Algebra

Algebra plays an essential role in many competitive exams. Following are some definitions and formulae one needs to know before starting basic algebra.

Algebraic expression: An algebraic expression comprises both numbers and variables together with at least one arithmetic operation.

Constant: A fixed quantity that doesn't change is known as a constant. Example: 2, 5/6.

Variable: A variable is a symbol that we assign to an unknown value. It is usually represented by letters such as *x*, *y*, or *t*.

Coefficient: The coefficient of a variable is the number that is placed in front of a variable.

Equation: An equation consists of two expressions separated by an equal sign. The expression on one side of the equal sign has the same value as the expression on the other side.

Algebraic fraction: An algebraic fraction is a fraction that contains an algebraic expression in its numerator and/or denominator. For Example: $4/(2x-3)$, $(3x-5x+3)/4$, $(3x-5)/(x+3)$

Basic Formulae

I. $(a + b)^2 = a^2 + b^2 + 2ab$.

II. $(a - b)^2 = a^2 + b^2 - 2ab$.

III. $(a^2 - b^2) = (a + b)(a - b)$.

IV. $(a + b)^2 - (a - b)^2 = 4ab$.

V. $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$.

VI. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$.

VII. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$.

VIII. $(a^3 + b^3) = (a + b)(a^2 + b^2 - ab)$.

IX. $(a^3 - b^3) = (a - b)(a^2 + b^2 + ab)$.

X. $(a + b + c)^2 = [a^2 + b^2 + c^2 + 2(ab + bc + ca)]$.

XI. $(a + b + c + \mathrm{d})^2 = [a^2 + b^2 + c^2 + \mathrm{d}^2 + 2a(b + c + \mathrm{d}) + 2b(c + \mathrm{d}) + 2c\mathrm{d}]$.

XII. $(a^3 + b^3 + c^3 - 3abc) = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$. And

If

$a + b + c = 0$, $a^3 + b^3 + c^3 = 3abc$.

XIII. $(x + a)(x + b) = x^2 + (a + b)x + ab$.

XIV. $(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$.

Solved Examples

Ex. 1: If $x + 1/x = 7$, what is the value of $x^2 + 1/x^2$?

Sol. $x + 1/x = 7$ (squaring both sides)

$\Rightarrow x^2 + 1/x^2 + 2*x*1/x = 49$

$\Rightarrow x^2 + 1/x^2 = 49 - 2 = 47$.

Systems of equations

Linear Equations in Two Variables

If an equation is written in form ax+by+c=0 where a, b and c are integers and the coefficients of x and y, i.e., a and b, are non-zero, it is a linear equation in two variables.

5x+4y=7 and −4x+7y=9 are examples of linear equations in two variables.

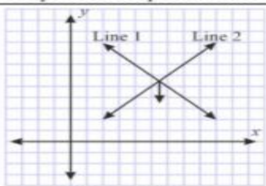
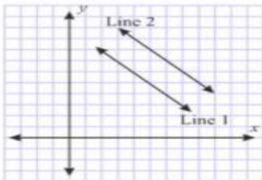
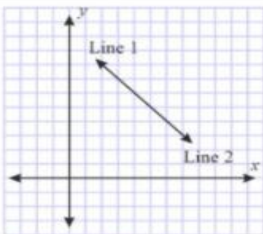
Solvability Conditions for a Pair of Two-Variable Simultaneous Linear Equations

Consider linear equation in two variables x and y

$a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$

Consistent System: If a system of simultaneous linear equations has at least one solution, it is said to be consistent.
1. When, $\mathbf{a_1/a_2 \neq b_1/b_2}$, we get a unique solution. In this case, the two lines representing the equations intersect each other.
2. When $\mathbf{a_1/a_2 = b_1/b_2 = c_1c_2}$, there are infinitely many solutions. In this case, the two lines representing the equations overlap each other.

In-consistent system: If a system of simultaneous linear equations has no solution, it is said to be inconsistent.
When $\mathbf{a_1/a_2 = b_1/b_2 \neq c_1/c_2}$, there is no solution. In this case, the two lines representing the equations are parallel to each other.

