-r^{\cancel{2}}\frac{d\theta}{dr}\right) = \cancel{r}\sin\theta.$$

$$-\frac{1+\cos\theta}{\sin\theta}\,d\theta = \frac{1}{r}dr.$$

$$-(\operatorname{cosec}\theta + \cot\theta)d\theta = \frac{1}{r}dr$$

$$-\int\operatorname{cosec}\theta\,d\theta - \int\cot\theta.d\theta = \int\frac{1}{r}dr$$

$$-\log(\operatorname{cosec}\theta - \cot\theta) - \log(\sin\theta) = \log r + \log c.$$

$$-\left[\log\frac{(\operatorname{cosec}\theta - \cot\theta)}{\cancel{\sin\theta}}(\sin\theta)\right] = \log r + \log c$$

$$\log\left(\frac{\operatorname{cosec}\theta - \cot\theta}{\cancel{\sin\theta}}\right)^{-1}\frac{1}{\sin\theta} = \log(r.c)$$

$$\frac{1}{(\sin\theta)}\frac{\cancel{\sin\theta}\,1}{(\operatorname{cosec}\theta - \cot\theta)} = c.r.$$

$$\frac{1}{\sin\theta}\cdot\frac{1}{\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}} = cr.$$

$$\left(\frac{1}{\sin\theta}\right)\frac{1}{\frac{1-\cos\theta}{\sin\theta}} = cr.$$

$$cr = \frac{1}{1-\cos\theta}$$

(or)

$$\frac{1}{cr} = 1-\cos\theta$$

$$\frac{1}{r} = c(1-\cos\theta)$$

(6) $r = a(1-\cos\theta)$

Sol:- $r = a(1-\cos\theta) \rightarrow ①$

diff. w.r.to 'θ'

$$\frac{dr}{d\theta} = a(0-(-\sin\theta))$$

$$\frac{dr}{d\theta} = a\cdot\sin\theta$$

$$a = \frac{1}{\sin\theta}\cdot\frac{dr}{d\theta}$$

from①, $r = \frac{1}{\sin\theta}\cdot\frac{dr}{d\theta}(1-\cos\theta)$

Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$.

$$\cancel{r} = \frac{1}{\sin\theta}\cdot(-r^{\cancel{2}})\frac{d\theta}{dr}(1-\cos\theta)$$

$$\frac{1}{r}dr = \frac{-(1-\cos\theta)}{\sin\theta}d\theta$$

$$-\frac{1}{r}dr = (\operatorname{cosec}\theta-\cot\theta)d\theta$$

$$-\int\frac{1}{r}dr = \int\operatorname{cosec}\theta\, d\theta - \int\cot\theta\, d\theta$$

$$-\log r = \log(\operatorname{cosec}\theta-\cot\theta) - \log(\sin\theta) + \log c$$

$$-\log r - \log c = \log\left(\frac{\operatorname{cosec}\theta-\cot\theta}{\sin\theta}\right)$$

$$-[\log(c\cdot r)] = \log(\operatorname{cosec}^2\theta - \operatorname{cosec}^2\theta\cdot\cos\theta)$$

$$\log(cr)^{-1} = \log\operatorname{cosec}^2\theta(1-\cos\theta)$$

$$\frac{1}{cr} = \operatorname{cosec}^2\theta(1-\cos\theta)$$

$$\frac{1}{r} = c\cdot\operatorname{cosec}^2\theta(1-\cos\theta)$$

(7) $r = a(1+\sin^2\theta)$

Sol:- $r = a(1+\sin^2\theta) \to ①$

diff. w. r. to 'θ'

$$\frac{dr}{d\theta} = a(0 + 2\sin\theta.\cos\theta)$$

$$\frac{dr}{d\theta} = 0 + 2a.\sin\theta.\cos\theta$$

$$\frac{dr}{d\theta} = a.\sin 2\theta$$

$$a = \frac{1}{\sin 2\theta}.\frac{dr}{d\theta}$$

from①,

$$r = \frac{1}{\sin 2\theta}.\frac{dr}{d\theta}(1+\sin^2\theta)$$

Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$\cancel{r} = \frac{1}{\sin 2\theta}(-r\cancel{^2})\frac{d\theta}{dr}(1+\sin^2\theta)$$

$$\frac{1}{r}dr = -\left(\frac{1+\sin^2\theta}{\sin 2\theta}\right)d\theta$$

$$-\frac{1}{r}dr = \left(\frac{1}{\sin 2\theta} + \frac{\sin^2\theta}{2\sin\theta\cos\theta}\right)d\theta$$

$$-\frac{1}{r}dr = \left(\operatorname{cosec}2\theta + \frac{1}{2}\tan\theta\right).d\theta$$

$$-\int\frac{1}{r}dr = \int\operatorname{cosec}2\theta\, d\theta + \frac{1}{2}\int\tan\theta\, d\theta$$

$$-\log r = \frac{\log(\operatorname{cosec}2\theta - \cot 2\theta)}{2} + \frac{1}{2}.\log(\sec\theta) + \log c$$

$$-\log r - \log c = \frac{1}{2}[\log(\operatorname{cosec}2\theta - \cot 2\theta)(\sec\theta)].$$

$$-2[\log r + \log c] = \log[(\operatorname{cosec}2\theta - \cot 2\theta)(\sec\theta)]$$

$$-2\log(rc) = \log(\operatorname{cosec}2\theta - \cot 2\theta)(\sec\theta)$$

$$\cancel{\log}(r.c)^{-2} = \cancel{\log}[(\operatorname{cosec}2\theta - \cot 2\theta)(\sec\theta)].$$

$$\frac{1}{r^2c^2} = \left(\frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta}\right)\sec\theta.$$

$$\frac{1}{r^2c} = \left(\frac{1-\cos 2\theta}{\sin 2\theta}\right).\sec\theta.$$

$$\frac{1}{r^2c} = \frac{\cancel{2}\sin^2\theta.}{\cancel{2}\sin\theta.\cos\theta}.\frac{1}{\cos\theta}.$$

$$\frac{1}{r^2} = c.\sec\theta.\tan\theta.$$

(8) $r^2 = a^2 \sin 2\theta$

Sol:- $\log r^2 = \log(a^2 \sin 2\theta)$

$2\log r = \log a^2 + \log \sin 2\theta$

$2\log r = 2.\log a + \log \sin 2\theta$

diff w.r.to 'θ'

$$2.\frac{1}{r}.\frac{dr}{d\theta} = 0 + \frac{1}{\sin 2\theta}(\cos 2\theta)2.$$

$$\cancel{2}.\frac{1}{r}\frac{dr}{d\theta} = \cot 2\theta.\cancel{2}$$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$\frac{1}{\cancel{r}}(-r^{\cancel{2}})\frac{d\theta}{dr} = \cot 2\theta.$$

$$-\frac{1}{\cot 2\theta}.d\theta = \frac{1}{r}dr.$$

$$-\int \tan 2\theta . d\theta = \int \frac{1}{r}dr.$$

$$-\frac{\log(\sec 2\theta)}{2} = \log r + \log c.$$

$$-\frac{1}{2}\log(\sec 2\theta) = \log r + \log c$$

$$\cancel{\log}(\sec 2\theta)^{-1/2} = \cancel{\log}(c.r)$$

$$\sec 2\theta = (cr)^{-2}.$$

$$\sec 2\theta = \frac{1}{c^2 r^2}$$

$$c.\sec 2\theta = \frac{1}{r^2}$$

(9) $r = a.\cos^2\theta.$

Sol:- $r = a.\cos^2\theta. \rightarrow ①$

diff. w.r.to 'θ'

$$\frac{dr}{d\theta} = a.2\cos\theta.(-\sin\theta)$$

$$\frac{dr}{d\theta} = a.-(\sin 2\theta)$$

$$a = \frac{-1}{\sin 2\theta}.\frac{dr}{d\theta}.$$

from ①, $r = \frac{-1}{\sin 2\theta}.\frac{dr}{d\theta}.\cos^2\theta.$

Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$\cancel{r} = \frac{-1}{\sin 2\theta}\cdot(-r\cancel{^2})\frac{d\theta}{dr}(\cos^2\theta)$$

$$\frac{1}{r}dr = \frac{\cos^2\theta}{\sin 2\theta}\,d\theta$$

$$\frac{1}{r}dr = \frac{\cos\cancel{^2}\theta}{2\sin\theta\cdot\cancel{\cos\theta}}\,d\theta$$

$$\frac{1}{r}dr = \frac{1}{2}\cot\theta\,d\theta$$

$$\int\frac{1}{r}dr = \frac{1}{2}\int\cot\theta\,d\theta$$

$$\log r = \frac{1}{2}\log(\sin\theta) + \log c$$

$$\log r - \log c = \frac{1}{2}\log(\sin\theta)$$

$$2\log\left(\frac{r}{c}\right) = \log(\sin\theta)$$

$$\cancel{\log}\frac{r^2}{c^2} = \cancel{\log}(\sin\theta)$$

$$r^2 = c\cdot\sin\theta$$

(10) $r = 2a(\sin\theta + \cos\theta)$

Sol:- $r = 2a(\sin\theta + \cos\theta) \rightarrow (1)$

diff. w. r. to 'θ'.

$$\frac{dr}{d\theta} = 2a(\cos\theta - \sin\theta)$$

$$2a = \frac{1}{\cos\theta - \sin\theta}\cdot\frac{dr}{d\theta}$$

from (1),

$$r = \frac{1}{\cos\theta - \sin\theta}\cdot\frac{dr}{d\theta}(\sin\theta + \cos\theta)$$

Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$.

$$\cancel{r} = \frac{1}{\cos\theta - \sin\theta}(-r\cancel{^2})\frac{d\theta}{dr}(\sin\theta + \cos\theta)$$

$$\frac{1}{r}dr = \frac{\cancel{-}(\sin\theta + \cos\theta)}{\cancel{-}(-\cos\theta + \sin\theta)}\,d\theta$$

$$\int\frac{1}{r}dr = \int\frac{\sin\theta + \cos\theta}{-\cos\theta + \sin\theta}\,d\theta$$

$$\log r = \log(\sin\theta - \cos\theta) + \log c$$

$$\cancel{\log} r = \cancel{\log}(\sin\theta - \cos\theta)\cdot c$$

$$r = c\cdot(\sin\theta - \cos\theta)$$

Thursday
10/10 Law of Natural Decay & Growth:

(4) In a certain culture of bacteria the rate of increases is proportional to the number present.
(a) If it is found that the number doubles in 4 hrs, How many may Expected at the end of 12 hrs.

Sol: We have, $y = c \cdot e^{Kt} \rightarrow ①$

Initially $t=0$ and $y = y_0$

from ①,
$$y_0 = c \cdot e^{K(0)}$$
$$= c \cdot e^{0}$$
$$y_0 = c(1)$$
$$\boxed{c = y_0}$$
$$y = y_0 e^{Kt} \rightarrow ②$$

$t = 4$ hrs and $y = 2y_0$
$$2y_0 = y_0 e^{K(4)}$$
$$e^{4K} = 2.$$
$$4K = \log 2.$$
$$K = \frac{1}{4} \cdot \log 2$$
$$\boxed{K = 0.17329}$$
$$y = y_0 e^{(0.17329)t} \rightarrow ③$$

And also $t = 12$, $y = ?$
$$y = y_0 e^{(0.17329)12}$$
$$y = y_0(8.0003076)$$
$$\boxed{y = 8y_0.}$$

(6)

Sol:- We have $y = ce^{Kt} \longrightarrow ①$

Initially $t=0$ and $y=y_0$

from①, $y_0 = c\cdot e^{K(0)}$

$= c\cdot e^{0}$

$y_0 = c(1)$

$\Rightarrow \boxed{c = y_0}$

$y = y_0 e^{Kt} \longrightarrow ②$

$t = 5$ hrs and $y = 3y_0$.

$3\cancel{y_0} = \cancel{y_0}\, e^{K(5)}$

$e^{5k} = 3$

$5k = \log 3.$

$k = \frac{1}{5}\log 3$

$\boxed{K = 0.21972}$

$y = y_0\cdot e^{(0.21972)t} \longrightarrow ③$

(a)
And also $t = 10$ hrs and $y = ?$

$y = y_0\, e^{(0.21972)10}.$

$y = y_0\,(8.999778807)$

$\boxed{y = 9y_0.}$

(b) $t = ?$ and $y = 10y_0$

$$10\cancel{y_0} = \cancel{y_0} \cdot e^{(0.21972)t}$$

$$e^{(0.21972)t} = 10.$$

$$(0.21972)t = \log 10$$

$$t = \frac{1}{0.21972} \log 10.$$

$$t = 10.4796335.$$

$\boxed{t = 10.48.}$ hrs.

$\boxed{t = 11 \;\; \cancel{hrs}}$ hrs.

(10) The rate of at which the bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hrs. In how many hours will it triple.

sol: We have $y = ce^{kt} \rightarrow (1)$

Initially $t = 0$ and $y = y_0$.

from (1),

$$y_0 = ce^{k(0)}$$

$$= ce^0$$

$$y_0 = c(1)$$

$$\Rightarrow \boxed{c = y_0}$$

$$y = y_0 e^{kt} \rightarrow (2)$$

$t = 2$ hrs and $y = 2y_0$.

$$2\cancel{y_0} = \cancel{y_0} \cdot e^{k(2)}$$

$$e^{2k} = 2$$

$$2k = \log 2$$

$$k = \frac{1}{2} \log 2.$$

$$k = 0.34657$$

$$\boxed{k = 0.3466}$$

$$y = y_0 \cdot e^{(0.3466)t} \rightarrow ③$$

And also $t = ?$ $y = 3y_0$.

$$3y_0 = y_0\, e^{(0.3466)t}$$

$$e^{(0.3466)t} = 3$$

$$(0.3466)t = \log 3$$

$$t = \frac{1}{0.3466} \log 3$$

$$t = 3.169683$$

$\boxed{t \cong 3}$ hrs.

* (a) The world population at the begining ~~__~~ 1970 was 3.6 billion. The weight of the earth is 6.586×10^{21} tones. If the population continuous to increase exponentially with a growth constant $K = 0.02$ and with time measure in years, In what year did the weight of the people equal to the weight of the earth, If we assume that the average person weight is 120 found.

decay

* (b) In a certain chemical reaction the rate of conversion of a substance, at time 't' is proportional to the quantity of the substance still untransformed at that instant. At the end of '1' hour 60 grams remain and at the end of '4' hours 21 grams. How many grams of the 1st substance was there initially.

Sol: Wehave by law of natural growth

$$y = c \cdot e^{Kt} \rightarrow ①$$

Initially $t = 0$ and $y = 3.6 \times 10^9$

$$3.6 \times 10^9 = c \cdot e^{K(0)}$$

$$3.6 \times 10^9 = c\, e^{(0)}$$

$$\boxed{c = 3.6 \times 10^9}$$

$$y = 3.6 \times 10^9\, e^{Kt} \rightarrow ②$$

Given that $k = 0.02$.

$$y = 3.6 \times 10^{9}\, e^{(0.02)t} \rightarrow ③$$

Weight of the Earth 6.586×10^{21} ~~Yards~~ tones.

Weight of the people $3.6 \times 10^{9}\, e^{(0.02)t} \times 120$ pounds

(1 ton = 2240 pounds)

weight of the Earth $6.586 \times 10^{21} \times 2240$. pounds.

weight of the people = weight of the Earth

$$3.6 \times 10^{9}\, e^{(0.02)t} \times 120 = 6.586 \times 10^{21} \times 2240$$

$$e^{(0.02)t} = \frac{6.586 \times 10^{\cancel{21}\,12} \times 224\cancel{0}}{3.6 \times \cancel{10^{9}} \times 12\cancel{0}}$$

$$= \frac{6.586 \times 10^{12} \times 224}{43.2} = \frac{1.475264 \times 10^{13}}{43.2}$$

$$e^{(0.02)t} = 3.414962963 \times 10^{13}$$

$$(0.02)t = \log(3.414962963 \times 10^{13})$$

$$(0.02)t = 31.16177286$$

$$t = \frac{31.16177286}{0.02}$$

$$t = 1558.088643$$

at $t = 1558 + 1970$

$= 3528$ year.

The rate of the population and weight of the earth are equal.

1.

We have $y = c \cdot e^{kt} \rightarrow ①$

Initially $t = 0$ and $y = 100$

from①, $100 = c \cdot e^{k(0)}$

$= c \cdot e^{0}$

$= c(1)$

$\Rightarrow \boxed{c = 100}$

$$y = 100 \cdot e^{Kt} \rightarrow ②$$

$t = 1$ and $y = 332$

$$332 = 100 \cdot e^{K(1)}$$

$1\frac{1}{2}$

$3/2$

$$e^K = \frac{332}{100}$$

$$e^K = 3.32$$

$$\boxed{K = \log(3.32)}$$

$$K = 1.19996$$

$$y = 100 \cdot e^{(1.19996)t} \rightarrow ③$$

And also. $t = 1\frac{1}{2}$ hour and $y = ?$

$$y = 100 \cdot e^{(1.19996)\,1.5}$$

$$y = 100 \times 6.0492$$

$$y = 604.92 \simeq \underline{605}.$$

(2) We have $y = c \cdot e^{kt} \rightarrow ①$

Initially $t = 0$ and $y = y_0$

$$y_0 = c\, e^{K(0)}$$

$$= c \cdot e^0$$

$$y_0 = c\,(1)$$

$$\boxed{c = y_0}$$

from ①,

$$y = y_0\, e^{Kt} \rightarrow ②$$

$t = 2$ and $y = 2y_0$

$$2y_0 = y_0\, e^{K(2)}$$

$$e^{2K} = 2$$

$$2K = \log 2$$

$$\boxed{K = \frac{1}{2} \log 2}$$

$y = y_0 \cdot e$

$\boxed{K = 0.34657}$

$y = y_0 \cdot e^{(0.34657)t} \rightarrow (3)$

And also $t = 8$ and $y = ?$

$y = y_0 \cdot e^{(0.34657)8}$

$y = y_0 (15.9995)$

$\boxed{y \cong 16 y_0}$

And also $t = ?$ and $y = 8y_0$.

$8\cancel{y_0} = \cancel{y_0} \cdot e^{(0.34657)t}$

$e^{(0.34657)t} = 8$

$(0.34657)\, t = \log 8$

$t = \frac{1}{0.34657} \log 8$

$t = 6.00006\,2157.$

$t \cong 6.1$.

$\boxed{t \cong 6 \text{ hours}}$

(5).

We have $y = ce^{kt} \rightarrow (1)$

Initially $t = 0$ and $y = y_0$

$y_0 = ce^{k(0)}$

$= c \cdot e^0$

$= c(1)$

$\Rightarrow \boxed{c = y_0}$

from (1),

$y = y_0 e^{kt} \rightarrow (2)$

$t = 50$ and $y = 2y_0$.

$2\cancel{y_0} = \cancel{y_0} \cdot e^{(50)k}$.

$$e^{K(50)} = 2$$

$$K(50) = \log 2$$

$$K = \frac{1}{50} \log 2$$

$$\boxed{K = 0.01386}$$

$$y = y_0 e^{(0.01386)t} \rightarrow ②$$

And also $t = ?$ and $y = 3y_0$.

$$3\cancel{y_0} = \cancel{y_0} \cdot e^{(0.01386)t}$$

$$e^{(0.01386)t} = 3$$

$$(0.01386)t = \log 3$$

$$t = \frac{1}{0.01386} \log 3$$

$$t = 79.2649$$

$$\boxed{t \cong 79 \text{ years}}$$

(8)

We have $y = ce^{Kt} \rightarrow ①$

Initially $t=0$ and $y=y_0$

$$y_0 = ce^{K(0)}$$

$$= c \cdot e^0$$

$$= c(1)$$

$$\boxed{c = y_0}$$

from ①, $y = y_0 e^{Kt} \rightarrow ②$

And $t = 3$ and $y = 2y_0$.

$$2\cancel{y_0} = \cancel{y_0} e^{K(3)}$$

$$e^{K(3)} = 2$$

$$K(3) = \log 2$$

$$K = \frac{1}{3} \log 2$$

$$\boxed{K = 0.23104}$$

$$y = y_0 \cdot e^{(0.23104).t} \rightarrow ③$$

And also $t = \cancel{8}\,15$ and $y = ?$

$$y = y_0 \cdot e^{(0.23104)15}$$

$$y = 31.99565\ y_0$$

$$\boxed{y \cong 32 y_0}$$

(9)

We have $y = c.e^{Kt} \rightarrow ①$

Initially $t = 0$ and $y = 100$.

$$100 = ce^{K(0)}$$

$$100 = c.e^{(0)}$$

$$100 = c.(1)$$

$$\boxed{c = 100}$$

from①, $y = 100\,e^{Kt} \rightarrow ②$

And $t = 12$ hours and $y = 400$.

$$4\cancel{00} = 1\cancel{00} \cdot e^{K(12)}$$

$$e^{K(12)} = 4$$

$$K(12) = \log 4$$

$$K = \frac{1}{12} \log 4$$

$$\boxed{K = 0.115524}$$

$$y = 100\,e^{(0.115524)t} \rightarrow ③$$

And also $t = 3$ and $y = ?$

$$y = 100\,e^{(0.115524)3}$$

$$y = 100 \times 1.41421$$

$$y = 141.421 \rightarrow \boxed{y \approx 141.}$$

Law of Natural Decay:

(4)

We have the law of natural decay is $y = ce^{-kt} \rightarrow (1)$

Initially $t=0$, $y=y_0$

$y_0 = ce^{-k(0)}$

$y_0 = ce^{(0)}$

$y_0 = c\,(1)$

$\Rightarrow \boxed{c = y_0}$

$y = y_0 e^{-kt} \rightarrow (2)$

$t = 1500$ and $y = \frac{y_0}{2}$

$\frac{\cancel{y_0}}{2} = \cancel{y_0} e^{-k(1500)}$

$\frac{1}{2} = e^{-k(1500)}$

$e^{-k(1500)} = 0.5$

$-k(1500) = \log(0.5)$

$k = \frac{-1}{1500}\log(0.5)$

$k = -(-4.620981204 \times 10^{-4})$

$k = 0.0004620981204$

$k = 0.000462$

$y = y_0 e^{-(0.000462)t} \rightarrow (3)$

(a) $t = 4500$ and $y = ?$

$y = y_0 e^{-(0.000462)(4500)}$

$y = y_0 (0.125055204)$

$y = 0.125\, y_0$

$y = 12.5\, y_0$.

(b) $t = ?$ and $y = \frac{1}{10} y_0$.

$$\frac{1}{10}\cancel{y_0} = \cancel{y_0} \cdot e^{-(0.000462)t}$$

$$e^{-(0.000462)t} = 0.1$$

$$-(0.000462)t = \log 0.1$$

$$t = \frac{-1}{0.000462} \log(0.1)$$

$$= -(-4983.950418)$$

$$= 4983.950418$$

$$\boxed{t \simeq 4984} \text{ years.}$$

(1) In a chemical reaction the rate of conversion of a substance at time 't' is proportional to

By law of natural decay,

we have $y = ce^{-kt} \rightarrow (1)$

Initially $t = 0$ and $y = y_0$.

$$y_0 = ce^{-k(0)}$$

$$= c.e^{(0)}$$

$$y_0 = c(1)$$

$$\Rightarrow \boxed{c = y_0}$$

$$y = y_0 e^{-kt} \rightarrow (2)$$

And $t = 1$ and $y = 60$ grams

~~and~~ $60 = y_0 \, e^{-k(1)}$

$$60 = y_0 \cdot e^{-k} \rightarrow (3)$$

And also $t = 4$ and $y = 21$ grams

$$21 = y_0 \, e^{-k(4)}$$

$$21 = y_0 \cdot e^{-4k} \rightarrow (4)$$

divide $(3)/(4)$

$$\Rightarrow \frac{\cancel{y_0} e^{-k}}{\cancel{y_0} \cdot e^{-4k}} = \frac{\cancel{60}\, 20}{\cancel{21}\, 7}$$

$$\frac{1}{e^{-3k}} = \frac{20}{7}$$

$e^{3K} = \frac{20}{7}$

$e^{3K} = 2.857142857.$

$3K = \log(2.857)$

$K = \frac{1}{3}\log(2.857)$

$K = 0.349924$

$\boxed{K = 0.3499}$

sub 'K' value in Eqn ③.

from ③, $60 = y_0 e^{-K}$

$60 = y_0 e^{-(0.3499)}$

$y_0 = \frac{1}{e^{-(0.3499)}} 60.$

$y_0 = e^{(0.3499)} .60$

$y_0 = 85.1355$

$\boxed{y_0 \cong 85 \text{ grams}}$

- If 30% of a radio active substance desappear in 10 days. How long will it take for 90% of its to disappear.

Sol: We have $y = ce^{-Kt} \rightarrow ①$

Initially $t=0$ and $y = y_0$

$y_0 = ce^{-K(0)}$

$= c.e^{(0)}$

$= c(1)$

$\Rightarrow \boxed{c = y_0}$

$y = y_0 e^{-Kt} \rightarrow ②$

And $t = 10$ and $y = 70\% . y_0$

$= \frac{70}{100} y_0$

$\frac{70}{100} \cancel{y_0} = \cancel{y_0} e^{-K(10)}$

$e^{-10K} = 0.7$

$$-10K = \log 0.7$$

$$K = \frac{-1}{10}\log(0.7)$$

$$K = 0.0356674994$$

$$\boxed{K = 0.0357}$$

$$y = y_0 \cdot e^{-(0.0357)t} \rightarrow ③$$

And also $t = ?$ and $y = 10\% \, y_0$

$$= \frac{10}{100} y_0$$

$$\frac{10}{100} y_0 = y_0 e^{-(0.0357)t}$$

$$e^{-(0.0357)t} = 0.1$$

$$-(0.0357)t = \log 0.1$$

$$t = \frac{-1}{0.0357}\log(0.1)$$

$$= -(-64.4981818)$$

$$t = 64.5$$

$$\boxed{t \cong 64 \text{ days}}$$

(3) Find the half-life of uranium, which disintegrates at a rate proportional to the amount present at any instant given that m_1 and m_2 grams of uranium are present at t_1 and t_2 respectively.

Sol: We have $y = ce^{-Kt} \rightarrow ①$

Initially $t=0$, $y = M$.

$$M = ce^{-K(0)}$$

$$= c \cdot e^{(0)}$$

$$M = c(1)$$

$$\boxed{c = M}$$

$$y = Me^{-Kt} \rightarrow ②$$

And $t = t_1$ and $y = m_1$, $t = t_2$ and $y = m_2$

$$m_1 = M e^{-Kt_1} \rightarrow ③$$

$$m_2 = M e^{-Kt_2} \rightarrow ④$$

$$\frac{③}{④} \Rightarrow \frac{\cancel{M} e^{-Kt_1}}{\cancel{M} e^{-Kt_2}} = \frac{m_1}{m_2}$$

$$\frac{e^{-Kt_1}}{e^{-Kt_2}} = \frac{m_1}{m_2}$$

$$e^{-Kt_1} \cdot e^{+Kt_2} = \frac{m_1}{m_2}$$

$$e^{Kt_2 - Kt_1} = \frac{m_1}{m_2}$$

$$e^{K(t_2-t_1)} = \frac{m_1}{m_2}$$

$$K(t_2-t_1) = \log \frac{m_1}{m_2}$$

$$\boxed{K = \frac{1}{t_2-t_1} \log \frac{m_1}{m_2}}$$

Sub 'K' in Equn ②.

$$y = M \cdot e^{-\left[\left(\frac{1}{t_2-t_1}\right) \log \frac{m_1}{m_2}\right] t} \longrightarrow ⑤$$

And also $t = ?$ and $y = \frac{M}{2}$

$$\frac{\cancel{M}}{2} = \cancel{M} \cdot e^{-\left(\frac{\log \frac{m_1}{m_2}}{t_2-t_1}\right) t}$$

$$e^{-\left(\frac{\log m_1/m_2}{t_2-t_1}\right) t} = \frac{1}{2}$$

$$-\left(\frac{\log \frac{m_1}{m_2}}{t_2-t_1}\right) t = \log 0.5$$

$$t = \frac{-1}{\frac{\log \frac{m_1}{m_2}}{t_2-t_1}} \log 0.5$$

$$t = \frac{-(t_2-t_1)}{\log \frac{m_1}{m_2}} \log 0.5$$

$$t = \frac{t_1-t_2}{\log \frac{m_1}{m_2}} \log 0.5$$

(OR)

$$t = \frac{(t_1 - t_2)\log\left(\frac{1}{2}\right)}{\log \frac{m_1}{m_2}}$$

$$t = \frac{(t_1 - t_2)\,\log(1) - \log(2)}{\log \frac{m_1}{m_2}}$$

$$= \frac{(t_1 - t_2)\,(0 - \log 2)}{\log \frac{m_1}{m_2}}$$

$$= \frac{(t_1 - t_2)\,(-\log 2)}{\log \frac{m_1}{m_2}}$$

$$= \frac{(t_2 - t_1)\log 2}{\log \frac{m_1}{m_2}}$$

(2)

We have, $y = ce^{-kt} \rightarrow (1)$

Initially $t=0$ and $y = y_0$.

$$y_0 = ce^{-k(0)}$$

$$y_0 = c\, e^{(0)}$$

$$= c(1)$$

$$\Rightarrow \boxed{c = y_0}$$

$$y = y_0 e^{-kt} \rightarrow (2)$$

(5) We have $y = c \cdot e^{-Kt} \rightarrow ①$

Initially $t=0$ and $y=10$.

$$10 = c \cdot e^{-K(0)}$$

$$10 = c\, e^{(0)}$$

$$= c(1)$$

$$\Rightarrow \boxed{c=10}$$

$$y = 10\, e^{-Kt} \rightarrow ②$$

and $t=1$ and $y=0.051$

$$0.051 = 10\, e^{-K(1)}$$

$$\frac{0.051}{10} = e^{-K}$$

$$-K = \log\left(\frac{0.051}{10}\right)$$

$$K = -\log\left(\frac{0.051}{10}\right)$$

$$K = -(-5.278514739)$$

$$\boxed{K = 5.279}$$

$$y = 10e^{-(5.279)t} \rightarrow ③$$

And also $y=5$ and $t=?$

$$\cancel{5} = \cancel{10}^{2} \cdot e^{-(5.279)t}$$

$$1/2 = e^{-(5.279)t}$$

$$\frac{1}{2} = e^{-(5.279)t}$$

$$e^{-(5.279)t} = \frac{1}{2}$$

$$-(5.279)t = \log(1/2)$$

$$t = \frac{-1}{5.279} \log(1/2)$$

$$t = -(-0.131302743)$$

$$\boxed{t = 0.1313}$$

(2) Tuesday 15/10.

Newton's Law of Coolinge

(3)

By Newton's Law of cooling,

we have $T = T_A + c.e^{-kt}$ $\rightarrow$ (1)

Initially $t=0$, $T = 100\,°c$ and $T_A = 40°c$.

$$100 = 40 + ce^{-k(0)}$$

$$100 - 40 = ce^{(0)}$$

$$60 = c(1)$$

$$\Rightarrow \boxed{c = 60}$$

from (1),

~~100 = 40.~~

$$T = 40 + 60e^{-kt} \rightarrow (2)$$

$t = 4$, $T = 60$

$$60 = 40 + 60e^{-k(4)}$$

$$60 - 40 = 60.e^{-4k}$$

$$\cancel{20} = \cancel{60}^{3} e^{-4k}$$

$$\frac{1}{3} = e^{-4k}$$

$$-4k = \log\frac{1}{3}$$

$$k = \frac{-1}{4} \log\frac{1}{3}$$

$K = \frac{-1}{4} \log(0.33)$

$K = -(-\cancel{0}0.277165656)$

$\boxed{K = 0.2772}$

$T = 40 + 60e^{-(0.2772)t} \rightarrow ③$

And also $t = ?$ $T = 50$

$50 = 40 + 60e^{-(0.2772)t}$

$1\cancel{0} = 6\cancel{0}e^{-(0.2772)t}$

$\frac{1}{6} = e^{-(0.2772)t}$

$-(0.2772)t = \log 1/6$

$t = \frac{-1}{0.2772} \log(1/6)$

$t = -(\cancel{-6.4641044242})$

$\boxed{t = 7 \text{ min}}$

By Newton's Law of Cooling,

we have $T = T_A + ce^{-Kt} \rightarrow ①$

Initially $t = 0$, $T = 370K$ and $T_A = 300K$.

$370 = 300 + ce^{-K(0)}$

$70 = ce^{(0)}$

$70 = c(1)$

$\boxed{C = 70}$

from①, $T = 300 + 70e^{-Kt} \rightarrow ②$

And $t = 15$ min, $T = 340K$

$340 = 300 + 70e^{-K(15)}$

$4\cancel{0} = 7\cancel{0}e^{-15K}$

$4/7 = e^{-15K}$

$\cancel{4/7 = 0.6}$

$-15K = \log 4/7$

$K = \frac{-1}{15} \log(0.6)$

$K = \cancel{-(-0.0634807747)}$ 0.037307719

$\cancel{K = 0.034}$ $\boxed{K = 0.0373}$

$$T = 300 + 70e^{-(0.0373)t} \rightarrow ③$$

And also $t = ?$ and $T = 310\,K$

$$310 = 300 + 70e^{-(0.0373)t}$$

$$1\cancel{0} = 7\cancel{0}e^{-(0.0373)t}$$

$$-(0.0373)t = \log(1/7)$$

$$t = \frac{-1}{0.0373}\log(1/7)$$

$$t = -(-52.1691729)$$

$$\boxed{t \cong 52\ \text{min.}}$$

(7)

By Newton's Law of Cooling,

we have $T = T_A + ce^{-kt} \rightarrow ①$

Initially $t = 0$, $T = 100°c$ and $T_A = 25°c$.

$$100 = 25 + ce^{-k(0)}$$

$$75 = ce^{(0)}$$

$$75 = c(1)$$

$$\boxed{c = 75}$$

from ①, $T = 25 + 75e^{-kt} \rightarrow ②$

$t = 10$ and $T = 80°c$

$$80 = 25 + 75e^{-k(10)}$$

$$80 - 25 = 75e^{-10k}$$

$$\cancel{55}^{11} = \cancel{75}^{15}e^{-10k}$$

$$11/15 = e^{-10k}$$

$$-10k = \log 11/15$$

$$k = \frac{-1}{10}\log(11/15)$$

$$k = -(-0.031015492)$$

$$\boxed{k = 0.031}$$

$$T = 25 + 75e^{-(0.031)t} \rightarrow ③$$

~~Anekartsu~~

(i) $t = 20$ and $T = ?$

$$T = 25 + 75 e^{-(0.031)20}$$

$$T = 25 + 75 \cdot (0.537944487)$$

$$T = 25 + 40.34583282$$

$$T = 25 + 40.346$$

$$T = 65.346$$

$$\boxed{T \cong 65°c}$$

(ii) $t = ?$ and $T = 40°c$

$$40 = 25 + 75 e^{-(0.031)t}$$

$$\cancel{15} = \overset{5}{\cancel{75}} e^{-(0.031)t}$$

$$1/5 = e^{-(0.031)t}$$

$$-(0.031)t = \log(1/5)$$

$$t = \frac{-1}{0.031} \log(1/5)$$

$$t = -(-51.9\cancel{1735201})$$

$$\boxed{t \cong 52 \text{ min}}$$

(8) By Newton's Law of Cooling,

we have $T = T_A + ce^{-kt} \rightarrow ①$

Initially $t=0$, $T = 80°c$ and $T_A = 30°c$.

$$80 = 30 + ce^{-k(0)}$$

$$50 = ce^{(0)}$$

$$50 = c(1)$$

$$\Rightarrow \boxed{c = 50}$$

from ①, $T = 30 + 50e^{-kt} \rightarrow ②$

and $t = 12$, $T = 60°c$.

$$60 = 30 + 50e^{-k(12)}$$

$$60 - 30 = 50e^{-k(12)}$$

$$3\cancel{0} = 5\cancel{0}e^{-12k}$$

$$3/5 = e^{-12K}$$

$$-12K = \log 3/5$$

$$K = \frac{-1}{12} \log 3/5$$

(1) By Newton's Law of cooling,

we have $T = T_A + Ce^{-Kt} \rightarrow ①$

Initially $t = 0$, $T = 100°c$ and $T_A = 20°c$

$$100 = 20 + Ce^{-K(0)}$$

$$100 - 20 = Ce^{-(0)k}$$

$$80 = C \cdot \cancel{e^{-10k}}_{(1)}$$

$$\Rightarrow \boxed{C = 80}$$

from ①, $T = 20 + 80e^{-Kt} \rightarrow ②$

And $t = 10$, $T = 25°c$

$$25 = 20 + 80e^{-K(10)}$$

$$25 - 20 = 80e^{-10k}$$

$$\cancel{5} = \cancel{80}^{16} e^{-10k}$$

$$\frac{1}{16} = e^{-10k}$$

$$-10k = \log(1/16)$$

$$K = \frac{-1}{10} \log(1/16)$$

$$K = -(-0.27725887 2)$$

$$\boxed{K = 0.28}$$

$$T = 20 + 80 \cdot e^{-(0.28)t} \rightarrow ③$$

And also $t = \frac{1}{2}$ hr (0.5) and $T = ?$

$$T = 20 + 80e^{-(0.28)(0.5)}$$

$$T = 20 + 80 \times 0.869358235$$

$$T = 20 + 69.54865883$$

$$T = 20 + 69$$

$$\boxed{T \cong 89°c}$$

~~And also~~

(2) By Newton's Law of cooling,

we have $T = T_A + ce^{-kt} \rightarrow (1)$

Initially $t=0, T = 75°C$ and $T_A = 25°C$.

$$75 = 25 + ce^{-k(0)}$$

$$75-25 = c \cdot e^{(0)}$$

$$50 = c(1)$$

$$\Rightarrow \boxed{c = 50}$$

from (1),

$$T = 25 + 50e^{-kt} \rightarrow (2)$$

$t = 10$ min, $\therefore T = 65°C$.

$$65 = 25 + 50e^{-k(10)}$$

$$65-25 = 50e^{-10k}$$

$$4\cancel{0} = 5\cancel{0}e^{-10k}$$

$$-10k = \log(4/5)$$

$$k = \frac{-1}{10}\log(4/5)$$

$$k = -(-0.022314355)$$

$$\boxed{k = 0.0223}$$

$$T = 25 + 50e^{-(0.0223)t} \rightarrow (3)$$

And also $t = 20$ min, $T = ?$

$$T = 25 + 50e^{-(0.0223)20}$$

$$T = 25 + 32.0091886$$

$$T = 25 + 32$$

$$\boxed{T \simeq 57}$$

And also $t = ?$ and $T = 55°C$

$$55 = 25 + 50e^{-(0.0223)t}$$

$$55-25 = 50e^{-(0.0223)t}$$

$$3\cancel{0} = 5\cancel{0}e^{-(0.0223)t}$$

$$t = \frac{-1}{0.0223}\log(3/5)$$

$$= -(-22.90697864)$$

$$\boxed{t \simeq 23}$$

(5) By Newton's Law of Cooling,

we have $T = T_A + ce^{-Kt} \rightarrow ①$

Initially $t=0$, $T = 100°c$; $T_A = 20°c$

$$100 = 20 + ce^{-K(0)}$$

$$100-20 = ce^{(0)}$$

$$80 = c(1)$$

$$\Rightarrow \boxed{c=80}$$

from ①,

$$T = 20 + 80e^{-Kt} \rightarrow ②$$

$t = 1$ min , $T = 60°c$

$$60 = 20 + 80e^{-K(1)}$$

$$60-20 = 80e^{-K}$$

$$4\cancel{0} = \cancel{80}e^{-K}$$

$$1/2 = e^{-K}$$

$$-K = \log(1/2)$$

$$K = -\log(1/2)$$

$$K = -(-0.69314718)$$

$$\boxed{K = 0.693}$$

$$T = 20 + 80e^{-(0.693)t} \rightarrow ③$$

And also $t = 2$min and $T = ?$

$$T = 20 + 80e^{-(0.693)2}$$

$$T = 20 + 20.00588809$$

$$T = 20 + 20$$

$$\boxed{T \cong 40}$$

(6) By Newton's Law of Cooling,

we have $T = T_A + ce^{-Kt} \rightarrow ①$

Initially $t=0$, $T = 100°c$ and $T_A = 30°c$

$$100 = 30 + ce^{-K(0)}$$

$$100-30 = c \cdot e^{(0)}$$

$$70 = c(1)$$

$$\boxed{c=70}$$

from ①, $T = 30 + 70e^{-Kt}$ → ②

$t = 10$ min, $T = 80°c$

$80 = 30 + 70e^{-K(10)}$

$80 - 30 = 70\, e^{-10K}$

$5\cancel{0} = 7\cancel{0}\, e^{-10K}$

$e^{-10K} = 5/7$

$-10K = \log(5/7)$

$K = \frac{-1}{10}\log(5/7)$

$K = -(-0.033647223)$

$\boxed{K = 0.034}$

$T = 30 + 70e^{-(0.034)t}$ → ③

And also $t = ?$ and $T = 40°c$

$40 = 30 + 70 \cdot e^{-(0.034)t}$

$1\cancel{0} = 7\cancel{0}\, e^{-(0.034)t}$

$e^{-(0.034)t} = (1/7)$

$-(0.034)t = \log(1/7)$

$t = \frac{-1}{0.034}\log(1/7)$

$t = -(-57.23265144)$

$\boxed{t \cong 57}$

(9) By Newton's Law of Cooling,

we have $T = T_A + Ce^{-Kt}$ → ①

Initially $t=0$, $T = 100$, $T_A = 15°c$.

$100 = 15 + Ce^{-K(0)}$

$85 = C\, e^{(0)}$

$85 = C(1)$

$\Rightarrow \boxed{C = 85}$

$T = 15 + 85e^{-Kt}$ → ②

$t = 5$ min , $T = 60°C$

$$60 = 15 + 85 e^{-K(5)}$$

$$45 = 85 e^{-5k}$$

$$e^{-5K} = \frac{45}{85}$$

$$e^{-5K} = 0.529411764$$

$$e^{-5K} = 0.53$$

$$-5k = \log(0.53)$$

$$K = \frac{-1}{5} \log(0.53)$$

$$K = -(-0.126975654)$$

$$\boxed{K = 0.13}$$

$$T = 15 + 85 e^{-(0.13)t} \rightarrow (3)$$

And also $t = 5$, $T = ?$

$$T = 15 + 85 e^{-(0.13)5}$$

$$T = 15 + 44.37389102$$

$$T = 15 + 44$$

$$\boxed{T \simeq 59}$$

(10) By Newton's Law of Cooling,

we have $T = T_A + C e^{-Kt} \rightarrow (1)$

Initially $t = 0$, $T = 110°C$, $T_A = 10°C$

$$110 = 10 + C e^{-K(0)}$$

$$100 = C e^{(0)}$$

$$100 = C(1)$$

$$\boxed{C = 100}$$

from (1), $T = 10 + 100 e^{-Kt} \longrightarrow (2)$

$t = 1$ hr. , $T = 60°C$

$$60 = 10 + 100 e^{-K(1)}$$

$$\cancel{50} = \cancel{100}^{2} e^{-K}$$

$$e^{-K} = 1/2$$

$$-k = \log(1/2)$$
$$k = -\log(1/2)$$
$$k = -(-0.69314718)$$
$$\boxed{k = 0.693}$$
$$T = 10 + 100\,e^{-(0.693)t} \rightarrow (3)$$

And also $t = ?$ $T = 30°C$

$$30 = 10 + 100\,e^{-(0.693)t}$$
$$\cancel{20} = \cancel{100}^{5}\,e^{-(0.693)t}$$
$$1/5 = e^{-(0.693)t}$$
$$-(0.693)t = \log(1/5)$$
$$t = \frac{-1}{0.693}\log(1/5)$$
$$t = -(-2.32242123)$$
$$\boxed{t \cong 2\,hr}$$

Tuesday
15/10 <u>Electrical circuits:</u>

(1) A constant electromotive force E volts is applied to a circuit containing a constant resistance 'R' ohms in series and a constant inductance 'N' henry's. If the initial current is '0'. Show that the current builds up to half of its maximum in $\frac{L\log 2}{R}$ sec.

(2) A resistance of 100 ohm's and inductance of 0.5 henry are connected in a series with a battery of 20 volts. Find the current in the circuit, if initially there is no current in the circuit.

(3) A voltage Ee^{-at} is applied at $t=0$ to a circuit containing inductance 'L' and resistance 'R'. Show that at any time t is $\frac{E}{R-aL}\left(e^{-at} - e^{-\frac{R}{L}t}\right)$.

(4) Solve the equn $L\frac{di}{dt} + Ri = 200\cos(300t)$. when $R = 100$, $L = 0.05$ and find 'i'. Given that $i = 0$ when $t = 0$, what value thus 'i' approach after a long time.

1. ~~By using Krichhoff's~~

By using Kirchoff's Law the equ^n of the LR circuit is $L\frac{di}{dt} + Ri = E$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \longrightarrow (1)$$

Equ^n (1) is in linear form $\frac{dy}{dx} + Py = Q$

$$P = \frac{R}{L} \quad \text{and} \quad Q = \frac{E}{L}$$

I.F $e^{\int P(t)dt}$

$$= e^{\int \frac{R}{L} dt}$$

$$= e^{\frac{R}{L}\int(1)dt}$$

$$= e^{\frac{R}{L}t}$$

Now the solution of equ^n (1) is

$$i \cdot e^{\frac{R}{L}t} = \int \frac{E}{L} e^{\frac{R}{L}t} + c$$

$$= \frac{E}{L}\int e^{\frac{R}{L}t} + c$$

$$= \frac{E}{\cancel{L}} \frac{e^{\frac{R}{L}t}}{R/\cancel{L}} + c$$

$$i \cdot e^{\frac{R}{L}t} = \frac{E}{R} e^{\frac{R}{L}t} + c$$

$$i \cdot \cancel{e^{R/L\,t}} = \cancel{e^{R/L\,t}}\left(\frac{E}{R} + c\,e^{-R/L\,t}\right)$$

$$i = \frac{E}{R} + c\,e^{-\frac{R}{L}t}$$

Initially $t=0$ and $i=0$

$$0 = \frac{E}{R} + c \cdot e^{-\frac{R}{L}(0)}$$

$$-\frac{E}{R} = c e^{(0)}$$

$$\frac{-E}{R} = c(1)$$

$$\Rightarrow \boxed{c = \frac{-E}{R}}$$

$$i = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L}t}$$

$$\boxed{i = \frac{E}{R}\left(1 - e^{-R/L\,t}\right)}$$

Given that $i = \frac{1}{2}\frac{E}{R}$, $t = ?$

$$\frac{1}{2}\frac{E}{R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$e^{-\frac{R}{L}t} = 1 - \frac{1}{2}$$

$$e^{-R/L\,t} = \frac{1}{2}$$

$$-\frac{R}{L}t = \log \frac{1}{2}$$

$$-t = \frac{-L}{R}(\log 1 - \log 2)$$

$$t = -\frac{L}{R}(0 - \log 2)$$

$$t = \frac{+L \log 2}{R} \text{ sec.}$$

③ By using Kirchoff's Law the equn of the LR circuit is $L\frac{di}{dt} + Ri = E$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at} \rightarrow ①$$

Equn① is in linear form

$$P = \frac{R}{L} \quad , \quad Q = \frac{E}{L}e^{-at}$$

$$\text{IF } e^{\int p(t)dt} = e^{\int \frac{R}{L}dt}$$

$$= e^{\frac{R}{L}\int(1)dt}$$

$$= e^{\frac{R}{L}t}$$

Now the soln of Equn① is

$$i \cdot e^{\frac{R}{L}t} = \int \frac{E}{L}e^{-at}e^{R/L\,t} + C$$

$$= \frac{E}{L}\frac{e^{R/L\,t}}{R/L} + C$$

$$= \frac{E}{R}e^{R/L\,t} + C$$

$$i \cdot e^{R/L\,t} = e^{R/L\,t}\left(\frac{E}{R} + C \cdot e^{-R/L\,t}\right)$$

$$\boxed{i = \frac{E}{R} + Ce^{-R/L\,t}}$$

initially ~~te~~, $t = 0$, $i = 0$.

$$0 = \frac{E}{R} + C\cdot e^{-\frac{R}{L}(0)}$$

$$-\frac{E}{R} = C e^{(0)}$$

$$\Rightarrow \boxed{C = \frac{-E}{R}}$$

$$i = \frac{E}{R} + -\frac{E}{R} e^{-R/L\, t}$$

$$\boxed{i = \frac{E}{R}\left(1 - e^{-R/L\, t}\right)}$$

Given that

$$i \cdot e^{R/L\, t} = \int \frac{E}{L} e^{-at} e^{R/L\, t}\, dt + C$$

$$= \frac{E}{L}\int e^{R/L\, t - at}\, dt + C$$

$$= \frac{E}{L}\int e^{(R/L - a)t}\, dt + C$$

$$= \frac{E}{L} \frac{e^{(R/L - a)t}}{(R/L - a)} + C$$

$$= \frac{E}{\cancel{L}} \frac{e^{(R/L - a)t}}{\left(\frac{R - aL}{\cancel{L}}\right)} + C$$

$$= \frac{E}{R - aL} e^{(R/L - a)t} + C$$

$$= \frac{E}{R - aL} e^{R/L\, t} \cdot e^{-at} + C$$

$$i \cdot \cancel{e^{R/L\, t}} = \cancel{e^{R/L\, t}}\left[\frac{E}{R - aL} e^{-at} + C\cdot e^{-R/L\, t}\right]$$

$$i = \left(\frac{E}{R - aL} e^{-at} + C\cdot e^{-R/L\, t}\right)$$

Initially $t = 0$ and $i = 0$

$$0 = \frac{E}{R - aL} e^{-a(0)} + C\cdot e^{-R/L(0)}$$

$$-\frac{E}{R - aL} (e)^{(0)} = C\cdot e^{(0)}$$

$$\frac{-E}{R - aL} = C(1)$$

$$\boxed{C = \frac{-E}{R - aL}}$$

$$i = \frac{E}{R-aL} \cdot e^{-at} - \frac{E}{R-aL} e^{-R/L\, t}$$

$$i = \frac{E}{R-aL} \left[e^{-at} - e^{-R/L\, t} \right]$$

(4) By using Kirchoff's Law the equⁿ of the LR circuit is $L\frac{di}{dt} + Ri = E$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

Given that $L\frac{di}{dt} + Ri = 200 \cdot \cos(300t)$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{200 \cdot \cos(300t)}{L} \rightarrow (1)$$

Given that $R = 100$, $L = 0.05$

$$\frac{di}{dt} + \frac{100}{0.05} i = \frac{200 \cdot \cos(300t)}{0.05}$$

$$\frac{di}{dt} + 2000\, i = 4000 \cdot \cos(300t) \rightarrow (1)$$

Equⁿ (1) is in linear form.

$P = 2000$ and $Q = 4000 \cos(300t)$.

I.F $e^{\int 2000 dt} = e^{2000 \int (1) dt} = e^{2000t}$

$$i \cdot e^{2000t} = \int 4000 \cdot \cos(300t)\, e^{2000t}\, dt + c$$

$$= 4000 \int \cos(300t) \cdot e^{2000t}\, dt + c.$$

$$= 4000 \int e^{2000t} \cos(300t) + c.$$

$$= 4000 \cdot \left[\frac{e^{(2000)t}}{(2000)^2 + (300)^2} \left[2000 \cos(300)t + 300 \cdot \sin(300)t \right] + c \right]$$

$$i \cdot e^{2000t} = \cancel{4000} \left[\frac{e^{(2000)t}}{4090\cancel{000}} \left[2000 \cos(300)t + 300 \sin(300)t \right] + c \right]$$

$$= \frac{4}{4090} e^{(2000)t} \left[2000 \cdot \cos(300)t + 300 \sin(300)t \right] + c$$

$$= e^{(2000)t} \left[\frac{4 \times 200\cancel{0}}{409\cancel{0}} \cos(300)t + \frac{4 \times 30\cancel{0}}{409\cancel{0}} \sin(300)t \right] + c$$

$$= e^{(2000)t} \left[\frac{40 \times 20}{409} \cos(300t) + \frac{40 \times 3}{409} \sin(300)t \right] + c$$

$$i \cdot e^{(2000)t} = e^{(2000)t} \frac{40}{409}\left\{\left[20 \cdot \cos(300)t + 3 \cdot \sin(300)t\right] + C e^{(2000)}\right.$$

$$i = \frac{40}{409}\left[20 \cos(300)t + 3 \sin(300)t\right] + C \cdot e^{-(2000)t}$$

Given that $i = 0$ and $t = 0$.

$$0 = \frac{40}{409}\left[20 \cdot \cos(300)(0) + 3 \cdot \sin(300)(0)\right] + C \cdot e^{-(2000)(0)}$$

$$0 = \frac{40}{409}\left(20 \cdot \cos(0) + 3 \cdot \sin(0)\right) + C \cdot e^{(0)}$$

$$0 = \frac{40}{409}\left(20(1) + 3(0)\right) + C(1)$$

$$0 = \frac{40}{409}(20 + 0) + C$$

$$\boxed{C = \frac{-40 \times 20}{409}}$$

$$i = \frac{40}{409}\left[20 \cos(300)t + 3 \cdot \sin(300)t\right] - \frac{40 \times 20}{409} e^{-(2000)t}$$

$$i = \frac{40}{409}\left[20 \cos(300)t + 3 \cdot \sin(300)t - 20 \cdot e^{-(2000)t}\right]$$

$$i = \frac{40}{409}\left[20\left(\cos(300)t - e^{-(2000)t}\right) + 3 \cdot \sin(300)t\right]$$

(2) By using Kerchoff's Law the equⁿ of the LR circuit is $L\frac{di}{dt} + Ri = E$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

Given that $R = 100$, $L = 0.5$, $E\ 20$

$$\frac{di}{dt} + \frac{100}{0.5} i = \frac{20}{0.5}$$

$$\frac{di}{dt} + 200 \cdot i = 40 \quad \rightarrow (1)$$

Equⁿ (1) is in linear form.

$P = 200$ and $Q = 40$

$$\text{I.F } e^{\int 200\, dt} = e^{200 \int (1) dt}$$

$$= e^{200t}$$

$$i \cdot e^{200t} = \int 40 \cdot e^{200t}\, dt + c$$

$$= 40\int e^{200t}\, dt + c$$

$$= 4\cancel{0}\, \frac{e^{200t}}{\underset{5}{\cancel{200}}} + c$$

$$i \cdot e^{200t} = \frac{1}{5} \cdot e^{200t} + c$$

$$i \cdot \cancel{e^{200t}} = \cancel{e^{200t}}\left(\frac{1}{5} + c \cdot e^{-200t}\right)$$

$$i = \frac{1}{5} + c \cdot e^{-200t}.$$

Initially $t=0$ and $i=0$

$$0 = \frac{1}{5} + c \cdot e^{-200(0)}$$

$$-\frac{1}{5} = c \cdot e^{(0)}$$

$$-1/5 = c(1)$$

$$\Rightarrow \boxed{c = -1/5}$$

$$i = \frac{1}{5} - \frac{1}{5} e^{-200t}$$

$$i = \frac{1}{5}\left(1 - e^{-200t}\right).$$

Law of Growth:

(3) We have $y = ce^{kt} \rightarrow (1)$

Initially $t=0$ and $y = N$

$$N = ce^{k(0)}$$

$$N = c\, e^{(0)}$$

$$= c(1)$$

$$\Rightarrow \boxed{c = N}$$

$$y = N \cdot e^{kt} \rightarrow (2)$$

and $t = 2$ and $y = 3N$.

$$3\cancel{N} = \cancel{N} e^{k(2)}$$

$$2k = \log 3$$

$$k = \frac{1}{2}\log 3$$

$$k = 0.549306144$$

$$\boxed{k = 0.549}$$

$$y = N \cdot e^{(0.549)t} \rightarrow (3)$$

And also $t = ?$ and $y = 100N$

$$100\cancel{N} = \cancel{N} e^{(0.549)t}$$

$$e^{(0.549)t} = 100$$

$$(0.549)\, t = \log 100$$

$$t = \frac{1}{0.549} \log(100)$$

$$t = 8.388288135$$

$$\boxed{t \simeq 8}$$

Higher Order Differential Equations

(3)

Solutions of Higher order Homogeneous Differential Equations:

H.O.D.E

$f(D)y = 0$ → Homogeneous H.D.E → solution → $y =$ complementary function (C.F)

$f(D)y = X$ → Non-Homogeneous H.DE → solution → $y =$ Complementary function + Particular Integral (C.F + P.I)

Solve the following Higher order Differential Equations:

① $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0.$

② $\frac{d^4y}{dx^4} + 13\frac{d^2y}{dx^2} + 36y = 0$

③ $\frac{d^4x}{dt^4} + 4x = 0.$

④ $(D^4 + 4)y = 0$

⑤ $y'' - 2y' + 10y = 0$ given $y(0) = 4$, $y'(0) = 1$

⑥ $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 8y = 0$ under the conditions $y(0) = 0$ and $y'(0) = 0$, $y''(0) = 2$.

⑦ $\frac{d^4x}{dt^4} = m^4x$. show that $x = c_1\cos mt + c_2\sin mt + c_3\cosh mt + c_4\sinh mt$.

⑧ $(D^3 + 1)y = 0.$

⑨ $(D^4 + 6D^3 + 11D^2 + 6D)y = 0.$

④ Given D.E is $(D^4+4)y=0 \rightarrow ①$

An A.E is $m^4+4=0$

$$(m^2)^2+(2)^2=0$$

$$(m^2+2)^2-2m^2(2)=0$$

$$(m^2+2)^2-4m^2=0$$

$$(m^2+2)^2-(2m)^2=0$$

$$(m^2+2+2m)(m^2+2-2m)=0$$

$m^2+2m+2=0$ and $m^2-2m+2=0$

$m=\dfrac{-2\pm\sqrt{4-8}}{2}$ and $m=\dfrac{2\pm\sqrt{4-8}}{2}$

$=\dfrac{-2\pm 2i}{2}$ $\qquad$ $=\dfrac{2\pm 2i}{2}$

$=\dfrac{\cancel{2}(-1\pm i)}{\cancel{2}}$ $\qquad$ $=\dfrac{\cancel{2}(1\pm i)}{\cancel{2}}$

$m=-1\pm i$ $\qquad$ $m=1\pm i.$

∴ The roots $-1\pm i$, $1\pm i$ are complex and distinct roots.

∴ The complementary function (C.F) is

$$e^{-1(x)}[c_1\cos x+c_2\sin x]+e^{1(x)}[c_3\cos x+c_4\sin x]$$

∴ The solution of Equⁿ① is $y=C.F$

$$y=e^{-x}[c_1\cos x+c_2\sin x]+e^{x}[c_3\cos x+c_4\sin x].$$

⑤ Given D.E is $y''-2y'+10y=0$

$$D^2y-2Dy+10y=0$$

$$(D^2-2D+10)y=0$$

An A.E is $m^2-2m+10=0$

$$m=\frac{2\pm\sqrt{4-40}}{2}$$

$$=\frac{2\pm\sqrt{-36}}{2}$$

$$= \frac{2 \pm 6i}{2}$$

$$= \frac{\cancel{2}(1 \pm 3i)}{\cancel{2}}$$

$$m = 1 \pm 3i$$

$\therefore$ The roots $1 \pm 3i$ are complex and distinct roots.

$\therefore$ The complementary function $= e^{(1)x}[c_1 \cos 3x + c_2 \sin 3x]$

$\therefore$ The solution is $y = C.F$

$$y = e^{x}[c_1 \cos 3x + c_2 \sin 3x] \rightarrow ①$$

Given that $y(0) = 4$ and $y'(0) = 1$

$x = 0;\ y = 4$ $\qquad$ $x = 0,\ y' = 1$

at $x = 0$, $y = 4$

from①, $\quad y = e^{0}[c_1 \cos 3(0) + c_2 \sin 3(0)]$

$$4 = 1[c_1 \cos 0 + c_2 \sin 0]$$

$$4 = c_1(1) + c_2(0)$$

$$\Rightarrow \boxed{c_1 = 4}$$

from①

at $x=0$, $\quad y' = e^{x}[c_1 \cos 3x + c_2 \sin 3x] + e^{x}[c_1(-\sin 3x)3 + c_2(\cos 3x)(3)]$

$$y' = e^{x}[c_1 \cos 3x + c_2 \sin 3x] + e^{x}[-3c_1 \sin 3x + 3c_2 \cos 3x]$$

at $x = 0$, $y' = 1$ and $c_1 = 4$

$$\therefore 1 = e^{0}[4 \cdot \cos 3(0) + c_2 \sin 3(0)] + e^{0}[-3(4) \sin 3(0) + 3c_2 \cos 3(0)]$$

$$1 = (1)[4(1) + c_2(0)] + (1)[-12(0) + 3c_2(1)]$$

$$1 = (4 + 0) + (0 + 3c_2)$$

$$1 = 4 + 3c_2$$

$$3c_2 = 1 - 4$$

$$3c_2 = -3$$

$$\boxed{c_2 = -1}$$

$$\therefore y = e^{x}[4 \cos 3x - \sin 3x]$$

⑥ $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 8y = 0$

$D^3y + 6D^2y + 12Dy + 8y = 0$

$(D^3+6D^2+12D+8)\,y = 0$

A.E is $m^3+6m^2+12m+8=0$

$(m+2)(m^2+4m+4)=0$

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 12 & 8 \\ & 0 & -2 & -8 & -8 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array}$$

$m+2=0,\quad m^2+4m+4=0$

$m=-2,\quad (m+2)(m+2)=0$

$m=-2,\ m=-2$

$\therefore$ The roots $-2,-2,-2$ are real and repeated roots.

Now, $C.F = c_1e^{-2x} + c_2e^{-2x}(x) + c_3e^{-2x}(x^2)$

$\therefore$ The solution is $y = C.F$

$y = c_1e^{-2x} + c_2e^{-2x}(x) + c_3e^{-2x}x^2$

$y = e^{-2x}[c_1 + c_2x + c_3x^2]$ $\rightarrow$ ①

Given that $y(0)=0,\ y'(0)=0$ and $y''(0)=2$.

at $x=0,\ y=0$.

$0 = e^{-2(0)}[c_1 + c_2(0) + c_3(0)^2]$

$0 = (1)[c_1+0+0]$

$\Rightarrow \boxed{c_1=0}$

from ①,

$y' = e^{-2x}(-2)[c_1+c_2x+c_3x^2] + e^{-2x}[0+c_2+2c_3x]$ $\rightarrow$ ②

at $x=0,\ y'=0$.

$0 = e^{-2(0)}(-2)[c_1+c_2(0)+c_3(0)] + e^{-2(0)}[c_2+2c_3(0)]$

$0 = (1)(-2)[0+0] + (1)[c_2+0]$

$0 = -2c_1 + c_2$

$\Rightarrow \boxed{c_2=0}$

from ②,

$$y'' = -2\cdot e^{-2x}(-2)[c_1 + c_2 x + c_3 x^2] + (-2)e^{-2x}[0 + c_2 + 2c_3 x]$$
$$+ -2e^{-2x}[c_2 + 2c_3 x] + e^{-2x}[0 + 2c_3]$$
$$= 4e^{-2x}[c_1 + c_2 x + c_3 x^2] - 2e^{-2x}(c_2 + 2c_3 x)$$
$$-2e^{-2x}(c_2 + 2c_3 x) + e^{-2x} 2c_3.$$

$$= 4e^{-2x}(c_1 + c_2 x + c_3 x^2) - 4e^{-2x}(c_2 + 2c_3 x) + 2\cdot e^{-2x} c_3$$

at $x=0$, $y''=2$

$$2 = 4e^{-2(0)}(c_2(0) + c_3(0)^2) - 4e^{-2(0)}[c_2 + 2c_3(0)] + 2e^{-2(0)} c_3$$

$$2 = 4(1)[0+0] - 4(1)[c_2 + 0] + 2(1)c_3$$

$$2 = 4(0) - 4c_2 + 2c_3 \qquad \underline{\text{Wrong}}$$

$$2 = 0 - 4(2) + 2c_3$$

$$2 = -8 + 2c_3$$

$$\cancel{2}c_3 = \cancel{10}\,5$$

$$\boxed{c_3 = 5}$$

$$y' = -2e^{-2x}\cdot[c_1 + c_2 x + c_3 x^2] + e^{-2x}[c_2 + 2c_3 x]$$

$$y' = e^{-2x}[-2c_1 - 2c_2 x - 2c_3 x^2 + c_2 + 2c_3 x]$$

$$y'' = e^{-2x}(-2)[-2c_1 - 2c_2 x - 2c_3 x^2 + c_2 + 2c_3 x] + e^{-2x}[0 - 2c_2 - 2c_3(2x) + 0 + 2c_3]$$

$$= -2e^{-2x}[-2c_1 - 2c_2 x - 2c_3 x^2 + c_2 + 2c_3 x] + e^{-2x}[-2c_2 - 4x c_3 + 2c_3]$$

at $x=0$, $y''=2$

$$2 = -2e^{-2(0)}[-2(0) - 2(0)x - 2c_3(0)^2 + 0 + 2c_3(0)] + e^{-2(0)}[-2(0) - 4(0)c_3 + 2c_3]$$

$$2 = -2(1)\cdot[0] + (1)[2c_3]$$

$$2 = 0 + 2c_3$$

$$\cancel{2} = \cancel{2}c_3 \Rightarrow \boxed{c_3 = 1}$$

$$\therefore c_1 = 0, \quad c_2 = 0, \quad c_3 = 1$$

from ①,

$$\therefore y = e^{-2x}[c_1 + c_2 x + c_3 x^2]$$
$$= e^{-2x}[0 + 0 + 1] \Rightarrow y = e^{-2x}$$

① $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0 \quad \rightarrow ①$

$D^3y - 7Dy - 6y = 0$

$(D^3 - 7D - 6)y = 0$

An Auxiliary equ^n is $m^3 - 7m - 6 = 0$

$(m+1)(m^2 - m - 6) = 0$

$$\begin{array}{c|cccc} -1 & 1 & 0 & -7 & -6 \\ & 0 & -1 & +1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$m+1 = 0$ and $m^2 - m - 6 = 0$

$\boxed{m = -1}$

$m^2 - 3m + 2m - 6 = 0$

$m(m-3) + 2(m-3) = 0$

$(m-3)(m+2) = 0$

$\boxed{m = -2, 3.}$

$\therefore m = -1, -2, 3.$

$\therefore$ The roots are real and distinct.

Now, $C.F = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

Now, the solution of equ^n ① is $y = C.F$

$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

② Given D.E is $\frac{d^4y}{dx^4} + 13\frac{d^2y}{dx^2} + 36y = 0$

$D^4y + 13D^2y + 36y = 0$

$(D^4 + 13D^2 + 36)y = 0$

An Auxiliary equ^n is $m^4 + 13m^2 + 36 = 0$

$$\begin{array}{c|ccccc} -1 & 1 & 0 & 13 & & 36 \\ & 0 & -1 & & & \\ \hline & 1 & -1 & & & \end{array}$$

(3). Given D.E is $\frac{d^4x}{dt^4} + 4x = 0. \rightarrow (1)$

$\left(\sqrt{2}\right)^4 \quad \sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2} \quad (\sqrt{2})^2(\sqrt{2})^4$

$$D^4x + 4x = 0$$

$$(D^4+4)x = 0.$$

An A.E is $m^4+4 = 0$

$$(m^2)^2 + (2)^2 = 0$$

$$(m^2+2)^2 - 2(2)m^2 = 0$$

$$(m^2+2)^2 - (2m)^2 = 0$$

$$(m^2+2+2m)(m^2+2-2m) = 0.$$

$$(m^2+2m+2)\ (m^2-2m+2) = 0$$

$$m = \frac{-2\pm\sqrt{4-8}}{2} \qquad m = \frac{2\pm\sqrt{4-8}}{2}$$

$$= \frac{-2\pm 2i}{2} \qquad = \frac{2\pm 2i}{2}$$

$$= \frac{\cancel{2}(-1\pm i)}{\cancel{2}} \qquad = \frac{\cancel{2}(1\pm i)}{\cancel{2}}$$

$$= -1\pm i \qquad = 1\pm i$$

$m = -1\pm i,\ 1\pm i.$

$\therefore$ The roots are complex and distinct.

Now the C.F $= e^{-t}[c_1\cos t + c_2\sin t] + e^{t}[c_3\cos t + c_4\sin t]$

$\therefore$ The solution of equ^n (1) is $x = C.F$

$$x = e^{-t}[c_1\cos t + c_2\sin t] + e^{t}[c_3\cos t + c_4\sin t]$$

4 (7) Given D.E is $\frac{d^4x}{dt^4} = m^4x. \rightarrow (1)$

$$D^4x = m^4x.$$

$$D^4x - m^4x = 0$$

$$x(D^4-m^4) = 0$$

An A.E is m^4-

(8) Given D.E is $(D^3+1)y=0 \rightarrow (1)$

An A.E is $m^3+1=0$

$$m^3+(1)^3=0$$

$$(m+1)^3-3m\cdot(m+1)=0$$

$$(m+1)\left[(m+1)^2-3m\right]=0$$

$$(m+1)\left[m^2+1+2m-3m\right]=0$$

$$(m+1)(m^2-m+1)=0$$

$$m=-1, \quad m=\frac{1\pm\sqrt{1-4}}{2}$$

$$=\frac{1\pm\sqrt{3}i}{2}$$

$$m=-1, \frac{1\pm\sqrt{3}i}{2}$$

∴ The roots are real, complex and distinct.

Now, $C.F = c_1e^{-x}+e^{\frac{1}{2}x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}\right)+c_2\sin\frac{\sqrt{3}}{2}\right]$

Now the solution of equ(1) is $y=C.F$

$$y=c_1e^{-x}+e^{\frac{1}{2}x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}\right)+c_2\sin\frac{\sqrt{3}}{2}\right]$$

(9) Given D.E is $(D^4+6D^3+11D^2+6D)y=0$

An A.E is $m^4+6m^3+11m^2+6m=0$

$$(m+1)(m+2)(m^2+3m)=0.$$

$$m+1=0, \quad m+2=0, \quad m^2+3m=0$$

$$m=-1, \quad m=-2, \quad m(m+3)=0$$

$$m=0, \quad m=-3.$$

∴ $m=0,-1,-2,-3.$

∴ The roots are real and distinct

Now, the $C.F = c_1e^{(0)x}+c_2e^{-x}+c_3e^{-2x}+c_4e^{-3x}$

∴ the solution of equ(1) is $y=C.F$

$$y=c_1e^{(0)x}+c_2e^{-x}+c_3e^{-2x}+c_4e^{-3x}$$

(1) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

sol: $D^3y - 6.D^2y + 11.Dy - 6y = 0$

$(D^3 - 6D^2 + 11D - 6)\, y = 0.$

$m^3 - 6m^2 + 11m - 6 = 0.$ (auxiliary equation)

$(m^2 - 5m + 6)(m-1) = 0$

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$m - 1 = 0$ and $m^2 - 5m + 6 = 0$

$m = 1$ $\qquad m^2 - 3m - 2m + 6 = 0$

$m(m-3) - 2(m-3) = 0$

$m = 2,\ m = 3.$

The roots are real and distinct.

$c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

$\therefore$ The solution is $y =$ C.F (complementary function)

$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

(2) $\frac{d^3y}{dx^3} - 8y = 0.$

sol: $D^3y - 8y = 0$

$(D^3 - 8)\, y = 0$

An auxiliary equⁿ is $m^3 - 8 = 0.$

$\cancel{m^3} = 8$

$m^3 = 2^3$

$\boxed{\cancel{m} = 2}$

$m^3 - 2^3 = 0$

$(m-2)(m^2 + 2m + 4) = 0.$

$m - 2 = 0$ and $m^2 + 2m + 4 = 0$

$\boxed{m = 2}$ $\qquad m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$m = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}i}{2}$$

$$= \frac{\cancel{2}(-1 \pm \sqrt{3}i)}{\cancel{2}}$$

$$\boxed{m = -1 \pm \sqrt{3}i.}$$

$m = 2, \quad -1+\sqrt{3}i, \quad -1-\sqrt{3}i.$

Now, Complementary function is

$$c_1 e^{2x} + c_2 e^{(-1+\sqrt{3}i)x} + c_3 e^{(-1-\sqrt{3}i)x}$$

$$= e^{-x}[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x] + c_3 e^{2x}.$$

Now the solution is $y = C.F$

$$y = e^{-x}[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x] + c_3 e^{2x}$$

<u>Non-Homogeneous Higher Order D.E:</u>

① ~~$\frac{d^3y}{dx^3} + y = 3+5e^x$~~

<u>Type-I</u>

~~$\frac{d^3y}{dx^3} + y = 3+5e^x$~~

② $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$

<u>Sol:-</u> $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x \longrightarrow ①$

Equn ① is a non-homogeneous H.O.D equn.

$D^2y + 4Dy + 5y = -2\cosh x$

$(D^2 + 4D + 5)y = -2\cosh x.$

An Auxiliary equn is $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm 2i}{2} = \frac{\cancel{2}(-2 \pm i)}{\cancel{2}}$$

$$m = -2 \pm i$$

$\therefore$ The roots are complex and distinct.

Now, $C.F = e^{-2x}[c_1 \cos x + c_2 \sin x]$

And Particular

The particular Integral of the Equ^n ①

$$P.I = \frac{1}{f(D)} X$$

$$= \frac{1}{D^2+4D+5} -2\cosh x$$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$$= \frac{1}{D^2+4D+5} -\cancel{2}\left(\frac{e^x+e^{-x}}{\cancel{2}}\right)$$

$$= -\left[\frac{+1}{D^2+4D+5}(e^x+e^{-x})\right]$$

$$= -\left[\frac{1}{D^2+4D+5}e^x + \frac{1}{D^2+4D+5}e^{-x}\right]$$

$$= -\left[\frac{1}{(1)^2+4(1)+5}e^x + \frac{1}{(-1)^2+4(-1)+5}e^{-x}\right]$$

$$= -\left[\frac{1}{1+4+5}e^x + \frac{e^{-x}}{1-4+5}\right]$$

$$= -\left[\frac{e^x}{10} + \frac{e^{-x}}{2}\right]$$

$$\cancel{= - \left[\frac{1+5}{10}\right]}$$

$$\cancel{= -\left[\frac{6}{10}\right]}$$

$$\cancel{P.I = -\frac{3}{5} e^x}$$

$\therefore$ The solution of Equ^n ① is $y = C.F + P.I$

$$y = e^{-2x}[c_1 \cos x + c_2 \sin x] - \frac{1}{10}e^x - \frac{1}{2}e^{-x}.$$

③

$$\frac{d^2y}{dx^2} - 4y = (1+e^x)^2 \rightarrow ①$$

$$D^2y - 4y = (1+e^x)^2$$

$$(D^2-4)y = 1+(e^x)^2+2e^x$$

An A.E is $m^2-4=0$

$$m^2-(2)^2=0$$

$$(m+2)(m-2)=0$$

$m = -2, 2.$

$\therefore$ The roots are real and distrinct.

