

Applied Algorithms

CSCI-B505 / INFO-I500

Lecture 12.

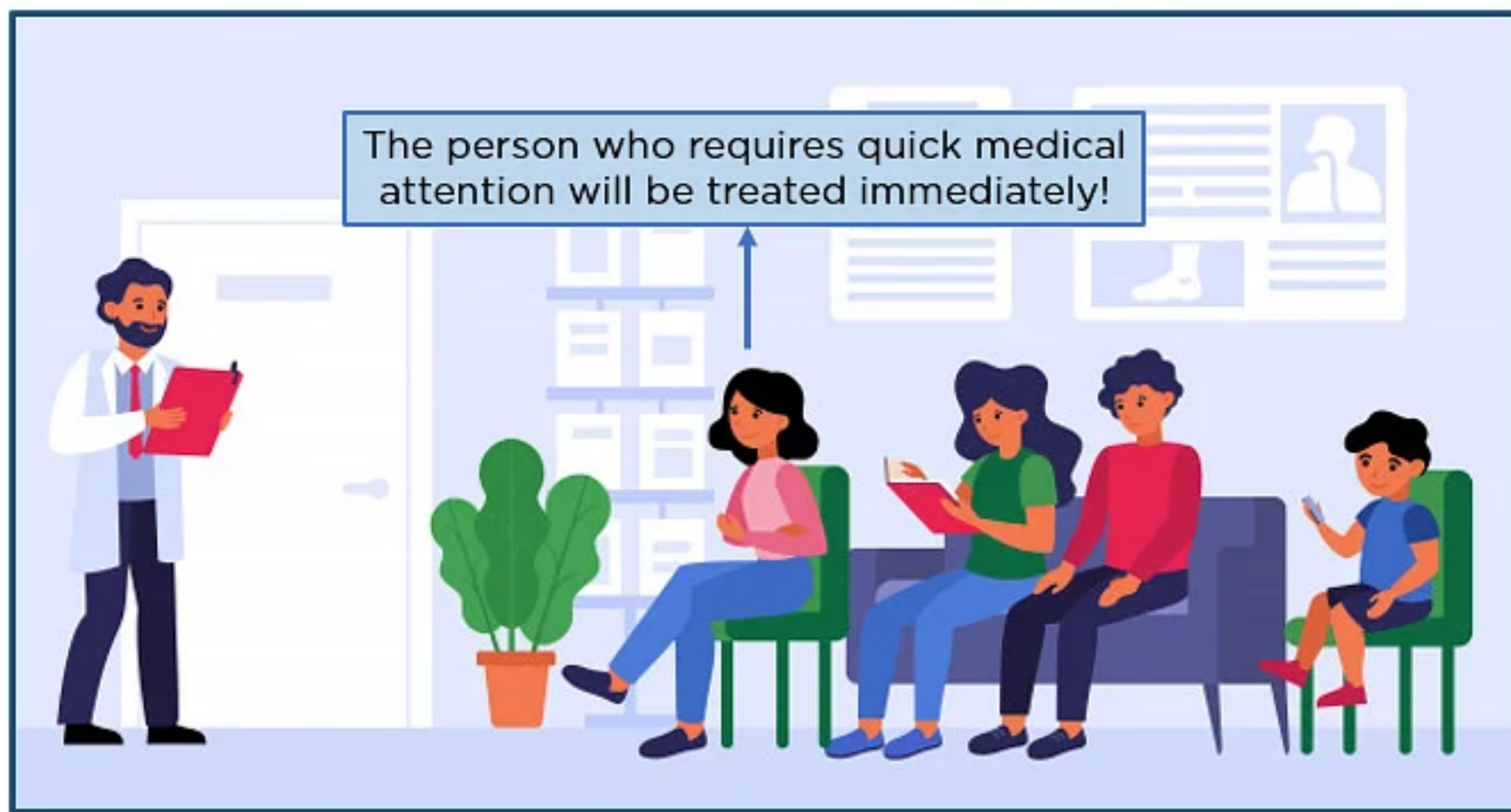
Priority Queue & Heap Data Structure

M. Oguzhan Kulekci, October 3/4, 2022

- Priority Queue
- Heap Data Structure
 - Insert, delete, update operations
 - Heap construction and sorting
- Some examples

Priority Queue

Hospital Emergency Queue



<https://www.simplilearn.com/tutorials/data-structure-tutorial/priority-queue-in-data-structure>

- Normal queue operations might not be adequate in some situations.
- Each item on the queue has a priority
- Higher priority means getting service earlier

- How to implement such priority queues ? *Arrays, linked-lists, skip-lists*

Heap

- How about using a **binary tree** to implement a priority queue ?

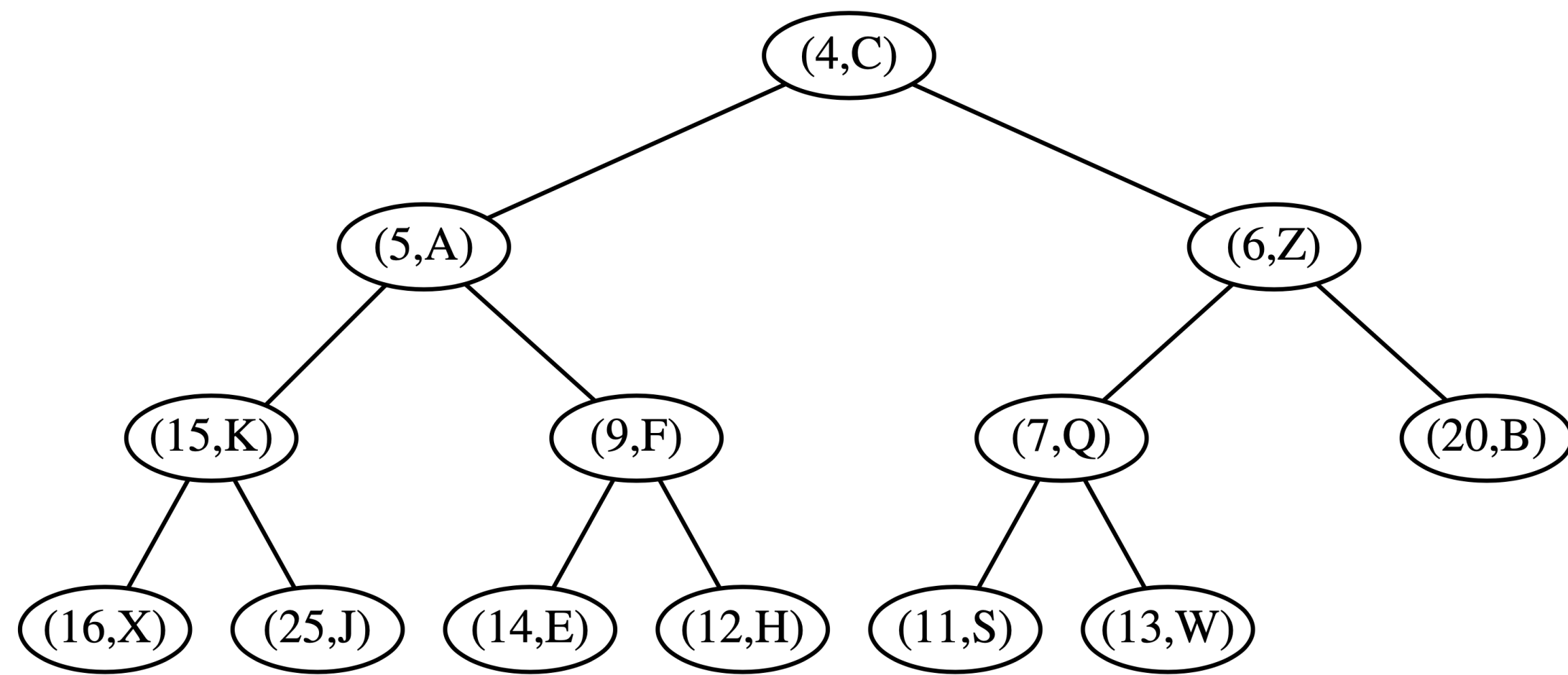


Figure 9.1: Example of a heap storing 13 entries with integer keys. The last position is the one storing entry (13, W).

