Applied Algorithms

CSCI-B505 / INFO-I500

Lecture 9.

Divide&Conquer Approach

M. Oguzhan Kulekci, September 21/22 10, 2022

- Divide-and-Conquer Approach
  - Maximal subarray problem
  - Master Theorem (or method) for solving D&C type recurrences
  - Karatsuba matrix multiplication
  - Closest pair in 2-d space

Divide-And-Conquer Approach

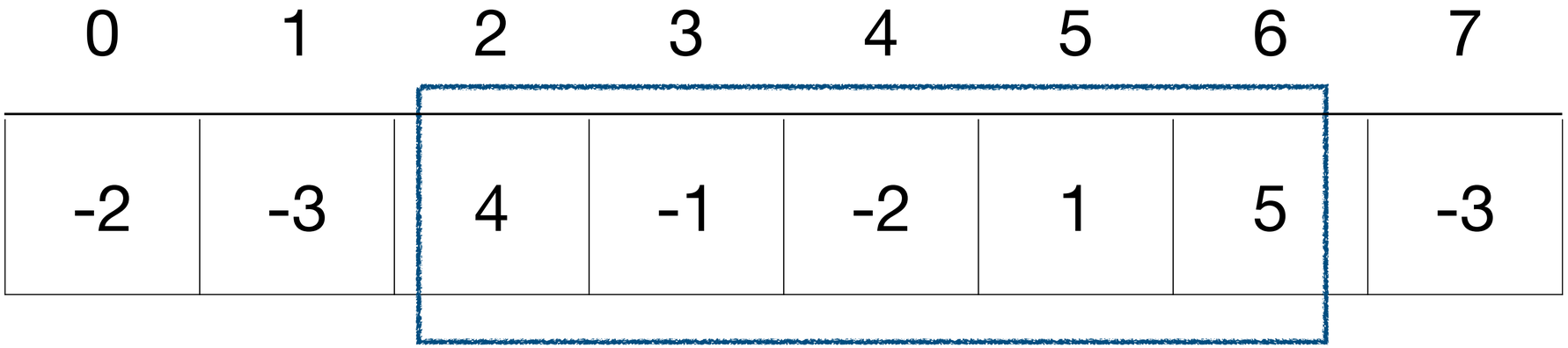
Given a problem,

- Partition it into small problems.
- Solve these small problems.
- Gather the results to arrive the final output.

Widely used in solving many computational challenges, e.g., quick sort, merge sort, tree/graph traversals, etc…

Divide-And-Conquer

Problem: Given an array $A[0..n-1]$ of **integers**, find the subarray $A[i\,..\,j]$ with the maximum sum.



Divide-And-Conquer

Problem: Given an array A[0..n-1] of integers, find the subarray A[i..j] with the maximum sum.

Naive or Brute-Force Solution:

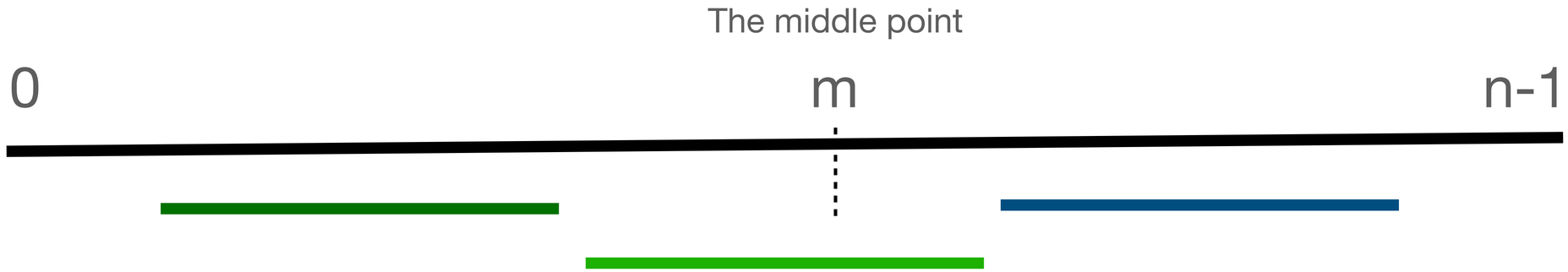
- How many different subarrays exist $\binom{n}{2} = n(n+1)/2 \in O(n^2)$.
- Generate all of them and choose the one with the maximum sum.

How to make it better than $O(n^2)$?

Divide-and-conquer ?