Type of solution	Conditions	Graphical Representation
Unique Solution (Consistent and Independent)	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	 Unique Solution (Consistent and independent)
No Solution (Inconsistent and Independent)	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	 No Solution (Inconsistent and independent)
Infinite Number of Solutions (Consistent and Dependent)	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	 Infinite number of solutions (consistent and dependent)

Solution of Linear Equations in Two Variables

Different approaches are used to solve linear equations. They are as follows:

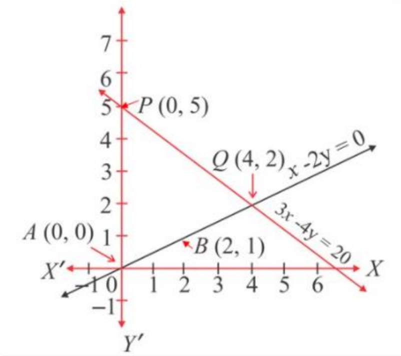
1. Graphical Method

The steps involved to solve the linear equations in two variables graphically are given as follows:

1. Take a graph paper and mark the scale.
2. Transform the equations in the form of y= mx + c.
3. Determine the points which satisfy the above equations and plot them on graph paper.
4. In intersecting lines, check for the point at which two lines intersect, which

is the solution to the given pair of linear equations.
5. There are infinitely many solutions for coincident lines but no solution for parallel lines.

Example: The graph for the system of linear equations 3x+4y=20 and x–2y=0 are drawn below:



Therefore, we see that the intersection point Q (4,2) gives the given system of linear equations.

Elimination Method

The following are the steps to be followed in the elimination method:

1. If the equations have the same coefficient for any variable, add or subtract them to cancel the variable.
2. If the coefficients are different, multiply one or both equations by a non-zero constant value to numerically balance the coefficient of one variable (x or y). To cancel the variable, add or subtract the equation.
3. To find the value of a single variable (x or y), solve the obtained equation.
4. To get the value of another variable, substitute this value of the variable (x or y) in any of the equations

Example: Solve the pair of linear equations x+y=5 and 2x–3y=4 by using the elimination method.
Solution: Given, x + y = 5 —— (i)

and $2x - 3y = 4$ —— (ii)
To make the coefficient of variable x the same, multiply the equation (i) with 2.
Thus, (i) × 2 ⇒ $2x + 2y = 10$ —– (iii)
Subtract equation (iii) from equation (ii),
$(2x-3y = 4) - (2x + 2y = 10) = -5y = -6$
$\Rightarrow y = 6/5$
Now, substitute the above value in the equation (i),
$\Rightarrow x + 6/5 = 5$
$\Rightarrow x = 5 - 6/5 = (25 - 6)/5$
$\Rightarrow x = 19/5$
So, the values of x,y are 19/5, 6/5 respectively.

3. Substitution Method

The following are the steps to solve the equation using the substitution method:

- ✓ Solve any one equation for either x or y. Let us say we write y in terms of x.
- ✓ Now, substitute the value of variable y obtained in step 1 in another equation.
- ✓ Solve the equation obtained in step 2 to get the value of the variable x.

Example:

Solve the system of linear equations 7x–15y=, x+2y=3 by using the substitution method.
Given, $7x-15y=2$ …….(i)
and $x+2y=3$ ……. (ii)
Consider the equation (ii), transpose the term 2y to R.H.S., we get
$\Rightarrow x = 3-2y$ $\Rightarrow x = 3-2y$ ………(iv)
Now, substitute the above value of x in the equation (i),
$\Rightarrow 7(3-2y)-15y = 2$ $\Rightarrow 7(3-2y)-15y = 2$
$\Rightarrow 21-14y-15y = 2$ $\Rightarrow 21-14y-15y = 2$
$\Rightarrow -29y = 2-21 = -19$
$\Rightarrow$ $y=19/29$
Substitute the value of y in the equation (iv),
$\Rightarrow x=3-2^{*}(19/29)$
$\Rightarrow x=87-38/29$

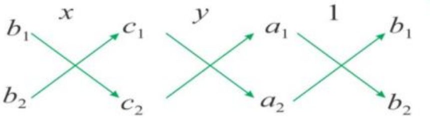
⇒x=49/29

Hence, the required solution is x=49/29, y=19/29

4. Cross-Multiplication Method

Consider $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$. Using the cross-multiplication method, we can solve the pair of equations as follows:

1. As indicated below, write the coefficients of the variables (x,y) as well as the constant values:



$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{b_2a_1 - b_1a_2}$$

$$x = \frac{b_1c_2 - b_2c_1}{b_2a_1 - b_1a_2} \text{ and } y = \frac{c_1a_2 - c_2a_1}{b_2a_1 - b_1a_2}$$

Example 1: Solve the following pair of linear equations by using the cross multiplication method.