Min-Heap

Properties (min-heap):

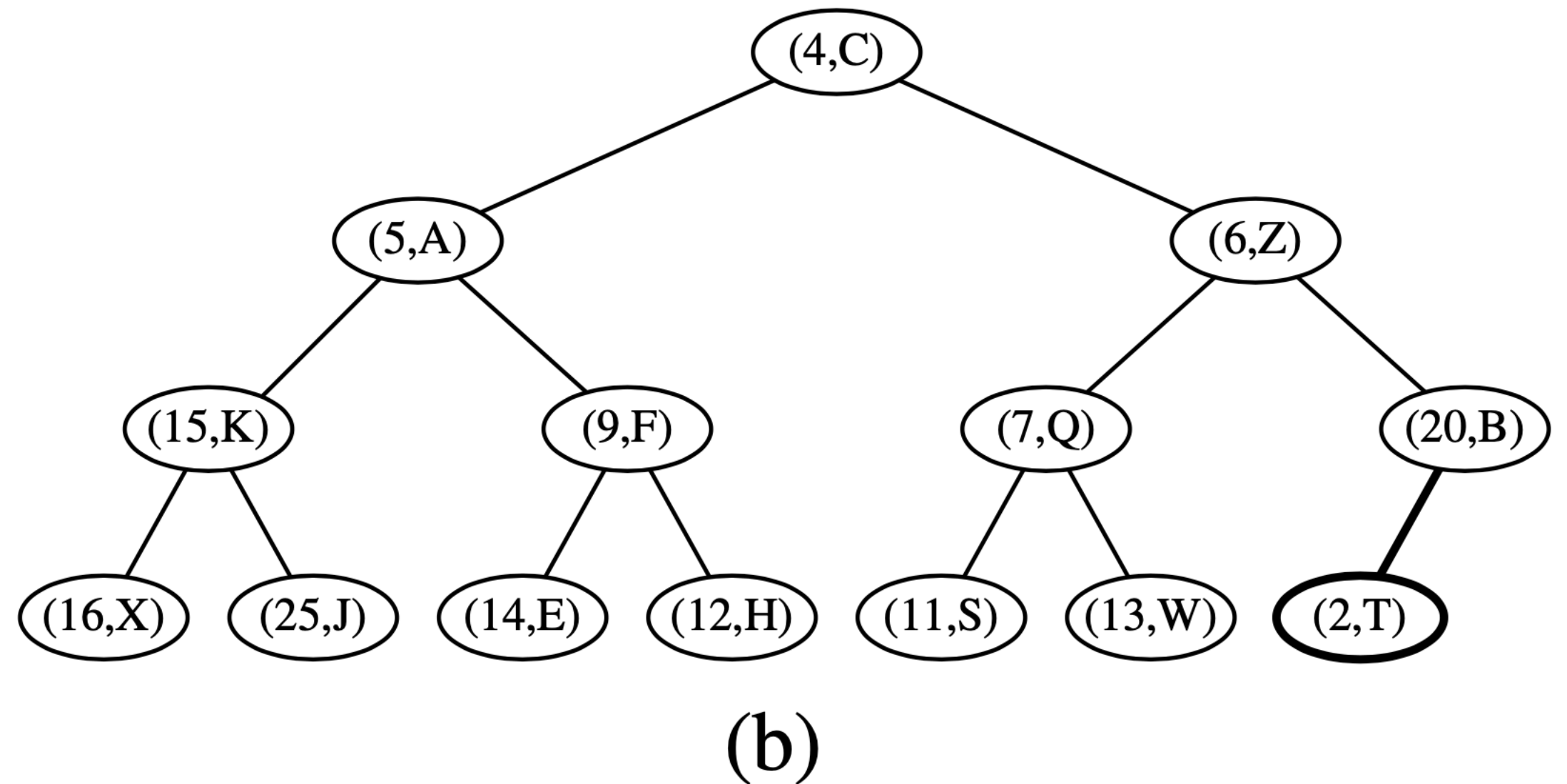
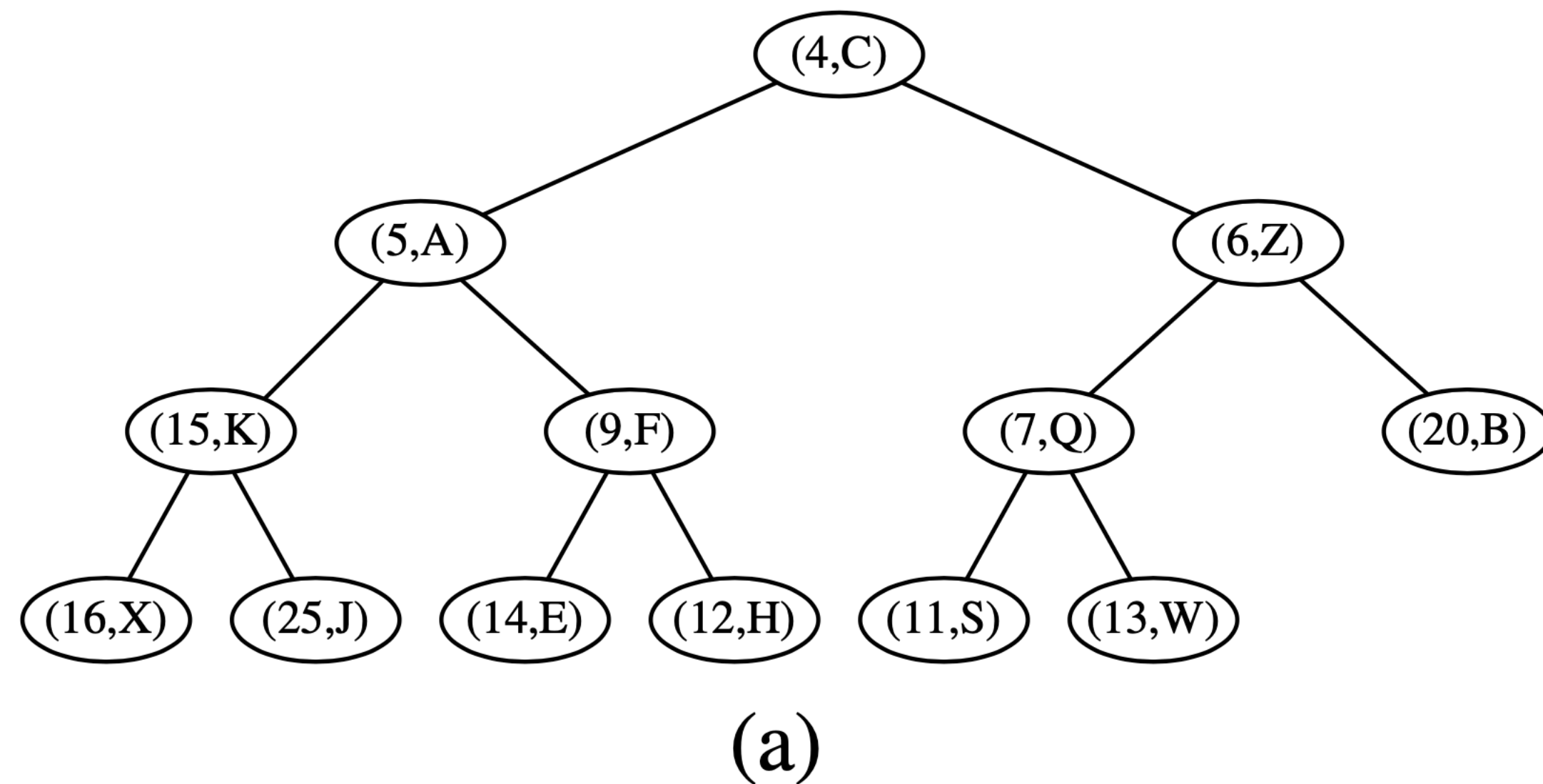
1. Every item in the tree except the root has an associated value that is ***greater*** than or equal to its parent.
2. The tree is *almost* complete, meaning all levels are fully occupied except the lowest level, which is filled from left to right

Max-Heap is similar...

Up-Heap Bubbling

- How to add an item ?