Now, The $C.F = c_1 e^{-2x} + c_2 e^{2x}$

Now the $P.I = \frac{1}{f(D)} (x)$

$$= \frac{1}{D^2-4} (1+e^x)^2$$

$$= \frac{1}{D^2-4} (1+e^{2x}+2e^x)$$

$$P.I = \frac{1}{D^2-4}(1) + \frac{1}{D^2-4} e^{2x} + \frac{1}{D^2-4} 2e^x.$$

$$= \frac{1}{D^2-4} e^{(0)x} + \frac{1}{D^2-4} e^{2x} + \frac{1}{D^2-4} 2e^x \longrightarrow ②$$

$$\cancel{= \frac{1}{0-4} e^{(0)x} + \frac{D}{D-4}}$$

$(P.I-1)$ $(P.I-2)$ $(P.I-3)$

$$PI_1 = \frac{1}{D^2-4} e^{(0)x} = \frac{1}{0-4} e^{(0)x} = \frac{-1}{4}.$$

$$PI_2 = \frac{1}{D^2-4} e^{2x}$$

$$= \frac{x}{2D-0} e^{2x} = \frac{x}{2(2)} e^{2x} = \frac{x}{4} e^{2x}$$

$$PI_3 = \frac{1}{D^2-4} 2e^x = \frac{2}{(1)^2-4} e^x = 2 \cdot \frac{1}{1-4} e^x = \frac{-2}{3} e^x.$$

Equ^n ②,

$$P.I = \frac{-1}{4} + \frac{x}{4} e^{2x} - \frac{2}{3} e^x.$$

Now the solution is $y = C.F + P.I$

$$y = c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{4} + \frac{x}{4} e^{2x} - \frac{2}{3} e^x.$$

⑧

$$(D+2) \cdot (D-1)^2 y = e^{-2x} + 2\sinh x \longrightarrow ①$$

$$(D+2)(D^2-1-2D) y = e^{-2x} + 2\sinh x$$

An A.E is $(m+2)(m-1)^2 = 0.$

$m+2=0$, $(m-1)^2 = 0.$

$m = -2$, $(m-1)(m-1) = 0$

$m = 1$

$\therefore m = 1, 1, -2$

$\therefore$ The roots are real and district, repeat.

Now, the C.F $= C_1 e^{x} + C_2 x.e^{x} + C_3 e^{-2x}$.

Now the Particular Integral $= \frac{1}{F(D)} (x)$

$$P.I = \frac{1}{(D+2)(D-1)^2} (e^{-2x} + 2\sinh x)$$

$$= \frac{1}{(D+2)(D-1)^2} \left[e^{-2x} + \cancel{2}\left(\frac{e^{x}-e^{-x}}{\cancel{2}}\right)\right]$$

$$= \frac{1}{(D+2)(D-1)^2} \left[e^{-2x} + e^{x} - e^{-x}\right]$$

$$= \underbrace{\frac{1}{(D+2)(D-1)^2} e^{-2x}}_{(PI_1)} + \underbrace{\frac{1}{(D+2)(D-1)^2} e^{x}}_{(PI_2)}$$

$$- \underbrace{\frac{1}{(D+2)(D-1)^2} e^{-x}}_{(PI_3)} \longrightarrow (2)$$

$$PI_1 = \frac{1}{(D+2)(D-1)^2} e^{-2x}$$

$$= \frac{x}{(1+0)\ 2(D-1)(1-0)} e^{-2x}$$

$$= \frac{x}{2(-2-1)} e^{-2x} = \frac{x}{-6} e^{-2x} = \frac{-x}{6} e^{-2x}$$

$$PI_2 = \frac{1}{(D+2)(D-1)^2} e^{x}$$

$$= \frac{x}{}$$

$$PI_1 = \frac{1}{(D+2)(D-1)^2} e^{-2x}$$

$$= \frac{x}{(D+2)\ 2(D-1) + (D-1)^2(1+0)} e^{-2x}$$

$$= \frac{x}{(-2+2)\ 2(-2-1) + (-2-1)^2} e^{-2x}$$

$$= \frac{x}{0 + (-3)^2} e^{-2x} = \frac{x}{9} e^{-2x}$$

$$PI_2 = \frac{1}{(D+2)(D-1)^2} e^{x}$$

$$= \frac{x}{(D+2).2(D-1) + (D-1)^2(1+0)} e^{x}$$

$$= \frac{x}{(1+2)\ 2(1-1) + (1-1)}$$

$$= \frac{x}{(D-1)\left[2(D+2) + (D-1)\right]} e^{x}$$

$$= \frac{x^2}{(D-1)[2(1+0)+(1-0)]+[2(D+2)+(D-1)](1-0)} e^x$$

$$= \frac{x^2}{(1-1)[2+1]+[2(1+3)+(1-1)](1)} e^x$$

$$= \frac{x^2}{0+2(3)+0} e^x$$

$$= \frac{x^2}{6} e^x.$$

$$PI_3 = \frac{1}{(D+2)(D-1)^2} e^{-x}$$

$$= \frac{1}{(-1+2)(-1-1)^2} e^{-x}$$

$$= \frac{1}{(1)(-2)^2} e^{-x} = \frac{1}{4} e^{-x}.$$

from ②,

$$P.I = \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}.$$

Now the solution of equⁿ ① is $y = C.F + P.I$

$$y = c_1 e^x + c_2 x e^x + c_3 e^{-2x} + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$$

⑨ Given D.E is $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x \rightarrow ①$

$$D^2y - 4y = \cosh(2x-1) + 3^x$$

$$(D^2-4)y = \cosh(2x-1) + 3^x$$

An AE is $m^2 - 4 = 0$

$$m^2 - (2)^2 = 0$$

$$(m+2)(m-2) = 0$$

$$m = -2, 2.$$

∴ The the roots are real and distinct.

Now, the C.F $= c_1 e^{-2x} + c_2 e^{2x}$.

Now, the particular Integral $= \frac{1}{F(D)} X.$

$$= \frac{1}{D^2-4}\left[\cosh(2x-1) + 3^x\right]$$

$$= \frac{1}{D^2-4}\cosh(2x-1) + \frac{1}{D^2-4} 3^x$$

$$= \frac{1}{D^2-4}\left[\cosh(2x)\cdot\cosh(1) - \sinh(2x)\cdot\sinh(1)\right] + \frac{1}{D^2-4} 3^x$$

$$= \frac{1}{D^2-4}\cosh(2x)\cosh(1) - \frac{1}{D^2-4}\sinh(2x)\cdot\sinh(1) + \frac{1}{D^2-4} 3^x$$

$$P.I = \cosh(1)\underbrace{\frac{1}{D^2-4}\cosh(2x)}_{PI_1} - \sinh(1)\underbrace{\frac{1}{D^2-4}\sinh(2x)}_{PI_2} + \underbrace{\frac{1}{D^2-4} 3^x}_{PI_3} \longrightarrow (2)$$

$\cosh(a \pm b) = \cosh a \cdot \cosh b \pm \sinh a \sinh b$

$\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b$

$$P.I_1 = \frac{1}{D^2-4}\cosh 2x$$

$$= \frac{1}{D^2-4}\frac{e^{2x}+e^{-2x}}{2}$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}e^{2x} + \frac{1}{D^2-4}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{x}{2D}e^{2x} + \frac{x}{2D}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{x}{4}e^{2x} + \frac{x}{-4}e^{-2x}\right]$$

$$= \frac{1}{2}\cdot\frac{x}{4}\left[e^{2x} - e^{-2x}\right]$$

$$= \frac{x}{4}\left(\frac{e^{2x}-e^{-2x}}{2}\right) = \underline{\frac{x}{4}\sinh(2x)}$$

$$PI_2 = \frac{1}{D^2-4}\sinh(2x)$$

$$= \frac{1}{D^2-4}\left(\frac{e^{2x}-e^{-2x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}e^{2x} - \frac{1}{D^2-4}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{x}{2D}e^{2x} - \frac{x}{2D}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{x}{4}e^{2x} - \left(\frac{x}{-4}\right)e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{x}{4}e^{2x} + \frac{x}{4}e^{-2x}\right]$$

$$= \frac{x}{4}\left[\frac{e^{2x}+e^{-2x}}{2}\right] = \frac{x}{4}\cosh(2x)$$

$$PI_3 = \frac{1}{D^2-4}3^x$$

$$= \frac{1}{D^2-4}e^{\log 3^x}$$

$$= \frac{1}{D^2-4}e^{x\log 3}$$

$$= \frac{1}{D^2-4}e^{(\log 3)x}$$

$$= \frac{1}{(\log 3)^2-4}e^{(\log 3)x}$$

$$= \frac{1}{(\log 3)^2-4}3^x.$$

$$P.I = \frac{x}{4}\overset{\cosh(1)}{\sinh(2x)} - \frac{x}{4}\overset{\sinh(1)}{\cosh(2x)} + \frac{1}{(\log 3)^2 - 4} 3^x$$

$$= \frac{x}{4}\left[\sinh(2x)\cosh(1) - \cosh(2x)\sinh(1)\right] + \frac{1}{(\log 3)^2 - 4} 3^x$$

$$= \frac{x}{4}\sinh(2x-1) + \frac{1}{(\log 3)^2 - 4} 3^x.$$

$\therefore$ the solution of equ^n ① is $y = C.F + P.I$

$$y = c_1 e^{-2x} + c_2 e^{2x} + \frac{x}{4}\sinh(2x-1) + \frac{1}{(\log 3)^2 - 4} \cdot 3^x.$$

Wednesday 30/10/19

Type - II

④ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$

sol:

$$D^2 y + 2Dy + y = e^{2x} - \cos^2 x.$$

$$(D^2 + 2D + 1)y = e^{2x} - \cos^2 x.$$

An A.E is $m^2 + 2m + 1 = 0$

$$m^2 + m + m + 1 = 0$$

$$m(m+1) + 1(m+1) = 0$$

$$(m+1)(m+1) = 0$$

$$\therefore m = -1, -1$$

$\therefore$ The roots are real and repeat.

Now, the $C.F = c_1 e^{-x} + c_2 x . e^{-x}$.

$$PI = \frac{1}{f(D)} X$$

$$= \frac{1}{D^2 + 2D + 1}(e^{2x} - \cos^2 x)$$

$$= \underbrace{\frac{1}{D^2 + 2D + 1} e^{2x}}_{PI_1} - \underbrace{\frac{1}{D^2 + 2D + 1}\cos^2 x}_{PI_2}$$

$$PI_1 = \frac{1}{D^2 + 2D + 1} e^{2x}$$

$$= \frac{1}{4 + 4 + 1} e^{2x} = \frac{1}{9} e^{2x}$$

$$PI_2 = \frac{1}{D^2+2D+1} \cos^2 x$$

$$\boxed{\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}}$$

$$= \frac{1}{D^2+2D+1}\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+2D+1}(1+\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+2D+1}(1) + \frac{1}{D^2+2D+1}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+2D+1} e^{(0)x} + \frac{1}{D^2+2D+1} \cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{0+0+1} e^{(0)x} + \frac{1}{-4+2D+1} \cos 2x\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2D-3} \cos 2x\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2D-3} \times \frac{2D+3}{2D+3} \cos 2x\right]$$

$$= \frac{1}{2}\left[1 + \frac{2D+3}{4D^2-9} \cos 2x\right]$$

$$= \frac{1}{2}\left[1 + \frac{2D+3}{4(-4)-9} \cos 2x\right]$$

$$= \frac{1}{2}\left[1 + \frac{2D+3}{-16-9} \cos 2x\right]$$

$$= \frac{1}{2}\left[1 + \frac{2D+3}{-25} \cos 2x\right]$$

$$= \frac{1}{2}\left[1 - \frac{2D+3}{25} \cos 2x\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{25}(2D\cos 2x + 3\cos 2x)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{25}(2 - \sin 2x\,(2) + 3\cos 2x)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{25}(-4\sin 2x + 3\cos 2x)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{25}(3\cos 2x - 4\sin 2x)\right]$$

$$= \frac{1}{2} - \frac{1}{50}(3\cos 2x - 4\sin 2x)$$

$$= \frac{1}{2} - \frac{3}{50}\cos 2x + \frac{2}{25}\sin 2x.$$

$$PI = \frac{1}{9}e^{2x} + \frac{1}{2} - \frac{3}{50}\cos 2x + \frac{2}{25}\sin 2x$$

Now, the solution is $y = CF + P.I$

$$y = \frac{1}{9}e^{2x} + \frac{1}{2} - \frac{3}{50}\cos 2x + \frac{2}{25}\sin 2x + C_1 e^{-x} + C_2 x e^{-x}$$

⑤ $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{-x} + \sin 2x$

Sol:-

$$D^3y + 2D^2y + Dy = e^{-x} + \sin 2x$$

$$(D^3 + 2D^2 + D)y = e^{-x} + \sin 2x$$

An. A.E is $m^3 + 2m^2 + m = 0$.

$$(m+1)(m^2+m) = 0$$

$$\begin{array}{r|cccc} -1 & 1 & 2 & 1 & 0 \\ & 0 & -1 & -1 & 0 \\ \hline & 1 & 1 & 0 & \boxed{0} \end{array}$$

$m+1=0 \qquad m(m+1)=0$

$m=-1, \qquad m+1=0$

$m=-1, \; m=0$

$\therefore m = -1, -1, 0$

$\therefore$ The roots are real and repeat.

Now, $C.F = C_1 e^{-x} + C_2 x e^{-x} + C_3 e^{(0)x}$.

$$PI = \frac{1}{D^3+2D^2+D}(e^{-x} + \sin 2x)$$

$$= \underbrace{\frac{1}{D^3+2D^2+D}e^{-x}}_{PI_1} + \underbrace{\frac{1}{D^3+2D^2+D}\sin 2x}_{PI_2}$$

$$PI_1 = \frac{1}{D^3+2D^2+D}e^{-x}$$

$$= \frac{x}{3D^2+4D+1}e^{-x}$$

$$= \frac{x^2}{6D+4}e^{-x}$$

$$= \frac{x^2}{-6+4}e^{-x} = \frac{-x^2}{2}e^{-x}$$

$$PI_2 = \frac{1}{D^3+2D^2+D}\sin 2x$$

$$= \frac{1}{D^2 \cdot D + 2D^2 + D}\sin 2x$$

$$= \frac{1}{(-4)D + 2(-4) + D}\sin 2x$$

$$= \frac{1}{-4D - 8 + D}\sin 2x$$

$$= \frac{1}{-3D-8} \sin 2x$$

$$= \frac{1}{-3D-8} \times \frac{-3D+8}{-3D+8} \sin 2x$$

$$= \frac{-3D+8}{9D^2-64} \sin 2x$$

$$= \frac{-3D+8}{9(-4)-64} \sin 2x$$

$$= \frac{-3D+8}{-36-64} \cdot \sin 2x$$

$$= \frac{\cancel{-}(3D-8)}{\cancel{-}100} \sin 2x$$

$$= \frac{1}{100}\left[3D \sin 2x - 8\sin 2x\right]$$

$$= \frac{1}{100}\left[3\cos 2x(2) - 8\sin 2x\right]$$

$$= \frac{\cancel{6}^{3}}{\cancel{100}_{50}} \cdot \cos 2x - \frac{\cancel{8}^{2}}{\cancel{100}_{25}} \sin 2x$$

$$= \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x.$$

$$PI = \frac{-x^2}{2}e^{-x} + \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$$

Now, the solution is $y = C.F + P.I$

$$y = c_1e^{-x} + c_2x \cdot e^{-x} + c_3e^{(0)x} - \frac{x^2}{2}e^{-x} + \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$$

⑥ $(D^2+D+1)y = (1+\sin x)^2$

Sol: $(D^2+D+1)y = 1 + \sin^2 x + 2\sin x$

An A.E is $m^2+m+1 = 0$

$$m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\therefore m = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\therefore m = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$$

$\therefore$ The roots are complex and distinct.

Now, $C.F = e^{-\frac{1}{2}x}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]$

$$PI = \frac{1}{D^2+D+1}\left(1+\sin^2 x + 2\sin x\right)$$

$$= \frac{1}{D^2+D+1}(1) + \frac{1}{D^2+D+1}\sin^2 x + \frac{1}{D^2+D+1} 2\sin x$$

$$PI_1 \qquad PI_2 \qquad PI_3 \longrightarrow (2)$$

$$PI_1 = \frac{1}{D^2+D+1} e^{(0)x}$$

$$= \frac{1}{0+0+1} e^{(0)x}$$

$$= (1) e^{(0)x}.$$

$$PI_2 = \frac{1}{D^2+D+1}\sin^2 x$$

$$= \frac{1}{D^2+D+1}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+D+1} - \frac{1}{D^2+D+1}.\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+D+1} e^{(0)x} - \frac{1}{D^2+D+1}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{0+0+1} e^{(0)x} - \frac{1}{-4+D+1}\cos 2x\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{D-3}\cos 2x\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{D-3}\times\frac{D+3}{D+3}\cos 2x\right]$$

$$= \frac{1}{2}\left[1 - \frac{D+3}{D^2-9}\cos 2x\right]$$

$$= \frac{1}{2}\left[1 - \frac{D+3}{-4-9}\cos 2x\right]$$

$$= \frac{1}{2}\left[1 - \frac{D+3}{-13}\cos 2x\right]$$

$$= \frac{1}{2}\left[1 + \frac{D+3}{13}.\cos 2x\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{13}\left(D\cos 2x + 3.\cos 2x\right)\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{13}\left(-\sin 2x(2) + 3\cos 2x\right)\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{13}\left(-2\sin 2x + 3\cos 2x\right)\right]$$

$$= \frac{1}{2} - \frac{\sin 2x}{13} + \frac{3}{2}\cos 2x.$$

$$PI_3 = \frac{1}{D^2+D+1} 2\sin x.$$

$$= 2 \cdot \frac{1}{D^2+D+1} \sin x$$

$$= 2 \frac{1}{-\cancel{1}+D+\cancel{1}} \sin x.$$

$$= 2 \cdot \frac{1}{D} \sin x.$$

$$= 2(-\cos x).$$

$$PI = 1 + \frac{1}{2} - \frac{1}{13}\sin 2x + \frac{3}{2}\cos 2x - 2\cos x.$$

Now, the solution of ~~eqn~~ is $y = C.F + P.I$

$$y = e^{-1/2 x}\left[C_1\cos\left(\frac{\sqrt{3}}{2}\right)x + C_2\sin\frac{\sqrt{3}}{2}x\right] + 1 + \frac{1}{2} - \frac{1}{13}\sin 2x + \frac{3}{2}\cos 2x - 2\cos x.$$

① $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x.$

Given D.E is $D^3y + D^2y + Dy + y = \sin 2x.$

$$(D^3 + D^2 + D + 1)y = \sin 2x.$$

An A.E is $m^3 + m^2 + m + 1 = 0.$

$$(m+1)(m^2+1) = 0.$$

$m+1 = 0,$ ~~$m^2 + (m)^2 = 0$~~ ~~$(m+1)^2 - 2m = 0$~~

$m = -1,$ $m = \pm i$

$\therefore$ The roots are real, complex and distinct.

$$C.F = C_1 e^{-x} + e^{(0)x}[C_1\cos x + C_2\sin x]$$

$$PI = \frac{1}{D^3+D^2+D+1}\sin 2x$$

$$= \frac{1}{-4D - 4 + D + 1}\sin 2x$$

$$= \frac{1}{-3D-3}\sin 2x$$

$$= \frac{1}{-3D-3} \times \frac{-3D+3}{-3D+3}\sin 2x$$

$$= \frac{-3D+3}{9D^2-9} \sin 2x$$

$$= \frac{-3D+3}{9(-4)-9} \sin 2x$$

$$= \frac{-3D+3}{-36-9} \sin 2x$$

$$= \frac{-3D+3}{-45} \sin 2x$$

$$= \frac{\cancel{-}(3D-3)}{\cancel{-}45} \sin 2x$$

$$= \frac{\cancel{3}(D-1)}{\cancel{45}_{15}} \sin 2x$$

$$= \frac{D-1}{15} \sin 2x$$

$$= \frac{1}{15}\left[D\cdot\sin 2x - \sin 2x\right]$$

$$= \frac{1}{15}\left[\cos 2x.(2) - \sin 2x\right]$$

$$= \frac{1}{15}\left[2\cos 2x - \sin 2x\right]$$

$$P.I. = \frac{1}{15}\left[2\cos 2x - \sin 2x\right]$$

$\therefore$ The solution of Equⁿ ① is $y = C.F + P.I$

$$y = c_1 e^{-x} + e^{(0)x}\left[c_1\cos x + c_2 \sin x\right] + \frac{1}{15}\left[2\cos 2x - \sin 2x\right]$$

② $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos 2x$

sol: $D^2y + Dy = \cos 2x.$

$(D^2 + D)\, y = \cos 2x.$

An A.E is $m^2 + m = 0$

$m(m+1) = 0$

$m = 0, \quad m = -1$

∴ The roots are real and ~~not~~ distrinct.

Now $C.F = c_1 e^{(0)x} + c_2 e^{-x}.$

$PI = \frac{1}{D^2+D} \cos 2x$

$= \frac{1}{-4+D} \cos 2x$

$= \frac{1}{-4+D} \times \frac{-4-D}{-4-D} \cos 2x$

$= \frac{-4-D}{16-D^2} \cos 2x$

$= \frac{-4-D}{16-(-4)}$

$= \frac{-4-D}{20} \cos 2x$

$= \frac{-1}{20} \left[4\cos 2x + D\cos 2x\right]$

$= \frac{-1}{20} \left[4\cos 2x + (-\sin 2x)\, 2\right]$

$= \frac{-1}{20} \left[4\cos 2x - 2\sin 2x\right]$

$= \frac{-1}{5} \cos 2x + \frac{1}{10} \sin 2x.$

Now the solution is $y = C.F + PI$

$y = c_1 e^{(0)x} + c_2 e^{-x} - \frac{1}{5} \cos 2x + \frac{1}{10} \sin 2x$

③ $(D^3+1)\, y = 2\cos^2 x.$

Given D.E. is $(D^3+1)\, y = 2\cos^2 x$

An A.E is $m^3 + 1 = 0$

$(m+1)(m^2 - m + 1) = 0$

$m = -1, \quad m = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

∴ The roots are real, ~~and~~ complex and distrinct.

Now, $C.F = c_1 e^{-x} + e^{-1/2 x}\left[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right]$

$$PI = \frac{1}{D^3+1} 2\cos^2 x$$

$$= 2\left[\frac{1}{D^3+1} \cos^2 x\right]$$

$$= 2\left[\frac{1}{D^3+1} \left(\frac{1+\cos 2x}{2}\right)\right]$$

$$= \frac{1}{D^3+1}(1) + \frac{1}{D^3+1} \cos 2x$$

$$= \frac{1}{D^3+1} e^{(0)x} + \frac{1}{D^3+1} \cdot \cos 2x$$

$$= \frac{1}{(0)0+1} e^{(0)x} + \frac{1}{-4D+1} \cdot \cos 2x$$

$$= 1 + \frac{1}{-4D+1} \times \frac{-4D-1}{-4D-1} \cos 2x$$

$$= 1 + \frac{-4D-1}{16D^2-1} \cos 2x.$$

$$= 1 - \frac{4D+1}{16D^2+1} \cos 2x.$$

$$= 1 - \frac{4D+1}{-64+1} \cos 2x$$

$$= 1 + \frac{4D+1}{63} \cos 2x$$

$$= 1 + \frac{4D}{63} \cos 2x + \frac{1}{63} \sin 2x$$

$$= 1 + \frac{4}{63}(-\sin 2x)(2) + \frac{1}{63} \sin 2x$$

$$= 1 - \frac{8}{63} \sin 2x + \frac{1}{63} \sin x.$$

~~the solution~~

$\therefore$ The solution of is $y = C.F + PI$

$$y = c_1 e^{-x} + e^{-1/2x}\left[\cos \frac{\sqrt{3}}{2}x + \sin \frac{\sqrt{3}}{2}x\right] + 1 - \frac{8}{63}\sin 2x + \frac{1}{63} \sin x.$$

(8) $(D^2-3D+2)y = 6e^{-3x} + \sin 2x$

Sol: Given D.E is $(D^2-3D+2)y = 6e^{-3x} + \sin 2x \rightarrow (1)$

~~An~~ A.E is $m^2-3m+2=0$

$$m^2-m-2m+2=0$$

$$m(m-1)-2(m-1)=0$$

$$(m-1)(m-2)=0$$

$$m = 1, 2$$

$\therefore$ The roots are real and distinct.

Now, $C.F = c_1 e^{x} + c_2 e^{2x}$

Now, the $PI = \frac{1}{D^2-3D+2}(6e^{-3x} + \sin 2x)$

$$= 6 \cdot \underbrace{\frac{1}{D^2-3D+2} e^{-3x}}_{PI_1} + \underbrace{\frac{1}{D^2-3D+2} \sin 2x}_{PI_2} \longrightarrow ②$$

$PI_1 = 6 \frac{1}{D^2-3D+2} e^{-3x}$

$= 6 \frac{1}{9+9+2} e^{-3x}$

$= \cancel{6}^{3} \frac{1}{\cancel{20}_{10}} e^{-3x} \quad = \underline{\underline{\frac{3}{10} e^{-3x}}}$

$PI_2 = \frac{1}{D^2-3D+2} \sin 2x.$

$= \frac{1}{-4-3D+2} \sin 2x.$

$= \frac{1}{-3D-2} \sin 2x$

$= \frac{1}{-3D-2} \times \frac{-3D+2}{-3D+2} \sin 2x$

$= \frac{-3D+2}{9D^2-4} \sin 2x$

$= \frac{-3D+2}{-36-4} \sin 2x = \frac{-3D+2}{-40} \sin 2x.$

$= \frac{-3}{-40}[D \sin 2x] + \frac{\cancel{2}}{\cancel{-40}_{20}}[\sin 2x]$

$= \frac{3}{\cancel{40}_{20}} \cos 2x(\cancel{2}) - \frac{1}{20} \sin 2x.$

$= \frac{3}{20} \cos 2x - \frac{1}{20} \sin 2x.$

from ②,

$PI = \frac{3}{10} e^{-3x} + \frac{3}{20} \cos 2x - \frac{1}{20} \sin 2x.$

$= \frac{1}{10}\left[3 \cdot e^{-3x} + \frac{3}{2} \cos 2x - \frac{1}{2} \sin 2x\right]$

$\therefore$ The solution of equⁿ① is $y = C.F + P.I$

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{10}\left[3 \cdot e^{-3x} + \frac{3}{2} \cos 2x - \frac{1}{2} \sin 2x\right]$$

⑨ $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$

Sol: Given D.E is $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$.

$$D^2y + 4y = e^x + \sin 2x.$$

$$(D^2+4)\,y = e^x + \sin 2x \quad \rightarrow ①$$

An AE is $m^2+4=0$

$$m^2 = -4$$

$$m = \frac{0 \pm \sqrt{0-16}}{2}$$

$$= \frac{\pm\sqrt{16}\,i}{2}$$

$$= \frac{\pm \cancel{4}^{2} i}{\cancel{2}}$$

$$m = \pm 2i$$

∴ The roots are complex and distrinct.

Now, the $C.F = e^{(0)x}[c_1\cos 2x + c_2 \sin 2x]$

Now, the $P.I = \frac{1}{D^2+4}(e^x + \sin 2x)$

$$= \underbrace{\frac{1}{D^2+4}e^x}_{P.I_1} + \underbrace{\frac{1}{D^2+4}\sin 2x}_{PI_2} \quad \rightarrow ②$$

$$\cancel{= \frac{1}{1+4}e^x + \frac{1}{-4+4}}$$

$$PI_1 = \frac{1}{D^2+4}e^x = \frac{1}{1+4}e^x = \frac{1}{5}e^x$$

$$PI_2 = \frac{1}{D^2+4}\sin 2x$$

$$= \frac{x}{2D}\sin 2x$$

$$= \frac{x}{2}\,\frac{1}{D}\sin 2x$$

$$= \frac{x}{2}\;\frac{-\cos 2x}{2}$$

$$= \frac{-x}{4}\cdot\cos 2x.$$

$$PI = \frac{1}{5}e^x - \frac{x}{4}\cos 2x.$$

Now the solution of Equn ① is $y = C.F + P.I$

$$y = e^{(0)x}\{c_1\cos 2x + c_2 \sin 2x\} + \frac{1}{5}e^x - \frac{x}{4}\cos 2x.$$

7 $(D^2-4D+3)y = \sin 3x \cos 2x$.

Given DE is $(D^2-4D+3)y = \sin 3x . \cos 2x$ $\rightarrow$ (1).

AE is $m^2-4m+3=0$

$m^2-m-3m+3=0$

$m(m-1)-3(m-1)=0$

$(m-1)(m-3)=0$

$m=1, 3$.

$\therefore$ The roots are real and distinct.

Now, the $C.F = c_1 e^{x} + c_2 e^{3x}$

$\boxed{\begin{array}{l} \sin A \cdot \cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)] \\ \cos A \cdot \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)] \\ \sin A \cdot \sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)] \end{array}}$

$P.I = \frac{1}{D^2-4D+3} \sin 3x . \cos 2x$

$= \frac{1}{D^2-4D+3} \frac{1}{2}[\sin 5x + \sin x]$

$= \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{D^2-4D+3}\sin x\right]$

$= \underbrace{\frac{1}{2}\frac{1}{D^2-4D+3}\sin 5x}_{PI_1} + \underbrace{\frac{1}{2}\frac{1}{D^2-4D+3}\sin x}_{PI_2} \rightarrow (2)$.

$PI_1 = \frac{1}{2}\frac{1}{D^2-4D+3}\sin 5x$

$= \frac{1}{2}\frac{1}{-25-4D+3}\sin 5x$

$= \frac{1}{2}\frac{1}{-4D-22}\sin 5x$

$= \frac{1}{2}\frac{1}{-4D-22} \times \frac{-4D+22}{-4D+22}\sin 5x$

$= \frac{1}{2}\frac{-4D+22}{16D^2-484}\sin 5x$.

$= \frac{1}{2}\frac{-4D+22}{16(-25)-484}\sin 5x$

$= \frac{1}{2}\frac{-4D+22}{-400-484}\sin 5x$

$= \frac{1}{2}\frac{\cancel{-}(4D-22)}{\cancel{-}884}\sin 5x$

$$= \frac{1}{884}\left[4\,(D\sin 5x) - 22\sin 5x\right]$$

$$= \frac{1}{884}\left[4.\cos 5x\,(5) - 22\sin 5x\right]$$

$$= \frac{\cancel{20}^{5}}{\cancel{884}_{221}}\cos 5x - \frac{\cancel{22}^{11}}{\cancel{884}_{442}}\sin 5x$$

$$= \frac{5}{221}\cos 5x - \frac{11}{442}\sin 5x.$$

$$PI_2 = \frac{1}{D^2-4D+3}\sin x$$

$$= \frac{1}{-1-4D+3}\sin x$$

$$= \frac{1}{-4D+2}\sin x = \frac{1}{-4D+2} \times \frac{-4D-2}{-4D-2}\sin x$$

$$= \frac{-4D+2}{16D^2-4}\sin x$$

$$= \frac{-4D-2}{16(-1)-4}\sin x$$

$$= \frac{-(4D+2)}{-16-4}\sin x$$

$$= \frac{+\cancel{2}(2D+1)}{+\cancel{20}_{10}}\sin x = \frac{1}{10}\left[2(D\sin x) + \sin x\right]$$

$$= \frac{1}{10}\left[2.\cos x + \sin x\right]$$

$$= \frac{2}{10}\cos x + \frac{1}{10}\sin x.$$

$$= \frac{1}{5}\cos x + \frac{1}{10}\sin x.$$

$$PI = \frac{1}{2}\left[\frac{5}{221}\cos 5x - \frac{11}{442}\sin 5x + \frac{1}{5}\cos x + \frac{1}{10}\sin x\right]$$

Now the solution of Equ^n ① is $y = C.F + P.I$

$$y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{2}\left[\frac{5}{221}\cos 5x - \frac{11}{442}\sin 5x + \frac{1}{5}\cos x + \frac{1}{10}\sin x\right]$$

(10) $\frac{d^3y}{dx^3} + y = \cos(2x-1)$

Given D.E is $\frac{d^3y}{dx^3} + y = \cos(2x-1)$

$$D^3y + y = \cos(2x-1)$$

$$(D^3+1)\,y = \cos(2x-1) \quad \rightarrow ①$$

An A.E is $m^3+1=0$.

$$(m+1)(m^2-m+1)=0$$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & 0 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

$$(m+1)(m^2-m+1)=0$$

$m+1=0$, $\quad m^2-m+1=0$

$m=-1$, $\quad m=\frac{1\pm\sqrt{1-4}}{2}$

$$=\frac{1\pm\sqrt{3}i}{2}$$

$\therefore m=-1,\ \frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$\therefore$ The roots are real, complex and distinct.

Now, the $C.F = c_1e^{-x}+e^{\frac{1}{2}x}\left[c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right]$

$$P.I=\frac{1}{D^3+1}\cos(2x-1)$$

$$=\frac{1}{D^3+1}\left[\cos 2x.\cos(1)+\sin 2x\sin(1)\right]$$

$$=\underbrace{\cos(1)\frac{1}{D^3+1}\cos 2x}_{PI_1}+\underbrace{\sin(1)\frac{1}{D^3+1}\sin 2x}_{PI_2}\longrightarrow (2)$$

$$PI_1=\cos(1)\frac{1}{D^3+1}\cos 2x$$

$$=\cos(1)\frac{1}{D^2\cdot D+1}\cos 2x$$

$$=\cos(1)\frac{1}{-4D+1}\cos 2x$$

$$=\cos(1)\frac{1}{-4D+1}\times\frac{-4D-1}{-4D-1}\cos 2x$$

$$=\cos(1)\frac{-4D-1}{16D^2-1}\cos 2x$$

$$=\cos(1)\frac{-(4D+1)}{16(-4)-1}\cos 2x$$

$$=\cos(1)\frac{+(4D+1)}{\cancel{-}65}\cos 2x$$

$$=\frac{\cos(1)}{65}\left[4(D\cos 2x)+\cos 2x\right]$$

$$=\frac{\cos(1)}{65}\left[4.(-\sin 2x)(2)+\cos 2x\right]$$

$$=\frac{\cos(1)}{65}\left[-8\sin 2x+\cos 2x\right]$$

$$PI_2 = \sin(1)\frac{1}{D^3+1}\sin 2x$$

$$= \sin(1)\frac{1}{-4D+1}\sin 2x$$

$$= \sin(1)\frac{1}{-4D+1}\times\frac{-4D-1}{-4D-1}\sin 2x$$

$$= \sin(1)\frac{-4D-1}{16D^2-1}\sin 2x$$

$$= \sin(1)\frac{-(4D+1)}{+16(4)-1}\sin 2x$$

$$= \sin(1)\frac{+(4D+1)}{+65}\sin 2x$$

$$= \frac{\sin(1)}{65}\left[4(D\sin 2x) + \sin 2x\right]$$

$$= \frac{\sin(1)}{65}\left[4.\ \cos 2x(2) + \sin 2x\right]$$

$$= \frac{\sin(1)}{65}\left[8\cos 2x + \sin 2x\right]$$

$$PI = \frac{\cos(1)}{65}\left(-8\sin 2x + \cos 2x\right) + \frac{\sin(1)}{65}\left(8\cos 2x + \sin 2x\right)$$

∴ The solution of equn is, $y = C.F + P.I$.

$$y = c_1e^{-x} + e^{-\frac{1}{2}x}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]$$

$$+ \frac{\cos(1)}{65}\left(-8\sin 2x + \cos 2x\right) + \frac{\sin(1)}{65}\left(8\cos 2x + \sin 2x\right)$$

Monday
4/4/2019

Type - III

① $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

Sol: Given D.E is $D^2y + Dy = x^2 + 2x + 4$.

$$(D^2 + D)y = x^2 + 2x + 4 \rightarrow ①$$

An A.E is $m^2 + m = 0$

$m(m+1) = 0$

$m = 0, -1$

∴ The roots are real and distinct.

Now, $C.F = c_1e^{(0)x} + c_2e^{-x}$

$$P.I = \frac{1}{D^2+D}(x^2 + 2x + 4)$$

$$= \frac{1}{D(D+1)}(x^2 + 2x + 4)$$

$$= \frac{1}{D(1+D^2)}(x^2+2x+4)$$

$$= \frac{1}{D}(1+D^2)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}[1-D^2+D^4-D^3+\cdots](x^2+2x+4)$$

$$= \frac{1}{D}[x^2+2x+4-(2x+2)+2]$$

$$= \frac{1}{D}[x^2+\cancel{2x}+4-\cancel{2x}-\cancel{2}+\cancel{2}]$$

$$= \frac{1}{D}(x^2+4)$$

$$P.I. = \frac{x^3}{3}+4x.$$

Now the solution of equ^n ① is $y = C.F + P.I$

$$y = c_1 e^{(0)x} + c_2 e^{-x} + \frac{x^3}{3} + 4x.$$

② $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1+x^2$

sol: Given D.E is $D^3y - D^2y - 6Dy = 1+x^2$

$$(D^3-D^2-6D)y = 1+x^2 \rightarrow ①$$

A.E is $m^3-m^2-6m = 0.$

$$\begin{array}{r|rrrr} 3 & 1 & -1 & -6 & 0 \\ & 0 & 3 & 6 & 0 \\ \hline & 1 & 2 & 0 & \boxed{0} \end{array}$$

$$(m-3)(m^2+2m) = 0$$

$$(m-3)\,m(m+2) = 0$$

$$m=0,\ m=-2,\ m=3.$$

∴ The roots are real and distinct.

$$C.F = c_1 e^{(0)x} + c_2 e^{-2x} + c_3 e^{3x}$$

$$P.I = \frac{1}{D^3-D^2-6D}(1+x^2)$$

$$= \frac{1}{+6D\left(\frac{D^2-D}{6}-1\right)}(1+x^2)$$

$$= \frac{1}{-6D\left[1-\left(\frac{D^2-D}{6}\right)\right]}(1+x^2)$$

$$= \frac{-1}{6D}\left[1-\left(\frac{D^2-D}{6}\right)\right]^{-1}(1+x^2)$$

$$= \frac{-1}{6D}\left[1 + \left(\frac{D^2-D}{6}\right) + \left(\frac{D^2-D}{6}\right)^2 + \cdots\right](1+x^2)$$

$$= \frac{-1}{6D}\left[(1+x^2) + \frac{D^2-D}{6}(1+x^2) + \left(\frac{D^2-D}{6}\right)^2(1+x^2)\right]$$

$$= \frac{-1}{6D}\left[1+x^2 + \frac{1}{6}\left[2-(0+2x)\right] + \left(\frac{D^4+D^2-2D^3}{36}\right)(1+x^2)\right]$$

$$= \frac{-1}{6D}\left[1+x^2+\frac{1}{6}(2-2x)\right] + \frac{1}{36}\left[0+2-0\right]\right]$$

$$= \frac{-1}{6D}\left[1+x^2+\cancel{\frac{2}{6}}^{\frac{1}{3}}(1-x)\right] + \cancel{\frac{1}{36}}_{18}(\cancel{2})\right]$$

$$= \frac{-1}{6D}\left[1+x^2+\frac{1-x}{3}\right] + \frac{1}{18}\right]$$

$$\cancel{= \frac{1}{6D}\left(x^2-x+2\right)+\frac{1}{18}}$$

$$\cancel{= \frac{-1}{6}\left[\frac{1}{D}(x^2) - \frac{1}{D}(x) + \frac{1}{D}(2)\right] + \frac{1}{D}\left(\frac{1}{18}\right)\right]}$$

$$\cancel{= \frac{-1}{6}\left[\frac{x^3}{3} - \frac{x^2}{2} + 2x\right] + \frac{1}{18}x\right]}$$

$$= \frac{-1}{6D}\left[1+x^2+\frac{1}{3}-\frac{x}{3}+\frac{1}{18}\right]$$

$$= \frac{-1}{6}\left[\frac{1}{D}(1) + \frac{1}{D}(x^2) - \frac{1}{D}\frac{1}{3} - \frac{1}{D}\frac{x}{3} + \frac{1}{D}\frac{1}{18}\right]$$

$$= \frac{-1}{6}\left[x + \frac{x^3}{3} - \frac{1}{3}x - \frac{1}{3}\frac{x^2}{2} + \frac{1}{18}x\right]$$

$$= \frac{-1}{6}\left[x + \frac{x^3}{3} - \frac{x}{3} - \frac{x^2}{6} + \frac{x}{18}\right]$$

$$= \frac{-1}{6}\left[\frac{18x+6x^3-6x-3x^2+x}{18}\right]$$

$$P.I. = \frac{-1}{108}\left(6x^3-3x^2+13x\right)$$

Now, the solution of Equ^n ① is $y = C.F + P.I$

$$y = c_1 e^{(0)x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{108}\left(6x^3-3x^2+13x\right)$$

⑤ $\frac{d^2y}{dx^2}$

sol:

PI

P

⑤ $\frac{d^2y}{dx^2} - 4y = x^2+2x$.

sol:- Given D.E is $D^2y - 4y = x^2+2x$

$$(D^2-4)\,y = x^2+2x \quad \rightarrow ①$$

The A.E is $m^2-4=0$.

$$(m+2)(m-2)=0$$

$$m = 2, -2$$

∴ The roots are real and distinct.

$$C.F = c_1e^{2x} + c_2e^{-2x}$$

$$PI = \frac{1}{D^2-4}(x^2+2x)$$

$$= \frac{1}{4\left(\frac{D^2}{4}-1\right)}(x^2+2x)$$

$$= \frac{1}{-4\left(1-\frac{D^2}{4}\right)}(x^2+2x)$$

$$= \frac{-1}{4}\left(1-\frac{D^2}{4}\right)^{-1}(x^2+2x)$$

$$= \frac{-1}{4}\left[1+\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+\cdots\right](x^2+2x)$$

$$= \frac{-1}{4}\left[(x^2+2x)+\frac{1}{4}D^2(x^2+2x)+\frac{1}{16}D^4(x^2+2x)\right]$$

$$= \frac{-1}{4}\left[x^2+2x+\frac{1}{4}(2)+0\right]$$

$$= \frac{-1}{4}\left[x^2+2x+\frac{1}{2}\right]$$

$$= \frac{-1}{4}\left[\frac{2x^2+4x+1}{2}\right]$$

$$P.I = \frac{-1}{8}\left[2x^2+4x+1\right]$$

Now, the solution of Equⁿ ① is $y = C.F + P.I$

$$y = c_1e^{2x}+c_2e^{-2x} - \frac{1}{8}\left[2x^2+4x+1\right].$$

⑧ $(D^3-D)z = 2y+1+4\cos y+2e^y$

Given D.E is $(D^3-D)z = 2y+1+4\cos y+2e^y \rightarrow ①$

An A.E is $m^3-m=0$

$m(m^2-1)=0$

$m(m+1)(m-1)=0$

$m=0,-1,1$

$\therefore$ The roots are real and distinct.

$C.F = c_1e^{(0)x}+c_2e^{-x}+c_3e^{x}$

$$P.I = \frac{1}{D^3-D}(2y+1+4\cos y+2e^y)$$

$$= \frac{1}{D^3-D}2y+\frac{1}{D^3-D}(1)+\frac{1}{D^3-D}4\cos y+\frac{1}{D^3-D}2e^y.$$

$$= 2\frac{1}{D^3-D}y+\frac{1}{D^3-D}e^{(0)y}+(4)\frac{1}{D^3-D}\cos y+2\cdot\frac{1}{D^3-D}e^y \rightarrow ②$$

PI_1 PI_2 PI_3 PI_4

$$PI_1 = 2\cdot\frac{1}{D^3-D}y$$ ③

$$= 2\frac{1}{D(D^2-1)}y$$

$$= \frac{2}{-D}\frac{1}{(1-D^2)}y = \frac{-2}{D}(1-D^2)^{-1}y$$

$$= \frac{-2}{D}[1+D^2+(D^2)^2+(D^2)^3+\cdots]y$$

$$= \frac{-2}{D}[y+D^2(y)+0+0]$$

$$= \frac{-2}{D}[y+0]$$

$$= \frac{-2}{D}(y)$$

$$= -2\frac{1}{D}(y)$$

$$= -\cancel{2}\cdot\frac{y^2}{\cancel{2}}$$

$$= \underline{-y^2}$$

$PI_2 = \frac{1}{D^3-D} e^{(0)y}$

$= \frac{y}{3D^2-1} e^{(0)y}$

$= \frac{y}{0-1} e^{(0)y} = \underline{-y}$

$PI_3 = 4\frac{1}{D^3-D} \cos y$

$= 4\frac{y}{3D^2-1} \cos y$

$= 4\frac{y}{3(-1)-1} \cos y$

$= \cancel{4}\frac{y}{-\cancel{4}} \cos y$

$= \underline{-y \cdot \cos y}$

$PI_4 = 2\frac{1}{D^3-D} e^y$

$= 2\frac{y}{3D^2-1} e^y$

$= 2\frac{y}{3(1)-1} e^y$

$= \cancel{2}\frac{y}{\cancel{2}} e^y$

$= \underline{y \cdot e^y}$

$PI = -y^2 - y - y\cos y + ye^y.$

Now the solution of equ^n ① is $z = C.F + P.I$

$z = c_1 e^{(0)x} + c_2 e^{-x} + c_3 e^x - y^2 - y - y\cos y + ye^y.$

③ $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2) \rightarrow ①$

An A.E is $(m-2)^2 = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$

$\therefore$ The roots are real and repeat.

$C.F = c_1 e^{2x} + c_1 x.e^{2x}.$

$P.I = \frac{1}{(D-2)^2} 8(e^{2x} + \sin 2x + x^2)$

$= 8 \cdot \frac{1}{2\left[\frac{D^2}{2}-1\right]^2}(e^{2x} + \sin 2x + x^2)$

$= -\frac{8}{2}\frac{1}{\left[1-\frac{D^2}{2}\right]^2}(e^{2x} + \sin 2x + x^2).$

$= -4\left[1-\frac{D^2}{2}\right]^{-2}(e^{2x} + \sin 2x + x^2)$

$= -4\{$

$$= 8\left[\underbrace{\frac{1}{(D-2)^2}e^{2x}}_{PI_1} + \underbrace{\frac{1}{(D-2)^2}\cdot\sin 2x}_{PI_2} + \underbrace{\frac{1}{(D-2)^2}\cdot x^2}_{PI_3}\right] \rightarrow (2)$$

$PI_1 = \frac{1}{(D-2)^2} e^{2x}$

$= \frac{x}{2(D-2)} e^{2x}$

$= \frac{x^2}{2(1)} e^{2x}$

$= \frac{x^2}{2} \cdot e^{2x}$

$PI_2 = \frac{1}{D^2 - 4D + 4} \sin 2x$

$= \frac{1}{\cancel{-4} - 4D + \cancel{4}} \sin 2x$

$= \frac{-1}{4} \frac{1}{D} \sin 2x$

$= \cancel{-}\frac{1}{4} \frac{(\cancel{-}\cos 2x)}{2}$

$= \frac{1}{8} \cos 2x$

$PI_3 = \frac{1}{(D-2)^2} x^2 = \frac{1}{-2\left(1-\frac{D^2}{2}\right)^2} x^2$

$= \frac{-1}{2}\left(1-\frac{D^2}{2}\right)^{-2} \cdot x^2$

$= \frac{-1}{2}\left[1 + \cancel{2}\frac{D^2}{\cancel{2}} + 3\left(\frac{D^2}{2}\right)^2 + \cdots\right] x^2$

$= \frac{-1}{2}\left[x^2 + D^2(x^2) + 0\right]$

$= \frac{-1}{2}\left[x^2 + 2\right]$

from (2),

$PI = 8\left[\frac{x^2}{2} e^{2x} + \frac{1}{8}\cos 2x - \frac{1}{2}(x^2+2)\right]$

$= \cancel{8}\left(\frac{4x^2 e^{2x} + \cos 2x - 4x^2 - 8}{\cancel{8}}\right)$

$= 4x^2 e^{2x} + \cos 2x - 4x^2 - 8$

Now the solution of equ(1) is $y = C.F + P.I$

$y = c_1 e^{2x} + c_2 \cdot x e^{2x} + 4x^2 e^{2x} + \cos 2x - 4x^2 - 8$

(6) $\frac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3$.

Given DE is $D^2y + y = e^{2x} + \cosh 2x + x^3$

$(D^2+1)\, y = e^{2x} + \cosh 2x + x^3 \rightarrow (1)$

An A.E is $m^2 + 1 = 0$

$m = \frac{-0 \pm \sqrt{0-4}}{2}$

$= \frac{\pm \cancel{2} i}{\cancel{2}}$

$= \pm i$

$\therefore$ The roots are ~~real~~ complex and distinct.

$$C.F = e^{(0)x}\left[c_1 \cos x + c_2 \sin x\right]$$

$$P.I = \frac{1}{D^2+1}\left[e^{2x} + \cosh 2x + x^3\right]$$

$$= \underbrace{\frac{1}{D^2+1} e^{2x}}_{PI_1} + \underbrace{\frac{1}{D^2+1} \cosh 2x}_{PI_2} + \underbrace{\frac{1}{D^2+1} x^3}_{PI_3}$$

$$PI_1 = \frac{1}{D^2+1} e^{2x}$$

$$= \frac{1}{4+1} e^{2x} = \frac{1}{5} e^{2x}$$

$$PI_2 = \frac{1}{D^2+1} \cosh 2x$$

$$= \frac{1}{D^2+1} \frac{e^{2x}+e^{-2x}}{2}$$

$$= \frac{1}{2}\left(\frac{1}{D^2+1} e^{2x} + \frac{1}{D^2+1} e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4+1} e^{2x} + \frac{1}{4+1} e^{-2x}\right)$$

$$= \frac{1}{2}\left[\frac{1}{5} e^{2x} + \frac{1}{5} e^{-2x}\right)$$

$$= \frac{1}{10}\left(e^{2x} + e^{-2x}\right)$$

$$PI_3 = \frac{1}{D^2+1} \cdot x^3$$

$$= \frac{1}{1+D^2} x^3$$

$$= (1+D^2)^{-1} x^3$$

$$= \left[1 - D + D^2 - D^3 + \dots\right] x^3$$

$$= x^3 - D(x^3) + D^2(x^3) - D^3(x^3)$$

$$= x^3 - 3x^2 + 6x - 6$$

from ③

$$PI = \frac{1}{5} e^{2x} + \frac{1}{10}\left(e^{2x} + e^{-2x}\right) + x^3 - 3x^2 + 6x - 6$$

Now the solution of equ^n ① is $y = C.F + P.I$

$$y = e^{(0)x}\left[c_1 \cos x + c_2 \sin x\right] + \frac{1}{5}\left[e^{2x} + \frac{1}{2}\left(e^{2x} + e^{-2x}\right)\right] + x^3 - 3x^2 + 6x - 6$$

⑦ $(D-1)^2(D+1)^2 y = \sin^2\frac{x}{2} + e^x + x$.

Sol:- Given D.E is $(D-1)^2(D+1)^2 = \sin^2 x/_2 + e^x + x \rightarrow (1)$

An A.E is $(m-1)^2(m+1)^2 = 0$

$m = 1, 1, \quad m = -1, -1$

$\therefore$ The roots are real and repeat.

$C.F = c_1 e^x + c_2 x.e^x + c_3 e^{-x} + c_4 x.e^{-x}$.

$PI = \frac{1}{(D-1)^2(D+1)^2}\left(\sin^2 x/_2 + e^x + x\right)$

$= \frac{1}{(D-1)^2(D+1)^2}\left[\frac{1-\cos x}{2} + e^x + x\right]$

$= \frac{1}{(D-1)^2(D+1)^2}\left(\frac{1-\cos x}{2}\right) + \frac{1}{(D-1)^2(D+1)^2}e^x + \frac{1}{(D-1)^2(D+1)^2}x$.

$= \underbrace{\frac{1}{2}\frac{1}{(D-1)^2(D+1)^2}(1)}_{PI_1} - \underbrace{\frac{1}{2}\frac{1}{(D-1)^2(D+1)^2}\cos x}_{PI_2} + \underbrace{\frac{1}{(D-1)^2(D+1)^2}e^x}_{PI_3} + \underbrace{\frac{1}{(D-1)^2(D+1)^2}x}_{PI_4} \rightarrow (2)$

$PI_1 = \frac{1}{2}\frac{1}{(D-1)^2(D+1)^2}e^{(0)x}$

$= \frac{1}{2}\frac{1}{(1)(1)}e^{(0)x} = \frac{1}{2}$.

$PI_2 = \frac{1}{2}\frac{1}{(D-1)^2(D+1)^2}\cos x$

$= \cancel{\frac{1}{2}\frac{x}{(D-1)^2 2(D+1) + (D+1)^2 2(D-1)}\cos x}$

$= \cancel{\frac{1}{2}\frac{x^2}{\quad}}$

$= \frac{1}{2}\frac{1}{[(D-1)(D+1)]^2}\cos x$

$= \frac{1}{2}\frac{1}{(D^2-1)^2}\cos x \qquad = \frac{1}{2}\frac{1}{(-1-1)^2}\cos x$

$= \cancel{\frac{1}{2}\frac{x}{2(D^2-1)2D}\cos x} \qquad = \frac{1}{2}\frac{1}{(-2)^2}\cos x$

$= \cancel{\frac{1}{8}\frac{x^2}{1(2D)+(D^2-1)(1)}\cos x} \qquad = \frac{1}{2}\cdot\frac{1}{4}\cos x$

$= \frac{1}{8}\cos x$

$= \cancel{\frac{1}{8}\frac{x^2}{(1)[2(1)]+(1-1)(1)}\cos x}$

$= \frac{x^2}{8} \frac{1}{(2+0)} \cos 3x$

$\Rightarrow \frac{x^2}{16} \cos 3x$

$PI_3 = \frac{1}{(D-1)^2(D+1)^2} e^x$

$\Rightarrow \frac{1}{(D^2-1)^2} e^x$

$= \frac{x}{2(D^2-1)}$

$= \frac{x}{(D-1)^2}$

$= \frac{1}{(D^2-1)^2} e^x$

$= \frac{x}{2(D^2-1)(2D)} e^x = \frac{1}{4} \frac{x}{D(D^2-1)} e^x$

$= \frac{1}{4} \frac{x^2}{D(2D-0) + (D^2-1)(1)} e^x$

$= \frac{1}{4} \frac{x^2}{(1)\ 2(1) + (0)(1)} e^x$

$= \frac{1}{4} \frac{x^2}{2} e^x \quad = \frac{x^2}{8} e^x$

$PI_4 = \frac{1}{(D-1)^2(D+1)^2} x$

$= \frac{1}{(D^2-1)^2} x = \frac{1}{(-1)(1-D^2)^2} x$

$= (1-D^2)^{-2} x$

$= [1 + 2(D^2) + 3(D^2)^2 + 4(D^2)^3 + 5(D^2)^4 + \dots] x$

$= (x + 2D^2x + 3D^4x + 4D^6x + \dots)$

$= x + 2(0) + 3(0) + 4(0) + \dots$

$= x + 0 + 0$

$= x$

$PI = \frac{1}{2} + \frac{1}{8}\cos 3x + \frac{x^2}{8}e^x + x$

Now the solution of Equ^n ① is $y = C.F + P.I$

$y = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x} + \frac{1}{2} + \frac{1}{8}\cos 3x + \frac{x^2}{8} e^x + x$

wednesday
6/11/19 Type-4.

⑦ $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2e^{3x} + \sin 2x.$

Sol:- Given D.E is $D^2y - 3Dy + 2y = x^2 e^{3x} + \sin 2x$

$$(D^2 - 3D + 2)y = x^2 e^{3x} + \sin 2x \longrightarrow ①$$

An AE is $m^2 - 3m + 2 = 0$

$m^2 - m - 2m + 2 = 0$

$m(m-1) - 2(m-1) = 0$

$(m-1)(m-2) = 0$

$m = 1, 2.$

∴ The roots are real and distinct.

$C.F = c_1 e^{x} + c_2 e^{2x}$

$$PI = \frac{1}{(D^2-3D+2)}(x \cdot e^{3x} + \sin 2x)$$

$$= \underbrace{e^{3x}\frac{1}{(D+3)^2 - 3(D+3) + 2}x^2}_{PI_1} + \underbrace{\frac{1}{(D)^2 - 3(D) + 2}\sin 2x.}_{PI_2}$$

$$PI_1 = e^{3x}\frac{1}{D^2 + \cancel{9} + 6D - 3D - \cancel{9} + 2}x^2$$

$$= e^{3x}\frac{1}{D^2+3D+2}x^2$$

$$= e^{3x}\frac{1}{2\left(\frac{D^2+3D}{2}+1\right)}x^2$$

$$= \frac{e^{3x}}{2}\frac{1}{1+\left(\frac{D^2+3D}{2}\right)}x^2$$

$$= \frac{e^{3x}}{2}\left(1+\frac{D^2+3D}{2}\right)^{-1}x^2$$

$$= \frac{e^{3x}}{2}\left[1 - \left(\frac{D^2+3D}{2}\right) + \left(\frac{D^2+3D}{2}\right)^2 - \left(\frac{D^2+3D}{2}\right)^3 + \cdots\right]x^2$$

$$= \frac{e^{3x}}{2}\left[x^2 - \left(\frac{D^2+3D}{2}\right)x^2 + \left(\frac{D^2+3D}{2}\right)^2 x^2 - \cdots\right]$$

$$= \frac{e^{3x}}{2}\left[x^2 - \frac{1}{2}\left[D^2(x^2) + 3D(x^2)\right] + \left(\frac{D^4+9D^2+6D^3}{4}\right)x^2\right]$$

$$= \frac{e^{3x}}{2}\left[x^2 - \frac{1}{2}\left[2 + 3(2x)\right] + \frac{1}{4}\left[D^4(x^2) + 9D^2(x^2) + 6D(x^2)\right]\right]$$

$$= \frac{e^{3x}}{2}\left[x^2 - \frac{1}{2}(2 + 6x) + \frac{1}{4}(0 + 9(2) + 6(2x))\right]$$

$$= \frac{e^{3x}}{2}\left[x^2 - 1 - 3x + \frac{1}{4}(18 + 12x)\right]$$

$$= \frac{e^{3x}}{2}\left(x^2 - 1 - 3x + \frac{1}{2}(9 + 6x)\right)$$

$$= \frac{e^{3x}}{2}\left[x^2 - 1 - 3x + \frac{9}{2} + \cancel{\frac{6x}{2}}\right)$$

$$= \frac{e^{3x}}{2}\left[x^2 - 1 - \cancel{3x} + \frac{9}{2} + \cancel{3x}\right]$$

$$= \frac{e^{3x}}{2}\left[x^2 - 1 + \frac{9}{2} - 3x\right]$$

$$-1 + \frac{9}{2} = \frac{-2+9}{2} = \frac{7}{2}$$

$$\cancel{= \frac{e^{3x}}{2}\frac{1}{2}\left[2x^2 - 2 + 9\right]}$$

$$\cancel{= \frac{e^{3x}}{4}(2x^2 + 7)}$$

$$= \frac{e^{3x}}{2}\left[x^2 - 3x + \frac{7}{2}\right)$$

$$PI_2 = \frac{1}{D^2 - 3D + 2}\sin 2x$$

$$= \frac{1}{-4 - 3D + 2}\sin 2x$$

$$= \frac{1}{-3D - 2}\sin 2x$$

$$= \frac{1}{-3D-2} \times \frac{-3D+2}{-3D+2}\sin 2x$$

$$= \frac{-3D+2}{9D^2 - 4}\sin 2x$$

$$= \frac{-3D+2}{9(-4) - 4}\sin 2x$$

$$= \frac{-3D+2}{-36-4}\sin 2x$$

$$= \frac{+(3D-2)}{+40}\sin 2x$$

$$= \left(\frac{3D-2}{40}\right)\sin 2x$$

$= \frac{1}{40}[3D \sin 2x - 2\sin 2x]$

$= \frac{1}{40}[3 \cdot \cos 2x (2) - 2\sin 2x]$

$= \frac{1}{40}[6\cos 2x - 2\sin 2x]$

$= \frac{\cancel{2}}{\cancel{40}_{20}}[3\cos 2x - \sin 2x]$

$= \frac{1}{20}[3\cos 2x - \sin 2x]$

$PI = \frac{e^{3x}}{2}[x^2 - 3x + \frac{7}{2}] + \frac{1}{20}[3\cos 2x - \sin 2x]$

Now the solution of equn ① is $y = C.F + P.I$

$y = c_1 e^x + c_2 e^{2x} + \frac{e^{3x}}{2}[x^2 - 3x + \frac{7}{2}] + \frac{1}{20}[3\cos 2x - \sin 2x]$

① $(D^2 - 4D + 3)y = e^x \cos 2x$

sol: Given D.E is $(D^2 - 4D + 3)y = e^x \cos 2x \rightarrow ①$

~~An~~ A.E is $m^2 - 4m + 3 = 0$

$m^2 - m - 3m + 3 = 0$

$m(m-1) - 3(m-1) = 0$

$(m-1)(m-3) = 0$

$m = 1, 3$.

∴ The roots are real and distinct.

$CF = c_1 e^x + c_2 e^{3x}$

$P.I = \frac{1}{D^2 - 4D + 3} e^x . \cos 2x$

$= e^x \frac{1}{(D+1)^2 - 4(D+1) + 3} \cos 2x$

$= e^x \frac{1}{D^2 + \cancel{1} + 2D - 4D - \cancel{4} + \cancel{3}} \cos 2x$

$= e^x \frac{1}{D^2 - 2D} \cos 2x$

$= e^x \frac{1}{-4 - 2D} \cos 2x$

$= e^x \frac{1}{-4 - 2D} \times \frac{-4 + 2D}{-4 + 2D} \cos 2x$

$$= e^{x} \frac{-4+2D}{16-4D^2} \cos 2x$$

$$= e^{x} \frac{-4+2D}{16-4(-4)} \cos 2x$$

$$= e^{x} \frac{-4+2D}{16+16} \cos 2x$$

$$= e^{x} \frac{2D-4}{32} \cos 2x$$

$$= \frac{e^{x}}{32} (2 \cdot D \cos 2x - 4 \cos 2x)$$

$$= \frac{e^{x}}{32} (2 \cdot (-\sin 2x)(2) - 4\cos 2x)$$

$$= \frac{e^{x}}{32} (-4 \sin 2x - 4 \cos 2x)$$

$$= \frac{-e^{x}}{8} (\sin 2x + \cos 2x)$$

Now the solution of equ^n ① is $y = C.F + P.I.$

$$y = c_1 e^{x} + c_2 e^{3x} + -\frac{e^{x}}{8} \sin 2x - \frac{e^{x}}{8} \cos 2x$$

② $(D^4 - 1) y = \cos x . \cosh x$

Sol:- Given D.E is $(D^4-1) y = \cos x . \cosh x \rightarrow ①$

An A.E is $m^4 - 1 = 0$

$$(m^2)^2 - (1)^2 = 0$$

$$(m^2+1)(m^2-1) = 0.$$

$$m^2+1 = 0, \quad m^2 - 1 = 0$$

$$m = \pm i, \quad m = \pm 1$$

∴ The roots are real, imaginary and distinct.

$$C.F = c_1 e^{x} + c_2 e^{-x} + e^{(0)x} [c_1 \cos x + c_2 \sin x]$$

$$P.I = \frac{1}{D^4 - 1} \cos x . \cosh x$$

$$= \frac{1}{D^4-1} \cos x \left(\frac{e^{x} + e^{-x}}{2} \right)$$

$$= \frac{1}{2}\frac{1}{D^4-1}\left(e^x.\cos x + e^{-x}.\cos x\right)$$

$$= \frac{1}{2}\Bigg[\underbrace{\frac{1}{D^4-1}e^x.\cos x}_{PI_1} + \underbrace{\frac{1}{D^4-1}.e^{-x}\cos x}_{PI_2}\Bigg] \rightarrow (2)$$

$$PI_1 = \frac{1}{D^4-1}\ e^x.\cos x$$

$$= e^x\frac{1}{(D+1)^4-1}.\cos x.$$

$$= e^x\frac{1}{[(D+1)^2]^2-1}\cos x$$

$$= e^x\frac{1}{(D^2+2D+1)^2-1}\cos x.$$

$$= e^x\frac{1}{(D^2)^2+4D^2+\cancel{1}+4D^3+4D+2D^2\cancel{-1}}\cos x$$

$$= e^x\frac{1}{(D^2)^2+6D^2+4D^3+4D}\cos x$$

$$= e^x\frac{1}{(-1)^2+6(-1)+4(-1)D+4D}\cos x$$

$$= e^x\frac{1}{1-6-\cancel{4D}+\cancel{4D}}\cos x$$

$$= e^x\frac{1}{-5}\cos x = \frac{-e^x}{5}\cos x$$

$$PI_2 = \frac{1}{(D^2+1)(D^2-1)}\ e^{-x}\cos x$$

$$= e^{-x}\frac{1}{[(D-1)^2+1][(D-1)^2-1]}\cos x$$

$$= e^{-x}\frac{1}{[D^2+1-2D+1](D^2+\cancel{1}-2D\cancel{-1})}\cos x$$

$$= e^{-x}\frac{1}{(D^2-2D+2)(D^2-2D)}\cos x$$

$$= e^{-x}\frac{1}{D^4-2D^3-2D^3+\cancel{4D^2}+2D^2-\cancel{4D}}\cos x$$

$$= e^{-x}\frac{1}{D^4-4D^3+6D^2-4D}\cos x.$$

$$= e^{-x} \frac{1}{(-1)^2 - 4(-1)D + 6(-1) - 4D} \cos x$$

$$= e^{-x} \frac{1}{1 + \cancel{4D} - 6 - \cancel{4D}} \cos x$$

$$= e^{-x} \cdot \frac{1}{-5} \cdot \cos x = \frac{-e^{-x}}{5} \cos x.$$

from ②,

$$P.I = -\frac{e^{x}}{5} \cos x - \frac{e^{-x}}{5}$$

Now the solution of equⁿ① is $y = C.F + P.I$

$$y = c_1 e^{x} + c_2 e^{-x} + e^{(0)x}[c_1 \cos x + c_2 \sin x] - \frac{e^{x}}{5} \cos x - \frac{e^{-x}}{5}.$$

③ $\frac{d^2y}{dx^2} - 4y = x.\sinh x$

sol: Given D-E is $D^2y - 4y = x.\sinh x$

$$(D^2 - 4)y = x.\sinh x \rightarrow ①$$

An Auxiliary Equation is $m^2 - 4 = 0$

$$m^2 - (2)^2 = 0$$

$$(m+2)(m-2) = 0$$

$$m = -2, 2.$$

∴ The roots are real and distinct.

$$C.F = c_1 e^{-2x} + c_2 e^{2x}$$

$$P.I = \frac{1}{D^2 - 4} x.\sinh x$$

$$= \frac{1}{D^2 - 4} x\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{D^2 - 4}\right)\left[x.e^{x} - x.e^{-x}\right]$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{D^2 - 4} x e^{x}}_{PI_1} - \underbrace{\frac{1}{D^2 - 4} e^{-x} x}_{PI_2}\right] \rightarrow ②$$

$$P.I_1 = \frac{1}{D^2 - 4} x.e^{x}$$

$$= e^{x} \frac{1}{(D+1)^2 - 4} x.$$

$$= e^{x} \frac{1}{4\left(1 - \frac{(D+1)^2}{4}\right)} x.$$

$$= -\frac{e^x}{4}\left[1-\frac{(D+1)^2}{4}\right]^{-1} x$$

$$= -\frac{e^x}{4}\left[1+\frac{(D+1)^2}{4}+\left(\frac{(D+1)^2}{4}\right)^2+\cdots\right] x$$

$$= -\frac{e^x}{4}\left[x+\frac{(D+1)^2}{4}x+\frac{((D+1)^2)^2}{16}x\right]$$

$$= -\frac{e^x}{4}\left[x+\frac{D^2+1+2D}{4}x+\frac{(D^2+2D+1)^2}{16}\cdot x\right]$$

$$= -\frac{e^x}{4}\left[x+\frac{1}{4}(D^2(x)+x+2Dx)+\left(\frac{D^4+4D^2+1+4D^3+4D+2D^2}{16}x\right)\right]$$

$$= -\frac{e^x}{4}\left[x+\frac{1}{4}[0+x+2]+\frac{1}{16}[0+0+x+0+4+0]\right]$$

$$= -\frac{e^x}{4}\left[x+\frac{x}{4}+\frac{\cancel{2}}{\cancel{4}2}+\frac{1}{16}(x+4)\right]$$

$$= -\frac{e^x}{4}\left[x+\frac{x}{4}+\frac{1}{2}+\frac{x}{16}+\frac{1}{4}\right]$$

$$= -\frac{e^x}{4}\left[\frac{16x+4x+8+x+4}{16}\right]$$

$$= -\frac{e^x}{4}\left(\frac{21x+12}{16}\right)$$

$$= -\frac{e^x}{4}\left(\frac{21x}{16}+\frac{\cancel{12}^{\,3}}{\cancel{16}_{\,4}}\right)$$

$$= -\frac{e^x}{4}\left(\frac{21x}{16}+\frac{3}{4}\right)$$

$$P.I_2 = \frac{1}{D^2-4}\, e^{-x}.x$$

$$= e^{-x}\frac{1}{(D-1)^2-4}\, x$$

$$= e^{-x}\frac{1}{D^2+1-2D-4}\, x$$

$$= e^{-x}\frac{1}{D^2-2D-3}\, x$$

$$= e^{-x}\frac{1}{-3\left(1-\left(\frac{D^2-2D}{3}\right)\right)}\, x$$

$$= -\frac{e^{-x}}{3}\left[1-\left(\frac{D^2-2D}{3}\right)\right]^{-1} x.$$

$$= \frac{-e^{x}}{3}\left[1+\left(\frac{D^2-2D}{3}\right)+\left(\frac{D^2-2D}{3}\right)^2+\cdots\right]x$$

$$= -\frac{e^{-x}}{3}\left[x+\left(\frac{D^2-2D}{3}\right)x+\frac{D^4+4D^2-4D^3}{9}\cdot x\right]$$

$$= -\frac{e^{-x}}{3}\left[x+\frac{1}{3}(D^2x-2Dx)+0\right]$$

$$= -\frac{e^{-x}}{3}\left[x+\frac{1}{3}(0-2)\right]$$

$$= -\frac{e^{-x}}{3}\left(x-\frac{2}{3}\right)$$

$$= -\frac{e^{-x}}{9}(3x-2)$$

$$P.I = \frac{-e^{x}}{4}\left(\frac{21x}{16}+\frac{3}{4}\right)-\frac{e^{-x}}{9}(3x-2)$$

Now the solution of Equ^n ① is $y = C.F + P.I$.

$$y = c_1e^{-2x}+c_2e^{2x}-\frac{e^{x}}{4}\left(\frac{21x}{16}+\frac{3}{4}\right)-\frac{e^{-x}}{9}(3x-2).$$

④ $\frac{d^2y}{dx^2}+y = x^2\sin 2x$

Sol: Given D.E is $D^2y+y = x^2\sin 2x$

$$(D^2+1)\,y = x^2\sin 2x \rightarrow ①$$

A.E is $m^2+1=0$

$m = \pm i$

∴ The roots are complex and distinct.

$$C.F = e^{(0)x}[c_1\cos x + c_2\sin x]$$

$$P.I = \frac{1}{D^2+1}x^2\sin 2x.$$

$$= I.P\left\{\frac{1}{D^2+1}x^2(\cos 2x+i\sin 2x)\right]$$

$$= I.P\left[\frac{1}{D^2+1}\cdot x^2\, e^{2ix}\right]$$

$$= I.P.e^{2ix}\left[\frac{1}{(D+2i)^2+1}\cdot x^2\right)$$

$$= I.P\; e^{2ix}\frac{1}{1+(D+2i)^2}x^2$$

$$= I.p\ e^{2ix}\left[1+(D+2i)^2\right]^{-1}\cdot x^2$$

$$= I.p\ e^{2ix}\left[1-(D+2i)^2+\left((D+2i)^2\right)^2+\cdots\right]x^2$$

$$= I.p\ e^{2ix}\left[x^2-(D^2+4i^2+4Di)\,x^2+(D^2+4i^2+4Di)^2\,x^2\right]$$

$$(D^2-4+4Di)$$

$$= I.p\ e^{2ix}\left[x^2-(D^2x^2-4x^2+4iDx^2)+\left[D^4+16+16D^2i^2-8D^2-32Di+8D^3i\right]x^2\right]$$

$$= I.p\ e^{2ix}\left[x^2-(2x-4x^2+4i(2x))+(0+16x^2-16(2)-8(2)-32i(2x)+0)\right]$$

$$= I.p\ e^{2ix}\left[x^2-2x+4x^2-8xi+16x^2-32-16-64xi\right]$$

$$P.I. = I.p\ e^{2ix}\left[21x^2-72xi-2x-48\right]$$

Now the solution of equ^n ① is $y = C.F + P.I.$

$$y = e^{(0)x}\left[c_1\cos x+c_2\sin x\right]+I.p\ e^{2ix}(21x^2-72xi-2x-48).$$

⑤ $(D^4+2D^2+1)\,y = x^2\cos x.$

Sol: Given D.E is $(D^4+2D^2+1)\,y = x^2\cos x \rightarrow ①$

A.E is $m^4+2m^2+1=0$

$$(m^2+1)^2=0$$

$$(m^2+1)(m^2+1)=0$$

$$m=\pm i,\quad m=\pm i$$

$\therefore$ The roots are complex and repeat.

$$C.F = e^{(0)x}\left[(c_1+c_2x)\cos x+(c_3+c_4x)\sin x\right]$$

$$P.I = \frac{1}{(D^2+1)^2}\cdot x^2\cos x$$

$$= R.p\left[\frac{1}{(D^2+1)^2}\cdot x^2\,(\cos x+i\sin x)\right]$$

$$= R.p\left[\frac{1}{(D^2+1)^2}\,x^2\,e^{ix}\right]$$

$$= R.p\ e^{ix}\left[\frac{1}{(D^2+1)^2}\,x^2\right]$$

$$= R.p\ e^{ix}\,\frac{1}{(1+D^2)^2}\,x^2$$

$= R.P\, e^{ix}\, (1+D^2)^{-2}.\, x^2$

$= R.P\, e^{ix}\, [1 - 2D^2 + 3(D^2)^2 + \cdots]\, x^2$

$= R.P\, e^{ix}\, [x^2 - 2D^2 x^2 + 3D^4 . x^2 + \cdots]$

$= R.p\; e^{ix} \dfrac{1}{((D+i)^2+1)^2}\, x^2$

$= R.p\; e^{ix} \dfrac{1}{[1+(D+i)^2]^2}\, x^2$

$= R.p\; e^{ix}\, [1+(D+i)^2]^{-2}.\, x^2$

$= R.p\; e^{ix}\, [1 - 2(D+i)^2 + 3((D+i)^2)^2 - 4((D+i)^2)^3 + \cdots]\, x^2$

$= R.p\; e^{ix}\, [x^2 - 2(D^2+i^2+2Di)x^2 + 3(D^2+i^2+2Di)^2 x^2]$

$(D^2-1+2Di)$

$= R.p\; e^{ix}\, [x^2 - 2(D^2x^2 - x^2 + 2Dix^2) + 3(D^4 + 1 + 4D^2i^2 - 2D^2 - 4Di + 4D^3i)\, x^2]$

$= R.p\; e^{ix}\, [x^2 - 2(2 - x^2 + 2i(2x)) + 3(0 + x^2 - 4(2) - 2(2) - 4i(2x) + 0)]$

$= R.p\; e^{ix}\, [x^2 - 4 + 2x^2 - 8xi + 3x^2 - 24 - 12 - 24xi]$

$P.I. = R.p\; e^{ix}\, [6x^2 - 32xi - 40]$

Now the solution of equ (1) is $y = C.F + P.I$

$y = e^{(0)x}[(c_1+c_2x)\cos x + (c_3+c_4x)\sin x] + R.p\; e^{ix}[6x^2 - 32xi - 40]$

(8) $\dfrac{d^4y}{dx^4} - y = e^x \cos x$

Solⁿ Given D.E is $D^4y - y = e^x \cos x$

$(D^4-1)\, y = e^x \cos x \rightarrow (1)$

An A.E is $m^4 - 1 = 0$

$(m^2)^2 - (1)^2 = 0$

$(m^2+1)(m^2-1) = 0$

$m = \pm i,\; m = \pm 1$

∴ The roots are real, complex and distinct.

