Number System

FACTORS

We will begin our discussion with the prime factor representation of a number. By prime factor representation, we mean breaking the number as a product of factors that are prime numbers. Here, we should explain to the students the process of breaking a number into its prime factors using a few examples.

For e.g. $24 = 2^3 \times 3^1$

In general, we try to see the highest powers of the commonly used prime numbers (starting with 2, 3, 5 etc) that divide the number and then individually break them into prime factors.

For example: $480 = 48 \times 10$

$$= 16 \times 3 \times 10$$
$$= 2^4 \times 3 \times 5 \times 2$$
$$= 2^5 \times 3 \times 5$$

1. When $N = a^p . b^q . c^r$ where a, b, c are prime. No. of factors of $N = (p + 1)(q + 1)(r + 1)$

Example 1. Consider a no. $N = 2^9 . 3^7 . 7^8 . 10^2$

(a) No. of factors?

Solution: It is tempting to arrive at $2160 = (9 + 1)(7 + 1)(8 + 1)(2 + 1)$ as the answer.

However, please note that 10 is a composite number and the formula is applicable only for prime numbers.

$\therefore$ 10^2 will need to be broken into prime factors as the formula is applicable only for the prime factor representation

Thus, $2160 = 2^{11} . 3^7 . 7^8 . 5^2$

No. of factors = 12 x 8 x 9 x 3 = 2592

(b) Number of prime factors?

Solution: The number of prime factors is the same as the number of distinct terms in the prime factor representation i.e., 4 (2, 3, 5 and 7). Any other factor will be some combination of these 4 factors and so cannot be a prime number.

(c) Number of odd factors?

Solution: If we take 2^n out, the combinations of other terms will be odd.

$\Rightarrow$Number of ways = $(7 + 1)(2 + 1)(8 + 1)$

(d) Number of even factors?

Solution: If we take a 2 out, create factors from the remaining number and multiply back 2 in the end, any such factor will be even.

$\Rightarrow$Number of ways = $(10 + 1)(7 + 1)(2 + 1)(8 + 1)$

2. When N = $a^p \cdot b^q \cdot c^r$ where a, b, c are prime, the number of ways expressing a number as a product of 2 factors = ½ (number of factors).

Example 2: In how many ways can 24 be written as the product of 2 factors?

Solution: $24 = 2^3 \times 3^1$

No. of factors = 4 x 2 = 8

No. of ways 24 can be written as the product of 2 factors = 8/2 = 4

3. **When N = $a^p \cdot b^q \cdot c^r$ where a, b, c are prime, the number of ways expressing a number as a product of 3 factors**

Example 3: In how many ways can 300 be written as the product of 3 factors?

Solution: Write 300 in the form of prime factorization form = $300 = 2^2 \times 3 \times 5^2$. Let us say, 300 is a product of 3 numbers a,b,c i.e.

$300 = a*b*c = 2^2 \times 3 \times 5^2$

a or b or c can be of the form, $a = 2^{x1} \times 3^{y1} \times 5^{z1}$

$b = 2^{x2} \times 3^{y2} \times 5^{z2}$

$c = 2^{x3} \times 3^{y3} \times 5^{z3}$

if we multiply a*b*c we see that,

$a*b*c = 2^{(x1+x2+x3)} \times 3^{(y1+y2+y3)} \times 5^{(z1+z2+z3)} = 2^2 \times 3 \times 5^2$,

which implies x1+x2+x3 =2, y1 + y2 +y3 = 1, z1+z2+z3 =2,

No of solutions for x1+x2+x3 =2,

${}^4C2$ = **6** ways, (no of variables + Value at right hand side of equation -1 = 4) Similarly,

No of solutions for y1+y2+y3 =2,

${}^3C2$ = **3** ways, (no of variables + Value at right hand side of equation -1 = 3)

Also, No of solutions for z1+z2+z3 =2,

${}^4C2$ = **6** ways, (no of variables + Value at right hand side of equation -1 = 4)

So, total ways is the product of all the ways of a,b,c = **6*3*6** = **108** ways in total.