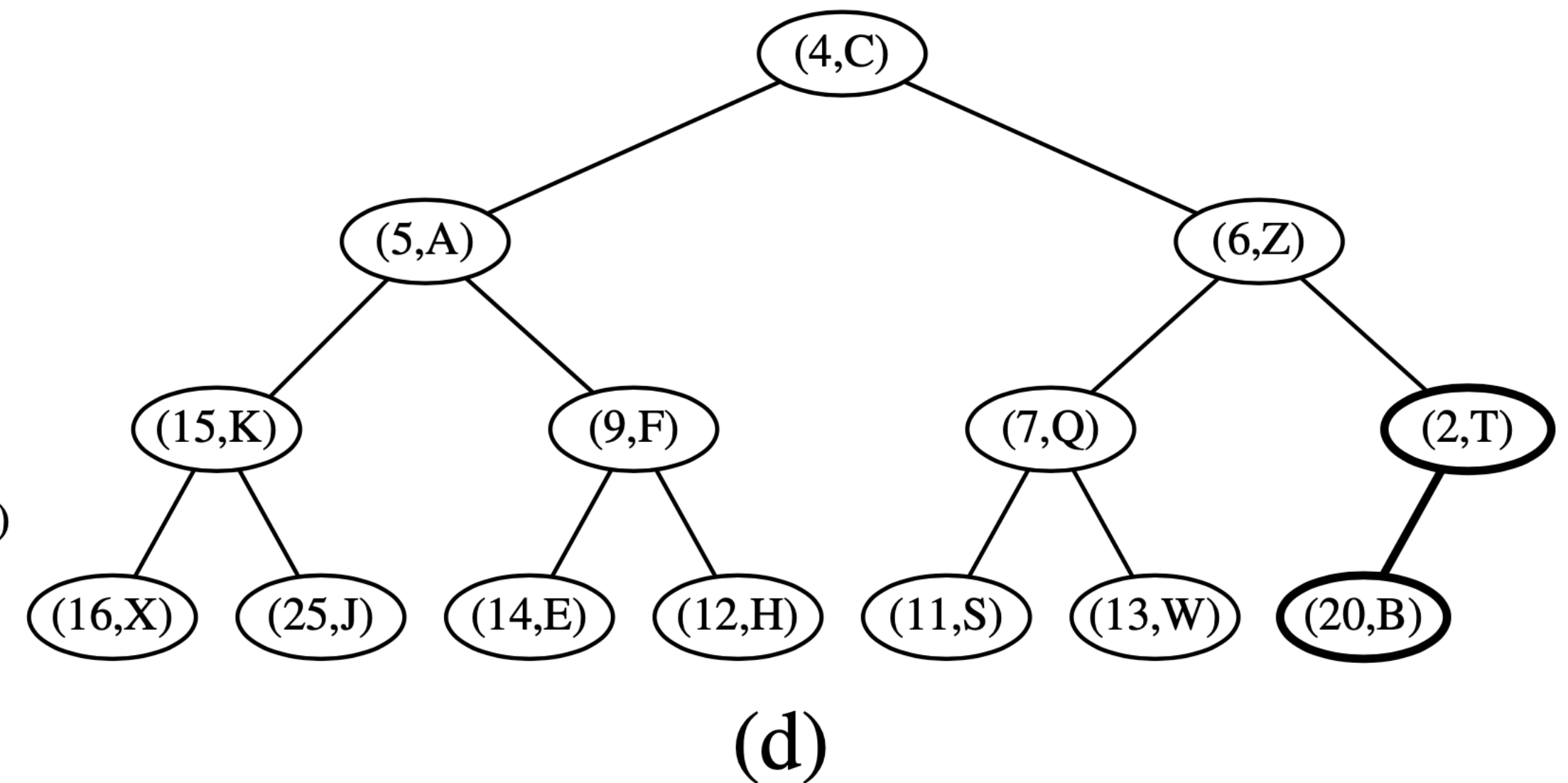
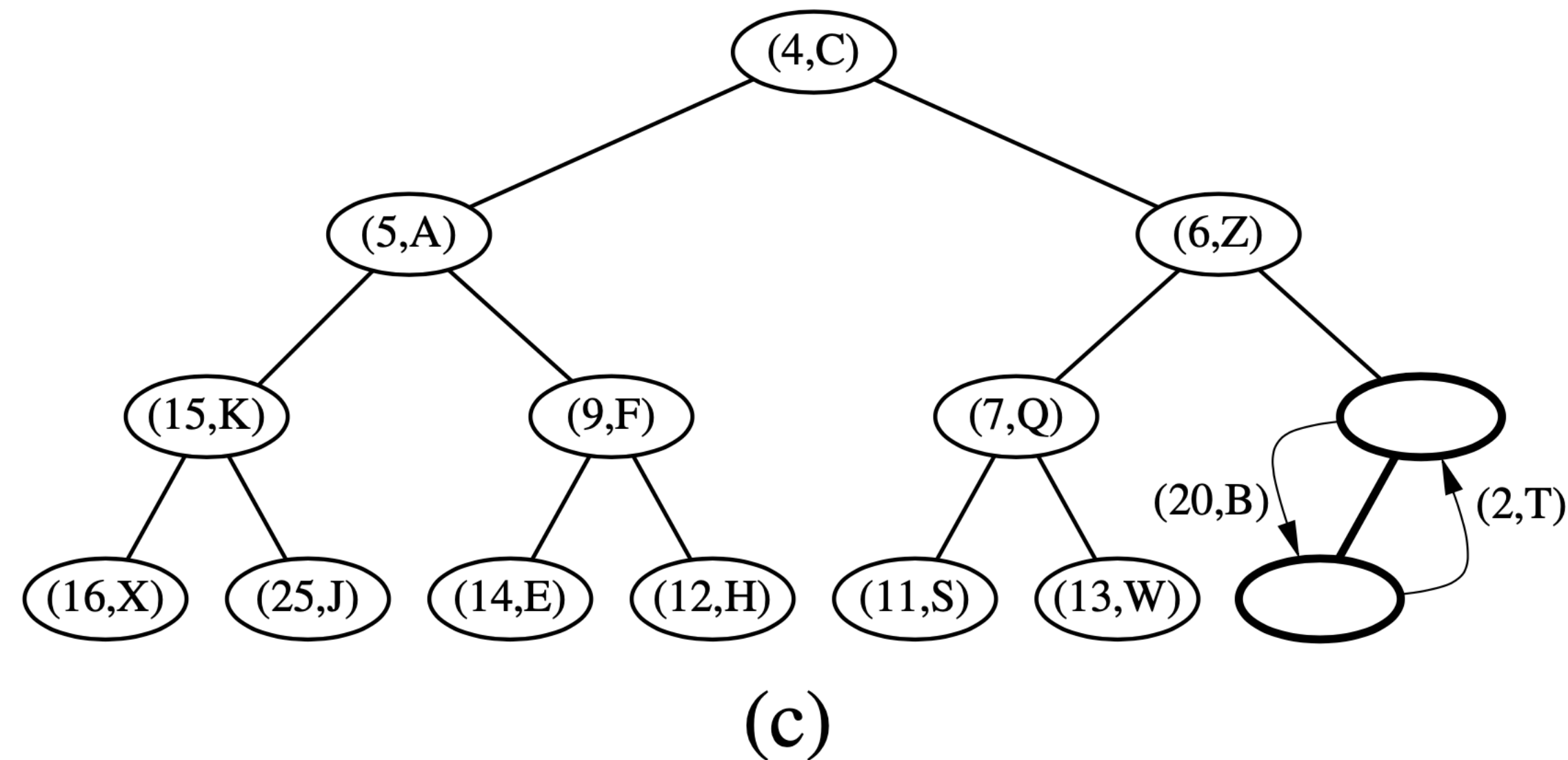
STEP 1. To keep tree **almost complete**, we initially locate the new item to the rightmost positions on the last level heap.



Up-Heap Bubbling

- How to add an item ?