$C.F = c_1 e^{-x} + c_2 e^{x} + e^{(0)x}[c_3 \cos x + c_4 \sin x]$

$$P.I = \frac{1}{D^4-1} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^4-1} \cdot \cos x$$

$$= \cancel{e^x \frac{1}{(D^2+1+2D)^2-1}} \cos x$$

$$= \cancel{e^x \frac{1}{(-1+1)^2-1}} \cos x$$

$$\neq \cancel{e^x \frac{1}{0-1}} \cos x$$

$$\cancel{P.I \neq -e^x \cos x}$$

$$= e^x \frac{1}{(D^2+1+2D)^2-1} \cos x$$

$$= e^x \frac{1}{D^4+\cancel{1}+4D^2+2D^2+4D+4D^3-\cancel{1}} \cos x$$

$$= e^x \frac{1}{(-1)(-1)+4(-1)+2(-1)+4D+4(-1)D} \cos x$$

$$= e^x \cdot \frac{1}{1-4-2+\cancel{4D}-\cancel{4D}} \cos x$$

$$= e^x \frac{1}{-5} \cos x$$

$$P.I = -\frac{e^x}{5} \cos x$$

Now the solution of Eqn ① is $y = C.F + P.I$

$$y = c_1 e^{-x} + c_2 e^x + e^{(0)x}\left[c_3 \cos x + c_4 \sin x\right] - \frac{e^x}{5} \cos x.$$

⑨ $(D^2-2D)y = e^x \sin x$

Given D.E is $(D^2-2D)y = e^x \sin x \rightarrow$ ①

An A.E is $m^2-2m=0$

$m(m-2)=0$

$m=0, 2.$

∴ The roots are real and distinct.

$$C.F = c_1 e^{(0)x} + c_2 e^{2x}$$

$$P.I = \frac{1}{D^2-2D} e^x \sin x$$

$$= e^x \frac{1}{(D+1)^2-2(D+1)} \sin x$$

$$= e^x \frac{1}{D^2+1+\cancel{2D}-\cancel{2D}-2} \sin x$$

$$= e^x \frac{1}{D^2-1} \sin x$$

$$= e^x \frac{1}{-1-1} \sin x$$

$$= e^x \frac{1}{-2} \sin x$$

$$P.I = -\frac{e^x}{2} \sin x$$

Now the solution of Eqn ① is $y = C.F + P.I$

$$y = c_1 e^{(0)x} + c_2 e^{2x} - \frac{e^x}{2} \sin x.$$

(10) $y'' - 2y' + 2y = x + e^{x}.\cos x$

sol:- Given D.E is $y'' - 2y' + 2y = x + e^{x}\cos x$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = x + e^{x}\cos x$$

$$D^2y - 2Dy + 2y = x + e^{x}\cos x$$

$$(D^2 - 2D + 2)y = x + e^{x}\cos x \rightarrow (1)$$

A.E is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$$m = 1 \pm i$$

∴ The roots are complex and distinct.

$$C.F = e^{x}[c_1\cos x + c_2\sin x]$$

$$P.I = \frac{1}{D^2-2D+2}(x + e^{x}\cos x) = \underbrace{\frac{1}{D^2-2D+2}x}_{PI_1} + \underbrace{\frac{1}{D^2-2D+2}e^{x}\cos x}_{PI_2} \rightarrow (2)$$

$$PI_1 = \frac{1}{2\left(\frac{D^2-2D}{2}+1\right)}(x \cancel{+ e^{x}\cos x})$$

$$= \frac{1}{2\left(1+\frac{D^2-2D}{2}\right)}(x)\cancel{+e^{x}\cos x}$$

$$= \frac{1}{2}\left[1+\left(\frac{D^2-2D}{2}\right)\right]^{-1}(x)\cancel{+e^{x}\cos x}$$

$$= \frac{1}{2}\left[1 - \left(\frac{D^2-2D}{2}\right) + \left(\frac{D^2-2D}{2}\right)^2 - \cdots\right]x.$$

$$= \frac{1}{2}\left[x - \left(\frac{D^2-2D}{2}\right)x + \left(\frac{D^4+4D^2-4D^3}{2}\right)x\right]$$

$$= \frac{1}{2}\left[x - \frac{1}{2}(D^2x - 2Dx) + 0\right]$$

$$= \frac{1}{2}\left[x - \frac{1}{2}(0-2)\right]$$

$$= \frac{1}{2}[x+1]$$

$$PI_2 = \frac{1}{D^2-2D+2}e^{x}.\cos x.$$

$$= e^{x}\frac{1}{(D+1)^2 - 2(D+1) + 2}\cos x$$

$$= e^x \frac{1}{D^2+1+\cancel{2D}-\cancel{2D}-\cancel{1}+\cancel{1}} \cos x$$

$$= e^x \frac{1}{D^2+1} \cos x$$

$$= e^x \frac{x}{2D} \cos x$$

$$= e^x \frac{x}{2(\ldots)}$$

$$= e^x \cdot \frac{x}{2} \cdot \frac{1}{D}(\cos x)$$

$$PI_2 = \frac{x \cdot e^x}{2} \sin x.$$

$$P.I = \frac{1}{2}(x+1) + \frac{x \cdot e^x}{2} \sin x.$$

Now the solution of Equ^n ① is $y = C.F + P.I$

$$y = e^x [c_1 \cos x + c_2 \sin x] + \frac{1}{2}(x+1) + \frac{x \cdot e^x}{2} \sin x.$$

(12) $\dfrac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x (\cos 2x)$

Sol: Given D.E is $D^2y + 2y = x^2 e^{3x} + e^x (\cos 2x)$

$$(D^2+2)\, y = x^2 e^{3x} + e^x \cos 2x \longrightarrow ①$$

An A.E is $m^2 + 2 = 0$

$$m^2 = -2$$

$$m = \sqrt{-2}$$

$$m = \pm\sqrt{2}\, i.$$

∴ The roots are complex and distinct.

$$C.F = e^{(0)x} \left[c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x \right]$$

$$P.I = \frac{1}{D^2+2} \left(x^2 \, e^{3x} + e^x \cos 2x \right)$$

$$= \underbrace{\frac{1}{D^2+2} x^2 e^{3x}}_{PI_1} + \underbrace{\frac{1}{D^2+2} e^x \cos 2x}_{PI_2} \longrightarrow ②$$

$$P.I_1 = \frac{1}{D^2+2} e^{3x} \cdot x^2$$

$$= e^{3x} \frac{1}{(D+3)^2+2} x^2$$

$$= e^{3x} \frac{1}{D^2+9+6D+2} x^2$$

$$= e^{3x} \frac{1}{D^2+6D+11} x^2$$

$$= e^{3x} \frac{1}{11\left(\frac{D^2+6D}{11}+1\right)} x^2$$

$$= e^{3x} \frac{1}{11\left(1+\frac{D^2+6D}{11}\right)} x^2$$

$$= \frac{e^{3x}}{11} \left(1+\frac{D^2+6D}{11}\right)^{-1} x^2$$

$$= \frac{e^{3x}}{11} \left[1-\left(\frac{D^2+6D}{11}\right)+\left(\frac{D^2+6D}{11}\right)^2 - \cdots\right] x^2$$

$$= \frac{e^{3x}}{11} \left[x^2-\left(\frac{D^2+6D}{11}\right)x^2+\left(\frac{D^2+6D}{11}\right)^2 x^2 - \cdots\right]$$

$$= \frac{e^{3x}}{11} \left[x^2-\frac{1}{11}\left(D^2x^2+6Dx^2\right)+\frac{1}{121}\left(D^4+36D^2+12D^3\right)x^2\right]$$

$$= \frac{e^{3x}}{11}\left[x^2-\frac{1}{11}\left(\cancel{D^2}\,2+6(2x)\right)+\frac{1}{121}\left(0+36(2)+0\right)\right]$$

$$= \frac{e^{3x}}{11}\left(x^2-\frac{2}{11}-\frac{12x}{11}+\frac{72}{121}\right)$$

$$= \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$$

$$P.I_2 = \frac{1}{D^2+2} e^{x} \cos 2x$$

$$= e^{x} \frac{1}{(D+1)^2+2} \cos 2x$$

$$= e^{x} \frac{1}{D^2+2D+3} \cos 2x$$

$$= e^{x} \frac{1}{-4+2D+3} \cos 2x$$

$$= e^{x} \frac{1}{2D-1} \cos 2x$$

$$= e^{x} \frac{1}{2D-1} \times \frac{2D+1}{2D+1} \cos 2x$$

$$= e^x \frac{2D+1}{4D^2-1} \cos 2x$$

$$= e^x \frac{2D+1}{4(-4)-1} \cos 2x$$

$$= e^x \frac{2D+1}{-17} \cos 2x$$

$$= \frac{-e^x}{17} (2 D\cos 2x + (1)\cos 2x)$$

$$= -\frac{e^x}{17} (2(-\sin 2x)2 + \cos 2x)$$

$$= -\frac{e^x}{17} (-4\sin 2x + \cos 2x)$$

$$= \frac{e^x}{17} (4\sin 2x - \cos 2x)$$

$$P.I = \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$$

Now the solution of equ^n (1) is $y = C.F + P.I$

$$y = e^{(0)x}\left[c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right] + \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$$

(11) $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x.$

Given D.E is $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$ $\longrightarrow$ (1)

An Auxiliary Equation is $m^3 + 2m^2 + m = 0$

$$m(m^2 + 2m + 1) = 0$$

$$m(m^2 + m + m + 1) = 0$$

$$m[m(m+1) + 1(m+1)] = 0$$

$$m(m+1)(m+1) = 0$$

$$m = 0, -1, -1$$

$\therefore$ The roots are real and repeat.

$$C.F = c_1 e^{(0)x} + c_2 e^{-x} + c_3 x.e^{-x}$$

$$P.I = \frac{1}{D^3 + 2D^2 + D} x^2 e^{2x} + \sin^2 x$$

$$= \frac{1}{D^3 + 2D^2 + D} x^2 . e^{2x} + \frac{1}{D^3 + 2D^2 + D} \cdot \frac{1 - \cos 2x}{2}$$

$$= \frac{1}{D^3 + 2D^2 + D} x^2 e^{2x} + \frac{1}{D^3 + 2D^2 + D}\frac{1}{2} - \frac{1}{D^3 + 2D^2 + D}\frac{1}{2}.\cos 2x \longrightarrow (2)$$

$$PI_1 = e^{2x} \cdot \frac{1}{(D+2)^3+2(D+2)^2+(D+2)} x^2$$

$$= e^{2x} \frac{1}{D^3+8+6D^2+12D+2D^2+8+8D+D+2} x^2$$

$$= e^{2x} \frac{1}{D^3+8D^2+21D+18} x^2$$

$$= e^{2x} \frac{1}{18\left(1+\frac{D^3+8D^2+21D}{18}\right)} x^2$$

$$= \frac{e^{2x}}{18}\left(1+\frac{D^3+8D^2+21D}{18}\right)^{-1} x^2$$

$$= \frac{e^{2x}}{18}\left[1-\frac{D^3+8D^2+21D}{18}+\left(\frac{D^3+8D^2+21D}{18}\right)^2 - \cdots\right] x^2$$

$$= \frac{e^{2x}}{18}\left[1-\frac{1}{18}\left[D^3+8D^2+21D\right]x^2 + \left[D^6+64D^4+441D^2+16D^5 + 336D^3+42D^4\right]x^2\right]$$

$$= \frac{e^{2x}}{18}\left[1-\frac{1}{18}\left[0+8(2)+21(2x)\right]+\left[0+0+441(2)+0+0+0\right]\right]$$

$$= \frac{e^{2x}}{18}\left[1-\frac{1}{18}\left[16+42x\right]+882\right]$$

$$= \frac{e^{2x}}{18}\left[1-\frac{\cancel{2}}{\cancel{18}_9}(8+21x)+882\right]$$

$$= \frac{e^{2x}}{18}\left[1-\frac{8}{9}-\frac{\cancel{21}^{7}x}{\cancel{9}_3}+882\right]$$

$$= \frac{e^{2x}}{18}\left[\frac{7939}{9}-\frac{7x}{3}\right]$$

$$1-\frac{8}{9}+882 = \frac{9-8+7938}{9} = \frac{7930+9}{9} = \frac{7939}{9}$$

$$PI_2 = \frac{1}{2}\frac{1}{D^3+2D^2+D} e^{(0)x}$$

$$= \frac{1}{2}\frac{x}{3D^2+4D+1} e^{(0)x}$$

$$= \frac{1}{2}\frac{x}{1} e^{(0)x} = \frac{x}{2}$$

$$PI_3 = \frac{1}{2}\frac{1}{D^3+2D^2+D}\cos 2x$$

$$= \frac{1}{2}\frac{1}{-4D-8+D}\cos 2x$$

$$= \frac{1}{2}\frac{1}{-8-3D}\cos 2x$$

$$= \frac{1}{2}\frac{1}{-8-3D}\times\frac{-8+3D}{-8+3D}\cos 2x$$

$$= \frac{1}{2}\frac{-8+3D}{64-9D^2}\cos 2x$$

$$=\frac{1}{2}\,\frac{-8+3D}{64+36}\cos 2x$$

$$=\frac{1}{2}\,\frac{-8+3D}{100}\cos 2x$$

$$=\frac{3D-8}{200}\cdot\cos 2x$$

$$=\left(\frac{3D}{200}-\frac{1}{25}\right)\cos 2x$$

from ①,

$$P.I=\frac{e^{2x}}{18}\left(\frac{7939}{9}-\frac{7x}{3}\right)+\frac{x}{2}-\left(\frac{3D}{200}-\frac{1}{25}\right)\cos 2x$$

Now the solution of equⁿ ① is $y=C.F+P.I$

$$y=c_1e^{(0)x}+c_2e^{-x}+c_3x.e^{-x}+\frac{e^{2x}}{18}\left(\frac{7939}{9}-\frac{7x}{3}\right)+\frac{x}{2}-\left(\frac{3D}{100}-\frac{1}{25}\right)\cos 2x$$

Formulas:

$$e^{i\theta}=\cos\theta+i\sin\theta$$

$$e^{i(2\theta)}=\cos 2\theta+i\sin 2\theta$$

$$e^{i(n\theta)}=\cos n\theta+i\sin n\theta$$

$$\sin^2 x=\frac{1-\cos 2x}{2}$$

$$\cos^2 x=\frac{1+\cos 2x}{2}$$

$$\sin A\cdot\cos B=\frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$$

$$\cos A\cdot\cos B=\frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

$$\sin A\cdot\sin B=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$

Type-IV

(4) $\frac{d^2y}{dx^2}+4y = x^2\sin 2x$.

Given D.E. is $\frac{d^2y}{dx^2}+4y = x^2\sin 2x$

$$D^2y+4y = x^2\sin 2x$$

$$(D^2+4)y = x^2\sin 2x \rightarrow (1)$$

An Auxiliary equation is $m^2+4=0$

$$m^2=-4$$

$$m=\pm 2i$$

$\therefore$ The roots are complex and distinct.

$$C.F = e^{(0)x}[c_1\cos 2x + c_2\sin 2x]$$

$$P.I = \frac{1}{D^2+4}x^2\sin 2x$$

$$= \frac{1}{D^2+4}x^2\ I.p\ e^{i(2x)}$$

$$= I.p\left[\frac{1}{D^2+4}x^2e^{i(2x)}\right]$$

$$= I.p\left[e^{2ix}\frac{1}{(D+2i)^2+4}x^2\right]$$

$$= I.p\left[e^{2ix}\frac{1}{D^2-\cancel{4}+4Di+\cancel{4}}x^2\right]$$

$$= I.p\left[e^{2ix}\frac{1}{4Di\left(\frac{D^{\cancel{2}}}{4\cancel{D}i}+1\right)}x^2\right]$$

$$= I.p\left[e^{2ix}\frac{1}{4Di\left(\frac{D}{4i}+1\right)}x^2\right]$$

$$= I.p\left[\frac{e^{2ix}}{4Di}\left(1+\frac{D}{4i}\right)^{-1}x^2\right]$$

$$= I.p\left[\frac{e^{2ix}}{4Di}\left(1-\frac{D}{4i}+\left(\frac{D}{4i}\right)^2-\left(\frac{D}{4i}\right)^3+\ldots\right)x^2\right]$$

$$= I.p\left[\frac{e^{2ix}}{4Di}\left(x^2-\frac{D}{4i}x^2+\frac{D^2}{(-16)}x^2\right]\right.$$

$$= I.p\left[\frac{e^{2ix}}{4Di}\left(x^2-\frac{1}{\cancel{4}_2 i}(\cancel{2}x)-\frac{1}{\cancel{16}_8}(\cancel{2})\right)\right]$$

$$= I.P\left[\frac{e^{2ix}}{4Di}\left(x^2 - \frac{x}{2i} - \frac{1}{8}\right)\right]$$

$$= I.P\left[\frac{e^{2ix}}{4Di}\times\frac{i}{i}\left(x^2 - \frac{xi}{2i^2} - \frac{1}{8}\right)\right]$$

$$= I.P\left[\frac{e^{2ix}}{-4D} i\left(x^2 + \frac{x}{2}i - \frac{1}{8}\right)\right]$$

$$= I.P\left[\frac{ie^{2ix}}{-4}\left(\frac{x^3}{3} + \frac{x^2}{4}i - \frac{1}{8}x\right)\right]$$

$$= I.P\left[\frac{ie^{2ix}}{-4}\left(\frac{x^3}{3} + \frac{x^2}{4}i - \frac{x}{8}\right)\right]$$

$$= I.P\left[\frac{i}{-4}(\cos 2x + i\sin 2x)\left(\frac{x^3}{3} + \frac{x^2}{4}i - \frac{x}{8}\right)\right]$$

$$= I.P\left[\frac{-i}{4}(\cos 2x + i\sin 2x)\left(\left(\frac{x^3}{3} - \frac{x}{8}\right) + i\left(\frac{x^2}{4}\right)\right)\right]$$

$$= I.P\left[\frac{-i}{4}\cos 2x.\left(\frac{x^3}{3} - \frac{x}{8}\right) + i\frac{x^2}{4}\cos 2x + i\sin 2x\left(\frac{x^3}{3} - \frac{x}{8}\right) - \frac{x^2}{4}\sin 2x\right]$$

$$= I.P\left[\frac{-i}{4}\left(\cos 2x\left(\frac{x^3}{3} - \frac{x}{8}\right) - \frac{x^2}{4}\sin 2x + i\left(\frac{x^2}{4}\cos 2x + \sin 2x\left(\frac{x^3}{3} - \frac{x}{8}\right)\right)\right)\right]$$

$$= I.P\left[\frac{-i}{4}\left(\cos 2x\left(\frac{x^3}{3} - \frac{x}{8}\right) - \frac{x^2}{4}\sin 2x\right) + \frac{1}{4}\left(\frac{x^2}{2}\cos 2x + \sin 2x\left(\frac{x^3}{8} - \frac{x}{8}\right)\right)\right]$$

$$= I.P\frac{1}{4}\left[\cancel{-}\frac{x^2}{2}\cos 2x + \sin 2x\left(\frac{x^3}{3} - \frac{x}{8}\right) - \frac{i}{4}\left[\cos 2x\left(\frac{x^3}{3} - \frac{x}{8}\right) - \frac{x^2}{4}\sin 2x\right]\right]$$

$$= \frac{-1}{4}.\cos 2x\left(\frac{x^3}{3} - \frac{x}{8}\right) - \frac{x^2}{4}\sin 2x$$

$$P.I = \frac{1}{4}\left[\frac{x^2}{4}\sin 2x - \left(\frac{x^3}{3} - \frac{x}{8}\right)\cos 2x\right]$$

Now the solution of equn ① is $y = C.F + P.I$

$$y = e^{(0)x}\left[c_1\cos 2x + c_2\sin 2x\right] + \frac{1}{4}\left[\frac{x^2}{4}\sin 2x - \left(\frac{x^3}{3} - \frac{x}{8}\right)\cos 2x\right]$$

⑤ $(D^4+2D^2+1)y = x^2\cos x$.

Given D.E is $(D^4+2D^2+1)y = x^2\cos x \to ①$

An Auxiliary equn is $m^4+2m^2+1=0$

$$(m^2+1)^2 = 0.$$

$$(m^2+1)(m^2+1) = 0$$

$$m=\pm i, \quad m=\pm i$$

∴ The roots are complex and repeated roots.

$$C.F = e^{(0)x}\left[c_1\cos x + c_2\sin x\right] + xe^{(0)x}\left[c_3\cos x + c_4\sin x\right]$$

$$= c_1\cos x + c_2\sin x + c_3 x\cos x + c_4 x\sin x$$

$$= (c_1+c_3x)\cos x + (c_2+c_4x)\sin x$$

$$P.I = \frac{1}{D^4+2D^2+1}\, x^2\cos x$$

$$\boxed{(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4}$$

$$= \frac{1}{(D^2+1)^2}\, x^2\cos x.$$

$$= \frac{1}{D^4+2D^2+1}\, x^2\cdot(R.P\; e^{ix})$$

$$= R.P\left[\frac{1}{D^4+2D^2+1}\, x^2\cdot e^{ix}\right]$$

$$= R.p\left[e^{ix}\,\frac{1}{(D+i)^4+2\cdot(D+i)^2+1}\, x^2\right]$$

$$= R.p\left[e^{ix}\,\frac{1}{D^4+4D^3i+6D^2i^2+4Di^3+i^4+2(D^2+i^2+2Di)+1}\, x^2\right]$$

$$= R.p\left[e^{ix}\,\frac{1}{D^4+4D^3i-6D^2-\cancel{4Di}+\cancel{1}+\cancel{2D^2}-\cancel{2}+\cancel{4Di}+\cancel{1}}\, x^2\right]$$

$$= R.p\left[e^{ix}\,\frac{1}{D^4-4D^2+4D^3i}\, x^2\right]$$

$$= R.p\left[e^{ix}\,\frac{1}{4D^2\left(\frac{D^4+4D^3i}{4D^2}-1\right)}\, x^2\right]$$

$$= R.p\left[\frac{e^{ix}}{-4D^2}\,\frac{1}{\left(1-\frac{D^2+4Di}{4D^2}\right)}\, x^2\right]$$

$$= R.P\left[\frac{-e^{ix}}{4D^2}\left(1-\frac{D^2+4Di}{4}\right)^{-1}x^2\right]$$

$$= R.P\left[\frac{-e^{ix}}{4D^2}\left[1+\left(\frac{D^2+4Di}{4}\right)+\left(\frac{D^2+4Di}{4}\right)^2+\cdots\right]x^2\right]$$

$$= R.P\left[\frac{-e^{ix}}{4D^2}\left(x^2+\frac{1}{4}\left(D^2x^2+4Dix^2\right)+\frac{1}{16}\left(D^4x^2+16D^2i^2x^2+8D^3ix^2\right)\right)\right]$$

$$= R.P\left[\frac{-e^{ix}}{4D^2}\left\{x^2+\frac{1}{4}\left[2+4i(2x)\right]\right\}+\frac{1}{16}\left(0-16(2)+0\right)\right]$$

$$= R.P\left[\frac{-e^{ix}}{4D^2}\left(x^2+\frac{2}{4}+\frac{8xi}{4}-\frac{1}{16}(32)\right)\right]$$

$$= R.P\left[\frac{-e^{ix}}{4D^2}\left(x^2-\frac{1}{2}+2xi-2\right)\right]$$

$$-\frac{1}{2}-2 = \frac{-1-4}{2} = \frac{-5}{2}$$

$$= R.P\left[\frac{-e^{ix}}{4D^2}\left(x^2+2xi-\frac{5}{2}\right)\right]$$

$$= R.P\left[\frac{-e^{ix}}{4D^2}\left(x^2+2xi-\frac{5}{2}\right)\right]$$

$$= R.P\left[\frac{-e^{ix}}{4}\frac{1}{D}\left(\frac{x^3}{3}+\cancel{2}i\frac{x^2}{\cancel{2}}-\frac{5}{2}x\right)\right]$$

$$= R.P\left[\frac{-e^{ix}}{4}\left(\frac{1}{3}\frac{x^4}{4}+i\frac{x^3}{3}-\frac{5}{2}\frac{x^2}{2}\right)\right]$$

$$= R.P\left[\frac{-e^{ix}}{4}\left(\frac{x^4}{12}+i\frac{x^3}{3}-\frac{5x^2}{4}\right)\right]$$

$$= R.P\left[\frac{-e^{ix}}{4}\left(\frac{x^4+4ix^3-15x^2}{12}\right)\right]$$

$$= R.P\left[\frac{-e^{ix}}{48}\left(x^4+4ix^3-15x^2\right)\right]$$

$$= R.P\left[\frac{-1}{48}(\cos x+i\sin x)(x^4+4ix^3-15x^2)\right]$$

$$= R.P\left[\frac{-1}{48}\left(\cos x(x^4-15x^2)+\cos x\cdot 4ix^3+i\sin x(x^4-15x^2)+i^2 4x^3\sin x\right)\right]$$

$$= R.P\left[\frac{-1}{48}\left(\cos x(x^4-15x^2)-4x^3\sin x\right)+i\left(\cos x\, 4x^3+\sin x(x^4-15x^2)\right)\right]$$

$$= R.P\left[\frac{-1}{48}\left(\cos x(x^4-15x^2)-4x^3\sin x\right)+i\left(4x^3\cos x+(x^4-15x^2)\sin x\right)\right]$$

$$P.I = \frac{-1}{48}\left[(x^4-15x^2)\cos x - 4x^3\sin x\right]$$

$$P.I. = \frac{1}{48}\left[4x^3\sin x - (x^4-15x^2)\cos x\right]$$

Now the solution of equ^n ① is $y = C.F + P.I$

$$y = (C_1 + C_3x)\cos x + (C_2 + C_4x)\sin x + \frac{1}{48}\left[4x^3\sin x - (x^4-15x^2)\cos x\right]$$

8/11/19
Saturday Type - V

② $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x.e^x \sin x$

Sol: Given D.E is $D^2y - 2Dy + y = x.e^x \sin x$

$$(D^2 - 2D + 1)y = xe^x \sin x \rightarrow ①$$

An Auxiliary Equ^n is $m^2 - 2m + 1 = 0$

$$m^2 - m - m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

∴ The roots are real and repeat.

$$C.F = C_1e^x + C_2x.e^x.$$

$$P.I = \frac{1}{D^2-2D+1} x.e^x \sin x$$

$$= \underbrace{x \frac{1}{(D-1)^2} e^x \sin x}_{PI_1} - \underbrace{\frac{2(D-1)}{[(D-1)^2]^2} e^x \sin x}_{PI_2} \rightarrow ②$$

$$PI_1 = x \frac{1}{D^2-2D+1} e^x \sin x$$

$$= x e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} \sin x$$

$$= x.e^x \frac{1}{D^2 + \cancel{1} + \cancel{2D} - \cancel{2D} - \cancel{2} + \cancel{1}} \sin x$$

$$= x.e^x \frac{1}{D^2} \sin x$$

$$= xe^x(-\sin x)$$

$$= -xe^x \sin x$$

$$PI_2 = \frac{2(D-1)}{(D^2-2D+1)^2} e^x \sin x$$

$$= 2(D-1) \frac{1}{D^4-4D^2+1-4D^3+2D^2-4D} e^x \sin x$$

$$= 2(D-1) \frac{1}{D^4-4D^3-2D^2-4D+1}$$

$$= 2(D-1) \frac{1}{(D^2-2D+1)^2} e^x \sin x$$

$$= 2(D-1) e^x \frac{1}{[(D+1)^2-2(D+1)+1]^2} \sin x$$

$$= 2(D-1) e^x \frac{1}{(D^2+2D+1-2D-2+1)^2} \sin x$$

$$= 2(D-1) e^x \frac{1}{(D^2)^2} \sin x$$

$$= 2(D-1) e^x \frac{1}{D^4} \sin x$$

$$= 2(D-1) e^x \sin x.$$

$$= 2e^x (D\sin x - \sin x)$$

$$PI_2 = 2e^x (\cos x - \sin x)$$

$$PI = -x e^x \sin x - 2e^x(\cos x - \sin x)$$

$$= -xe^x \sin x - 2e^x \cos x + 2e^x \sin x$$

$$= e^x(2\sin x - x\sin x - 2\cos x)$$

$$= 2e^x \sin x - xe^x \sin x - 2e^x \cos x$$

$$= 2e^x \sin x + e^x(-x\sin x - 2\cos x)$$

Now the solution of equ① is $y = C.F + P.I$

$$y = c_1 e^x + c_2 x e^x + 2e^x \sin x + e^x(-x\sin x - 2\cos x)$$

⑤ $(D^2-1)y = x\sin x + (1+x^2)e^x$.

Sol:- Given D.E is $(D^2-1)y = x\sin x + (1+x^2)e^x$ → ①

An Auxiliary equn is $m^2-1=0$

$m^2=1$

$m=\pm 1$

∴ The roots are real and distinct.

$C.F = c_1 e^x + c_2 e^{-x}$

$$P.I = \frac{1}{D^2-1}\left[x\sin x + (1+x^2)e^x\right]$$

$$= \underbrace{\frac{1}{D^2-1} x\sin x}_{PI_1} + \underbrace{\frac{1}{D^2-1} (1+\cancel{x^2}) e^x}_{PI_2} + \underbrace{\frac{1}{D^2-1} x^2 e^x}_{PI_3} \rightarrow ②$$

$$PI_1 = \frac{1}{D^2-1} x\sin x$$

$$= x\frac{1}{D^2-1}\sin x - \frac{2D}{(D^2-1)^2}\sin x$$

$$= x\cdot\frac{1}{-1-1}\sin x - 2D\frac{1}{(-1-1)^2}\sin x$$

$$= \frac{x}{-2}\sin x - \cancel{2}D\frac{1}{\cancel{4}\,2}\sin x$$

$$= -\frac{x}{2}\sin x - \frac{1}{2}\cos x$$

$$PI_2 = \frac{1}{D^2-1}\cdot e^x$$

$$= \frac{x}{2D}e^x$$

$$= \frac{x}{2(1)}e^x = \frac{x}{2}e^x$$

$$PI_3 = \frac{1}{D^2-1}x^2 e^x$$

$$= e^x\frac{1}{(D+1)^2-1}x^2$$

$$= e^x\frac{1}{D^2+\cancel{1}+2D-\cancel{1}}x^2$$

$$= e^x\frac{1}{2D\left(1+\frac{D}{2}\right)}x^2$$

$$= \frac{e^x}{2D}\left(1+\frac{D}{2}\right)^{-1}x^2$$

$$= \frac{e^x}{2D}\left(1-\frac{D}{2}+\left(\frac{D}{2}\right)^2-\cdots\right)x^2$$

$$= \frac{e^x}{2D}\left(x^2 - \frac{D}{2}x^2 + \frac{D^2}{4}x^2\right)$$

$$= \frac{e^x}{2D}\left[x^2 - \frac{1}{2}(2x) + \frac{1}{4}(2)\right]$$

$$= \frac{e^x}{2D}\left(x^2 - x + \frac{1}{2}\right)$$

$$= \frac{e^x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\right)$$

from ②,

$$PI = -\frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{x}{2}e^x + \frac{e^x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\right)$$

Now the solution of equⁿ ① is $y = C.F + P.I$

$$y = c_1 e^x + c_2 e^{-x} - \frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{x}{2}e^x + \frac{e^x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\right)$$

⑦ $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x \sin x$

sol: Given D.E is $D^2y + 3Dy + 2y = xe^x \sin x$

$$(D^2 + 3D + 2)y = xe^x \sin x \quad \rightarrow ①$$

An Auxiliary equⁿ is $m^2 + 3m + 2 = 0$

$$m^2 + m + 2m + 2 = 0$$

$$m(m+1) + 2(m+1) = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

∴ The roots are real and distinct.

$$C.F = c_1 e^{-x} + c_2 e^{-2x}.$$

$$P.I = \frac{1}{D^2 + 3D + 2} xe^x \sin x$$

$$= \frac{1}{D^2 + 3D + 2} e^x (x \sin x)$$

$$= e^x \frac{1}{(D+1)^2 + 3(D+1) + 2} x \sin x$$

$$= e^x \frac{1}{D^2 + 2D + 1 + 3D + 3 + 2} x \sin x$$

$$= e^x \frac{1}{D^2 + 5D + 6} x \sin x$$

$$= e^x \left[x \frac{1}{D^2 + 5D + 6} \sin x - \frac{2D + 5}{(D^2 + 5D + 6)^2} \sin x\right]$$

$$= e^x \left[x \frac{1}{-1+5D+6} \sin x - \frac{2D+5}{(-1+5D+6)^2} \sin x \right]$$

$$= e^x \left[x \frac{1}{5D+5} \sin x - \frac{2D+5}{(5D+5)^2} \sin x \right]$$

$$= e^x \left[x \frac{1}{5(D+1)} \sin x - \frac{2D+5}{25(D+1)^2} \sin x \right]$$

$$= e^x \left[\frac{x}{5} \frac{1}{D+1} \times \frac{D-1}{D-1} \sin x - \frac{2D+5}{25} \cancel{\frac{1}{D+1} \times \frac{D+1}{D+1} \sin x} \frac{1}{D^2+2D+1} \sin x \right]$$

$$= e^x \left[\frac{x}{5} \frac{D-1}{D^2-1} \sin x - \frac{2D+5}{25} \frac{1}{\cancel{-1}+2D+\cancel{1}} \sin x \right]$$

$$= e^x \left[\frac{x}{5} \frac{D-1}{-1-1} \sin x - \frac{2D+5}{25} \frac{1}{2D} \sin x \right]$$

$$= e^x \left[\frac{-x}{10} (D \cdot \sin x - \sin x) - \frac{2D+5}{50} (-\cos x) \right]$$

$$= e^x \left[\frac{-x}{10} (\cos x - \sin x) + \frac{\cancel{2}}{\cancel{50}_{25}} (-\sin x) + \frac{\cancel{5}}{\cancel{50}_{10}} \cos x \right]$$

$$= e^x \left[\frac{x}{10} (\sin x - \cos x) - \frac{1}{25} \sin x + \frac{1}{10} \cos x \right]$$

$$= e^x \left[\frac{x}{10} \sin x - \frac{x}{10} \cos x - \frac{1}{25} \sin x + \frac{1}{10} \cos x \right]$$

$$= e^x \left[\frac{1}{10} \cos x (1-x) + \frac{1}{5} \sin x \left(\frac{x}{2} - \frac{1}{5} \right) \right]$$

$$= e^x \left[\frac{1}{10} \cos x (1-x) + \frac{1}{5} \sin x \left(\frac{5x-2}{10} \right) \right]$$

$$= e^x \left[\frac{1}{10} \cos x (1-x) + \frac{1}{50} \sin x (5x-2) \right]$$

$$P.I = \frac{e^x}{10} \left[\cos x (1-x) + \frac{1}{5} \sin x (5x-2) \right]$$

Now the solution of equⁿ ① is $y = C.F + P.I$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{e^x}{10} \left[\cos x (1-x) + \frac{1}{5} \sin x (5x-2) \right].$$

① $(D^2-4)\, y = x \cos 2x$

Sol:- Given D.E is $(D^2-4)\, y = x\cos 2x \rightarrow ①$

An Auxiliary equⁿ is $m^2-4=0$

$$m^2-2^2=0$$

$$(m+2)\,(m-2)=0$$

$$m=2,-2.$$

∴ The roots are real and distinct.

$$C.F = c_1 e^{2x} + c_2 e^{-2x}$$

$$P.I = \frac{1}{D^2-4}\, x\,(\cos 2x)$$

$$= x.\frac{1}{D^2-4}\cos 2x - \frac{2D}{(D^2-4)^2}\cos 2x$$

$$= x.\frac{1}{-4-4}\cos 2x - \frac{2D}{(-4-4)^2}\cos 2x$$

$$= x\,\frac{1}{-8}\cos 2x - \frac{\cancel{2}D}{\cancel{64}_{32}}\cos 2x$$

$$= \frac{-x}{8}\cos 2x - \frac{1}{32}D(\cos 2x)$$

$$= \frac{-x}{8}\cos 2x - \frac{1}{\cancel{32}_{16}}(-\sin 2x)\cancel{2}$$

$$P.I = -\frac{x}{8}\cos 2x + \frac{1}{16}\sin 2x$$

Now the solution of equⁿ① is $y = C.F + P.I$

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{8}\cos 2x + \frac{1}{16}\sin 2x.$$

③ $\frac{d^2y}{dx^2} + 4y = x\sin x$

Sol:- Given D.E is $\frac{d^2y}{dx^2} + 4y = x\sin x$

$$D^2y + 4y = x\sin x$$

$$(D^2+4)\, y = x\sin x \rightarrow ①$$

An Auxiliary equⁿ is $m^2+4=0$

$$m^2=-4$$

$$m=\pm 2i$$

∴ The roots are complex and distinct.

$C.F = e^{(0)x}\left[c_1\cos 2x + c_2\sin 2x\right]$

$P.I = \frac{1}{D^2+4} \cdot x\sin x$

$= x\frac{1}{D^2+4}\sin x - \frac{2D}{(D^2+4)^2}\sin x$

$= x\frac{1}{-1+4}\sin x - \frac{2D}{(-1+4)^2}\sin x$

$= \frac{x}{3}\sin x - \frac{2D}{9}\cdot\sin x$

$P.I = \frac{x}{3}\cdot\sin x - \frac{2}{9}\cos x.$

Now the solution of equⁿ ① is $y = C.F + P.I$

$y = e^{(0)x}\left[c_1\cos 2x + c_2\sin 2x\right] + \frac{x}{3}\sin x - \frac{2}{9}\cos x.$

④ $\frac{d^2y}{dx^2} - 9y = x\cos 2x$

Sol:- Given D.E is $D^2y - 9y = x\cos 2x$

$(D^2-9)\,y = x\cos 2x \rightarrow ①$

An Auxiliary equⁿ is $m^2-9=0$

$m^2 - 3^2 = 0$

$(m-3)(m+3) = 0$

$m = 3, -3.$

∴ The roots are real and distinct.

$C.F = c_1e^{3x} + c_2e^{-3x}.$

$P.I = \frac{1}{D^2-9}\cdot x\cdot\cos 2x$

$= x\cdot\frac{1}{D^2-9}\cos 2x - \frac{2D}{(D^2-9)^2}\cos 2x$

$= x\frac{1}{-4-9}\cos 2x - \frac{2D}{(-4-9)^2}\cos 2x$

$= x\frac{1}{-13}\cos 2x - \frac{2D}{+169}\cos 2x$

$= \frac{-x}{13}\cos 2x - \frac{2}{169}(-\sin 2x)\,2$

$= \frac{-x}{13}\cos 2x + \frac{4}{169}\sin 2x$

$\begin{array}{r} 13\times 13 \\ \hline 39 \\ 13 \\ \hline 169 \\ \hline \end{array}$

Now the solution of equⁿ ① is $y = CF + PI$

$y = c_1e^{3x} + c_2e^{-3x} - \frac{x}{13}\cos 2x + \frac{4}{169}\sin 2x.$

⑥ $(D^2-1)y = x\sin 3x + \cos x$

Sol:-

Given D.E is $(D^2-1)y = x\sin 3x + \cos x$. $\rightarrow$ ①

An Auxiliary equn is $m^2-1=0$

$(m+1)(m-1)=0$

$m=1,-1$

∴ The roots are real and distinct.

$C.F = c_1e^{x} + c_2e^{-x}$.

$$P.I = \frac{1}{D^2-1}(x\sin 3x + \cos x)$$

$$= \underbrace{\frac{1}{D^2-1}x\sin 3x}_{PI_1} + \underbrace{\frac{1}{D^2-1}\cos x}_{PI_2} \rightarrow ②$$

$$PI_1 = \frac{1}{D^2-1}x\sin 3x$$

$$= x\frac{1}{D^2-1}\sin 3x - \frac{2D}{(D^2-1)^2}\sin 3x$$

$$= x\frac{1}{-9-1}\sin 3x - \frac{2D}{(-9-1)^2}\sin 3x$$

$$= x\frac{1}{-10}\sin 3x - \frac{\cancel{2}D}{\cancel{100}\,50}\sin 3x$$

$$= \frac{-x}{10}\sin 3x - \frac{1}{50}\cos 3x\,(3)$$

$$= \frac{-x}{10}\sin 3x - \frac{3}{50}\cos 3x.$$

$$PI_2 = \frac{1}{D^2-1}\cos x.$$

$$= \frac{1}{-1-1}\cos x$$

$$= \frac{1}{-2}\cos x$$

$$= \frac{-1}{2}\cos x$$

from ②,

$$P.I = \frac{-x}{10}\sin 3x - \frac{3}{50}\cos 3x - \frac{1}{2}\cos x.$$

Now the solution of equn ① is $y = C.F + P.I$

$$y = c_1e^{x} + c_2e^{-x} - \frac{x}{10}\sin 3x - \frac{3}{50}\cos 3x - \frac{1}{2}\cos x.$$

11/11/2019
Monday

General Method:

③ $\frac{d^2y}{dx^2} + a^2y = \sec ax.$

Sol:- Given D.E is $D^2y + a^2y = \sec ax$

$(D^2+a^2)y = \sec ax \rightarrow ①$

An Auxiliary Equn is $m^2+a^2=0$

$m^2 = -a^2$

$m = \pm ai$

∴ The roots are complex and distinct.

$C.F = e^{(0)x}[c_1\cos ax + c_2\sin ax]$

$P.I = \frac{1}{D^2+a^2}\sec ax$

$= \frac{1}{(D+ai)(D-ai)}\sec ax$

$= \frac{1}{2ai}\left(\frac{1}{D-ai} - \frac{1}{D+ai}\right)\sec ax$

$= \frac{1}{2ai}\left\{\underbrace{\frac{1}{D-ai}\sec ax}_{PI_1} - \underbrace{\frac{1}{D+ai}\sec ax}_{PI_2}\right\} \rightarrow ②$

$PI_1 = \frac{1}{D-ai}\sec ax$

$= e^{iax}\int \sec ax \cdot e^{-iax}\,dx$

$= e^{iax}\left[\int \sec ax\,(\cos ax - i\sin ax)\,dx\right]$

$= e^{iax}\left[\int \sec ax\cos ax\,dx - i\int \sin ax\cdot\sec ax\,dx\right]$

$= e^{iax}\left[\int (1)\,dx - i\int \tan ax\,dx\right]$

$= e^{iax}\left(x - i\,\frac{\log(\sec ax)}{a}\right)$

$= e^{iax}\left[x - \frac{i}{a}\log(\sec ax)\right]$

$PI_2 = \frac{1}{D+ai}\sec ax$

$= \frac{1}{[D-(-ai)]}\sec ax$

$$= e^{-iax}\int \sec ax\, e^{iax}\,dx$$

$$= e^{-iax}\int \sec ax.(\cos ax + i\sin ax)\,dx$$

$$= e^{-iax}\int \sec ax.\cos ax\,dx + i\int \sec ax.\sin ax.\,dx$$

$$= e^{-iax}\int (1)\,dx + i\int \tan ax\,dx$$

$$= e^{-iax}\left[x + i\frac{\log(\sec ax)}{a}\right]$$

$$= e^{-iax}\left[x + \frac{i}{a}\log(\sec ax)\right]$$

$$P.I = \frac{1}{2ai}\left[e^{iax}\left[x - \frac{i}{a}\log(\sec ax)\right] - e^{-iax}\left[x + \frac{i}{a}\log(\sec ax)\right]\right]$$

$$= \frac{1}{2ai}\left[e^{iax}.x - e^{iax}\frac{i}{a}\log(\sec ax) - e^{-iax}x - e^{-iax}\frac{i}{a}\log(\sec ax)\right]$$

$$= \frac{1}{2ai}\left[x\left(e^{iax} - e^{-iax}\right) - \frac{i}{a}\log(\sec ax)\left(e^{iax} + e^{-iax}\right)\right]$$

$$= \frac{1}{2ai}\left[x.2i\sin ax - \frac{i}{a}\log(\sec ax)\,2\cos ax\right]$$

$$P.I = \frac{x}{a}\sin ax - \frac{1}{a^2}\log(\sec ax)\cos ax.$$

Now the solution of equ^n (1) is $y = C.F + P.I$

$$y = e^{(0)x}(c_1\cos ax + c_2\sin ax) + \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax.\log(\sec ax).$$

(5) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}.$

Sol:- Given D.E is $D^2y + 3Dy + 2y = e^{e^x}$

$$(D^2 + 3D + 2)y = e^{e^x} \rightarrow (1)$$

An Auxiliary equ^n is $m^2 + 3m + 2 = 0$

$$m^2 + m + 2m + 2 = 0$$

$$m(m+1) + 2(m+1) = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2.$$

$\therefore$ The roots are real and distinct.

$$C.F = c_1 e^{-x} + c_2 e^{-2x}$$

$$P.I = \frac{1}{(D+1)(D+2)} e^{e^x}$$

$$= \frac{1}{2}\left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^x}$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{D+1} e^{e^x}}_{PI_1} - \underbrace{\frac{1}{D+2} e^{e^x}}_{PI_2}\right] \rightarrow ②$$

$$PI_1 = \frac{1}{D+1} e^{e^x}$$

$$= \frac{1}{D-(-1)} e^{e^x}$$

$$= e^{-x}\int e^{e^x} e^{x} dx$$

$$= e^{-x}\int e^{e^x} e^{x} dx$$

$$\boxed{\begin{matrix} e^x = t \\ e^x dx = dt \end{matrix}}$$

$$= e^{-x}\int e^t dt$$

$$= e^{-x} e^t$$

$$= e^{-x} e^{e^x}$$

$$PI_2 = \frac{1}{D+2} e^{e^x}$$

$$= \frac{1}{D-(-2)} e^{e^x}$$

$$= e^{-2x}\int e^{e^x} e^{2x} dx$$

$$= e^{-2x}\int e^{e^x} e^x e^x dx$$

$$= e^{-2x}\int e^t \cdot t \, dt$$

$$= e^{-2x} e^t (t-1)$$

$$= e^{-2x} e^{e^x} (e^x-1)$$

$$PI = \frac{1}{2}\left[e^{-x} e^{e^x} - e^{-2x} e^{e^x}(e^x-1)\right]$$

$$= \frac{1}{2}\left[e^{-x} e^{e^x} - e^{-2x} e^{e^x} e^x + e^{-2x} e^{e^x}\right]$$

$$= \frac{1}{2}\left[e^{-x} e^{e^x} - e^{-2x} e^{e^x} e^x + e^{-x} e^{-x} e^{e^x}\right]$$

$$= \frac{1}{2}\left\{e^{-x} e^{e^x} - e^{-x} \cancel{e^{-x}} e^{e^x} \cancel{e^x} + e^{-2x} e^{e^x}\right]$$

$$= \frac{1}{2}\left[\cancel{e^{-x} e^{e^x}} - \cancel{e^{-x} e^{e^x}} + e^{-2x} e^{e^x}\right]$$

$$= \frac{1}{2} e^{-2x} e^{e^x}$$

④ $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$.

Sol: Given D.E is $D^2y + 4y = 4\tan 2x$

$$(D^2+4)y = 4\tan 2x \rightarrow ①$$

An Auxiliary eqn is $m^2+4=0$

$$m^2 = -4$$

$$m = \pm 2i$$

∴ The roots are complex and distinct.

$$C.F = e^{(0)x}\left[C_1\cos 2x + C_2\sin 2x\right]$$

$$P.I = \frac{1}{D^2+4}\, 4\tan 2x$$

$$= \frac{1}{(D-2i)(D+2i)}\, 4\tan 2x$$

$$= \frac{1}{4i}\left(\frac{1}{D-2i} - \frac{1}{D+2i}\right) 4\tan 2x$$

$$= \frac{1}{i}\left(\frac{1}{D-2i} - \frac{1}{D+2i}\right)\tan 2x$$

$$= \frac{1}{i}\Big[\underbrace{\frac{1}{D-2i}\tan 2x}_{PI_1} - \underbrace{\frac{1}{D+2i}\tan 2x}_{PI_2}\Big] \rightarrow ②$$

$$PI_1 = \frac{1}{D-2i}\tan 2x$$

$$= e^{2ix}\int \tan 2x\, e^{-2ix}\,dx$$

$$= e^{2ix}\int \tan 2x \cdot e^{-i(2x)}\,dx$$

$$= e^{2ix}\int \tan 2x\cdot(\cos 2x - i\sin 2x)\,dx$$

$$\cancel{D-2i} = e^{2ix}\int \sin 2x\cdot dx - i\int \tan 2x\cdot \sin 2x\,dx$$

$-2i$
$$= e^{2ix}\left[\left(-\frac{\cos 2x}{2}\right) - i\int \frac{\sin^2(2x)}{\cos(2x)}\,dx\right]$$

$$= e^{+2ix}\left[-\frac{\cos 2x}{2} - i\int \frac{1-\cos^2 2x}{\cos 2x}\,dx\right]$$

$$\cancel{2i - 2i} = e^{2ix}\left[-\frac{\cos 2x}{2} - i\int \frac{1}{\cos 2x}\,dx + i\int \cos 2x\,dx\right.$$

$$= e^{2ix}\left[-\frac{\cos 2x}{2} - i\,\frac{\log(\sec 2x+\tan 2x)}{2} + i\,\frac{\sin(2x)}{2}\right.$$

$$= e^{2ix}\left(-\frac{\cos 2x}{2} - i\,\frac{\log(\sec 2x+\tan 2x)}{2} + i\,\frac{\sin 2x}{2}\right)$$

$$P.I_2 = \frac{1}{D+2i}\tan 2x$$

$$= \frac{1}{D-(-2i)}\tan 2x$$

$\cos ax + i\sin ax$
$\cos ax - i\sin ax$
$+$

$$= e^{-2ix}\int \tan 2x \cdot e^{2ix}\,dx$$

$$= e^{-2ix}\int \tan 2x\,(\cos 2x + i\sin 2x)\,dx$$

$$= e^{-2ix}\int (\tan 2x\cdot\cos 2x + i\tan 2x\cdot\sin 2x)\,dx$$

$$= e^{-2ix}\int \sin 2x\,dx + i\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$= e^{-2ix}\int -\frac{\cos 2x}{2} + i\int \frac{1}{\cos 2x}\,dx - i\int \frac{\cos^2 2x}{\cancel{\cos 2x}}\,dx$$

$$= e^{-2ix}\left[\frac{-\cos 2x}{2} + i\frac{\log(\sec 2x + \tan 2x)}{2} - i\frac{\sin 2x}{2}\right]$$

$$P.I = \frac{1}{i}\left[e^{2ix}\frac{-\cos 2x}{2} + \frac{i\log(\sec 2x + \tan 2x)}{2}e^{2ix} - i\frac{\sin 2x}{2}e^{2ix}\right]$$

$$-\left[e^{-2ix}\frac{-\cos 2x}{2} + i\frac{\log(\sec 2x + \tan 2x)}{2}e^{-2ix} - \frac{i}{2}\sin 2x\, e^{-2ix}\right]$$

log

$$= \frac{1}{i}\left[\frac{-\cos 2x}{2}2i\sin 2x + \frac{i}{2}\log(\sec 2x + \tan 2x)\,2\cos 2x - i\frac{\sin 2x}{2}\,2\cos 2x\right.$$

$$\begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$\cancel{\cos x} - i\sin x - \cancel{\cos x} - i\sin x$

$\cos x + \cancel{i\sin x} + \cos x - \cancel{i\sin x}$

$2\cos x$

$$-e^{-x}2e^{-2x} + e^{-x}\cdot e^{-2x}$$

$$-2e^{-3x} + e^{-3x}$$

$$-e^{-3x}$$

$$C.F = e^{(0)x}\left(c_1\cos 2x + c_2\sin 2x\right).$$

$$P.I = \frac{1}{D^2+4}\, 4\tan 2x$$

$$= 4\,\frac{1}{D^2+4}\,\tan 2x.$$

$$= 4\,\frac{1}{(D+2i)(D-2i)}\,\tan 2x$$

$$= 4\,\frac{-1}{4i}\left(\frac{1}{D+2i} - \frac{1}{D-2i}\right)\tan 2x$$

$$= \frac{-1}{i}\left(\frac{1}{D+2i} - \frac{1}{D-2i}\right)\tan 2x$$

$$= \frac{-1}{i}\left(\underbrace{\frac{1}{D+2i}\tan 2x}_{P.I_1} - \underbrace{\frac{1}{D-2i}\tan 2x}_{PI_2}\right) \rightarrow (2)$$

$$PI_1 = \frac{1}{D+2i}\tan 2x$$

$$= \frac{1}{D-(-2i)}\tan 2x$$

$$= e^{-2ix}\int \tan 2x \cdot e^{2ix}\,dx$$

$$= e^{-2ix}\int \tan 2x\,(\cos 2x + i\sin 2x)\,dx$$

$$= e^{-2ix}\int \frac{\sin 2x}{\cancel{\cos 2x}}\,\cancel{\cos 2x}\,dx + i\int \frac{\sin 2x}{\cos 2x}\sin 2x\,dx$$

$$= e^{-2ix}\left[\frac{-\cos 2x}{2} + i\int \frac{\sin^2 2x}{\cos 2x}\,dx\right]$$

$$= e^{-2ix}\left[\frac{-\cos 2x}{2} + i\int \frac{1-\cos^2 2x}{\cos 2x}\,dx\right]$$

$$= e^{-2ix}\left[\frac{-\cos 2x}{2} + i\int \sec 2x\,dx - i\int \cos 2x\,dx\right.$$

$$= e^{-2ix}\left[\frac{-\cos 2x}{2} + i\,\frac{\log(\sec 2x + \tan 2x)}{2} - i\,\frac{\sin 2x}{2}\right]$$

$$= e^{-2ix}\left[\frac{-1}{2}\cos 2x + \frac{i}{2}\log(\sec 2x + \tan 2x) - \frac{i}{2}\sin 2x\right]$$

$$PI_2 = \frac{1}{D-2i}\tan 2x$$

$$= e^{2ix}\int \tan 2x \cdot e^{-2ix}\,dx$$

$$= e^{2ix}\int \tan 2x\,(\cos 2x - i\sin 2x)\,dx$$

$$= e^{2ix}\int \sin 2x\,dx - i\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$= e^{2ix}\left[\frac{-\cos 2x}{2} - i\int \frac{1-\cos^2 2x}{\cos 2x}\,dx\right]$$

$= e^{2ix}\left[-\frac{\cos 2x}{2} - i\int \sec 2x\,dx + i\int \cos 2x\,dx.\right.$

$= e^{2ix}\left[\frac{-1}{2}\cos 2x - i\,\frac{\log(\sec 2x + \tan 2x)}{2} + i\,\frac{\sin 2x}{2}\right]$

$= e^{2ix}\left[\frac{-1}{2}\cos 2x - \frac{i}{2}\log(\sec 2x + \tan 2x) + \frac{i}{2}\sin 2x\right]$

$P.I = \frac{-1}{i}\left[e^{-2ix}\left[\frac{-1}{2}\cos 2x + \frac{i}{2}\log(\sec 2x + \tan 2x) - \frac{i}{2}\sin 2x\right] - \right.$
$\left. e^{2ix}\left[\frac{-1}{2}\cos 2x - \frac{i}{2}\log(\sec 2x + \tan 2x) + \frac{i}{2}\sin 2x\right]\right]$

$= \frac{-1}{i}\left[e^{-2ix}\frac{-1}{2}\cos 2x + \frac{i}{2}e^{-2ix}\log(\sec 2x + \tan 2x) - \frac{i}{2}e^{-2ix}\sin 2x\right.$
$\left. + e^{2ix}\frac{1}{2}\cos 2x + \frac{i}{2}e^{2ix}\log(\sec 2x + \tan 2x) - \frac{i}{2}e^{+2ix}\sin 2x\right]$

$= \frac{-1}{i}\left[\frac{1}{2}\cos 2x\left[e^{2ix} - e^{-2ix}\right] + \frac{i}{2}\log(\sec 2x + \tan 2x)\left[e^{2ix} + e^{-2ix}\right]\right.$
$\left. - \frac{i}{2}\sin 2x\left[e^{2ix} + e^{-2ix}\right]\right]$

$= \frac{-1}{i}\left[\frac{1}{\cancel{2}}\cos 2x\,(\cancel{2}i\sin 2x) + \frac{i}{\cancel{2}}\log(\sec 2x + \tan 2x)\,\cancel{2}\cos 2x\right.$
$\left. - \frac{i}{\cancel{2}}\sin 2x\,\cancel{2}\cos 2x\right]$

$= -\cancel{\cos 2x\,\sin 2x} - \log(\sec 2x + \tan 2x)\cos 2x + \cancel{\sin 2x\,\cos 2x}.$

$P.I = -\log(\sec 2x + \tan 2x)$

Now the solution of Equⁿ ① is $y = C.F + P.I$

$$y = e^{(0)x}\left[c_1\cos 2x + c_2\sin 2x\right] - \log(\sec 2x + \tan 2x)$$

① $\frac{d^2y}{dx^2} + a^2y = \tan ax$

Sol: Given D.E is $D^2y + a^2y = \tan ax$

$$(D^2 + a^2)y = \tan ax \rightarrow ①$$

An Auxiliary Equⁿ is $m^2 + a^2 = 0$

$$m^2 = -a^2$$

$$m = \pm ai$$

∴ The roots are complex and distinct.

$$C.F = e^{(0)x}\left[c_1\cos ax + c_2\sin ax\right]$$

Tuesday 12/11/2019

Method of variation of Parameter:

② $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$.

Given D.E is $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

$$D^2y - 6Dy + 9y = \frac{e^{3x}}{x^2}$$

$$(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2} \rightarrow ①$$

An Auxiliary eqn is $m^2 - 6m + 9 = 0$

$$(m-3)^2 = 0$$

$$(m-3)(m-3) = 0$$

$$m = 3, 3.$$

∴ The roots are real and repeat.

$$C.F = c_1 e^{3x} + c_2 x e^{3x}$$

Let us take $y_1 = e^{3x}$ and $y_2 = xe^{3x}$

The P.I is of the form $P.I = u_1 y_1 + u_2 y_2$

$u_1 = -\int$

$$P.I = \frac{1}{D^2 + a^2} \operatorname{Tan} ax$$

$$= \frac{1}{(D+ai)(D-ai)} \operatorname{Tan} ax$$

$$= \frac{-1}{2ai}\left(\frac{1}{D+ai} - \frac{1}{D-ai}\right) \operatorname{Tan} ax$$

$$= \frac{-1}{2ai}\Big[\underbrace{\frac{1}{D+ai}\operatorname{Tan} ax}_{PI_1} - \underbrace{\frac{1}{D-ai}\tan ax}_{PI_2}\Big] \rightarrow ②$$

$$PI_1 = \frac{1}{D+ai}\operatorname{Tan} ax$$

$$= \frac{1}{D-(-ai)}\operatorname{Tan} ax$$

$$= e^{-aix}\int \operatorname{Tan} ax \, e^{aix} \, dx$$

$$= e^{-aix}\int \tan ax(\cos ax + i\sin ax)\, dx$$

$$= e^{-aix}\int \frac{\sin ax}{\cos ax}\cos ax \, dx + i\int \frac{\sin^2 ax}{\cos ax}\, dx$$

$$= e^{-aix}\left[-\frac{\cos ax}{a} + i\int \sec ax\, dx - i\int \cos ax\, dx\right]$$

$$= e^{-aix}\left[\frac{-1}{a}\cos ax + i\,\frac{\log(\sec ax + \tan ax)}{a} - i\cdot\frac{\sin ax}{a}\right]$$

$$= e^{-aix}\left[\frac{-1}{a}\cos ax + \frac{i}{a}\log(\sec ax + \tan ax) - \frac{i}{a}\sin ax\right]$$

$$PI_2 = \frac{1}{D-ai}\tan ax$$

$$= e^{aix}\int \tan ax\, e^{-aix}\, dx.$$

$$= e^{aix}\int \tan ax\,(\cos ax - i\sin ax)\cdot dx$$

$$= e^{aix}\int \frac{\sin ax}{\cos ax}\cos ax - i\int\frac{\sin^2 ax}{\cos ax}\, dx$$

$$= e^{aix}\left[-\frac{\cos ax}{a} - i\int \sec ax\, dx + i\int \cos ax\, dx\right]$$

$$= e^{aix}\left[\frac{-1}{a}\cos ax - i\,\frac{\log(\sec ax + \tan ax)}{a} + i\,\frac{\sin ax}{a}\right]$$

$$= e^{aix}\left[\frac{-1}{a}\cos ax - \frac{i}{a}\log(\sec ax + \tan ax) + \frac{i}{a}(\sin ax)\right]$$

$$P.I = \frac{-1}{2ai}\Big[e^{-aix}\left[\frac{-1}{a}\cos ax + \frac{i}{a}\log(\sec ax + \tan ax) - \frac{i}{a}\sin ax\right] - e^{aix}\left[\frac{-1}{a}\cos ax - \frac{i}{a}\log(\sec ax + \tan ax) + \frac{i}{a}\sin ax\right]\Big]$$

$$= \frac{-1}{2ai}\Big[\frac{-1}{a}e^{-aix}\cos ax + \frac{i}{a}e^{-aix}\log(\sec ax + \tan ax) - \frac{i}{a}e^{-aix}\sin ax + \frac{1}{a}e^{aix}\cos ax + \frac{i}{a}e^{aix}\log(\sec ax + \tan ax) - \frac{i}{a}e^{aix}\sin ax\Big]$$

$$= \frac{-1}{2ai}\Big[\frac{1}{a}\cos ax\,(e^{aix} - e^{-aix}) + \frac{i}{a}\log(\sec ax + \tan ax)\,(e^{aix} + e^{-aix}) - \frac{i}{a}\sin ax\,[e^{aix} + e^{-aix}]\Big]$$

$$= \frac{-1}{2ai}\Big[\frac{1}{a}\cos ax\; 2i\sin ax + \frac{i}{a}\log(\sec ax + \tan ax)\,2\cos ax\,\cdot - \frac{i}{a}\sin ax\; 2\cos ax\Big]$$

$$= \frac{-1}{a^2}\sin ax\,\cancel{\cos ax} - \frac{1}{a^2}\cdot\log(\sec ax + \tan ax) + \frac{1}{a^2}\cdot\cancel{\sin ax\cos ax}$$

$$P.I = \frac{-1}{a^2}\log(\sec ax + \tan ax)$$

Now the solution of Eqn ① is $y = C.F + P.I$

$$y = e^{(0)x}[c_1\cos ax + c_2\sin ax] - \frac{1}{a^2}\log(\sec ax + \tan ax)$$

② $\frac{d^2y}{dx^2} + y = \cancel{\cos ax}\ \text{cosec}\, x$

Sol: Given D.E is $\frac{d^2y}{dx^2} + y = \text{cosec}\, x$

$D^2y + y = \text{cosec}\, x$

$(D^2+1)y = \text{cosec}\, x \rightarrow ①$

An Auxiliary equn is $m^2+1=0$

$m^2 = -1$

$m = \pm i$

∴ The roots are complex and distinct.

$C.F = e^{(0)x}[c_1 \cos x + c_2 \sin x]$

$P.I = \frac{1}{D^2+1} \cancel{\cos ax}\ \text{cosec}\, x$

$= \frac{1}{(D+i)(D-i)} \cos ax\ \text{cosec}\, x$

$= \left[\frac{1}{D+i} - \frac{1}{D-i}\right] \cos ax.\ \text{cosec}\, x$

$= \frac{-1}{2i}\left[\frac{1}{D+i} - \frac{1}{D-i}\right] \cancel{\cos ax}\ \text{cosec}\, x$

$= \frac{-1}{2i}\left[\underbrace{\frac{1}{D+i} \text{cosec}\, x}_{PI_1} - \underbrace{\frac{1}{D-i} \text{cosec}\, x}_{PI_2}\right] \rightarrow ②$

$PI_1 = \frac{1}{D-(-i)} \text{cosec}\, x$

$= e^{-ix}\int \text{cosec}\, x.\ e^{ix}\, dx$

$= e^{-ix}\int \text{cosec}\, x\ (\cos x + i \sin x)\, dx$

$= e^{-ix}\int \frac{1}{\sin x} \cos x + i\int \cancel{\frac{1}{\sin x}}.\cancel{\sin x}\, dx$

$= e^{-ix}\int \cot x\, dx + i\int (1)\, dx$

$= e^{-ix}\left[\log(\sin x) + i x\right]$

$PI_2 = \frac{1}{D-i} \text{cosec}\, x$

$= e^{ix}\int \text{cosec}\, x. e^{-ix}\, dx$

$= e^{ix}\int \text{cosec}\, x\ (\cos x - i\sin x)\, dx$

$= e^{ix}\int \cot x\, dx - i\int (1)\, dx$

$$= e^{ix}\left[\log(\sin x) - ix\right]$$

$$P.I = \frac{-1}{2i}\left[e^{-ix}\left[\log(\sin x) + ix\right] - e^{ix}\left[\log(\sin x) - ix\right]\right]$$

$$= \frac{-1}{2i}\left[e^{-ix}\log(\sin x) + ixe^{-ix} - e^{ix}\log(\sin x) + e^{ix}\,ix\right]$$

$$= \frac{-1}{2i}\left[\log(\sin x)\left(e^{-ix} - e^{ix}\right) + ix\left(e^{ix} + e^{-ix}\right)\right]$$

$$= \frac{-1}{2i}\left[\log(\sin x)(-2i\sin x) + ix\, 2\cos x\right]$$

$$= +\log(\sin x)\cdot\sin x - x\cdot\cos x$$

$$P.I = \sin x \cdot \log(\sin x) - x\cdot\cos x$$

Now the solution of equ^n ① is $y = C.F + P.I$

$$y = e^{(0)x}\left[c_1\cos x + c_2\sin x\right] + \sin x\cdot\log(\sin x) - x\cos x.$$

---

* M.O.V.O.P :- continuous:

where $v_1 = -\int \frac{y_2 X}{W}\,dx$ and $v_2 = -\int \frac{y_1 X}{W}\,dx$

Wronskin $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} \end{vmatrix}$$

$xe^{3x}(3) + e^{3x}$

$$= \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}(1+3x) \end{vmatrix}$$

$$= e^{3x}\cdot e^{3x}(1+3x) - 3xe^{3x}\cdot e^{3x}$$

$$= e^{6x} + \cancel{3xe^{6x}} - \cancel{3xe^{6x}}$$

$$\boxed{W = e^{6x}}$$

$$U_1 = -\int \frac{\cancel{x}e^{3x} \cdot \frac{e^{3x}}{x\cancel{y}}}{e^{6x}} dx \qquad U_2 = -\int \frac{e^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}} dx$$

$$= -\int \frac{\frac{1}{x}\cancel{e^{6x}}}{\cancel{e^{6x}}} dx \qquad = -\int \frac{1}{x^2} dx$$

$$= -\int x^{-2} dx$$

$$= -\int \frac{1}{x} dx \qquad = -\left(\frac{x^{-1}}{-1}\right)$$

$$= -\log x \qquad = \frac{1}{x}.$$

Now the $P.I = -\log x\, e^{3x} + \frac{1}{\cancel{x}} \cdot \cancel{x} e^{3x} = e^{3x}(1-\log x)$

~~Now the solution of equ n① is $y = C.F + P.I$~~

Now the solution of equ^{n}① is $y = C.F + P.I$

$y = c_1 e^{3x} + c_2 x e^{3x} + e^{3x}(1-\log x)$.

④ $y'' - 2y' + y = e^x \log x$.

Given D.E is ~~$m^2 - 2m + m$~~

$D^2y - 2Dy + y = e^x \log x$,

$(D^2 - 2D + 1)y = e^x \log x \rightarrow$ ①

An Auxiliary equ^n is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$.

∴ The roots are real and complex.

$C.F = c_1 e^x + c_2 x e^x$.

Let us take $y_1 = e^x$, $y_2 = xe^x$

The P.I is of the form $P.I = U_1 y_1 + U_2 y_2$

where $U_1 = -\int \frac{y_2 X}{W} dx$ and $U_2 = -\int \frac{y_1 X}{W} dx$

Wronskin $W = \begin{vmatrix} e^x & xe^x \\ e^x & x.e^x + e^x \end{vmatrix}$.

$= \begin{vmatrix} e^x & xe^x \\ e^x & e^x(1+x) \end{vmatrix}$

$$= e^{x} e^{x}(1+x) - e^{x}. x e^{x}$$

$$= e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}}$$

$$\boxed{W = e^{2x}}$$

$$U_1 = -\int \frac{\cancel{xe^{x}}.\cancel{e^{x}} \log x}{\cancel{e^{2x}}} dx \qquad U_2 = \int \frac{\cancel{e^{x}.e^{x}} \log x}{\cancel{e^{2x}}} dx$$

$$= -\int x \log x \, dx \qquad = -\int \log x \, dx$$

$$= -\left[\log x . \frac{x^2}{2} - \int \frac{1}{\cancel{x}} \frac{x^{\cancel{2}}}{2} dx\right] \qquad = -(x \log x - x)$$

$$= -\left[\frac{x^2}{2} \log x - \frac{1}{2} \int x \, dx\right] \qquad = -x \log x + x$$

$$= -\left[\frac{x^2}{2} \log x - \frac{1}{2} \frac{x^2}{2}\right)$$

$$= -\frac{x^2}{2} \log x + \frac{x^2}{4}$$

$$P.I = \left(-\frac{x^2}{2} \log x + \frac{x^2}{4}\right) e^{x} + (-x \log x + x) x . e^{x}$$

$$= -\frac{x^2}{2} \log x . e^{x} + \frac{x^2}{4} e^{x} + - x \log x \, x e^{x} + x . x e^{x}$$

$$= -\frac{x^2}{2} \log x . e^{x} + \frac{x^2}{4} e^{x} - x^2 \log x \, e^{x} + x^2 e^{x}.$$

$$= \log x . e^{x}\left(-\frac{x^2}{2} - x^2\right) + x^2 e^{x}\left(\frac{1}{4} + 1\right)$$

$$= \log x . e^{x}\left(\frac{-x^2 - 2x^2}{2}\right) + x^2 e^{x}\left(\frac{1+4}{4}\right)$$

$$P.I = e^{x}. \log x \left(\frac{-3x^2}{2}\right) + x^2 e^{x}\left(\frac{5}{4}\right)$$

Now the solution of Eqn ① is $y = C.F + P.I$

$$y = c_1 e^{x} + c_2 x e^{x} - e^{x} \log x . \left(\frac{3x^2}{2}\right) + \frac{5}{4} x^2 e^{x}.$$

⑥ $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$.

Given D-E is $D^2y + y = \frac{1}{1+\sin x}$

$$(D^2+1) y = \frac{1}{1+\sin x} \rightarrow ①$$

An Auxiliary Eqn is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$\therefore$ The roots are complex and distinct.

$C.F = e^{(0)x}[c_1 \cos x + c_2 \sin x]$

Let us take $y_1 = \cos x$ and $y_2 = \sin x$.

The P.I is of the form $P.I = u_1 y_1 + u_2 y_2$

Where $u_1 = -\int \frac{y_2 X}{W} dx$ and $u_2 = \int \frac{y_1 X}{W} dx$

$$\text{Wronskin } (W) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos x \cdot \cos x + \sin x \cdot \sin x$$

$$= \cos^2 x + \sin^2 x$$

$$\boxed{W = 1}$$

$$u_1 = -\int \frac{\sin x \cdot \frac{1}{1+\sin x}}{1} dx$$

$$= -\int \sin x \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$

$$= -\int \sin x \left(\frac{1-\sin x}{1-\sin^2 x}\right) dx$$

$$= -\int \sin x \left(\frac{1-\sin x}{\cos^2 x}\right) dx$$

$$= -\int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\right) dx$$

$$= -\int \sec x \cdot \tan x \, dx + \int \tan^2 x \, dx$$

$$= -\sec x + \int (\sec^2 x - 1) dx$$

$$= -\sec x + \int \sec^2 x \, dx - \int (1) dx$$

$$= -\sec x + \tan x - x.$$

$$u_2 = -\int \frac{\cos x \cdot \frac{1}{1+\sin x}}{1} dx$$

$$= -\int \cos x \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}$$

$$= -\int \cos x \cdot \left(\frac{1-\sin x}{\cos^2 x}\right) dx$$

$$= -\int \sec x \, dx + \int \tan x \, dx$$

$$= -\log(\sec x + \tan x) + \log(\sec$$

$$P.I = (-\sec x + \tan x - x)\cos x + [-\log(\sec x + \tan x) + \log(\sec x)]\sin x$$

$$= -\sec x \cdot \cos x + \tan x \cdot \cos x - x\cos x - \log(\sec x + \tan x)\sin x + \log(\sec x)\sin x$$

$$= -1 + \cos x \cdot \tan x - x\cos x - \sin x[\log(\sec x + \tan x) - \log(\sec x)]$$

Now the solution of equ^n ① is $y = C.F + P.I$

$$y = e^{(0)x}\left[c_1\cos x + c_2\sin x\right] + \cos x\tan x - (x\cos x + 1)$$
$$- \sin x\left[\log(\sec x + \tan x) - \log(\sec x)\right]$$

$$= e^{(0)x}\left[c_1\cos x + c_2\sin x\right] + \sin x - (x\cos x + 1) - \sin x \cdot \log\left(\frac{\sec x + \tan x}{\tan x}\right)$$

⑩ $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = \frac{1}{x^3}e^{-3x}$

sol: Given D.E is $D^2y + 6Dy + 9y = \frac{1}{x^3}e^{-3x}$

$$(D^2 + 6D + 9)y = \frac{1}{x^3}e^{-3x} \rightarrow ①$$

An Auxiliary equ^n is $m^2 + 6m + 9 = 0$

$$(m+3)^2 = 0$$
$$(m+3)(m+3) = 0$$
$$m = -3, -3$$

$\therefore$ The roots are real and repeat.

$C.F = c_1 e^{-3x} + c_2 x e^{-3x}$.

Let us take $y_1 = e^{-3x}$, $y_2 = xe^{-3x}$.