4. Sum of the factors of a number N (= $a^p \cdot b^q \cdot c^r \ldots$) = $\left(\frac{a^{p+1}-1}{a-1}\right) \times \left(\frac{b^{q+1}-1}{b-1}\right) \times \left(\frac{c^{r+1}-1}{c-1}\right)$

Example 4: What is the sum of the factors of 48?

Solution. N = 48 = 2^4 x 3^1

$$=\frac{2^5-1}{2-1}\times\frac{3^2-1}{3-1} = 124$$

5. Product of the factors of a number N (= a^p . b^q. c^r. .) = $N^{1/2\ (\text{number of factors})}$

Example 5: What is the product of the factors of 72?

Solution: 72 = 2^3 x 3^2

No. of Factors = 4x 3 = 12

Product of all factors of 72 = $(72)^{1/2(12)}$ = $(72)^6$

HIGHEST POWER OF A NUMBER k IN n!

This is used to identify the highest power of a number k in n!.
Let us first spend a few minutes on understanding factorials. Factorial of a whole number n is defined as the product of all natural numbers from 1 to n; so, n! = n(n – 1)(n – 2) …… 4.3.2.1

By definition, the factorial of
0 is 1. As an example
1! = 1
3! = 3 × 2 × 1 = 6
5! = 5 × 4 × 3 × 2 × 1 = 120 etc

Now let us consider the following questions

Example : Find the highest power of 2 in 10!.
Solution: 10! = 1 × 2 × 3 × 22 × 5 × 2 × 3 × 7×23 × 32× 2× 5 = $2^8 \times 3^4 \times 5^2 \times 7^1$
The highest power of 2 in 10! = 8. Use successive division method: n

$$\begin{array}{r|l} 2 & 10 \\ \hline 2 & 5 \\ \hline 2 & 2 \\ \hline & 1 \end{array} \quad = 5+2+1 = 8$$

Example: Find the highest power of 6 in 10!.

Solution: As “6” is not a prime number, factorize 6 = 2 x 3; The highest power of 6 is the lower of the highest powers of 2 and 3. Find the highest powers of 2 and 3 in 10!

Consider the minimum of these 2 values.

The highest power of 2 in 10! is 8

The highest power of 3 in 10! is 4

Highest power of 6 in 10! is min(8,4) = 4

Example:

Find the highest power of 8 in 20!.

Solution: As $8 = 2^3$, find the highest powers of 2 in 20! Divide this number by 3. Take the integral portion. The highest power of 2 in 20! is 18

2	20
2	10
2	5
2	2
	1

$n = 10 + 5 + 2 + 1 = 18$

The highest power of 8 in 20! is = 6

Example: Find the highest power of 24 in 40!.

Solution: $24 = 2^3 \times 3^1$

Highest power of 2 in 40! = 38

Highest power of 3 in 40! = 18

Highest power of 2^3 in 40! = 38/3 ~ 12

Highest power of 24 in 40! = min (12, 18) = 12

NUMBERS OF TRAILING 0

Now the number of zeroes is the same as the highest power of 10 in the number which is the same as the lower of the highest powers of 5 and 2

Example: Find the number of zeros at the end of the product of the first 50
(1) Natural Nos.
(2) Odd Nos
(3) Even Nos.
(4) Prime nos.
(5) Multiples of 3
(6) Multiples of 5

Solution: (1) Highest power of 10 in 50! Same as the highest power of 5 in 50!. Thus, no of trailing 0= 12
(2) There are no 2s; we need both a 5 and a 2 to form a ten. Thus, no of trailing 0=0
(3) N = $(1 \times 2 \times 3 \times \ldots \times 50)\ 2^{50}$, the highest powers of 2 and 5 are 97 and 12 respectively. Thus, no of trailing 0 = 12
(4) Exactly 1-5 and exactly 1-2. Thus, no of trailing 0 =1
(5) N = $(1\times 2 \times 3 \times \ldots\ldots 50)\ 3^{50}$
The highest powers of 2 and 5 are 47 and 12 respectively. Thus, no of trailing 0 = 12
(6) N = $(1\times 2 \times 3 \times \ldots\ldots 50)\ 5^{50}$
The highest powers of 2 and 5 are 47 and 62 respectively. Thus, no of trailing 0 = 47