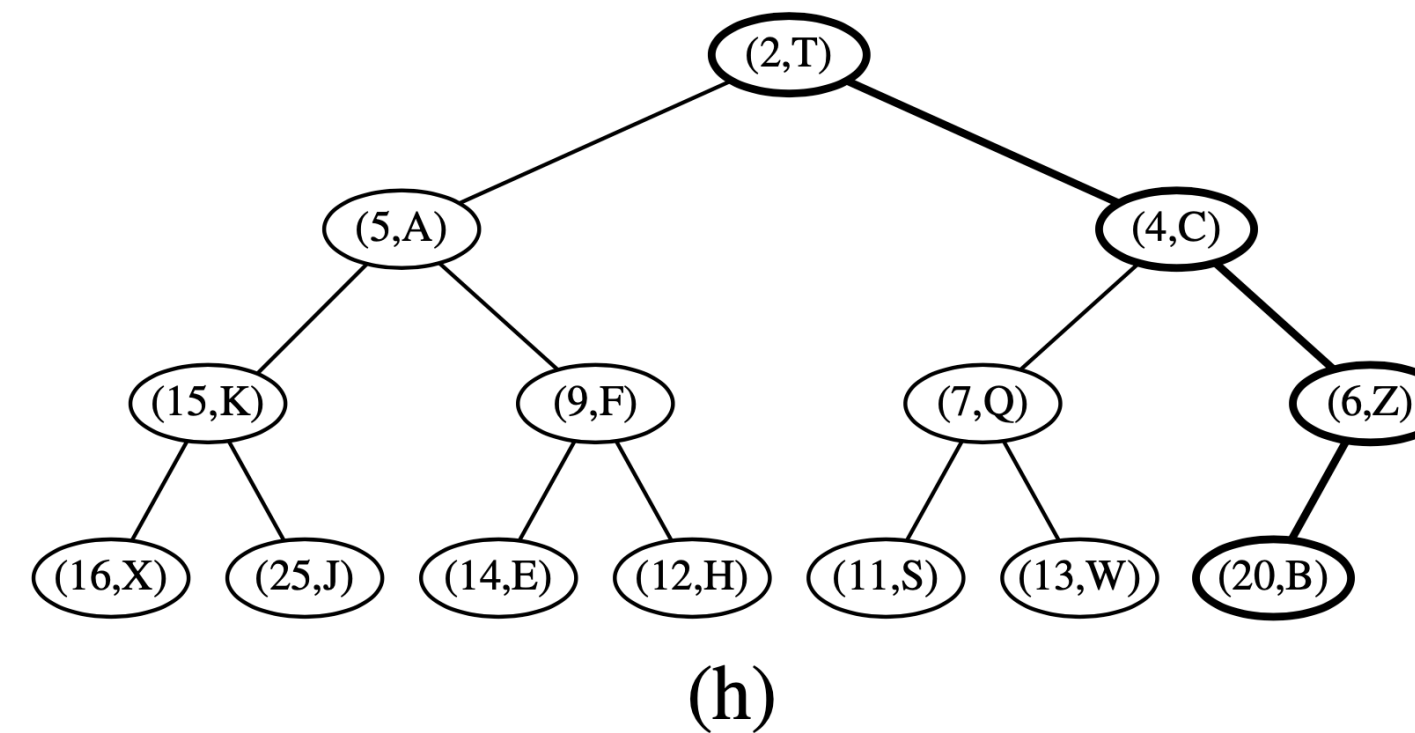
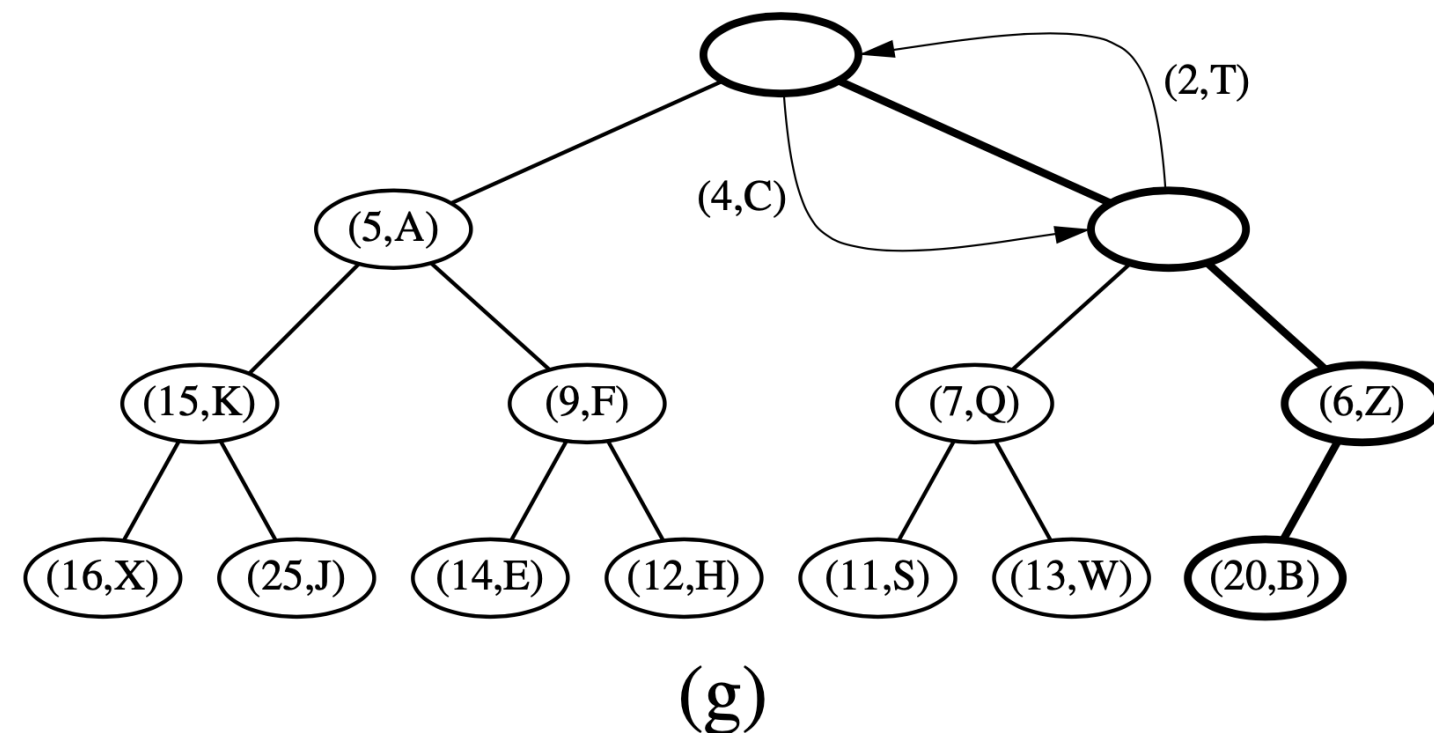
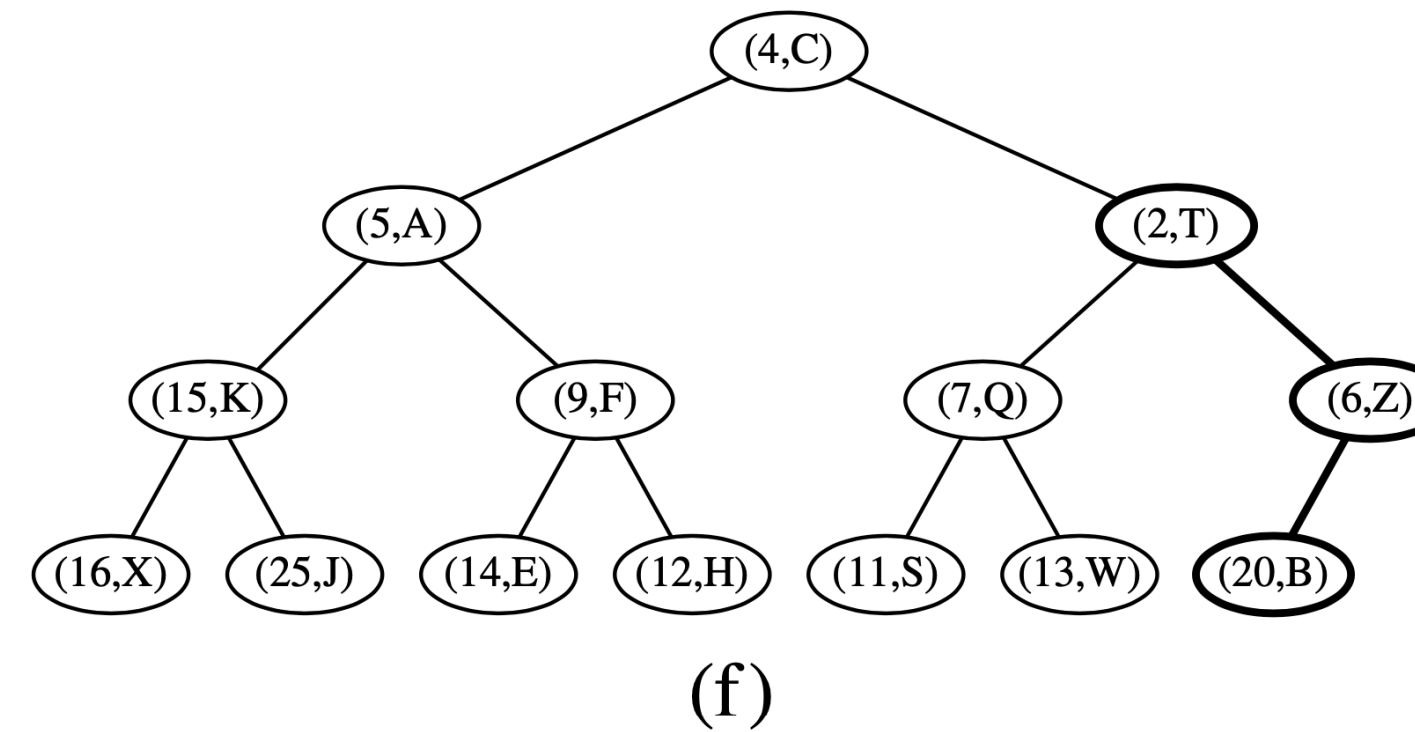
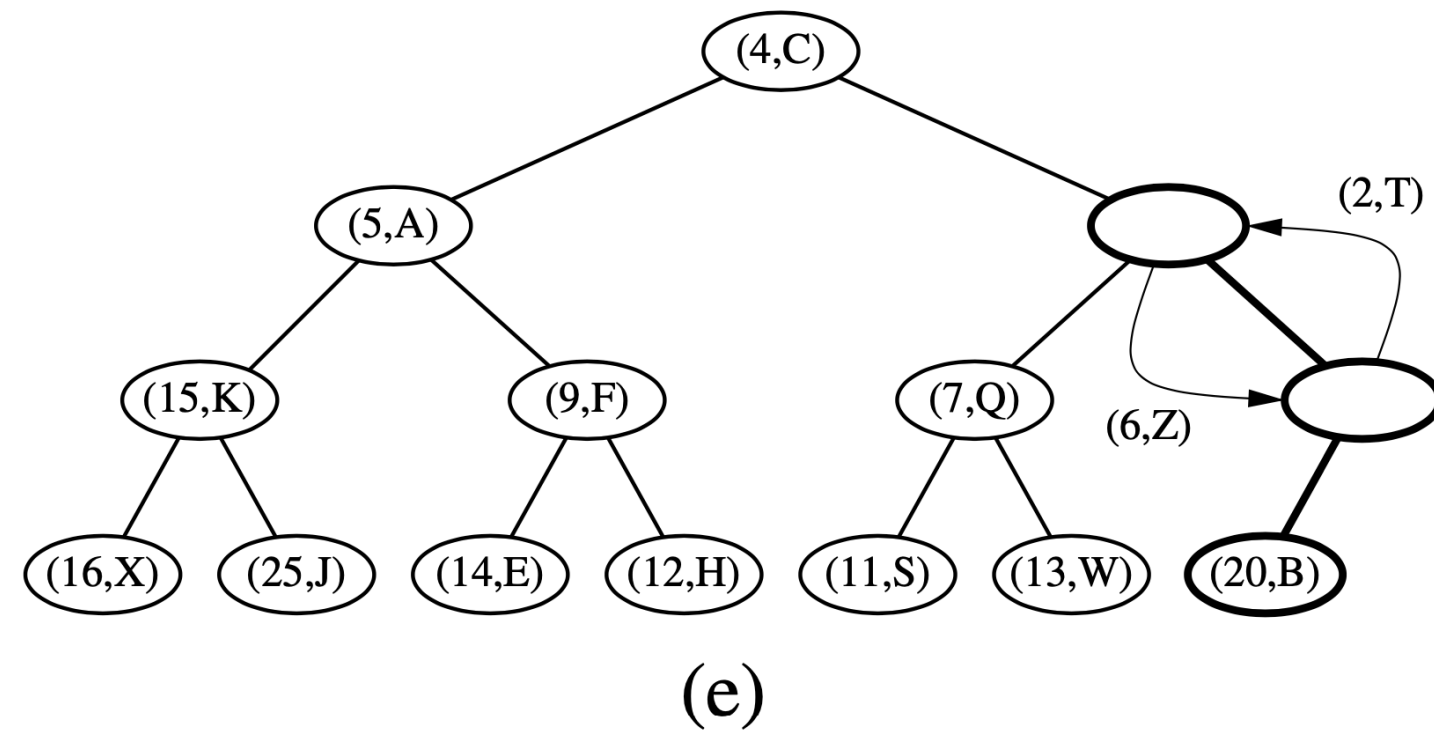
STEP 2. To maintain the heap property, we check the path **up** to the root iteratively and swap the nodes if necessary.