The P.I is of the form $P.I = U_1 y_1 + U_2 y_2$

where $U_1 = -\int\frac{y_2 X}{W}dx$, $U_2 = \int\frac{y_1 X}{W}dx$

$$\text{Wronskin } (W) = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & x.e^{-3x}(-3) + e^{-3x} \end{vmatrix}$$

$$= \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x}(-3x+1) \end{vmatrix}$$

$$= e^{-3x}.e^{-3x}(-3x+1) + 3x.e^{-3x}.e^{-3x}$$

$$= \cancel{-3x.e^{-6x}} + e^{-6x} + \cancel{3xe^{-6x}}$$

$$\boxed{W = e^{-6x}}$$

$$u_1 = -\int \frac{\cancel{x} e^{-3x} \cdot \frac{1}{x^{\cancel{3}2}} e^{-3x}}{\cancel{e^{-6x}}} dx \qquad u_2 = -\int \frac{e^{-3x} \cdot \frac{1}{x^3} e^{-3x}}{\cancel{e^{-6x}}} dx$$

$$= -\int x^{-2} dx \qquad\qquad = -\int x^{-3} dx$$

$$= -\left(\frac{x^{-1}}{-1}\right) \qquad\qquad = -\left(\frac{x^{-2}}{-2}\right)$$

$$= \frac{1}{x} \qquad\qquad = \frac{1}{2x^2}$$

$$P.I = \frac{1}{x} e^{-3x} + \frac{1}{2x^{\cancel{2}}} \cdot \cancel{x} \cdot e^{-3x} = \frac{1}{x} e^{-3x} + \frac{1}{2x} e^{-3x}$$

Now the solution of equ^n ① is $y = C.F + P.I$

$$y = c_1 e^{-3x} + c_2 x e^{-3x} + e^{-3x} \frac{1}{x}\left(1 + \frac{1}{2}\right)$$

$$y = c_1 e^{-3x} + c_2 x e^{-3x} + \frac{e^{-3x}}{x}(3/2).$$

① $\frac{d^2y}{dx^2} + 4y = 4\sec^2 2x.$

Sol:- Given DE is $D^2y + 4y = 4\sec^2 2x$

$$(D^2+4)y = 4\sec^2 2x \rightarrow ①$$

An Auxiliary equ^n is $m^2+4=0$

$$m^2 = -4$$

$$m = \pm 2i$$

∴ The roots are complex and distinct

$$C.F = e^{(0)x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

Let us take $y_1 = \cos 2x$, $y_2 = \sin 2x$

The P.I is of the form $P.I = u_1 y_1 + u_2 y_2$

where $u_1 = -\int \frac{y_2 X}{W} dx$ and $u_2 = -\int \frac{y_1 X}{W} dx$

$$\text{Wronskian value } (W) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$= 2\cos^2 2x + 2\sin^2 2x$

$= 2[\cos^2 2x + \sin^2 2x]$

$= 2(1)$

$\boxed{W = 2}$

$U_1 = -\int \frac{\sin 2x \cdot \cancel{4}^{2} \sec^2 2x}{\cancel{2}} dx$

$= -2\int \sin 2x \, (1 + \tan^2 2x) \, dx$

$= -2\left[\int \sin 2x \, dx + \int \sin 2x \tan^2 2x \, dx\right]$

$= -2$

$= -2\int \sin 2x \frac{1}{\cos^2 2x} dx$

$= -2\int \tan 2x \cdot \sec 2x \, dx$

$= -\cancel{2} \frac{\sec 2x}{\cancel{2}} = \underline{\underline{-\sec 2x}}$

$U_2 = -\int \frac{\cos 2x \cdot \cancel{4}^{2} \sec^2 2x}{\cancel{2}} dx$

$= -2\int \cancel{\cos 2x} \cdot \frac{1}{\cos^2 2x} dx$

$= -2\int \sec 2x \, dx$

$= -\cancel{2} \frac{\log(\sec 2x + \tan 2x)}{\cancel{2}}$

$= \underline{\underline{-\log(\sec 2x + \tan 2x)}}$

$P.I = -\sec 2x \cos 2x + (-\log(\sec 2x + \tan 2x)\sin 2x)$

$= -1 - \log(\sec 2x + \tan 2x) \sin 2x$

$= -[\sin 2x \cdot \log(\sec 2x + \tan 2x) + 1]$

Now the solution of Equn ① is $y = C.F + P.I$

$y = e^{(0)x}[c_1 \cos 2x + c_2 \sin 2x] - [\sin 2x \cdot \log(\sec 2x + \tan 2x) + 1]$

③ $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$

Sol:- Given D.E is $D^2y + y = \operatorname{cosec} x$

$(D^2+1)y = \operatorname{cosec} x \rightarrow ①$

An Auxiliary Equn is $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

∴ The roots are complex and distinct.

$C.F = e^{(0)x}[c_1\cos x + c_2\sin x]$

Let us take $y_1 = \cos x$, $y_2 = \sin x$.

The P.I is of the form $P.I = u_1y_1 + u_2y_2$.

where $u_1 = -\int \frac{y_2 X}{W}\,dx$ and $u_2 = -\int \frac{y_1 X}{W}\,dx$

Wronskin value $(W) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x$

$\boxed{W = 1}$

$u_1 = -\int \frac{\sin x \operatorname{cosec} x}{1}\,dx$

$= -\int (1)\,dx$

$= -x$.

$u_2 = -\int \frac{\cos x \cdot \operatorname{cosec} x}{1}\,dx$

$= -\int \cot x\,dx$

$= -\log(\sin x)$

$P.I = -x \cdot \cos x - \log(\sin x) \cdot \sin x$

$= -[x\cos x - \sin x \cdot \log(\sin x)]$

Now the solution of Equn ① is $y = C.F + P.I$

$y = e^{(0)x}[c_1\cos x + c_2\sin x] - [x\cos x - \sin x \cdot \log(\sin x)]$

⑤ $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

Sol: Given D.E is $D^2y - y = \frac{2}{1+e^x}$

$(D^2-1)\,y = \frac{2}{1+e^x} \rightarrow ①$

An Auxiliary Equn is $m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

$\therefore$ The roots are real and distinct.

$C.F = c_1 e^{x} + c_2 e^{-x}$

Let us take $y_1 = e^x$, $y_2 = e^{-x}$

The P.I is of the form $P.I = u_1y_1 + u_2y_2$

where $v_1 = -\int \frac{y_2 X}{W}\,dx$ and $v_2 = -\int \frac{y_1 X}{W}\,dx$.

Wronskin value $= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$

$= -\cancel{e^x}\cdot\cancel{e^{-x}} - \cancel{e^x}\cdot\cancel{e^{-x}}$

$= -1-1$

$\boxed{W = -2}$

$v_1 = -\int \frac{e^{-x}\cdot\frac{\cancel{2}}{1+e^x}}{-\cancel{2}}\,dx$

$= \int \frac{e^{-x}}{1+e^x}\,dx$

$= \int \cancel{e^{-x}}\cdot\frac{1}{\cancel{1+e^x}}\,dx$

$\int \frac{1}{\cancel{e^x+1e^{2x}}}\,dx$

$= \int \frac{e^{-x}\cdot e^{-x}}{1+e^{-x}}\,dx$ $\boxed{\begin{array}{l} 1+e^{-x}=t \\ -e^{-x}dx=dt \\ e^{-x}dx=-dt \end{array}}$

$= \int \frac{t-1}{t}\cdot(-dt)$

$= -\int\left(1-\frac{1}{t}\right)dt$

$= -\int(1)\,dt + \int\frac{1}{t}\,dt$

$= -t + \log t$

$= -(1+e^{-x}) + \log(1+e^{-x})$

$v_2 = -\int \frac{e^x\cdot\frac{\cancel{2}}{1+e^x}}{-\cancel{2}}\,dx$

$= \int \frac{e^x}{1+e^x}\,dx$

$= \int \frac{\cancel{e^{-x}}\cdot e^x}{e^{-x}+1}\,dx$

$= \frac{\log(e^{-x}+1)}{-e^{-x}}\ \cancel{(\times\, e^{-x})}$

$= -e^x\cdot\log(e^{-x}+1)$

$P.I = \left[-(1+e^{-x}) + \log(1+e^{-x})\right]e^x + \left[-e^x\cdot\log(e^{-x}+1)\right]e^{-x}$

$= -e^x - \cancel{e^{-x}}\cancel{e^x} + e^x\cdot\log(1+e^{-x}) - \cancel{e^x}\cdot\log(e^{-x}+1)\,\cancel{e^{-x}}$

$= -e^x - 1 + e^x\log(1+e^{-x}) - \log(1+e^{-x})$

$= -e^x\left(1-\log(1+e^{-x})\right) - 1\left(1+\log(1+e^{-x})\right)$

Now the solution of equn ① is $y = C.F + P.I$

$y = c_1e^x + c_2e^{-x} - e^x\left(1-\log(1+e^{-x})\right) - \left(1+\log(1+e^{-x})\right)$

⑦ $\frac{d^2y}{dx^2} + y = \operatorname{Tan} x.$

Sol:- Given D.E is $D^2y + y = \operatorname{Tan} x$

$$(D^2+1)y = \operatorname{Tan} x \rightarrow ①$$

An Auxiliary equⁿ is $m^2+1=0$

$$m^2 = -1$$

$$m = \pm i$$

∴ The roots are complex and distinct.

$$C.F = e^{(0)x}[c_1 \cos x + c_2 \sin x]$$

Let us take $y_1 = \cos x$ and $y_2 = \sin x$

The P.I of is of the form $P.I = u_1 y_1 + u_2 y_2$

where $u_1 = -\int \frac{y_2 X}{W} dx$ and $u_2 = -\int \frac{y_1 X}{W} dx$

Wronskin value $(W) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$$= \cos^2 x + \sin^2 x$$

$$\boxed{W=1}$$

$u_1 = -\int \frac{\sin x \cdot \operatorname{Tan} x}{1} dx$

$= -\int \frac{\sin^2 x}{\cos x} dx$

$= -\int \frac{1-\cos^2 x}{\cos x} dx$

$= -\int \sec x\, dx + \int \cos x\, dx$

$= -\log(\sec x + \operatorname{Tan} x) + \sin x$

$u_2 = -\int \frac{\cos x \cdot \operatorname{Tan} x}{1} dx$

$= -\int \cos x \cdot \frac{\sin x}{\cos x} dx$

$= -\int \sin x\, dx$

$= -(-\cos x)$

$= \cos x$

$P.I = \{-\log(\sec x + \operatorname{Tan} x) + \sin x\} \cos x + \cos x \sin x$

$= -\log(\sec x + \operatorname{Tan} x) + \sin x \cdot \cos x + \sin x \cdot \cos x$

$= 2\sin x \cos x - \log(\sec x + \operatorname{Tan} x)$

$P.I = \sin 2x - \log(\sec x + \tan x)$

Now the solution of equⁿ① is $y = C.F + P.I$

$$y = e^{(0)x}[c_1 \cos x + c_2 \sin x] + \sin 2x - \log(\sec x + \tan x)$$

⑧ $y'' + y = \sec^2 x.$

sol:- Given D.E is $D^2y + y = \sec^2 x.$

$$(D^2+1)\, y = \sec^2 x \rightarrow ①$$

An Auxiliary equⁿ is $m^2+1=0$

$$m^2 = -1$$

$$m = \pm i$$

∴ The roots are complex and distinct.

$$C.F = e^{(0)x}\left[c_1 \cos x + c_2 \sin x\right]$$

Let us take $y_1 = \cos x$, $y_2 = \sin x$

The P.I is of the form $P.I = U_1 y_1 + U_2 y_2$

where $U_1 = -\int \frac{y_2 X}{W} dx$ and $U_2 = -\int \frac{y_1 X}{W} dx$

$$\text{Wronskin value } (W) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$\boxed{W = 1}$$

$$U_1 = -\int \frac{\sin x \sec^2 x}{1} dx$$

$$= -\int \sin x (1+\tan^2 x)\, dx$$

$$= -\int \sin x\, dx - \int \sin x \cdot \frac{\sin^2 x}{\cos^2 x} dx$$

$$= -(-\cos x) - \int \sin x \left(\frac{1-\cos^2 x}{\cos^2 x}\right) dx$$

$$= \cos x - \int \sin x \cdot \frac{1}{\cos^2 x} dx + \int \sin x\, dx$$

$$= \cos x - \int \sec x \cdot \tan x\, dx + (-\cos x)$$

$$= \cancel{\cos x} - \sec x - \cancel{\cos x}$$

$$= -\sec x.$$

(or)

$$\boxed{\begin{aligned} U_1 &= -\int \sin x \cdot \sec^2 x\, dx \\ &= -\int \sec x \cdot \tan x\, dx \\ &= -\sec x \end{aligned}}$$

$$U_2 = -\int \frac{\cos x \cdot \sec^2 x}{1} dx$$

$$= -\int \cancel{\cos x} \cdot \frac{1}{\cos^{\cancel{2}} x} \cdot dx$$

$$= -\int \sec x \cdot dx$$

$$= -\log(\sec x + \tan x)$$

$$P.I = -\sec x \cdot \cos x + \left[-\log(\sec x + \tan x)\right] \sin x$$

$$= -\left[1 + \sin x \cdot \log(\sec x + \tan x)\right]$$

Now the solution of equⁿ① is $y = C.F + P.I$

$$y = e^{(0)x}\left[c_1 \cos x + c_2 \sin x\right] - \left[1 + \sin x \cdot \log(\sec x + \tan x)\right]$$

⑨ $\frac{d^2y}{dx^2} + y = x\sin x$.

Sol: Given D.E is $D^2y + y = x\sin x$

$$(D^2+1)y = x\sin x \rightarrow ①$$

An Auxiliary eqn is $m^2+1=0$

$$m^2 = -1$$

$$m = \pm i$$

∴ The roots are ~~real~~ complex and distinct.

$$C.F = e^{0x}\left[c_1\cos x + c_2\sin x\right]$$

Let us take $y_1 = \cos x$, $y_2 = \sin x$

The P.I is of the form $P.I = u_1y_1 + u_2y_2$

where $u_1 = -\int \frac{y_2 X}{W}dx$ and $u_2 = -\int \frac{y_1 X}{W}dx$

Wronskin value $(W) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$$= \cos^2 x + \sin^2 x$$

$$\boxed{W = 1}$$

$$u_1 = -\int \frac{\sin x \cdot (x\sin x)}{1}dx$$

$$= -\int x\cdot\sin^2 x\, dx$$

$$= -\left[x\cdot\frac{(\sin x)^3}{3} - \int \frac{(\sin x)^3}{3}dx\right]$$

$$= -\left[x\cdot\frac{(\sin x)^3}{3} - \frac{1}{3}\int \sin^3 x\, dx\right]$$

$$= -\left[x\cdot\frac{(\sin x)^3}{3} - \frac{1}{3}\int\left(\frac{3\sin x - \sin 3x}{4}\right)dx\right]$$

$$= -\left[x\cdot\frac{(\sin x)^3}{3} - \frac{1}{3}\times\frac{3}{4}\int \sin x\, dx + \frac{1}{12}\int \sin 3x\, dx\right]$$

$$= -\left[\frac{x}{3}(\sin x)^3 - \frac{1}{4}(-\cos x) + \frac{1}{12}\left(\frac{-\cos 3x}{3}\right)\right]$$

$$= -\frac{x}{3}(\sin x)^3 + \frac{1}{4}\cos x - \frac{1}{36}\cos 3x$$

$$u_2 = -\int \frac{\cos x \cdot x\sin x}{1}dx$$

$$= -\frac{1}{2}\int x\cdot \sin 2x\, dx$$

$$= \frac{-1}{2}\left[x\cdot\left(-\frac{\cos 2x}{2}\right) - \int (1)\left(\frac{\cos 2x}{2}\right) dx\right]$$

$$= \frac{-1}{2}\left[\frac{-x}{2}\cos 2x + \frac{1}{2}\int \cos 2x\, dx\right]$$

$$= \frac{-1}{2}\left[\frac{-x}{2}\cos 2x + \frac{1}{2}\frac{\sin 2x}{2}\right]$$

$$= \frac{-1}{2}\left[\frac{-x}{2}\cdot\cos 2x + \frac{1}{4}\sin 2x\right]$$

$$= \frac{1}{4}\left[x\cdot\cos 2x - \frac{1}{2}\sin 2x\right]$$

D I: $\sin 2x$, 1 $\to$ $-\frac{\cos 2x}{2}$, 0 $\to$ $-\frac{\sin 2x}{4}$

$$P.I = \left(\cancel{-\frac{x}{3}(\sin x)^3 + \frac{1}{4}\cos x + \frac{1}{36}\cos 3x}\right)\cancel{\cos x} + \cancel{\frac{1}{4}(x\cos 2x - \frac{1}{2}\sin 2x)\sin x}$$

$$\cancel{= -\frac{x}{3}(\sin x)^3}$$

$$\cancel{= -\frac{x}{3}(\sin x)^3 + \frac{1}{4}\cos x + \frac{1}{4}\cos^2 x - \frac{1}{36}(\cos 3x)}$$

$$\cancel{+\frac{1}{4}x\cos 2x\cdot\sin x - \frac{1}{8}\sin 2x\cdot\sin x}$$

$$\cancel{= -\frac{x}{3}(\sin x)^3\cdot\cos x}$$

Now ~~π~~

$$u_1 = -\int \sin x\cdot x\cdot\sin x\, dx$$

$$= -\int x\cdot\sin^2 x = -\int x\left(\frac{1-\cos 2x}{2}\right) = -\int \frac{x}{2}dx + \int x\frac{\cos 2x}{2}dx$$

$$= -\frac{1}{2}\int x\, dx + \frac{1}{2}\int \cos 2x\cdot x\, dx \quad \cancel{= -\frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{2}\frac{\sin 2x}{2}}$$

D I: x, $\cos 2x$; 1 $\to$ $\frac{\sin 2x}{2}$; 0 $\to$ $-\frac{\cos 2x}{4}$

$$= -\frac{1}{2}\frac{x^2}{2} + \frac{1}{2}\left(x\cdot\frac{\sin 2x}{2} + \int\frac{\cos 2x}{4}dx\right) \quad \cancel{-\frac{x^2}{4} + \frac{1}{4}\sin 2x}$$

$$= -\frac{x^2}{4} + \frac{1}{2}\left[\frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right]$$

$$= -\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x.$$

$$P.I = \left(-\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x\right)\cos x + \frac{1}{4}\left(x\cos 2x - \frac{1}{2}\sin 2x\right)\sin x$$

Now the solution of equ^n ① is $y = C.F + P.I$

$$y = e^{(0)x}\left[c_1\cos x + c_2\sin x\right] + \left(-\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x\right)\cos x$$
$$+ \frac{1}{4}\left(x\cos 2x - \frac{1}{2}\sin 2x\right)\sin x.$$

14/11 thurs

Applications of Higher order d.e:

①

The equation of the L.C.R circuit is

$$L\frac{d^2q}{dt^2}+R\cdot\frac{dq}{dt}+\frac{q}{C}=0.$$

Since $L=0.1$, $R=20$, $C=25\times10^{-6}$

$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=0$$

$$\frac{d^2q}{dt^2}+\frac{20}{0.1}\frac{dq}{dt}+\frac{q}{(0.1)(25\times10^{-6})}=0$$

$$\frac{d^2q}{dt^2}+200\frac{dq}{dt}+400000\,q=0. \rightarrow ①$$

Equⁿ ① is Higher order homogeneous d.e.

∴ The solution is $q=$ complementary function.

$$D^2q+200Dq+400000q=0$$

$$(D^2+200D+400000)q=0.$$

An Auxiliary equⁿ is $m^2+200m+400000=0$.

$$m=\frac{-200\pm\sqrt{(200)^2-4(1)400000}}{2(1)}$$

$$=\frac{-200\pm\sqrt{40000-1600000}}{2}$$

$$=\frac{-200\pm\sqrt{-1560000}}{2} \qquad \because \boxed{1248.9996}$$

$$=\frac{-200\pm1249i}{2}$$

$$m=-100\pm624.5i$$

∴ The roots are complex and distinct.

$$C.F=e^{-100t}\left[c_1\cos(624.5)t+c_2\sin(624.5)t\right]$$

Now the solution of equⁿ ① is $q=C.F$

$$q=e^{-100t}\left[c_1\cos(624.5)t+c_2\sin(624.5)t\right]$$

Given that at $t=0$, $q=0.05$, $i=0$

at $t=0$, $q=0.05$

$$0.05 = e^{-100(0)}\left[c_1 \cos(624.5)0 + c_2 \sin(624.5)0\right]$$

$$0.05 = e^{(0)}\left[c_1(1) + c_2(0)\right]$$

$$\boxed{c_1 = 0.05}$$

$$i = \frac{dq}{dt} = e^{-100t}(-100)\left[c_1\cos(624.5)t + c_2\sin(624.5)t\right].$$

$$+ e^{-100t}\left[c_1(-\sin(624.5)t)(624.5) + c_2\cos(624.5)t \cdot (624.5)\right]$$

at $t=0$, $i=0$

$$0 = e^{-100(0)}(-100)\left[c_1 \cos(624.5)0 + c_2 \sin(624.5)0\right]$$
$$+ e^{-100t}\left[-c_1 \sin[(624.5)0](624.5) + c_2 \cos[(624.5)0].(624.5)\right]$$

$$0 = -100.\left[c_1(1) + c_2(0)\right] + e^{-100(0)}(0 + c_2\, 624.5)$$

$$0 = -c_1^{100} + c_2 . 624.5$$

$$0 = -0.05^{(100)} + c_2\, 624.5 \Rightarrow 0 = 5 + c_2\, 624.5$$

$\cancel{c_2\, 624.5 = 0.05}$ $\qquad c_2 = \frac{-5}{624.5}$

$\cancel{c_2 = \frac{0.05}{624.5}}$ $\qquad c_2 = 0.008006405$

$$\boxed{c_2 = 0.008}$$

③ The eqn of the L.C.R. circuit is

$$L\frac{d^2q}{dt^2} + R.\frac{dq}{dt} + \frac{q}{C} = E \sin wt$$

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E}{L}\sin wt$$

$$\frac{d^2q}{dt^2} + 2s\frac{dq}{dt} + w^2 q = \frac{E}{L}\sin wt. \rightarrow ①$$

where $w^2 = \frac{1}{LC}$

$2s = \frac{R}{L}$

$$D^2 q + 2S Dq + \omega^2 q = \frac{E}{L}\sin \omega t$$

$$(D^2 + 2DS + \omega^2) q = \frac{E}{L}\sin \omega t$$

An auxiliary eqn is $m^2 + 2Sm + \omega^2 = 0$

$$m = \frac{-2S \pm \sqrt{4S^2 - 4(1)\omega^2}}{2}$$

$$= \frac{-2S \pm \sqrt{4S^2 - 4\omega^2}}{2}$$

$$= \frac{\cancel{2}(-S \pm \sqrt{S^2 - \omega^2})}{\cancel{2}}$$

$$= -S \pm \sqrt{S^2 - \omega^2}$$

We have $R^2 < \frac{4L}{C}$

$$\frac{R^2}{4L} < \frac{1}{C}$$

$$\frac{R^2}{4L^2} < \frac{1}{LC}$$

$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$$

$$\boxed{S^2 - \omega^2 < 0}$$

$$m = -S \pm \sqrt{S^2 - \omega^2}\, i$$

∴ The roots are complex and distinct.

Let $p = \sqrt{S^2 - \omega^2}$

$$m = -S \pm p i$$

$$C.F = e^{-St}[C_1 \cos pt + C_2 \sin pt].$$

The particular integral is of the form $= \frac{1}{f(D)} X$

$$= \frac{1}{D^2 + 2DS + \omega^2} \cdot \frac{E}{L} \sin \omega t.$$

$$= \frac{E}{L} \frac{1}{D^2 + 2SD + \omega^2} \sin \omega t$$

$$= \frac{E}{L} \frac{1}{\cancel{-\omega^2} + 2SD + \cancel{\omega^2}} \sin \omega t$$

$$= \frac{E}{L} \cdot \frac{1}{2S}\left(\frac{1}{D} \sin \omega t\right)$$

$= \frac{E}{2LS}\left(\frac{-\cos\omega t}{\omega}\right)$

$P.I = \frac{-E}{2LS\omega}(\cos\omega t)$.

$\boxed{2S = \frac{R}{L} \Rightarrow S = \frac{R}{2L} \quad \boxed{R = 2LS}}$

$= \frac{-E}{R\omega}(\cos\omega t)$

Now the solution for equn ① is $q = C.F + P.I$

$$q = e^{-St}[c_1\cos pt + c_2 \sin pt] - \frac{E}{R\omega}(\cos\omega t) \rightarrow ②$$

we have $t=0$, $q=0$

$$0 = e^{-S(0)}[c_1 \cos p(0) + c_2 \sin p(0)] - \frac{E}{R\omega}\cos\omega(0)$$

$$0 = (1)[c_1 (1) + c_2(0)] - \frac{E}{R\omega}(1)$$

$$\boxed{c_1 = \frac{E}{R\omega}}$$

$$i = \frac{dq}{dt} = e^{-St}(-S)[c_1\cos pt + c_2 \sin pt] + e^{-St}[c_1(-\sin pt(p)) + c_2 \cos pt(p)]$$

$$i = -Se^{-St}[c_1 \cos pt + c_2 \sin pt] + e^{-St}[-pc_1 \sin pt + pc_2 \cos pt] + \frac{E}{R\omega}\sin\omega t(\omega) + \frac{E}{R}\sin\omega t$$

we have $t=0$, $i=0$

$$0 = -Se^{-S(0)}[c_1 \cos p(0) + c_2 \sin p(0)] + e^{-S(0)}[-pc_1 \sin p(0) + pc_2 \cos p(0)] + \frac{E}{R}\sin\omega(0)$$

$$0 = -S(1)[c_1(1) + c_2(0)] + (1)[pc_1(0) + pc_2(1)] + \frac{E}{R}(0)$$

$$0 = -Sc_1 + pc_2$$

$$0 = -S\frac{E}{R\omega} + pc_2$$

$$pc_2 = \frac{SE}{R\omega}$$

$$\boxed{c_2 = \frac{SE}{pR\omega}}$$

from (2),

$$q = e^{-St}\,[c_1 \cos pt + c_2 \sin pt] - \frac{E}{R\omega}\cos\omega t.$$

$$= e^{-St}\left[\frac{E}{R\omega}\cos pt + \frac{ES}{PR\omega}\sin pt\right] - \frac{E}{R\omega}\cos\omega t$$

$$= \frac{E}{R\omega}\left[e^{-St}\left(\cos pt + \frac{S}{p}\sin pt\right) - \cos\omega t\right]$$

$$= \frac{E}{R\omega} \quad -\cos\omega t + e^{\frac{-Rt}{2L}}$$

$$q = \frac{E}{R\omega}\left[-\cos\omega t + e^{\frac{-Rt}{2L}}\left(\cos pt + \frac{R}{2Lp}\sin pt\right)\right]$$

$$i = \frac{dq}{dt} = e^{-St}(-S)\,[c_1\cos pt + c_2 \sin pt] + e^{-St}[c_1(-\sin pt)p + c_2(\cos pt$$

$$- \frac{E}{R\omega}(-\sin\omega t)\,\omega$$

$$i = \frac{dq}{dt} = \frac{E}{R\omega}\left[\sin\omega t\;\omega + e^{-R/2L\,t}\cdot\left(\frac{-R}{2L}\right)\left(\cos pt + \frac{R}{2Lp}\sin pt\right)\right.$$

$$\left. + e^{\frac{-Rt}{2L}}\left[-\sin pt(p) + \frac{R}{2Lp}\cos(pt)\,p\right]\right]$$

$$= \frac{E}{R\omega}\left[\omega\cdot\sin\omega t - e^{\frac{-Rt}{2L}}\left(\frac{R}{2L}\right)\left(\cos pt + \frac{R}{2Lp}\sin pt\right)\right.$$

$$\left. - e^{\frac{-Rt}{2L}}\,p\cdot\sin pt + e^{\frac{-Rt}{2L}}\cdot\frac{R}{2L}\cdot\cos pt\right]$$

$$= \frac{E}{R\omega}\left[\omega\cdot\sin\omega t - e^{\frac{-Rt}{2L}}\cancel{\frac{R}{2L}\cos pt} - e^{\frac{-Rt}{2L}}\cdot\frac{R}{2L}\cdot\frac{R}{2Lp}\sin pt\right.$$

$$\left. - e^{\frac{-Rt}{2L}}\,p\cdot\sin pt + e^{\frac{-Rt}{2L}}\cdot\cancel{\frac{R}{2L}\cos pt}\right]$$

$$= \frac{E}{R\omega}\left[\omega\cdot\sin\omega t - e^{\frac{-Rt}{2L}}\cdot\frac{R^2}{4L^2p}\sin pt - e^{\frac{-Rt}{2L}}\cdot p\cdot\sin pt\right]$$

$$= \frac{E}{R\omega}\left[\omega\cdot\sin\omega t - e^{\frac{-Rt}{2L}}\sin pt\left(\frac{S^2}{P} + P\right)\right]$$

$$= \frac{E}{R\omega}\left[\omega\cdot\sin\omega t - e^{\frac{-Rt}{2L}}\sin pt\left(\frac{S^2 + P^2}{P}\right)\right]$$

$$= \frac{E}{R\omega}\left[\omega\cdot\sin\omega t - e^{\frac{-Rt}{2L}}\sin pt\left(\frac{\cancel{S^2} + \omega^2 - \cancel{S^2}}{P}\right)\right]$$

$$= \frac{E}{R\omega}\left[\omega\cdot\sin\omega t - e^{\frac{-Rt}{2L}}\sin pt\left(\frac{\omega^2}{P}\right)\right]$$

$$= \frac{E}{R\omega} \omega \left[\sin\omega t - e^{\frac{-Rt}{2L}} \frac{\omega}{p} \cdot \sin pt\right]$$

$$i = \frac{E}{R}\left[\sin\omega t - e^{\frac{-Rt}{2L}} \frac{1}{p\sqrt{LC}} \sin pt\right]$$

④

The eqn of the L.C.R circuit is

$$L\frac{d^2q}{dt^2} + R \cdot \frac{dq}{dt} + \frac{q}{C} = E\sin\omega t$$

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E}{L}\sin\omega t$$

$$\frac{d^2q}{dt^2}$$

② An uncharged condenser --

Given that,

The eqn of the L.C.R circuit is

$$L\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{C} = E\sin\frac{t}{\sqrt{LC}}$$

Given that resistance is negligible.

then, $L\frac{d^2q}{dt^2} + \frac{q}{C} = E\sin\frac{t}{\sqrt{LC}}$

$$\frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{E}{L}\sin\frac{t}{\sqrt{LC}}$$

$$D^2q + \omega^2 q = \frac{E}{L}\sin\frac{t}{\sqrt{LC}}$$

$$(D^2+\omega^2)\, q = \frac{E}{L}\sin\frac{t}{\sqrt{LC}} \rightarrow ①$$

An Auxiliary eqn is

$$m^2+\omega^2=0$$

$$m^2=-\omega^2$$

$$m = \pm\omega i$$

∴ The roots are complex and distinct.

$$C.F = e^{(0)t}\left[c_1\cos\omega t + c_2\sin\omega t\right]$$

$$P.I = \frac{1}{D^2+\omega^2} \frac{E}{L}\sin\frac{t}{\sqrt{LC}}$$

$$= \frac{E}{L} \cdot \frac{1}{D^2+\omega^2}\sin\omega t.$$

$$= \frac{E}{L}\frac{t}{2D+0}\sin\omega t$$

$$= \frac{Et}{2L}\frac{1}{D}\sin\omega t$$

$$= \frac{Et}{2L}\frac{-\cos\omega t}{\omega}$$

$$P.I = \frac{-Et}{2L\omega}\cos\omega t$$

Now the solution of equⁿ① is $q = C.F + P.I$

$$q = C_1\cos\omega t + C_2\sin\omega t - \frac{Et}{2L\omega}\cos\omega t. \rightarrow ②$$

At $t=0$, $q=0$

$$0 = C_1\cos\omega(0) + C_2\sin\omega(0) - \frac{E(0)}{2L\omega}\cos\omega(0)$$

$$0 = C_1(1) + C_2(0) - 0$$

$$\Rightarrow \boxed{C_1 = 0}$$

from ①,

$$q = C_2\sin\omega t - \frac{Et}{2L\omega}\cos\omega t. \rightarrow ③$$

$$i = \frac{dq}{dt} = C_2\cos\omega t(\omega) - \frac{E}{2L\omega}\left[t.(-\sin\omega t)\omega + \cos\omega t(1)\right]$$

$$i = C_2\omega.\cos\omega t + \frac{Et}{2L}\sin\omega t - \frac{E}{2L\omega}\cos\omega t$$

At $t=0$, $i=0$

$$0 = C_2\omega.\cos\omega(0) + \frac{E(0)}{2L}\sin\omega(0) - \frac{E}{2L\omega}\cos\omega(0)$$

$$0 = C_2\omega + 0 - \frac{E}{2L\omega}$$

$$C_2\omega = \frac{E}{2L\omega} \Rightarrow \boxed{C_2 = \frac{E}{2L\omega^2}}$$

from ③,

$$q = \frac{E}{2L\omega^2}\sin\omega t - \frac{Et}{2L\omega}\cos\omega t.$$

$$= \frac{E\cancel{L}C}{2\cancel{L}}\sin\frac{t}{\sqrt{LC}} - \frac{Et\sqrt{LC}}{2L}\cos\omega t.$$

$$= \frac{EC}{2}\left[\sin\frac{t}{\sqrt{LC}} - \frac{t\sqrt{LC}}{LC}\cos\frac{t}{\sqrt{LC}}\right]$$

$$\boxed{q = \frac{EC}{2}\left[\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right]}$$

* Simple Harmonic Motion

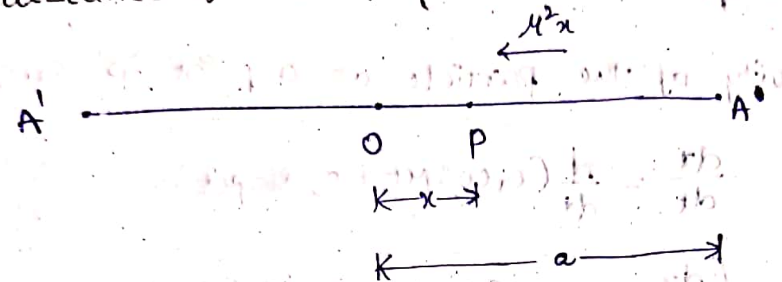
Max^m distance = amplitude

a^2 = Intensity

velocity $(v) = \frac{dx}{dt}$

acceleration $(a) = \frac{dv}{dt}$

*① A particle is said to execute S.H.M if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.



→ Let 'O' be the fixed point in the line AA'.

→ Let 'P' be the position of the particle at any time 't'.

→ Where $\boxed{OP = x}$

→ Since the acceleration is always directed towards the point 'O', i.e, the acceleration is in the direction opposite to that in which 'x' increases.

→ Therefore, the equ^n of the motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu^2 x.$$

(or)

$$\frac{d^2x}{dt^2} + \mu^2 x = 0$$

(or)

$$\boxed{D^2 x + \mu^2 x = 0}$$

$$\boxed{(D^2 + \mu^2)x = 0} \longrightarrow ①$$

where $D = \frac{d}{dt}$

→ It is a linear differential equ^n with constant co-efficient.

i.e., $D^2 + \mu^2 = 0$ $\boxed{x \neq 0}$

→ $D^2 = -\mu^2$

→ $\boxed{D = \pm \mu i}$

∴ The solution of equ^n (1) is

$$\boxed{x = c_1 \cos \mu t + c_2 \sin \mu t.} \rightarrow (2)$$

∴ The velocity of the particle at a point 'p' can be written as $\frac{dx}{dt} = \frac{d}{dt}(c_1 \cos \mu t + c_2 \sin \mu t)$

$$v = \boxed{\frac{dx}{dt} = -c_1 \mu \sin \mu t + c_2 \mu \cos \mu t} \rightarrow (3)$$

→ If the particle starts from the rest at 'A', where OA = a.

→ Therefore from (2)

At $t = 0$, $x = a$

$a = c_1 \cos \mu(0) + c_2 \sin \mu t.$

$= c_1 (1) + c_2 (0)$

→ $\boxed{a = c_1}$

→ a or c_1 from (3) At $t = 0$, $v = 0$, $\frac{dx}{dt} = 0$

$$v = \frac{dx}{dt} = -c_1 \mu \sin \mu(0) + c_2 \mu \cos \mu(0)$$

$$\frac{dx}{dt} = -c_1 \mu (0) + c_2 \mu (1)$$

Substitution 'c_1' and 'c_2' value in (1)

$$* \boxed{x = a \cos \mu t} \longrightarrow (4)$$

$\therefore$ velocity $= \frac{dx}{dt} = -a\mu \sin \mu t \rightarrow (5)$

$$v = \frac{dx}{dt} = -a\mu\sqrt{1-\cos^2\mu t}$$

Let $\cos \mu t = \frac{x}{a}$. then above eqn can be written

as $= -a\mu\sqrt{1-\cos^2\mu t}$

$$\frac{dx}{dt} = -a\mu\sqrt{1-\frac{x^2}{a^2}}$$

$$\frac{dx}{dt} = -\cancel{a}\mu\sqrt{\frac{a^2-x^2}{\cancel{a^2}}}$$

$$\frac{dx}{dt} = -\mu\sqrt{a^2-x^2} \rightarrow (6)$$

Time Period:

The time taken for one perfect oscillation is called time period. which is denoted T.

→ The time period can be written as $\boxed{T = \frac{2\pi}{\mu}}$

Frequency of the oscillators:

The no. of oscillations fore second is called frequency of the oscillator.

→ which is denoted by n $\quad n = \frac{1}{T}$

$$n = \frac{1}{2\pi/\mu}$$

$$\boxed{n = \frac{\mu}{2\pi}}$$

① A particle is executing S.H.M ...

Sol: Given amplitude $= 20\,cm$

time $(T) = 4$ seconds

We know that, $T = \frac{2\pi}{\mu}$

$$4 = \frac{2\pi}{\mu}$$

$$\boxed{\mu = \pi/2}$$

We know that, $x = a\cos\mu t$

Case(i)

At $x_1 = 5cm, \mu = \pi/2, a = 20\,cm$

$x_1 = a\cos\mu t$

$5 = 20\cos\frac{\pi}{2}t$

$1/4 = \cos\frac{\pi}{2}t$

$\cos^{-1}(1/4) = \frac{\pi}{2}t$

$\boxed{t_1 = \frac{2}{\pi}\cos^{-1}1/4}$

Case(ii),

At $x_2 = 15cm, \mu = \pi/2, a = 20\,c$

$x_2 = a\cos\mu t$

$15 = 20\cos\frac{\pi}{2}t$

$3/4 = \cos\frac{\pi}{2}t$

$\cos^{-1}(3/4) = \frac{\pi}{2}t$

$\boxed{t_2 = \frac{2}{\pi}\cos^{-1}3/4}$

$\therefore t_2 - t_1 = \frac{2}{\pi}\cos^{-1}3/4 - \frac{2}{\pi}\cos^{-1}1/4$

$= \frac{2}{\pi}\left[\cos^{-1}(3/4) - \cos^{-1}(1/4)\right]$

$= \frac{2}{180}[41.40962211 - 75.52248781]$

$= \frac{1}{90}(-34.11286557)$

$= -0.1379$

$\boxed{t_2 - t_1 \approx -0.38}$ seconds

② A particle moving in a straight line...

Sol: Given $x = a\cos\mu t$.

We know that the velocity, $V = -a\mu\sin\mu t$

(or)

$$V = -\mu\sqrt{a^2 - x^2}$$

case(i)

At displacement $= x_1$

velocity $= v_1$

$v_1 = -\mu\sqrt{a^2 - x_1^2}$

$v_1^2 = \mu^2(a^2 - x_1^2)$

case(ii) At displacement $= x_2$

velocity $= v_2$

$v_2 = +\mu\sqrt{a^2 - x_2^2}$

$v_2^2 = \mu^2(a^2 - x_2^2)$

$\therefore v_2^2 - v_1^2 = \mu^2(a^2 - x_2^2) - \mu^2(a^2 - x_1^2)$

$= \cancel{a^2\mu^2} - \mu^2 x_2^2 - \cancel{\mu^2 a^2} + x_1^2\mu^2$

$v_2^2 - v_1^2 = \mu^2(x_1^2 - x_2^2)$

$$\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2} = \mu^2$$

$$\mu = \left(\sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}\right)$$

We know that Time period $T = \frac{2\pi}{\mu}$.

$$T = \frac{2 \times \pi}{\sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}}$$

$$T = 2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$$

③ At the end of the three successive seconds. the distances of a point moving with S.H.M from its mean position are x_1, x_2, x_3 respectively. Show that the time of a complete oscillation is $\frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}$

Sol: Given that x_1, x_2, x_3 are the distances.

Let at the positions the times can be taken as t, $t+1$, $t+2$ seconds respectively.

We know that, $x = a\cos\mu t$

$x_1 = a\cos\mu t \rightarrow ①$

$x_2 = a\cos\mu(t+1) \rightarrow ②$

$x_3 = a\cos\mu(t+2) \rightarrow ③$

By adding equ^n ① & ③

We get $x_1 + x_3 = a\cos\mu t + a\cos\mu(t+2)$

$$x_1+x_3 = a\left[\cos\mu t+\cos\mu(t+2)\right]$$

$$x_1+x_3 = a\ 2\cos\left(\frac{\mu t+\mu(t+2)}{2}\right)\cos\left(\frac{\mu t-\mu(t+2)}{2}\right)$$

$$= 2a\cos\left(\frac{\mu t+\mu t+2\mu}{2}\right)\cos\left(\frac{\cancel{\mu t}-\cancel{\mu t}-2\mu}{2}\right)$$

$$= 2a\cos\left(\frac{2\mu t+2\mu}{2}\right)\cos\left(\frac{-\cancel{2}\mu}{\cancel{2}}\right)$$

$$= 2a\cos\left(\frac{\cancel{2}\mu(t+1)}{\cancel{2}}\right)\cos(-\mu)$$

$$x_1+x_3 = 2a\cos\mu(t+1)\cdot\cos\mu$$

$$x_1+x_3 = 2\cdot\cos\mu\left[a\cos\mu(t+1)\right]$$

$$x_1+x_3 = 2\cos\mu\ x_2$$

$$\cos\mu = \frac{x_1+x_3}{2x_2}$$

$$\mu = \cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)$$

We know that,

$$\text{Time period } (T) = \frac{2\pi}{\mu}$$

$$T = \frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}$$

④ A particle is executing S.H.M. ....

sol: Given amplitude = 5 meters.

time (T) = 4 seconds

W.K.T, $T = \frac{2\pi}{\mu}$

$$4 = \frac{2\pi}{\mu}$$

$$\boxed{\mu = \pi/2}$$

W.K.T, $x = a\cos\mu t_1$

case(i) At $x_1 = 4$ m, $\mu = \pi/2$, $a = 5$ m

$$x_1 = a\cos\mu t_1$$

$$4 = 5\cos\pi/2\ t_1$$

case(ii) At $x_2 = 2$ m, $\mu = \pi/2$, $a = 5$

$$x_2 = a\cos\mu t_2$$

$$2 = 5\cos\pi/2\ t_2$$

$4/5 = \cos \pi/2\, t$ $\qquad$ $2/5 = \cos \pi/2 \cdot t_2$

$\cos^{-1}(4/5) = \pi/2\, t$ $\qquad$ $\cos^{-1}(2/5) = \pi/2\, t_2$

$t_1 = \frac{2}{\pi} \cos^{-1}(4/5)$ $\qquad$ $t_2 = \frac{2}{\pi} \cos^{-1}(2/5)$

$\therefore t_2 - t_1 = \frac{2}{\pi} \cos^{-1}(2/5) - \frac{2}{\pi} \cos^{-1}(4/5)$

$= \frac{2}{180} \left[\cos^{-1}(2/5) - \cos^{-1}(4/5) \right]$

$= \frac{2}{180} \left[66.42 - 36.86 \right]$

$= \frac{2}{180} \times 29.56$

$= 0.3284$

$\boxed{t_2 - t_1 \cong 0.33}$ seconds.

(5) At the end of the three successive seconds, - -

sol: Given that $x_1 = 1$, $x_2 = 5$, $x_3 = 5$

Time period $(T) = \frac{2\pi}{\theta}$

Let at the positions the times can be taken as,

t, $t+1$, $t+2$ seconds respectively.

W.K.T, $x = a \cos \mu t$

$x_1 = a \cos \mu t \Rightarrow 1 = a \cos \mu t \rightarrow (1)$

$x_2 = a \cos \mu (t+1) \Rightarrow 5 = a \cos \mu (t+1) \rightarrow (2)$

$x_3 = a \cos \mu (t+2) \Rightarrow 5 = a \cos \mu (t+2) \rightarrow (3)$

By adding eqn (1) & (3)

$1+5 = a \cos \mu t + a \cos \mu (t+2)$

$6 = a \left[\cos \mu t + \cos \mu (t+2) \right]$

$6 = a\, 2 \cos \left(\frac{\mu t + \mu (t+2)}{2} \right) \cos \left(\frac{\mu t - \mu (t+2)}{2} \right)$

$6 = 2a \cos \left(\frac{\mu t + \mu t + 2\mu}{2} \right) \cos \left(\frac{\cancel{\mu t} - \cancel{\mu t} - 2\mu}{2} \right)$

$6 = 2a \cos \left(\frac{2\mu t + 2\mu}{2} \right) \cos \left(\frac{-\cancel{2}\mu}{\cancel{2}} \right)$

$3 = a \cos \frac{\cancel{2}(\mu t + \mu)}{\cancel{2}} \cos(-\mu)$

$3 = a \cos \mu (t+1) \cos \mu$

$3 = 5 \cos \mu$

$\cos \mu = \frac{3}{5}$.

$$\boxed{\therefore \cos \theta = \frac{3}{5}}$$

unit - 4

16/11/19 saturday

Partial Derivatives

1. Homogeneous function, Euler's Theorem, Total derivatives, chain rule, Jackobean, Functionally dependents;
Taylor's and ~~Machlarries~~ Maclaurin's Expansions with two variables.

Applications: Maxima and minima with constants and without constants, Lagranges

(I)

③ If $U = \mathrm{Tan}^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ (or) prove that $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \sin 2U$.

Sol:-

Given $U = \mathrm{Tan}^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$$\mathrm{Tan}\,U = \frac{x^3+y^3}{x+y}$$

$$\mathrm{Tan}\,U = \frac{x^{\cancel{3}\,2}\left(1+\frac{y^3}{x^3}\right)}{\cancel{x}\left(1+\frac{y}{x}\right)}$$

$$\mathrm{Tan}\,U = x^2\left[\frac{1+\left(\frac{y}{x}\right)^3}{\left(1+\frac{y}{x}\right)}\right]$$

$$\mathrm{Tan}\,U = x^2 \cdot f\left(\frac{y}{x}\right)$$

$\rightarrow$ Tanu is homogeneous of degree "2".

By Euler's theorem,

$$x \cdot \frac{\partial \mathrm{Tan}u}{\partial x} + y \cdot \frac{\partial \mathrm{Tan}u}{\partial y} = 2 \cdot \mathrm{Tan}U$$

$$x \cdot \sec^2 U \frac{\partial U}{\partial x} + y \sec^2 U \frac{\partial U}{\partial y} = 2 \cdot \mathrm{Tan}U$$

$$\sec^2 U\left(x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y}\right) = 2 \cdot \mathrm{Tan}U$$

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \frac{2 \cdot \mathrm{Tan}U}{\sec^2 U}$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 2 \cdot \frac{\sin U}{\cos U} \times \cos^2 U$$

$$\boxed{x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \sin 2U}$$

④ If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -3\tan u$

sol:

Given $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$

$$u = \sin^{-1}\left(\frac{x\left(1+2\frac{y}{x}+3\frac{z}{x}\right)}{x^{4}\sqrt{1+\frac{y^8}{x^8}+\frac{z^8}{x^8}}}\right)$$

$$\sin u = x^{-3}\left[\frac{1+2\cdot\frac{y}{x}+3\left(\frac{z}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^8+\left(\frac{z}{x}\right)^8}}\right]$$

$$\sin u = x^{-3}\cdot f\left(\frac{y}{x}, \frac{z}{x}\right)$$

$\therefore \sin u$ is homogeneous of degree "-3".

By Euler's theorem,

$$x\cdot\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} + z\frac{\partial \sin u}{\partial z} = -3\sin u$$

$$x\cdot\cos u\cdot\frac{\partial u}{\partial x} + y\cdot\cos u\frac{\partial u}{\partial y} + z\cdot\cos u\frac{\partial u}{\partial z} = -3\sin u$$

$$\cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z}\right) = -3\sin u$$

$$x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z} = -3\frac{\sin u}{\cos u}$$

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z} = -3\tan u.$$

⑤ $u = \log\left(\frac{x^4+y^4}{x+y}\right)$ show that $x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 3$.

sol

Given $u = \log\left(\frac{x^4+y^4}{x+y}\right)$

$$e^u = \frac{x^{4}\left(1+\frac{y^4}{x^4}\right)}{x\left(1+\frac{y}{x}\right)}$$

$$e^u = x^3\left(\frac{1+(y/x)^4}{1+y/x}\right)$$

$$e^u = x^3\cdot f\left(\frac{y}{x}\right)$$

$\therefore e^u$ is homogeneous of degree "3".

By Euler's theorem,

$$x\cdot\frac{\partial e^u}{\partial x} + y\cdot\frac{\partial e^u}{\partial y} = 3\cdot e^u$$

$$x \cdot e^{u} \frac{\partial u}{\partial x} + y e^{u} \frac{\partial u}{\partial y} = 3 \cdot e^{u}$$

$$e^{u}\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 3 \cdot e^{u}$$

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3.}$$

(7) $u = x f\left(\frac{y}{x}\right)$ prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$.

Given $u = x f\left(\frac{y}{x}\right)$

$\therefore$ u is the homogeneous of degree "1".

By Euler's theorem,

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \cdot u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = (1) u.$$

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u}$$

(1) $u = (x^{1/2} + y^{1/2})(x^n + y^n)$ verify the Euler's theorem.

Given $u = (x^{1/2} + y^{1/2})(x^n + y^n)$

$$u = x^{1/2}\left(1 + \frac{y^{1/2}}{x^{1/2}} \right) x^n \left(1 + \frac{y^n}{x^n} \right)$$

$$= x^{n+1/2}\left[\left(1 + \left(\frac{y}{x}\right)^{1/2} \right) \left(1 + \left(\frac{y}{x}\right)^n \right) \right]$$

$$u = x^{n+1/2} f\left(\frac{y}{x}\right)$$

$\therefore$ u is the homogeneous of degree "$n + \frac{1}{2}$".

By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \cdot u$$

$$\boxed{x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right) u}$$

We have to prove that $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right) \cdot u$.

$$\frac{\partial}{\partial x}(u) = \frac{\partial}{\partial x}\left[(x^{1/2} + y^{1/2})(x^n + y^n) \right]$$

$$= (x^{1/2} + y^{1/2})(n x^{n-1} + 0) + (x^n + y^n)\left(\frac{1}{2} x^{-1/2} + 0 \right)$$

$$= (x^{1/2} + y^{1/2}) n x^{n-1} + (x^n + y^n) \frac{1}{2} x^{-1/2}$$

$$x \cdot \frac{\partial u}{\partial x} = n \cdot x^n (x^{1/2} + y^{1/2}) + \frac{1}{2} x^{1/2} (x^n + y^n)$$

Similarly, $\frac{\partial u}{\partial y} = n.y^{n-1}(x^{1/2}+y^{1/2}) + \frac{1}{2}y^{-1/2}(x^n+y^n)$

$$y.\frac{\partial u}{\partial y} = n.y^n(x^{1/2}+y^{1/2}) + \frac{1}{2}y^{1/2}(x^n+y^n)$$

L.H.S

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$= n.x^n(x^{1/2}+y^{1/2}) + \frac{1}{2}x^{1/2}(x^n+y^n) + ny^n(x^{1/2}+y^{1/2}) + \frac{1}{2}y^{1/2}(x^n+y^n)$

$= n(x^{1/2}+y^{1/2})(x^n+y^n) + \frac{1}{2}(x^n+y^n)(x^{1/2}+y^{1/2})$

$= (x^n+y^n)(x^{1/2}+y^{1/2})(n+\frac{1}{2})$

$= (n+\frac{1}{2})u$

$= R.H.S$

$\therefore$ Euler's theorem verified.

② $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$. Verify the Euler's theorem.

Given $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$= \operatorname{cosec}^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$u = x^0\left[\operatorname{cosec}^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$u = x^0 f\left(\frac{y}{x}\right)$$

$\therefore u$ is homogeneous of degree "0".

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n.u$$
$$= (0)u. = 0.$$

We have to prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

L.H.S

$\frac{\partial}{\partial x}(u) = \frac{\partial}{\partial x}\left[\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right]$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}(y)\left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{y}\frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} + \frac{-y}{x^2}\frac{1}{\frac{x^2+y^2}{x^2}}$$

$$= \frac{1}{\cancel{y}\frac{\sqrt{y^2-x^2}}{\cancel{y}}} + -\frac{y}{x^2+y^2}$$

$$\frac{\partial U}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\Rightarrow x \cdot \frac{\partial U}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}.$$

$$\frac{\partial}{\partial y}(U) = \frac{\partial}{\partial y}\left[\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\, x\left(\frac{-1}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= \frac{-x}{\cancel{y^2}\frac{\sqrt{y^2-x^2}}{\cancel{y}}} + \frac{1}{\cancel{x}\cdot\left(\frac{x^2+y^2}{x^{\cancel{2}}}\right)}$$

$$\frac{\partial U}{\partial y} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\Rightarrow y\frac{\partial U}{\partial y} = \frac{-x\cancel{y}}{\cancel{y}\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$= \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

L.H.S

$$x\cdot\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}$$

$$= \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$= 0$

$=$ R.H.S

$\therefore$ Euler's theorem verified.

(6) $U = \log\left(\frac{x^2+y^2}{xy}\right)$ verify the Euler's theorem.

Sol: Given $U = \log\left(\frac{x^2+y^2}{xy}\right)$

$$e^U = \frac{x^{\cancel{2}}\left(1+\frac{y^2}{x^2}\right)}{\cancel{x}y}$$

$$e^U = \frac{x\left(1+\left(\frac{y}{x}\right)^2\right)}{x\cdot(y/x)}$$

$$e^U = x^0 f\left(\frac{y}{x}\right)$$

$\therefore e^U$ is homogeneous of degree '0'.

By Euler's theorem, $x\cdot\frac{\partial U}{\partial x} + y\cdot\frac{\partial U}{\partial y} = n\cdot U$
$= (0)U = 0$

We have to prove that, $x\frac{dU}{dx} + y\cdot\frac{dU}{dy} = 0.$

$$\frac{d}{dx}(U) = \frac{d}{dx}\left[\log\left(\frac{x^2+y^2}{xy}\right)\right]$$

$$= \frac{1}{\frac{x^2+y^2}{xy}}\left[\frac{xy(2x+0) - (x^2+y^2)\cdot y}{(xy)^2}\right]$$

$$= \frac{\cancel{xy}}{x^2+y^2}\left[\frac{xy(2x) - (x^2+y^2)y}{(xy)\cancel{^2}}\right]$$

$$= \frac{1}{x^2+y^2}\left[\frac{2x^2y - x^2y - y^3}{xy}\right]$$

$$= \frac{1}{x^2+y^2}\left(\frac{x^2y - y^3}{xy}\right)$$

$$= \frac{1}{x^2+y^2}\frac{\cancel{y}(x^2-y^2)}{x\cancel{y}}$$

$$\frac{dU}{dx} = \frac{x^2-y^2}{x(x^2+y^2)}$$

$$\Rightarrow x\cdot\frac{dU}{dx} = \frac{\cancel{x}\cdot(x^2-y^2)}{\cancel{x}(x^2+y^2)} = \frac{x^2-y^2}{x^2+y^2}$$

$$\frac{d}{dy}(U) = \frac{d}{dx}\left[\log\left(\frac{x^2+y^2}{xy}\right)\right]$$

$$= \frac{1}{\frac{x^2+y^2}{\cancel{xy}}}\left[\frac{xy(0+2y) - (x^2+y^2)x}{(xy)\cancel{^2}}\right]$$

$$= \frac{1}{x^2+y^2}\left[\frac{2xy^2 - x^3 - xy^2}{xy}\right]$$

$$= \frac{1}{x^2+y^2}\left(\frac{xy^2 - x^3}{xy}\right)$$

$$= \frac{1}{x^2+y^2}\frac{\cancel{x}(y^2-x^2)}{\cancel{x}y}$$

$$\frac{dU}{dy} = \frac{y^2-x^2}{y(x^2+y^2)}$$

$$\Rightarrow y\cdot\frac{dU}{dy} = (\cancel{y})\cdot\frac{y^2-x^2}{\cancel{y}(x^2+y^2)} = \frac{y^2-x^2}{x^2+y^2}$$

<u>L.H.S</u>

$$x\frac{dU}{dx} + y\cdot\frac{dU}{dy}$$

$$= \frac{x^2-y^2}{x^2+y^2} + \frac{y^2-x^2}{x^2+y^2}$$

$$= \frac{\cancel{x^2}-\cancel{y^2}+\cancel{y^2}-\cancel{x^2}}{x^2+y^2}$$

$$= \frac{0}{x^2+y^2} = 0. = R.H.S$$

$\therefore$ Euler's theorem verified.

⑧ $U = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$. verify the Euler's theorem.

sol: Given $U = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

$$U = \frac{x^{1/4}\left[1+\frac{y^{1/4}}{x^{1/4}}\right]}{x^{1/5}\left[1+\frac{y^{1/5}}{x^{1/5}}\right]}$$

$$U = x^{1/4}.x^{-1/5}\left[\frac{1+(y/x)^{1/4}}{1+(y/x)^{1/5}}\right]$$

$$U = x^{1/4-1/5}\left[\frac{1+(y/x)^{1/4}}{1+(y/x)^{1/5}}\right]$$

$$U = x^{\frac{5-4}{20}}\left[\frac{1+(y/x)^{1/4}}{1+(y/x)^{1/5}}\right]$$

$$U = x^{1/20} f\left(\frac{y}{x}\right)$$

$\therefore U$ is homogeneous of degree '$1/20$'.

By Euler's theorem, $x.\dfrac{\partial U}{\partial x} + y.\dfrac{\partial U}{\partial y} = n.U$

$$x.\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \frac{1}{20}U.$$

We have to prove that,

$$x\frac{\partial U}{\partial x} + y.\frac{\partial U}{\partial y} = \frac{1}{20}U.$$

$$\frac{\partial}{\partial x}(U) = \frac{\partial}{\partial x}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$$

$$= \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}.x^{1/4-1}+0\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{1/5-1}+0\right)}{(x^{1/5}+y^{1/5})^2}$$

$$\frac{\partial U}{\partial x} = \frac{\frac{1}{4}x^{-3/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}.x^{-4/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$\Rightarrow x.\frac{\partial U}{\partial x} = \frac{\frac{1}{4}x^{-3/4+1}(x^{1/5}+y^{1/5}) - \frac{1}{5}x^{-4/5+1}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{1/4\, x^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$\frac{\partial}{\partial y}(U) = \frac{\partial}{\partial y}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$$

$$= \frac{(x^{1/5}+y^{1/5})\left(0+\frac{1}{4}y^{1/4-1}\right) - (x^{1/4}+y^{1/4})\left(0+\frac{1}{5}y^{1/5-1}\right)}{(x^{1/5}+y^{1/5})^2}$$

$$\frac{\partial U}{\partial y} = \frac{\frac{1}{4}\cdot y^{-3/4}\cdot(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{-4/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$\Rightarrow y\frac{\partial U}{\partial y} = \frac{\frac{1}{4}y^{-3/4+1}(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{-4/5+1}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{\frac{1}{4}y^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

<u>L.H.S</u>

$$x\cdot\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}$$

$$= \frac{\frac{1}{4}x^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2} + \frac{\frac{1}{4}y^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{\frac{1}{4}x^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4}) + \frac{1}{4}y^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{\frac{1}{4}(x^{1/5}+y^{1/5})[x^{1/4}+y^{1/4}] - \frac{1}{5}(x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{(x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})\left(\frac{1}{4}-\frac{1}{5}\right)}{(x^{1/5}+y^{1/5})^2} = \frac{(x^{1/4}+y^{1/4})\cancel{(x^{1/5}+y^{1/5})}\left(\frac{1}{20}\right)}{(x^{1/5}+y^{1/5})^{\cancel{2}}}$$

$$= \cancel{x^{1/4}}\ \frac{1}{20}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$$

$$= \frac{1}{20}U.$$

$= R.H.S$

$\therefore$ Euler's theorem verified.

22/11/2019
Friday (II)

(9) If $U = \frac{x^2y}{x+y}$ show that $x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial y \partial x} = 2\cdot\frac{\partial U}{\partial x}$.

Sol:-

Given $U = \frac{x^2y}{x+y}$

$U = \frac{x^2y}{x(1+\frac{y}{x})} = x^2\left[\frac{\frac{y}{x}}{1+\frac{y}{x}}\right]$

$U = x^2\left[\frac{y/x}{1+y/x}\right]$

$\therefore U$ is homogeneous of degree "2".

By Euler's theorem $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = 2U$.

diff. w. r. to 'x' partially

$(1)\frac{\partial U}{\partial x} + x\frac{\partial^2 U}{\partial x^2} + (0)\frac{\partial U}{\partial y} + y\cdot\frac{\partial^2 U}{\partial x \partial y} = 2\frac{\partial U}{\partial x}$

$\frac{\partial U}{\partial x} + x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial x\partial y} = 2\cdot\frac{\partial U}{\partial x}$

$x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial x \partial y} = 2\frac{\partial U}{\partial x} - \frac{\partial U}{\partial x}$

$$\boxed{x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial x\partial y} = 2\frac{\partial U}{\partial x}}$$

(16) If $U = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ prove that $x^2\frac{\partial^2 U}{\partial x^2} + 2xy\frac{\partial^2 U}{\partial x\partial y} + y^2\frac{\partial^2 U}{\partial y^2} =$

$\sin 4U - \sin 2U = 2\cos 3U \sin U$

Sol:

Given $U = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$\tan U = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left(1+\frac{y}{x}\right)}$

$\tan U = x^2\left[\frac{1+(y/x)^3}{1+(y/x)}\right]$

$\therefore \tan U$ is homogeneous of degree "2".

By Euler's theorem, $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = n\cdot U$

$x\frac{\partial}{\partial x}(\tan U) + y\cdot\frac{\partial}{\partial y}(\tan U) = 2\tan U$ → (1)

$x\cdot\sec^2 U\cdot\frac{\partial U}{\partial x} + y\sec^2 U\cdot\frac{\partial U}{\partial y} = 2\tan U$

$$\sec^2 u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=2\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2.\frac{\sin u}{\cos u}\times\cos^2 u.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u \longrightarrow (2)$$

diff. w. r. to 'x' partially.

$$(1)\frac{\partial u}{\partial x}+x.\frac{\partial^2 u}{\partial x^2}+y.\frac{\partial^2 u}{\partial x.\partial y}=\cos 2u\,(2)\frac{\partial u}{\partial x}.$$

$$\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\frac{\partial u}{\partial x}$$

$$\cancel{x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u.\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}}$$

$$\cancel{x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(2\cos 2u-1)\frac{\partial u}{\partial x}}$$

$$x\frac{\partial u}{\partial x}+x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\cdot x\frac{\partial u}{\partial x}\rightarrow (3)$$

from (2),

$$\text{lly},\quad y.\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+xy\frac{\partial^2 u}{\partial y\partial x}=2\cos u.y.\frac{\partial u}{\partial y}\rightarrow(4)$$

Adding (3) & (4)

$$x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=2\cos u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)-\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\ \sin 2u-\sin 2u.$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=\cancel{\sin 2u\,(2\cos 2u-}$$
$$\sin 4u-\sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=2\cos\left(\frac{4u+2u}{2}\right).\sin\left(\frac{4u-2u}{2}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\cos 3u\sin u.$$

(17) If $u=\tan^{-1}\left(\frac{y^2}{x}\right)$ show that $x^2.\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\sin 2u.\sin^2 u.$

Sol: Given $u=\tan^{-1}\left(\frac{y^2}{x}\right)$.

$$\tan u=\frac{y^2}{x}.$$

$\tan u = \frac{xy^2}{x^2}$

$\tan u = x\left(\frac{y}{x}\right)^2 \Rightarrow \tan u = x \cdot f\left(\frac{y}{x}\right)$

$\therefore \tan u$ is homogeneous of degree "1".

By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

$$x \cdot \frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \tan u. \rightarrow (1)$$

$$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y\sec^2 u \cdot \frac{\partial u}{\partial y} = \tan u.$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cancel{\cos u}} \times \cos^{\cancel{2}} u$$

$$x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin u \cdot \cos u.$$

$$x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u. \rightarrow (2)$$

diff. w. r. to "x" partially

$$(1)\frac{\partial u}{\partial x} + x \cdot \frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{\cancel{2}}\cos 2u(\cancel{2})\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x} + x \cdot \frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \cos 2u\frac{\partial u}{\partial x}$$

$$x \cdot \frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = x \cdot \cos 2u \cdot \frac{\partial u}{\partial x} \rightarrow (3)$$

from (2),

Ily,
$$y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = y\cos 2u\frac{\partial u}{\partial x} \rightarrow (4)$$

$(3) + (4) \Rightarrow$

$$x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}(\cos 2u - 1)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\sin 2u \cdot (\cos 2u - 1)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\sin 2u\,(\cancel{1} - 2\sin^2 u - \cancel{1})$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{1}{\cancel{2}}\sin 2u\sin^2 u(\cancel{2})$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u \cdot \sin^2 u.$$

(18) If $u = (x^2+y^2)^{1/3}$. Show that $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{-2u}{9}$.

(19) If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$. Then evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

(20) If $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$. Evaluate $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y}$.

(10) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ Prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4\cos^3 u}$

sol:-

Given $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\sin u = \frac{x(1+y/x)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$\sin u = x \cdot x^{-1/2}\left[\frac{1+y/x}{1+\sqrt{y/x}}\right]$$

$$\sin u = x^{1/2} f\left(\frac{y}{x}\right)$$

$\therefore \sin u$ is homogeneous of degree "$\frac{1}{2}$".

By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot u$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u \rightarrow (1)$$

$$x \cdot \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \rightarrow (2)$$

diff w.r.to "x" partially.

$$(1)\frac{\partial u}{\partial x} + x\cdot\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u \cdot \frac{\partial u}{\partial x}$$

$$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}x\cdot\sec^2 u\cdot\frac{\partial u}{\partial x} \rightarrow (3)$$

from (2)

lly,

$$y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u \cdot y\frac{\partial u}{\partial y} \rightarrow (4)$$

(3) + (4)

$$\rightarrow x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \left[\frac{1}{2}\sec^2 u - 1\right]\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{1}{2}\sec^2 u - 1\right)\frac{1}{2}\tan u.$$

$$= \frac{1}{4}\frac{1}{\cos^2 u}\cdot\frac{\sin u}{\cos u} - \frac{1}{2}\tan u$$

$$= \frac{1}{4}\frac{\sin u}{\cos^3 u} - \frac{1}{2}\frac{\sin u}{\cos u}$$

$$= \frac{\sin u - 2\sin u\cos^2 u}{4\cos^3 u}$$

$$= \frac{\sin u\,(1-2\cos^2 u)}{4\cos^3 u}$$

$$= \frac{-\sin u\,(2\cos^2 u - 1)}{4\cos^3 u}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{-\sin u\cos 2u}{4\cos^3 u}.$$

(12) If $f(x,y) = \sqrt{x^2-y^2}\,\sin^{-1}\left(\frac{y}{x}\right)$, prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x,y)$.

Given $f(x,y) = \sqrt{x^2-y^2}\,\sin^{-1}\left(\frac{y}{x}\right)$

$$f(x,y) = x^1\sqrt{1-\left(\frac{y}{x}\right)^2}\,\sin^{-1}\left(\frac{y}{x}\right)$$

$$f(x,y) = x^1 f\left(\frac{y}{x}\right)$$

$\therefore$ f is homogeneous of degree "1".

By using Euler's theorem, $x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot u$

$$x\cdot\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = (1)\,f(x,y)$$

$$x\cdot\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x,y).$$

(13) If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$. Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

Given $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$\cos u = \frac{x\,(1+y/x)}{\sqrt{x}\,\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$\cos u = x\cdot x^{-1/2}\left(\frac{1+y/x}{1-\sqrt{y/x}}\right)$$

$$\cos u = x^{1/2} f\left(\frac{y}{x}\right)$$

$\therefore \cos u$ is homogeneous of degree "$\frac{1}{2}$".

By using Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u$

$$x\cdot\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u \quad \to (1)$$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u.$$

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

$$\boxed{x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.}$$

(14) If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ show that $\frac{\partial u}{\partial x} = -\frac{y}{x}\cdot\frac{\partial u}{\partial y}$

Sol:- Given $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$

$$\sin u = \frac{\cancel{\sqrt{x}}(1-\sqrt{y}/\sqrt{x})}{\cancel{\sqrt{x}}(1+\sqrt{y}/\sqrt{x})}$$

$$\sin u = x^0\left[\frac{1-\sqrt{y/x}}{1+\sqrt{y/x}}\right]$$

$$\sin u = x^0\cdot f(y/x)$$

$\therefore \sin u$ is homogeneous of degree "0".

By Euler's theorem, $x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = n\cdot u$

$$x\cdot\frac{\partial}{\partial x}(\sin u) + y\cdot\frac{\partial}{\partial y}(\sin u) = 0.$$

$$x\cdot\cos u\,\frac{\partial u}{\partial x} + y\cos u\,\frac{\partial u}{\partial y} = 0.$$

$$x\cdot\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = 0$$

$$x\,\frac{\partial u}{\partial x} = -y\,\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{-y}{x}\,\frac{\partial u}{\partial y}}$$

(15) Show that $x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$ where $\log u = \frac{x^3+y^3}{3x+4y}$

Sol:- Given $\log u = \frac{x^3+y^3}{3x+4y}$

$$\log u = \frac{x^{\cancel{3}2}(1+y^3/x^3)}{\cancel{x}(3+4(y/x))}$$

$$\log u = x^2\left[\frac{1+(y/x)^3}{3+4(y/x)}\right]$$

$$\log u = x^2\cdot f(y/x)$$

$\therefore \log u$ is homogeneous of degree "2".

By Euler's theorem, $x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot u$

$$x\,\frac{\partial}{\partial x}(\log u) + y\,\frac{\partial}{\partial y}(\log u) = 2\cdot\log u$$

$$x \cdot \frac{1}{u} \cdot \frac{\partial u}{\partial x} + y \frac{1}{u} \frac{\partial u}{\partial y} = 2 \log u$$

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u \log u}$$

(18) If $u = (x^2+y^2)^{1/3}$ show that $x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \frac{-2u}{9}$

Sol:- Given $u = (x^2+y^2)^{1/3}$

$$u = \left[x^2 (1 + y^2/x^2)\right]^{1/3}$$

$$u = x^{2/3} \left(1 + (y/x)^2\right)^{1/3}$$

$$u = x^{2/3} \cdot f(y/x)$$

$\therefore u$ is homogeneous of degree "2/3".

By Euler's theorem, $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \cdot u$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{2}{3} u \rightarrow (1)$$

diff. w. r. to 'x' partially

$$(1) \frac{\partial u}{\partial x} + x \cdot \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{2}{3} \frac{\partial u}{\partial x}$$

$$x \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = \frac{2}{3} \cdot x \frac{\partial u}{\partial x} \rightarrow (2)$$

Illy $y \cdot \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial x \partial y} = \frac{2}{3} y \cdot \frac{\partial u}{\partial x} \rightarrow (3)$

(2)+(3)

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2}{3}\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{2}{3} - 1\right)\left(x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{2-3}{3}\right) \frac{2}{3} u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \frac{-2u}{9}.$$

(19)

Given $u = x^2 \cdot \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}(x/y)$

$$u = x^2 \tan^{-1}(y/x) - y^2 \cot^{-1}(y/x)$$

$$u = x^2 \left[\tan^{-1}(y/x) - \left(\frac{y}{x}\right)^2 \cot^{-1}(y/x)\right]$$

$$u = x^2 \cdot f(y/x).$$

$\therefore u$ is homogeneous of degree "2".

By Euler's theorem, $x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = n \cdot U$

$$x \cdot \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = 2U \rightarrow (1)$$

diff. w. r. to 'x' partially

$$(1) \frac{\partial U}{\partial x} + x \cdot \frac{\partial^2 U}{\partial x^2} + y \cdot \frac{\partial^2 U}{\partial x \partial y} = 2 \cdot \frac{\partial U}{\partial x}.$$

$$x \cdot \frac{\partial U}{\partial x} + x^2 \frac{\partial^2 U}{\partial x^2} + xy \frac{\partial^2 U}{\partial x \partial y} = 2x \frac{\partial U}{\partial x} \rightarrow (2)$$

lly, $y \cdot \frac{\partial U}{\partial y} + y^2 \frac{\partial^2 U}{\partial y^2} + xy \frac{\partial^2 U}{\partial x \partial y} = 2y \frac{\partial U}{\partial x} \rightarrow (3)$

(2) + (3)

$$\Rightarrow x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = 2\left(x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y}\right)$$

$$x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} = x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y}$$

$$x^2 \frac{\partial^2 U}{\partial x^2} + y^2 \frac{\partial^2 U}{\partial y^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} = 2U.$$

(20)

Given $U = \operatorname{cosec}^{-1} \left[\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right]^{1/2}$

$$\operatorname{cosec} \cdot U = \left[\frac{x^{1/2}(1 + y^{1/2}/x^{1/2})}{x^{1/3}(1 + y^{1/3}/x^{1/3})}\right]^{1/2}$$

$$\operatorname{cosec} U = \frac{x^{1/4}}{x^{1/6}} \left[\frac{1 + (y/x)^{1/2}}{1 + (y/x)^{1/3}}\right]^{1/2}$$

$$\operatorname{cosec} U = x^{1/4} \cdot x^{-1/6} f(y/x)$$

$$\operatorname{cosec} U = x^{1/12} f(y/x)$$

$2 \lfloor 6, 4$
$3, 2$

$\frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$

$\therefore \operatorname{cosec} U$ is homogeneous of degree '$1/12$'.

By Euler's theorem, $x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = n \cdot U$

$$x \cdot \frac{\partial}{\partial x}(\operatorname{cosec} U) + y \frac{\partial}{\partial y}(\operatorname{cosec} U) = \frac{1}{12} \cdot \operatorname{cosec} U.$$

$$-x \cdot \operatorname{cosec} U \cdot \cot U \cdot \frac{\partial U}{\partial x} + y(-\operatorname{cosec} U \cdot \cot U) \frac{\partial U}{\partial y} = \frac{1}{12} \operatorname{cosec} U.$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \frac{1}{12} \frac{\cancel{\operatorname{cosec} U}}{-\cancel{\operatorname{cosec} U} \cdot \cot U}$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \frac{-1}{12} \tan U \rightarrow (1)$$

diff. w.r.to "x" partially,

$$(1)\frac{\partial u}{\partial x} + x.\frac{\partial^2 u}{\partial x^2} + y.\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12}\sec^2 u.\frac{\partial u}{\partial x}$$

$$x.\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12}\sec^2 u.\, x.\frac{\partial u}{\partial x} \rightarrow (2)$$

lly, $$y.\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12}\sec^2 u.\, y.\frac{\partial u}{\partial y} \rightarrow (3)$$

(2)+(3)

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \left[\frac{-1}{12}\sec^2 u - 1\right]\left(x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= \left(\frac{-1}{12}\sec^2 u - 1\right)\left(\frac{-1}{12}\tan u\right)$$

$$= \frac{1}{144}.\frac{\sin u}{\cos^3 u} + \frac{1}{12}.\frac{\sin u}{\cos u}$$

$$= \frac{\sin u + 12\sin u\cos^2 u}{144.\cos^3 u}$$

$$= \frac{\sin u(1 + 12\cos^2 u)}{144.\cos^3 u}$$

$$= \frac{\sin u\,[1 + 12(1-\sin^2 u)]}{144\cos^3 u}$$

$$= \frac{\sin u\,[1 + 12 - 12\sin^2 u]}{144\cos^3 u}$$

$$= \frac{11\sin u - 12\sin^3 u}{144.\cos^3 u}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{11}{144}.\frac{\sin u}{\cos^3 u} - \frac{1}{12}\tan^3 u.$$

22/11/19 Friday

Total Derivative and Chain Rule:

$$U \begin{array}{c} x \\ y \end{array} t \qquad\qquad U \begin{array}{c} x \\ y \end{array} r,\theta.$$

(1) If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$. show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.

sol: Given $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$ $\qquad U \begin{array}{c} x \\ y \end{array} t$

By using Total Derivative

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\sin^{-1}(x-y) = \frac{1}{\sqrt{1-(x-y)^2}}(1-0) = \frac{1}{\sqrt{1-(x-y)^2}}.$$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left[\sin^{-1}(x-y)\right] = \frac{1}{\sqrt{1-(x-y)^2}}(0-1) = \frac{-1}{\sqrt{1-(x-y)^2}}$$

$$\frac{dx}{dt} = \frac{d}{dt}(3t) = 3 \quad , \quad \frac{dy}{dt} = \frac{d}{dt}(4t^3) = 12t^2.$$

$$\frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}(3) + \frac{-1}{\sqrt{1-(x-y)^2}}(12t^2)$$

$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3\cancel{t}-12t^2}{\sqrt{1-x^2-y^2+2xy}}$$

$$= \frac{3\cancel{t}-12t^2}{\sqrt{1-9t^2-16t^6+24t^4}}$$

$$= \frac{3(1-4t^2)}{\sqrt{-16t^6+24t^4-9t^2+1}}$$

$$= \frac{3(1-4t^2)}{\sqrt{-16x^3+24x^2-9x+1}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-x)(1-4x)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-x}\,(1-4x)}$$

$$= \frac{3\cancel{(1-4t^2)}}{\sqrt{1-t^2}\,\cancel{(1-4t^2)}}$$

$$= \frac{3}{\sqrt{1-t^2}}$$

$$\begin{array}{r|rrrr} 1 & -16 & 24 & -9 & 1 \\ & 0 & -16 & 8 & -1 \\ \hline & -16 & 8 & -1 & \boxed{0} \end{array}$$

$(x-1)(-16x^2+8x-1)=0$

$(x-1)[-(16x^2-8x+1)]=0$

$(1-x)(4x-1)^2=0$

(10) If $u = \tan^{-1}(y/x)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ then find $\frac{du}{dt}$.