Divide-And-Conquer

Problem: Given an array $A[0..n-1]$ of integers, find the subarray $A[i\,..\,j]$ with the maximum sum.



Maximum subarray $A[i,j]$ is on the left-half if $0 \le i \le j < m$

Maximum subarray $A[i,j]$ is on the left-half if $m < i \le j < n$

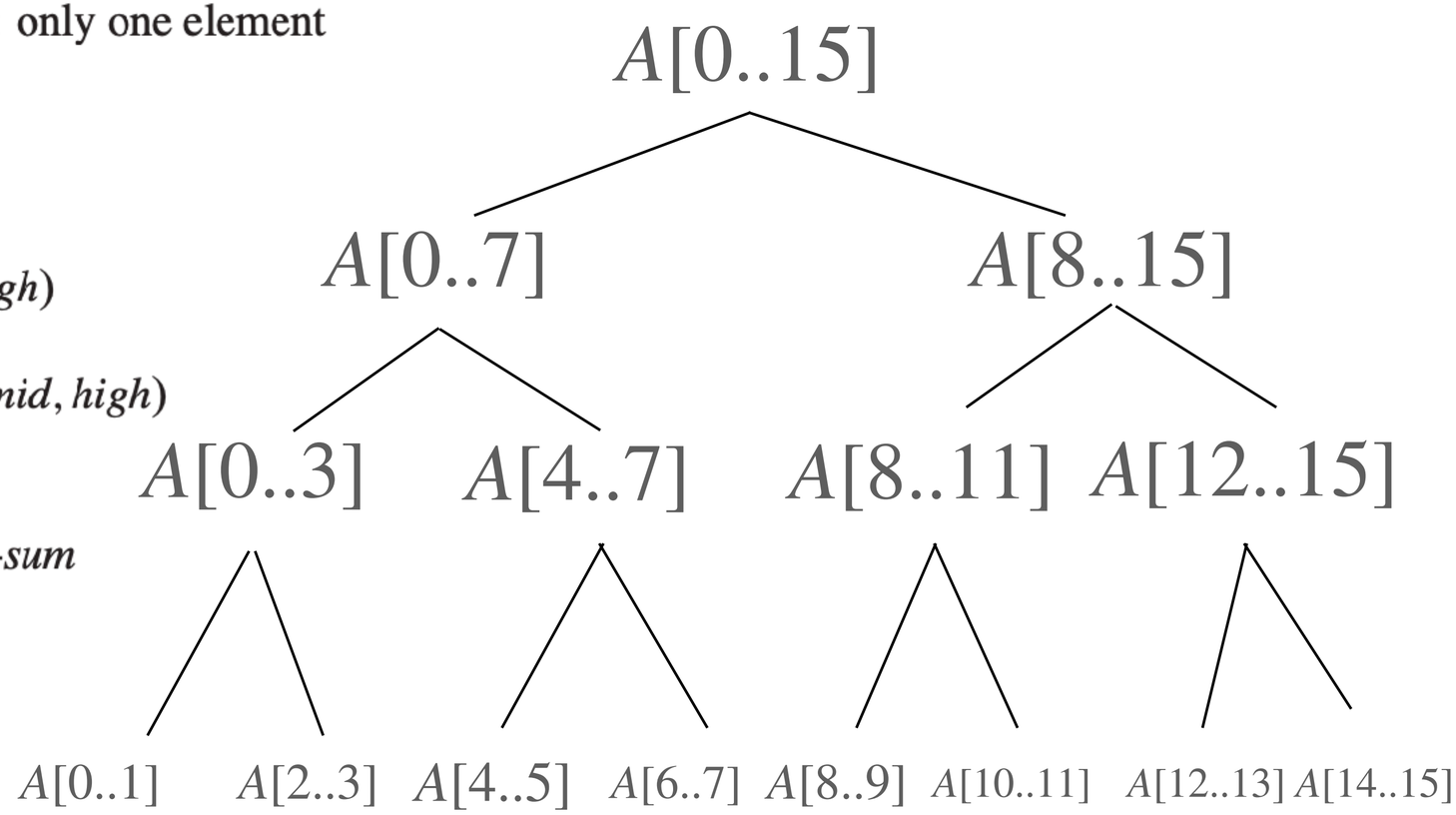
Maximum subarray $A[i,j]$ passes over m if $0 \le i < m \le j \le n$

Find left, half and central maximal subarrays and choose the largest one !

Divide-And-Conquer

Find left, half and central maximal subarrays and choose the largest one !