Up-Heap Bubbling

- How to add an item ?

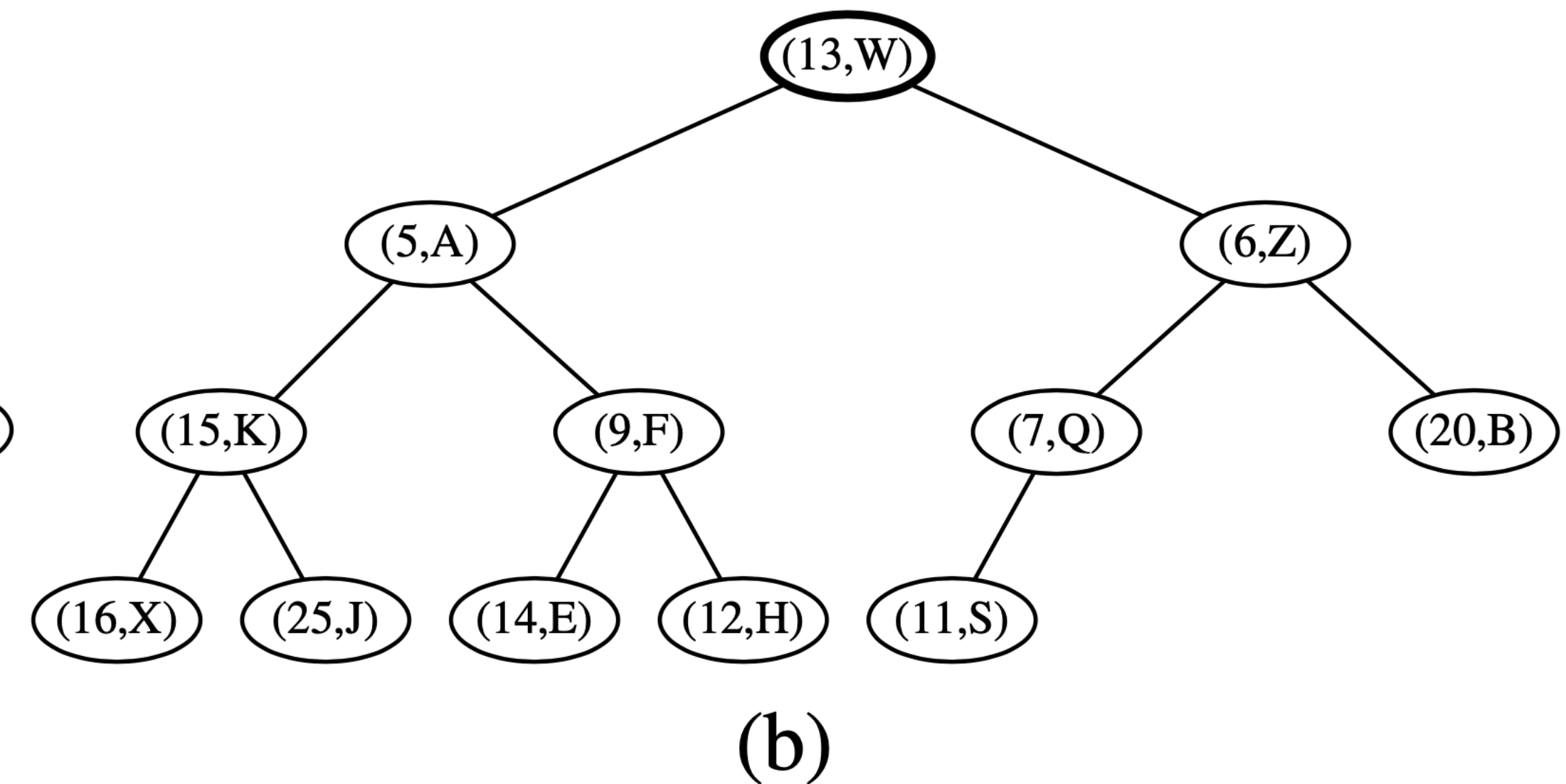
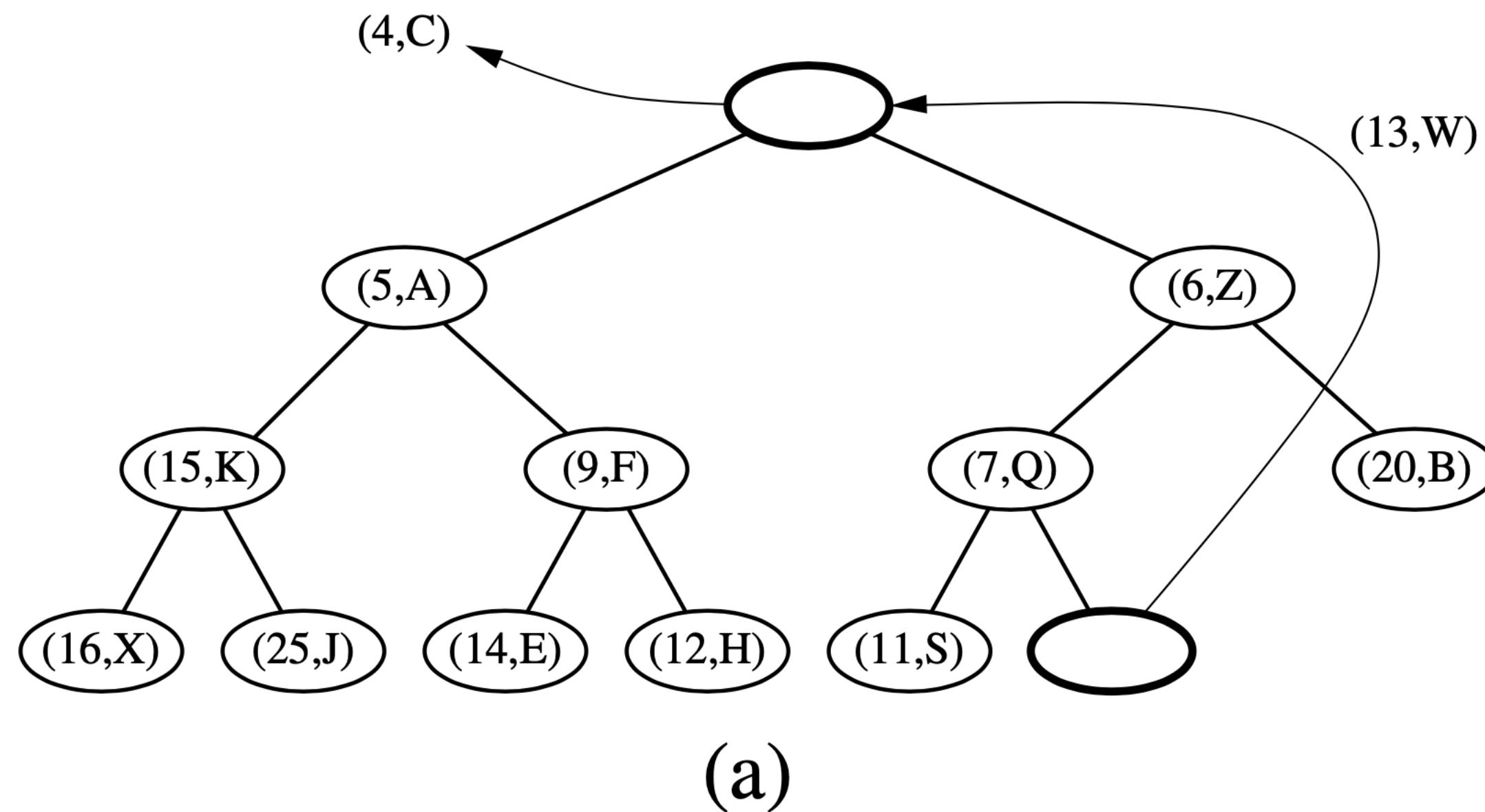
STEP 2. To maintain the heap property, we check the path **up** to the root iteratively and swap the nodes if necessary.



Down-Heap Bubbling

- How to remove the root item, which is the highest priority?