3x – 4y = 2

y – 2x = 7

Solution:

We can rewrite the above-given equation as:

3x – 4y = 2

–2x + y = 7

By the method of the cross multiplication,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{b_2a_1 - b_1a_2}$$

When we substitute the values from the above given equation,

$$\frac{x}{28+2} = \frac{y}{4+21} = \frac{1}{13-8}$$

$$\frac{x}{30} = \frac{x}{25} = -15$$

Hence, x = -6 and y = -5

Solved Examples

Q.1. Solve 2(5x–2)=4x+8.

Ans: The given equation is 2(5x–2)=4x+8

$\Rightarrow 10x-4=4x+8$

$\Rightarrow 10x-4x=8+4$

$\Rightarrow 6x=12$

$\Rightarrow x=2$

Hence, the value of x is 2.

Q.2. Solve (7y+2)/5=(6y–5)/11

Ans: The given equation is (7y+2)/5=(6y–5)/11

$\Rightarrow 11(7y+2)=5(6y-5)$

$\Rightarrow 77y+22=30y-25$

$\Rightarrow 77y-30y=-25-22$

$\Rightarrow 45y=-45$

$\Rightarrow y=-1$

Hence, the value of y is −1.

Q.3. Solve the pair of linear equations in two variables graphically x–2y=−4 and −3x+6y=0.

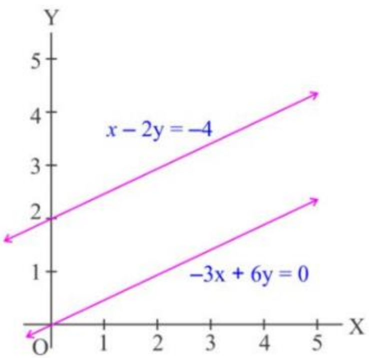
Ans: Given, x–2y=−4 and −3x+6y=0.

For the first equation x–2y=−4

x	0	4	2
y	2	4	3

For the second equation −3x+6y=0

x	0	2	4
y	0	1	2



From the graph, we can say that two lines are non-intersecting lines or parallel lines. So, the given linear equation has no solution.

Q.4 Check if the given pair of linear equations is consistent or not?
3x+2y=15; 6x+4y=10
Ans: Compare the given lines 3x+2y=15; 6x+4y=10 with the standard pair of linear equations $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$.
Thus, $a_1=3$, $b_1=2$, $c_1=-15$, $a_2=6$, $b_2=4$, $c_2=-10$
Comparing the ratio of the coefficient of the linear equations, then
$\Rightarrow a_1/a_2 = 3/6 = 1/2$ and $b_1/b_2 = 2/4 = 1/2$ and $c_1/c_2 = 15/10 = 3/2$
Two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are said to be inconsistent lines. If $a_1/a_2 = b_1/b_2 \neq c_1/c_2$
And, here $a_1/a_2 = b_1/b_2 \neq c_1/c_2$. So the given lines are inconsistent

Q.5 Twenty years ago, Mikkel's age was one-third of what it is now. What is Mikkel's present age?
Ans. We can write the given information by using a linear equation in one variable. Let Mikkel's present age be x years. Twenty years ago, Mikkel's age was (x-20) years. According to the given information,
$x - 20 = x/3$
$\Rightarrow 3(x - 20) = x$
$\Rightarrow 3x - 60 = x$
$\Rightarrow 3x - x = 60$
$\Rightarrow 2x = 60$
$\Rightarrow x = 60/2$
$\Rightarrow x = 30$
Therefore, the present age of Mikkel is 30 years.

Quadratic Equations

Usually, the quadratic equation is represented in the form of $ax^2+bx+c=0$, where x is the variable and a, b, c are the real numbers & $a \neq 0$. Here, a and b are the coefficients of x^2 and x, respectively. So, basically, a quadratic equation is a polynomial whose highest degree is 2. Let us see some examples:

$3x^2+x+1$, where a=3, b=1, c=1
$9x^2-11x+5$, where a=9, b=-11, c=5

Roots of Quadratic Equations:

If we solve any quadratic equation, then the value we obtain are called the roots of the equation. Since the degree of the quadratic equation is two, therefore we get here two solutions and hence two roots.

There are different methods to find the roots of quadratic equation, such as:

- ✓ Factorization
- ✓ Completing the square
- ✓ Using quadratic formula

Quadratic Equation Formula:

The quadratic formula to find the roots of the quadratic equation is given by:

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where $b^2 - 4ac$ is called the discriminant of the equation.