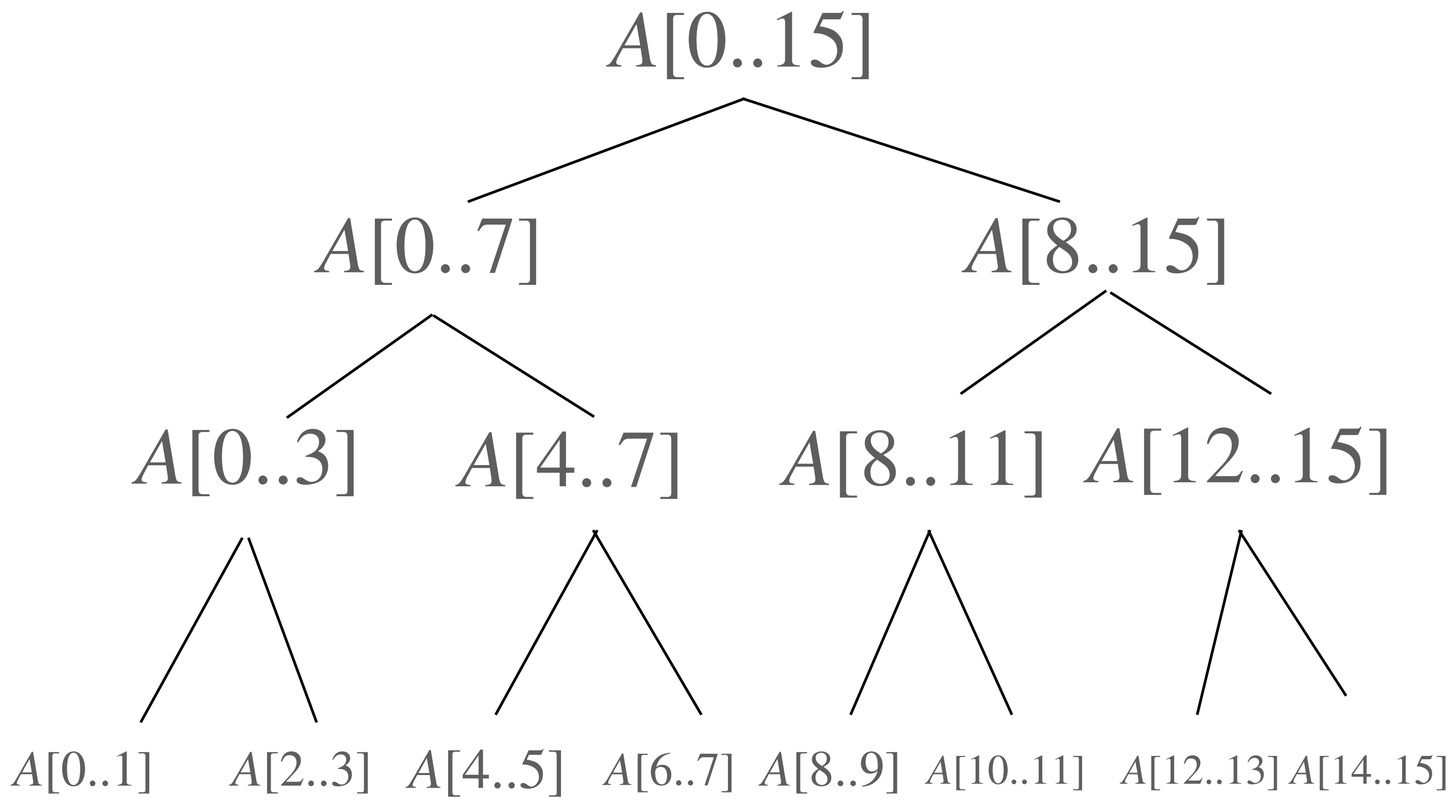
```
FIND-MAXIMUM-SUBARRAY(A, low, high)
 1  if high == low
 2       return (low, high, A[low])          // base case: only one element
 3  else mid = ⌊(low + high)/2⌋
 4       (left-low, left-high, left-sum) =
             FIND-MAXIMUM-SUBARRAY(A, low, mid)
 5       (right-low, right-high, right-sum) =
             FIND-MAXIMUM-SUBARRAY(A, mid + 1, high)
 6       (cross-low, cross-high, cross-sum) =
             FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
 7       if left-sum ≥ right-sum and left-sum ≥ cross-sum
 8            return (left-low, left-high, left-sum)
 9       elseif right-sum ≥ left-sum and right-sum ≥ cross-sum
10            return (right-low, right-high, right-sum)
11       else return (cross-low, cross-high, cross-sum)
```