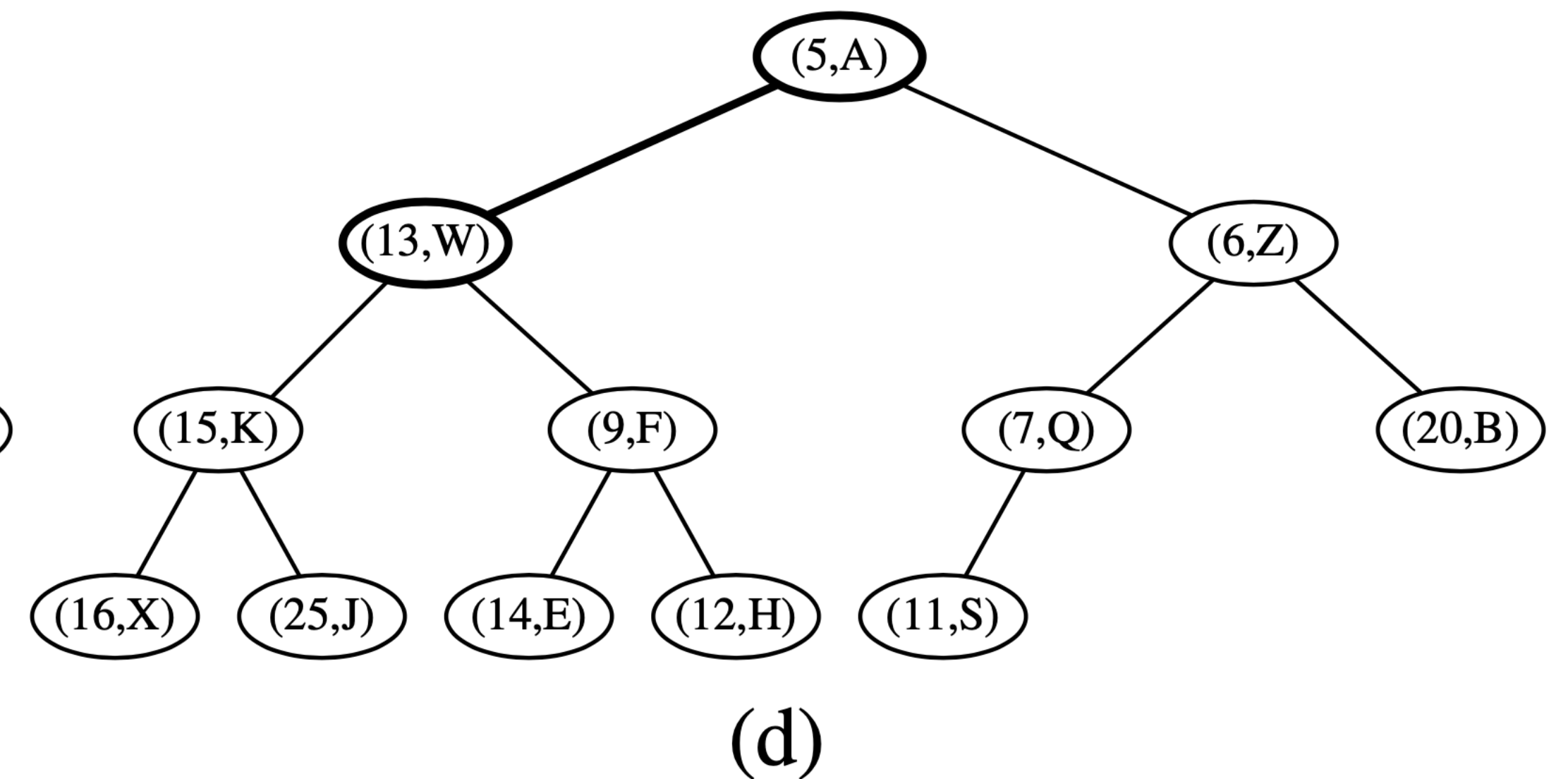
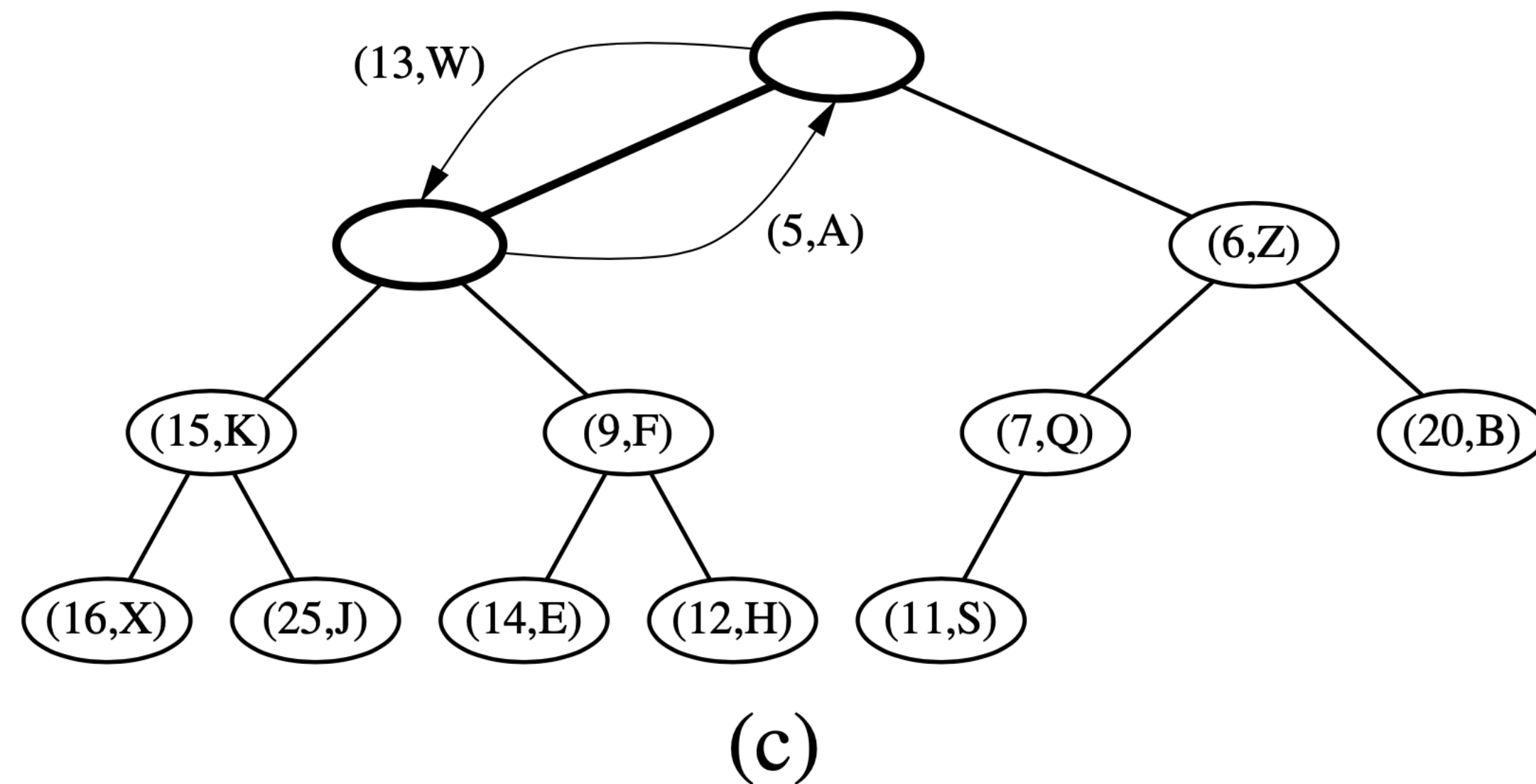
STEP 1. To keep tree **almost complete**, remove the root and move the right-most item on the last level to the root position.



Down-Heap Bubbling

- How to remove the root item, which is the highest priority?

