Applied Algorithms

CSCI-B505 / INFO-I500

Lecture 11.

Dynamic Programming - II

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- Dynamic Programming
  - Longest Increasing Sequence
  - Subset Sum
  - Ordered Partitioning

Dynamic Programming

- Find the minimum or maximum of a combinatorial challenge (combinatorial opt.)
- Exhaustive search guarantees the optimum, but very expensive
- Greedy approach (!) is more reasonable, but no guarantees (*in general*)
- Dynamic programming aims to compute the optimum with a good complexity by storing the results of some prior computations for the sake of some others later.
- **DP is particularly useful when there is a reductive solution but with significant overlaps between the recursive steps.**

Longest Increasing Subsequence in an Array

$$S = \langle\ 2, 4, 3, 5, 1, 7, 6, 9, 8\ \rangle$$

- **Not** longest increasing **run**, but **subsequence**, e.g., (2,4) is a **run**, (2,5,7,9) is a **subsequence**
- What do we need to decide on the current position? How?
  1. The longest increasing subsequence length of the previous position
  2. The last element information
- If we define L_i as the length of the longest increasing run of $\langle s_1, s_2, \ldots, s_i \rangle$ **ending** at s_i then (2) is automatically included.

$$L_0 = 0 \qquad L_i = 1 + \max_{\substack{0 \le j < i \\ s_j < s_i}} L_j,$$

Longest Increasing Subsequence in an Array

Index i	1	2	3	4	5	6	7	8	9
Sequence s_i	2	4	3	5	1	7	6	9	8
Length L_i	1	2	2	3	1	4	4	5	5
Predecessor p_i	–	1	1	2	–	4	4	6	6

$$
\begin{aligned}
L_0 &= 0 \\
L_i &= 1 + \max_{\substack{0 \le j < i \\ s_j < s_i}} L_j,
\end{aligned}
$$

- Computing L_i needs to investigate all previous positions. If they were cached, it will be a linear operation. However, the total process is quadratic as we need this linear operation on all positions.

- Reporting the sequence beyond its length requires also maintaining the predecessor array, which marks the j value in the equation of L_i above.

Longest Increasing Subsequence in an Array

- A second solution of DP for this problem is something akin to **longest common subsequence**.
- Align the input sequence with its sorted version.

		1	2	3	4	5	6	7	8	9
	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	1	1	1	1	1	1
4	0	0	1	1	2	2	2	2	2	2
3	0	0	1	2	2	2	2	2	2	2
5	0	0	1	2	2	3	3	3	3	3
1	0	1	1	2	2	3	3	3	3	3
7	0	1	1	2	2	3	3	4	4	4
6	0	1	1	2	2	3	4	4	4	4
9	0	1	1	2	2	3	4	4	4	5
8	0	1	1	2	2	3	4	4	5	**5**

Subset Sum (Unordered Partitioning)

Let $S = \{\ s_1, s_2, s_3, \ldots, s_n\ \}$ be a set of integers. Is there a subset of S, whose elements sum up to a queried value k ?

The number of subsets is 2^n. Therefore, the exhaustive search is exponential.

The problem is NP-complete.

We will be examining the **pseudo-polinomial time (?)** dynamic programming solution.

Subset Sum

Let $S = \{ s_1, s_2, s_3, \ldots, s_n \}$ be a set of integers. Is there a subset of S, whose elements sum up to a queried value k ?

- Let $T_{n,k}$ denote whether there is such a subset or not.
  - If there is a subset of $\{ s_1, s_2, s_3, \ldots, s_{n-1} \}$ summing up to k, which we can show with $T_{n-1,k}$, then $T_{n,k}$ is true
  - **OR**, if there is a subset of $\{ s_1, s_2, s_3, \ldots, s_{n-1} \}$summing up to $k - s_n$, which we can show with $T_{n-1,k-s_n}$, then $T_{n,k}$ is true.
- Therefore, $T_{n,k} = T_{n-1,k} \vee T_{n-1,k-s_n}$.

Subset Sum

$S = \{\ 1, 2, 4, 8\ \}, k = 11$

i	s_i	0	1	2	3	4	5	6	7	8	9	10	11
0	0	T	F	F	F	F	F	F	F	F	F	F	F
1	1	T	T	F	F	F	F	F	F	F	F	F	F
2	2	T	T	T	T	F	F	F	F	F	F	F	F
3	4	T	T	T	T	T	T	T	T	F	F	F	F
4	8	T	T	T	T	T	T	T	T	T	T	T	T

```
bool sum[MAXN+1][MAXSUM+1];      /* table of realizable sums */
int parent[MAXN+1][MAXSUM+1];    /* table of parent pointers */

bool subset_sum(int s[], int n, int k) {
    int i, j;                    /* counters */

    sum[0][0] = true;
    parent[0][0] = NIL;

    for (i = 1; i <= k; i++) {
        sum[0][i] = false;
        parent[0][i] = NIL;
    }

    for (i = 1; i <= n; i++) {   /* build table */
        for (j = 0; j <= k; j++) {
            sum[i][j] = sum[i-1][j];
            parent[i][j] = NIL;

            if ((j >= s[i-1]) && (sum[i-1][j-s[i-1]]==true)) {
            sum[i][j] = true;
            parent[i][j] = j-s[i-1];
        }
    }
}

return(sum[n][k]);
}
```

$$T_{i,j} = T_{i-1,j} \vee T_{i-1,j-s_n}$$

The matrix above shows which sums are possible and which are not.