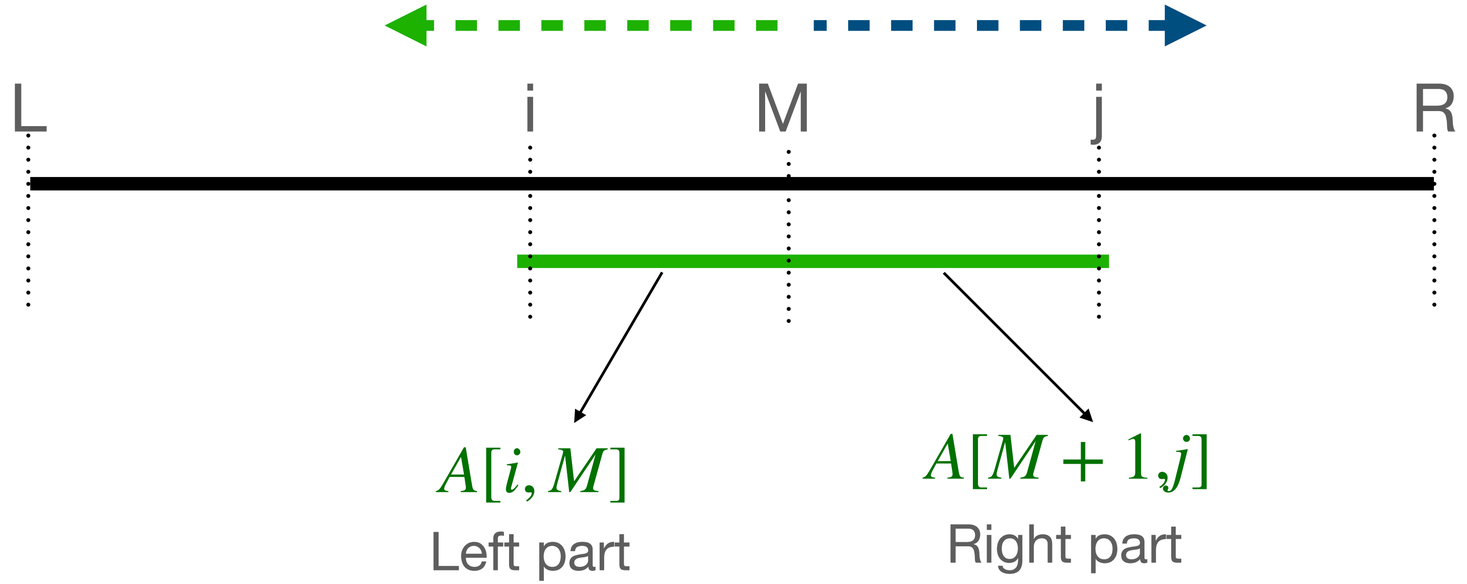
Divide-And-Conquer

What is the complexity ?

- At each node, the cost is central maximal subarray computation plus the recursions.
- We assume calling the recursion is free.
- **Then finding the maximal subarray is the main cost ?**