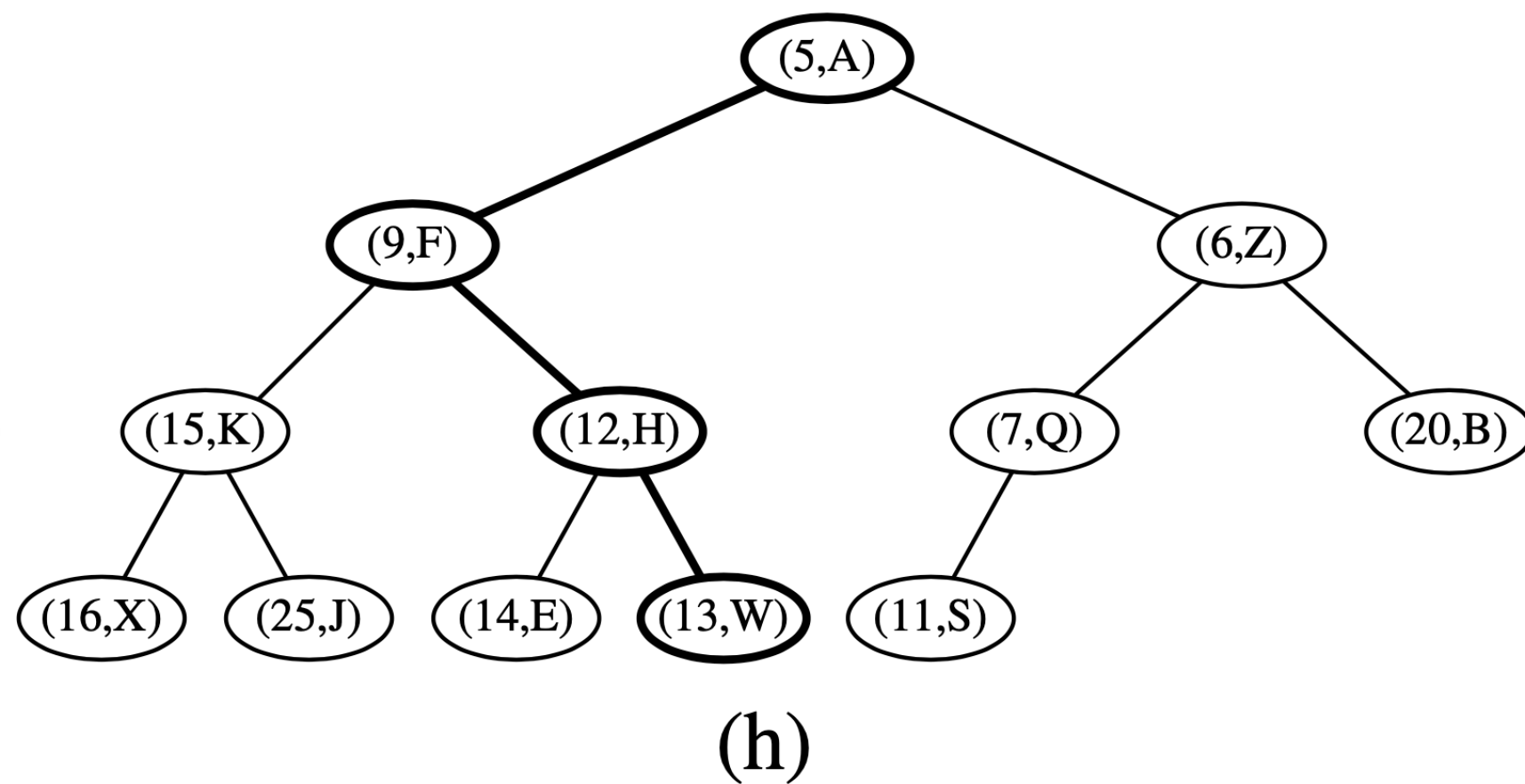
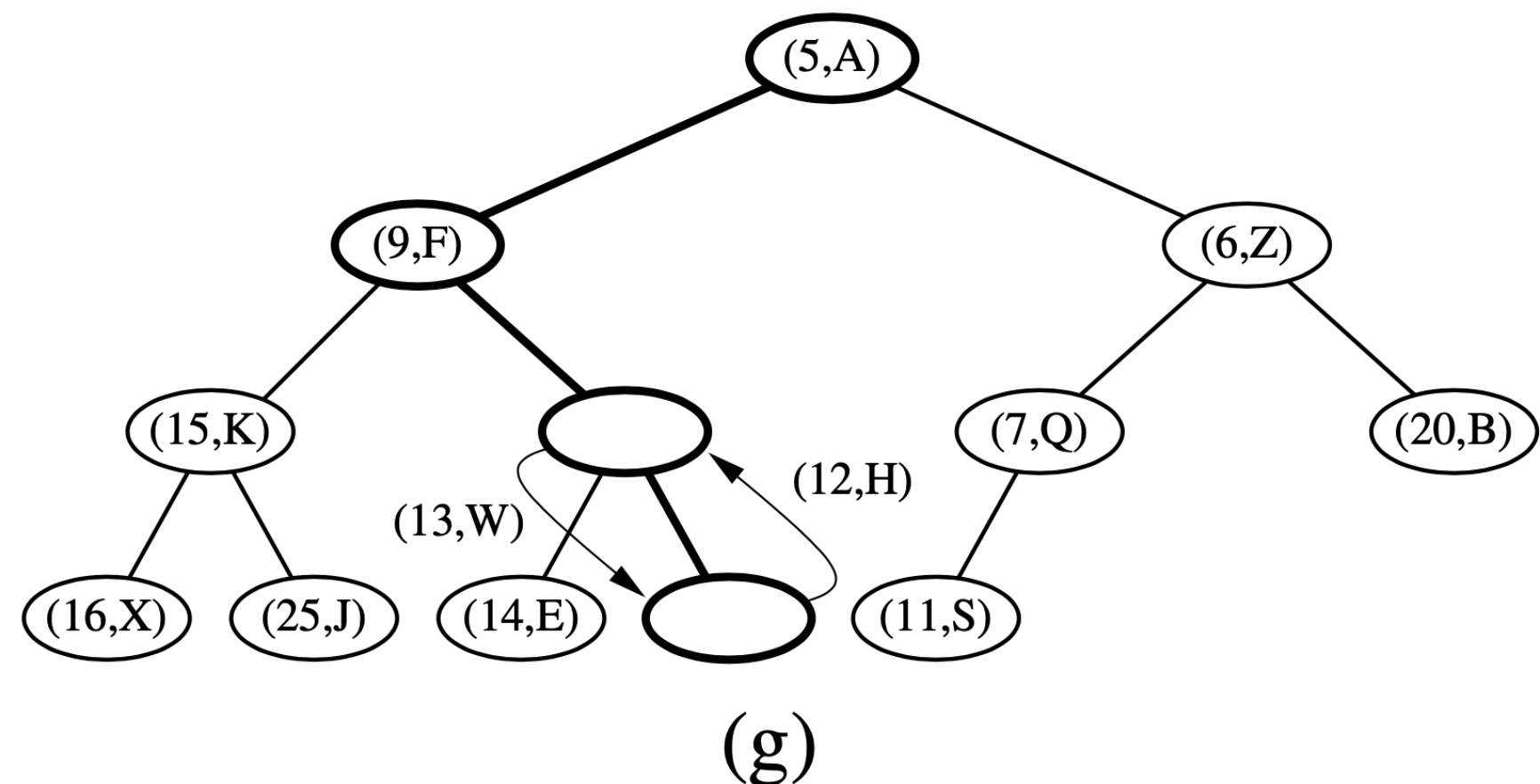
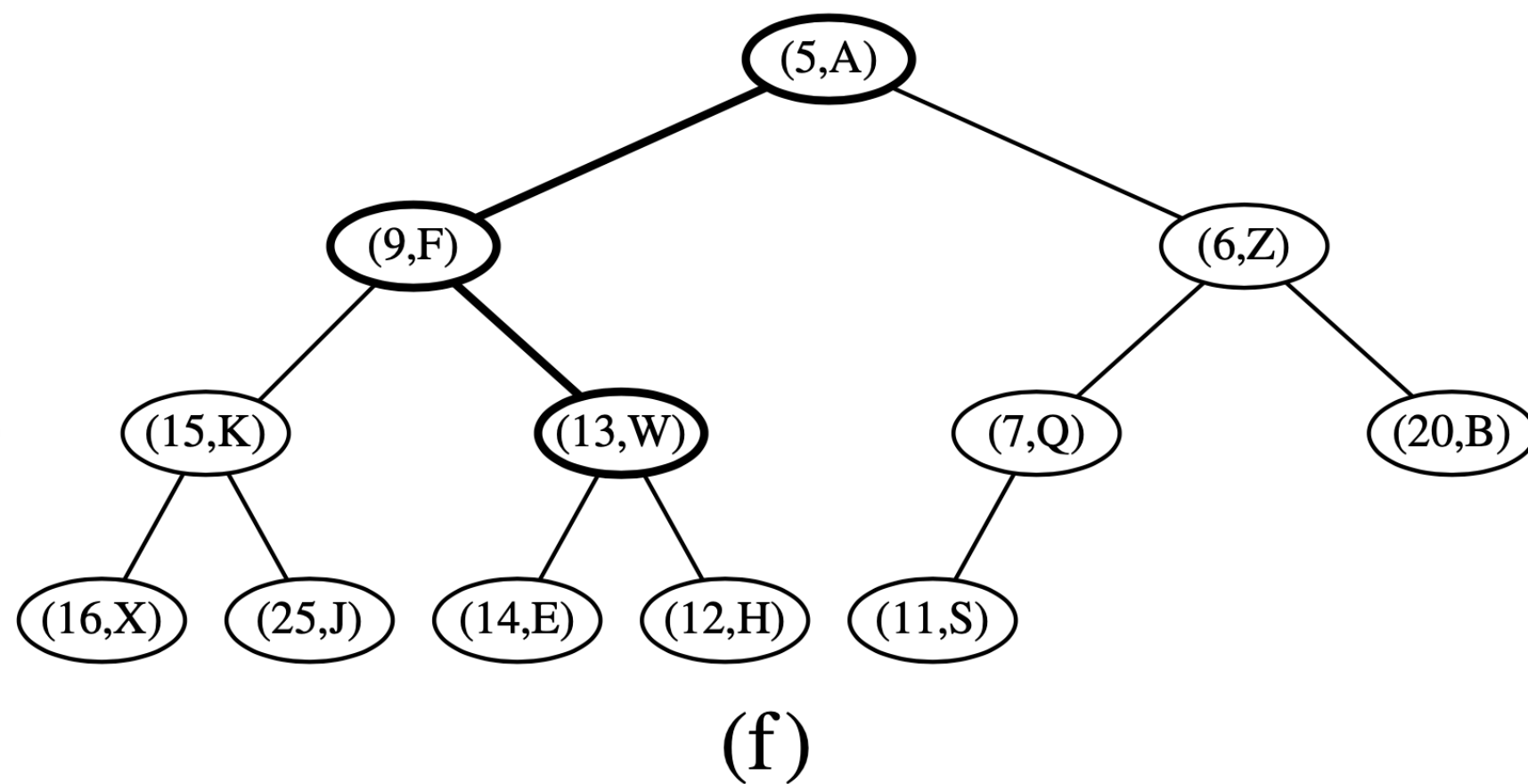
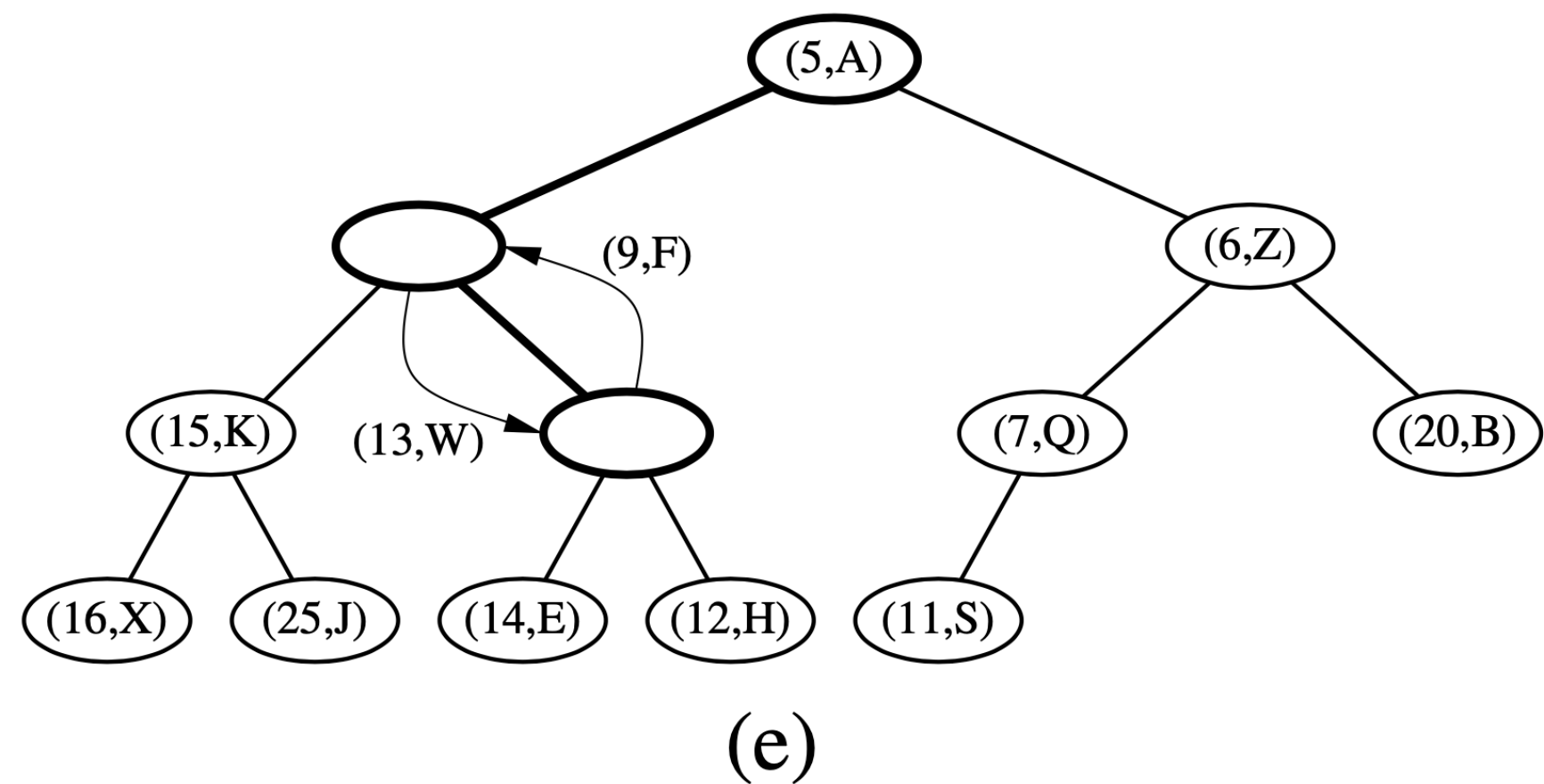
STEP 2. To maintain the heap property, we check the path **down** to the last level iteratively and swap the nodes if necessary.