THE LAST DIGIT OF ANY POWER OF A NUMBER N

The last digits of the powers of any number follow a cyclic pattern - i.e., they repeat after a certain number of steps. If we find out after how many steps the last digit of the powers of a number repeat, then we can find out the last digit of any power of any number. Let us look at the powers of 2.
Last digit of 2^1 is 2
Last digit of 2^2 is 4
Last digit of 2^3 is 8
Last digit of 2^4 is 6
Last digit of 2^5 is 2

Since the last digit of 2^5 is the same as the last digit of 2^1, then onwards the last digit will start repeating, i.e. digits of 2^5, 2^6, 2^7, 2^8 will be the same as those of 2^1, 2^2, 2^3, 2^4 respectively. Then the last digit of 2^9 is again the same as the last digit of 2^1 and so on. So, we have been able to identify that for powers of 2, the last digits repeat after every 4 steps. In other words whenever the power is a multiple of 4, the last digit of that number will be the same as the last digit of 2^4.
Suppose we want to find out the last digit of 2^{67}, we should look at a multiple of 4 which is less than or equal to the power 67. Since 64 is a multiple of 4, the last digit of 2^{64} will be the same as the last digit of 2^4. Then the last digits of 2^{65}, 2^{66}, 2^{67} will be the same as the last digits of 2^1, 2^2, 2^3 respectively. Hence the last digit of 2^{67} is the same as the last digit of 2^3 i.e. 8.

Similarly, we can find out the last digit of 3^{74} by writing the pattern of the powers of 3. Last digit of 3^1 is 3
Last digit of 3^2 is 9
Last digit of 3^3 is 7
Last digit of 3^4 is 1

Last digit of 3^5 is 3

The last digit repeats after 4 steps (like in the case of the powers of 2)

To find the last digit of 3^{74}, we look for a multiple of 4 which is less than or equal to 74. Since 72 is a multiple of 4, the last digit of 3^{72} will be the same as that of 3^4. Hence the last digit of 3^{74} will be the same as the last digit of 3^2, i.e. 9.

Here are the power cycles of cell digits 1 – 0:

1	1
2	2 4 8 6
3	3 9 7 1

4	4 6
5	5
6	6
7	7 9 3 1
8	8 4 2 6
9	9 1
0	0

A. The cyclicity of 1, 5, 6 and 0 is 1. So, for any power, their last digit remains the same.

B. The cyclicity of 4 and 9 is 2. So, for odd powers 4 would be four and for even powers it would be 6. Similarly, 9 would be 9 and 1 for odd and even powers respectively.

C. The cyclicity of 2, 3, 7 and 8 is 4. So, to find the last digit of their powers, we need to divide the last 2 digits of the power term by 4. If the remainder is 1, we raise the base to power 1, similarly for remainders 2 and 3. If, however, the remainder is zero, we need to take the 4th power of the base.

D. The last digit of any multiplication or addition or power depends only on the last digits of the base terms. The numbers preceding the last digit have no bearing on the last digit.

Example: Find the last digit of

(a) 2^{143} (b) 13^{47}

(b) 6712344567 (d) 1914568

Solution: (a) $2^{143} = 2^{4 \times 35 + 3}$ i.e. $2^3 = 8$

(b) $13^{47} = 3^{4 \times 11 + 3}$ i.e. $3^3 = 7$

(c) 6712344567 = 712344567 = 767 = 74 x 16 + 3 = 73 = 3

(d) The last digit for 9 depends first on the parity of power. It's 9 for odd and 1 for even powers. As the power is even, it ends with 1

Example Find the last digit of

(a) $13^{27} + 27^{13}$

(b) $113^{39} \times 225^{25}$

Solution: (a) $13^{27} = 3^{4 \times 6 + 3}$ i.e. $3^3 = 7$, $27^{13} = 7^{4 \times 3 + 1}$ i.e. $7^1 = 7$

Unit digit = 7+7 = 4

(b) $113^{39} = 3^{4 \times 9 + 3}$ i.e. $3^3 = 7$

$225^{25} = 5^{25} = 5$

Unit Digit = 7 x 5 = 5

Remainders

A. Pattern Method

Consider an example - "2^{59} is to be divided by 9" Let us find the pattern that remainder follows when various powers of 2 are divided by 9.