Divide-And-Conquer



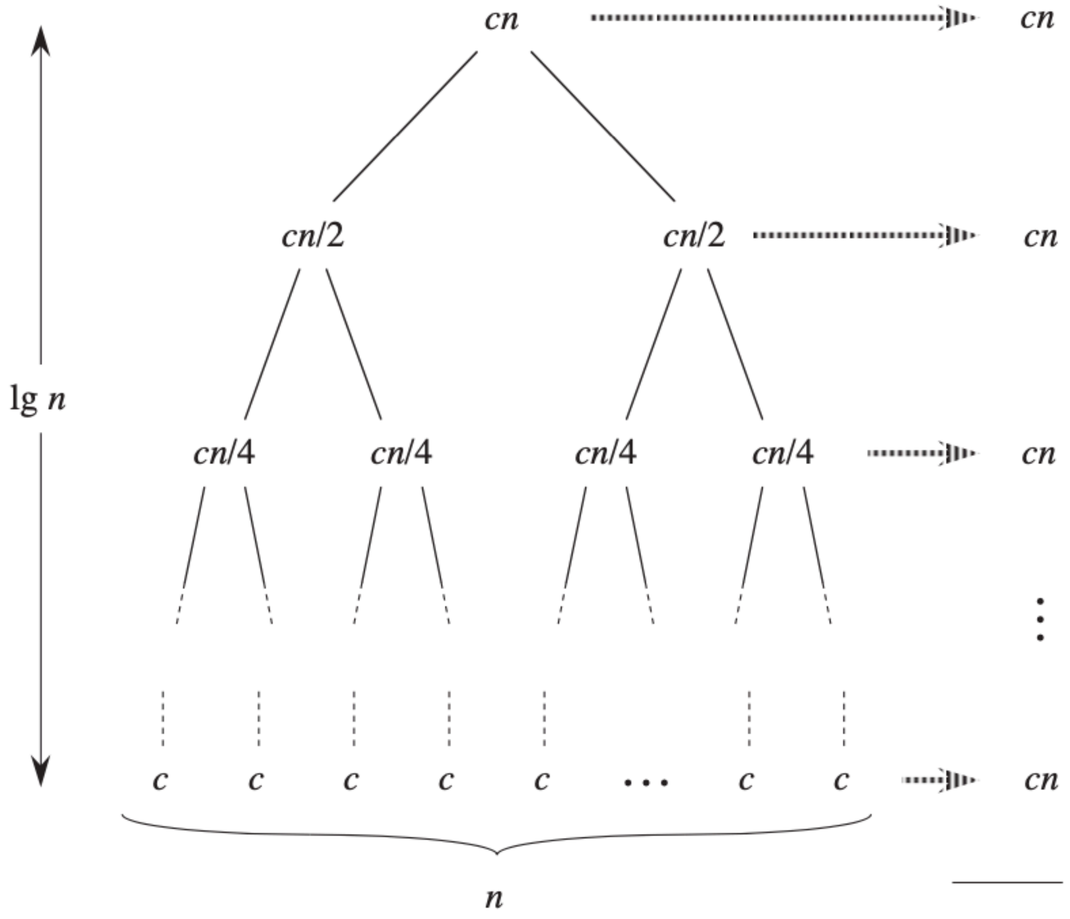

```
FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
1   left-sum = −∞
2   sum = 0
3   for i = mid downto low
4       sum = sum + A[i]
5       if sum > left-sum
6           left-sum = sum
7           max-left = i
8   right-sum = −∞
9   sum = 0
10  for j = mid + 1 to high
11      sum = sum + A[j]
12      if sum > right-sum
13          right-sum = sum
14          max-right = j
15  return (max-left, max-right, left-sum + right-sum)
```

- The central maximal subarray has a left part A[i,M] and a right part A[M+1,j].
- Would it be possible to find a better subarray $A[i', M]$ for $L \leq i' < i$???
- Would it be possible to find a better subarray $A[M+1, j']$ for $j < j' \leq R$???
- **Therefore, detecting the central maximal subarray is $O(n)$?**

Divide-And-Conquer

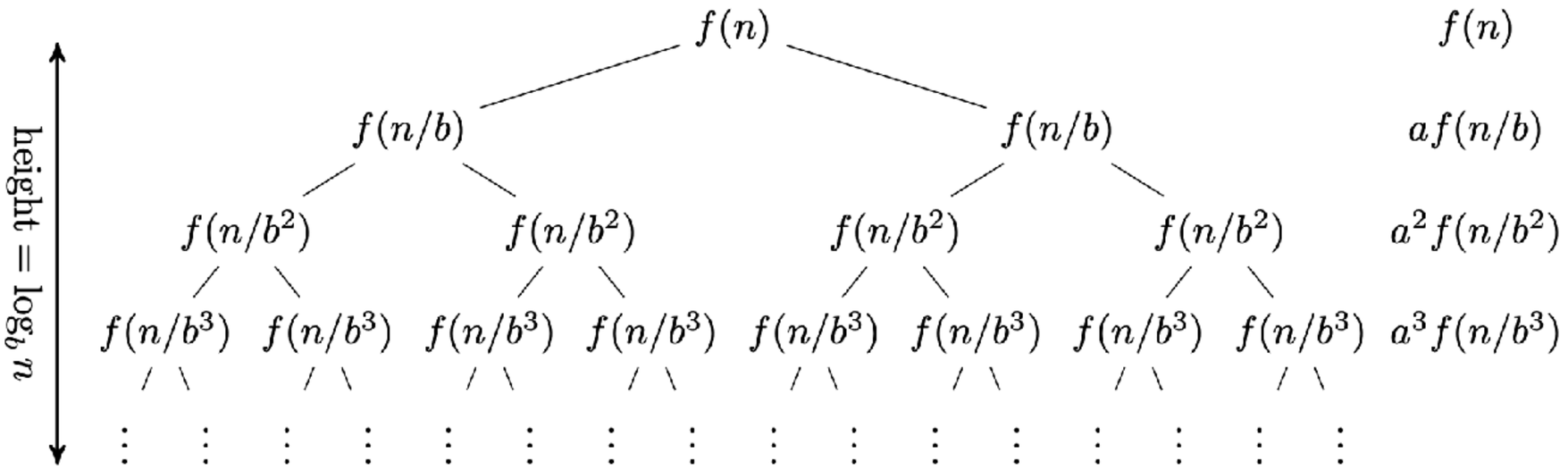
What is the complexity ?

