① Big Omega notation : Prove that $g(n) = n^3 + n^2 + 4n$ is $\Omega(n^3)$

$g(n) \geq c \cdot n^3$

$g(n) = n^3 + 2n^2 + 4n$

For finding constants c and n_0

$n^3 + 2n^2 + 4n \geq cn^3$

Divide both sides with n^3

$1 + \frac{2n^2}{n^3} + \frac{4n}{n^3} \geq c$

$1 + \frac{2}{n} + \frac{4}{n^2} \geq c$

Here $\frac{2}{n}$ and $\frac{4}{n^2}$ approaches 0

$1 + \frac{2}{n} + \frac{4}{n^2} \approx 1$

Example $c = 1/2$

$1 + \frac{2}{n} + \frac{4}{n^2} \geq \frac{1}{2}$

$1 + \frac{2}{n} + \frac{4}{n^2} \geq 1$

$1 + \frac{2}{n} + \frac{4}{n^2} \geq 1/2$

Thus, $g(n) = n^3 + 2n^2 + 4n$ is indeeded $\Omega(n^3)$

② Big theta notation : Determine whether $h(n) = 4n^2 + 3n$ is $\Theta(n^2)$ or not

$c_1 n^2 \leq h(n) \leq c_2 n^2$

In upper bound $h(n)$ is $O(n^2)$

In lower bound $h(n)$ is $\Omega(n^2)$.

Upper bound $(O(n^2))$

$h(n) = 4n^2+3n$

$h(n) \leq c_2 n^2$

$4n^2+3n \leq c_2 n^2 \Rightarrow 4n^2+3n \leq 5n^2$

lets $c_2 = 5$

Divide both sides by n^2

$4 + \frac{3}{n} \leq 5$

$h(n) = 4n^2+3n$ is $O(n^2)$ $(c_2 = 5, n_0 = 1)$

Lower bound:-

$h(n) = 4n^2+3n$

$h(n) \geq c_1 n^2$

$4n^2+3n \geq c_1 n^2$

lets $c_1 = 4 \Rightarrow 4n^2+3n \geq 4n^2$

Divide both sides by n^2

$4 + \frac{3}{n} \geq 4$

$h(n) = 4n^2+3n$ $(c_1 = 4, n_0 = 1)$

$h(n) = 4n^2+3n$ is $\Theta(n^2)$

③ Let $f(n) = n3 - 2n^2 + n$ and $g(n) - n_2$ show whether $f(n) = \Omega(g(n))$ is true or flase and justify your answer.

$f(n) \geq c \cdot g(n)$

substuting $f(n)$ and $g(n)$ into this inequality we get.

$n^3 - 2n^2 + n \geq c \cdot (-n^2)$

Find c and n_0 holds $n \geq n_0$

$n^3 - 2n^2 + n \geq -cn^2$

$n^3 - 2n^2 + n + cn^2 \geq 0$

$n^3 + (c-2)n^2 + n \geq 0$

$n^3 + (c-2)n^2 + n \geq 0$ $(n^3 \geq 0)$

$n^3 + (1-2)n^2 + n = n^3 - n^2 + n \geq 0$.

$f(n) = n^3 - 2n^2 + n$ is $\Omega(g(n)) = \Omega(-n^2)$

Therefore the statement $f(n) = \Omega\, g((n))$ is True.

(4) Determine whether $h(n) = n\log n + n$ is $\Theta(n\log n)$ Prove a rigorous proof for your conclusion.

$c_1 n \log n \leq h(n) \leq c_2 n \log n$

Upper bound:-

$h(n) \leq c_2 n \log n$

$h(n) = n \log n + n$

$n\log n + n \leq c_2 n \log n$

Divide both sides by $n\log n$

$1 + \frac{n}{n\log n} \leq 2$

$1 + \frac{1}{\log n} \leq c_2$

$1 + \frac{1}{\log n} \leq 2$

Then $h(n)$ is $O(n\log n)$

Lower bound:-

$h(n) \geq c_1 n\log n$

$h(n) = n\log n + n$

$n\log n + n \geq c_1 n\log n$

Divide both sides by $n\log n$

$1 + \frac{n}{n\log n} \geq c_1$

$1 + \frac{1}{\log n} \geq c_1$

$1 + \frac{1}{\log n} \geq 1$

$\frac{1}{\log n} \geq 0$

$h(n)$ is $\Omega(n\log n)$ $(c_1 = 1, n_0 = 1)$

$h(n) = n\log n + n$ is $\Theta(n\log n)$

5. Solve the following recurrence relations and Find the order of growth of solutions.

$T(n) = 4T(n/2) + n^2, T(1) = 1$

$T(n) = 4T(n/2) + n^2, T(1) = 1$

$T(n) = aT(n/b) + f(n)$

$a = 4, b = 2 \quad f(n) = n^2$

Applying master theorem

$T(n) = aT(n/b) + f(n)$

$f(n) = O(n^{\log_b a - \epsilon})$

$f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

$f(n) = \Omega[n^{\log_b a + \epsilon}]$, Then $T(n) = f(n)$

Calculating $\log_b^a$:

$\log_b^a = \log_2 4 = 2$

$f(n) = n^2 = \Theta(n^2)$ (comparing $f(n)$ with $n^{\log_b^a}$)

$f(n) = \Theta(n^2) = \Theta(n^{\log_b^a})$,

$T(n) = 4T(n/2) + n^2$

$T(n) = \Theta(n^{\log_b^a} \log n) = \Theta(n^2 \log n)$

order of growth

$T(n) = 4T(n/2) + n^2$ with $T(1) = 1$

is $\Theta(n^2 \log n)$