The final line is to be repeated with the new numbers replaced. Remainder when 21 is divided by 9 is 2

___________22________9_____4
___________23________9_____8
___________24________9_____7
___________25________9_____5
___________26________9_____1

Once you have obtained a 1 as remainder, the same cycle will start repeating. Now we observe that there are 6 terms in the cycle (called power cycle) of 2. We can write 2^{59} as $(2^6)^9$. 2^5 This is obtained as follows

$$\begin{array}{r|l} & 9 \\ 6 & 59 \\ & \underline{54} \\ & 5 \end{array} \Rightarrow 2^{59} = (2^{6})^{9} . 2^{5}$$

Once we have obtained the cycle, the remainder when $(2^6)^9$ is divided by 9 would be 1 as it completes 9 cycles. And, the remainder from 25 would be 5 (as seen above). So the overall remainder is the product of the 2 remainders.

2^{59} / 9 = 1 x 5 = 5.

Please also note that:

(a) The division cycle for successive powers can be stopped when we obtain a +1 or even a –1.

(b) You need not calculate the exact value of powers each time you need to find the remainder. The remainder for a^n divided by b is the same as the remainder for a^{n-1} multiplied by a. For example

2^1 / 11 = 2

2^2 / 11 = 4

2^3 / 11 = 8

2^4 / 11 = 8 x 2 = 16 & 16/11 = 5

2^5 / 11 = 5 x 2 = 10

and so on.

Example: The faculty may now ask the students to calculate the remainders for 2^{59} divided by 3,5 etc.

Solution: 2^{59} / 3 = 2; 2^{59} / 5 = 3

Fermat's Little theorem:

The following formula will be extremely useful in solving problems related to finding the remainders and must be explained with ample examples (preferably the same questions that have been discussed before using the Pattern method)

Remainder when a(p–1) divided by p will be 1, when "p" is a prime number and 'a' is not a multiple of "p".

Example: Find the remainder when 7^{70} is divided by 11.

$$\frac{7^{11-1}}{11} = 1$$

Therefore, 7^{70} / 11 = 1

Example: Find the remainder when 10^{400} is divided by 199.

Solution: 10^{400} / 199 = 10^4 $(10^{198})^2$ / 199 = 10^4 / 199 = 50

Euler's Theorem

Euler's theorem states that if p and n are co-prime positive integers, then

Remainder $(P^{\phi(n)} / n) = 1$ where $N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)$............ $\phi(n) =$

where $N = a^p . b^q . c^r . . .$

Example: Find the remainder of $(7^{100}) / 66$

Solution: As you can see, 7 and 66 are co-prime to each other.

Therefore, Φ (66) = 66 x (1-1/2) x (1-1/3) x

(1-1/11) = 20 So, remainder of $(7^{100})/ 66 = 1$

Wilson's Theorem

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If p is prime, then $(p-1)! + 1$ is a multiple of p. For e.g. $(2-1)! + 1$ is divisible by 2, $(5-1)! + 1 = 25$ which is divisible by 5 etc. The method would be handy when we need to solve problems involving a factorial in the numerator. To understand the depth of the application of Wilson's theorem,
let us begin with a simple example

Example: **Find the remainder when 10! is divided by 11.**
Solution: Since $10! + 1 = 11k$, the remainder when 10! is divided by 11 has to be -1, which is the same as $-1 + 11 = 10$.

Example: Find the remainder when 9! is divided by 11.
Solution: Since $10! + 1 = 11k$, the remainder when 10! is divided by 11 has to be -1, which is the same as $-1 + 11 = 10$

Now $10! = 9!\,(10)$

So, $10! \mid 11 = (9! \mid 11)\,(10 \mid 11)$

or $10 = (9! \mid 11)\,.\,10$

or $(9! \mid 11) = 1$