- At each node, the cost is central maximal subarray computation plus the recursions (that are free).
- **Total cost of maximal subarray detection is $O(n \log n)$ with the recursion.**



Divide-and-Conquer

$$T(n) = aT(n/b) + f(n)$$



https://www.cs.cornell.edu/courses/cs3110/2012sp/lectures/lec20-master/mm-proof.pdf

- Recursively split problem of size n into problems of size n/b until the problem size reaches the unit size 1.
- We can visualize it with a recursion tree that shows the resursion $T(n) = aT(n/b) + f(n)$.
- On this tree each node includes a cost that is required to be done to join the parent. For instance, the cost of finding the maximal subarray crossing the central point in our example.

Solution of Divide-and-Conquer Type Recursions

$$T(n) = aT(n/b) + f(n)$$

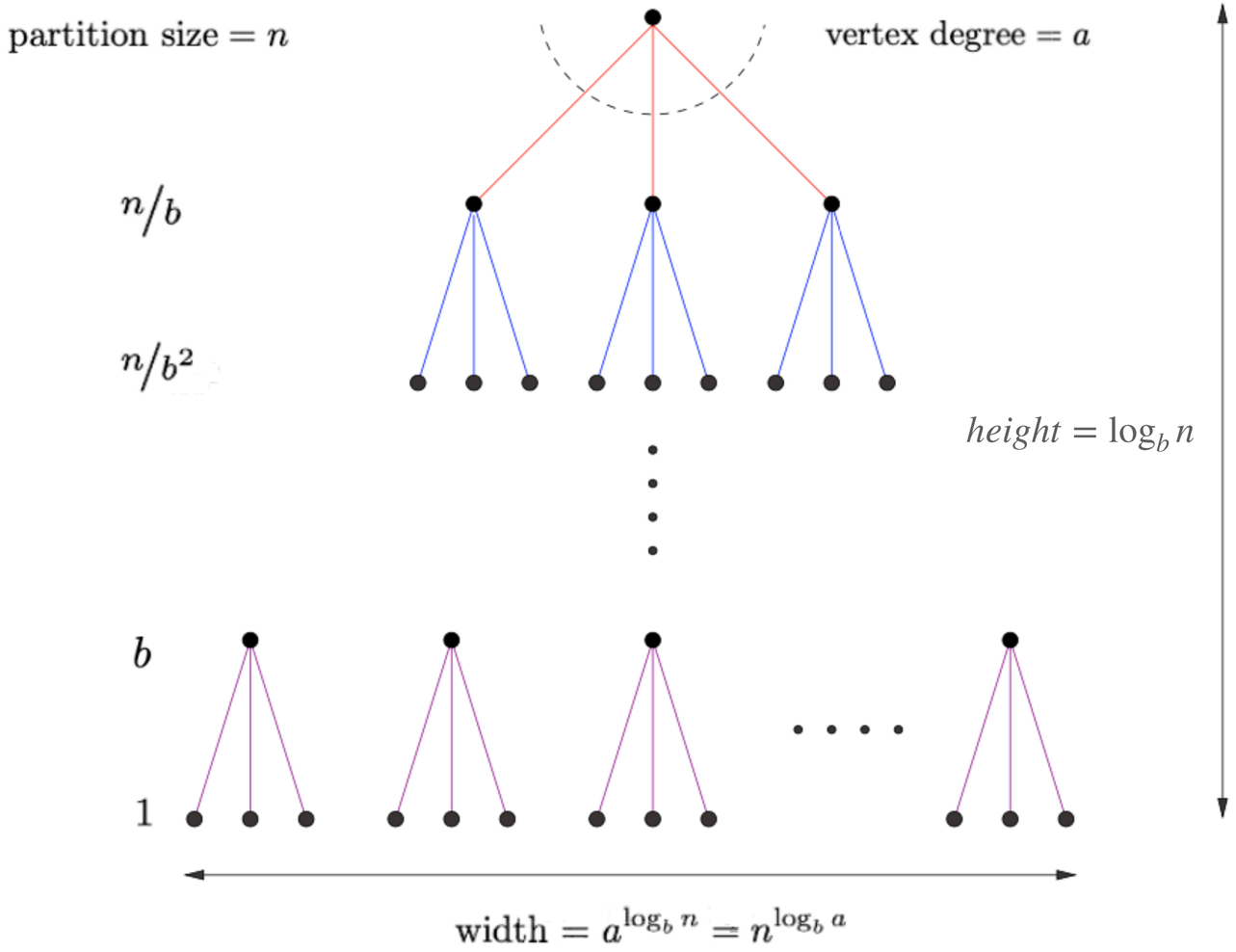
Master Method:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some $c < 1$, then $T(n) = \Theta(f(n))$.