Sol: Given $u = \tan^{-1}(y/x)$

$u \begin{matrix} x \\ y \end{matrix} t$

By using Total Derivative,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}.$$

$$\frac{\partial}{\partial x}\left[\tan^{-1}(y/x)\right] = \frac{1}{1+\frac{y^2}{x^2}}\, y\left(\frac{-1}{x^2}\right) = \frac{-y}{\cancel{x^2}}\cdot\frac{1}{\frac{x^2+y^2}{\cancel{x^2}}} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left[\tan^{-1}(y/x)\right] = \frac{1}{1+y^2/x^2}\,\frac{1}{x} = \frac{1}{\cancel{x}}\cdot\frac{1}{\frac{x^2+y^2}{x^{\cancel{2}}}} = \frac{x}{x^2+y^2}.$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^t - e^{-t}) = e^t - e^{-t}(-1) = e^t + e^{-t}.$$

$$\frac{dy}{dt} = \frac{d}{dt}(e^t + e^{-t}) = e^t + e^{-t}(-1) = e^t - e^{-t}$$

$$\frac{du}{dt} = \frac{-y}{x^2+y^2}(e^t + e^{-t}) + \frac{x}{x^2+y^2}(e^t - e^{-t})$$

$$= \frac{-y(y) + x(x)}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

$$= \frac{(e^t - e^{-t})^2 - (e^t + e^{-t})^2}{(e^t - e^{-t})^2 + (e^t + e^{-t})^2}$$

$$= \frac{e^{2t} + e^{-2t} - 2 - e^{2t} - e^{-2t} - 2}{e^{2t} + e^{-2t} - \cancel{2} + e^{2t} + e^{-2t} + \cancel{2}}$$

$$= \frac{-4^{2}}{\cancel{2}(e^{2t} + e^{-2t})}$$

$$= \frac{-1}{\frac{e^{2t} + e^{-2t}}{2}} = \frac{-1}{\cosh 2t} = -\operatorname{sech} 2t.$$

(14) If $u = f(x^2 + 2yz, y^2 + 2zx)$ prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$

Sol: Given $u = f(x^2 + 2yz, y^2 + 2zx)$

$U = f(r, s)$ where $r = x^2 + 2yz$, $s = y^2 + 2zx$

By using chain Rule,

$$U \begin{cases} r \\ s \end{cases} x, y, z$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{\partial f}{\partial r}; \frac{\partial u}{\partial s} = \frac{\partial f}{\partial s}} \qquad \boxed{U = f(r, s)}$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2yz) = 2x + 0 = 2x.$$

$$\Rightarrow \frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2yz) = (0 + 2z) = 2z.$$

$$\Rightarrow \frac{\partial r}{\partial z} = \frac{\partial}{\partial z}(x^2 + 2yz) = (0 + 2y) = 2y$$

$$\Rightarrow \frac{\partial s}{\partial x} = \frac{\partial}{\partial x}(y^2 + 2zx) = (0 + 2z) = 2z.$$

$$\Rightarrow \frac{\partial s}{\partial y} = \frac{\partial}{\partial y}(y^2 + 2zx) = (2y + 0) = 2y$$

$$\Rightarrow \frac{\partial s}{\partial z} = \frac{\partial}{\partial z}(y^2 + 2zx) = (0 + 2x) = 2x.$$

$$\frac{\partial U}{\partial x} = \frac{\partial f}{\partial r}(2x) + \frac{\partial f}{\partial s}(2z)$$

$$\frac{\partial U}{\partial y} = \frac{\partial f}{\partial r}(2z) + \frac{\partial f}{\partial s}(2y)$$

$$\frac{\partial U}{\partial z} = \frac{\partial f}{\partial r}(2y) + \frac{\partial f}{\partial s}(2x)$$

Now, $(y^2-zx)\frac{\partial U}{\partial x} + (x^2-yz)\frac{\partial U}{\partial y} + (z^2-xy)\frac{\partial U}{\partial z}$

$$= (y^2-zx)\left(\frac{\partial f}{\partial r}2x + \frac{\partial f}{\partial s}2z\right) + (x^2-yz)\left(\frac{\partial f}{\partial r}2z + \frac{\partial f}{\partial s}2y\right) + (z^2-xy)\left(\frac{\partial f}{\partial r}2y + \frac{\partial f}{\partial s}2x\right)$$

$$= \cancel{2xy^2\frac{\partial f}{\partial r}} - \cancel{2x^2z\frac{\partial f}{\partial r}} + \cancel{2zy^2\frac{\partial f}{\partial s}} - \cancel{2z^2x\frac{\partial f}{\partial s}} + \cancel{2x^2z\frac{\partial f}{\partial r}} - \cancel{2z^2y\frac{\partial f}{\partial r}} + \cancel{2yx^2\frac{\partial f}{\partial s}} - \cancel{2y^2z\frac{\partial f}{\partial s}} + \cancel{2yz^2\frac{\partial f}{\partial r}} - \cancel{2xy^2\frac{\partial f}{\partial r}} + \cancel{2xz^2\frac{\partial f}{\partial s}} - \cancel{2x^2y\frac{\partial f}{\partial s}}$$

$= 0.$

② If z is a function of x and y where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$. show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

Sol:- Given $z = f(x,y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$.

By using chain Rule, $z \begin{cases} x \\ y \end{cases} \to u, v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}$$

$\cancel{\frac{\partial z}{\partial x}}$

$\frac{\partial x}{\partial u} = e^u + \cancel{e^{-v}}(0)$
$= e^u$

$\frac{\partial x}{\partial v} = 0 + e^{-v}(-1)$
$= -e^{-v}$

$\frac{\partial y}{\partial u} = e^{-u}(-1) - 0 = -e^{-u}$

$\frac{\partial y}{\partial v} = 0 - e^v = -e^v$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}(e^u) + \frac{\partial f}{\partial y}(-e^{-u}) = \frac{\partial f}{\partial x}e^u - \frac{\partial f}{\partial y}e^{-u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}(-e^{-v}) + \frac{\partial f}{\partial y}(-e^v) = -\left[\frac{\partial f}{\partial x}e^{-v} + \frac{\partial f}{\partial y}e^v\right]$$

$$\therefore \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}e^{u} - \frac{\partial f}{\partial y}e^{-u} + \frac{\partial f}{\partial x}e^{-v} + \frac{\partial f}{\partial y}e^{v}$$

$$= (e^{u}+e^{-v})\frac{\partial f}{\partial x} + (e^{v}-e^{-u})\frac{\partial f}{\partial y}$$

$$= (e^{u}+e^{-v})\frac{\partial f}{\partial x} - (e^{-u}-e^{v})\frac{\partial f}{\partial y}$$

$$= x\cdot\frac{\partial z}{\partial x} - y\cdot\frac{\partial z}{\partial y}.$$

(3) If $u = f(y-z,\ z-x,\ x-y)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Given $u = f(y-z,\ z-x,\ x-y)$

$u = f(a,b,c)$

Where $a = y-z$, $b = z-x$, $c = x-y$

By using chain Rule, $u \begin{matrix} a \\ b \\ c \end{matrix} x\,y\,z$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial x} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial x} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial y} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial y} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a}\cdot\frac{\partial a}{\partial z} + \frac{\partial u}{\partial b}\cdot\frac{\partial b}{\partial z} + \frac{\partial u}{\partial c}\cdot\frac{\partial c}{\partial z}$$

$$\frac{\partial u}{\partial a} = \frac{\partial f}{\partial a}, \quad \frac{\partial u}{\partial b} = \frac{\partial f}{\partial b}, \quad \frac{\partial u}{\partial c} = \frac{\partial f}{\partial c}$$

$\frac{\partial a}{\partial x} = \frac{\partial}{\partial x}(y-z) = 0$	$\frac{\partial b}{\partial x} = \frac{\partial}{\partial x}(z-x) = -1$	$\frac{\partial c}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1$
$\frac{\partial a}{\partial y} = \frac{\partial}{\partial y}(y-z) = 1$	$\frac{\partial b}{\partial y} = \frac{\partial}{\partial y}(z-x) = 0$	$\frac{\partial c}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1$
$\frac{\partial a}{\partial z} = \frac{\partial}{\partial z}(y-z) = -1$	$\frac{\partial b}{\partial z} = \frac{\partial}{\partial z}(z-x) = 1$	$\frac{\partial c}{\partial z} = \frac{\partial}{\partial z}(x-y) = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial a}(0) + \frac{\partial f}{\partial b}(-1) + \frac{\partial f}{\partial c}(1) = -\frac{\partial f}{\partial b} + \frac{\partial f}{\partial c}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial a}(1) + \frac{\partial f}{\partial b}(0) + \frac{\partial f}{\partial c}(-1) = \frac{\partial f}{\partial a} - \frac{\partial f}{\partial c}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial a}(-1) + \frac{\partial f}{\partial b}(1) + \frac{\partial f}{\partial c}(0) = -\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= -\cancel{\frac{\partial f}{\partial b}} + \cancel{\frac{\partial f}{\partial c}} + \cancel{\frac{\partial f}{\partial a}} - \cancel{\frac{\partial f}{\partial c}} - \cancel{\frac{\partial f}{\partial a}} + \cancel{\frac{\partial f}{\partial b}}$$

$$= 0.$$

④ If $w = f(x,y)$, $x = r\cos\theta$; $y = r\sin\theta$. Show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Sol:- Given $w = f(x,y)$

and $x = r\cos\theta$, $y = r\sin\theta$.

By using chain Rule, $w < \begin{matrix} x \\ y \end{matrix} > r, \theta$.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x}, \quad \frac{\partial w}{\partial y} = \frac{\partial f}{\partial y}$$

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta. \quad \Big| \quad \frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta.$$

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cdot\cos\theta) = r(-\sin\theta) = -r\sin\theta \quad \Big| \quad \frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta.$$

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}(\cos\theta) + \frac{\partial f}{\partial y}\sin\theta. \rightarrow ①$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta. \rightarrow ②$$

$$① \Rightarrow \left(\frac{\partial w}{\partial r}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\sin^2\theta + \cancel{2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\sin\theta\cdot\cos\theta}.$$

$$② \Rightarrow \left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 r^2\sin^2\theta + \left(\frac{\partial f}{\partial y}\right)^2 r^2\cos^2\theta - 2r^2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\sin\theta\cos\theta.$$

$$\left(\frac{\partial w}{\partial \theta}\right)^2 = r^2\left[\left(\frac{\partial f}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\cos^2\theta - 2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\sin\theta\cos\theta\right]$$

$$\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\cos^2\theta - \cancel{2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\sin\theta\cos\theta}$$

①+③ $\Rightarrow$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2[\cos^2\theta + \sin^2\theta] + \left(\frac{\partial f}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta) \rightarrow ③$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

⑤ If f is the function u, v and $u = x^2 - y^2$, $v = 2xy$, then show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2}\right)$

Sol:- Given ~~f(u,v)~~ $f = \theta(u,v)$

$u = x^2 - y^2$; $v = 2xy$

By using chain Rule, $f < \begin{matrix} u \\ v \end{matrix} > xy$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial U}\cdot\frac{\partial U}{\partial x} + \frac{\partial f}{\partial V}\cdot\frac{\partial V}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial U}\cdot\frac{\partial U}{\partial y} + \frac{\partial f}{\partial V}\cdot\frac{\partial V}{\partial y}$$

$$\frac{\partial f}{\partial U} = \frac{\partial \theta}{\partial U}, \quad \frac{\partial f}{\partial V} = \frac{\partial \theta}{\partial V}$$

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(x^2-y^2) = 2x \qquad \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y$$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}(x^2-y^2) = -2y \qquad \frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$$

$$\frac{\partial f}{\partial x} = \frac{\partial \theta}{\partial U}(2x) + \frac{\partial \theta}{\partial V}(2y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial \theta}{\partial U}(-2y) + \frac{\partial \theta}{\partial V}(2x)$$

$$\cancel{\frac{\partial f}{\partial x} = 2\frac{\partial f}{\partial U}x + 2\cdot\frac{\partial f}{\partial V}y}$$

~~diff. w. r. to "x" partially~~

$$\cancel{\frac{\partial^2 \theta}{\partial x^2} = 2\frac{\partial f}{\partial U}(1) + 2x\frac{\partial^2 f}{\partial U\partial x} + 2y\frac{\partial^2 f}{\partial V\partial x} + 2\frac{\partial f}{\partial V}(0)}$$

$$\cancel{= 2\cdot\frac{\partial f}{\partial U} + 2x\frac{\partial^2 f}{\partial x\cdot\partial U} + 2y\frac{\partial^2 f}{\partial V\partial x}}$$

$$\cancel{\frac{\partial^2 \theta}{\partial y^2} = (-2y)\frac{\partial^2 f}{\partial U\partial y} + 2\frac{\partial f}{\partial U}(1) + 2x\cdot\frac{\partial^2 f}{\partial V^2}}$$

$$\frac{\partial f}{\partial x} = 2x\cdot\frac{\partial \theta}{\partial U} + 2y\frac{\partial \theta}{\partial V} \rightarrow (1)$$

$$\frac{\partial f}{\partial x} = 2\left[x\cdot\frac{\partial \theta}{\partial U} + y\cdot\frac{\partial \theta}{\partial V}\right]$$

$$\frac{\partial f}{\partial x} = 2\left[x\cdot\frac{\partial}{\partial U} + y\cdot\frac{\partial}{\partial V}\right]\theta$$

$$\frac{\partial}{\partial x} = 2\left[x\cdot\frac{\partial}{\partial U} + y\cdot\frac{\partial}{\partial V}\right] \rightarrow (2)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= 2\left[x\cdot\frac{\partial}{\partial U} + y\cdot\frac{\partial}{\partial V}\right] 2\left(x\cdot\frac{\partial \theta}{\partial U} + y\cdot\frac{\partial \theta}{\partial V}\right) \quad [\because \text{from (1) \& (2)}]$$

$$= 4\cdot\left(x^2\cdot\frac{\partial^2 \theta}{\partial U^2} + xy\cdot\frac{\partial^2 \theta}{\partial U\cdot\partial V} + xy\frac{\partial^2 \theta}{\partial U\cdot\partial V} + y^2\cdot\frac{\partial^2 \theta}{\partial V^2}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = 4\left(x^2\cdot\frac{\partial^2 \theta}{\partial U^2} + 2xy\frac{\partial^2 \theta}{\partial U\cdot\partial V} + y^2\frac{\partial^2 \theta}{\partial V^2}\right) \rightarrow (3)$$

⑥ If $U = x^2+y^2+z^2$ and $x = e^{2t}$, $y = e^{2t}\cdot\cos 3t$, $z = e^{2t}\sin 3t$ find $\frac{du}{dt}$.

Sol: Given $U = x^2+y^2+z^2$

and $x = e^{2t}$, $y = e^{2t}\cdot\cos 3t$, $z = e^{2t}\sin 3t$

By using Total Derivatives $U < x, y, z > t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}.$$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2) = 2x$ | $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2+z^2) = 2y$ | $\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(x^2+y^2+z^2) = 2z$.

$\frac{dx}{dt} = \frac{d}{dt}(e^{2t}) = 2\cdot e^{2t}$

$\frac{dy}{dt} = \frac{d}{dt}(e^{2t}\cos 3t) = e^{2t}(-\sin 3t)3 + \cos 3t\, e^{2t}(2)$
$= -3e^{2t}\sin 3t + 2e^{2t}\cos 3t$

$\frac{dz}{dt} = \frac{d}{dt}(e^{2t}\sin 3t) = e^{2t}\cos 3t(3) + \sin 3t(e^{2t})2$
$= 3e^{2t}\cos 3t + 2e^{2t}\sin 3t$

$$\frac{du}{dt} = 2x(2\cdot e^{2t}) + 2y(-3e^{2t}\sin 3t + 2\cdot e^{2t}\cos 3t) + 2z(3e^{2t}\cos 3t + 2e^{2t}\sin 3t)$$

$= 4x\cdot e^{2t} - 6y e^{2t}\sin 3t + 4ye^{2t}\cos 3t + 6ze^{2t}\cos 3t + 4ze^{2t}\sin 3t.$

$= 4x\cdot e^{2t} - e^{2t}\sin 3t(6y - 4z) + e^{2t}\cos 3t(4y + 6z)$

$= 4x\cdot e^{2t} - e^{2t}\sin 3t(6e^{2t}\cos 3t - 4e^{2t}\sin 3t) + e^{2t}\cos 3t(4e^{2t}\cos 3t + 6e^{2t}\sin 3t)$

$= 4e^{2t}e^{2t} - \cancel{6e^{4t}\sin 3t\cos 3t} + 4e^{4t}\sin^2 3t + 4e^{4t}\cos^2 3t + \cancel{6e^{4t}\sin 3t\cos 3t}$

$= 4\cdot e^{4t}(1 + \underline{\sin^2 3t + \cos^2 3t})$

$= 4\cdot e^{4t}(1+1) = 4\cdot e^{4t}(2) = \underline{8\cdot e^{4t}}$

⑦ If $U = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ then find $\frac{du}{dt}$.

Given $u = \sin\left(\frac{x}{y}\right)$

$x = e^t$, $y = t^2$

By using Total Derivative, $U < x, y - t$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$\frac{\partial u}{\partial x} = \cos\frac{x}{y}\left(\frac{1}{y}\right)$ | $\frac{\partial u}{\partial y} = \cos\left(\frac{x}{y}\right)x\left(\frac{-1}{y^2}\right)$

$\frac{dx}{dt} = e^t$ | $\frac{dy}{dt} = 2t$.

$$\frac{du}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)e^t - \frac{x}{y^2}\cos\left(\frac{x}{y}\right)2t.$$

$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right) - \frac{e^t}{t^4}\cos\left(\frac{e^t}{t^2}\right)2t$

$= \frac{e^t}{t^2}\left\{\cos\left(\frac{e^t}{t^2}\right)\left[1-\frac{2t}{t^2}\right]\right.$

$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right)\left(\frac{t^2-2t}{t^2}\right)$

$\frac{du}{dt} = \frac{e^t(t^2-2t)}{t^4}\cos\left(\frac{e^t}{t^2}\right) \Rightarrow \frac{du}{dt} = \frac{e^t(t-2)}{t^3}\cdot\cos\left(\frac{e^t}{t^2}\right)$

⑧ If $u = x^3+y^3$ where $x = a\cos t$, $y = b\sin t$ find $\frac{du}{dt}$.

Given $u = x^3+y^3$, $x = a\cos t$, $y = b\sin t$.

By using Total Derivative, $u < {x \atop y} > t$

$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$

$\frac{\partial u}{\partial x} = 3x^2$ $\quad\quad \frac{\partial u}{\partial y} = 3y^2$

$\frac{dx}{dt} = -a\sin t$ $\quad\quad \frac{dy}{dt} = b\cos t$

$\frac{du}{dt} = 3x^2(-a\sin t) + 3y^2\cdot b\cos t$

$= -3(x^2 a\sin t - y^2 b\cos t)$

$= -3(a^2\cos^2 t\cdot a\sin t - b^2\sin^2 t\cdot b\cdot\cos t)$

$= -3(a^3\cdot\sin t\cdot\cos^2 t - b^3\sin^2 t\cdot\cos t)$

$= -3\sin t\cdot\cos t\,(a^3\cos t - b^3\sin t)$

$= \frac{-3}{2}\cdot\sin 2t\,(a^3\cos t - b^3\sin t)$

⑨ If $z = u^2+v^2$, $u = r\cos\theta$, $v = r\sin\theta$. find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$.

Sol:- Given $z = u^2+v^2 = f(u,v)$

$u = r\cos\theta$. $v = r\sin\theta$.

By using chain Rule, $z < {u \atop v} > r,\theta$

$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial r} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial r}$

$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial \theta} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial \theta}$

$\frac{\partial z}{\partial u} = 2u$	$\frac{\partial z}{\partial v} = 2v$
$\frac{\partial u}{\partial r} = \cos\theta$	$\frac{\partial u}{\partial \theta} = r(-\sin\theta)$
$\frac{\partial v}{\partial r} = \sin\theta$	$\frac{\partial v}{\partial \theta} = r\cos\theta$

$$\frac{\partial z}{\partial r} = 2u\cos\theta + 2v(-r\sin\theta)$$
$$= 2(r\cos\theta)\cos\theta + -2(r\sin\theta)(-r\sin\theta)$$
$$= 2r\cos^2\theta + 2r^2\sin^2\theta.$$
$$= 2r(\cos^2\theta + r.\sin^2\theta)$$

$$\frac{\partial z}{\partial \theta} = 2u\sin\theta + 2v(r\cos\theta)$$
$$= 2(r\cos\theta)(\sin\theta) + 2(r\sin\theta)r\cos\theta$$
$$= r.2\sin\theta.\cos\theta + r^2.2\sin\theta.\cos\theta$$
$$= r.\sin 2\theta + r^2.\sin 2\theta$$
$$= r\sin 2\theta(1+r)$$

(10) If $u = \tan^{-1}(y/x)$; $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ find $\frac{du}{dt}$.

Given $u = \tan^{-1}(y/x)$

$x = e^t - e^{-t}$.

(11) If $z = \log(u^2+v)$, $u = e^{x^2+y^2}$, $v = x^2+y$ find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

Given $z = \log(u^2+v)$

$u = e^{x^2+y^2}$, $v = x^2+y$

By using chain Rule, $z < \begin{matrix} u \\ v \end{matrix} > x, y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

$$\frac{\partial z}{\partial u} = \frac{1}{u^2+v}(2u) \qquad \frac{\partial z}{\partial v} = \frac{1}{u^2+v}(1)$$
$$\frac{\partial u}{\partial x} = e^{x^2+y^2}(2x) \qquad \frac{\partial u}{\partial y} = e^{x^2+y^2}(2y)$$
$$\frac{\partial v}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 1$$

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v}\cdot e^{x^2+y^2}(2x) + \frac{1}{u^2+v}(2x)$$
$$= \frac{2x}{u^2+v}(2u.e^{x^2+y^2}+1)$$
$$= \frac{2x}{(e^{x^2+y^2})^2 + x^2+y}(2.e^{x^2+y^2}.e^{(x^2+y^2)}+1)$$
$$= \frac{2x}{e^{2(x^2+y^2)} + x^2+y}\left[2.e^{2(x^2+y^2)}+1\right]$$

$$\frac{\partial z}{\partial y} = \frac{2u}{u^2+v} e^{x^2+y^2}(2y) + \frac{1}{u^2+v}(1)$$

$$= \frac{4yu \cdot e^{x^2+y^2}}{u^2+v} + \frac{1}{u^2+v}$$

$$= \frac{4y\, e^{x^2+y^2} \cdot e^{x^2+y^2} + 1}{(e^{x^2+y^2})^2 + x^2+y}$$

$$= \frac{4y \cdot e^{2(x^2+y^2)} + 1}{e^{2(x^2+y^2)} + x^2+y}$$

(12) If $u = f(r,s,t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ prove that,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Given $u = f(r,s,t)$

$r = \frac{x}{y}$, $s = \frac{y}{z}$; $t = \frac{z}{x}$

By using chain Rule, $u \langle r, s, t \rangle\, x, y, z$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial r}$	$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial s}$	$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial t}$
$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}$	$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = x\left(\frac{-1}{y^2}\right)$	$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x}{y}\right) = 0$
$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{z}\right) = 0$	$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{z}\right) = \frac{1}{z}$	$\frac{\partial s}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{z}\right) = y\left(\frac{-1}{z^2}\right)$
$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}\left(\frac{z}{x}\right) = z\left(\frac{-1}{x^2}\right)$	$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left(\frac{z}{x}\right) = 0$	$\frac{\partial t}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z}{x}\right) = \frac{1}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}\left(\frac{-z}{x^2}\right) = \frac{1}{y}\cdot\frac{\partial f}{\partial r} - \frac{z}{x^2}\cdot\frac{\partial f}{\partial t}$$

$$\Rightarrow x\cdot\frac{\partial u}{\partial x} = \frac{x}{y}\cdot\frac{\partial f}{\partial r} - \frac{z}{x}\cdot\frac{\partial f}{\partial t} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial f}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial f}{\partial t}(0)$$

$$\Rightarrow y\cdot\frac{\partial u}{\partial y} = -\frac{x}{y}\cdot\frac{\partial f}{\partial r} + \frac{y}{z}\cdot\frac{\partial f}{\partial s} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\cdot(0) + \frac{\partial f}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial f}{\partial t}\left(\frac{1}{x}\right)$$

$\Rightarrow z\cdot\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial f}{\partial s} + \frac{z}{x}\cdot\frac{\partial f}{\partial t} \rightarrow ③$

Adding ①+②+③

$\Rightarrow x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z}$

$= \frac{x}{y}\cancel{\frac{\partial f}{\partial r}} - \frac{z}{x}\cancel{\frac{\partial f}{\partial t}} - \frac{x}{y}\cancel{\frac{\partial f}{\partial r}} + \frac{y}{z}\cancel{\frac{\partial f}{\partial s}} - \frac{y}{z}\cancel{\frac{\partial f}{\partial s}} + \frac{z}{x}\cdot\cancel{\frac{\partial f}{\partial t}}$

$= 0.$

⑮ If $u=f(r,s)$, $r=x+y$, $s=x-y$ show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = 2\cdot\frac{\partial u}{\partial r}$.

Given $u=f(r,s)$

$r=x+y$, $s=x-y$

By using chain Rule, $u \langle {r \atop s} \rangle x,y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial r}$ | $\frac{\partial u}{\partial s} = \frac{\partial f}{\partial s}$

$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x+y) = 1$ | $\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(x+y) = 1$

$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1$ | $\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}(1) + \frac{\partial f}{\partial s}(1) = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}(1) + \frac{\partial f}{\partial s}(-1) = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial s}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial f}{\partial r} + \cancel{\frac{\partial f}{\partial s}} + \frac{\partial f}{\partial r} - \cancel{\frac{\partial f}{\partial s}}$$

$$= 2\cdot\frac{\partial f}{\partial r}$$

$$= 2\cdot\frac{\partial u}{\partial r}$$

⑬ If $u=f[2x-3y, (3y-4z), 4z-2x]$ Prove that

$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$

Given $u=f(2x-3y, 3y-4z, 4z-2x)$

$u=f(r,s,t)$

where $r = 2x-3y$, $s = 3y-4z$, $t = 4z-2x$.

By using chain Rule, $u \begin{matrix} r \\ s \\ t \end{matrix} x, y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}.$$

$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial r}$	$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial s}$	$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial t}$
$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(2x-3y) = 2$	$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(2x-3y) = -3$	$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}(2x-3y) = 0$
$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}(3y-4z) = 0$	$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}(3y-4z) = 3$	$\frac{\partial s}{\partial z} = \frac{\partial}{\partial z}(3y-4z) = -4$
$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}(4z-2x) = -2$	$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}(4z-2x) = 0$	$\frac{\partial t}{\partial z} = \frac{\partial}{\partial z}(4z-2x) = 4$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}(2) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}(-2)$$

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t} \rightarrow (1).$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}(-3) + \frac{\partial f}{\partial s}(3) + \frac{\partial f}{\partial t}(0)$$

$$\Rightarrow \frac{1}{3}\frac{\partial u}{\partial y} = -\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}(0) + \frac{\partial f}{\partial s}(-4) + \frac{\partial f}{\partial t}(4)$$

$$\rightarrow \frac{1}{4}\frac{\partial u}{\partial z} = -\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \rightarrow (3)$$

Adding (1) + (2) + (3)

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$$

$$= \cancel{\frac{\partial f}{\partial r}} - \cancel{\frac{\partial f}{\partial t}} - \cancel{\frac{\partial f}{\partial r}} + \cancel{\frac{\partial f}{\partial s}} - \cancel{\frac{\partial f}{\partial s}} + \cancel{\frac{\partial f}{\partial t}}$$

$= 0.$

$$\therefore \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$$

⑤ → Continuous :-

$$\frac{\partial f}{\partial y} = -2y\frac{\partial \theta}{\partial u} + 2x\frac{\partial \theta}{\partial v} \rightarrow ④$$

$$\frac{\partial f}{\partial y} = 2\left(x\cdot\frac{\partial \theta}{\partial v} - y\frac{\partial \theta}{\partial u}\right)$$

$$\frac{\partial f}{\partial y} = 2\cdot\left(x\frac{\partial}{\partial v} - y\frac{\partial}{\partial u}\right)\theta$$

$$\frac{\partial}{\partial y} = 2\left(x\cdot\frac{\partial}{\partial v} - y\frac{\partial}{\partial u}\right) \rightarrow ⑤$$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= 2\left(x\frac{\partial}{\partial v} - y\cdot\frac{\partial}{\partial u}\right) 2\left(x\cdot\frac{\partial \theta}{\partial v} - y\frac{\partial \theta}{\partial u}\right)$$

$$= 4\left[x^2\frac{\partial^2 \theta}{\partial v^2} - xy\frac{\partial^2 \theta}{\partial u\cdot\partial v} - xy\frac{\partial^2 \theta}{\partial u\cdot\partial v} + y^2\cdot\frac{\partial^2 \theta}{\partial u^2}\right] \rightarrow ⑥$$

Adding ③ + ⑥

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$= 4\left[x^2\frac{\partial^2 \theta}{\partial u^2} + 2xy\frac{\partial^2 \theta}{\partial u\cdot\partial v} + y^2\cdot\frac{\partial^2 \theta}{\partial v^2}\right] + 4\left[x^2\cdot\frac{\partial^2 \theta}{\partial v^2} - 2xy\frac{\partial^2 \theta}{\partial u\cdot\partial v} + y^2\frac{\partial^2 \theta}{\partial u^2}\right]$$

$$= 4\left[x^2\frac{\partial^2 \theta}{\partial u^2} + \cancel{2xy\frac{\partial^2 \theta}{\partial u\cdot\partial v}} + y^2\frac{\partial^2 \theta}{\partial v^2} + x^2\frac{\partial^2 \theta}{\partial v^2} - \cancel{2xy\frac{\partial^2 \theta}{\partial u\cdot\partial v}} + y^2\cdot\frac{\partial^2 \theta}{\partial u^2}\right]$$

$$= 4\left[\frac{\partial^2 \theta}{\partial u^2}\cdot(x^2+y^2) + \frac{\partial^2 \theta}{\partial v^2}(x^2+y^2)\right]$$

$$= 4(x^2+y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right)$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right)$$

23/11/19
Saturday

Implicit Function:

① If $z = \sqrt{x^2+y^2}$ and $x^3+y^3+3axy = 5a^2$. Find the value of $\frac{dz}{dx}$ when $x = y = a$.

Given $z = \sqrt{x^2+y^2}$, $x^3+y^3+3axy = 5a^2$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$z < \begin{matrix} x \\ y \end{matrix} > x$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$z=\sqrt{x^2+y^2}$$

$$\frac{\partial z}{\partial x}=\frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x)=\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}y)=\frac{y}{\sqrt{x^2+y^2}}$$

Given $x^3+y^3+3axy-5a^2=0$

differentiate with r.to 'x'.

$$3x^2+3y^2\frac{dy}{dx}+3a\left[(1)y+x.\frac{dy}{dx}\right]=0$$

$$x^2+y^2\frac{dy}{dx}+ay+ax.\frac{dy}{dx}=0$$

$$(y^2+ax)\frac{dy}{dx}=-(x^2+ay)$$

$$\frac{dy}{dx}=\frac{-(x^2+ay)}{y^2+ax}$$

$$\therefore \frac{dz}{dx}=\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}\left(\frac{-(x^2+ay)}{y^2+ax}\right)$$

$$=\frac{x}{\sqrt{x^2+y^2}}-\frac{y(x^2+ay)}{\sqrt{x^2+y^2}(y^2+ax)}$$

at $x=y=a$

$$=\frac{x(y^2+ax)-y(x^2+ay)}{\sqrt{x^2+y^2}(y^2+ax)}$$

$$\frac{dz}{dx}=\frac{(a-a)a^2-(a-a)a^2}{\sqrt{a^2+a^2}(a^2+a^2)}$$

$$=\frac{xy^2+ax^2-x^2y-ay^2}{\sqrt{x^2+y^2}(y^2+ax)}$$

$$=\frac{0-0}{\sqrt{2a^2}(2a^2)}$$

$$=\frac{(x-a)y^2+(a-y)x^2}{\sqrt{x^2+y^2}(y^2+ax)}$$

$$\boxed{\frac{dz}{dx}=0.}$$

$$\frac{dz}{dx}=\frac{(x-a)y^2-(y-a)x^2}{\sqrt{x^2+y^2}(y^2+ax)}$$

② If $U=x\log(xy)$ where $x^3+y^3+3xy=1$ find $\frac{dU}{dx}$.

Given $U=x.\log(xy)$

$U<{x \atop y}=x.$

$$\frac{dU}{dx}=\frac{\partial U}{\partial x}.\frac{dx}{dx}+\frac{\partial U}{\partial y}.\frac{dy}{dx}$$

$$\frac{dU}{dx}=\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}.\frac{dy}{dx}$$

$U = x . \log(xy)$

$$\frac{\partial U}{\partial x} = x\left(\frac{1}{xy}\right)(y) + \log(xy)(1)$$

$$= 1 + \log(xy)$$

$$\frac{\partial U}{\partial y} = x . \frac{1}{xy}(x) = \frac{x}{y}.$$

Given $x^3 + y^3 + 3xy = 1$

diff. w. r. to 'x'.

$$3x^2 + 3y^2 \frac{dy}{dx} + 3\left[x . \frac{dy}{dx} + y(1)\right] = 0$$

$$x^2 + y^2 . \frac{dy}{dx} + x . \frac{dy}{dx} + y = 0$$

$$(y^2 + x)\frac{dy}{dx} = -(x^2 + y)$$

$$\frac{dy}{dx} = \frac{-(x^2+y)}{y^2+x}.$$

$$\frac{du}{dx} = 1 + \log(xy) + \frac{x}{y} . \left(-\frac{(x^2+y)}{y^2+x}\right)$$

$$= 1 + \log(xy) - \frac{x(x^2+y)}{y(y^2+x)}$$

$$= \log(xy) + \frac{y^3 + \cancel{xy} - x^3 - \cancel{xy}}{y(y^2+x)}$$

$$= \log(xy) + \cancel{y^3 - x}$$

③ If $z = x^2 y$ and $x^2 + xy + y^2 = 1$, find $\frac{dz}{dx}$.

Given $z = x^2 y$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} . \frac{dx}{dx} + \frac{\partial z}{\partial y} . \frac{dy}{dx}$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} . \frac{dy}{dx}$$

$$z = x^2 y$$

$$\frac{\partial z}{\partial x} = y(2x) = 2xy$$

$$\frac{\partial z}{\partial y} = x^2(1) = x^2$$

Given $x^2 + xy + y^2 = 1$

diff. w. r. to x!

$$2x + \cancel{2}(x \cdot \frac{dy}{dx} + y(1)) + 2y = 0$$

$$2x + x \cdot \frac{dy}{dx} + 3y = 0$$

$$x \cdot \frac{dy}{dx} = -(2x+3y)$$

$$\frac{dy}{dx} = \frac{-(2x+3y)}{x}$$

$$\frac{dz}{dx} = 2xy + x^2 \frac{-(2x+3y)}{\cancel{x}}$$

$$= 2xy - x(2x+3y)$$

$$= 2xy - 2x^2 - 3xy = -2x^2 - xy.$$

$$= -(2x^2 + xy)$$

⑤ If $x^y = y^x$ then find $\frac{dy}{dx}$.

Given $x^y = y^x$

$$x^y - y^x = 0$$

$$f(x,y) = x^y - y^x \longrightarrow ①$$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

differentiate Equ① w.r.to 'x' partially.

$$\Rightarrow \frac{\partial f}{\partial x} = y \cdot x^{y-1} \cancel{\cdot x^y \log x} \cancel{= y \cdot x^{y-1}} - y^x \cdot \log y.$$

diff. w.r.to "y" Partially.

$$\Rightarrow \frac{\partial f}{\partial y} = x^y \cdot \log x - x \cdot y^{x-1}$$

$$\therefore \frac{dy}{dx} = -\frac{y \cdot x^{y-1} - y^x \cdot \log y}{x^y \cdot \log x - x \cdot y^{x-1}}$$

⑨ Find $\frac{dy}{dx}$ when $(\cos x)^y = (\sin y)^x$

Given $(\cos x)^y = (\sin y)^x$

$$(\cos x)^y - (\sin y)^x = 0.$$

$$f(x,y) = (\cos x)^y - (\sin y)^x \longrightarrow ①$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

diff. equⁿ ① w. r. to 'x' partially,

$$\Rightarrow \frac{df}{dx} = y(\cos x)^{y-1}(-\sin x) - \sin y^{x}.\log \sin y \,(\cos y)\cos$$

$$= -y\sin x(\cos x)^{y-1} - \cos y.(\sin y)^{x}.\log \sin y$$

diff. equⁿ ① w. respect to 'y' partially.

$$\Rightarrow \frac{df}{dy} = (\cos x)^{y}.\log \cos x.(-\sin x) - x.(\sin y)^{x-1}\cos y$$

$$= -\sin x(\cos x)^{y}.\log(\cos x) - x.\cos y.(\sin y)^{x-1}$$

$$\therefore \frac{dy}{dx} = \frac{-[-y\sin x.(\cos x)^{y-1} - \cos y.(\sin y)^{x}.\log \sin y]}{-[\sin x(\cos x)^{y}\log(\cos x) + x.\cos y(\sin y)^{x-1}]}$$

$$= -\frac{(y\sin x(\cos x)^{y-1} + \cos y(\sin y)^{x}.\log \sin y)}{\sin x.(\cos x)^{y}.\log \cos x + x.\cos y.(\sin y)^{x-1}}$$

$\therefore \frac{dy}{dx} \neq$

diff. equⁿ ① w. r. to 'x' partially

$$\Rightarrow \frac{df}{dx} = y(\cos x)^{y-1}(-\sin x) - (\sin y)^{x}.\log \sin y$$

diff. equⁿ ① w. r. to 'y' partially.

$$\Rightarrow \frac{df}{dy} = (\cos x)^{y}.\log(\cos x) - x.(\sin y)^{x-1}.\cos y$$

$$\therefore \frac{dy}{dx} = -\frac{[-y\sin x.(\cos x)^{y-1} - (\sin y)^{x}.\log(\sin y)]}{(\cos x)^{y}.\log(\cos x) - x.\frac{(\sin y)^{x}}{\sin y}.\cos y}$$

$$= \frac{y.\tan x.(\cos x)^{y} + (\cos x)^{y}.\log \sin y}{(\cos x)^{y}\log(\cos x) - x(\sin y)^{x}.\cot y}$$

$$= \frac{\cancel{(\cos x)^{y}}[y\tan x + \cancel{y}.\log(\sin y)]}{\cancel{(\cos x)^{y}}[\log \cos x - x.\cot y]}$$

$$= \frac{y\tan x + \log(\sin y)}{\log(\cos x) - x.\cot y}$$

④ If $x^3+3x^2y+6xy^2+y^3=1$. Find $\frac{dy}{dx}$

Given that $x^3+3x^2y+6xy^2+y^3=1$

$x^3+3x^2y+6xy^2+y^3-1=0$

$f(x,y) = x^3+3x^2y+6xy^2+y^3-1 \rightarrow ①$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

diff. equⁿ① w.r. to 'x' partially

$\Rightarrow \frac{\partial f}{\partial x} = 3x^2+3y(2x)+6y^2(1)+0-0$

$= 3x^2+6xy+6y^2$

diff. equⁿ① w.r. to 'y' partially.

$\Rightarrow \frac{\partial f}{\partial y} = 0+3x^2(1)+6x(2y)+3y^2-0$

$= 3x^2+12xy+3y^2$

$$\therefore \frac{dy}{dx} = \frac{-(3x^2+6xy+6y^2)}{3x^2+12xy+3y^2}$$

$$= \frac{-\cancel{3}(x^2+2xy+2y^2)}{\cancel{3}(x^2+4xy+y^2)}$$

$$= \frac{-(x^2+2xy+2y^2)}{x^2+4xy+y^2}$$

⑥ If $x^3+y^3-3axy=0$. Find $\frac{dy}{dx}$.

Given that $x^3+y^3-3axy=0$

$f(x,y) = x^3+y^3-3axy \rightarrow ①$

diff. equⁿ① w.r. to 'x' partially.

$\frac{\partial f}{\partial x} = 3x^2+0-3ay(1) = 3x^2-3ay$

diff. w.r. to 'y' partially.

$\frac{\partial f}{\partial y} = 0+3y^2-3ax(1) = 3y^2-3ax$.

$$\therefore \frac{dy}{dx} = \frac{-(3x^2-3ay)}{(3y^2-3ax)} = \frac{-(x^2-ay)}{y^2-ax}.$$

(7) ~~prove that~~ find $\frac{dy}{dx}$ If $y^3-3ax^2+x^3=0$.

sol:

Given that $y^3-3ax^2+x^3=0$

$$f(x,y)=y^3-3ax^2+x^3 \rightarrow (1)$$

diff. equ^n (1) w. r. to 'x' partially.

$$\frac{\partial f}{\partial x}=0-3a(2x)+3x^2 = 3x^2-6ax$$

diff. equ^n (1) w. r. to 'y' partially.

$$\frac{\partial f}{\partial y}=3y^2-0+0 = 3y^2$$

$$\therefore \frac{dy}{dx}=\frac{-(3x^2-6ax)}{3y^2}=\frac{-\cancel{3}(x^2-2ax)}{\cancel{3}y^2}=\frac{2ax-x^2}{y^2}$$

(8) find $\frac{dy}{dx}$ when $x^y+y^x=c$.

Given that $x^y+y^x=c$

$$x^y+y^x-c=0$$

$$f(x,y)=x^y+y^x-c \rightarrow (1)$$

diff. equ^n (1) w. r. to 'x' partially.

$$\frac{\partial f}{\partial x}=y\cdot x^{y-1}+y^x\log y-0 = yx^{y-1}+y^x.\log y$$

diff. equ^n (1) w. r. to 'y' partially

$$\frac{\partial f}{\partial y}=x^y.\log x+x.y^{x-1}-0=x^y\log x+x.y^{x-1}$$

$$\therefore \frac{dy}{dx}=\frac{-(yx^{y-1}+y^x.\log y)}{x^y\log x+x.y^{x-1}}$$

26/11/19 Tuesday

Taylor's (Expansion) theorem:

* Expand the following functions.

① $f(x,y) = e^x \sin y$

By Maclaurin's expansion,

$$f(x,y) = f(0,0) + [x \cdot f_x(0,0) + y \cdot f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)] + \cdots$$

Now, $f(x,y) = e^x \sin y$

$\Rightarrow f(0,0) = e^0 \cdot \sin(0) = (1)(0) = 0.$

$\Rightarrow f_x = \frac{\partial f}{\partial x} = \sin y \cdot e^x \Rightarrow f_x(0,0) = \sin(0) e^0 = 0.$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = e^x \cdot \cos y \Rightarrow f_y(0,0) = e^0 \cdot \cos(0) = 1$

$\Rightarrow f_{xx} = \frac{\partial^2 f}{\partial x^2} = \sin y \cdot e^x \Rightarrow f_{xx}(0,0) = \sin(0)\, e^{(0)} = 0$

$\Rightarrow f_{yy} = \frac{\partial^2 f}{\partial y^2} = e^x \cdot (-\sin y) \Rightarrow f_{yy}(0,0) = -e^0 \sin(0) = 0.$

$\Rightarrow f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = e^x \cdot \cos y \Rightarrow f_{xy}(0,0) = e^0 \cdot \cos 0 = 1$

$\therefore e^x \sin y = 0 + [x(0) + y(1)] + \frac{1}{2!}[x^2(0) + y^2(0) + 2xy(1)] + \cdots$

$= 0 + 0 + y + 0 + 0 + \frac{1}{\cancel{2}}(\cancel{2}xy) + \cdots$

$= y + xy + \cdots$

② $f(x,y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ up to third degree terms. Hence compute $f(1.1, 0.9)$ approximately.

By Taylor's expansion at the (a,b) is

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + (y-a)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)] +$$

$$\frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + (y-b)^3 f_{yyy}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b)] + \cdots$$

at $(1,1)$.

$$f(x,y) = f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + (y-1)^2 f_{yy}(1,1) + 2(x-1)(y-1) f_{xy}(1,1)] + \frac{1}{3!}[(x-1)^3 f_{xxx}(1,1) + (y-1)^3 f_{yyy}(1,1) + 3(x-1)^2(y-1) \cdot f_{xxy}(1,1) + 3(x-1)(y-1)^2 f_{xyy}(1,1)] + \longrightarrow ①$$

We have $f(x,y) = \tan^{-1}(y/x)$

$\Rightarrow f(1,1) = \tan^{-1}(1/1) = \tan^{-1}(1) = \pi/4$.

$f_x = \frac{df}{dx} = \frac{1}{1+(y/x)^2} y\left(\frac{-1}{x^2}\right) = \frac{-y}{y^2+x^2} \Rightarrow f_x(1,1) = \frac{-1}{1+1} = \frac{-1}{2}$

$f_y = \frac{df}{dy} = \frac{1}{1+y^2/x^2} \cdot \frac{1}{x}(1) = \frac{x}{y^2+x^2} \rightarrow f_y(1,1) = \frac{1}{1+1} = \frac{1}{2}$.

$f_{xx} = \frac{d^2f}{dx^2} = (-y)\frac{-1}{(x^2+y^2)^2}(2x) \Rightarrow f_{xx}(1,1) = -1\,\frac{-1}{(1+1)^2}(2(1))$

$= \frac{\cancel{2}}{\cancel{4}_2} = \frac{1}{2}$

$f_{yy} = \frac{d^2f}{dy^2} = x\cdot\frac{-1}{(x^2+y^2)^2}(2y) = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1) = \frac{\cancel{2}}{(2)^2} = \frac{-1}{2}$.

$f_{xy} = \frac{d^2f}{dxdy} = \cancel{\frac{1}{1+(y/x)^2}(y)\left(\frac{-1}{x^2}\right)}\,\frac{(y^2+x^2)(-1)-(-y)(2y+0)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$

$= \cancel{\frac{-(x^2+y^2+2xy)}{(x^2+y^2)^2}} \neq \cancel{\frac{-(x+y)^2}{(x^2+y^2)^2}}$

$\Rightarrow f_{xy}(1,1) = \cancel{\frac{-(1+1)^2}{(1+1)^2}} \neq \cancel{\frac{-4}{-8}}$

$\Rightarrow f_{xy}(1,1) = \frac{-1-1+2}{(1+1)^2} = 0$.

$f_{xxx} = \frac{(x^2+y^2)^2(2y)(1) - 2xy\; 2(x^2+y^2)(2x)}{[(x^2+y^2)^2]^2}$

$= \frac{2y(x^2+y^2)^2 - 8x^2y\cdot(x^2+y^2)}{(x^2+y^2)^4}$

$\Rightarrow f_{xxx}(1,1) = \frac{2(1)(1+1)^2 - 8(1)(1)(1+1)}{(1+1)^4} = \frac{8-16}{16} = \frac{-8}{16} = \frac{-1}{2}$.

$f_{yyy} = \frac{(x^2+y^2)^2(-2x)(1) + 2xy\; 2(x^2+y^2)(0+2y)}{((x^2+y^2)^2)^2}$

$= \frac{-2x(x^2+y^2)^2 + 8xy^2(x^2+y^2)}{(x^2+y^2)^4}$

$\Rightarrow f_{yyy}(1,1) = \frac{-2(4)+8(1+1)}{16} = \frac{-8+16}{16} = \frac{8}{16} = \frac{1}{2}$.

$f_{xxy} = 2x\left[\frac{(x^2+y^2)^2(1) - y\cdot 2(x^2+y^2)\,2y}{[(x^2+y^2)^2]^2}\right]$

$= 2x\left[\frac{(x^2+y^2) - 4y^2}{(x^2+y^2)^3}\right]$

$\Rightarrow f_{xxy}(1,1) = \cancel{x}\left[\frac{2-4}{8_y}\right] = \frac{-2}{4} = \frac{-1}{2}.$

$|||^{ly} = f_{xyy}(1,1) = \frac{1}{2}.$

from (1),

$\tan^{-1}(y/x) = \pi/4 + \left[(x-1)\left(\frac{-1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right] + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + (y-1)^2\left(\frac{-1}{2}\right) + 2(x-1)(y-1)(0)\right]$

$f(x,y) =$

$+ \frac{1}{3!}\left[(x-1)^3 \cdot \cancel{d}\left(\frac{-1}{2}\right) + (y-1)^3\left(\frac{1}{2}\right) + 3(x-1)^2(y-1)\left(-\frac{1}{2}\right) + 3(x-1)(y-1)^2 \cdot \cancel{8}\left(\frac{1}{2}\right)\right]$

$= \frac{\pi}{4} + \frac{1}{2}\left[-(x-1) + (y-1)\right] + \frac{1}{2!}\frac{1}{2}\left[(x-1)^2 \mp (y-1)^2\right] + \frac{1}{3!}\frac{1}{2}\left[(x-1)^3\right.$
$\left. + (y-1)^3 + 3(x-1)^2(y-1) + 3(x-1)(y-1)^2\right].$

$= \frac{\pi}{4} + \frac{1}{2}\left(-(x-1) + (y-1)\right) + \frac{1}{4}\left[(x-1)^2 \mp (y-1)^2\right] + \frac{1}{12}\left[(x-1)^3 + (y-1)^3\right.$
$\left. + 3(x-1)^2(y-1) + 3(x-1)(y-1)^2\right] + \cdots$

$f(1.1, 0.9) = \frac{\pi}{4} + \frac{1}{2}\left[-(1.1-1) + (0.9-1)\right] + \frac{1}{4}\left[(1.1-1)^2 \mp (0.9-1)^2\right] + \frac{1}{12}\left[(1.1-1)^3\right.$
$\left. + (0.9-1)^3 + 3(1.1-1)^2(0.9-1) + 3(1.1-1)(0.9-1)^2\right]$

$= \frac{3.14}{4} + \frac{1}{2}\left[-0.2\right] + \frac{1}{4}\left[0.02\right] + \frac{1}{12}\left\{-3.009\right\}$

$= 0.785 - 0.1 + \frac{1}{4}(0) + \frac{1}{12}\left[0.001 \mp 0.001 + 3(\right.$

$= 0.68533.$

④ $f(x,y) = e^x \log(1+x)$

Sol:- $f(x,y) = e^x \log(1+x)$

$f(x,y) = f(0,0) + \left[x \cdot f_x(0,0) + y \cdot f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

We have,

$f(x,y) = e^y \log(1+x) \Rightarrow f(0,0) = e^0 \cdot [\log(1)] = 0.$

$f_x = \frac{\partial f}{\partial x} = e^y \frac{1}{1+x} \Rightarrow f_x(0,0) = e^0 \frac{1}{1+0} = 1$

$f_y = \frac{\partial f}{\partial y} = \log(1+x)e^y \Rightarrow f_y(0,0) = 0.$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = e^y \frac{-1}{(1+x)^2} \Rightarrow f_{xx}(0,0) = e^0 \frac{-1}{(1+0)^2} = -1.$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{1}{1+x} e^y \Rightarrow f_{xy}(0,0) = \frac{1}{1+0} e^{(0)} = 1.$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \log(1+x) e^y \Rightarrow f_{yy}(0,0) = 0.$

$$e^y \log(1+x) = 0 + [x(1) + y(0)] + \frac{1}{2!}[x^2(-1) + 2xy(1) + y^2(0)] + \cdots$$

$$= x + \frac{1}{2}(-x^2 + 2xy) + \cdots$$

$$= x - \frac{x^2}{2} + xy + \cdots$$

(3) $f(x,y) = e^x \log(1+y)$.

Given $f(x,y) = e^x \log(1+y)$

By Maclaurin's expansion

$$f(x,y) = f(0,0) + [x \cdot f_x(0,0) + y\, f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy\, f_{xy}(0,0)] + \cdots \rightarrow (1)$$

$f(x,y) = e^x \log(1+y) \Rightarrow f(0,0) = e^0 \log(1+0) = 0.$

$f_x = \frac{\partial f}{\partial x} = \log(1+y)\, e^x \Rightarrow f_x(0,0) = \log(1+0)\, e^0 = 0.$

$f_y = \frac{\partial f}{\partial y} = e^x \cdot \frac{1}{1+y} \Rightarrow f_y(0,0) = e^0 \frac{1}{1+0} = 1.$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \log(1+y)\, e^x \Rightarrow f_{xx}(0,0) = \log(1+0)\, e^0 = 0$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = e^x \cdot \frac{-1}{(1+y)^2} \Rightarrow f_{yy}(0,0) = e^0 \cdot \frac{-1}{(1+0)^2} = -1$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = e^x \cdot \frac{1}{1+y} \Rightarrow f_{xy}(0,0) = e^0 \frac{1}{1+0} = 1$

from (1),

$$e^x \log(1+y) = 0 + [x \cdot (0) + y(1)] + \frac{1}{2!}[x^2(0) + y^2(-1) + 2xy(1)] + \cdots$$

$$= y + \frac{1}{2}[-y^2 + 2xy] + \cdots$$

$$= y - \frac{y^2}{2} + xy + \cdots$$

(9) Expand $x^2y + 3y - 2$ in power of $(x-1)$ and $(y+2)$ using Taylor's theorem.

By Taylor's expansion,

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots$$

$$= f(1,-2) + [(x-1) f_x(1,-2) + (y+2) f_y(1,-2)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,-2) + 2(x-1)(y+2) f_{xy}(1,-2) + (y+2)^2 f_{yy}(1,-2)] + \cdots \rightarrow (1)$$

We have $f(x,y) = x^2y + 3y - 2 \Rightarrow f(1,-2) = -2-6-2 = -10$

$f_x = \frac{\partial f}{\partial x} = y(2x)+0-0 \Rightarrow f_x(1,-2) = -4$

$f_y = \frac{\partial f}{\partial y} = x^2(1) + 3(1) - 0 \Rightarrow f_y(1,-2) = 1+3 = 4$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2y(1) \Rightarrow f_{xx}(1,-2) = -4$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 2x(1) \Rightarrow f_{xy}(1,-2) = 2$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 0+0 \Rightarrow f_{yy}(1,-2) = 0.$

from (1),

$$x^2y+3y-2 = -10 + [(x-1)(-4) + (y+2)(+4)] + \frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)] + \cdots$$

$$= -10 - 4[(x-1)-(y+2)] + \frac{4}{2}[(x-1)^2 - (x-1)(y+2)] + \cdots$$

$$= -10 - 4[(x-1)-(y+2)] - 2[(x-1)^2 - (x-1)(y+2)] + \cdots$$

(8) Show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$

and hence deduce that $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$

By Maclaurin's expansion,

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

We have $f(x) = \log(1+e^x) \Rightarrow f(0) = \log(1+e^0) = \log 2.$

$f'(x) = \frac{1}{1+e^x} \cdot e^x \Rightarrow f'(0) = \frac{e^0}{1+e^0} = \frac{1}{2}$

$f''(x) = \frac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2} = \frac{e^x + \cancel{e^{2x}} - \cancel{e^{2x}}}{(1+e^x)^2} \Rightarrow f''(0) = \frac{e^0}{(1+e^0)^2} = \frac{1}{4}$

$f'''(x) = \frac{(1+e^x)^2 e^x - e^x \cdot 2(1+e^x)e^x}{[(1+e^x)^2]^2} = \frac{\cancel{(1+e^x)}[(1+e^x)e^x - 2e^{2x}]}{(1+e^x)^{\cancel{4}\,3}}$

$= \frac{e^x + e^{2x} - 2e^{2x}}{(1+e^x)^3} = \frac{e^x - e^{2x}}{(1+e^x)^3}$

$\Rightarrow f'''(0) = \frac{e^0 - e^0}{(1+e^0)^3} = 0.$

$f^{iv}(x) = \frac{(1+e^x)^3\{e^x - e^{2x}(2)\} - (e^x - e^{2x})\, 3(1+e^x)^2 e^x}{[(1+e^x)^3]^2}$

$$f^{IV}(x) = \frac{(1+e^x)^2\left[1+e^x(e^x-2e^{2x}) - 3e^x(e^x-e^{2x})\right]}{(1+e^x)^4}$$

$$\Rightarrow f^{IV}(0) = \frac{(1+e^0)(e^0-2\cdot e^0) - 3\cdot e^0(e^0-e^0)}{(1+e^0)^4}$$

$$= \frac{2\cdot(1-2) - 3(1)(1-1)}{(1+1)^4} = \frac{-2-0}{16} = \frac{-2}{16} = \frac{-1}{8}.$$

$$\log(1+e^x) = \log 2 + x\cdot\frac{1}{2} + \frac{x^2}{2!}\cdot\frac{1}{4} + \frac{x^3}{3!}(0) + \frac{x^4}{4!}\left(\frac{-1}{8}\right) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$$

diff. w. r. to 'x'

$$\frac{1}{1+e^x}\cdot e^x = 0 + \frac{1}{2} + \frac{1}{8}(2x) - \frac{4x^3}{192_{48}} + \cdots$$

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

5) $f(x,y) = e^{xy}$ in powers of $(x-1)$ and $(y-1)$.

By Taylor's expansion,

$$f(x,y) = f(1,1) + \left[(x-1)f_x(1,1) + (y-1)f_y(1,1)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + (y-1)^2 f_{yy}(1,1) + 2(x-1)(y-1)f_{xy}(1,1)\right] + \cdots$$

We have $f(x,y) = e^{xy} \Rightarrow f(1,1) = e^{(1)(1)} = e.$

$$f_x = \frac{\partial f}{\partial x} = e^{xy}(y) \Rightarrow f_x(1,1) = e^{(1)(1)}(1) = e.$$

$$f_y = \frac{\partial f}{\partial y} = e^{xy}(x) \Rightarrow f_y(1,1) = e^{(1)(1)}(1) = e.$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = y\cdot e^{xy}(y) \Rightarrow f_{xx}(1,1) = (1)e^{(1)(1)}(1) = e.$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = x\cdot e^{xy}(x) \Rightarrow f_{yy}(1,1) = (1)e^{(1)(1)}(1) = e.$$

$$f_{xy} = \frac{\partial^2 f}{\partial x\partial y} = e^{xy}(1) + y\cdot e^{xy}(x) \Rightarrow f_{xy}(1,1) = e + e = 2e$$

$$e^{xy} = e + \left[(x-1)e + (y-1)e\right] + \frac{1}{2!}\left[(x-1)^2 e + (y-1)^2 e + 2(x-1)(y-1)2e\right] + \cdots$$

$$= e + e\left[(x-1) + (y-1)\right] + \frac{e}{2!}\left[(x-1)^2 + (y-1)^2 + 4(x-1)(y-1)\right] + \cdots$$

⑥ $f(x,y) = e^x \cos y$ about $(1, \pi/4)$.

By Taylor's Expansion,

$$f(x,y) = f(1,\pi/4) + \left[(x-1) f_x(1,\pi/4) + (y-\pi/4) f_y(1,\pi/4)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,\pi/4)\right.$$
$$\left. + 2(x-1)(y-\pi/4) f_{xy}(1,\pi/4) + (y-\pi/4)^2 f_{yy}(1,\pi/4)\right] + \cdots$$

We have $f(x,y) = e^x \cos y \Rightarrow f(1,\pi/4) = e^1 \cos \pi/4 = \frac{e}{\sqrt{2}}$.

$f_x = \frac{\partial f}{\partial x} = \cos y \cdot e^x \Rightarrow f_x(1,\pi/4) = \cos\pi/4 \, e^{(1)} = \frac{e}{\sqrt{2}}$

$f_y = \frac{\partial f}{\partial y} = e^x \cdot (-\sin y) \Rightarrow f_y(1,\pi/4) = -e^{(1)} \sin \pi/4 = \frac{-e}{\sqrt{2}}$.

$f_{xx} = \cos y \cdot e^x \Rightarrow f_{xx}(1,\pi/4) = \cos \pi/4 \cdot e^{(1)} = \frac{e}{\sqrt{2}}$

$f_{xy} = e^x(-\sin y) \Rightarrow f_{xy}(1,\pi/4) = -e^{(1)} \sin \pi/4 = \frac{-e}{\sqrt{2}}$.

$f_{yy} = -e^x \cdot \cos y \Rightarrow f_{yy}(1,\pi/4) = -e^{(1)} \cos \pi/4 = \frac{-e}{\sqrt{2}}$

$$e^x \cdot \cos y = \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + (y-\pi/4)\left(\frac{-e}{\sqrt{2}}\right)\right] + \frac{1}{2!}\left[(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)(y-\pi/4)\left(\frac{-e}{\sqrt{2}}\right)\right.$$
$$\left. + (y-\pi/4)^2\left(\frac{-e}{\sqrt{2}}\right)\right] + \cdots$$

$$= \frac{e}{\sqrt{2}} + \frac{e}{\sqrt{2}}\left[(x-1) - (y-\pi/4)\right] + \frac{1}{2!}\frac{e}{\sqrt{2}}\left[(x-1)^2 - 2(x-1)(y-\pi/4) - (y-\pi/4)^2\right] + \cdots$$

$$= \frac{e}{\sqrt{2}} + \frac{e}{\sqrt{2}}\left[(x-1) - (y-\pi/4)\right] + \frac{e}{2\sqrt{2}}\left[(x-1)^2 - 2(x-1)(y-\pi/4) - (y-\pi/4)^2\right] + \cdots$$

⑦ $f(x,y) = \sin xy$ in powers of $(x-1)$ and $(y-\pi/2)$ up to second degree terms.

By Taylor's Expansion,

$$f(x,y) = f(1,\pi/2) + \left[(x-1) f_x(1,\pi/2) + (y-\pi/2) f_y(1,\pi/2)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,\pi/2)\right.$$
$$\left. + 2(x-1)(y-\pi/2) f_{xy}(1,\pi/2) + (y-\pi/2)^2 f_{yy}(1,\pi/2)\right] + \cdots$$

We have,

$f(x,y) = \sin xy \Rightarrow f(1,\pi/2) = \sin \pi/2 = 1$

$f_x = \frac{\partial f}{\partial x} = \cos xy \, (y) \Rightarrow f_x(1,\pi/2) = (\pi/2) \cos \pi/2 = 0$.

$f_y = \frac{\partial f}{\partial y} = \cos xy (x) \Rightarrow f_y(1,\pi/2) = (1) \cos \pi/2 = 0$.

$f_{xx} = y \cdot (-\sin xy)(y) \Rightarrow f_{xx}(1,\pi/2) = \pi/2 \cdot \pi/2 (-\sin \pi/2) = \frac{\pi^2}{4}(-1) = \frac{-\pi^2}{4}$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = y.(-\sin xy)(x) + \cos xy\,(1)$

$\Rightarrow f_{xy}(1, \pi/2) = \pi/2 - \sin \pi/2 (1) + \cos \pi/2$

$= -\pi/2 (1) + 0 = -\pi/2$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = x.(-\sin xy)(x) = (1) - \sin \pi/2 (1) = -1$

$\sin xy = 1 + [(x-1)0 + (y-\pi/2)0] + \frac{1}{2!}\left[(x-1)^2(-\pi^2/4) + 2(x-1)(y-\pi/2)(-\pi/2) + (y-\pi/2)^2(-1)\right] + \ldots$

$= 1 + [0+0] + \frac{1}{2}\left[(x-1)^2(-\pi^2/4) \mp 0(x-1)(y-\pi/2)\pi/2 + -(y-\pi/2)^2\right] + \ldots$

$\sin xy = 1 \mp \frac{1}{2}\left[(x-1)^2 \pi^2/4 + (x-1)(y-\pi/2)\pi/2 + (y-\pi/2)^2\right] + \ldots$

28/11/19
Thursday

Jacobian:

② If $x = r\sin\theta\cos\phi$, $y = r\sin\theta.\sin\phi$, $z = r\cos\theta$. show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.$$

Sol:- $x = r\sin\theta\cos\phi$ $\quad y = r\sin\theta.\sin\phi$ $\quad z = r\cos\theta$

$x, y, z \rightarrow r\ \theta\ \phi$

$$\frac{\partial(x\,y\,z)}{\partial(r\,\theta\,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$x = r\sin\theta.\cos\phi$

$\frac{\partial x}{\partial r} = \sin\theta.\cos\phi(1)$

$\frac{\partial x}{\partial \theta} = r\cos\phi\ \cos\theta)$

$\frac{\partial x}{\partial \phi} = r\sin\theta.(-\sin\phi)$

$y = r\sin\theta.\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta.\sin\phi$

$\frac{\partial y}{\partial \theta} = r\sin\phi.\cos\theta$

$\frac{\partial y}{\partial \phi} = r\sin\theta.\cos\phi$

$z = r\cos\theta$

$\frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial z}{\partial \theta} = r(-\sin\theta)$

$\frac{\partial z}{\partial \phi} = 0.$

$$\frac{\partial(x\,y\,z)}{\partial(r\,\theta\,\phi)} = \begin{array}{ccc} (+) & (-) & (+) \end{array}\begin{vmatrix} \sin\theta.\cos\phi & r\cos\theta\cos\phi & -r\sin\theta.\sin\phi \\ \sin\theta.\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta\cdot\cos\phi\,[0 + r^2\sin^2\theta\cdot\cos\phi] - r\cos\theta\cdot\cos\phi\,[0 - r\sin\theta\cos\theta\cos\phi]$

$- r\sin\theta\cdot\sin\phi\,[-r\sin^2\theta\cdot\sin\phi - r\cdot\cos^2\theta\cdot\sin\phi]$

$= r^2\sin^3\theta\cdot\cos^2\phi + r^2\sin\theta\cdot\cos^2\theta\cdot\cos^2\phi - r\sin\theta\cdot\sin\phi\,\{(-r\sin\phi)[\sin^2\theta + \cos^2\theta]\}$

$= r^2\sin^3\theta\cdot\cos^2\phi + r^2\sin\theta\cdot\cos^2\theta\cdot\cos^2\phi + r^2\sin\theta\cdot\sin^2\phi$

$= r^2\sin\theta\cdot\cos^2\phi\,[\sin^2\theta + \cos^2\theta] + r^2\sin\theta\cdot\sin^2\phi$

$= r^2\sin\theta\cdot\cos^2\phi\,(1) + r^2\sin\theta\cdot\sin^2\phi$

$= r^2\sin\theta\,[\cos^2\phi + \sin^2\phi]$

$= r^2\sin\theta.$

③ If $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$ show that $\dfrac{\partial(uvw)}{\partial(xyz)} = 0$.

Sol: $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$ $\qquad uvw \leftarrow \begin{matrix}x\\y\\z\end{matrix}$ (or) $\begin{matrix}u\\v\\w\end{matrix} \rightarrow xyz$

$$\frac{\partial(uvw)}{\partial(xyz)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = \dfrac{x}{y-z}$

$\dfrac{\partial u}{\partial x} = \dfrac{1}{y-z}$

$\dfrac{\partial u}{\partial y} = x\cdot\dfrac{-1}{(y-z)^2}$

$\dfrac{\partial u}{\partial z} = x\cdot\dfrac{-1}{(y-z)^2}(0-1)$

$= \dfrac{x}{(y-z)^2}$

$v = \dfrac{y}{z-x}$

$\dfrac{\partial v}{\partial x} = y\cdot\dfrac{-1}{(z-x)^2}(-1)$

$= \dfrac{y}{(z-x)^2}$

$\dfrac{\partial v}{\partial y} = \dfrac{1}{z-x}$

$\dfrac{\partial v}{\partial z} = y\dfrac{-1}{(z-x)^2}$

$w = \dfrac{z}{x-y}$

$\dfrac{\partial w}{\partial x} = z\,\dfrac{-1}{(x-y)^2}$

$\dfrac{\partial w}{\partial y} = z\,\dfrac{-1}{(x-y)^2}(-1) = \dfrac{z}{(x-y)^2}$

$\dfrac{\partial w}{\partial z} = \dfrac{1}{x-y}$

$$\frac{\partial(uvw)}{\partial(xyz)} = \begin{array}{ccc} (+) & (-) & (+) \end{array} \begin{vmatrix} \frac{1}{y-z} & \frac{-x}{(y-z)^2} & \frac{x}{(y-z)^2} \\ \frac{y}{(z-x)^2} & \frac{1}{z-x} & \frac{-y}{(z-x)^2} \\ \frac{-z}{(x-y)^2} & \frac{z}{(x-y)^2} & \frac{1}{x-y} \end{vmatrix}$$

$= \dfrac{1}{y-z}\left[\dfrac{1}{(x-y)(z-x)} + \dfrac{yz}{(x-y)^2(z-x)^2}\right] + \dfrac{x}{(y-z)^2}\left[\dfrac{y}{(x-y)(z-x)^2}\right.$

$\left. - \dfrac{zy}{(x-y)^2(z-x)^2}\right] + \dfrac{x}{(y-z)^2}\left[\dfrac{yz}{(z-x)^2(x-y)^2} + \dfrac{z}{(x-y)^2(z-x)}\right]$

$$= \frac{1}{y-z}\cdot\frac{1}{x-y}\cdot\frac{1}{z-x}\left[1+\frac{yz}{(x-y)(z-x)}\right] + \frac{xy}{(x-y)(y-z)^2(z-x)^2}\left[1-\frac{z}{x-y}\right]$$

$$+ \frac{xz}{(y-z)^2(z-x)(x-y)^2}\left[\frac{y}{z-x}+1\right]$$

$$= \frac{1}{(x-y)(y-z)(z-x)}\left[\frac{(x-y)(z-x)+yz}{(x-y)(z-x)}\right] + \frac{xy}{(x-y)(y-z)^2(z-x)^2}\left[\frac{x-y-z}{x-y}\right]$$

$$+ \frac{xz}{(y-z)^2(z-x)(x-y)^2}\left[\frac{y+z-x}{z-x}\right]$$

$$= \frac{1}{(x-y)^2(y-z)(z-x)^2}\left[xz-x^2-yz+xy+yz\right] + \frac{xy}{(x-y)^2(y-z)^2(z-x)^2}(x-y-z)$$

$$+ \frac{xz}{(y-z)^2(z-x)^2(x-y)^2}[y+z-x]$$

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2}\left[(y-z)(xz-x^2+xy)+xy(x-y-z)+xz(y+z-x)\right]$$

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2}\left[xyz-x^2y+xy^2-xz^2+zx^2-xyz+x^2y-xy^2-xyz+xyz+xz^2-x^2z\right]$$

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2}(0)$$

$$=0.$$

① If $r=\sqrt{x^2+y^2}$. $\theta=\tan^{-1}(y/x)$. evaluate $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Sol:- $r=\sqrt{x^2+y^2}$ $\quad \theta=\tan^{-1}(y/x)$

$\tan\theta = y/x$

$r,\theta < \begin{matrix} x \\ y \end{matrix}$

$\begin{matrix} r \\ \theta \end{matrix} > xy$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x) \qquad \frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\,y\left(\frac{-1}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}y) \qquad \frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{\sqrt{x^2+y^2}(x^2+y^2)} + \frac{y^2}{\sqrt{x^2+y^2}(x^2+y^2)}$$

$$= \frac{\cancel{x^2+y^2}}{\sqrt{x^2+y^2}\,\cancel{(x^2+y^2)}}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\sqrt{x^2+y^2}}$$

⑫ If $U = \frac{yz}{x}$, $V = \frac{zx}{y}$, $\omega = \frac{xy}{z}$ show that $\frac{\partial(xyz)}{\partial(UVW)} = \frac{1}{4}$.

$UVW \left< \begin{matrix} x \\ y \\ z \end{matrix} \right.$

$$\frac{\partial(U,V,W)}{\partial(x,yz)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial z} \end{vmatrix}$$

$\frac{\partial U}{\partial x} = yz\left(\frac{-1}{x^2}\right)$ | $\frac{\partial V}{\partial x} = \frac{z}{y}$ | $\frac{\partial W}{\partial x} = \frac{y}{z}$

$\frac{\partial U}{\partial y} = \frac{z}{x}$ | $\frac{\partial V}{\partial y} = zx\left(\frac{-1}{y^2}\right)$ | $\frac{\partial W}{\partial y} = \frac{x}{z}$

$\frac{\partial U}{\partial z} = \frac{y}{x}$ | $\frac{\partial V}{\partial z} = \frac{x}{y}$ | $\frac{\partial W}{\partial z} = xy\left(\frac{-1}{z^2}\right)$

$$\frac{\partial(UVW)}{\partial(xyz)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ y/z & x/z & \frac{-xy}{z^2} \end{vmatrix}$$

$$\ne \frac{-yz}{x^2}\left[\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right]$$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{zx}{x^2} & \frac{xy}{x^2} \\ \frac{zy}{y^2} & \frac{-zx}{y^2} & \frac{xy}{y^2} \\ \frac{yz}{z^2} & \frac{xz}{z^2} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{(yz)(zx)(xy)}{x^2\,y^2\,z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{\cancel{x^2y^2z^2}}{\cancel{x^2y^2z^2}}\left[-1(1-1) - 1(-1-1) + 1(1+1)\right]$$

$$= 0 + 2 + 2 = 4 \Rightarrow \boxed{\frac{\partial(xyz)}{\partial(UVW)} = \frac{1}{4}}$$

We know that,

$$\frac{\partial(u\,v\,w)}{\partial(x\,y\,z)} \cdot \frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = 1$$

$$4 \cdot \frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = 1$$

$$\boxed{\frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = 1/4}$$

(14) $u = x+y+z$; $uv = y+z$; $uvw = z$ show that $\dfrac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = u^2v$.