Based on the discriminant value, there are three possible conditions, which defines the nature of roots as follows:

- two distinct real roots, if $b^2 - 4ac > 0$
- two equal real roots, if $b^2 - 4ac = 0$
- no real roots, if $b^2 - 4ac < 0$

Example 1. Rahul and Rohan have 45 marbles together. After losing 5 marbles each, the product of the number of marbles they both have now is 124. How to find out how many marbles they had to start with.

Solution: Say, the number of marbles Rahul had be x.

Then the number of marbles Rohan had = 45 – x.

The number of marbles left with Rahul after losing 5 marbles = x – 5

The number of marbles left with Rohan after losing 5 marbles = 45 – x – 5 = 40 – x

The product of number of marbles = 124

$\Rightarrow (x – 5) (40 – x) = 124$

$\Rightarrow 40x – x^2 – 200 + 5x = 124$

$\Rightarrow - x^2 + 45x – 200 = 124$

$\Rightarrow x^2 – 45x + 324 = 0$

This represents the quadratic equation. Hence by solving the given equation for x, we get;

x = 36 and x = 9

So, the number of marbles Rahul had is 36 and Rohan had 9 or vice versa.

Example 2. Check if x(x + 1) + 8 = (x + 2) (x – 2) is in the form of a quadratic equation.

Solution: Given, x (x + 1) + 8 = (x + 2) (x – 2)

$\Rightarrow x^2+x+8 = x^2-22$ [By algebraic identities]

⇒Cancel x2 on both sides.

$\Rightarrow x+8= -4$

$\Rightarrow x+12=0$

Since, this expression is not in the form of ax^2+bx+c, hence it is not a quadratic equation.

Example3. Find the roots of the equation $2x^2 – 5x + 3 = 0$ using factorization.

Solution: Given, $2x^2 – 5x + 3 = 0$

$\Rightarrow 2x^2 – 2x-3x+3 = 0$

$\Rightarrow 2x(x-1)-3(x-1) = 0$

$\Rightarrow (2x-3)\ (x-1) = 0$

$\Rightarrow 2x-3 = 0;\ x = 3/2$

$\Rightarrow (x-1) = 0;\ x=1$

Therefore, 3/2 and 1 are the roots of the given equation.

Example 4. Solve the quadratic equation $2x^2 + x – 300 = 0$ using factorization.
Solution: $2x^2 + x – 300 = 0$

$\Rightarrow 2x^2 – 24x + 25x – 300 = 0$

$\Rightarrow 2x\ (x – 12) + 25\ (x – 12) = 0$

$\Rightarrow (x – 12)\ (2x + 25) = 0$

$\Rightarrow x-12=0;\ x=12$

$\Rightarrow (2x+25) = 0;\ x=-25/2 = -12.5$

Therefore, 12 and -12.5 are two roots of the given equation.

Example5. Solve the equation $x^2+4x-5=0$.
Solution: $x^2 + 4x – 5 = 0$

$\Rightarrow x^2-1x+5x-5 = 0$

$\Rightarrow x(x-1) +5(x-1) =0$

$\Rightarrow (x-1)\ (x+5) =0$

Hence, $(x-1) =0$, and $(x+5) =0$

$\Rightarrow x-1 = 0$

$\Rightarrow x = 1$

similarly, $x+5 = 0$

$\Rightarrow x =-5$.

Therefore, $x=-5$ & $x=1$

Example 6. Solve the quadratic equation $2x^2 + x – 528 = 0$, using the quadratic formula.
Solution: If we compare it with standard equation, $ax^2+bx+c = 0$
$a=2$, $b=1$ and $c=-528$

Hence, by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now putting the values of a, b and c.

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2(-528)}}{2 \cdot 2}$$

⇒x=64/4 or x=-66/4

⇒x=16 or x=-33/2

Example7. Find the roots of $x^2 + 4x + 5 = 0$, if any exist, using the quadratic formula.

Solution: To check whether there are real roots available for the quadratic equation, we need to find the discriminant value.

⇒D = b^2-4ac = 42– 4(1)(5) = 16-20 = -4

Since, the square root of -4 will not give a real number. Hence there are no real roots for the given equation.

Example 8. Find the discriminant of the equation: $3x^2-2x+⅓ = 0$.

Solution: Here, a = 3, b=-2 and c=⅓

Hence, discriminant, D = $b^2 – 4ac$

⇒D = (-2)*2-4(3)*(⅓)

⇒D = 4-4

⇒D=0