You can check the proof from Cormen's book.

- In our example $T(n) = T(n/2) + f(n)$, $f(n) \in \Theta(n)$ is the cost of detecting the central maximal subarray (why $\Theta(n)$?).
- Therefore, $T(n) = \Theta(n \log n)$ due to case 2 with $a = 2, \ \ b = 2$.

Master Theorem



Case 1: Too many leaves
Case 2: Equal amount of work at each level
Case 3: Aggregating the nodes to the parent dominates everything

At each case compare $f(n)$ with $n^{\log_b a}$.
Whichever dominates determine the solution

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some $c < 1$, then $T(n) = \Theta(f(n))$.

$$T(n) = aT(n/b) + f(n)$$

Fast n-bit Multiplication

Karatsuba Algorithm

Multiplying two n-digit numbers takes $O(n^2)$ time with the classic way since it takes n^2 multiplication and $O(n)$ additions.

$$
\begin{array}{rr}
 & a_3 a_2 a_1 \\
 & b_3 b_2 b_1 \\
\times & \hline \\
 & b_1 \times a_3 a_2 a_1 \\
 & b_2 0 \times a_3 a_2 a_1 \\
+ & b_3 00 \times a_3 a_2 a_1 \\
\hline
\end{array}
$$

1234 x 4567 = (7 x1234) +
(60 x 1234) +
(500 x 1234) +
(4000 x 1234) = 5635678

Would it be possible to achieve it with less multiplications?

Fast n-bit Multiplication

Karatsuba Algorithm

Let's split each n-digit into two n/2 digit numbers and assume $w = 10^{(n/2)}$.

$A = 1234 \rightarrow (a_1 = 12, a_2 = 34) : 1234 = a_1 \cdot w + a_2$ $\quad$ $B = 4567 \rightarrow (b_1 = 45, b_2 = 67) : 4567 = b_1 \cdot w + b_2$

$$A \cdot B = (a_1 w + a_2)(b_1 w + b_2) = a_1 b_1 w^2 + a_1 b_2 w + a_2 b_1 w + a_2 b_2$$

- Multiplications with w are simply padding with zeros, so doesn't matter.
- Now we need 4 $(n/2)$ multiplications with $f(n) \in O(n)$ additions. Therefore, according to first case (why?) of master theorem:

$$T(n) = 4T(n/2) + O(n) \rightarrow T(n) \in O(n^{\log_2 4}) = \Theta(n^2)$$

No improvement over the n-digit case.

Fast n-digit Multiplication

Karatsuba Algorithm

- Kolmogorov conjectured n-digit multiplication is $\Omega(n^2)$ time. However, Karatsuba showed its possible by computing the product differently, which makes it in $\Theta(n^{1.585})$, although with a penalty of a bit more additions, not hurting the complexity.

$$q_1 = a_1 \cdot b_1 \qquad q_2 = (a_1 + b_1)(a_2 + b_2) \qquad q_3 = a_2 \cdot b_2$$

$$A \cdot B = q_1 + (q_2 - q_1 - q_3)w + q_3 w^2$$

According to the **first case** of the master theorem again

$$T(n) = 3T(n/2) + O(n) \rightarrow T(n) \in \theta(n^{\log_2 3}) = \Theta(n^{1.585})$$

Closest Pair

Given a set of (x, y) points, find the closest pair.

- *Same idea used in maximal subarray problem makes sense*

1. Sort all points according to x dimension and also in y direction separately.
2. Recursion to find the closest pair on the left and on the right of the middle x.
3. Say, the minimum of these two pairs return a distance of δ.
4. Consider the (x_i, y_i) tuples with $(middle - \delta) \leq x \leq (middle + \delta)$. Notice that others will absolutely have a distance larger than δ.
5. On this group, each data point will be compared at most 7 points (**not obvious**). Therefore, finding the middle cost is linear time with some good management.
6. $T(n) = 2T(n/2) + O(n)$, by the second rule of master theorem, that turns out to $T(n) \in O(n \log n)$.