$u = x+y+z$ $\qquad$ $uv = y+z$ $\qquad$ $uvw = z$

$u = x+uv$ $\qquad$ $uv = y+uvw$ $\qquad$ $z = uvw.$

$x = u-uv$ $\qquad$ $y = uv-uvw$

$$x\ y\ z \left< \begin{matrix} u \\ v \\ w \end{matrix} \right.$$

$$\frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = u-uv$

$\frac{\partial x}{\partial u} = 1-v$

$\frac{\partial x}{\partial v} = 0-u$

$\frac{\partial x}{\partial w} = 0$

$y = uv-uvw$

$\frac{\partial y}{\partial u} = v-vw$

$\frac{\partial y}{\partial v} = u-uw$

$\frac{\partial y}{\partial w} = 0-uv$

$z = uvw$

$\frac{\partial z}{\partial u} = vw$

$\frac{\partial z}{\partial v} = uw$

$\frac{\partial z}{\partial w} = uv$

$$\frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$=(1-v)\left[(u-uw)uv + u^2vw\right] + u\left[(v-vw)uv + uv^2w\right] + 0$$

$$=(1-v)\left[u^2v - \cancel{u^2vw} + \cancel{u^2vw}\right] + u\left\{uv^2 - \cancel{uv^2w} + \cancel{uv^2w}\right\}$$

$$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$$

$$= u^2v.$$

(16) $y_1 = 1 - x_1$; $y_2 = x_1(1-x_2)$; $y_3 = x_1 x_2 (1-x_3)$ find $\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)}$.

Sol:- $y_1 = 1 - x_1$ $\quad y_2 = x_1 - x_1 x_2$ $\quad y_3 = x_1 x_2 - x_1 x_2 x_3$

$$y_1 y_2 y_3 \left\langle \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right.$$

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$y_1 = 1 - x_1$

$$\frac{\partial y_1}{\partial x_1} = -1 \qquad \frac{\partial y_1}{\partial x_2} = 0 \qquad \frac{\partial y_1}{\partial x_3} = 0$$

$y_2 = x_1 - x_1 x_2$

$$\frac{\partial y_2}{\partial x_1} = 1 - x_2 \qquad \frac{\partial y_2}{\partial x_2} = 0 - x_1 \qquad \frac{\partial y_2}{\partial x_3} = 0$$

$y_3 = x_1 x_2 - x_1 x_2 x_3$

$$\frac{\partial y_3}{\partial x_1} = x_2 - x_2 x_3 \qquad \frac{\partial y_3}{\partial x_2} = x_1 - x_1 x_3 \qquad \frac{\partial y_3}{\partial x_3} = 0 - x_1 x_2$$

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = \begin{vmatrix} -1 & 0 & 0 \\ 1-x_2 & -x_1 & 0 \\ x_2 - x_2 x_3 & x_1 - x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= -1(x_1^2 x_2 - 0) - 0 + 0$$

$$= -x_1^2 x_2$$

(17) $U = x+y+z$; $U^2 V = y+z$; $U^3 W = z$ prove that $\frac{\partial(UVW)}{\partial(x y z)} = U^{-5}$.

Sol:- $U = x+y+z$ $\quad U^2 V = y+z$ $\quad U^3 W = z$

$U = x + U^2 V$ $\quad U^2 V = y + U^3 W$ $\quad z = U^3 W$

$x = U - U^2 V$ $\quad y = U^2 V - U^3 W$

$$x y z \left\langle \begin{matrix} U \\ V \\ W \end{matrix} \right.$$

$$\frac{\partial(x y z)}{\partial(UVW)} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} & \frac{\partial x}{\partial W} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} & \frac{\partial y}{\partial W} \\ \frac{\partial z}{\partial U} & \frac{\partial z}{\partial V} & \frac{\partial z}{\partial W} \end{vmatrix}$$

$x = U - U^2 V$

$$\frac{\partial x}{\partial U} = 1 - V \cdot 2U$$

$y = U^2 V - U^3 W$

$$\frac{\partial y}{\partial U} = 2UV - 3U^2 W$$

$z = U^3 W$

$$\frac{\partial z}{\partial U} = W \cdot 3U^2$$

$\frac{\partial x}{\partial v} = 0 - u^2$ | $\frac{\partial y}{\partial v} = u^2 - 0$ | $\frac{\partial z}{\partial v} = 0$

$\frac{\partial x}{\partial w} = 0$ | $\frac{\partial y}{\partial w} = 0 - u^3$ | $\frac{\partial z}{\partial w} = u^3$

$$\frac{\partial(xyz)}{\partial(uvw)} = \begin{vmatrix} 1-2uv & -u^2 & 0 \\ 2uv-3u^2w & u^2 & -u^3 \\ 3u^2w & 0 & u^3 \end{vmatrix} \quad R_1 \to R_1+R_2+R_3$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2uv-3u^2w & u^2 & -u^3 \\ 3u^2w & 0 & u^3 \end{vmatrix}$$

$$= u^3 \begin{vmatrix} 1 & 0 & 0 \\ 2uv-3u^2w & u^2 & -1 \\ 3u^2w & 0 & 1 \end{vmatrix}$$

$$= u^3 [1(u^2+0) - 0 + 0]$$

$$= u^3 (v^2)$$

$$\frac{\partial(xyz)}{\partial(uvw)} = u^5.$$

We know that, $\frac{\partial(u \cdot v \cdot w)}{\partial(xyz)} \cdot \frac{\partial(xyz)}{\partial(uvw)} = 1$

$$\frac{\partial(uvw)}{\partial(xyz)} \cdot u^5 = 1$$

$$\frac{\partial(uvw)}{\partial(xyz)} = \frac{1}{u^5}$$

$$\boxed{\frac{\partial(uvw)}{\partial(xyz)} = u^{-5}}$$

(18) If $u^3+v^3 = x+y$; $u^2+v^2 = x^3+y^3$ prove that $\frac{\partial(uv)}{\partial(xy)}$

Sol: Let us take $f_1 = u^3+v^3-x-y$

$f_2 = u^2+v^2-x^3-y^3$

$f_1 = u^3+v^3-x-y$ | $f_2 = u^2+v^2-x^3-y^3$

$\frac{\partial f_1}{\partial u} = 3u^2$ | $\frac{\partial f_2}{\partial u} = 2u$

$\frac{\partial f_1}{\partial v} = 3v^2$ | $\frac{\partial f_2}{\partial v} = 2v$

$$\frac{\partial f_1}{\partial x} = -1 \qquad \frac{\partial f_2}{\partial x} = -3x^2$$

$$\frac{\partial f_1}{\partial y} = -1 \qquad \frac{\partial f_2}{\partial y} = -3y^2$$

We know that $\frac{\partial(UV)}{\partial(xy)} = (-1)^2 \dfrac{\frac{\partial(f_1 f_2)}{\partial(xy)}}{\frac{\partial(f_1 f_2)}{\partial(UV)}}$

$$\frac{\partial(f_1 f_2)}{\partial(xy)} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} \qquad \frac{\partial(f_1 f_2)}{\partial(UV)} = \begin{vmatrix} 3U^2 & 3V^2 \\ 2U & 2V \end{vmatrix}$$

$$= 6U^2V - 6UV^2$$

$$= 3y^2 - 3x^2$$

$$\frac{\partial(UV)}{\partial(xy)} = \frac{3y^2 - 3x^2}{6U^2V - 6UV^2} = \frac{1}{2}\frac{(y^2 - x^2)}{(U^2V - UV^2)}$$

④ If $U = x(1-y)$, $V = xy$ prove that $\frac{\partial(UV)}{\partial(xy)} \times \frac{\partial(xy)}{\partial(UV)} = 1$.

$U = x(1-y) \quad V = xy$ $\qquad UV < \begin{matrix} x \\ y \end{matrix}$

$$J = \frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} \qquad \frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix}$$

$U = x(1-y)$ | $V = xy$

$\frac{\partial U}{\partial x} = 1-y$ | $\frac{\partial V}{\partial x} = y$

$\frac{\partial U}{\partial y} = -x$ | $\frac{\partial V}{\partial y} = x$

$$= (1-y)x + xy$$
$$= x - \cancel{xy} + \cancel{xy}$$
$$= x$$

$U = x - xy \qquad V = xy$

$U = x - V \qquad y = \frac{V}{x}$

$x = U + V \qquad y = \frac{V}{U+V}$

$xy < \begin{matrix} U \\ V \end{matrix}$

$$J^1 = \frac{\partial(xy)}{\partial(UV)} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix}$$

$x = U + V$ | $y = \frac{V}{U+V}$

$\frac{\partial x}{\partial U} = 1$ | $\frac{\partial y}{\partial U} = V \frac{-1}{(U+V)^2} = \frac{-V}{(U+V)^2}$

$\frac{\partial x}{\partial V} = 1$ | $\frac{\partial y}{\partial V} = \frac{(U+V)(1) - V(0+1)}{(U+V)^2} = \frac{U}{(U+V)^2}$

$$\frac{\partial(xy)}{\partial(uv)} = \begin{vmatrix} 1 & 1 \\ \frac{-v}{(u+v)^2} & \frac{u}{(u+v)^2} \end{vmatrix}$$

$$= \frac{u}{(u+v)^2} + \frac{v}{(u+v)^2}$$

$$= \frac{u+v}{(u+v)^2} = \frac{1}{u+v} = \frac{1}{x-xy+xy} = \frac{1}{x}$$

$$J \cdot J' = x \cdot \frac{1}{x} = 1$$

$$\therefore \frac{\partial(uv)}{\partial(xy)} \cdot \frac{\partial(xy)}{\partial(uv)} = 1$$

⑦ If $x = r\cos\theta$, $y = r\sin\theta$ show that $\frac{\partial(xy)}{\partial(r\theta)} \cdot \frac{\partial(r\theta)}{\partial(xy)} = 1$

Sol:- $x = r\cos\theta \quad y = r\sin\theta \qquad xy < {r \atop \theta}$

$$J = \frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta$

$\frac{\partial x}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = r(-\sin\theta)$

$y = r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial y}{\partial \theta} = r \cdot \cos\theta$

$$\frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$= r(\cos^2\theta + \sin^2\theta)$

$= r(1) = \underline{r}$

$x = r\cos\theta \qquad y = r\sin\theta.$

S.O.B. $\qquad$ S.O.B

$x^2 = r^2\cos^2\theta \qquad y^2 = r^2\sin^2\theta$

$x^2+y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$

$y/x = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \tan\theta.$

$\theta = \tan^{-1}(y/x)$

$r\theta < {x \atop y}$

$$J' = \frac{\partial(r\theta)}{\partial(xy)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cancel{2}x$

$= \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} (\cancel{2}y)$

$= \frac{y}{\sqrt{x^2+y^2}}$

$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2} \cdot y\left(\frac{-1}{x^2}\right)$

$= \frac{-y}{x^2+y^2}$

$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2} \frac{1}{x}$

$= \frac{x}{x^2+y^2}$

$$J' = \frac{\partial(r\theta)}{\partial(xy)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{+y}{\sqrt{x^2+y^2}} \\ \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{\sqrt{x^2+y^2}\,(x^2+y^2)} + \frac{y^2}{\sqrt{x^2+y^2}\,(x^2+y^2)}$$

$$= \frac{\cancel{x^2+y^2}}{\sqrt{x^2+y^2}\,\cancel{(x^2+y^2)}}$$

$$= \frac{1}{r}$$

$$\therefore J \cdot J' = \cancel{r} \cdot \frac{1}{\cancel{r}} = 1$$

$$\therefore \frac{\partial(x\,y)}{\partial(r\theta)} \cdot \frac{\partial(r\theta)}{\partial(xy)} = 1$$

⑤ If $x = UV$; $y = \frac{U}{V}$ prove that $\frac{\partial(xy)}{\partial(UV)} \times \frac{\partial(UV)}{\partial(xy)} = 1$.

sol: $x = UV \quad y = \frac{U}{V}$ $\qquad x\,y < \frac{U}{V}$

$$J = \frac{\partial(xy)}{\partial(UV)} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix} \qquad \frac{\partial(xy)}{\partial(UV)} = \begin{vmatrix} V & U \\ \frac{1}{V} & \frac{-U}{V^2} \end{vmatrix}$$

$x = UV$

$\frac{\partial x}{\partial U} = V$

$\frac{\partial x}{\partial V} = U$

$y = \frac{U}{V}$

$\frac{\partial y}{\partial U} = \frac{1}{V}$

$\frac{\partial y}{\partial V} = U\frac{-1}{V^2}$

$$= \frac{-\cancel{V}U}{V^{\cancel{2}}} - \frac{U}{V}$$

$$J = \frac{-2U}{V}$$

$x = UV$

$U = \frac{x}{V}$

$U = \frac{x}{U/y}$

$U = \frac{xy}{U}$

$U^2 = xy \Rightarrow U = \sqrt{xy}$

$y = \frac{U}{V}$

$V = \frac{U}{y} \Rightarrow V = \frac{x/V}{y}$

$V = \frac{x}{yV}$

$V^2 = \frac{x}{y} \Rightarrow V = \frac{\sqrt{x}}{\sqrt{y}}$

$UV < \frac{x}{y}$

$$J' = \frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix}$$

$U = \sqrt{xy}$

$\frac{\partial U}{\partial x} = \frac{1}{2\sqrt{xy}}\,y$

$\frac{\partial U}{\partial y} = \frac{1}{2\sqrt{xy}}\,x$

$V = \sqrt{\frac{x}{y}}$

$\frac{\partial V}{\partial x} = \frac{1}{\sqrt{y}}\,\frac{1}{2\sqrt{x}}$

$\frac{\partial V}{\partial y} = \frac{-\sqrt{x}}{2y\sqrt{y}}$

$$\frac{\partial(uv)}{\partial(xy)} = \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ \frac{1}{2\sqrt{xy}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix}$$

$$= \frac{-\cancel{y}\cancel{\sqrt{x}}}{4\cancel{y}\cancel{\sqrt{x}}y} - \frac{x}{4(\sqrt{xy})^2}$$

$$= \frac{-1}{4y} - \frac{\cancel{x}}{4\cancel{x}y}$$

$$= \frac{-1}{4y} - \frac{1}{4y} = \frac{-\cancel{2}}{\cancel{4}_2 y} = \frac{-1}{2y}$$

$$= \frac{-1}{2u/v} = \frac{-v}{2u}.$$

$$J \cdot J' = \frac{-\cancel{2u}}{\cancel{v}} \times \frac{-\cancel{v}}{\cancel{2u}}$$

$$= 1$$

$$\therefore \frac{\partial(xy)}{\partial(uv)} \times \frac{\partial(uv)}{\partial(xy)} = 1$$

⑥ If $x = r\cos\theta$, $y = r\sin\theta$ show that $\frac{\partial(xy)}{\partial(r\theta)} = r$.

Sol:

$x = r\cos\theta \qquad y = r\sin\theta \qquad x\,y < \begin{matrix} r \\ \theta \end{matrix}$

$$J = \frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} \end{vmatrix}$$

$x = r\cos\theta$

$\frac{\partial x}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial\theta} = r(-\sin\theta)$

$y = r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial y}{\partial\theta} = r\cos\theta$

$$\frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$\boxed{\frac{\partial(xy)}{\partial(r\theta)} = r.}$$

$x = r\cos\theta. \qquad y = r\sin\theta$

$x^2 = r^2\cos^2\theta \qquad y^2 = r^2\sin^2\theta$

$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$x^2+y^2 = r^2$

$r = \sqrt{x^2+y^2}$

$y/x = \frac{r\sin\theta}{r\cos\theta}$

$\tan\theta = y/x \Rightarrow \theta = \tan^{-1}(y/x)$

$r,\theta < \begin{matrix} x \\ y \end{matrix}$

$$J' = \frac{\partial(r\theta)}{\partial(xy)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix}$$

Extra :

$r = \sqrt{x^2+y^2}$

$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x)$

$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}y)$

$\theta = \tan^{-1}(y/x)$

$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\, y\left(\frac{-1}{x^2}\right) = \frac{-y}{x^2+y^2}$

$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{x}{x^2+y^2}$

$$\frac{\partial(r\theta)}{\partial(xy)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{x^2+y^2\sqrt{x^2+y^2}} + \frac{y^2}{x^2+y^2\sqrt{x^2+y^2}}$$

$$= \frac{\cancel{x^2+y^2}}{\cancel{x^2+y^2}\sqrt{x^2+y^2}} = \frac{1}{r}.$$

⑧. If $x = r\cos\theta$; $y = r\sin\theta$, $z = z$ evaluate $\frac{\partial(xyz)}{\partial(r\theta z)}$

$x = r\cos\theta$, $y = r\sin\theta$ $z = z$

$xyz < \begin{matrix} r \\ \theta \\ z \end{matrix}$

$$\frac{\partial(xyz)}{\partial(r\theta z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$x = r\cos\theta$	$y = r\sin\theta$	$z = z$
$\frac{\partial x}{\partial r} = \cos\theta$	$\frac{\partial y}{\partial r} = \sin\theta$	$\frac{\partial z}{\partial r} = 0$
$\frac{\partial x}{\partial \theta} = r(-\sin\theta)$	$\frac{\partial y}{\partial \theta} = r\cos\theta$	$\frac{\partial z}{\partial \theta} = 0$
$\frac{\partial x}{\partial z} = 0$	$\frac{\partial y}{\partial z} = 0$	$\frac{\partial z}{\partial z} = 1$

$$\frac{\partial(xyz)}{\partial(r\theta z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta[r\cos\theta - 0] + r\sin\theta[\sin\theta - 0]$$

$$= r\cos^2\theta + r\sin^2\theta.$$

$$= r[\cos^2\theta + \sin^2\theta]$$

$$= r.$$

⑨ If $U = 2xy$ $V = x^2 - y^2$ $x = r\cos\theta$, $y = r\sin\theta$ evaluate $\frac{\partial(UV)}{\partial(r\theta)}$.

Sol: $U = 2xy$. $V = x^2 - y^2$ $x = r\cos\theta$, $y = r\sin\theta$.

$UV < \begin{matrix} x \\ y \end{matrix} > r\theta.$

$$\frac{\partial(UV)}{\partial(r\theta)} = \frac{\partial(UV)}{\partial(xy)} \cdot \frac{\partial(xy)}{\partial(r\theta)}$$

$$\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}$$

$U = 2xy$ | $V = x^2 - y^2$

$\frac{\partial U}{\partial x} = 2y$ | $\frac{\partial V}{\partial x} = 2x$

$\frac{\partial U}{\partial y} = 2x$ | $\frac{\partial V}{\partial y} = -2y$

$$= -4y^2 - 4x^2$$
$$= \underline{-4(x^2+y^2)}.$$

$$\frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$x = r\cos\theta$ | $y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$ | $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = r(-\sin\theta)$ | $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$= r\cos^2\theta + r\sin^2\theta$$
$$= r(\cos^2\theta + \sin^2\theta)$$
$$= \underline{r}$$
$$= \sqrt{x^2+y^2}$$

$$\therefore \frac{\partial(UV)}{\partial(r\theta)} = -4(x^2+y^2)\sqrt{x^2+y^2} = \underline{-4(x^2+y^2)^{3/2}}$$
$$(\text{or})$$
$$= -4(r^2)^{3/2} = \underline{-4r^3}$$

(10) If $x = \sqrt{vw}$ $y = \sqrt{wu}$ $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$ $w = r\cos\theta$. Then evaluate $\frac{\partial(xyz)}{\partial(r\theta\phi)}$

$$\frac{\partial(xyz)}{\partial(r\theta\phi)} = \frac{\partial(xyz)}{\partial(uvw)} \cdot \frac{\partial(uvw)}{\partial(r\theta\phi)}$$

$\cancel{xyz}$

$$\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\} \to uvw \to r\theta\phi$$

$$\frac{\partial(xyz)}{\partial(uvw)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = \sqrt{vw}$ | $y = \sqrt{wu}$ | $z = \sqrt{uv}$

$\frac{\partial x}{\partial u} = 0$ | $\frac{\partial y}{\partial u} = \frac{1}{2\sqrt{wu}}$ | $\frac{\partial z}{\partial u} = \frac{1}{2\sqrt{uv}}$

$\frac{\partial x}{\partial v} = \frac{1}{2\sqrt{vw}}$ | $\frac{\partial y}{\partial v} = 0$ | $\frac{\partial z}{\partial v} = \frac{1}{2\sqrt{uv}}$

$\frac{\partial x}{\partial w} = \frac{1}{2\sqrt{vw}}$ | $\frac{\partial y}{\partial w} = \frac{1}{2\sqrt{wu}}$ | $\frac{\partial z}{\partial w} = 0$

$$\frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = \begin{vmatrix} 0 & \frac{1}{2\sqrt{vw}} & \frac{1}{2\sqrt{vw}} \\ \frac{1}{2\sqrt{wu}} & 0 & \frac{1}{2\sqrt{wu}} \\ \frac{1}{2\sqrt{uv}} & \frac{1}{2\sqrt{uv}} & 0 \end{vmatrix}$$

$$= \frac{1}{2\sqrt{vw}} \cdot \frac{1}{2\sqrt{wu}} \cdot \frac{1}{2\sqrt{uv}} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \frac{1}{8uvw}\left[-1(\cdot 0-1)+1(1-0)\right]$$

$$= \frac{1}{8uvw}(\cdot 1+1)$$

$$= \frac{1}{\cancel{8}_4 uvw}(\cancel{2}) = \frac{1}{4uvw}.$$

$$\frac{\partial(uvw)}{\partial(r\theta\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix}$$

$u = r\sin\theta\cos\phi$

$\frac{\partial u}{\partial r} = \sin\theta\cdot\cos\phi$

$\frac{\partial u}{\partial \theta} = r\cos\phi\cos\theta$

$\frac{\partial u}{\partial \phi} = r\sin\theta(-\sin\phi)$

$v = r\sin\theta\sin\phi$

$\frac{\partial v}{\partial r} = \sin\theta\sin\phi$

$\frac{\partial v}{\partial \theta} = r\sin\phi\cos\theta$

$\frac{\partial v}{\partial \phi} = r\sin\theta\cos\phi$

$w = r\cos\theta.$

$\frac{\partial w}{\partial r} = \cos\theta$

$\frac{\partial w}{\partial \theta} = r(-\sin\theta)$

$\frac{\partial w}{\partial \phi} = 0.$

$$\frac{\partial(uvw)}{\partial(r\theta\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & -r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r\cdot r \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \sin\theta\sin\phi & -\cos\theta\sin\phi & \sin\theta\cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\left[\sin\theta\cos\phi(0+\sin^2\theta\cdot\cos\phi) - \cos\theta\cos\phi(0-\sin\theta\cos\theta\cos\phi) - \sin\theta\sin\phi(-\sin^2\theta\sin\phi + \cos^2\theta\sin\phi)\right]$$

$$= r^2\left[\sin^3\theta\cdot\cos^2\phi + \sin\theta\cos^2\theta\cos^2\phi + \sin^3\theta\cdot\sin^2\phi - \sin\theta\cos^2\theta\sin^2\phi\right]$$

$$= r^2\left[\sin^3\theta(\cos^2\phi+\sin^2\phi) + \sin\theta\cos^2\theta(\cos^2\phi-\sin^2\phi)\right]$$

$$= r^2\left[\sin^3\theta + \sin\theta\cdot\cos^2\theta.\right.$$

(11) $y_1 = \dfrac{x_2 x_3}{x_1}$; $y_2 = \dfrac{x_3 x_1}{x_2}$; $y_3 = \dfrac{x_1 x_2}{x_3}$ show that $\dfrac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = 4$.

Sol:-

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial y_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$y_1, y_2, y_3 < x_1, x_2, x_3$

$y_1 = \dfrac{x_2 x_3}{x_1}$	$y_2 = \dfrac{x_3 x_1}{x_2}$	$y_3 = \dfrac{x_1 x_2}{x_3}$
$\dfrac{\partial y_1}{\partial x_1} = x_2 x_3\left(\dfrac{-1}{x_1^2}\right)$	$\dfrac{\partial y_2}{\partial x_1} = \dfrac{x_3}{x_2}$	$\dfrac{\partial y_3}{\partial x_1} = \dfrac{x_2}{x_3}$
$\dfrac{\partial y_1}{\partial x_2} = \dfrac{x_3}{x_1}$	$\dfrac{\partial y_2}{\partial x_2} = x_3 x_1\left(\dfrac{-1}{(x_2)^2}\right)$	$\dfrac{\partial y_3}{\partial x_2} = \dfrac{x_1}{x_3}$
$\dfrac{\partial y_1}{\partial x_3} = \dfrac{x_2}{x_1}$	$\dfrac{\partial y_2}{\partial x_3} = \dfrac{x_1}{x_2}$	$\dfrac{\partial y_3}{\partial x_3} = x_1 x_2\left(\dfrac{-1}{x_3^2}\right)$

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_1 x_3}{x_1^2} & \frac{x_1 x_2}{x_1^2} \\ \frac{x_2 x_3}{x_2^2} & \frac{-x_1 x_3}{x_2^2} & \frac{x_1 x_2}{x_2^2} \\ \frac{x_2 x_3}{x_3^2} & \frac{x_1 x_3}{x_3^2} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2}\cdot\frac{1}{x_2^2}\cdot\frac{1}{x_3^2}(x_2x_3)(x_1x_3)(x_1x_2)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{\cancel{x_1^2 x_2^2 x_3^2}}{\cancel{x_1^2 x_2^2 x_3^2}}\left[-1(1-1)-1(-1-1)+1(1+1)\right]$$

$$= -1(0)-1(-2)+1(2)$$

$$= 0+2+2$$

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1, x_2 x_3)} = \underline{\underline{4}}$$

(13) $U = \frac{y^2}{2x}$, $V = \frac{x^2+y^2}{2x}$ find $\frac{\partial(UV)}{\partial(xy)}$

$$\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} \qquad UV < \begin{matrix} x \\ y \end{matrix}$$

$U = \frac{y^2}{2x}$

$\frac{\partial U}{\partial x} = \frac{y^2}{2}\left(\frac{-1}{x^2}\right)$

$\frac{\partial U}{\partial y} = \frac{1}{\cancel{2}x}(\cancel{2}y) = y/x$

$V = \frac{x^2+y^2}{2x}$

$\frac{\partial V}{\partial x} = \frac{2x(2x+0)-(x^2+y^2)2}{(2x)^2} = \frac{4x^2-2x^2-2y^2}{4x^2} = \frac{\cancel{2}(x^2-y^2)}{\cancel{4}_2 x^2} = \frac{x^2-y^2}{2x^2}$

$\frac{\partial V}{\partial y} = \frac{1}{2x}(0+2y) = \frac{1}{\cancel{2}x}(\cancel{2}y) = y/x$

$$\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{-y^2}{2x^2} & y/x \\ \frac{x^2-y^2}{2x^2} & y/x \end{vmatrix}$$

$$= \frac{1}{2x^2}\cdot y/x \begin{vmatrix} -y^2 & 1 \\ x^2-y^2 & 1 \end{vmatrix}$$

$$= \frac{y}{2x^3}\left[-\cancel{y^2}-x^2+\cancel{y^2}\right]$$

$$= \frac{-\cancel{x^2}y}{2x^{\cancel{3}}} = \underline{\underline{\frac{-y}{2x}}}$$

(15) $U = xyz$, $V = xy+yz+zx$, $W = x+y+z$ show that

$$\frac{\partial(UVW)}{\partial(xyz)} = (x-y)(y-z)(z-x).$$

$$UVW \leftarrow \begin{matrix} x \\ y \\ z \end{matrix}$$

$$\frac{\partial(UVW)}{\partial(xyz)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial z} \end{vmatrix}$$

$U = xyz$ | $V = xy+yz+zx$ | $W = x+y+z$

$$\frac{\partial U}{\partial x} = yz \qquad \frac{\partial V}{\partial x} = y+0+z = y+z \qquad \frac{\partial W}{\partial x} = 1+0+0 = 1$$

$$\frac{\partial U}{\partial y} = xz \qquad \frac{\partial V}{\partial y} = x+z+0 = x+z \qquad \frac{\partial W}{\partial y} = 0+1+0 = 1$$

$$\frac{\partial U}{\partial z} = xy \qquad \frac{\partial V}{\partial z} = 0+y+x = x+y \qquad \frac{\partial W}{\partial z} = 0+0+1 = 1$$

$$\frac{\partial(UVW)}{\partial(xyz)} = \begin{vmatrix} yz & zx & xy \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(z+\cancel{x}-\cancel{x}-y) - zx(\cancel{y}+z-x-\cancel{y}) + xy(y+\cancel{z}-\cancel{z}-x)$$

$$= yz(z-y) - zx(z-x) + xy(y-x)$$

$$= yz^2 - y^2z - z^2x + zx^2 + xy^2 - x^2y$$

$$= x^2($$

(19) If $x^2+y^2+u^2-v^2=0$ and $uv+xy=0$, prove that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2-y^2}{u^2+v^2}$

Let us take $f_1 = x^2+y^2+u^2-v^2$, $f_2 = uv+xy$.

$f_1 f_2$ $u\ v\ x\ y$

$$\frac{\partial f_1}{\partial x} = 2x \qquad \frac{\partial f_2}{\partial x} = y$$

$$\frac{\partial f_1}{\partial y} = 2y \qquad \frac{\partial f_2}{\partial y} = x$$

$$\frac{\partial f_1}{\partial u} = 2u \qquad \frac{\partial f_2}{\partial u} = v$$

$$\frac{\partial f_1}{\partial v} = -2v \qquad \frac{\partial f_2}{\partial v} = u$$

We know that,

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix}$$

$$= 2x^2 - 2y^2$$

$$= 2(x^2-y^2)$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}$$

$$= 2u^2 + 2v^2$$

$$= 2(u^2+v^2)$$

$$\therefore \frac{\partial(UV)}{\partial(xy)} = (-1)^2 \frac{\cancel{\partial}(x^2-y^2)}{\cancel{\partial}(U^2+V^2)} = \frac{x^2-y^2}{U^2+V^2}.$$

3/12/2019
Tuesday

Functional Dependence

② If $U = \frac{x+y}{1-xy}$ and $V = \tan^{-1}x + \tan^{-1}y$.

$$J = \frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix}$$

$U, V \to xy.$

$$U = \frac{x+y}{1-xy}$$

$$\frac{\partial U}{\partial x} = \frac{(1-xy)(1)-(x+y)(0-y)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$= \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial U}{\partial y} = \frac{(1-xy)(1)-(x+y)(0-x)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2}$$

$$= \frac{1+x^2}{(1-xy)^2}.$$

$$V = \tan^{-1}x + \tan^{-1}y$$

$$\frac{\partial V}{\partial x} = \frac{1}{1+x^2} + 0.$$

$$= \frac{1}{1+x^2}$$

$$\frac{\partial V}{\partial y} = \cancel{\tan^{-1}x}\cdot 0 + \frac{1}{1+y^2}$$

$$= \frac{1}{1+y^2}.$$

$$\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2}\begin{vmatrix} 1+y^2 & 1+x^2 \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2}\left[1-1\right]$$

$$= \frac{1}{(1-xy)^2}(0).$$

$$\boxed{\frac{\partial(UV)}{\partial(xy)} = 0.}$$

$\therefore$ U and V are functionally dependent.

That is, there is a relation b/w U and V.

$V = \tan^{-1}x + \tan^{-1}y$

$\tan V = \tan[\tan^{-1}x + \tan^{-1}y]$

$= \dfrac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x).\tan(\tan^{-1}y)}$

$= \dfrac{x+y}{1-xy}$

$\tan V = U$

② If $u = x+y+z$, $u^2v = y+z$, $u^3w = z$.

$u = x+y+z \qquad u^2v = y+z \qquad u^3w = z$

$u = x + u^2v \qquad u^2v = y + u^3w \qquad z = u^3w.$

$x = u - u^2v \qquad y = u^2v - u^3w$

$$J = \frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\rangle u, v, w$

$x = u - u^2v$

$\frac{\partial x}{\partial u} = 1 - v(2u) = 1 - 2uv$

$\frac{\partial x}{\partial v} = 0 - u^2(1) = -u^2$

$\frac{\partial x}{\partial w} = 0.$

$y = u^2v - u^3w$

$\frac{\partial y}{\partial u} = v(2u) - w(3u^2) = 2uv - 3u^2w$

$\frac{\partial y}{\partial v} = u^2(1) - 0 = u^2$

$\frac{\partial y}{\partial w} = 0 - u^3 = -u^3$

$z = u^3w$

$\frac{\partial z}{\partial u} = w(3u^2) = 3u^2w$

$\frac{\partial z}{\partial v} = 0$

$\frac{\partial z}{\partial w} = u^3(1) = u^3$

$$J = \frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = \begin{vmatrix} 1-2uv & -u^2 & 0 \\ 2uv-3u^2w & u^2 & -u^3 \\ 3u^2w & 0 & u^3 \end{vmatrix}$$

$$= u^2.u^3 \begin{vmatrix} 1-2uv & -1 & 0 \\ 2uv-3u^2w & 1 & -1 \\ 3u^2w & 0 & 1 \end{vmatrix}$$

$$= u^5 \left[1 - 2uv(1+0) + 1(2uv - 3u^2w + 3u^2w) + 0 \right]$$

$$= u^5 \left[1 - \cancel{2uv} + \cancel{2uv} - \cancel{3u^2w} + \cancel{3u^2w} \right]$$

$$\frac{\partial(xyz)}{\partial(uvw)} = u^5.$$

$\therefore$ x,y,z are not functionally dependent.

Hence there is no relation between x, y and z.

(4) If $u = \frac{x-y}{x+y}$, $v = \frac{xy}{(x+y)^2}$

Sol:- $u = \frac{x-y}{x+y}$ $v = \frac{xy}{(x+y)^2}$

$$J = \frac{\partial(u\ v)}{\partial(xy)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$\begin{matrix} u \\ v \end{matrix} > xy$

$$u = \frac{x-y}{x+y}$$

$$\frac{\partial u}{\partial x} = \frac{(x+y)(1-0) - (x-y)(1+0)}{(x+y)^2}$$

$$= \frac{\cancel{x}+y-\cancel{x}+y}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x+y)(0-1) - (x-y)(0+1)}{(x+y)^2}$$

$$= \frac{-x\cancel{-y} - x\cancel{+y}}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

$$v = \frac{xy}{(x+y)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x+y)^2 y - xy\,2(x+y)}{[(x+y)^2]^2}$$

$$= \frac{x^2y + y^3 + \cancel{2xy^2} - 2x^2y - \cancel{2xy^2}}{(x+y)^4}$$

$$= \frac{y^3 - x^2y}{(x+y)^4}$$

$$\frac{\partial v}{\partial y} = \frac{(x+y)^2 x - xy\,2(x+y)}{((x+y)^2)^2}$$

$$= \frac{(x^2+y^2+2xy)x - 2x^2y - 2xy^2}{(x+y)^4}$$

$$= \frac{x^3 + xy^2 + \cancel{2x^2y} - \cancel{2x^2y} - 2xy^2}{(x+y)^4}$$

$$= \frac{x^3 - xy^2}{(x+y)^4}$$

$$\frac{\partial(uv)}{\partial(xy)} = \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{-2x}{(x+y)^2} \\ \frac{y(y^2-x^2)}{(x+y)^4} & \frac{x(x^2-y^2)}{(x+y)^4} \end{vmatrix}$$

$$= \frac{2x\cdot y}{(x+y)^2} \begin{vmatrix} 1 & -1 \\ \frac{y^2-x^2}{(x+y)^2} & \frac{x^2-y^2}{(x+y)^2} \end{vmatrix}$$

$$= \frac{2xy}{(x+y)^2} \left[\frac{x^2-y^2}{(x+y)^2} + \frac{y^2-x^2}{(x+y)^2} \right]$$

$$= \frac{2xy}{(x+y)^2} \left[\frac{\cancel{x^2}-\cancel{y^2}+\cancel{y^2}-\cancel{x^2}}{(x+y)^2} \right]$$

$$= \frac{2xy}{(x+y)^2}(0)$$

$$\boxed{\therefore \frac{\partial(u\,v)}{\partial(x\,y)} = 0}$$

⑤ $u = xy + yz + zx$, $v = x^2+y^2+z^2$, $w = x+y+z$.

$$J = \frac{\partial(u v w)}{\partial(x y z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u, v, w \leftarrow x, y, z$

$u = xy+yz+zx$	$v = x^2+y^2+z^2$	$w = x+y+z$
$\frac{\partial u}{\partial x} = y+0+z$	$\frac{\partial v}{\partial x} = 2x$	$\frac{\partial w}{\partial x} = 1$
$\frac{\partial u}{\partial y} = x+z$	$\frac{\partial v}{\partial y} = 2y$	$\frac{\partial w}{\partial y} = 1$
$\frac{\partial u}{\partial z} = y+x$	$\frac{\partial v}{\partial z} = 2z$	$\frac{\partial w}{\partial z} = 1$

$$\frac{\partial(u v w)}{\partial(x y z)} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$= y+z(2y-2z) - (x+z)(2x-2z) + (y+x)(2x-2y)$

$= 2y^2 - 2yz + 2yz - 2z^2 - 2x^2 + 2zx - 2zx + 2z^2 + 2xy - 2y^2 + 2x^2 - 2xy$

$= 0.$

$$\boxed{\therefore \frac{\partial(u\,v\,w)}{\partial(x y z)} = 0}$$

① If $U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $V = \sin^{-1}x + \sin^{-1}y$. Show that U, V are functionally dependent.

Sol:-

$$J = \frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix}$$

$U, V < {x \atop y}$

$$U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\frac{\partial U}{\partial x} = \sqrt{1-y^2} + y.\frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x)$$

$$= \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$\frac{\partial U}{\partial y} = x.\frac{1}{\cancel{2}\sqrt{1-y^2}}(-\cancel{2}y) + \sqrt{1-x^2}$$

$$= \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$V = \sin^{-1}x + \sin^{-1}y$$

$$\frac{\partial V}{\partial x} = \frac{1}{\sqrt{1-x^2}} + 0$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\partial V}{\partial y} = 0 + \frac{1}{\sqrt{1-y^2}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$$\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= \frac{1}{\sqrt{1-x^2}}\,\frac{1}{\sqrt{1-y^2}} \begin{vmatrix} \sqrt{1-x^2}.\sqrt{1-y^2} - xy & -xy + \sqrt{1-x^2}\sqrt{1-y^2} \\ 1 & 1 \end{vmatrix}$$

$$= \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\left[\sqrt{1-y^2-\cancel{x^2}+x^2y^2} - \cancel{xy} + \cancel{xy} - \sqrt{1-y^2-\cancel{x^2}+x^2y^2}\right]$$

$$= \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,(0)$$

$$= 0.$$

$$\boxed{\therefore \frac{\partial(UV)}{\partial(xy)} = 0}$$

$\therefore$ U,V are functionally dependent.

i.e., there is a relation b/w U and V.

$U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ $\qquad x = \sin y \Rightarrow y = \sin^{-1}x$

$\qquad\qquad\qquad\qquad\qquad\qquad y = \sin x \Rightarrow x = \sin^{-1}y$

$= \sin y\sqrt{1-\sin^2 x} + \sin x\sqrt{1-\sin^2 y}$

$= \sin y \cdot \cos x + \sin x \cdot \cos y$

$= \sin(x+y)$

$= \sin(\sin^{-1}y + \sin^{-1}x)$

$\boxed{U = \sin V}$

Maxima And Minima: (without constraints)

② $x^3y^2(1-x-y)$

sol:- Let $f(x,y) = x^3y^2(1-x-y)$

$f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = y^2(3x^2) - y^2 4x^3 - y^3(3x^2)$

$= 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$\frac{\partial f}{\partial y} = x^3(2y) - x^4(2y) - x^3 3y^2$

$= 2x^3y - 2x^4y - 3x^3y^2$

we have $\frac{\partial f}{\partial x} = 0$, $\qquad \frac{\partial f}{\partial y} = 0$

$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ $\qquad 2x^3y - 2x^4y - 3x^3y^2 = 0$

$x^2y^2(3-4x-3y) = 0$ $\qquad x^3y(2-2x-3y) = 0$

$x=0, y=0, 4x+3y-3=0$ $\qquad x=0, y=0, (2x+3y-2)=0$

if $x=0$, $2x+3y-2=0$ $\qquad$ if $y=0$, $2x+3y-2=0$

$3y-2=0$ $\qquad 2x-2=0$

$\boxed{y=2/3}$ $\qquad \boxed{x=1}$

$(0, 2/3)$ $\qquad (1,0)$

if $4x+3y-3=0$, $x=0$

$3y-3=0$

$\boxed{y=1}$

$(0,1)$

if $4x+3y-3=0$, $y=0$

$4x-3=0$

$\boxed{x=3/4}$

$(3/4,0)$

if $4x+3y-3=0$, $2x+3y-2=0$

$$\begin{array}{r} 4x+3y-3=0 \\ 2x+3y-2=0 \\ - \quad - \quad + \\ \hline 2x-1=0 \end{array}$$

$\boxed{x=1/2}$

$4(1/2)+3y-3=0$

$2+3y-3=0$

$\boxed{y=1/3}$

$(1/2, 1/3)$

$\therefore$ The stationary points are $(0,2/3)$, $(1,0)$, $(0,1)$, $(3/4,0)$, $(1/2,1/3)$.

$$r=\frac{\partial^2 f}{\partial x^2}=6xy^2-12x^2y^2-6xy^3$$

$$s=\frac{\partial^2 f}{\partial x\,\partial y}=6x^2y-8x^3y-9x^2y^2$$

$$t=\frac{\partial^2 f}{\partial y^2}=2x^3-2x^4-6x^3y$$

At the point $(0,2/3)$

$r=0$, $s=0$, $t=0$, $rt-s^2=0$.

At the point $(1,0)$

$r=0$, $s=0$, $t=0$, $rt-s^2=0$.

At the point $(0,1)$

$r=0$, $s=0$, $t=0$, $rt-s^2=0$

At the point $(3/4,0)$

$r=0$, $s=0$, $t=2(3/4)^3-2(3/4)^4$, $rt-s^2=0$

$=\frac{27}{128}$.

At the point $(1/2, 1/3)$

$r=6(1/2)(1/3)^2-12(1/2)^2(1/3)^2-6(1/2)(1/3)^3$

$=\cancel{\frac{1}{3}}-\cancel{\frac{1}{3}}-\frac{1}{9} \quad = -\frac{1}{9}$

$s=6(1/2)^2(1/3)-8(1/2)^3(1/3)-9(1/2)^2(1/3)^2$

$=\frac{1}{2}-\frac{1}{3}-\frac{1}{4} \quad = -\frac{1}{12}$

$t = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3)$

$= \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}.$

$\Rightarrow rt - s^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(\frac{-1}{12}\right)^2$

$= \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$

$rt - s^2 > 0, \quad r = -\frac{1}{9} < 0.$

$\therefore$ The function has maximum at the point $(1/2, 1/3)$.

Maximum value is $f = x^3y^2(1-x-y)$

$= (1/2)^3(1/3)^2(1 - 1/2 - 1/3)$

$= \frac{1}{72}\left(\frac{6-3-2}{6}\right)$

$= \frac{1}{72}\left(\frac{1}{6}\right) = \frac{1}{432}$

④ $\sin x + \sin y + \sin(x+y)$

Let $f(x,y) = \sin x + \sin y + \sin(x+y)$

$\frac{\partial f}{\partial x} = \cos x + \cos(x+y)$

$\frac{\partial f}{\partial y} = \cos y + \cos(x+y)$

We have $\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$

$\cos x + \cos(x+y) = 0$	$\cos y + \cos(x+y) = 0$
$2\cos\left(\frac{x+x+y}{2}\right)\cos\left(\frac{x-x-y}{2}\right) = 0$	$2\cos\left(\frac{y+x+y}{2}\right)\cos\left(\frac{y-x-y}{2}\right) = 0$
$\cos\left(\frac{2x+y}{2}\right)\cdot\cos(-y/2) = 0$	$\cos\left(\frac{x+2y}{2}\right)\cdot\cos(-x/2) = 0$
$\cos\frac{2x+y}{2} = 0, \quad \cos y/2 = 0$	$\cos\left(\frac{x+2y}{2}\right) = 0 \quad \cos x/2 = 0$
$\frac{2x+y}{2} = \cos^{-1}(0) \quad y/2 = \cos^{-1}(0)$	$\frac{x+2y}{2} = \cos^{-1}(0) \quad x/2 = \cos^{-1}(0)$
$\frac{2x+y}{2} = \frac{\pi}{2}, \frac{3\pi}{2}\ldots \quad y/2 = \pi/2, 3\pi/2\ldots$	$\frac{x+2y}{2} = \pi/2, 3\pi/2\ldots \quad x/2 = \pi/2, 3\pi/2\ldots$
$2x+y = \pi, 3\pi\ldots \quad y = \pi, 3\pi\ldots$	$x+2y = \pi, 3\pi\ldots \quad x = \pi, 3\pi\ldots$

$2x+y = \pi, \quad 2x+y = 3\pi, \quad y = \pi, \quad y = 3\pi$

$x+2y = \pi, \quad x+2y = 3\pi, \quad x = \pi, \quad x = 3\pi.$

if $2x+y=\pi$, $x+2y=2\pi$

$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$2x+y=\pi$
$2x+4y=2\pi$
$-3y=-\pi$
$\boxed{y=\pi/3}$

$2x+\pi/3=\pi$
$2x=\pi-\pi/3=2\pi/3$
$\boxed{x=\pi/3}$

if $2x+y=\pi$, $x+2y=3\pi$

$\left(-\frac{\pi}{3}, \frac{5\pi}{3}\right)$

if $2x+y=3\pi$, $x=3\pi$
$6\pi+y=3\pi \Rightarrow \boxed{y=-3\pi}$
$(3\pi,-3\pi)$

if $2x+y=\pi$, $x=\pi$

$2\pi+y=\pi$ $\quad$ $2x-\pi=\pi$
$\boxed{y=-\pi}$ $\quad$ $2x=2\pi$
$\boxed{x=\pi}$

$(\pi,-\pi)$

if $2x+y=\pi$, $x=3\pi$
$6\pi+y=\pi \Rightarrow \boxed{y=-5\pi}$
$2x-5\pi=\pi \Rightarrow 2x=6\pi$
$\boxed{x=3\pi}$
$(3\pi,-5\pi)$

if $2x+y=3\pi$, $x+2y=\pi$

$\left(\frac{5\pi}{3}, -\frac{\pi}{3}\right)$

if $2x+y=3\pi$, $x+2y=3\pi$

(π,π)

if $2x+y=3\pi$, $x=\pi$

(π,π)

if $y=\pi$, $x+2y=\pi$
$x+2\pi=\pi \Rightarrow \boxed{x=-\pi}$
$(-\pi,\pi)$

if $y=\pi$, $x+2y=3\pi$
$x+2\pi=3\pi \Rightarrow \boxed{x=\pi}$
(π,π)

if $y=3\pi$, $x+2y=\pi$
$x+6\pi=\pi \Rightarrow \boxed{x=-5\pi}$
$(-5\pi,3\pi)$

if $y=3\pi$, $x+2y=3\pi$
$x+6\pi=3\pi \Rightarrow \boxed{x=-3\pi}$
$(-3\pi,3\pi)$

$\therefore$ The stationary points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $\left(-\frac{\pi}{3}, \frac{5\pi}{3}\right)$, $(\pi,-\pi)$, $(3\pi,-5\pi)$, $\left(\frac{5\pi}{3}, -\frac{\pi}{3}\right)$
(π,π), (π,π), $(-\pi,\pi)$, (π,π), $(-5\pi,3\pi)$, $(-3\pi,3\pi)$, $(3\pi,-3\pi)$

$r=\frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y) = -\sin x - \sin(x+y)$

$s=\frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$t=\frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$r=-\sin\frac{\pi}{3} - \sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)$
$= -\frac{\sqrt{3}}{2} - \sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2} - \sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$$s = -\sin(\pi/3+\pi/3) = -\sin 2\pi/3 = -\sin\pi/3 = -\frac{\sqrt{3}}{2}.$$

$$t = -\sin\pi/3 - \sin(\pi/3+\pi/3)$$

$$= -\frac{\sqrt{3}}{2} - \sin 2\pi/3 = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}.$$

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= 3 - \frac{3}{4}$$

$$= \frac{12-3}{4} = \frac{9}{4} > 0.$$

$\therefore rs - t^2 > 0$, $\because r < 0$. $\because \boxed{r = -\sqrt{3}.}$

$\therefore$ The function has maximum at point $(\pi/3, \pi/3)$.

$\therefore$ Maximum value $f = \sin x + \sin y + \sin(x+y)$

$$= \sin\pi/3 + \sin\pi/3 + \sin(\pi/3+\pi/3)$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \sin 2\pi/3.$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \sin\pi/3.$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2}.$$

At $(-\pi/3, 5\pi/3)$

$$r = -\sin(-\pi/3) - \sin(-\pi/3+5\pi/3)$$

$$= \sin\pi/3 - \sin(4\pi/3)$$

$$= \frac{\sqrt{3}}{2} - \sin(\pi+\pi/3)$$

$$= \frac{\sqrt{3}}{2} + \sin\pi/3$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$s = -\sin(-\pi/3+5\pi/3)$$

$$= -\sin(4\pi/3)$$

$$= -\sin(\pi+\pi/3)$$

$$= \sin\pi/3$$

$$= \frac{\sqrt{3}}{2}$$

$$t = -\sin(5\pi/3) - \sin(-\pi/3+5\pi/3)$$

$$= -\sin(2\pi-\pi/3) - \sin(4\pi/3)$$

$$= \sin\pi/3 - \sin(\pi+\pi/3)$$

$$= \sin\pi/3 + \sin\pi/3 = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}$$

$rt - s^2$

$= (\sqrt{3})(\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2$

$= 3 - \frac{3}{4} = \frac{12-3}{4} = \frac{9}{4} > 0.$

$\therefore rt - s^2 > 0$, $r = \sqrt{3} > 0.$

$\therefore$ The function has minimum at the point $(-\pi/3, 5\pi/3)$

$\therefore$ Minimum value $f = \sin x + \sin y + \sin(x+y)$

$= \sin(-\pi/3) + \sin(5\pi/3) + \sin(-\pi/3 + 5\pi/3)$

$= -\sin \pi/3 + \sin(2\pi - \pi/3) + \sin(4\pi/3)$

$= -\frac{\sqrt{3}}{2} - \sin \pi/3 + \sin(\pi + \pi/3)$

$= \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - \sin \pi/3$

$= \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$

$= \frac{-3\sqrt{3}}{2}.$

At the points $(\pi, -\pi)$, $(3\pi, -5\pi)$, (π, π), $(\pi\ \pi)$, $(-\pi, \pi)$, (π, π), $(-5\pi, 3\pi)$ $(-3\pi, 3\pi)$, $(3\pi, -3\pi)$.

$rt - s^2 = 0.$

$\therefore$ We need further investigation.

(7) $xy + \frac{a^3}{x} + \frac{a^3}{y}$.

Let $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$

$\frac{\partial f}{\partial x} = y + a^3\left(\frac{-1}{x^2}\right) + 0 = y - \frac{a^3}{x^2}$

$\frac{\partial f}{\partial y} = x + 0 + a^3\left(\frac{-1}{y^2}\right) = x - \frac{a^3}{y^2}.$

we have $\frac{\partial f}{\partial x} = 0$ $\qquad \frac{\partial f}{\partial y} = 0$

$y - \frac{a^3}{x^2} = 0$ $\qquad x - \frac{a^3}{y^2} = 0.$

$y = \frac{a^3}{x^2}$ $\qquad x = \frac{a^3}{y^2}$

Sub y value in $x = \frac{a^3}{y^2}$

$$x - \frac{a^3}{\left(\frac{a^3}{x^2}\right)^2} = 0$$

$$x - \frac{\cancel{a^3}}{(a^3)^{\cancel{2}}} x^4 = 0$$

$$x - \frac{x^4}{a^3} = 0$$

$$a^3x - x^4 = 0$$

$$x(a^3 - x^3) = 0$$

$[a^3 - b^3 = (a-b)(a^2+ab+b^2)]$

$$x = 0, \ (a^3 - x^3) = 0$$

$$x = 0, \ (a - x) = 0$$

$$\boxed{x = a}$$

Sub $x = a$ in $y = \frac{a^3}{x^2}$

$$y = \frac{a^{\cancel{3}}}{\cancel{a^2}}$$

$$\boxed{y = a}$$

$\therefore x = a, \ y = a.$

$\therefore$ The stationary point is (a, a).

At (a, a) $r = \frac{d^2f}{dx^2} = \frac{a^3}{x^{\cancel{4}}{}_3}(2\cancel{x}) = \frac{2a^3}{x^3}$

$$s = \frac{d^2f}{dxdy} = 1$$

$$t = \frac{d^2f}{dy^2} = \frac{a^3}{y^{\cancel{4}}{}_3}(2\cancel{y}) = \frac{2a^3}{y^3}$$

At the point (a, a)

$$r = \frac{2\cancel{a^3}}{\cancel{a^3}} = 2, \quad s = 1, \quad t = \frac{2\cancel{a^3}}{\cancel{a^3}} = 2.$$

$$rt - s^2$$

$$= (2)(2) - (1)^2$$

$$= 4 - 1 = 3 > 0.$$

$rt - s^2 > 0, \quad r = 2 > 0.$

$\therefore$ The function has minimum value at the point (a, a).

$\therefore$ Minimum value is $f = (a)(a) + \frac{a^3}{a} + \frac{a^3}{a}$

$$= a^2 + a^2 + a^2$$

$$= \underline{3a^2}$$

5

Multiple Integrals and their Applications

5.1 INTRODUCTION TO DEFINITE INTEGRALS AND DOUBLE INTEGRALS

Definite Integrals

The concept of definite integral

$$\int_a^b f(x)\,dx \qquad \ldots(1)$$

is physically the area under a curve $y = f(x)$, (say), the x-axis and the two ordinates $x = a$ and $x = b$. It is defined as the limit of the sum

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots + f(x_n)\delta x_n$$

when $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots, \delta x_n$ tends to zero.

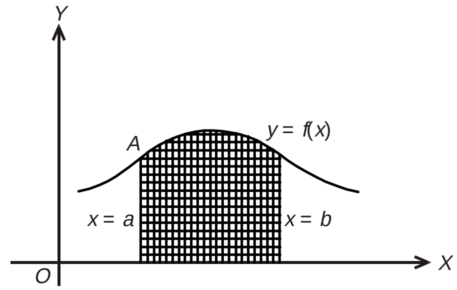


Fig. 5.1

Here $\delta x_1, \delta x_2, \ldots, \delta x_n$ are n subdivisions into which the range of integration has been divided and $x_1, x_2, \ldots, x_n$ are the values of x lying respectively in the Ist, 2nd, …, nth subintervals.

Double Integrals

A double integral is the counter part of the above definition in two dimensions.

Let $f(x, y)$ be a single valued and bounded function of two independent variables x and y defined in a closed region A in xy plane. Let A be divided into n elementary areas $\delta A_1, \delta A_2, \ldots, \delta A_n$.

Let (x_r, y_r) be any point inside the rth elementary area δA_r.

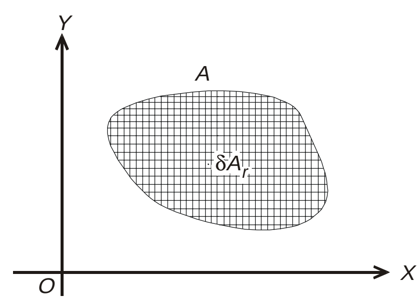


Fig. 5.2

Consider the sum

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \ldots + f(x_n, y_n)\delta A_n = \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \qquad \ldots(2)$$

Then the limit of the sum (2), if exists, as $n \to \infty$ and each sub-elementary area approaches to zero, is termed as '*double integral*' of $f(x, y)$ over the region A and expressed as $\iint_A f(x, y)\,dA$.

Thus $$\iint_A f(x,y)\,dA = \underset{\substack{n\to\infty \\ \delta A_r \to 0}}{Lt} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \ldots(3)$$

Observations: Double integrals are of limited use if they are evaluated as the limit of the sum. However, they are very useful for physical problems when they are evaluated by treating as successive single integrals.

Further just as the definite integral (1) can be interpreted as an area, similarly the double integrals (3) can be interpreted as a volume (see Figs. 5.1 and 5.2).

5.2 EVALUATION OF DOUBLE INTEGRAL

Evaluation of double integral $\iint_R f(x,y)\,dx\,dy$ is discussed under following three possible cases:

Case I: *When the region R is bounded by two continuous curves y = ψ(x) and y = ϕ(x) and the two lines (ordinates) x = a and x = b.*

In such a case, integration is first performed with respect to y keeping x as a constant and then the resulting integral is integrated within the limits $x = a$ and $x = b$.

Mathematically expressed as:

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{x=b}\left(\int_{y=\phi(x)}^{y=\Psi(x)} f(x,y)\,dy\right)dx$$

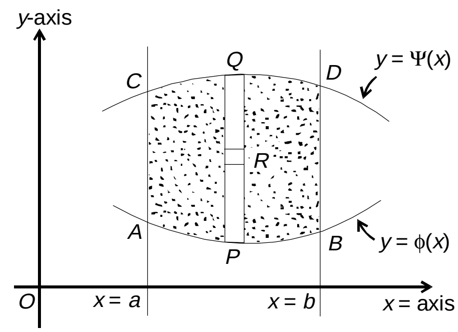


Fig. 5.3

Geometrically the process is shown in Fig. 5.3, where integration is carried out from inner rectangle (i.e., along the one edge of the 'vertical strip *PQ*' from *P* to *Q*) to the outer rectangle.

Case 2: *When the region R is bounded by two continuous curves x = ϕ(y) and x = Ψ(y) and the two lines (abscissa) y = a and y = b.*

In such a case, integration is first performed with respect to x. keeping y as a constant and then the resulting integral is integrated between the two limits $y = a$ and $y = b$.

Mathematically expressed as:

$$\iint_R f(x,y)\,dx\,dy = \int_{y=a}^{y=b}\left(\int_{x=\theta(y)}^{x=\Psi(y)} f(x,y)\,dx\right)dy$$

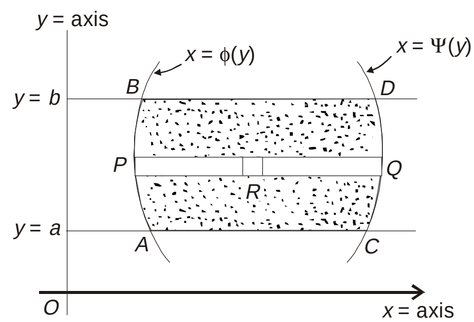


Fig. 5.4

Geometrically the process is shown in Fig. 5.4, where integration is carried out from inner rectangle (*i.e.*, along the one edge of the horizontal strip *PQ* from P to Q) to the outer rectangle.

Case 3: *When both pairs of limits are constants, the region of integration is the rectangle ABCD (say).*

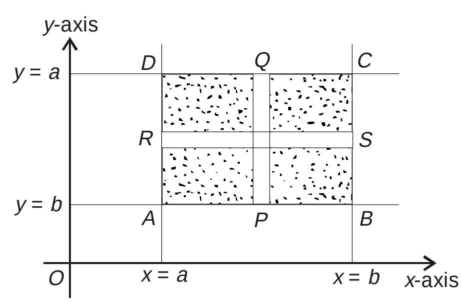


Fig. 5.5

In this case, it is immaterial whether $f(x, y)$ is integrated first with respect to x or y, the result is unaltered in both the cases (Fig. 5.5).

Observations: While calculating double integral, in either case, we proceed outwards from the innermost integration and this concept can be generalized to repeated integrals with three or more variable also.

Example 1: Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2+y^2)}\,dydx$ [Madras 2000; Rajasthan 2005].

Solution: Clearly, here $y = f(x)$ varies from 0 to $\sqrt{1+x^2}$ and finally x (as an independent variable) goes between 0 to 1.

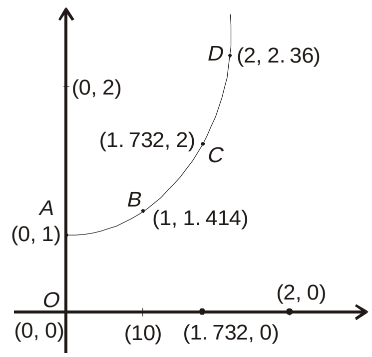


Fig. 5.6

$$I = \int_0^1\left(\int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}\,dy\right)dx$$

$$= \int_0^1\left(\int_0^{\sqrt{1+x^2}} \frac{1}{a^2+y^2}\,dy\right)dx\,,\ a^2 = (1+x^2)$$

$$= \int_0^1\left(\frac{1}{a}\tan^{-1}\frac{y}{a}\right)_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left(\tan^{-1}\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \tan^{-1}0\right)dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left(\frac{\pi}{4}-0\right)dx = \frac{\pi}{4}\left[\log\left\{x+\sqrt{1+x^2}\right\}\right]_0^1$$

$$= \frac{\pi}{4}\log\left(1+\sqrt{2}\right)$$

Example 2: Evaluate $\iint e^{2x+3y}\,dxdy$ over the triangle bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.

Solution: Here the region of integration is the triangle $OABO$ as the line $x + y = 1$ intersects the axes at points (1, 0) and (0, 1). Thus, precisely the region R (say) can be expressed as:

$0 \le x \le 1,\ 0 \le y \le 1 - x$ (Fig 5.7).

$$\therefore \qquad I = \iint_R e^{2x+3y}\,dxdy$$

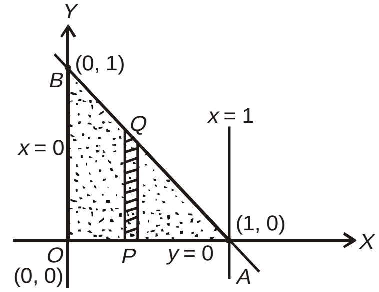


Fig. 5.7

$$= \int_0^1\left(\int_0^{1-x} e^{2x+3y}\,dy\right)dx$$

$$= \int_0^1\left[\frac{1}{3}e^{2x+3y}\right]_0^{1-x} dx$$

$$= \frac{1}{3}\int_0^1 \left(e^{3-x} - e^{2x}\right)dx$$

$$= \frac{1}{3}\left[\frac{e^{3-x}}{-1} - \frac{e^{2x}}{2}\right]_0^1$$

$$= \frac{-1}{3}\left[\left(e^2 + \frac{e^2}{2}\right) - \left(e^3 + \frac{1}{2}\right)\right]$$

$$= \frac{1}{6}\left[2e^3 - 3e^2 + 1\right] = \frac{1}{6}\left[(2e+1)(e-1)^2\right].$$

Example 3: Evaluate the integral $\iint_R xy(x+y)dxdy$ over the area between the curves $y = x^2$ and $y = x$.

Solution: We have $y = x^2$ and $y = x$ which implies $x^2 - x = 0$ i.e. either $x = 0$ or $x = 1$

Further, if $x = 0$ then $y = 0$; if $x = 1$ then $y = 1$. Means the two curves intersect at points (0, 0), (1, 1).

$\therefore$ The region R of integration is doted and can be expressed as: $0 \le x \le 1$, $x^2 \le y \le x$.

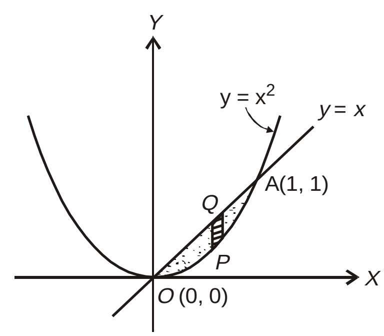


Fig. 5.8

$$\therefore \quad \iint_R xy(x+y)dxdy = \int_0^1 \left(\int_{x^2}^x xy(x+y)dy\right)dx$$

$$= \int_0^1 \left\{\left(x^2\frac{y^2}{2} + x\frac{y^3}{3}\right)_{x^2}^x\right\}dx$$

$$= \int_0^1 \left\{\left(\frac{x^4}{2} + \frac{x^4}{3}\right) - \left(\frac{x^6}{2} + \frac{x^7}{3}\right)\right\}dx$$

$$= \int_0^1 \left(\frac{5}{6}x^4 - \frac{1}{2}x^6 - \frac{1}{3}x^7\right)dx$$

$$= \left[\frac{5}{6}\times\frac{x^5}{5} - \frac{1}{2}\frac{x^7}{7} - \frac{1}{3}\frac{x^8}{8}\right]_0^1 = \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}$$

Example 4: Evaluate $\iint (x+y)^2\,dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

[UP Tech. 2004, 05; KUK, 2009]

Solution: For the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the region of integration can be considered as bounded by the curves $y=-b\sqrt{1-\frac{x^2}{a^2}}$, $y=b\sqrt{1-\frac{x^2}{a^2}}$ and finally x goes from $-a$ to a

$\therefore$ $$I=\iint(x+y)^2\,dx\,dy=\int_{-a}^{a}\left(\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}\left(x^2+y^2+2xy\right)dy\right)dx$$

$$I=\int_{-a}^{a}\left(\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}\left(x^2+y^2\right)dy\right)dx$$

[Here $\int 2xy\,dy=0$ as it has the same integral value for both limits i.e., the term xy, which is an odd function of y, on integration gives a zero value.]

$$I=4\int_0^a\left(\int_0^{b\sqrt{1-x^2/a^2}}\left(x^2+y^2\right)dy\right)dx$$

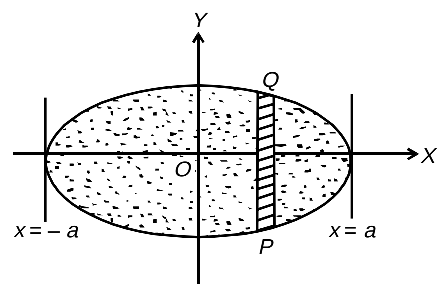


Fig. 5.9

$$I=4\int_0^a\left[x^2y+\frac{y^3}{3}\right]_0^{b\sqrt{1-x^2/a^2}}dx$$

$\Rightarrow$ $$I=4\int_0^a\left[x^2b\left(1-\frac{x^2}{a^2}\right)^{1/2}+\frac{b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}\right]dx$$

On putting $x=a\sin\theta$, $dx=a\cos\theta\,d\theta$; we get

$$I=4b\int_0^{\pi/2}\left(\left(a^2\sin^2\theta\cos\theta\right)+\frac{b^3}{3}\cos^3\theta\right)a\cos\theta\,d\theta$$

$$=4ab\int_0^{\pi/2}\left(a^2\sin^2\theta\cos^2\theta+\frac{b^3}{3}\cos^4\theta\right)d\theta$$

Now using formula $\displaystyle\int_0^{\pi/2}\sin^p x\cos^q x\,dx=\frac{\frac{1}{2}\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

and $\displaystyle\int_0^{\pi/2}\cos^n x\,dx=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{\sqrt{\pi}}{2}$, (in particular when $p=0$, $q=n$)

$$\iint(x+y)^2\,dx\,dy=4ab\left\{a^2\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{2\Gamma 3}+\frac{b^2}{3}\frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{2\Gamma 3}\right\}$$

$$= 4ab\left\{a^2\frac{\frac{\sqrt{\pi}}{2}\,\frac{\sqrt{\pi}}{2}}{2.2.1}+\frac{b^2}{3}\,\frac{\frac{3}{2}\,\frac{\sqrt{\pi}}{2}\sqrt{\pi}}{2.2.1}\right\} \qquad \left|\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right.$$

$$= 4ab\left\{\frac{\pi a^2}{16}+\frac{\pi b^2}{16}\right\}=\frac{\pi ab\left(a^2+b^2\right)}{4}$$

ASSIGNMENT 1

1. Evaluate $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{\left(1-x^2\right)\left(1-y^2\right)}}$

2. Evaluate $\iint_R xy\,dx\,dy,$ where A is the domain bounded by the x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$. [M.D.U., 2000]

3. Evaluate $\iint e^{ax+by}dy\,dx$, where R is the area of the triangle $x = 0$, $y = 0$, $ax + by = 1$ ($a > 0$, $b > 0$). [Hint: See example 2]

4. Prove that $\int_1^2\int_3^1\left(xy+e^y\right)dy\,dx=\int_3^1\int_1^2\left(xy+e^y\right)dx\,dy$.

5. Show that $\int_0^1 dx\int_0^1\frac{x-y}{\left(x+y\right)^3}dy \neq \int_0^1 dy\int_0^1\frac{x-y}{\left(x+y\right)^3}dx$.

6. Evaluate $\int_0^\infty\int_0^\infty e^{-x^2\left(1+y^2\right)}x\,dx\,dy$ [Hint: Put $x^2(1 + y^2) = t$, taking y as const.]

5.3 CHANGE OF ORDER OF INTEGRATION IN DOUBLE INTEGRALS

The concept of change of order of integration evolved to help in handling typical integrals occurring in evaluation of double integrals.

When the limits of given integral $\int_a^b\cdot\int_{y=\phi(x)}^{y=\Psi(x)} f\left(x,y\right)dy\,dx$ are clearly drawn and the region of integration is demarcated, then we can well change the order of integration be performing integration first with respect to x as a function of y (along the horizontal strip PQ from P to Q) and then with respect to y from c to d.

Mathematically expressed as:

$$I=\int_c^d\int_{x=\phi(y)}^{x=\Psi(y)} f\left(x,y\right)dx\,dy.$$

Sometimes the demarcated region may have to be split into two-to-three parts (as the case may be) for defining new limits for each region in the changed order.

Example 5: Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dy dx$ by changing the order of integration.

[KUK, 2000; NIT Kurukshetra, 2010]

Solution: In the above integral, y on vertical strip (say PQ) varies as a function of x and then the strip slides between $x = 0$ to $x = 1$.

Here $y = 0$ is the x-axis and $y = \sqrt{1-x^2}$ *i.e.*, $x^2 + y^2 = 1$ is the circle.

In the changed order, the strip becomes $P'Q'$, P' resting on the curve $x = 0$, Q' on the circle $x = \sqrt{1-y^2}$ and finally the strip $P'Q'$ sliding between $y = 0$ to $y = 1$.

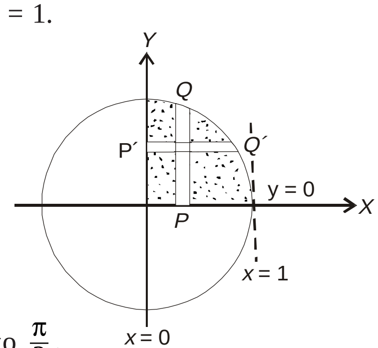


Fig. 5.10

$$\therefore \qquad I = \int_0^1 y^2 \left(\int_0^{\sqrt{1-y^2}} dx \right) dy$$

$$I = \int_0^1 y^2 \left[x\right]_0^{\sqrt{1-y^2}} dy$$

$$I = \int_0^1 y^2 \left(1 - y^2\right)^{\frac{1}{2}} dx$$

Substitute $y = \sin\theta$, so that $dy = \cos\theta\, d\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

$$I = \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta$$

$$I = \frac{(2-1)\cdot(2-1)}{4\cdot 2}\frac{\pi}{2} = \frac{\pi}{16}$$

$$\left[\because \int_0^{\frac{\pi}{2}} \sin^p\theta \cos\theta\, d\theta = \frac{(p-1)(p-3)\ldots(q-1)(q-3)}{(p+q)(p+q-2)\ldots\ldots} \times \frac{\pi}{2}, \text{ only if both } p \text{ and } q \text{ are + ve even integers}\right]$$

Example 6: Evaluate $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy dx$ by changing the order of integration.

[M.D.U. 2000; PTU, 2009]

Solution: In the given integral, over the vertical strip PQ (say), if y changes as a function of x such that P lies on the curve $y = \frac{x^2}{4a}$ and Q lies on the curve $y = 2\sqrt{ax}$ and finally the strip slides between $x = 0$ to $x = 4a$.

Here the curve $y = \frac{x^2}{4a}$ *i.e.* $x^2 = 4ay$ is a parabola with

$$\left.\begin{array}{lll} y = 0 & \text{implying} & x = 0 \\ y = 4a & \text{implying} & x = \pm 4a \end{array}\right\}$$

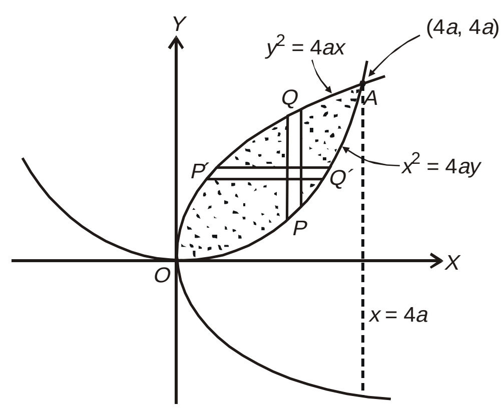


Fig. 5.11

i.e., it passes through (0, 0) $(4a, 4a)$, $(-4a, 4a)$.

Likewise, the curve $y = 2\sqrt{ax}$ or $y^2 = 4ax$ is also a parabola with

$$x = 0 \Rightarrow y = 0 \text{ and } x = 4a \Rightarrow y = \pm 4a$$

i.e., it passes through (0, 0), $(4a, 4a)$, $(4a, -4a)$.

Clearly the two curves are bounded at (0, 0) and $(4a, 4a)$.

$\therefore$ On changing the order of integration over the strip $P'Q'$, x changes as a function of y such that P' lies on the curve $y^2 = 4ax$ and Q' lies on the curve $x^2 = 4ay$ and finally $P'Q'$ slides between $y = 0$ to $y = 4a$.

whence
$$I = \int_0^{4a}\left(\int_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}} dx\right)dy$$

$$= \int_0^{4a}[x]_{\frac{y^2}{4a}}^{2\sqrt{ay}}\, dy$$

$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= \left[2\sqrt{a}\,\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{12a}\right]_0^{4a} = \frac{4\sqrt{a}}{3}(4a)^{\frac{3}{2}} - \frac{1}{12a}(4a)^3$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

Example 7: Evaluate $\int_0^a\int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}}(x^2 + y^2)\,dxdy$ by changing the order of integration.

Solution: In the given integral $\int_0^a\int_{x/a}^{\sqrt{x/a}}(x^2 + a^2)\,dx\,dy$, y varies along vertical strip PQ as a function of x and finally x as an independent variable varies from $x = 0$ to $x = a$.

Here $y = x/a$ *i.e.* $x = ay$ is a straight line and $y = \sqrt{x/a}$, i.e. $x = ay^2$ is a parabola.

For $x = ay$; $x = 0 \Rightarrow y = 0$ and $x = a \Rightarrow y = 1$.

Means the straight line passes through (0, 0), $(a, 1)$.

For $x = ay^2$; $x = 0 \Rightarrow y = 0$ and $x = a \Rightarrow y = \pm 1$.

Means the parabola passes through (0, 0), $(a, 1)$, $(a, -1)$,.

Further, the two curves $x = ay$ and $x = ay^2$ intersect at common points (0, 0) and $(a, 1)$.