But, what is the elements of the subset that gives this reachable sums ?
This is resolved by keeping track of the **parent** of each cell

Subset Sum

```
void report_subset(int n, int k) {
    if (k == 0) {
        return;
    }

    if (parent[n][k] == NIL) {
        report_subset(n-1,k);
    }
    else {
        report_subset(n-1,parent[n][k]);
          printf(" %d ",k-parent[n][k]);
    }
}
```

What is the elements of the subset that gives this reachable sums ?

This is resolved by keeping track of the **parent** of each cell

$S = \{ 1, 2, 4, 8 \}, k = 11$

i	s_i	0	1	2	3	4	5	6	7	8	9	10	11
0	0	T	F	F	F	F	F	F	F	F	F	F	F
1	1	T	T	F	F	F	F	F	F	F	F	F	F
2	2	T	T	T	T	F	F	F	F	F	F	F	F
3	4	T	T	T	T	T	T	T	T	F	F	F	F
4	8	T	T	T	T	T	T	T	T	T	T	T	T

$$T_{i,j} = T_{i-1,j} \vee T_{i-1,j-s_n}$$

i	s_i	0	1	2	3	4	5	6	7	8	9	10	11
0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	-1	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	2	-1	-1	0	1	-1	-1	-1	-1	-1	-1	-1	-1
3	4	-1	-1	-1	-1	0	1	2	3	-1	-1	-1	-1
4	8	-1	-1	-1	-1	-1	-1	-1	-1	0	1	2	3

report_subset(4,11)
report_subset(3,3)
report_subset(2,3)
report_subset(1,1)
report_subset(0,0)

1+2+8 = 11

report_subset(3,6)

report_subset(2,2)

report_subset(1,0)

2+4 =6

Ordered Partitioning Problem

Given an array of n positive integers as $\langle s_1, s_2, \ldots, s_n \rangle$, split this array into k partitions such that the sum of the integers in the partitions will be as balanced as possible, **which can be stated as the largest sum of integers in those partitions will be minimum.**

$k = 3$ $\quad S = \langle 100{,}200{,}300{,}400{,}500{,}600{,}700{,}800{,}900 \rangle$

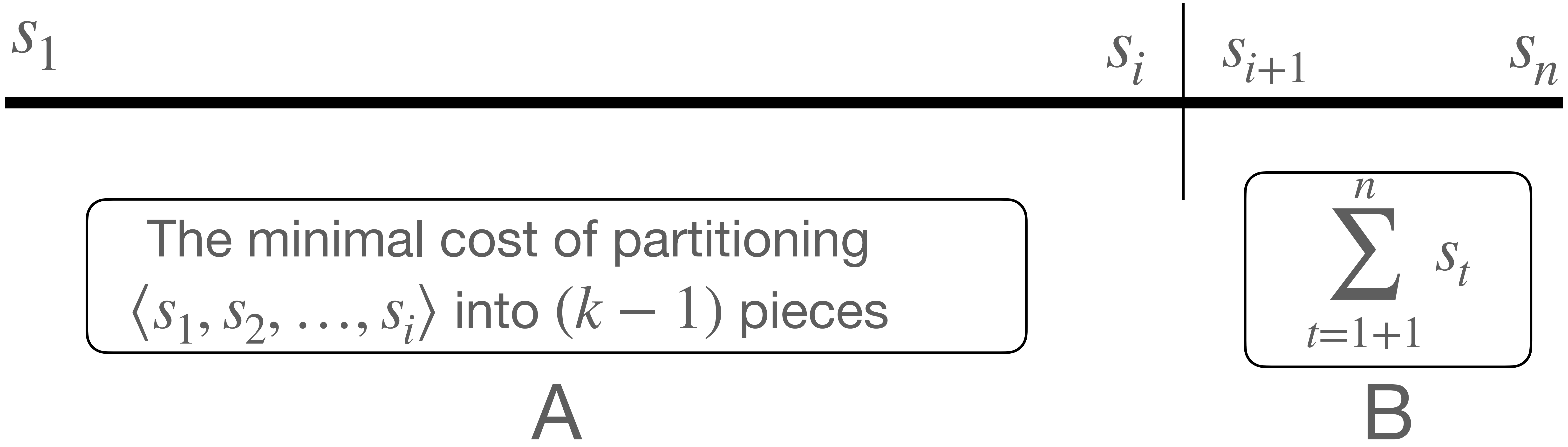
300 2500 1700 Maximum is 2500

600 1500 2400 Maximum is 2400, so better than 2500

What are the best positions for the (k-1) dividers so that we get the most balanced solution.

Important notice: Rearrangement is not allowed !!!