Closest Pair

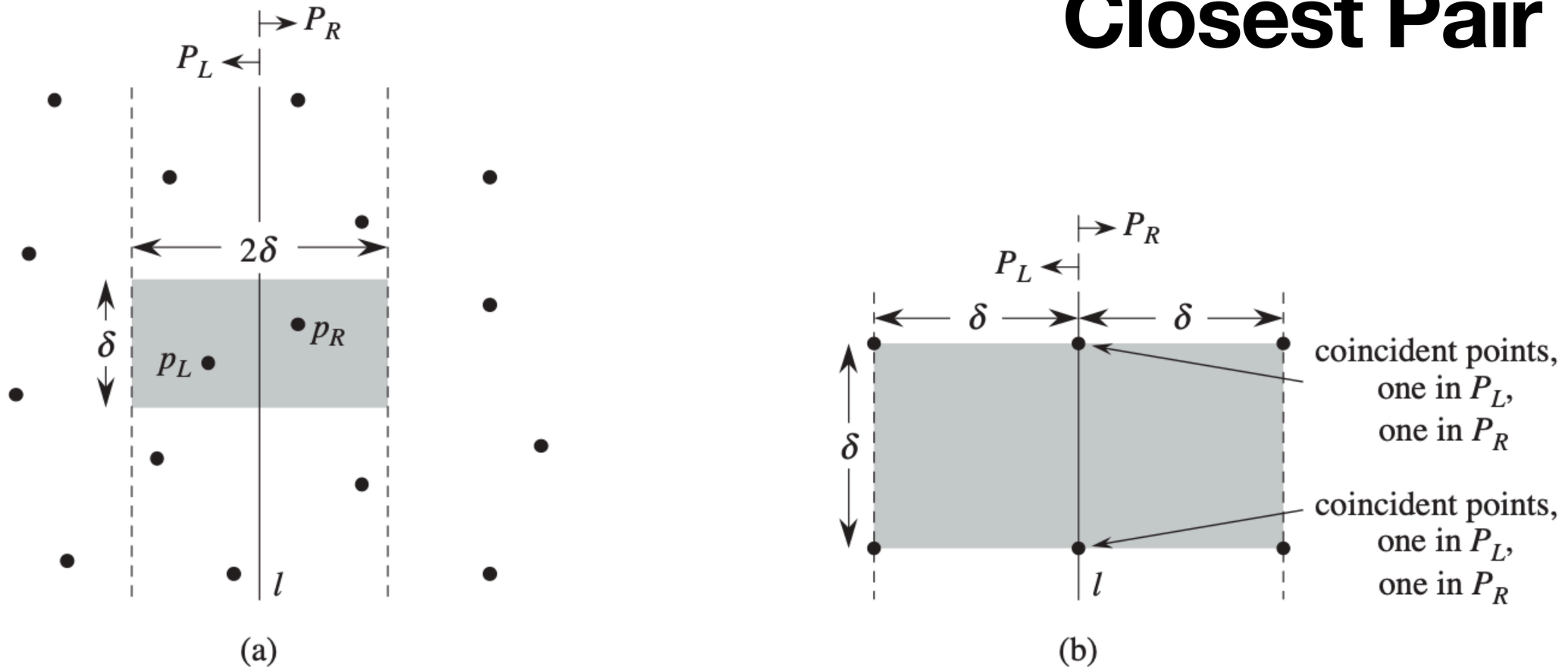


Figure 33.11 Key concepts in the proof that the closest-pair algorithm needs to check only 7 points following each point in the array Y'. **(a)** If $p_L \in P_L$ and $p_R \in P_R$ are less than δ units apart, they must reside within a $\delta \times 2\delta$ rectangle centered at line l. **(b)** How 4 points that are pairwise at least δ units apart can all reside within a $\delta \times \delta$ square. On the left are 4 points in P_L, and on the right are 4 points in P_R. The $\delta \times 2\delta$ rectangle can contain 8 points if the points shown on line l are actually pairs of coincident points with one point in P_L and one in P_R.

Reading assignment

- Read the recursion and divide-and-conquer chapters from the text books, particularly from Cormen and Skiena, I suggest. Also you can refer chapter 33.4 from Cormen for a deep explanation of closest pair problem.