On changing the order of integration,

$$\int_0^a\int_{x/a}^{\sqrt{x/a}}(x^2 + y^2)\,dxdy = \int_{y=0}^{y=1}\left(\int_{x=ay^2}^{x=ay}(x^2 + y^2)\,dxdy\right)$$

(at P')

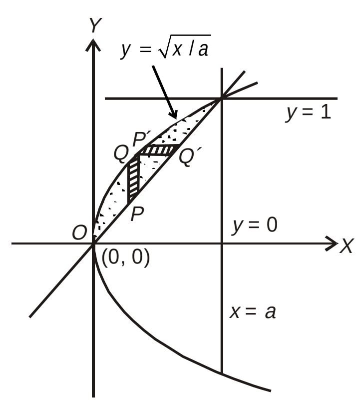


Fig. 5.12

$$I = \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_{ay^2}^{ay} dy$$

$$= \int_0^1 \left[\left(\frac{(ay)^3}{3} + ay.y^2\right) - \left(\frac{1}{3}(ay^2)^3 + ay^2 \cdot y^2\right)\right] dy$$

$$= \int_0^1 \left[\left(\frac{a^3}{3} + a\right)y^3 - \frac{a^3}{3}y^6 - ay^4\right] dy$$

$$= \left\{\left(\frac{a^3}{3} + a\right)\frac{y^4}{4} - \frac{a^3}{3}\frac{y^7}{7} - \frac{ay^5}{5}\right\}_0^1$$

$$= \left\{\left(\frac{a^3}{3\times 4} - \frac{a^3}{3\times 7}\right) + \left(\frac{a}{4} - \frac{a}{5}\right)\right\}$$

$$= \frac{a^3}{28} + \frac{a}{20} = \frac{a}{140}\left(5a^2 + 7\right).$$

Example 8: Evaluate $\displaystyle\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}}\, dy\, dx.$ **[SVTU, 2006]**

Solution: In the above integral, y on the vertical strip (say PQ) varies as a function of x and then the strip slides between $x = 0$ to $x = a$.

Here the curve $y = \sqrt{ax}$ *i.e.*, $y^2 = ax$ is the parabola and the curve $y = a$ is the straight line.

On the parabola, $x = 0 \Rightarrow y = 0$; $x = a \Rightarrow y = \pm a$ *i.e.*, the parabola passes through points (0, 0), (a, a) and (a, – a).

On changing the order of integration,

$$I = \int_0^a \left(\int_{\substack{x=0\\(at\ P')}}^{x=\frac{y^2}{a}} \frac{y^2}{\sqrt{y^4 - a^2x^2}}\, dx\right) dy$$

$$= \int_0^a \left(\int_0^{\frac{y^2}{a}} \frac{y^2}{a} \frac{1}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}\, dx\right) dy$$

$$= \int_0^a \frac{y^2}{a}\left[\sin^{-1}\frac{x}{\left(\frac{y^2}{a}\right)}\right]_0^{\frac{y^2}{a}} dy$$

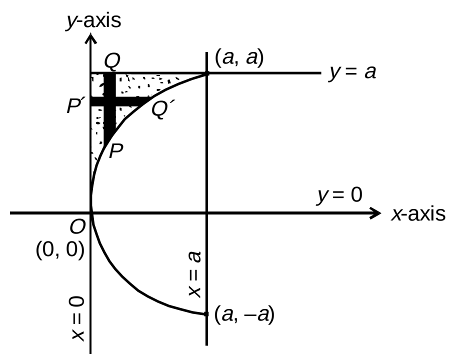


Fig. 5.13

$$= \int_0^a \frac{y^2}{a}\left[\sin^{-1}1 - \sin^{-1}0\right] dy$$

$$= \int_0^a \frac{y^2}{a}\frac{\pi}{2} dy = \frac{\pi}{2a}\frac{y^3}{3}\bigg|_0^a = \frac{\pi a^2}{6}.$$

Example 9: Change the order of integration of $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$ and hence evaluate the same.
[KUK, 2002; Cochin, 2005; PTU, 2005; UP Tech, 2005; SVTU, 2007]

Solution: In the given integral $\int_0^1\left(\int_{x^2}^{2-x} xydy\right)dx,$ on the vertical strip PQ(say), y varies as a function of x and finally x as an independent variable, varies from 0 to 1.

Here the curve $y = x^2$ is a parabola with

$$\left.\begin{array}{lll} y = 0 & \text{implying} & x = 0 \\ y = 1 & \text{implying} & x = \pm 1 \end{array}\right\}$$

i.e., it passes through (0, 0), (1, 1), (– 1, 1).

Likewise, the curve $y = 2 - x$ is straight line with

$$\left.\begin{array}{l} y = 0 \Rightarrow x = 2 \\ y = 1 \Rightarrow x = 1 \\ y = 2 \Rightarrow x = 0 \end{array}\right\}$$

i.e. it passes though (1, 1), (2, 0) and (0, 2)

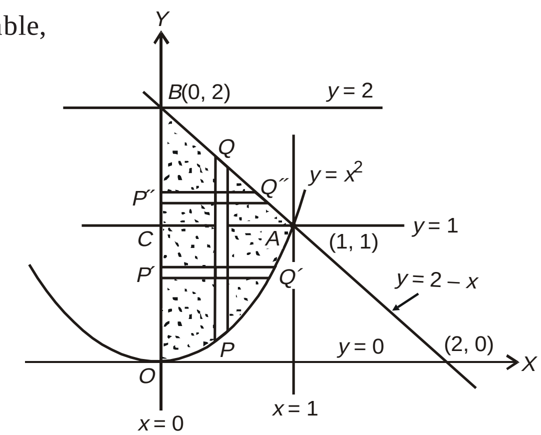


Fig. 5.14

On changing the order integration, the area $OABO$ is divided into two parts $OACO$ and $ABCA$. In the area $OACO$, on the strip $P'Q'$, x changes as a function of y from $x = 0$ to $x = \sqrt{y}$. Finally y goes from $y = 0$ to $y = 1$.

Likewise in the area ABCA, over the strip p”Q”, x changes as a function of y from $x = 0$ to $x = 2 - y$ and finally the strip P”Q” slides between $y = 1$ to $y = 2$.

$$\therefore \qquad \int_0^1\left(\int_0^{\sqrt{y}} xy\,dx\right)dy + \int_1^2\left(\int_0^{2-y} xy\,dx\right)dy$$

$$= \int_0^1\left(y\frac{x^2}{2}\bigg|_0^{\sqrt{y}}\right)dy + \int_1^2\left(y\frac{x^2}{2}\bigg|_0^{2-y}\right)dy$$

$$= \int_0^1 \frac{y^2}{2}dy + \int_1^2 \frac{y(2-y)^2}{2}dy$$

$$= \frac{1}{6} + \frac{1}{2}\left(2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right)_1^2$$

$$I = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Example 10: Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy dx$ **by changing order of integration.**

[KUK, 2000; MDU, 2003; JNTU, 2005; NIT Kurukshetra, 2008]

Soluton: Clearly over the strip PQ, y varies as a function of x such that P lies on the curve $y = x$ and Q lies on the curve $y = \sqrt{2 - x^2}$ and PQ slides between ordinates $x = 0$ and $x = 1$.

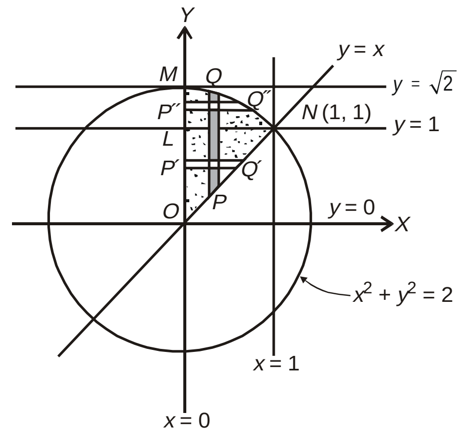


Fig. 5.15

The curves are $y = x$, a straight line and $y = \sqrt{2 - x^2}$, i.e. $x^2 + y^2 = 2$, a circle.

The common points of intersection of the two are (0, 0) and (1, 1).

On changing the order of integration, the same region $ONMO$ is divided into two parts $ONLO$ and $LNML$ with horizontal strips $P'Q'$ and $P''Q''$ sliding between $y = 0$ to $y = 1$ and $y = 1$ to $y = \sqrt{2}$ respectively.

whence
$$I = \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

Now the exp. $\frac{x}{x^2+y^2} = \frac{d}{dx}\left(x^2+y^2\right)^{\frac{1}{2}}$

$\therefore$
$$I = \int_0^1 \left[\left(x^2+y^2\right)^{\frac{1}{2}}\right]_0^y dy + \int_1^{\sqrt{2}} \left[\left(x^2+y^2\right)^{\frac{1}{2}}\right]_0^{\sqrt{2-y^2}} dy$$

$$I = \int_0^1 \left[\left(x^2+y^2\right)^{\frac{1}{2}}\right]_0^y dy + \int_1^{\sqrt{2}} \left[\left(x^2+y^2\right)^{\frac{1}{2}}\right]_0^{\sqrt{2-y^2}} dy$$

$$= \left(\sqrt{2}-1\right)\frac{y^2}{2}\Bigg|_0^1 + \left(\sqrt{2}y - \frac{y^2}{2}\right)_0^{\sqrt{2}} = \frac{1}{2}\left(\sqrt{2}-1\right)$$

Example 11: Evaluate $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dy\, dx$ **by changing the order of integration.**

Solution: Given $\int_{y=o}^{y=a} \left(\int_{x=a-\sqrt{a^2-y^2}}^{x=a+\sqrt{a^2-y^2}} dx \right) dy$

Clearly in the region under consideration, strip PQ is horizontal with point P lying on the curve $x = a - \sqrt{a^2 - y^2}$ and point Q lying on the curve $x = a + \sqrt{a^2 - y^2}$ and finally this strip slides between two abscissa $y = 0$ and $y = a$ as shown in Fig 5.16.

Now, for changing the order of integration, the region of integration under consideration is same but this time the strip is $P'Q'$ (vertical) which is a function of x with extremities P' and Q' at $y = 0$ and $y = \sqrt{2ax - x^2}$ respectively and slides between $x = 0$ and $x = 2a$.

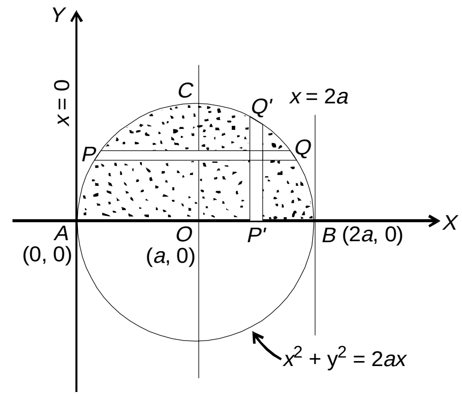


Fig. 5.16

Thus
$$I = \int_0^{2a}\left(\int_0^{\sqrt{2ax-x^2}} dy\right)dx = \int_0^{2a}\left[y\right]_0^{\sqrt{2ax-x^2}} dx$$
$$= \int_0^{2a}\sqrt{2ax - x^2}\,dx = \int_0^{2a}\sqrt{x}\sqrt{2a - x}\,dx$$

Take $\sqrt{x} = \sqrt{2a}\sin\theta$ so that $dx = 4a\sin\theta\cos\theta\,d\theta$,

Also, For $x = 0$, $\theta = 0$ and for $x = 2a$, $\theta = \dfrac{\pi}{2}$

Therefore,
$$I = \int_0^{\frac{\pi}{2}} \sqrt{2a}\sin\theta \cdot \sqrt{2a - 2a\sin^2\theta} \cdot 4a\sin\theta \cdot \cos\theta\, d\theta$$
$$= 8a^2\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\, d\theta = 8a^2 \cdot \frac{(2-1)(2-1)}{4(4-2)}\frac{\pi}{2} = \frac{\pi a^2}{2}$$
$$\left(\text{using}\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta = \frac{(p-1)(p-3)\ldots(q-1)(q-3)\ldots}{(p+q)(p+q-2)\ldots\ldots\ldots\ldots\ldots\ldots}\frac{\pi}{2},\right.$$
$$\left. p \text{ and } q \text{ both positive even integers}\right)$$

Example 12: Changing the order of integration, evaluate $\int_0^3\int_1^{\sqrt{4-y}}(x+y)\,dx\,dy$.
[MDU, 2001; Delhi, 2002; Anna, 2003; VTU, 2005]

Solution: Clearly in the given form of integral, x changes as a function of y (viz. $x = f(y)$ and y as an independent variable changes from 0 to 3.

Thus, the two curves are the straight line $x = 1$ and the parabola, $x = \sqrt{4-y}$ and the common area under consideration is ABQCA.

For changing the order of integration, we need to convert the horizontal strip PQ to a vertical strip P'Q' over which y changes as a function of x and it slides for values of $x = 1$ to $x = 2$ as shown in Fig. 5.17.

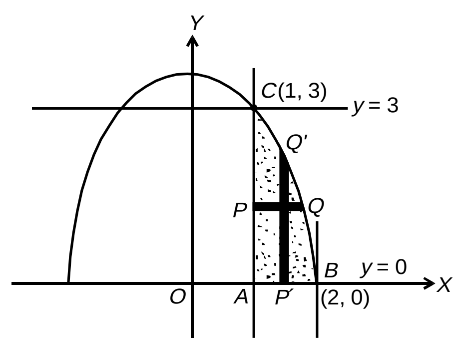


Fig. 5.17

$$\therefore \qquad I = \int_1^2\left(\int_0^{(4-x^2)}(x+y)\,dy\right)dx = \int_1^2\left[xy + \frac{y^2}{2}\right]_0^{4-x^2} dx$$

$$= \int_1^2 \left[x\left(4 - x^2\right) + \frac{\left(4 - x^2\right)^2}{2} \right] dx$$

$$= \int_1^2 \left[x\left(4 - x^2\right) + \left(8 + \frac{x^4}{2} - 4x^2\right) \right] dx$$

$$= \left[2x^2 - \frac{x^4}{4} + 8x + \frac{x^5}{10} - \frac{4}{3}x^3 \right]_1^2$$

$$= 2\left(2^2 - 1^2\right) - \frac{1}{4}\left(2^4 - 1^4\right) + 8(2-1) + \frac{1}{10}\left(2^5 - 1^5\right) - \frac{4}{3}\left(2^3 - 1^3\right)$$

$$= 6 - \frac{15}{4} + 8 + \frac{31}{10} - \frac{28}{3} = \frac{241}{60}.$$

Example 13: Evaluate $\int_0^{\frac{a}{2}} \int_0^{\sqrt{a^2 - y^2}} \log\left(x^2 + y^2\right) dx\, dy \left(a > 0\right)$ **changing the order of integration.**

[MDU, 2001]

Solution: Over the strip PQ (say), x changes as a function of y such that P lies on the curve $x = y$ and Q lies on the curve $x = \sqrt{a^2 - y^2}$ and the strip PQ slides between $y = 0$ to $y = \frac{a}{\sqrt{2}}$.

Here the curves, $x = y$ is a straight line

and
$$\left.\begin{array}{l} x = 0 \quad \Rightarrow y = 0 \\ x = \frac{a}{\sqrt{2}} \Rightarrow y = \frac{a}{\sqrt{2}} \end{array}\right\}$$

i.e. it passes through (0, 0) and $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$

Also $x = \sqrt{a^2 - y^2}$, i.e. $x^2 + y^2 = a^2$ is a circle with centre (0, 0) and radius a.

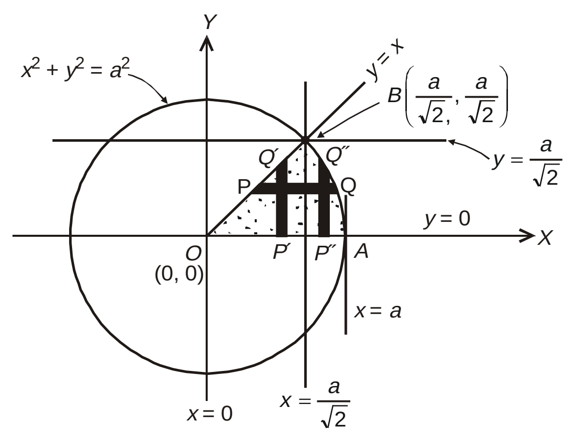


Fig. 5.18

Thus, the two curves intersect at $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$.

On changing the order of integration, the same region $OABO$ is divided into two parts with vertical strips $P'Q'$ and $P''Q''$ sliding between $x = 0$ to $x = \frac{a}{\sqrt{2}}$ and $x = \frac{a}{\sqrt{2}}$ to $x = a$ respectively.

Whence,
$$I = \int_0^{a/\sqrt{2}} \left(\int_0^x \log\left(x^2 + y^2\right) \cdot dy \right) dx + \int_{a/\sqrt{2}}^a \left(\int_0^{\sqrt{a^2 - x^2}} \log\left(x^2 + y^2\right) \cdot 1\, dy \right) dx \quad \ldots(1)$$

Now,

$$\int \underset{\text{Ist Function}}{\log\left(x^2+y^2\right)}\,\underset{\text{IInd Function}}{1}\,dy = \left[\log\left(x^2+y^2\right)\cdot y - \int \frac{1}{x^2+y^2}\,2y\cdot y\,dy\right]$$

$$= \left[y\log\left(x^2+y^2\right) - 2\int \frac{y^2+x^2-x^2}{x^2+y^2}\,dy\right]$$

$$= \left[y\log\left(x^2+y^2\right) - 2y + 2x^2\int \frac{1}{\left(x^2+y^2\right)}\,dy\right]$$

$$= \left[y\log\left(x^2+y^2\right) - 2y + 2x^2\left(\frac{1}{x}\tan^{-1}\frac{y}{x}\right)\right] \qquad \ldots(2)$$

On using (2),

$$I_1 = \int_0^{a/\sqrt{2}} \left[y\log\left(x^2+y^2\right) - 2y + 2x\left(\tan^{-1}\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_0^{a/\sqrt{2}} \left[x\log 2x^2 - 2x + 2x\tan^{-1}1\right]dx$$

$$= \int_0^{a/\sqrt{2}} \left[x\log 2x^2 - 2x + 2x\frac{\pi}{4}\right]dx$$

$$= \int_0^{a/\sqrt{2}} x\log 2x^2 dx + 2\left(\frac{\pi}{4}-1\right)\int_0^{a/\sqrt{2}} x\,dx$$

For first part, let $2x^2 = t$ so that $4x\,dx = dt$ and limits are $t = 0$ and $t = a^2$.

$$\therefore \qquad I_1 = \int_0^{a^2} \log t\cdot\frac{dt}{4} + 2\left(\frac{\pi}{4}-1\right)\left|\frac{x^2}{2}\right|_0^{a/\sqrt{2}}$$

$$= \frac{1}{4}t\left(\log t - 1\right)\Big|_0^{a^2} + \left(\frac{\pi}{4}-1\right)\frac{a^2}{2}, \text{ (By parts with } \log t = \log t\cdot 1)$$

$$= \frac{a^2}{4}\left(\log a^2 - 1\right) + \frac{\pi a^2}{8} - \frac{a^2}{2} \qquad \ldots(3)$$

Agian, using (2),

$$I_2 = \int_{a/\sqrt{2}}^{a} \left[y\log\left(x^2+y^2\right) - 2y + 2x\left(\tan^{-1}\frac{y}{x}\right)\right]_0^{\sqrt{a^2-x^2}} dx \qquad \ldots(4)$$

$$\Rightarrow \qquad = \int_{a/\sqrt{2}}^{a} \left[\sqrt{a^2-x^2}\log a^2 - 2\sqrt{a^2-x^2} + 2x\tan^{-1}\frac{\sqrt{a^2-x^2}}{x}\right]dx$$

Let $x = a\sin\theta$ so that $dx = a\cos\theta\, d\theta$ and limits, $\frac{\pi}{4}$ to $\frac{\pi}{2}$

$$\therefore \qquad I_2 = \int_{\pi/4}^{\pi/2}\left[\left(\log a^2 - 2\right)\sqrt{a^2 - a^2\sin^2\theta} + 2a\sin\theta\tan^{-1}\frac{\sqrt{a^2 - a^2\sin^2\theta}}{a\sin\theta}\right]a\cos\theta\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} a^2\left(\log a^2 - 2\right)\cos^2\theta\, d\theta + a^2\int_{\pi/4}^{\pi/2} 2\sin\theta\cos\theta\tan^{-1}(\cot\theta)\, d\theta$$

$$= a^2\left(\log a^2 - 2\right)\int_{\pi/4}^{\pi/2}\frac{(1+\cos 2\theta)}{2}\, d\theta + a^2\int_{\pi/4}^{\pi/2}\sin 2\theta\tan^{-1}\left(\tan\left(\frac{\pi}{2} - \theta\right)\right)d\theta$$

$$= \frac{a^2}{2}\left(\log a^2 - 2\right)\left[\theta + \frac{\sin 2\theta}{4}\right]_{\pi/4}^{\pi/2} + a^2\int_{\pi/4}^{\pi/2}\left(\frac{\pi}{2} - \theta\right)\sin 2\theta\, d\theta$$

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$$= \frac{a^2}{2}\left(\log a^2 - 2\right)\left[\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - \frac{1}{2}\right] + a^2\left[\left(\frac{\pi}{2} - \theta\right)\left(\frac{-\cos 2\theta}{2}\right)\Big|_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2}(-1)\left(\frac{-\cos 2\theta}{2}\right)d\theta\right]$$

$$I_2 = \frac{a^2}{2}\left(\log a^2 - 2\right)\left(\frac{\pi}{4} - \frac{1}{2}\right) - \frac{a^2}{2}\int_{\pi/4}^{\pi/2}\cos 2\theta\, d\theta,$$ $\left(\frac{\pi}{2} - \theta\right)\left(\frac{-\cos 2\theta}{2}\right)$ is zero for both the limits)

$$= \left(\frac{\pi a^2}{8}\log a^2 - \frac{\pi a^2}{4} + \frac{a^2}{2} - \frac{a^2}{4}\log a^2\right) - \frac{a^2}{4}(\sin 2\theta)_{\pi/4}^{\pi/2}$$

$$= \left(\frac{\pi a^2}{8}\log a^2 - \frac{\pi a^2}{4} + \frac{a^2}{2} - \frac{a^2}{4}\log a^2\right) + \frac{a^2}{4} \qquad \ldots(5)$$

On using results (3) and (5), we get

$$I = I_1 + I_2$$

$$= \left(\frac{a^2}{4}\log a^2 - \frac{a^2}{4} + \frac{\pi a^2}{8} - \frac{a^2}{2}\right) + \left(\frac{\pi a^2}{8}\log a^2 - \frac{\pi a^2}{4} + \frac{a^2}{2} - \frac{a^2}{4}\log a^2 + \frac{a^2}{4}\right)$$

$$= \frac{\pi a^2}{8}\log a^2 - \frac{\pi a^2}{8} = \frac{\pi a^2}{8}\left(\log a^2 - 1\right)$$

$$= \frac{\pi a^2}{8}\left(2\log a - 1\right) = \frac{\pi a^2}{4}\left(\log a - \frac{1}{2}\right).$$

Example 14: Evaluate by changing the order of integration. $\int_0^{\infty}\int_0^{x} xe^{-x^2/y}\, dx\, dy$

[VTU, 2004; UP Tech., 2005; SVTU, 2006; KUK, 2007; NIT Kurukshetra, 2007]

Solution: We write $\int_0^\infty \int_0^x xe^{-x^2/y}dxdy = \int_{x=0(=a)}^{x=\infty(=b)} \int_{y=f_1(x)=0}^{y=f_2(x)=x} xe^{-x^2/y}dxdy$

Here first integration is performed along the vertical strip with y as a function of x and then x is bounded between $x = 0$ to $x = \infty$.

We need to change, x as a function of y and finally the limits of y. Thus the desired geometry is as follows:

In this case, the strip PQ changes to $P'Q'$ with x as function of y, $x_1 = y$ and $x_2 = \infty$ and finally y varies from 0 to ∞.

Therefore Integtral

$$I = \int_0^\infty \int_y^\infty xe^{-x^2/y}dxdy$$

Put $x^2 = t$ so that $2x\,dx = dt$ Further, for $\left.\begin{array}{l} x = y,\ \ t = y^2, \\ x = \infty,\ \ t = \infty \end{array}\right\}$

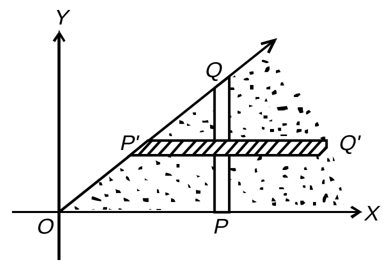

Fig. 5.19

$$I = \int_0^\infty \int_{y^2}^\infty e^{-t/y}\frac{dt}{2}dy,$$

$$= \frac{1}{2}\int_0^\infty \left(\left| \frac{e^{-t/y}}{-1/y} \right|_{y^2}^\infty \right) dy$$

$$= \int_0^\infty -\frac{y}{2}\left[0 - e^{-y}\right]dy$$

$$= \int_0^\infty \frac{ye^{-y}}{2}dy \qquad \text{(By parts)}$$

$$= \frac{1}{2}\left[y\left(\frac{e^{-y}}{-1}\right)\right]_0^\infty - \int_0^\infty 1\frac{e^{-y}}{-1}dy\Bigg]_0^\infty$$

$$= \frac{1}{2}\left[-ye^{-y} - e^{-y}\right]_0^\infty$$

$$= \frac{1}{2}\left[(0) - (0-1)\right] = \frac{1}{2}.$$

Example 15: Evaluate the integral $\int_0^\infty \int_x^\infty \frac{e^{-y}}{-y}dy\,dx$.

[NIT Jalandhar, 2004, 2005; VTU, 2007]

Soluton: In the given integral, integration is performed first with respect to y (as a function of x along the vertical strip say PQ, from P to Q) and then with respect to x from 0 to ∞.

On changing the order, of integration integration is performed first along the horizontal strip $P'Q'$ (x as a function of y) from P' to Q' and finally this strip $P'Q'$ slides between the limits $y = 0$ to $y = \infty$.

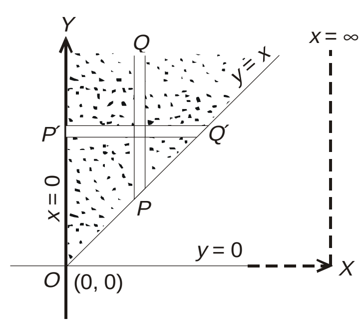

Fig. 5.20

$$\therefore \qquad I = \int_0^{\infty} \frac{e^{-y}}{y}\left(\int_0^{y} dx\right)dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{y}(y)\,dy = \int_0^{\infty} e^{-y}dy$$

$$= \left.\frac{e^{-y}}{-1}\right|_0^{\infty} = -1\left(\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right)$$

$$= -1(0-1) = 1$$

Example 16: Change the order of integration in the double integral $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dx\,dy$.

[Rajasthan, 2006; KUK, 2004-05]

Solution: Clearly from the expressions given above, the region of integration is described by a line which starts from $x = 0$ and moving parallel to itself goes over to $x = 2a$, and the extremities of the moving line lie on the parts of the circle $x^2 + y^2 - 2ax = 0$ the parabola $y^2 = 2ax$ in the first quadrant.

For change and of order of integration, we need to consider the same region as describe by a line moving parallel to x-axis instead of Y-axis.

In this way, the domain of integration is divided into three sub-regions I, II, III to each of which corresponds a double integral.

Thus, we get

$$\int_0^{2a}\int_{\sqrt{x^2-2ax}}^{\sqrt{2ax}} f(x,y)\,dydx = \underbrace{\int_0^{a}\int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x,y)\,dydx}_{\text{Part I}}$$

$$+\underbrace{\int_0^{a}\int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dydx}_{\text{Part II}} + \underbrace{\int_a^{2a}\int_{y^2/2a}^{2a} f(x,y)\,dydx}_{\text{Part III}}$$

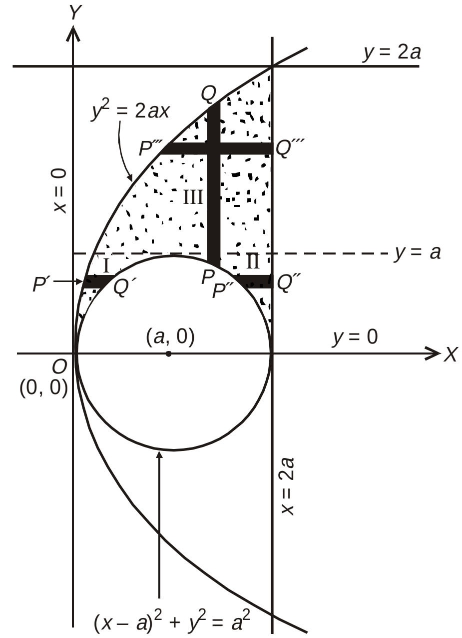

Fig. 5.21

Example 17: Find the area bounded by the lines $y = \sin x$, $y = \cos x$ and $x = 0$.

Solution: See Fig 5.22.

Clearly the desired area is the doted portion where along the strip PQ, P lies on the curve $y = \sin x$ and Q lies on the curve $y = \cos x$ and finally the strip slides between the ordinates $x = 0$ and $x = \frac{\pi}{4}$.

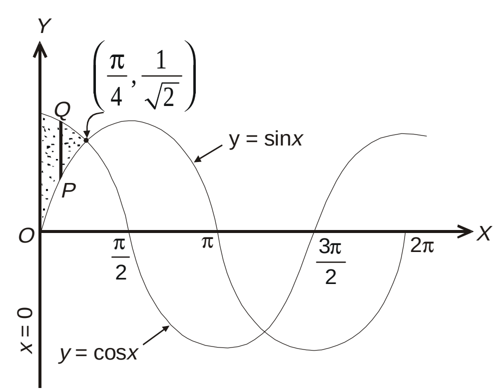

Fig. 5.22

$$\therefore \qquad \iint_R dx\,dy = \int_0^{\frac{\pi}{4}}\left(\int_{\sin x}^{\cos x} dy\right)dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx$$

$$= (\sin x + \cos x)_0^{\pi/4}$$

$$= \left(\frac{1}{\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} - 1\right).$$

$$= \left(\sqrt{2} - 1\right)$$

ASSIGNMENT 2

1. Change the order of integration $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dxdy$
2. Change the order integration in the integral $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$
3. Change the order of integration in $\int_0^{a\cdot\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$
4. Change the order of integration in $\int_0^a \int_{mx}^{lx} f(x,y)dxdy$ **[PTU, 2008]**

5.4 EVALUATION OF DOUBLE INTEGRAL IN POLAR COORDINATES

To evaluate $\int_{\theta=\alpha}^{\theta=\beta} \int_{r=\phi(\theta)}^{r=\Psi(\theta)} f(r,\theta)\ dr\ d\theta$, we first integrate with respect to r between the limits $r = \phi(\theta)$ to $r = \psi(\theta)$ keeping θ as a constant and then the resulting expression is integrated with respect to θ from $\theta = \alpha$ to $\theta = \beta$.

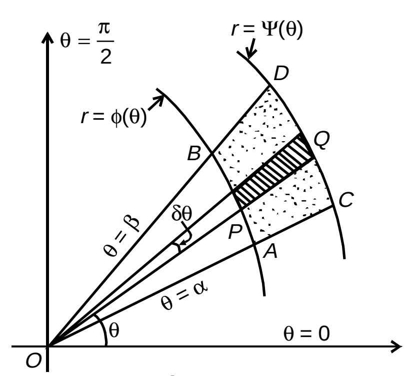

Fig. 5.23

Geometrical Illustration: Let AB and CD be the two continuous curves $r = \phi(\theta)$ and $r = \Psi(\theta)$ bounded between the lines $\theta = \alpha$ and $\theta = \beta$ so that $ABDC$ is the required region of integration.

Let PQ be a radial strip of angular thickness $\delta\theta$ when OP makes an angle θ with the initial line.

Here $\int_{r=\phi(\theta)}^{r=\Psi(\theta)} f(r,\theta)dr$ refers to the integration with respect to r along the radial strip PQ and then integration with respect to θ means rotation of this strip PQ from AC to CD.

Example 18: Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cardiod $r = a\,(1 - \cos\theta)$ above the initial line.

Solution: The region of integration under consideration is the cardiod $r = a(1 - \cos\theta)$ above the initial line.

In the cardiod $\quad r = a(1 - \cos\theta)$; for $\left.\begin{array}{ll}\theta = 0, & r = 0,\\ \theta = \dfrac{\pi}{2}, & r = a,\\ \theta = \pi, & r = 2a\end{array}\right\}$

As clear from the geometry along the radial strip OP, r (as a function of θ) varies from $r = 0$ to $r = a(1 - \cos\theta)$ and then this strip slides from $\theta = 0$ to $\theta = \pi$ for covering the area above the initial line.

Hence

$$I = \int_0^{\pi}\left(\int_0^{r=a(1-\cos\theta)} r\,dr\right)\sin\theta\, d\theta$$

$$= \int_0^{\pi}\left(\left.\frac{r^2}{2}\right|_0^{a(1-\cos\theta)}\right)\sin\theta\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}(1 - \cos\theta)^2 \sin\theta\, d\theta$$

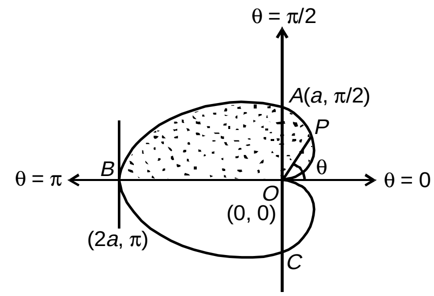

Fig. 5.24

$$= \frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3}\right]_0^{\pi}, \quad \left[\because \int f^n(x)\, f'(x)\,dx = \frac{f^{n+1}(x)}{n+1}\right]$$

$$= \frac{a^2}{6}\left[(1 - \cos\pi)^3 - (1 - \cos 0)\right] = \frac{a^2}{6}[8 - 0] = \frac{4a^2}{3}.$$

Example 19: Show that $\iint_R r^2 \sin\theta\, dr\, d\theta = \dfrac{2a^3}{3}$, where R is the semi circle $r = 2a\cos\theta$ above the initial line.

Solution: The region R of integration is the semi-circle $r = 2a\cos\theta$ above the initial line.

For the circle $\quad \left.\begin{array}{l} r = 2a\cos\theta,\ \theta = 0 \ \Rightarrow\ r = 2a\\ \qquad\qquad \theta = \dfrac{\pi}{2} \ \Rightarrow\ r = 0\end{array}\right\}$

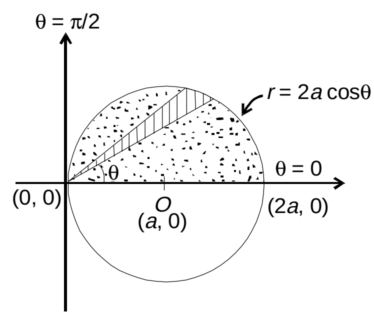

Fig. 5.25

Otherwise also, $\quad r = 2a\cos\theta \ \Rightarrow\ r^2 = 2ar\cos\theta$

$$x^2 + y^2 = 2ax$$

$$(x^2 - 2ax + a^2) + y^2 = a^2$$

$$(x - a)^2 + (y - 0)^2 = a^2$$

i.e., it is the circle with centre $(a, 0)$ and radius $r = a$

Hence the desired area $\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2 \sin\theta\, dr\, d\theta$

$$= \int_0^{\frac{\pi}{2}}\left(\int_0^{2a\cos\theta} r^2 dr\right)\sin\theta d\theta$$

$$= \int_0^{\pi/2}\left(\left|\frac{r^3}{3}\right|_0^{2a\cos\theta}\right)\sin\theta d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2}(2a)^3\cos^3\theta\sin\theta\, d\theta$$

$$= \frac{-8a^3}{3}\left(\frac{\cos^4\theta}{4}\right)_0^{\pi/2}, \quad \text{using } \int f^n(x)\, f'(x)dx = \frac{f^{n+1}(x)}{n+1}$$

$$= \frac{2a^3}{3}.$$

Example 20: Evaluate $\iint \frac{r\, dr\, d\theta}{\sqrt{a^2 + r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$.

[KUK, 2000; MDU, 2006]

Solution: The lemniscate is bounded for $r = 0$ implying $\theta = \pm\frac{\pi}{4}$ and maximum value of r is a.

See Fig. 5.26, in one complete loop, r varies from 0 to $r = a\sqrt{\cos 2\theta}$ and the radial strip slides between $\theta = -\frac{\pi}{4}$ to $\frac{\pi}{4}$.

Hence the desired area

$$A = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{r}{\left(a^2 + r^2\right)^{1/2}}\, dr\, d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left(\int_0^{a\sqrt{\cos 2\theta}} d\left(a^2 + r^2\right)^{1/2} dr\right)d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left(a^2 + r^2\right)^{1/2}\Big|_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\left(a^2 + a^2\cos 2\theta\right)^{1/2} - a\right]d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}\left(\sqrt{2}\cos\theta - 1\right)d\theta$$

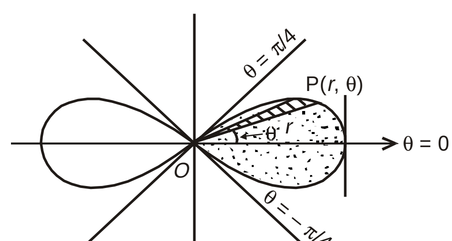

Fig. 5.26

$$= 2a\int_0^{\pi/4}\left(\sqrt{2}\cos\theta - 1\right)d\theta$$

$$= 2a\left[\left(\sqrt{2}\sin\theta - \theta\right)_0^{\pi/4}\right]$$

$$= 2a\left[\sqrt{2}\,\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] = 2a\left(1 - \frac{\pi}{4}\right).$$

Example 21: Evaluate $\iint r^3 dr\, d\theta$, over the area included between the circles $r = 2a\cos\theta$ and $r = 2b\cos\theta$ $(b < a)$. **[KUK, 2004]**

Solution: Given $r = 2a\cos\theta$ or $r^2 = 2ar\cos\theta$

$$x^2 + y^2 = 2ax$$

$$(x + a)^2 + (y - 0)^2 = a^2$$

i.e this curve represents the circle with centre $(a, 0)$ and radius a.

Likewise, $r = 2b\cos\theta$ represents the circle with centre $(b, 0)$ and radius b.

We need to calculate the area bounded between the two circles, where over the radial strip PQ, r varies from circle $r = 2b\cos\theta$ to $r = 2a\cos\theta$ and finally θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Thus, the given integral $\iint_R r^3 dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2b\cos\theta}^{2a\cos\theta} r^3 dr\, d\theta$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_{2b\cos\theta}^{2a\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2}\left[\left(2a\cos\theta\right)^4 - \left(2b\cos\theta\right)^4\right]d\theta$$

$$= 4\left(a^4 - b^4\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 8\left(a^4 - b^4\right)\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 8\left(a^4 - b^4\right)\frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2}$$

$$= \frac{3}{2}\pi\left(a^4 - b^4\right).$$

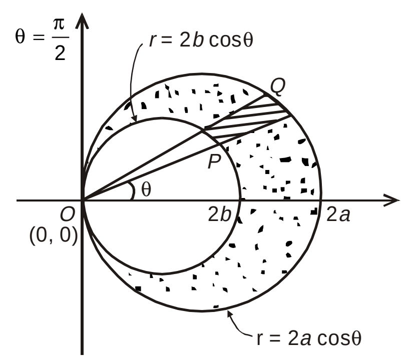

Fig 5. 27

Particular Case: When $r = 2\cos\theta$ and $r = 4\cos\theta$ *i.e.*, $a = 2$ and $b = 1$, then

$$I = \frac{3}{2}\pi\left(a^4 - b^4\right) = \frac{3}{2}\pi\left(2^4 - 1^4\right) = \frac{45\pi}{2} \text{ units}.$$

ASSIGNMENT 3

1. Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the area of the caridod $r = a(1 + \cos\theta)$ above the initial line.

$$\left[\textbf{Hint: } I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta\, dr\, d\theta\right]$$

2. Evaluate $\iint r^3 dr\, d\theta$, over the area included between the circles $r = 2a\cos\theta$ and $r = 2b\cos\theta$ $(b > a)$. **[Madras, 2006]**

$$\left[\textbf{Hint: } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_{r=2a\cos\theta}^{r=2b\cos\theta} r^3 dr\right)d\theta\right]$$ (See Fig. 5.27 with a and b interchanged)

3. Find by double integration, the area lying inside the cardiod $r = a(1 + \cos\theta)$ and outside the parabola $r(1 + \cos\theta) = a$. **[NIT Kurukshetra, 2008]**

$$\left[\textbf{Hint: } 2\int_0^{\pi/2}\left(\int_{\frac{a}{1+\cos\theta}}^{a(1+\cos\theta)} r dr\right)d\theta\right]$$

5.5 CHANGE OF ORDER OF INTERGRATION IN DOUBLE INTEGRAL IN POLAR COORDINATES

In the integral $\int_{\theta=\alpha}^{\theta=\beta}\int_{r=\phi(\theta)}^{r=\Psi(\theta)} f(r,\theta)\, dr\, d\theta$, interation is first performed with respect to r along a 'radial strip' and then this trip slides between two values of $\theta = \alpha$ to $\theta = \beta$.

In the changed order, integration is first performed with respect to θ (as a function of r along a 'circular arc') keeping r constant and then integrate the resulting integral with respect to r between two values $r = a$ to $r = b$ (say)

Mathematically expressed as

$$\int_{\theta=\alpha}^{\theta=\beta}\int_{r=\phi(\theta)}^{r=\Psi(\theta)} f(r,\theta)\, dr\, d\theta = I = \int_{r=a}^{r=b}\int_{\theta=f(r)}^{\theta=\eta(r)} f(r,\theta)\, d\theta\, dr$$

Example 22: Change the order of integration in the integral $\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\, dr\, d\theta$

Solution: Here, integration is first performed with respect to r (as a function of θ) along a **radial strip** ***OP*** (say) from $r = 0$ to $r = 2a\cos\theta$ and finally this radial strip slides between $\theta = 0$ to $\theta = \dfrac{\pi}{2}$.

Curve $r = 2a\cos\theta \Rightarrow r^2 = 2ar\cos\theta$

$\Rightarrow$ $x^2 + y^2 = 2ax \Rightarrow (x - a)^2 + y^2 = a^2$

i.e., it is circle with centre $(a, 0)$ and radius a.

On changing the order of integration, we have to first integrate with respect to θ (as a function of r) along

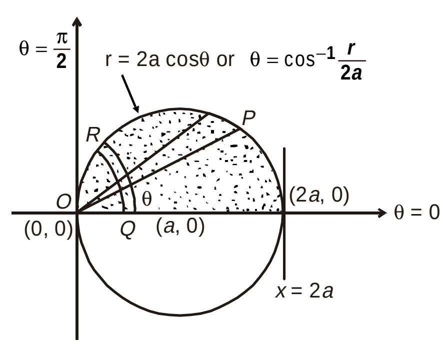

Fig. 5.28

the **'circular strip' QR** (say) with pt. Q on the curve $\theta = 0$ and pt. R on the curve $\theta = \cos^{-1}\dfrac{r}{2a}$ and finally r varies from 0 to $2a$.

$$\therefore \qquad I = \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} f(r,\theta)\,dr\,d\theta = \int_0^{2a}\left(\int_0^{\cos^{-1}\frac{r}{2a}} f(r,\theta)\,d\theta\right)dr$$

Example 23: Sketch the region of integration $\int_a^{ae^{\frac{\pi}{4}}}\int_{2\log\frac{r}{a}}^{\pi/2} f(r,\theta)\,r\,dr\,d\theta$ and change the order of integration.

Solution: Double integral $\int_0^{ae^{\pi/4}}\int_{2\log\frac{r}{a}}^{\pi/2} f(r,\theta)\,r\,dr\,d\theta$ is identical to $\int_{r=\alpha}^{r=\beta}\int_{\theta=f_1(r)}^{\theta=f_2(r)} f(r,\theta)\,r\,dr\,d\theta,$ whence integration is first performed with respect to θ as a function of r *i.e.*, $\theta = f(r)$ along the 'circular strip' PQ (say) with point P on the curve $\theta = 2\log\dfrac{r}{a}$ and point Q on the curve $\theta = \dfrac{\pi}{2}$ and finally this strip slides between between $r = a$ to $r = ae^{\pi/4}$. (See Fig. 5.29).

The curve $\theta = 2\log\dfrac{r}{a}$ implies $\dfrac{\theta}{2} = \log\dfrac{r}{a}$

$$e^{\theta/2} = \frac{r}{a} \quad \text{or} \quad r = ae^{\theta/2}$$

Now on changing the order, the integration is first performed with respect to r as a function of θ viz. $r = f(\theta)$ along the 'radial strip' PQ (say) and finally this strip slides between $\theta = 0$ to $\theta = \dfrac{\pi}{2}$. (Fig. 5.30).

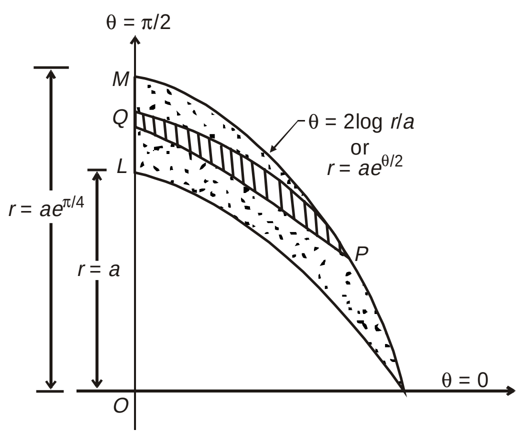

Fig. 5.29

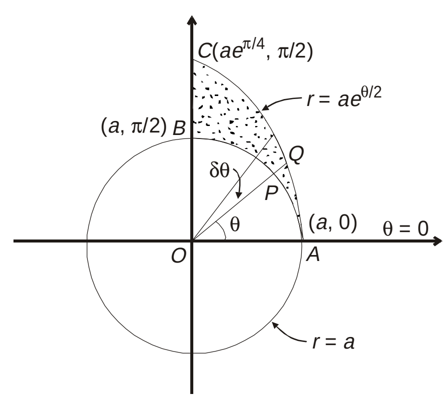

Fig. 5.30

$$\therefore \qquad I = \int_{\theta=0}^{\pi/2}\left(\int_{r=a}^{r=ae^{\theta/2}} f(r,\theta)\,r\,dr\right)d\theta$$

5.6. AREA ENCLOSED BY PLANE CURVES

1. **Cartesian Coordinates:** Consider the area bounded by the two continuous curves $y = \phi(x)$ and $y = \Psi(x)$ and the two ordinates $x = a$, $x = b$ (Fig. 5.31).

Now divide this area into vertical strips each of width δx.

Let $R(x, y)$ and $S(x + \delta x, y + \delta y)$ be the two neigbouring points, then the area of the elementary shaded portion (i.e., small rectangle) = $\delta x \delta y$

But all the such small rectangles on this strip PQ are of the same width δx and y changes as a function of x from $y = \phi(x)$ to $y = \Psi(x)$

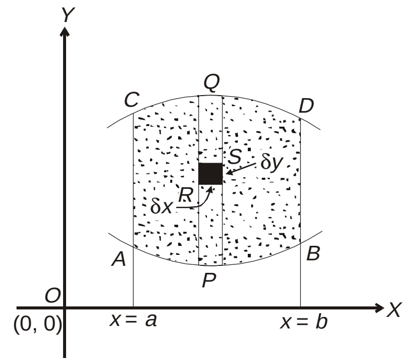

Fig. 5.31

$\therefore$ The area of the strip $PQ = \underset{\delta y \to 0}{Lt} \sum \delta x \delta y = \delta x \underset{\delta y \to 0}{Lt} \sum_{\phi(x)}^{\Psi(x)} dy = \delta x \int_{\phi(x)}^{\Psi(x)} dy$

Now on adding such strips from $x = a$, we get the desired area $ABCD$,

$$\underset{\delta y \to 0}{Lt} \sum_{\phi(x)}^{\Psi(x)} \delta x \int_{\phi(x)}^{\Psi(x)} dy = \int_a^b dx \int_{\phi(x)}^{\Psi(x)} dy = \int_a^b \int_{\phi(x)}^{\Psi(x)} dx dy$$

Likewise taking horizontal strip $P'Q'$ (say) as shown, the area $ABCD$ is given by

$$\int_{y=a}^{y=b} \int_{x=\phi(y)}^{x=\Psi(y)} dx\, dy$$

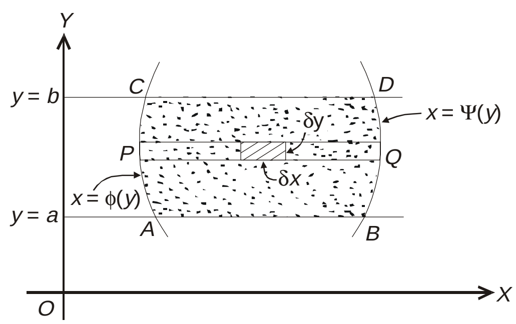

Fig. 5.32

2 **Polar Coordinates:** Let R be the region enclosed by a polar curve with $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ as two neighbouring points in it.

Let $PP'QQ'$ be the circular area with radii OP and OQ equal to r and $r + \delta r$ respectively.

Here the area of the curvilinear rectangle is approximately

$= PP' \cdot PQ' = \delta r \cdot r \sin \delta\theta = \delta r \cdot r\delta\theta = r\, \delta r\, \delta\theta.$

If the whole region R is divided into such small curvilinear rectangles then the limit of the sum $\Sigma r \delta r \delta\theta$ taken over R is the area A enclosed by the curve.

i.e., $$A = \underset{\substack{\delta r \to 0 \\ \delta\theta \to 0}}{Lt} \sum r \delta r\, \delta\theta = \iint_R r dr\, d\theta$$

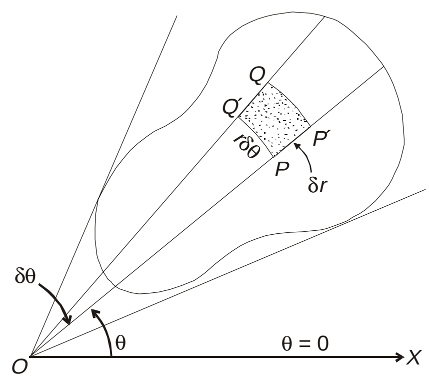

Fig. 5.33

Example 24: Find by double integration, the area lying between the curves $y = 2 - x^2$ and $y = x$.

Solution: The given curve $y = 2 - x^2$ is a parabola.

where in
$$\left.\begin{array}{rcl} x=0 & \Rightarrow & y=0 \\ x=1 & \Rightarrow & y=1 \\ x=2 & \Rightarrow & y=-2 \\ x=-1 & \Rightarrow & y=1 \\ x=-2 & \Rightarrow & y=-2 \end{array}\right\}$$

i.e., it passes through points (0, 2), (1, 1), (2, – 2), (– 1, 1), (– 2, – 2).

Likewise, the curve $y = x$ is a straight line

where
$$\left.\begin{array}{rcl} y=0 & \Rightarrow & x=0 \\ y=1 & \Rightarrow & x=1 \\ y=-2 & \Rightarrow & x=-2 \end{array}\right\}$$

i.e., it passes through (0, 0), (1, 1), (– 2, – 2)

Now for the two curves $y = x$ and $y = 2 - x^2$ to intersect, $x = 2 - x^2$ or $x^2 + x - 2 = 0$ i.e., $x = 1, -2$ which in turn implies $y = 1, -2$ respectively.

Thus, the two curves intersect at (1, 1) and (–2, –2),

Clearly, the area need to be required is *ABCDA*.

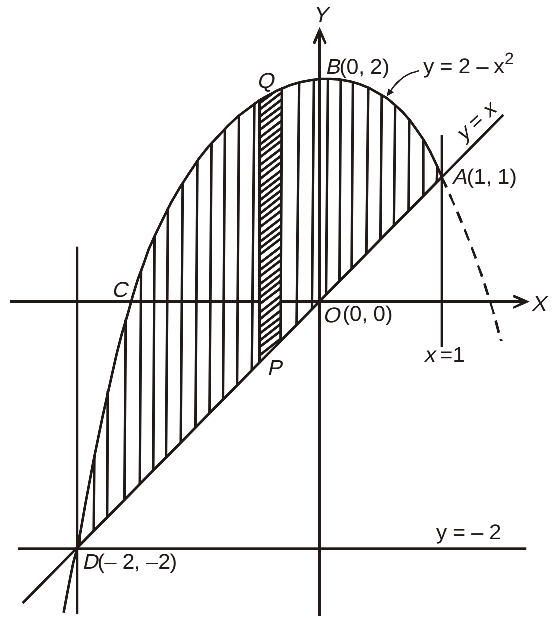

Fig. 5.34

$$\therefore \qquad A = \int_{-2}^{1}\left(\int_{x}^{2-x^2} dy\right)dx = \int_{-2}^{1}\left(2 - x^2 - x\right)dx$$

$$= \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1} = \frac{9}{2} \text{ units.}$$

Example 25: Find by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$. [KUK, 2001]

Solution: For the given curve $y = 4x - x^2$;

$$\left.\begin{array}{rcl} x=0 & \Rightarrow & y=0 \\ x=1 & \Rightarrow & y=2 \\ x=2 & \Rightarrow & y=4 \\ x=3 & \Rightarrow & y=3 \\ x=4 & \Rightarrow & y=0 \end{array}\right\}$$

i.e. it passes through the points (0, 0), (1, 2), (3, 3) and (4, 0).

Likewise, the curve $y = x$ passes through (0, 0) and (3, 3), and hence, (0, 0) and (3, 3) are the common points.

Otherwise also putting $y = x$ into $y = 4x - x^2$, we get $x = 4x - x^2 \Rightarrow x = 0, 3$.

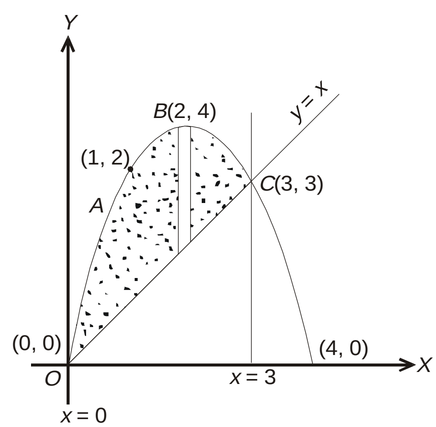

Fig. 5.35

See Fig. 5.35, *OABCO* is the area bounded by the two curves $y = x$ and $y = 4x - x^2$

$\therefore$ Area $$OABCO = \int_0^3 \int_x^{4x-x^2} dy\, dx$$

$$= \int_0^3 [y]_x^{4x-x^2} dx$$

$$= \int_0^3 (4x - x^2 - x)\, dx = \left[3\frac{x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{9}{2} \text{ units}$$

Example 26: Calculate the area of the region bounded by the curves $y = \dfrac{3x}{x^2+2}$ and $4y = x^2$

[JNTU, 2005]

Solution: The curve $4y = x^2$ is a parabola

where $\left.\begin{aligned} y = 0 &\Rightarrow x = 0, \\ y = 1 &\Rightarrow x = \pm 2 \end{aligned}\right\}$ *i.e.*, it passes through (–2, 1), (0, 0), (2, 1).

Likewise, for the curve $y = \dfrac{3x}{x^2 + 2}$

$$\left.\begin{aligned} y = 0 &\Rightarrow x = 0 \\ y = 1 &\Rightarrow x = 1, 2 \\ x = -1 &\Rightarrow y = -1 \end{aligned}\right\}$$

Hence it passes through points (0, 0), (1, 1), (2, 1), (–1, –1).

Also for the curve $(x^2 + 2)\, y = 3x$, $y = 0$ (i.e. X-axis) is an asymptote.

For the points of intersection of the two curves $y = \dfrac{3x}{x^2 + 2}$ and $4y = x^2$

we write $$\frac{3x}{x^2 + 2} = \frac{x^2}{4} \quad \text{or} \quad x^2 (x^2 + 2) = 12x$$

Then $x = 0 \Rightarrow y = 0$

$x = 2 \Rightarrow y = 1$

i.e. (0, 0) and (2, 1) are the two points of intersection.

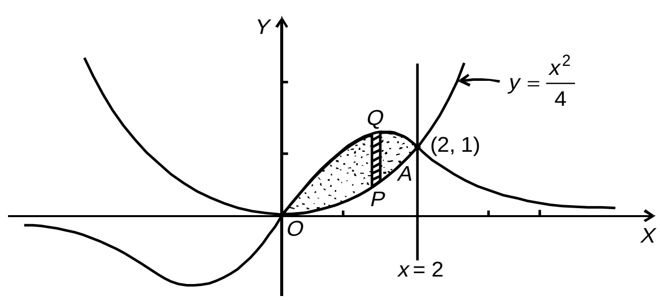

Fig. 5.36

The area under consideration,

$$A = \int_0^2 \left(\int_{y=\frac{x^2}{4}}^{y=\frac{3x}{x^2+2}} dy \right) dx = \int_0^2 \left[\frac{3x}{x^2+2} - \frac{x^2}{4} \right] dx$$

$$= \left[\frac{3}{2} \log\left(x^2+2\right) - \frac{x^3}{12} \right]_0^2$$

$$= \frac{3}{2}\left(\log 6 - \log 2\right) - \frac{2}{3} = \log 3^{\frac{3}{2}} - \frac{2}{3}.$$

Example 27: Find by the double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardiod $r = a(1 - \cos\theta)$. **[KUK 2005; NIT Kurukshetra 2007]**

Soluton: The area enclosed inside the circle $r = a\sin\theta$ and the cardiod $r = a(1 - \cos\theta)$ is shown as doted one.

For the radial strip PQ, r varies from $r = a(1 - \cos\theta)$ to $r = a\sin\theta$ and finally θ varies in between 0 to $\frac{\pi}{2}$.

For the circle $r = a\sin\theta$

$$\left.\begin{aligned} \theta = 0 &\Rightarrow r = 0 \\ \theta = \frac{\pi}{2} &\Rightarrow r = a \\ \theta = \pi &\Rightarrow r = 0 \end{aligned}\right\}$$

Likewise for the cardiod $r = a(1 - \cos\theta)$;

$$\left.\begin{aligned} \theta = 0 &\Rightarrow r = 0 \\ \theta = \frac{\pi}{2} &\Rightarrow r = a \\ \theta = \pi &\Rightarrow r = 2a \end{aligned}\right\}$$

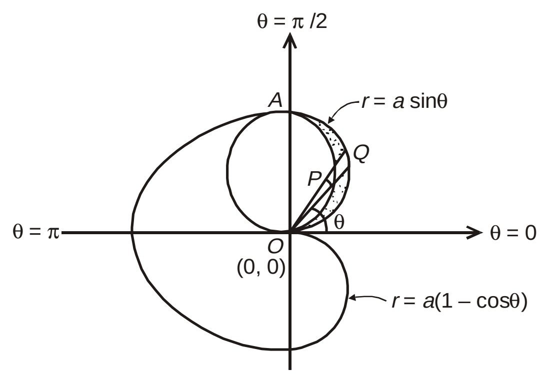

Fig. 5.37

Thus, the two curves intersect at $\theta = 0$ and $\theta = \frac{\pi}{2}$.

$$\therefore \qquad A = \int_0^{\frac{\pi}{2}} \int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left. \frac{r^2}{2} \right|_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left[\sin^2\theta - \left(1 + \cos^2\theta - 2\cos\theta\right) \right] d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} \left[-\cos 2\theta - 1 + 2\cos\theta \right] d\theta, \text{ since } (\sin^2\theta - \cos^2\theta) = -\cos 2\theta$$

$$= \frac{a^2}{2}\left[\frac{-\sin 2\theta}{2} - \theta + 2\sin\theta\right]_0^{\pi/2} = a^2\left(1 - \frac{\pi}{4}\right).$$

Example 28: Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote $r = a\sec\theta$. **[NIT Kurukshetra, 2007]**

Solution: As the given crave $r = a(\sec\theta + \cos\theta)$ *i.e.*, $r = a\left(\frac{1}{\cos\theta} + \cos\theta\right)$ contains cosine terms only and hence it is symmetrical about the initial axis.

Further, for $\theta = 0$, $r = 2a$ and, r goes on decreasing above and below the initial axis as θ approaches to $\frac{\pi}{2}$ and at $\theta = \frac{\pi}{2}$, $r = \infty$.

Clearly, the required area is the doted region in which r varies along the radial strip from $r = a\sec\theta$ to $r = a(\sec\theta + \cos\theta)$ and finally strip slides between $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

$$\therefore \quad A = 2\int_0^{\frac{\pi}{2}} \int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2} \left[\left(\frac{1+\cos^2\theta}{\cos\theta}\right)^2 - \left(\frac{1}{\cos\theta}\right)^2\right] d\theta$$

$$= a^2\int_0^{\pi/2} \left(\cos^2\theta + 2\right) d\theta$$

$$= a^2\int_0^{\pi/2} \frac{(5 + \cos 2\theta)}{2}\, d\theta$$

$$= \frac{a^2}{2}\left[5\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{5\pi a^2}{4}.$$

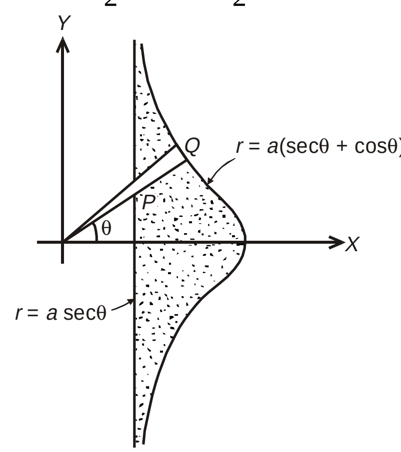

Fig. 5.38

ASSIGNMENT 4

1. Show by double integration, the area bounded between the parabola $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$. **[MDU, 2003; NIT Kurukshetra, 2010]**
2. Using double integration, find the area enclosed by the curves, $y^2 = x^3$ and $y = x$. **[PTU, 2005]**

Example 29: Find by double integration, the area of laminiscate $r^2 = a^2\cos 2\theta$. **[Madras, 2000]**

Solution: As the given curve $r^2 = a^2\cos 2\theta$ contains cosine terms only and hence it is symmetrical about the initial axis.

Further the curve lies wholly inside the circle $r = a$, since the maximum value of $|\cos\theta|$ is 1.

Also, no portion of the curve lies between $\theta = \frac{\pi}{4}$ to $\theta = \frac{3\pi}{4}$ and the extended axis.

See the geometry, for one loop, the curve is bounded between $\theta = -\frac{\pi}{4}$ to $\frac{\pi}{4}$

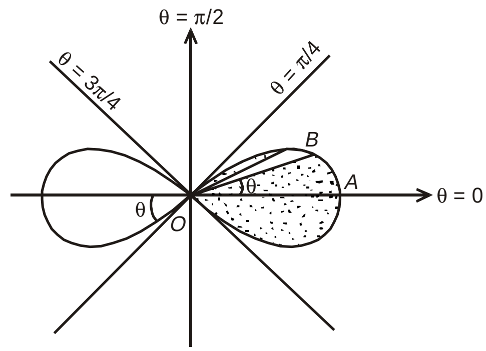

Fig. 5.39

$$\therefore \qquad \text{Area} = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{r=0}^{r=\sqrt{a^2\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4} \frac{r^2}{2}\Big|_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2a^2\int_0^{\pi/4} \cos 2\theta\, d\theta = 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = a^2$$

5.7 CHANGE OF VARIABLE IN DOUBLE INTEGRAL

The concept of change of variable had evolved to facilitate the evaluation of some typical integrals.

Case 1: General change from one set of variable (x, y) to another set of variables (u, v).

If it is desirable to change the variables in double integral $\iint_R f(x,y)\,dA$ by making $x = \phi(u, v)$ and $y = \Psi(u, v)$, the expression dA (the elementary area $\delta x \delta y$ in R_{xy}) in terms of u and v is given by

$$dA = \left|J\left(\frac{x,y}{u,v}\right)\right| du\,dv, \qquad J\left(\frac{x,y}{u,v}\right) \neq 0$$

J is the **Jacobian** (transformation coefficient) or **functional determinant.**

$$\therefore \qquad \iint_R f(x,y)\,dx\,dy = \iint_R F(u,v)\,J\left(\frac{x,y}{u,v}\right) du\,dv$$

Case 2: From Cartesian to Polar Coordinates: In transforming to polar coordinates by means of $x = r\cos\theta$ and $y = r\sin\theta$,

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$\therefore \qquad dA = r\,dr\,d\theta \quad \text{and} \quad \iint_R f(x,y)\,dx\,dy = \iint_{R'} F(r,\theta)\,r\,dr\,d\theta$$

Example 30: Evaluate $\iint_R (x+y)^2 dx\,dy$ where R is the parallelogram in the xy plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation $u = x + y$, $v = x - 2y$.
[KUK, 2000]

Solution: R_{xy} is the region bounded by the parallelogram $ABCD$ in the xy plane which on transformation becomes R'_{uv} *i.e.*, the region bounded by the rectangle $PQRS$, as shown in the Figs. 5.40 and 5.41 respectively.

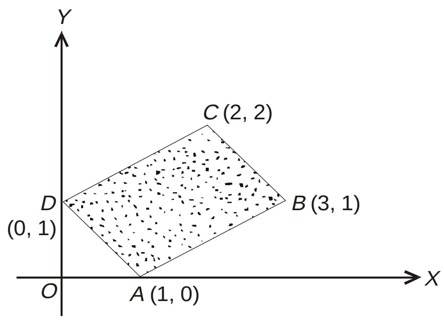

Fig. 5.40

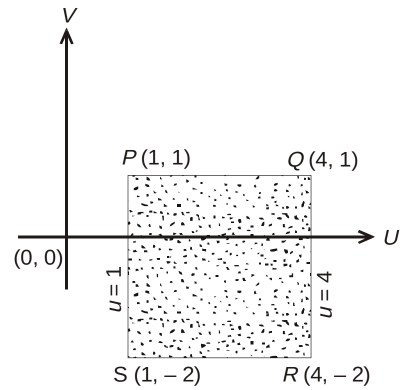

Fig. 5.41

With $\left.\begin{array}{l} u = x + y \\ v = x - 2y \end{array}\right\}$, A (1, 0) transforms to $\left.\begin{array}{l} u = 1 + 0 = 1 \\ v = 1 - 0 = 1 \end{array}\right\}$ i.e., $P(1, 1)$

$B(3, 1)$ transforms to $Q(4, 1)$

$C(2, 2)$ transforms to $R(4, -2)$

$D(0, 1)$ transforms to $S(1, -2)$

and $$J\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

Hence the given integral $\iint_R u^2 \frac{1}{3} du\,dv$

$$= \int_1^4 \int_{-2}^1 \frac{1}{3} u^2 du\,dv = \frac{1}{3}\int_1^4 [v]_{-2}^1 u^2 du$$

$$= \frac{1}{3} \times (1 + 2) \int_1^4 u^2 du$$

$$I = \left(\left| \frac{u^3}{3} \right|_1^4 \right) = \frac{63}{3} = 21 \text{ units}$$

Example 31: Using transformation $x + y = u$, $y = uv$, show that

$$\int_0^1\int_0^{1-x} e^{\left(\frac{y}{x+y}\right)} dxdy = \frac{1}{2}(e-1).$$ **[PTU, 2003]**

Solution: Clearly $y = f(x)$ represents curves $y = 0$ and $y = 1 - x$, and x which is an independent variable changes from $x = 0$ to $x = 1$. Thus, the area $OABO$ bounded between the two curves $y = 0$ and $x + y = 1$ and the two ordinates $x = 0$ and $x = 1$ is shown in Fig. 5.42.

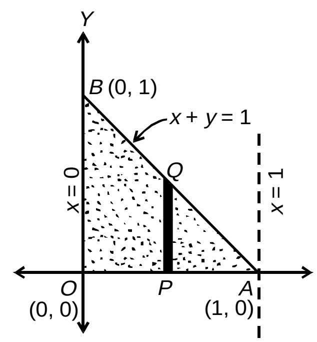

Fig. 5.42

On using transformation,

$$x + y = u \quad \Rightarrow \quad x = u(1 - v)$$
$$y = uv \quad \Rightarrow \quad y = uv$$

Now point O(0, 0) implies $0 = u(1 - v)$ ...(1)

and $0 = uv$ ...(2)

From (2), either $u = 0$ or $v = 0$ or both zero. From (1), we get

$u = 0$, $v = 1$

Hence $(x, y) = (0, 0)$ transforms to $(u, v) = (0, 0), (0, 1)$

Point A(1, 0), implies $1 = u(1 - v)$ ...(3)

and $0 = uv$ ...(4)

From (4) either $u = 0$ or $v = 0$, If $v = 0$ then from (3) we have $u = 1$, again if $u = 0$, equation (3) is inconsistent.

Hence, $A(1, 0)$ transforms to (1, 0), i.e. itself.

From Point B(0, 1), we get $0 = u(1 - v)$ ...(5)

and $1 = vu$ ...(6)

From (5), either $u = 0$ or $v = 1$

If $u = 0$, equation (6) becomes inconsistent.

If $v = 1$, the equation (6) gives $u = 1$.

Hence (0, 1) transform to (1, 1). See Fig. 5.43.

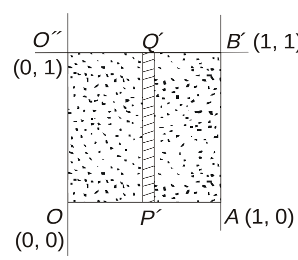

Fig. 5.43

Hence

$$\int_0^1\int_0^{1-x} e^{\left(\frac{y}{x+y}\right)} dxdy = \int_0^1\int_0^1 ue^v du\, dv \quad \text{where} \quad J = \frac{\partial(x,y)}{\partial(u,v)} = u$$

$$= \int_0^1 u\left(\int_0^1 e^v dv\right)du = \int_0^1 u\cdot(e-1)du = (e-1)\frac{u^2}{2}\Bigg|_0^1 = \frac{1}{2}(e-1)$$

Example 32: Evaluate the integral $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dxdy$ by transforming to polar coordinates.

Solution: Here the curves $x = \frac{y^2}{4a}$ or $y^2 = 4ax$ is parabola passing through (0, 0), ($4a$, $4a$).

Likewise the curve $x = y$ is a straight line passing through points (0, 0) ($4a$, $4a$).

Hence the two curves intersect at (0, 0), ($4a$, $4a$).

In the given form of the integral, x changes (as a function of y) from $x = \frac{y^2}{4a}$ to $x = y$ and finally y as an independent variable varies from $y = 0$ to $y = 4a$.

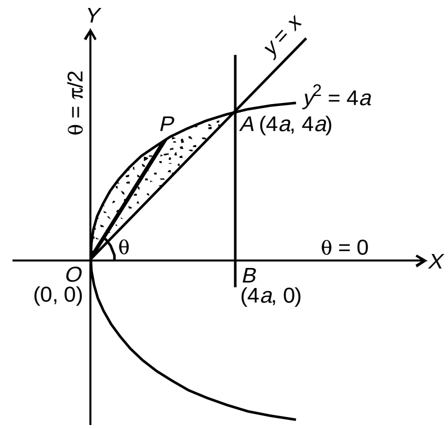

Fig. 5.44

For transformation to polar coordinates, we take

$$x = r\cos\theta,\ y = r\sin\theta \text{ and } J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$

The parabola $y^2 = 4ax$ implies $r^2\sin^2\theta = 4ar\cos\theta$ so that r(as a function of θ) varies from $r = 0$ to $r = \frac{4a\cos\theta}{\sin^2\theta}$ and θ varies from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

Therefore, on transformation the integral becomes

$$I = \int_{\pi/4}^{\pi/2}\int_0^{r=\frac{4a\cos\theta}{\sin^2\theta}} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \cos 2\theta \cdot \left[\frac{r^2}{2}\right]_0^{\frac{4a\cos 0}{\sin^2\theta}} d\theta$$

$$= \int_{\pi/4}^{\pi/2} (1 - 2\sin^2\theta)\frac{16a^2}{2}\frac{\cos^2\theta}{\sin^4\theta}\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} \frac{(1 - 2\sin^2\theta)(1 - \sin^2\theta)}{\sin^4\theta}\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} \frac{[1 - 3\sin^2\theta + 2\sin^4\theta]}{\sin^4\theta}\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} \left[\operatorname{cosec}^2\theta(1 + \cot^2\theta) - 3\operatorname{cosec}^2\theta + 2\right] d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} \left[\cot^2\theta\operatorname{cosec}^2\theta - 2\operatorname{cosec}^2\theta + 2\right] d\theta$$

$$= 8a^2\left[\int_{\pi/4}^{\pi/2}\cot^2\theta\,\mathrm{cosec}^2\theta\,d\theta + 2(\cot\theta)_{\pi/4}^{\pi/2} + (2\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

Let $\cot\theta = t$ so that $-\mathrm{cosec}^2\theta\, d\theta = dt$. Limits for $\left.\begin{array}{l}\theta = \dfrac{\pi}{4}, t = 1\\ \theta = \dfrac{\pi}{2}, t = 0\end{array}\right\}$

$$= 8a^2\left[\int_1^0 -t^2 dt + 2(0-1) + \frac{\pi}{2}\right] = 8a\left[\left|-\frac{t^3}{3}\right|_1^0 - 2 + \frac{\pi}{2}\right]$$

$$= 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right).$$

Example 33: Evaluate the integral $\int_0^a\int_{x/a}^{\sqrt{x/a}}(x^2+y^2)\,dxdy$ by changing to polar coordinates.

Solution: The above integral has already been discussed under change of order of integration in cartesian co-ordinate system, Example 7.

For transforming any point $P(x, y)$ of cartesian coordinate to polar coordinates $P(r, \theta)$, we take $x = r\cos\theta$, $y = r\sin\theta$ and $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = r.$

The parabola $y^2 = \dfrac{x}{a}$ implies $r^2\sin^2\theta = \dfrac{r\cos\theta}{a}$ *i.e.*, $r\left(r\sin^2\theta - \dfrac{\cos\theta}{a}\right) = 0$

$\Rightarrow$ either $r = 0$ or $r = \dfrac{\cos\theta}{a\sin^2\theta}$

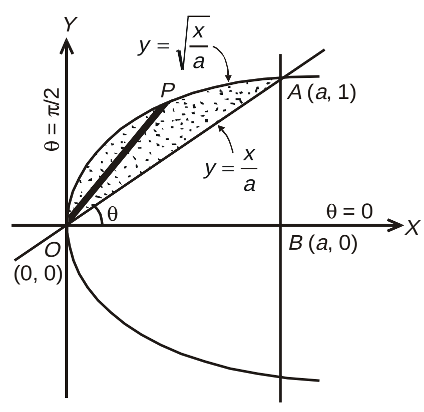

Fig. 5.45

Limits, for the curve $y = \dfrac{x}{a}$,

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{BA}{OB} = \tan^{-1}\frac{1}{a}$$

and for the curve $y = \sqrt{\dfrac{x}{a}}$

$$\theta = \tan^{-1}\frac{0}{a} = \frac{\pi}{2}$$

Here r (as a function of θ) varies from 0 to $\dfrac{\cos\theta}{a\sin^2\theta}$ and θ changes from $\tan^{-1}\dfrac{1}{a}$ to $\dfrac{\pi}{2}$.

Therefore, the integral,

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} \left(x^2 + y^2\right)$$

transforms to. $$I = \int_{\tan^{-1}\left(\frac{1}{a}\right)}^{\pi/2} \left(\int_0^{\left(\frac{\cos\theta}{a\sin^2\theta}\right)} r^3 dr \right) d\theta$$

$$I = \int_{\cot^{-1}(a)}^{\pi/2} \int_0^{r=\left(\frac{\cos\theta}{a\sin^2\theta}\right)} dr\, d\theta$$

$$= \frac{1}{4}\int_{\cot^{-1}a}^{\pi/2} \frac{\cos^4\theta}{a^4(\sin^4\theta)^2} d\theta$$

$$\Rightarrow \qquad I = \frac{1}{4a^4} \int_{\cot^{-1}a}^{\pi/2} \cot^4\theta\left(1+\cot^2\theta\right)\operatorname{cosec}^2\theta\, d\theta$$

Let $\cot\theta = t$ so that $\operatorname{cosec}^2\theta\, d\theta = dt\,(-1)$ and $\left.\begin{array}{l} \theta = \cot^{-1} a \Rightarrow t = a \\ \theta = \dfrac{\pi}{2} \qquad \Rightarrow t = 0 \end{array}\right\}$

$$\therefore \qquad I = \frac{1}{4a^4}\int_a^0 t^4\left(1+t^2\right)\left(-dt\right)$$

$$I = \frac{1}{4a^4}\int_0^a \left[t^4 + t^6\right] dt = \frac{1}{4a^4}\left[\frac{t^5}{5} + \frac{t^7}{7}\right]_0^a$$

$$I = \left(\frac{a}{20} + \frac{a^3}{28}\right).$$

Example 34: Evaluate $\int xy(x^2+y^2)^{\frac{n}{2}} dxdy$ over the positive quadrant of $x^2 + y^2 = 4$, supposing $n + 3 > 0$. **[SVTU, 2007]**

Solution: The double integral is to be evaluated over the area enclosed by the positive quadrant of the circle $x^2 + y^2 = 4$, whose centre is (0, 0) and radius 2.