Ordered Partitioning Problem



- The cost of placing the **(k-1)th** divider between i and $(i+1)$ is the maximum of A and B.
- Notice that A is indeed the same problem as partitioning $\langle s_1, s_2, \ldots, s_i \rangle$ into $(k-1)$ pieces
- If $M[n,k]$ is the minimum cost of partitioning $\langle s_1, s_2, \ldots, s_n \rangle$ into k pieces, then we can formulate

$$M[n,k] = \min_{i=1}^{n} \left(\max(M[i,k-1], \sum_{j=i+1}^{n} s_j) \right)$$

$$\boxed{\begin{aligned} M[1,k] &= s_1, \text{ for all } k > 0 \\ M[n,1] &= \sum_{i=1}^{n} s_i \end{aligned}}$$

Ordered Partitioning Problem

$$M[n,k] = \min_{i=1}^{n} \left(\max(M[i,k-1], \sum_{j=i+1}^{n} s_j) \right)$$

$S = \langle s_1, s_2, \ldots, s_9 \rangle = \langle 1,2,3,4,5,6,7,8,9 \rangle, \quad k = 3$

	1	2	3
1	1	1	1
2	3	2	2
3	6	3	3
4	10	6	4
5	15	9	6
6	21	11	9
7	28	15	11
8	36	21	15
9	45	24	17

$M[1,k] = s_1$

$M[n,1] = \sum_{i=1}^{n} s_i$

$$M[9,3] = min\Big(max(M[1,2], \sum_{j=2}^{9} s_j),$$

$$max(M[2,2], \sum_{j=3}^{9} s_j),$$

$$\vdots$$

$$max(M[7,2], \sum_{j=7}^{9} s_j),$$

$$\vdots$$

$$max(M[9,2], \sum_{j=10}^{9} s_j) \Big)$$

$max(15,8+9) = 17$

Means $\langle s_1, .. s_7 \rangle$ is divided into 2 with a cost 15, and $\langle s_8, s_9 \rangle$ is the last partition

Ordered Partitioning Problem

$S = \langle s_1, s_2, \ldots, s_9 \rangle = \langle 1,2,3,4,5,6,7,8,9 \rangle, \quad k = 3$

```
void partition(int s[], int n, int k) {
    int p[MAXN+1];              /* prefix sums array */
    int m[MAXN+1][MAXK+1];      /* DP table for values */
    int d[MAXN+1][MAXK+1];      /* DP table for dividers */
    int cost;                   /* test split cost */
    int i,j,x;                  /* counters */

    p[0] = 0;                   /* construct prefix sums */
    for (i = 1; i <= n; i++) {
        p[i] = p[i-1] + s[i];
    }

    for (i = 1; i <= n; i++) {
        m[i][1] = p[i];     /* initialize boundaries */
    }

    for (j = 1; j <= k; j++) {
        m[1][j] = s[1];
    }

    for (i = 2; i <= n; i++) {   /* evaluate main recurrence */
        for (j = 2; j <= k; j++) {
            m[i][j] = MAXINT;
            for (x = 1; x <= (i-1); x++) {
                cost = max(m[x][j-1], p[i]-p[x]);
                if (m[i][j] > cost) {
                    m[i][j] = cost;
                    d[i][j] = x;
                }
            }
        }
    }
    reconstruct_partition(s, d, n, k);     /* print book partition */
}
```

M	k		
s	1	2	3
1	1	1	1
2	3	2	2
3	6	3	3
4	10	6	4
5	15	9	6
6	21	11	9
7	28	15	11
8	36	21	15
9	45	24	17

D	k		
s	1	2	3
1	–	–	–
2	–	1	1
3	–	2	2
4	–	3	3
5	–	3	4
6	–	4	5
7	–	5	6
8	–	5	6
9	–	6	7

$O(kn^2)$-time, $O(kn)$-space,

Ordered Partitioning Problem

M	k				D	k		
s	1	2	3		s	1	2	3
1	1	1	1		1	–	–	–
2	3	2	2		2	–	1	1
3	6	3	3		3	–	2	2
4	10	6	4		4	–	3	3
5	15	9	6		5	–	3	4
6	21	11	9		6	–	4	5
7	28	15	11		7	–	5	6
8	36	21	15		8	–	5	6
9	45	24	17		9	–	6	7

```c
void reconstruct_partition(int s[],int d[MAXN+1][MAXK+1], int n, int k) {
    if (k == 1) {
        print_books(s, 1, n);
    } else {
        reconstruct_partition(s, d, d[n][k], k-1);
        print_books(s, d[n][k]+1, n);
    }
}


void print_books(int s[], int start, int end) {
    int i;    /* counter */

    printf("\{");
    for (i = start; i <= end; i++) {
        printf(" %d ", s[i]);
    }
    printf("}\n");
}
```

$$s_1, s_2, s_3, s_4, s_5 \mid s_6, s_7 \mid s_8, s_9$$

If we want to construct the partitions, we need to save the divider information for each cell, and then backtrack the optimum solution.

Reading assignment

- Read the Dynamic Programming chapters from the text books, particularly from Cormen and Skiena.