Let $x = r\cos\theta$, $y = r\sin\theta$, so that $x^2 + y^2 = r^2$.

Therefore on transformation to polar co-ordinates,

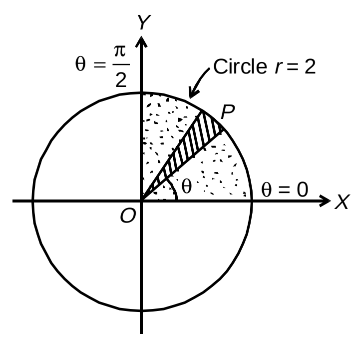

Fig. 5.46

$$I = \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2} r\cos\theta\, r\sin\theta\, r^n |J|\, dr\, d\theta,$$

$$= \int_0^{\pi/2}\int_0^2 \left(r^{n+3}dr\right)\sin\theta\cos\theta\, d\theta, \quad \left(J = \frac{\partial(x,y)}{\partial(r,\theta)} = r\right)$$

$$= \int_0^{\pi/2} \left(\frac{r^{n+4}}{n+4}\right)_0^2 \sin\theta\cos\theta\, d\theta$$

$$= \frac{2^{n+4}}{n+4}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta$$

$$= \frac{2^{n+4}}{(n+4)} \cdot \left|\frac{\sin^2\theta}{2}\right|_0^{\pi/2}, \text{ using } \int f'(x)\, f(x)\, dx = \frac{f^2(x)}{2}$$

$$= \frac{2^{n+3}}{(n+4)}, (n+3) > 0.$$

Example 35: Transform to cartesian coordinates and hence evaluate the $\int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta\, dr d\theta$.
[NIT Kurukshetra, 2007]

Solution: Clearly the region of integration is the area enclosed by the circle $r = 0$, $r = a$ between $\theta = 0$ to $\theta = \pi$.

Here $$I = \int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{a} r\sin\theta \cdot r\cos\theta \cdot r\, dr\, d\theta$$

On using transformation $x = r\cos\theta$, $y = r\sin\theta$,

$$I = \int_{-a}^{a}\int_0^{y=\sqrt{a^2-x^2}} xy\, dx\, dy$$

$$= \int_{-a}^{a} x\left(\int_0^{\sqrt{a^2-x^2}} y\, dy\right) dx$$

$$= \int_{-a}^{a} \left(\frac{y^2}{2}\right)\Bigg|_0^{\sqrt{a^2-x^2}} x\, dx$$

$$= \frac{1}{2}\int_{-a}^{a} x\left(a^2 - x^2\right) dx$$

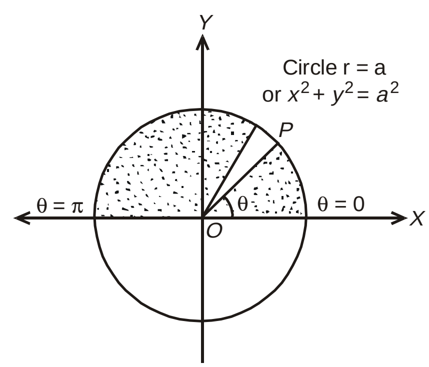

Fig. 5.47

As x and x^3 both are odd functions, therefore net value on integration of the above integral is zero.

i.e. $$I = \frac{1}{2}\int_{-a}^{a} \left(a^2 x - x^3\right) dx = 0.$$

ASSIGNMENTS 5

Evaluate the following integrals by changing to polar coordinates:

(1) $\int_0^{a}\int_0^{\sqrt{a^2-y^2}} (x^2 + y^2)\, dx\, dy$ (2) $\int_0^{a}\int_y^{a} \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$

(3) $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx\,dy$ (4) $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$ **[MDU, 2001]**

5.8 TRIPLE INTEGRAL (PHYSICAL SIGNIFICANCE)

The triple integral is defined in a manner entirely analogous to the definition of the double integral.

Let $F(x, y, z)$ be a function of three independent variables x, y, z defined at every point in a region of space V bounded by the surface S. Divided V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$ and let (x_r, y_r, z_r) be any point inside the rth sub division δV_r. Then, the limit of the sum

$$\sum_{r=1}^{n} F(x_r, y_r, z_r)\delta v_r\ , \qquad \ldots(1)$$

if exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the 'triple integral' of $R(x, y, z)$ over the region V, and is denoted by

$$\iiint F(x, y, z)\,dV \qquad \ldots(2)$$

In order to express triple integral in the 'integrated' form, V is considered to be sub-divided by planes parallel to the three coordinate planes. The volume V may then be considered as the sum of a number of vertical columns extending from the lower surface say, $z = f_1(x, y)$ to the upper surface say, $z = f_2(x, y)$ with base as the elementary areas δA_r over a region R in the xy-plance when all the columns in V are taken.

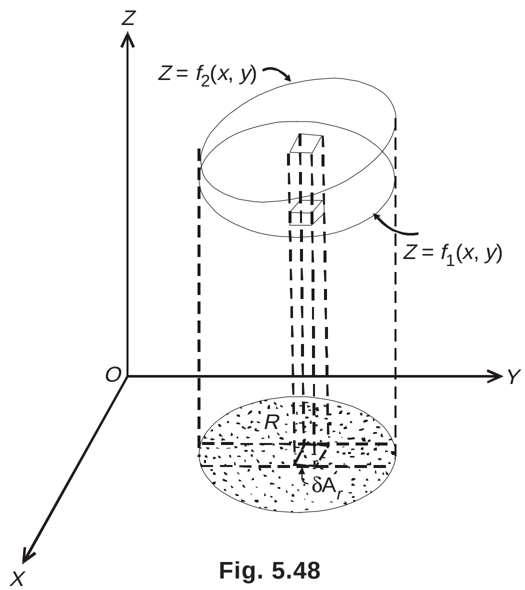

Fig. 5.48

On summing up the elementary cuboids in the same vertical columns first and then taking the sum for all the columns in V, it becomes

$$\sum_r \left[\sum_r F(x_r, y_r, z_r)\delta z\right]\delta A_r \quad \ldots(3)$$

with the pt. (x_r, y_r, z_r) in the rth cuboid over the element δA_r.

When δA_r and δz tend to zero, we can write (3) as

$$\int_R\left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\,dz\right] dA$$

Note: An ellipsoid, a rectangular parallelopiped and a tetrahedron are regular three dimensional regions.

5.9. EVALUATION OF TRIPLE INTEGRALS

For evaluation purpose, $\iiint_V F(x, y, z)\,dV$ $\qquad \ldots(1)$

is expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} F(x, y, z)\, dz\, dy\, dx \qquad \ldots(2)$$

where in the order of integration depends upon the limits.

If the limits z_1 and z_2 be the functions of (x, y); y_1 and y_2 be the functions of x and x_1, x_2 be constant, then

$$I = \int_{x=a}^{x=b}\left(\int_{y=\phi_1(x)}^{y=\phi_2(x)}\left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\right) dy\right) dx \qquad \ldots(3)$$

which shows that the first $F(x, y, z)$ is integrated with respect to z keeping x and y constant between the limits $z = f_1(x, y)$ to $z = f_2(x, y)$. The resultant which is a function of x, y is integrated with respect to y keeping x constant between the limits $y = f_1(x)$ to $y = f_2(x)$. Finally, the integrand is evaluated with respect to x between the limits $x = a$ to $x = b$.

Note: This order can accordingly be changed depending upon the comfort of integration.

Example 36: Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$. **[KUK, 2000, 2009]**

Solution: On integrating first with respect to z, keeping x and y constants, we get

$$I = \int_0^a\int_0^x \left[e^{(x+y)+z}\right]_0^{(x+y)} dy\, dx, \qquad \text{[Here } (x + y) = a, \text{ (say), like some constant]}$$

$$= \int_0^a\int_0^x \left[e^{(x+y)+(x+y)} - e^{(x+y)+0}\right] dy dx$$

$$= \int_0^a\int_0^x \left[e^{2(x+y)} - e^{(x+y)}\right] dy dx$$

$$= \int_0^a \left[\frac{e^{2x+2y}}{2} - \frac{e^{x+y}}{1}\right]_0^x dx, \quad \text{(Integrating with respect to } y\text{, keeping } x \text{ constant)}$$

$$= \int_0^a \left[\left(\frac{e^{4x}}{2} - \frac{e^{2x}}{1}\right) - \left(\frac{e^{2x}}{2} - \frac{e^{x}}{1}\right)\right] dx$$

On integrating with respect to x,

$$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + \frac{e^{x}}{1}\right]_0^a$$

$$= \left(\frac{e^{4a}}{8} - \frac{e^{2a}}{2} - \frac{e^{2a}}{4} + e^a\right) - \left(\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right)$$

$$\Rightarrow \qquad I = \left(\frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}\right).$$

Example 37: Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\, dr\, d\theta\, dz$. **[VTU, 2007; NIT Kurukshetra, 2007, 2010]**

Solution: On integrating with respect to z first keeping r and θ constants, we get

$$I = \int_0^{\pi/2}\int_0^{a\sin\theta} (z)_0^{\frac{a^2-r^2}{a}}\, r\,dr\,d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\int_0^{a\sin\theta}\left(a^2 - r^2\right) r\,dr\,d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\left(a^2\frac{r^2}{2} - \frac{r^4}{4}\right)_0^{a\sin\theta} d\theta, \quad \text{(On integrating with respect to r)}$$

$$= \frac{1}{a}\int_0^{\pi/2}\left(\frac{a^2\cdot a^2\sin^2\theta}{2} - \frac{a^4\sin^4\theta}{4}\right)d\theta$$

$$= \frac{a^3}{4}\int_0^{\frac{\pi}{2}}\left[2\sin^2\theta - \sin^4\theta\right]d\theta$$

$$= \frac{a^3}{4}\left[2\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\right],$$

$$\int_0^{\pi/2}\sin^p x\,dx = \frac{(p-1)\cdot(p-3)\ldots}{(p)\cdot(p-2)\ldots}\times\left(\frac{\pi}{2};\text{only if } p \text{ is even}\right)$$

$\therefore$

$$I = \frac{a^3}{4}\left[\frac{\pi}{2}\left(1 - \frac{3}{8}\right)\right] = \frac{5\pi a^3}{64}$$

Example 38: Evaluate $\displaystyle\iint_0^{\log y}\int_1^{e^x} \log z\,dz\,dy\,dx$. **[MDU, 2005; KUK, 2004, 05]**

Solution: $\displaystyle\int_1^e\int_0^{\log y}\left(\int_1^{e^x}\log z\,dz\right)dx\,dy$

[Here $z = f(x, y)$ with $z_1 = 1$ and $z_2 = e^{x+0y}$

$$= \int_1^e\int_0^{\log y}\left(\int_1^{e^x}\underset{\text{Ist fun.}}{\log z}\cdot\underset{\text{IInd fun.}}{1}\right)dz\,dx\,dy$$

$$= \int_1^e\int_0^{\log y}\left[\log z\times z - \int z\frac{1}{z}dz\right]_1^{e^x} dx\,dy$$

$$= \int_1^e\int_0^{\log y}\left[\left(e^x\log e^x - 1\cdot\log 1\right) - (z)_1^{e^x}\right]dx\,dy$$

$$= \int_1^e \left(\int_0^{\log y} \left[xe^x - \left(e^x - 1 \right) \right] dx \right) dy$$

$$= \int_1^e \int_0^{\log y} \left(\left[(x-1)e^x + 1 \right] dx \right) dy$$

$$= \int_1^e \left[xe^x - 2e^x + x \right]_0^{\log y} dy$$

$$= \int_1^e \left[\underset{\substack{\text{I}\\ \text{function}}}{\left(y+1 \right)} \cdot \underset{\substack{\text{II}\\ \text{function}}}{\log y} + 2\left(1-y \right) \right] dy$$

On integrating by parts,

$$I = \left[\log y \times \left(\frac{y^2}{2} + y \right) \Bigg|_1^e - \int_1^e \frac{1}{y} \cdot \left(\frac{y^2}{2} + y \right) dy + \left(2y - \frac{2y^2}{2} \right) \Bigg|_1^e \right]$$

$$= \left[(\log e)\left(\frac{e^2}{2} + e \right) - \log 1 \cdot \left(\frac{1}{2} + 1 \right) - \int_1^e \left(\frac{y}{2} + 1 \right) dy + \left(2e - e^2 \right) - \left(2 - 1 \right) \right]$$

$$= \left[\frac{e^2}{2} + e - \left(\frac{y^2}{4} + y \right) \Bigg|_1^e + 2e - e^2 - 1 \right]$$

$$= \left[\frac{e^2}{2} + e - \frac{e^2}{4} - e + \frac{1}{4} + 1 + 2e - e^2 - 1 \right]$$

$$= \left[\frac{1}{4}\left(1 + 8e - 3e^2 \right) \right].$$

Example 39: Evaluate $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} \left(x + y + z \right) dx\, dy\, dz.$ **[JNTU, 2000; Cochin, 2005]**

Solution: Integrating first with respect to y, keeping x and z constant,

$$I = \int_{-1}^{1} \int_0^z \left(\left[xy + \frac{y^2}{2} + yz \right]_{x-z}^{x+z} \right) dx\, dz$$

$$= \int_{-1}^{1} \left(\int_0^2 \left(4zx + 2z^2 \right) dx \right) dz$$

$$= \int_{-1}^{1} \left[4z \frac{x^2}{2} + 2 \cdot z^2 \cdot x \right]_0^z dz$$

$$= \int_{-1}^{1} \left[4z \cdot \frac{z^2}{2} + 2z^2 \cdot z \right] dz$$

$$= 4\int_{-1}^{1} z^3 dz = 4 \frac{z^4}{4} \Bigg|_{-1}^{1} = 0$$

ASSIGNMENT 6

Evaluate the following integrals:

(1) $\int_0^1\int_0^2\int_1^2 x^2yz\,dxdydz$ (2) $\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}\left(x^2+y^2+z^2\right)dxdydz$ **[VTU, 2000]**

(3) $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\,dx\,dz$ (4) $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}dzdydx$ **[NIT Kurukshetra, 2008]**

5.10 VOLUME AS A DOUBLE INTEGRAL

(Geometrical Interpretation of the Double Integral)

One of the most obvious use of double integral is the determination of volume of solids *viz*. 'volume between two surfaces'.

If $f(x, y)$ is a continuous and single valued function defined over the region R in the xy-plane with $z = f(x, y)$ as the equation of the surface. Let Γ be the closed curve which encloses R. Clearly, the surface R (*viz*. $z = f(x, y)$) is the orthogonal projection of S(*viz* $z = F(x, y)$) in the xy-plane.

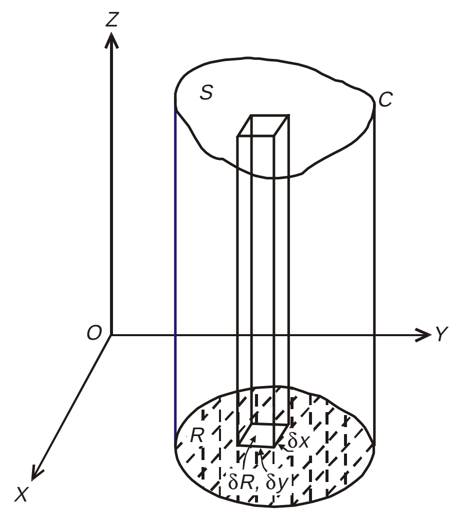

Fig. 5.49

Divided R into elementary rectangles of area $\delta x\delta y$ by drawing lines parallel to the axis of x and y. On each of these rectangles errect prisms having their lengths parallel to the z-axis. The volume of each such prism is $z\delta x\,\delta y$.

(Division of R is performed with the lines $x = x_i$ (i = 1, 2, …, m) and $y = y_j$(j = 1, 2, …, n). Through each line $x = x_i$, pass a plane parallel to yz-plane, and through each line $y = y_j$, pass a plance parallel to xz-plane. The rectangle ΔR_{ij} whose area is $\Delta A_{ij} = \Delta x_i\ \Delta y_j$ will be the base of a rectangle prism of height $f(x_{ij}, h_{ij})$, whose volume is approximately equal to the volume between the surface and the xy-plane $x = x_i - 1$, $x = x_i$; $y = y_i - 1$ $y = y_i$. Then $\sum_{\substack{i=1\\j=1}}^{n} f\left(\xi_{ij},\eta_{ij}\right)\Delta x_i\cdot\Delta y_j$ gives an approximate value for volume V of the prism of the cylinder enclosed between $z = f(x, y)$ and the xy-plane.

The volume V is the limit of the sum of each elementary volume $z\ \delta x\delta y$.

$$\therefore \qquad V = \underset{\substack{\delta x\to 0\\ \delta y\to 0}}{Lt}\sum\sum z\,\delta x\,\delta y = \iint_R z\,dx\,dy = \iint_R f\left(x,y\right)dA$$

***Note*:** In cyllidrical co-ordinates, the equation of the surface becomes $z = f(r, \theta)$, elementary area $dA = r\ dr\ d\theta$ and volume $= \iint_R f\left(r,\theta\right)r\,dr\,d\theta$

Problems on Volume of a Solid with the Help of Double Integral

Example 40: Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the co-ordinate planes. [Burdwan, 2003]

Solution: Given, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \;\Rightarrow\; z=f(x,y)=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$ ...(1)

If $f(x, y)$ is a continuous and single valued function over the region R (see Fig. 5. 50) in the xy plane, then $z = f(x, y)$ is the equation of the surface. Let C be the closed curve that is the boundary of R. Using R as a base, construct a cylinder having elements parallel to the z-axis. This cylinder intersects $z = f(x, y)$ in a curve Γ, whose projection on the xy-plane is C.

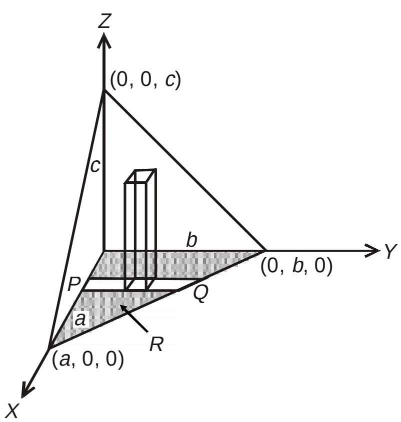

Fig. 5.50

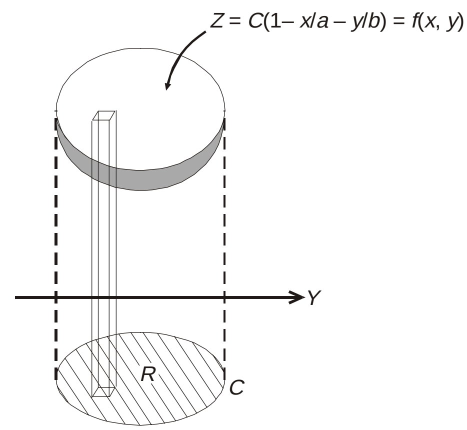

Fig. 5.51

The equation of the surface under which the region whose volume is required, may be written in the form (1) *i.e.*, $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$.

Hence the volume of the region $=\iint_R a\,dA=\iint_R c\left(1-\frac{x}{a}-\frac{y}{b}\right)dx\,dy$

The equation of the inter-section of the given surface with xy-plane is

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \text{...(2)}$$

If the prisms are summed first in the y-direction they will be summed from $y = 0$ to the line $y=b\left(1-\frac{x}{a}\right)$

Therefore,
$$V=\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$=c\int_0^a\left(y-\frac{xy}{a}-\frac{y^2}{2b}\right)\Bigg|_0^{b(1-x/a)}dx$$

$$= c\int_0^a b\left(\frac{1}{2} - \frac{x}{a} + \frac{x^2}{2a^2}\right)dx$$

$$= cb\left[\frac{x}{2} - \frac{x^2}{2a} + \frac{x^3}{6a^2}\right]_0^a$$

$$= bc\left[\frac{a}{2} - \frac{a^2}{2a} + \frac{a^3}{6a^2}\right] = \frac{abc}{6}.$$

Example 41: Prove that the volume enclosed between the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$ is $\frac{128a^2}{15}$.

Solution: Let V be required volume which is enclosed by the cylinder $x^2 + y^2 = 2ax$ and the paraboloid $z^2 = 2ax$.

Only half of the volume is shown in Fig 5.52.

Now, it is evident from that $z = \sqrt{2ax}$ is to be evaluated over the circle $x^2 + y^2 = 2ax$(with centre at $(a, 0)$ and radius a.

Here y varies from $-\sqrt{2ax - x^2}$ to $\sqrt{2ax - x^2}$ on the circle $x^2 + y^2 = 2ax$ and finally x varies from $x = 0$ to $x = 2a$

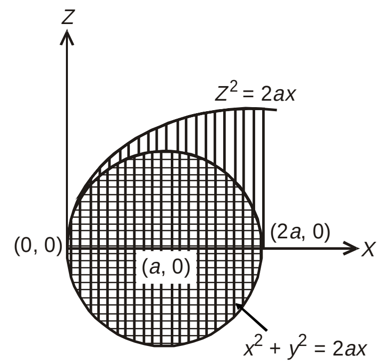

Fig. 5.52

$$\therefore \quad V = 2\int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} [z]\,dx\,dy \quad \text{as } z = f(x, y)$$

$$= 2\int_0^{2a}\left(2\cdot\int_0^{\sqrt{2ax-x^2}} \sqrt{2ax}\right)dy\,dx$$

$$= 4\int_0^{2a}\sqrt{2ax}\left(\int_0^{\sqrt{2ax-x^2}} dy\right)dx$$

$$= 4\int_0^{2a}\sqrt{2ax}\,|y|_0^{\sqrt{2ax-x^2}}\,dx = 4\int_0^{2a}\sqrt{2ax}\sqrt{2ax-x^2}\,dx$$

$$= 4\sqrt{2a}\int_0^{2a} x\sqrt{2a-x}\,dx$$

Let $x = 2a\sin^2\theta$, so that $dx = 4a\sin\theta\cos\theta\, d\theta$. Further, for $\left.\begin{array}{r} x = 0,\ \theta = 0 \\ x = 2a,\ \theta = \frac{\pi}{2}. \end{array}\right\}$

$$\therefore \quad V = 4\sqrt{2a}\int_0^{\pi/2} 2a\sin^2\theta\sqrt{2a}\cos\theta\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 64a^3\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta$$

$$= 64\,a^3\,\frac{(p-1)(p-3)\ldots(q-1)(q-3)\ldots}{(p+q)(p+q-2)\ldots}\cdot 1,\;\; p=3,\;\; q=2$$

$$= 64\,a^3\,\frac{(3-1)1}{5\cdot 3} = \frac{128\,a^3}{15}\cdot$$

Problems based on Volume as a Double Integral in Cylindrical Coordinates

Example 42: Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution: In cartesian co-ordinates, the section of the given hyperboloid $x^2 + y^2 - z^2 = 1$ in the xy plane ($z = 0$) is the circle $x^2 + y^2 = 1$, where as at the top and at the bottom end (along the z-axis *i.e.*, $z = \pm\sqrt{3}$) it shares common boundary with the circle $x^2 + y^2 = 4$ (Fig. 5.53 and 5.54).

Here we need to calculate the volume bounded by the two bodies (*i.e.*, the volume of shaded portion of the geometry).

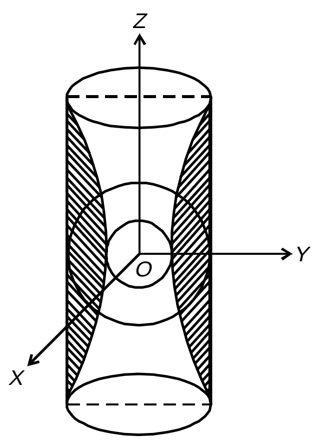

Fig. 5.53

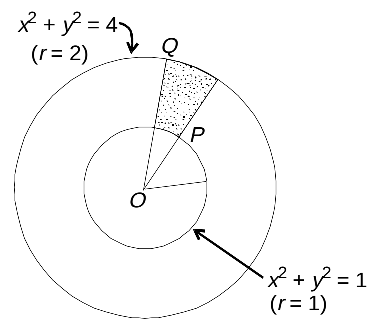

Fig. 5.54

(Best example of this geometry is a *solid damroo* in a *concentric long hollow drum*.)

In cylindrical polar coordinates, we see that here r varies from $r = 1$ to $r = 2$ and θ varies from 0 to 2π.

$$\therefore \qquad V = 2\left[\iint z\,dx\,dy\right] = 2\left[\iint f(r,\theta)\,r\,dr\,d\theta\right]$$

$$= 2\left[\int_0^{2\pi}\int_1^2 \sqrt{r^2-1}\,r\,dr\,d\theta\right] \qquad (\because \;\; x^2 + y^2 - z^2 - 1 \Rightarrow z = \sqrt{x^2+y^2-1}\,)$$

$$= 2\int_0^{2\pi}\left(\int_1^2 \frac{1}{3}\,d\left(r^2-1\right)^{\frac{3}{2}}\right)d\theta$$

$$= 2\int_0^{2\pi} \left. \frac{(r^2-1)^{\frac{3}{2}}}{3} \right|_1^2 d\theta$$

$$= 2\sqrt{3}\int_0^{2\pi} d\theta = 4\pi\sqrt{3}.$$

Example 43: Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$. **[KUK, 2000; MDU, 2002; Cochin, 2005; SVTU, 2007]**

Solution: From Fig. 5.55, it is very clear that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xy-plane.

To cover the shaded portion, x varies from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$ and y varies from – 2 to 2. Hence the desired volume,

$$V = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} z\,dx\,dy$$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{-2}^{2}(4-y)\left(\int_{0}^{\sqrt{4-y^2}} dx\right)dy$$

$$= 2\int_{-2}^{2}(4-y)\sqrt{4-y^2}\,dy$$

$$= 2\int_{-2}^{2}\left[4\sqrt{4-y^2} - y\sqrt{4-y^2}\right]dy$$

$$= 8\int_{-2}^{2}\sqrt{4-y^2}\,dy - 0$$

(The second term vanishes as the integrand is an odd function)

$$= 8\left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right]_{-2}^{2} = 16\pi.$$

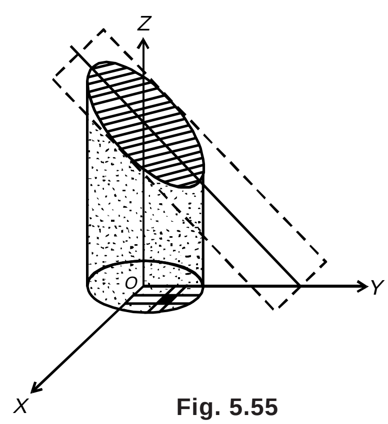

Fig. 5.55

ASSIGNMENT 7

1. Find the volume enclosed by the coordinate planes and the portion of the plane $lx + my + nz = 1$ lying in the first quadrant.
2. Obtain the volume bounded by the surface $z = c\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)$ and the quadrant of the elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

   [**Hint:** Use elliptic polar coordinates $x = ar\cos\theta$, $y = br\sin\theta$]

5.11 VOLUME AS A TRIPLE INTEGRAL

Divide the given solid by planes parallel to the coordinate plane into rectangular parallelopiped of elementary volume $\delta x \delta y \delta z$.

Then the total volume V is the limit of the sum of all elementary volume i.e.,

$$V = \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{Lt} \sum\sum\sum \delta x\, \delta y\, \delta z = \iiint dx\, dy\, dx$$

Problems based on Volume as a Triple Integral in cartesian Coordinate System

Example 44: Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Solution: The sections of the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ are the circles $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ in xy and xz plane respectively.

Here in the picture, one-eighth part of the required volume (covered in the 1st octant) is shown.

Clearly, in the common region, z varies from 0 to $\sqrt{a^2 - x^2}$ *i.e.*, $\sqrt{a^2 - 1x^2 - 0y^2}$, and x and y vary on the circle $x^2 + y^2 = a^2$.

The required volume

$$\therefore \quad V = 8\int_0^a \int_{y_1=0}^{y_2=\sqrt{a^2-x^2}} \int_{z_1=0}^{z_2=\sqrt{a^2-x^2-0y^2}} dz\, dy\, dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \left(z\Big|_0^{\sqrt{a^2-x^2}} \right) dy\, dx$$

$$= 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dy \right) dx$$

$$= 8\int_0^a \left(\sqrt{a^2 - x^2} \right) \left(\int_0^{\sqrt{a^2-x^2}} dy \right) dx$$

$$= 8\int_0^a \sqrt{a^2 - x^2} \left(\sqrt{a^2 - x^2} - 0 \right) dx$$

$$= 8\int_0^a \left(a^2 - x^2 \right) dx = 8\left[\left(a^2 x - \frac{x^3}{3} \right)\Bigg|_0^a \right]$$

$$= 8\left(a^3 - \frac{a^3}{3} \right) = \frac{16a^3}{3}.$$

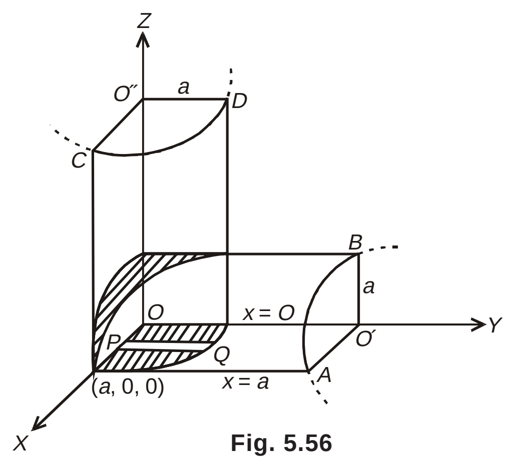

Fig. 5.56

Example 45: Find the volume bounded by the xy plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$.

Solution: Let $V(x, y, z)$ be the desired volume enclosed laterally by the cylinder $x^2 + y^2 = 1$ (in the xy-plane) and on the top, by the plane $x + y + z = 3$ (= a say).

Clearly, the limits of z are from 0 (on the xy-plane) to $z = (3 - x - y)$ and x and y vary on the circle $x^2 + y^2 = 1$

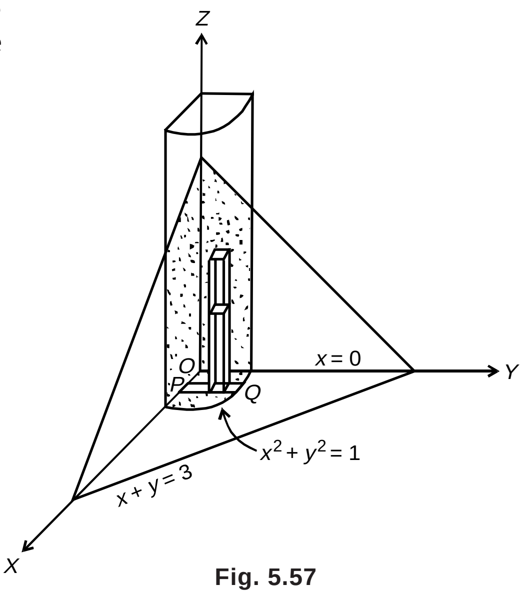

Fig. 5.57

$$\therefore \qquad V(x, y, z) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{3-x-y} dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{1\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(z\Big|_0^{(3-x-y)}\right)dy\,dx$$

$$= \int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(3 - x - y)\,dy\right)dx$$

$$= \int_{-1}^{1}\left[3y - xy - \frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$\Rightarrow \qquad I = \int_{-1}^{1}\left(6 \times \sqrt{1 - x^2} - 2x\sqrt{1 - x^2}\right)dx$$

On taking $x = \sin\theta$, we get $dx = d\theta$; $\left.\begin{array}{l}\text{For } x = -1, \theta = -\dfrac{\pi}{2} \\ \text{For } x = 1, \quad \theta = \dfrac{\pi}{2}\end{array}\right\}$

Thus,

$$V = \int_{-\pi/2}^{\pi/2}\left(6\sqrt{1 - \sin^2\theta} - 2\sin\theta\sqrt{1 - \sin^2\theta}\right)\cos\theta\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left(6\cos^2\theta - 2\sin\theta\cos^2\theta\right)d\theta$$

$$= 6 \times 2\underbrace{\int_{0}^{\pi/2}\cos^2\theta\,d\theta}_{\text{Ist}} - 2\underbrace{\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta\,d\theta}_{\text{IInd}}$$

$$= 12\frac{(2 - 1)}{2}\cdot\frac{\pi}{2} + 2\frac{\cos^3\theta}{3}\Bigg|_{-\pi/2}^{\pi/2} = 3\pi + \frac{2}{3} \times 0 = 3\pi$$

Using $\displaystyle\int_0^{\pi/2}\cos^p\theta\,d\theta = \frac{(p-1)(p-3)\ldots}{p(p-2)\ldots} \times \left(\frac{\pi}{2}, \text{ only if } p \text{ is even}\right)$ and

$$\int f'(x)\, f^n(x)\, dx = \frac{f^{n+1}(x)}{n+1}$$ for Ist and IInd integral respectively

Example 46: Find the volume bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

[MDU, 2000; KUK, 2001; Kottayam, 2005; PTU, 2006]

Solution: Considering the symmetry, the desired volume is 8 times the volume of the ellipsoid into the positive octant.

The ellipsoid cuts the XOY plane in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $z = 0$.

Therefore, the required volume lies between the ellipsoid

$$z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

and the plane XOY (*i.e.*, $z = 0$) and is bounded on the sides by the planes $x = 0$ and $y = 0$

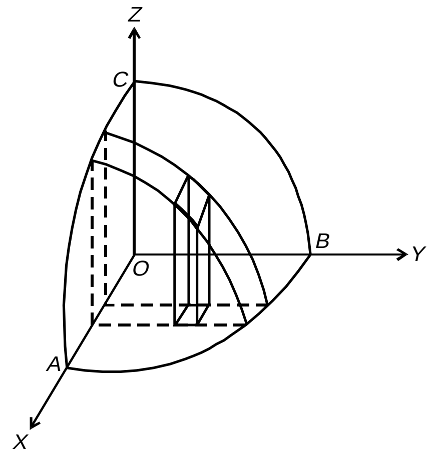

Fig. 5.58

Hence,
$$V = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\, dx$$

$$= 8\int_0^a \left(\int_0^{\alpha} \frac{c}{b}\sqrt{\alpha^2-y^2}\, dy\right) dx \qquad \left(\text{taking } \sqrt{\left(1-\frac{x^2}{a^2}\right)} = \frac{\alpha}{b}\right)$$

$$V = 8\frac{c}{b}\int_0^a \left[\frac{y\sqrt{\alpha^2-y^2}}{2}+\frac{\alpha^2}{2}\sin^{-1}\frac{y}{\alpha}\right]_0^{\alpha} dx$$

$$\left(\text{Using formula } \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\tan^{-1}\frac{x}{a}\right)$$

$$= 8\frac{c}{b}\int_0^a \left[0+\frac{\alpha^2}{2}\sin^{-1}1\right] dx$$

$$= \frac{4c}{b}\int_0^a \frac{\pi}{2}\alpha^2 dx = \frac{2\pi c}{b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right) dx, \quad \alpha = b\sqrt{1-\frac{x^2}{b^2}}$$

$$= 2\pi bc\left[x-\frac{1}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= \frac{4}{3}\pi abc.$$

Example 47: Evaluate the integral $\iint \dfrac{dxdydz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$ taken throughout the volume of the sphere. **[MDU, 2000]**

Solution: Here for the given sphere $x^2 + y^2 + z^2 = a^2$, any of the three variables x, y, z can be expressed in term of the other two, say $z = \pm\sqrt{a^2 - x^2 - y^2}$.

In the xy-plane, the projection of the sphere is the circle $x^2 + y^2 = a^2$.

Thus,
$$I = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$$

$$= 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz}{\sqrt{\alpha^2 - z^2}} \right) dy \right) dx, \ \alpha^2 = (a^2 - x^2 - y^2)$$

$$= 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\sin^{-1}\frac{z}{\alpha} \right)_0^{\alpha} dy \right) dx$$

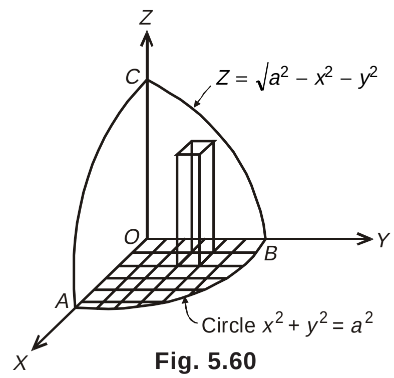

Fig. 5.60

$$= 8\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} \left(\sin^{-1}1 - \sin^{-1}0 \right) dy \right) dx$$

$$= 8\frac{\pi}{2}\int_0^a \left(\int_0^{\sqrt{a^2-x^2}} dy \right) dx = 4\pi\int_0^a \left(y\Big|_0^{\sqrt{a^2-x^2}} \right) dx$$

$$= 4\pi\int_0^a \sqrt{a^2 - x^2}\,dx = 4\pi\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a$$

$$= 4\pi\left[0 + \frac{a^2}{2}\frac{\pi}{2} \right] \quad I = \pi^2 a^2.$$

Example 48: Evaluate $\int (x + y + z)\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Solution: The integration is over the region R(shaded portion) bounded by the plane $x = 0$, $y = 0$, $z = 0$ and the plane $x + y + z = 1$.

The area OAB, in xy plane is bounded by the lines $x + y = 1$, $x = 0$, $y = 0$

Hence for any pt. (x, y) within this triangle, z goes from xy plane to plane ABC (*viz*. the surface of the tetrahedron) or in other words, z changes from $z = 0$ to $z = 1 - x - y$. Likewise in plane xy, y as a function x varies from $y = 0$ to $y = 1 - x$ and finally x varies from 0 to 1.

whence,
$$I = \underset{(over\,R)}{\iiint} (x + y + z)\,dx\,dy\,dz$$

$$= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} (x + y + z)\,dz \right) dy \right) dx$$

$$= \int_0^a \int_0^{1-x} \left((x+y)z + \frac{z^2}{2} \right) \Bigg|_0^{1-x-y} dy\,dx$$

$$= \int_0^a \int_0^{1-x} \left[(x+y)(1-x-y) + \frac{(1-x-y)^2}{2} \right] dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)(1+x+y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}\left[1-(x+y)^2\right] dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left[y - \frac{(x+y)^3}{3} \right]_0^{1-x} dx,$$

$$= \frac{1}{2}\int_0^1 \left[(1-x) - \left(\frac{1}{3} - \frac{x^3}{3} \right) \right] dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{12} \right]_0^1$$

$$= \frac{1}{2}\left[\frac{2}{3} - \frac{1}{2} + \frac{1}{12} \right] = \frac{1}{8}$$

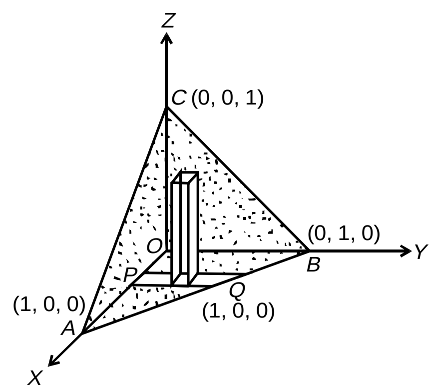

Fig. 5.61

ASSIGNMENT 8

1. Find the volume of the tetrahedron bounded by co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, by using triple integration **[KUK, 2002]**
2. Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.

5.12. VOLUMES OF SOLIDS OF REVOLUTION AS A DOUBLE INTEGRAL

Let P(x, y) be any point in a region R enclosing an elementary area $dx\ dy$ around it. This elementary area on revolution about x-axis form a ring of volume,

$$\delta V = \pi[(y + \delta y)^2 - y^2]\ \delta x = 2\pi y \delta x \delta y \quad \ldots(1)$$

Hence the total volume of the solid formed by revolution of this region R about x-axis is,

$$V = \iint_R 2\pi y\,dx\,dy \quad \ldots(2)$$

Similarly, if the same region is revolved about y-axis, then the required volume becomes

$$V = \iint_R 2\pi x\,dx\,dy \quad \ldots(3)$$

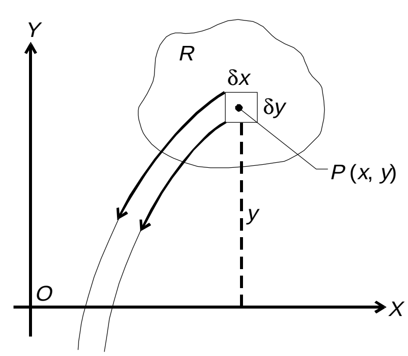

Fig. 5.62

Expressions for above volume in polar coordinates **about the initial** line and **about the pole** are $\iint_R 2\pi r^2 \sin\theta\, dr\, d\theta$ and $\iint_R 2\pi r^2 \cos\theta\, dr\, d\theta$ respectively.

Example 49: Find by double integration, the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about y-axis.

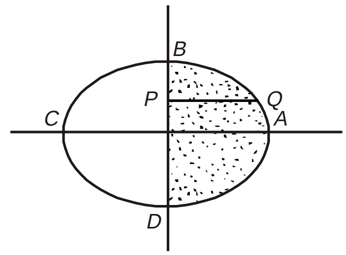

Fig. 5.63

Solution: As the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is symmetrical about the y-axis, the volume generated by the left and the right halves overlap.

Hence we shall consider the revolution of the right-half ABD for which x-varies from 0 to $a\sqrt{1-\frac{y^2}{b^2}}$ and y-varies from $-b$ to b.

$$\therefore \qquad V = \int_{-b}^{b}\int_{0}^{\frac{a}{b}\sqrt{b^2-y^2}} 2\pi x\, dx\, dy$$

$$= 2\pi\int_{-b}^{b}\left[\frac{x^2}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy = \frac{\pi a^2}{b^2}\int_{-b}^{b}\left(b^2-y^2\right)dy$$

$$= 2\pi\frac{a^2}{b^2}\int_0^b\left(b^2-y^2\right)dy = \frac{2\pi a^2}{b^2}\left[b^2y-\frac{y^3}{3}\right]_0^b$$

$$= \frac{4}{3}\pi a^2 b.$$

Example 50: The area bounded by the parabola $y^2 = 4x$ and the straight lines $x = 1$ and $y = 0$, in the first quadrant is revolved about the line $y = 2$. Find by double integration the volume of the solid generated.

Solution: Draw the standard parabola $y^2 = 4x$ to which the straight line $y = 2$ meets in the point $P(1, 2)$, Fig. 5.64.

Now the dotted portion *i.e.*, the area enclosed by parabola, the line $x = 1$ and $y = 0$ is revolved about the line $y = 2$.

$\therefore$ The required volume,

$$V = \int_0^1\int_0^{2\sqrt{x}} 2\pi\left(2-y\right)dx\, dy$$

$$= 2\pi\int_0^1\left[2y-\frac{y^2}{2}\right]_0^{2\sqrt{x}} dx = 2\pi\int_0^1\left(4\sqrt{x}-2x\right)dx$$

$$= 2\pi\left[\frac{8}{3}x^{3/2}-x^2\right]_0^1 = \frac{10\pi}{3}$$

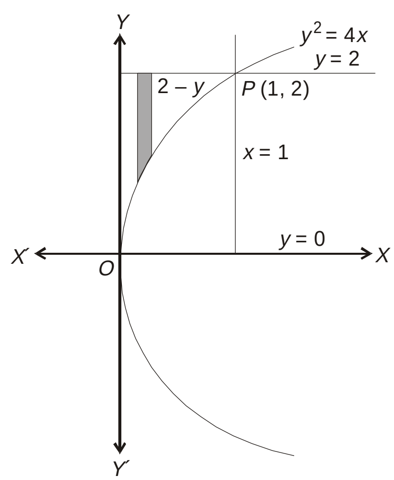

Fig. 5.64

Example 51: Calculate by double integration, the volume generated by the revolution of the cardiod $r = a(1 - \cos\theta)$ about its axis. [KUK, 2007, 2009]

Soluton: On considering the upper half of the cardiod, because due to symmetry the lower half generates the same volume.

$$\therefore \quad V = \int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= 2\pi\int_0^{\pi}\left|\frac{r^3}{3}\right|_0^{a(1-\cos\theta)} \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi}(1-\cos\theta)^3 \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3}\left|\frac{(1-\cos\theta)^4}{4}\right|_0^{\pi} = \frac{8\pi a^3}{3}.$$

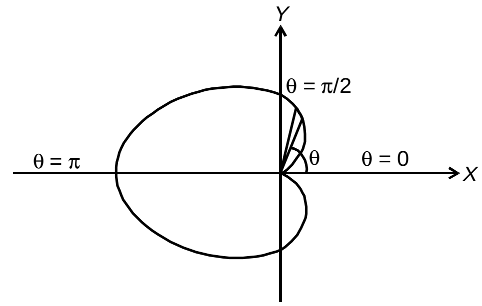

Fig. 5.65

Example 52: By using double integral, show that volume generated by revolution of cardiod $r = a(1 + \cos\theta)$ about the initial line is $\frac{8}{3}\pi a^3$.

Solution: The required volume

$$= \int_0^{\pi}\int_0^{a(1+\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= 2\pi\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} \sin\theta\, d\theta$$

$$= 2\pi\int_0^{\pi} a^3(1+\cos\theta)^3 \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3}\left[-\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi}$$

$$= -\frac{2\pi a^3}{3}\left[0 - \frac{2^4}{4}\right] = \frac{8\pi a^3}{3}.$$

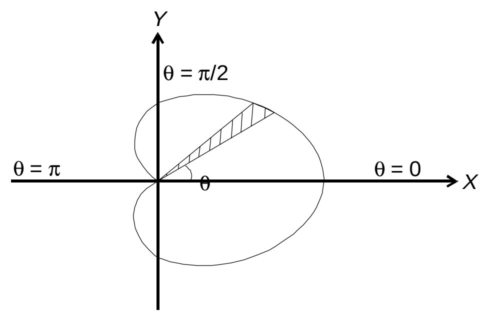

Fig. 5.66

ASSIGNMENT 9

1. Find by double integration the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the X-axis.
2. Find the volume generated by revolving a quadrant of the circle $x^2 + y^2 = a^2$, about its diameter.
3. Find the volume generated by the revolution of the curve $y^2(2a - x) = x^3$, about its asymptote through four right angles.
4. Find the volume of the solid obtained by the revolution of the leminiscate $r^2 = a^2\cos 2\theta$ about the initial line. **[Jammu Univ., 2002]**

5.13. CHANGE OF VARIABLE IN TRIPLE INTEGRAL

For transforming elementary area or the volume from one sets of coordinate to another, the necessary role of 'Jacobian' or 'functional determinant' comes into picture.

(a) Triple Integral Under General Transformation

Here $\iiint\limits_{R(x,y,z)} f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{R'(u,v,w)} F(u,v,w)\,|J|\,du\,dv\,dw$; where $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} (\neq 0)$ ...(1)

Since in the case of three variables $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ be continuous together with their first partial derivatives, the Jacobian of u, v, w with respect to x, y, z is defined by

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} & \dfrac{\partial w}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} & \dfrac{\partial w}{\partial y} \\ \dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial z} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

(b) Triple Integral in Cylindrical Coordinates

Here $\iiint\limits_{R} f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{R'} F(r,\theta,z)|J|\,dr\,d\theta\,dz$, where $|J| = r$

The position of a point P in space in cylindrical coordinates is determined by the three numbers r, θ, z where r and θ are polar co-ordinates of the projection of the point P on the xy-plane and z is the z coordinate of P *i.e.*, distance of the point (P) from the xy-plane with the plus sign if the point (P) lies above the xy-plane, and minus sign if below the xy-plane (Fig. 5.67).

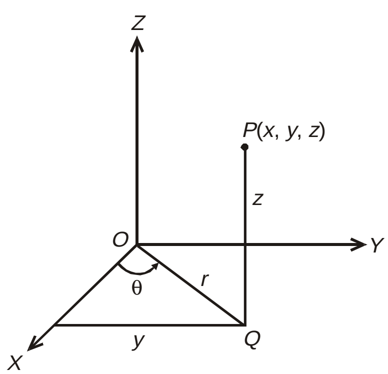

Fig. 5.67

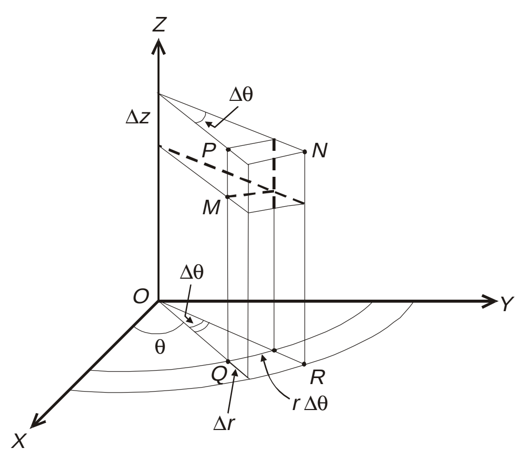

Fig. 5.68

In this case, divide the given three dimensional region R' (r, θ, z) into elementary volumes by coordinate surfaces $r = r_i$, $\theta = \theta_j$, $z = z_k$ (viz. half plane adjoining z-axis, circular cylinder axis coincides with Z-zxis, planes perpenducular to z-axis). The curvilinear 'prism' shown in Fig. 5. 68 is a volume element of which elementary base area is $r\,\Delta r\Delta\theta$ and height Δz, so that $\Delta v = r\,\Delta r\,\Delta\theta\,\Delta z$.

Here θ is the angle between OQ and the positive x-axis, r is the distance OQ and z is the distance QP. From the Fig. 5.62, it is evident that

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z \text{ and so that,}$$

$$J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \qquad \ldots(2)$$

Hence, the triple integral of the function F(r, θ, z) over R´ becomes

$$V = \iiint_{R'(r,\theta,z)} F(r,\theta,z)\, r\, dr\, d\theta dz \qquad \ldots(3)$$

(*c*) Triple Integral in Spherical Polar Coordinates

Here $\quad V = \iiint_R f(x,y,z)\,dxdydz = \iiint_R F(r,\theta,\phi)\,|J|\,drd\theta d\phi$, where $|J| = r^2\sin\theta$

The position of a point P in space in spherical coordinates is determined by the three variables r, θ, ϕ where r is the distance of the point (P) from the origin and so called radius vector, θ is the angle between the radius vector on the xy-plane and the x-axis to count from this axis in a positive sense viz. counter-clockwise.

For any point in space in spherical coordinates, we have

$$0 \le r \le \infty,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi.$$

Divide the region 'R' into elementary volumes ΔV by coordinate surfaces, r = constant (sphere), θ = constant (conic surfaces with vertices at the origin), ϕ = constant (half planes passing through the Z-axis).

To within infinitesimal of higher order, the volume element Δv may be considered a parallelopiped with edges of length Δr, $r\,\Delta\theta$, $r\sin\theta\,\Delta\phi$. Then the volume element becomes $\Delta V = r^2\sin\theta\,\Delta r\,\Delta\theta\,\Delta\phi$.

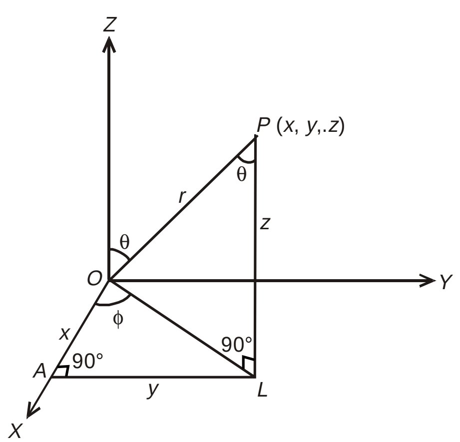

Fig. 5.69

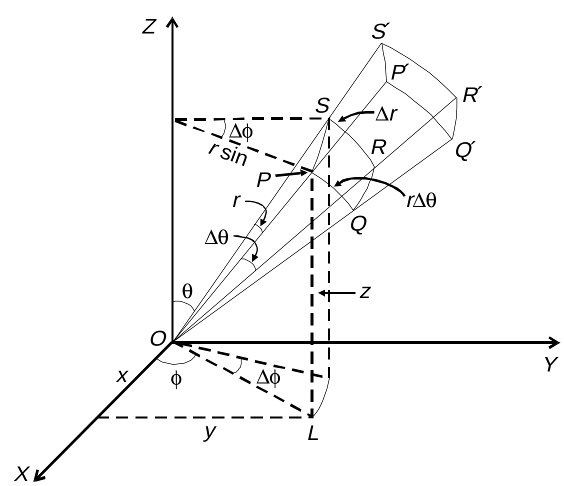

Fig. 5.70

For calculation purpose, it is evident from the Fig. 5.69 that in triangles, OAL and OPL,

$$x = \text{OL}\cos\phi = \text{OP}\cos(90-\theta)\,.\,\cos\phi = r\sin\theta\cos\phi,$$
$$y = \text{OL}\sin\phi = \text{OP}\sin\theta\,.\,\sin\phi = r\sin\theta\sin\phi,$$
$$z = r\cos\theta.$$

Thus, $$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix} = r^2\sin\theta$$

Problems Volume as a Triple Integral in Cylindrical Co-ordinates

Example 53: Find the volume intercepted between the paraboloid $x^2 + y^2 = 2az$ and the cylinder $x^2 + y^2 - 2ax = 0$.

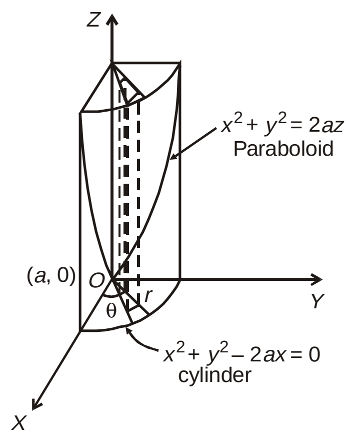

Fig. 5.71

Solution: Let V be required volume of the cylinder $x^2 + y^2 - 2ax = 0$ intercepted by the paraboloid $x^2 + y^2 = 2az$.

Transforming the given system of equations to polar-cylindrical co-ordinates.

Let $\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}$ so that $V(x, y, z) = V(r, \theta, z)$

By above substitution the equation of the paraboloid becomes $r^2 = 2az \;\Rightarrow\; z = \dfrac{r^2}{2a}$ and the cylinder $x^2 + y^2 = 2ax$ gives $r^2 - 2ar\cos\theta = 0 \;\Rightarrow\; r(r - 2a\cos\theta) = 0$ with $r = 0$ and $r = 2a\cos\theta$.

Thus, it is clear from the Fig. 5.71 that z varies from 0 to $\dfrac{r^2}{2a}$ and r as a function of θ varies from 0 to $2a\cos\theta$ with θ as limits 0 to 2π. Geometry clearly shows the volume covered under the +ve octant only, i.e. $\dfrac{1}{4}$th of the full volume.

$$\underset{(x,y,z)}{V} = \underset{(r,\theta,z)}{V'} = 4\int_0^{\theta=\pi/2}\int_{r=0}^{r=2a\cos\theta}\int_{z=0}^{z=r^2/2a} r\,dz\,dr\,d\theta, \text{ as } |J| = r$$

$$= 4\int_0^{\pi/2}\left(\int_0^{2a\cos\theta} r\left[z\right]_0^{r^2/2a} r\,dr\right)d\theta$$

$$= 4\int_0^{\pi/2}\left(\int_0^{2a\cos\theta}\frac{r^3}{2a}dr\right)d\theta$$

$$= 4\frac{1}{2a}\int_0^{\pi/2}\left.\frac{r^4}{4}\right|_0^{2a\cos\theta} d\theta$$

$$= 4\frac{1}{2a}\int_0^{\pi/2}\frac{2^4a^4}{4}\cos^4\theta\,d\theta$$

$$= 2^3a^3\frac{(4-1)(4-3)}{4\times 2}\frac{\pi}{2}$$

$$= \frac{3\pi a^3}{2}.$$

Example 54: Find the volume of the region bounded by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = b^2$. Also find the integral in case when $a = 2$ and $b = 2$.

Solution: On using the cylindrical polar co-ordinates (r, θ, z) with $x = r\cos\theta$, $y = r\sin\theta$, so that the equations of the cylinder and that of the paraboloid are $r = b$ and $z = \frac{r^2}{a}$ respectively.

See Fig. 5.72, only one-fourth of the common volume is shown.

Hence in the common region, z varies from $z = 0$ to $z = \frac{r^2}{a}$ and r and θ varies on the circle from 0 to b and 0 to $\frac{\pi}{2}$ respectively.

$\therefore$ The desired volume

$$V = 4\int_0^{\pi/2}\int_0^b\int_0^{r^2/a} r\,dr\,d\theta\,dz$$

$$= 4\int_0^{\pi/2}\left(\int_0^b r\,dr\left(\int_0^{r^2/a} dz\right)\right)d\theta$$

$$= 4\int_0^{\pi/2}\left(\int_0^b r\left(\frac{r^2}{a}\right)dr\right)d\theta$$

$$= \frac{4}{a}\int_0^{\pi/2}\left(\left.\frac{r^4}{4}\right|_0^b\right)d\theta$$

$$= \frac{4}{a}\times\frac{b^4}{4}\left.\theta\right|_0^{\pi/2} = \frac{\pi b^2}{2a}$$

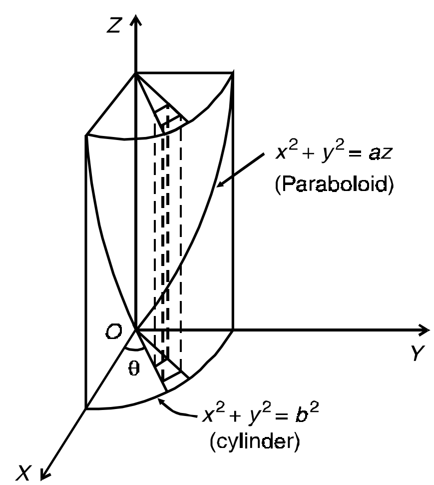

Fig. 5.72

As a particular case, when $a = 2$, $b = 2$, then

$$V = \frac{\pi(2)^4}{2\times 2} = 4\pi$$

Problmes on Volume in Polar Spherical Co-ordinates

Example 55: Find the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cone $x^2 + y^2 = z^2$

OR

Find the volume cut by the cone $x^2 + y^2 = z^2$ from the sphere $x^2 + y^2 + z^2 = a^2$.

[NIT Kurukshetra, 2010]

Solution: For the given sphere, $x^2 + y^2 + z^2 = a^2$ and the cone $x^2 + y^2 = z^2$, the centre of the sphere is (0, 0, 0) and the vertex of the cone is origin. Therefore, the volume common to the two bodies is symmetrical about the plane $z = 0$, i.e. the required volume, $V = 2\iiint dxdydz$

In spherical co-ordinates, we have $\left.\begin{array}{l} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{array}\right\}; J = r^2\sin\theta$

Thus, $x^2 + y^2 + z^2 = a^2$ becomes $r^2 = a^2$ *i.e.*, $r = a$

and $x^2 + y^2 = z^2$ becomes $r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = r^2\cos^2\theta$

i.e., $\sin^2\theta = \cos^2\theta$ *i.e.* $\theta = \pi/4$.

Clearly, the volume shown in the figure (Fig. 5.73) is one-fourth, i.e. in first quadrant only and, in the common region,

$$\left.\begin{array}{l} r \text{ varies from 0 to } a, \\ \theta \text{ varies from 0 to } \dfrac{\pi}{4}, \\ \phi \text{ varies from 0 to } \dfrac{\pi}{2} \end{array}\right\}$$

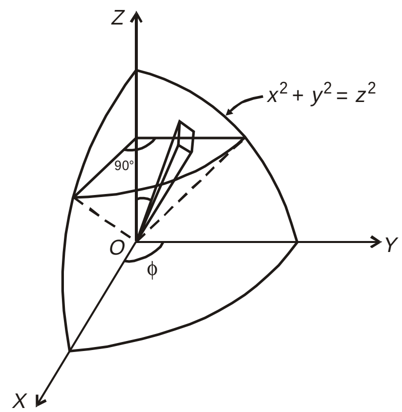

Fig. 5.73

Hence the required volume,

$$V = 2\left[4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^a r^2\sin\theta\, dr\, d\theta\, d\phi\right]$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/4}\left(\int_0^a r^2 dr\right)\sin\theta\, d\theta\, d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/4}\left(\frac{r^3}{3}\right)_0^a \sin\theta\, d\theta\, d\phi$$

$$= \frac{8}{3}a^3\int_0^{\pi/2}[-\cos\theta]_0^{\pi/4}\, d\phi$$

$$= \frac{8}{3}a^3\left(1 - \frac{1}{\sqrt{2}}\right)\int_0^{\pi/2} d\phi$$

$$= \frac{4\pi a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

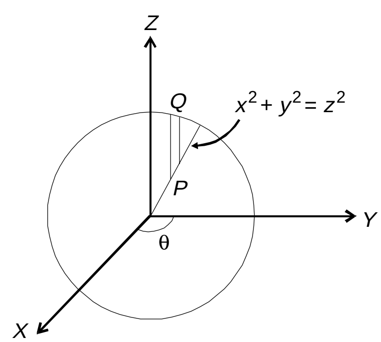

Fig. 5.74

Alternately: In polar-cylindrical co-ordinates, intersection of the two curves $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 = z^2$ results in $z^2 + z^2 = a^2$ or $z^2 = \frac{a^2}{2}$.

Further, $x^2 + y^2 = a^2 - z^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2} \Rightarrow r = \frac{a}{\sqrt{2}}$, i.e. r varies from 0 to $\frac{a}{\sqrt{2}}$

Hence, $$V = 2\int_0^{2\pi}\int_0^{a/\sqrt{2}}\left(\sqrt{a^2 - r^2} - r\right) r\, dr\, d\theta$$

| $\because$ P lies on the cone whereas Q lies on the sphere as a function of (r, θ)

$$= 2\int_0^{a/\sqrt{2}} \left(r\sqrt{a^2 - r^2} - r^2\right)\left(\int_0^{2\pi} d\theta\right) dr$$

$$= 4\pi\left[-\frac{1}{3}\left(a^2 - r^2\right)^{3/2} - \frac{r^3}{3}\right]_0^{\frac{a}{\sqrt{2}}} \quad \left[\text{since } r\left(a^2 - r^2\right)^{\frac{1}{2}} = \frac{-1}{3}\left(-3r\left(a^2 - r^2\right)^{\frac{1}{2}}\right) = \frac{-1}{3} d\left(a^2 - r^2\right)^{\frac{3}{2}}\right]$$

$$= 4\pi\left[-\frac{1}{3}\frac{a^3}{2\sqrt{2}} - \frac{1}{3}\frac{a^3}{2\sqrt{2}} + \frac{a^3}{3}\right]$$

$$= \frac{4\pi a^3}{3}\left[1 - \frac{1}{\sqrt{2}}\right]$$

Example 56: By changing to shperical polar co-ordinate system, prove that $\displaystyle\iiint_V \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dx\, dy\, dz = \frac{\pi}{4} abc$ **where** $V = \left\{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\right\}$

Solution: Taking $\left.\begin{array}{l} \frac{x}{a} = u, \\ \frac{y}{b} = v, \\ \frac{z}{c} = w \end{array}\right\}$, so that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \quad \Rightarrow \quad u^2 + v^2 + w^2 \le 1$

Now transformation co-efficient,

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$\therefore$
$$V = \iiint_{V(x,y,z)} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dx\, dy\, dz$$

$$= \iiint_{V'(u,v,w)} \sqrt{1 - u^2 - v^2 - w^2}\, (abc)\, du\, dv\, dw$$

To transform to polar spherical co-rodinate system, let $\left.\begin{array}{l} u = r\sin\theta\cos\phi, \\ v = r\sin\theta\sin\phi, \\ w = r\cos\theta \end{array}\right\}$

Then $V'_{(u,\, v,\, w)} = \{(u, v, w) : u^2 + v^2 + w^2 \le 1,\ u \ge 0,\ v \ge 0,\ w \ge 0\}$ reduces to

$V''_{(r,\, \theta,\, \phi)} = \{r^2 \le 1 \quad i.e., \quad 0 \le r \le 1,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi\}$

$\therefore$
$$= \iiint_{V'(u,v,w)} \sqrt{1 - u^2 - v^2 - w^2}\, abc\, du\, dv\, dw$$

$$= \iiint\limits_{V''(r,\theta,\phi)} abc\sqrt{1-r^2}\,|J|\,dr\,d\theta\,d\phi \qquad \text{where } |J| = r^2\sin\theta$$

$$\Rightarrow \qquad V''_{(r,\theta,\phi)} = abc\int_{\phi=0}^{\phi=2\pi}\left(\int_0^{\pi}\left(\int_0^1\sqrt{1-r^2}\,r^2dr\right)\sin\theta\,d\theta\right)d\phi$$

Now put $r = \sin t$ so that $dr = \cos t\,dt$ and for $\left.\begin{aligned} r = 0,\ t = 0,\\ r = 1,\ t = \frac{\pi}{2}\end{aligned}\right\}$

$$\therefore \qquad V''_{(r,\theta,\phi)} = abc\int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{\pi/2}\cos t\sin^2 t\cos t\,dt\right)\sin\theta\,d\theta\right)d\phi$$

$$= abc\int_0^{2\pi}\left(\int_0^{\pi}\left[\frac{(2-1)\cdot(2-1)}{(2+2)(4-2)}\frac{\pi}{2}\right]\sin\theta\,d\theta\right)d\phi$$

$$= abc\int_0^{2\pi}\left(\int_0^{\pi}\left(\frac{1}{4}\frac{1}{2}\frac{\pi}{2}\right)\sin\theta\,d\theta\right)d\phi$$

$$= \frac{\pi abc}{16}\int_0^{2\pi}[-\cos\theta]_0^{\pi}\,d\phi$$

$$= \frac{\pi abc}{16}\int_0^{2\pi}2\,d\phi = \frac{\pi abc}{8}\int_0^{2\pi}d\phi = \frac{\pi^2 abc}{4}.$$

Example 57: By change of variable in polar co-ordinate, prove that

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}.$$

OR

Evaluate the integral being extended to octant of the sphere $x^2 + y^2 + z^2 = 1$.

OR

Evaluate above integral by changing to polar spherical co-ordinate system.

Solution: Simple Evaluation:

$$I = \int_0^1 dx\int_0^{\sqrt{1-x^2}}dy\int_0^{\sqrt{1-x^2-y^2}}\frac{dz}{\sqrt{1-x^2-y^2-z^2}}$$

Treating $\dfrac{1}{\sqrt{(1-x^2-y^2)-z^2}}$ as $\dfrac{1}{\sqrt{a^2-z^2}}$

$$I = \int_0^1 dx\int_0^{\sqrt{1-x^2}}\left(\left|\sin^{-1}\frac{z}{a}\right|_0^{\sqrt{1-x^2-y^2}}\right)dy$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} \left(\left| \sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}} \right|_0^{\sqrt{1-x^2-y^2}} \right) dy, \text{ as } \quad a = \sqrt{1-x^2-y^2}$$

$$= \int_0^1 dx \int_0^{\sqrt{1-x^2}} \left(\frac{\pi}{2} - 0 \right) dy$$

$$= \frac{\pi}{2} \int_0^1 \left((y)_0^{\sqrt{1-x^2}} \right) dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2} \sin^{-1} x \right]_0^1, \text{ using } \int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$= \frac{\pi}{2} \left[0 + \frac{1}{2} \frac{\pi}{2} \right] = \frac{\pi^2}{8}$$

By change of variable to polar spherical co-ordinates, the region of integration

$$V = \{(x, y, z);\ x^2 + y^2 + z^2 \le 1;\ x \ge 0,\ z \ge 0,\ y \ge 0.\}$$

becomes $$I = \left\{ (r, \theta, \phi);\ r^2 \le 1, \text{ i.e. } 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2} \right\}$$

where $$\left. \begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta \end{aligned} \right\}$$

Now $$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \text{coefficient of transformation} = r^2 \sin\theta.$$

whence $$\iiint_V \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{r^2 \sin\theta}{\sqrt{1-r^2}}\, dr\, d\theta\, d\phi$$

$$I = \int_0^{\pi/2} d\phi \int_0^{\pi/2} \left(\sin\theta \left(\int_0^1 \frac{r^2}{\sqrt{1-r^2}} \right) dr \right) d\theta$$

Let $r = \sin t$ so that $dr = \cos t\, dt$. Further, when $\left. \begin{aligned} r = 0,\ t = 0, \\ r = 1,\ t = \frac{\pi}{2} \end{aligned} \right\}$

$\therefore$ $$I = \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{\pi/2} \frac{\sin^2 t}{\cos t} \cdot \cos t\, dt$$

$$= \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin\theta \left[\frac{1}{2} \cdot \frac{\pi}{2} \right];$$

$$= \frac{\pi}{4}\int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta$$

$$= \frac{\pi}{4}\int_0^{\pi/2} d\phi \left(-\cos\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{4}\phi\Big|_0^{\pi/2} = \frac{\pi^2}{8}.$$

Example 58: Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ by changing to polar co-ordinates. **[PTU, 2007]**

Solution: We discuss this problem under change of variables.

Take $\dfrac{x}{a} = X,\ \dfrac{y}{b} = Y,\ \dfrac{z}{c} = Z$ so that $J = \dfrac{\partial(x, y, z)}{\partial(X, Y, Z)} = abc$

$\therefore$ The required volume,

$$V = \iiint dx\, dy\, dz = \iiint |J|\, dX\, dY\, dZ$$

$= abc\iiint dX\, dY dZ$, taken throughout the sphere $X^2 + Y^2 + Z^2 = 1$.

Change this new system (X, Y, Z) to spherical polar co-ordinates (r, θ, ϕ) by taking

$$\left.\begin{aligned} X &= r\sin\theta\cos\phi, \\ Y &= r\sin\theta\sin\phi, \\ Z &= r\cos\theta \end{aligned}\right\} \text{ so that } J' = \frac{\partial(X, Y, Z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta,$$

$$V = abc\iiint |J|\, dr\, d\theta\, d\phi = abc \iiint r^2 \sin\theta\, dr\, d\theta\, d\phi$$

taken throughout the sphere $r^2 \le 1$, i.e. $0 \le r \le 1,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$

On considering the symmetry,

$$V = abc \cdot 8\int_0^{\pi/2}\left(\int_0^{\pi/2}\left(\int_0^1 r^2 dr\right)\sin\theta\, d\theta\right) d\phi$$

$$= 8\,abc\int_0^{\pi/2}\left(\int_0^{\pi/2} \frac{r^3}{3}\Big|_0^1 \sin\theta\, d\theta\right) d\phi$$

$$= \frac{8}{3}abc\int_0^{\pi/2} \left[-\cos\theta\right]_0^{\pi/2} d\phi$$

$$= \frac{8}{3}abc\int_0^{\pi/2} 1 \cdot d\phi$$

$$= \frac{8}{3}abc\phi\Big|_0^{\pi/2} = \frac{8}{3}abc\frac{\pi}{2} = \frac{4}{3}\pi abc$$

Miscellaneous Problem

Example 59: Evaluate the surface integral $I = \iint_S \left(x^3 dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy\right)$.

where S is the surface bounded by $z = 0$, $z = b$, $x^2 + y^2 = a^2$.

OR

By transformation to a triple Integral, evaluate $I = \iint_S \left(x^3 dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy\right)$, where S is the surface bounded by $z = 0$, $z = b$, $x^2 + y^2 = a^2$.

Solution: On making use of Green's Theorem,

$$I = \int_{-a}^{a}\int_{0}^{b}\left(\sqrt{a^2 - y^2}\right)^3 dz\, dy - \int_{-a}^{a}\int_{0}^{b}\left(-\sqrt{a^2 - y^2}\right)^3 dz\, dy$$

$$+\int_{-a}^{a}\int_{0}^{b} x^2\sqrt{a^2 - x^2}\, dz dx - \int_{-a}^{a}\int_{-a}^{a} x^2\left(-\sqrt{a^2 - x^2}\right) dz\, dx$$

$$+\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\left(a^2 - y^2\right) b\, dx\, dy - \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 0\, dx\, dy$$

Using Divergence Theorem,

$$I = \iiint_V \left(3x^2 + x^2 + x^2\right) dx\, dy\, dz$$

$$= 4\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \left(\int_0^b dz\right) dy\right] 5x^2\, dx$$

$$= 4\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} b dy\right] 5x^2\, dx$$

$$= 20\, b\int_0^a x^2\sqrt{a^2 - x^2}\, dx$$

$$= \frac{5}{4}\pi a^4 b\ .$$

***Note*:** As direct calculation of the integral may prove to be instructive. The evaluation of the integral can be carried out by calculating the sum of the integrals evaluated over the projections of the surface S on the co-ordinate planes. Thus, which upon evaluation is seen to check with the result already obtained. It should be noted that the angles α, β, γ are mode by the exterior normals in the +ve direction of the co-ordinate axes.

ANSWERS

Assignmet 1

1. $\left(\frac{\pi^2}{4}\right)$

2. $\frac{a^4}{3}$

3. $\frac{1}{ab}$

6. $\frac{\pi}{4}$

Assignment 2

1. $\int_0^a\left(\int_0^x \frac{x}{x^2+y^2}\,dy\right)dx$

3. $\int_a^{a\sin\alpha}\int_0^{y\cdot\cos\alpha} f(x,y)\,dx\,dy + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

2. $\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$

4. $\int_0^{ma}\int_{\frac{y}{l}}^{\frac{y}{m}} f(x,y)\,dx\,dy + \int_{ma}^{la} f(x,y)\,dx\,dy$

Assignment 3

1. $\frac{4a^2}{3}$ **2.** $\frac{3}{2}\pi\left(b^4-a^4\right)$ **3.** $a^2\left(\frac{3}{4}\pi+\frac{4}{3}\right)$

Assignment 4

2. $\frac{1}{10}$ sq. units

Assignment 5

1. $\frac{\pi a^4}{8}$ units

2. $\frac{a^3}{12}(\pi+2)$ units

3. $\frac{2\pi}{9}$ units

4. $\frac{\pi}{4}$ units

Assignment 6

1. 1

2. $\frac{8}{9}a^3bc\left(3+2ab^2+2ac^2\right)$

3. 8π

4. $\frac{8}{9}\log 2-\frac{19}{9}$

Assignment 7

1. $\frac{1}{6lmn}$

2. $abc\left(\frac{\pi}{4}-\frac{13}{24}\right)$

Assignment 8

1. $abc/6$

2. $\frac{3\pi a^3}{2}$

Assignment 9

1. $\frac{4\pi ab^2}{3}$

2. $\frac{2}{3}\pi a^2$

3. $2\pi^2 a^3$

4. $\frac{\pi a^3}{4}\left\{\frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)-\frac{1}{3}\right\}$