Down-Heap Bubbling

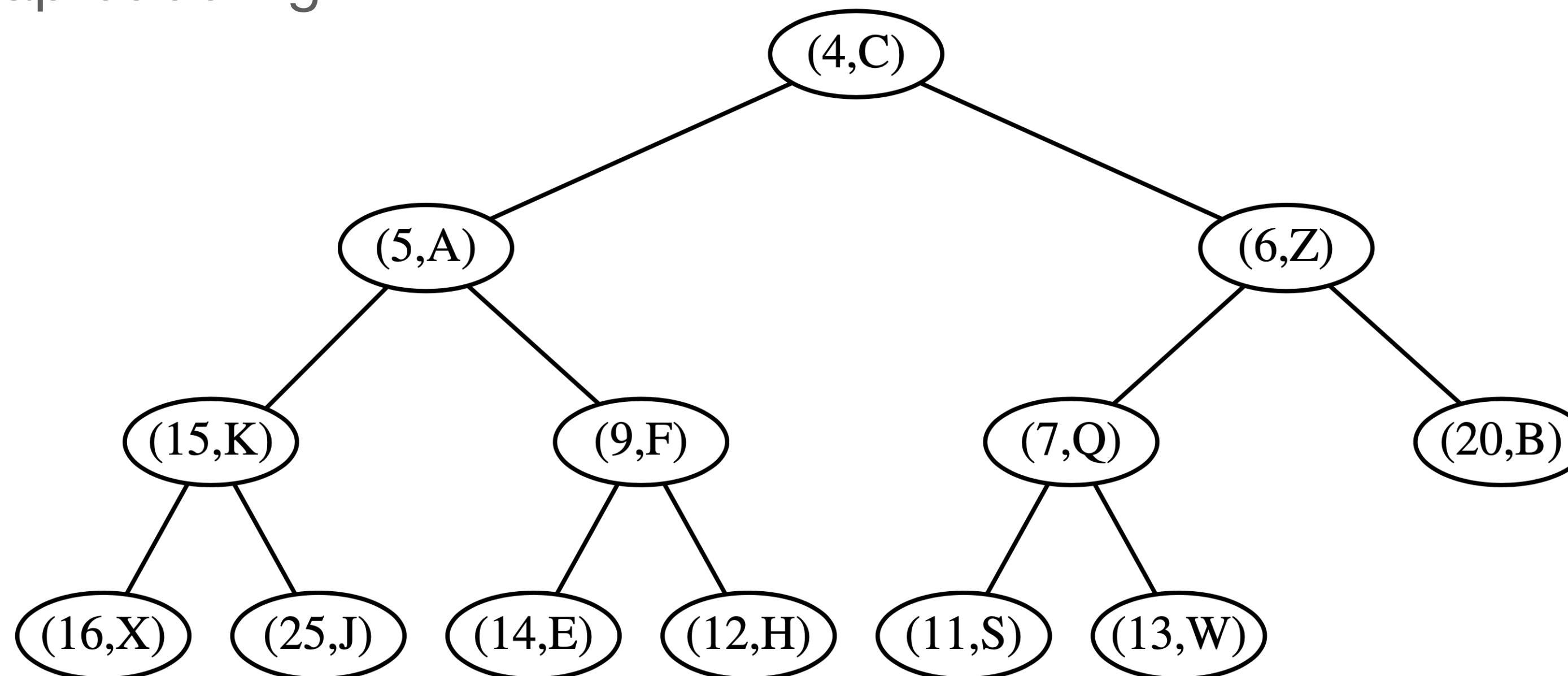
- How to remove the root item, which is the highest priority?

STEP 2. To maintain the heap property, we check the path **down** to the last level iteratively and swap the nodes if necessary.



Adaptable Priority Queue

- How to handle removing an item in the heap or changing its priority?
 - Assume the location of the item is provided !
 - **Update:** If decreasing the value, then check up-heap bubbling.
If increasing the value, then check down-heap bubbling.
 - **Remove:** Move the rightmost item of the last row to the desired location, check up-heap or down-heap bubbling



Implementing the Heap Data Structure

- Remember the array implementation of a binary tree
- A heap is nothing other than an array, and no worries about vacancies in ordinary binary-tree-implementing arrays due to “**almost complete**” property.
- We need traversal up and down, who is the parent, right-child, left-child ?
- Easy arithmetic operations depending on 0- or 1-based array implementation.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|----|---|---|----|----|----|----|----|----|----|
| 4 | 5 | 6 | 15 | 9 | 7 | 20 | 16 | 25 | 14 | 12 | 11 | 13 |

parent(i): $\lfloor i/2 \rfloor$

left-child(i): $i \cdot 2$

right-child(i): $i \cdot 2 + 1$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|----|---|---|----|----|----|----|----|----|----|
| 4 | 5 | 6 | 15 | 9 | 7 | 20 | 16 | 25 | 14 | 12 | 11 | 13 |

parent(i): $\lfloor (i - 1)/2 \rfloor$

left-child(i): $i \cdot 2 + 1$

right-child(i): $i \cdot 2 + 2$

Heap Construction

- Given an array, how to make it a heap?
- Can be done in $O(n)$ -time !
- Bottom-up heap construction: All the items after a location is already a heap!

.....
: Assume, w.l.g, $n = 2^{h+1} - 1$, :
: for some h , e.g., 1,3,7,15,31,... :
: A complete binary tree :
.....

$a[1]$ $a[(n+1)/2 - 1]$ $a[(n+1)/2]$ $a[n]$

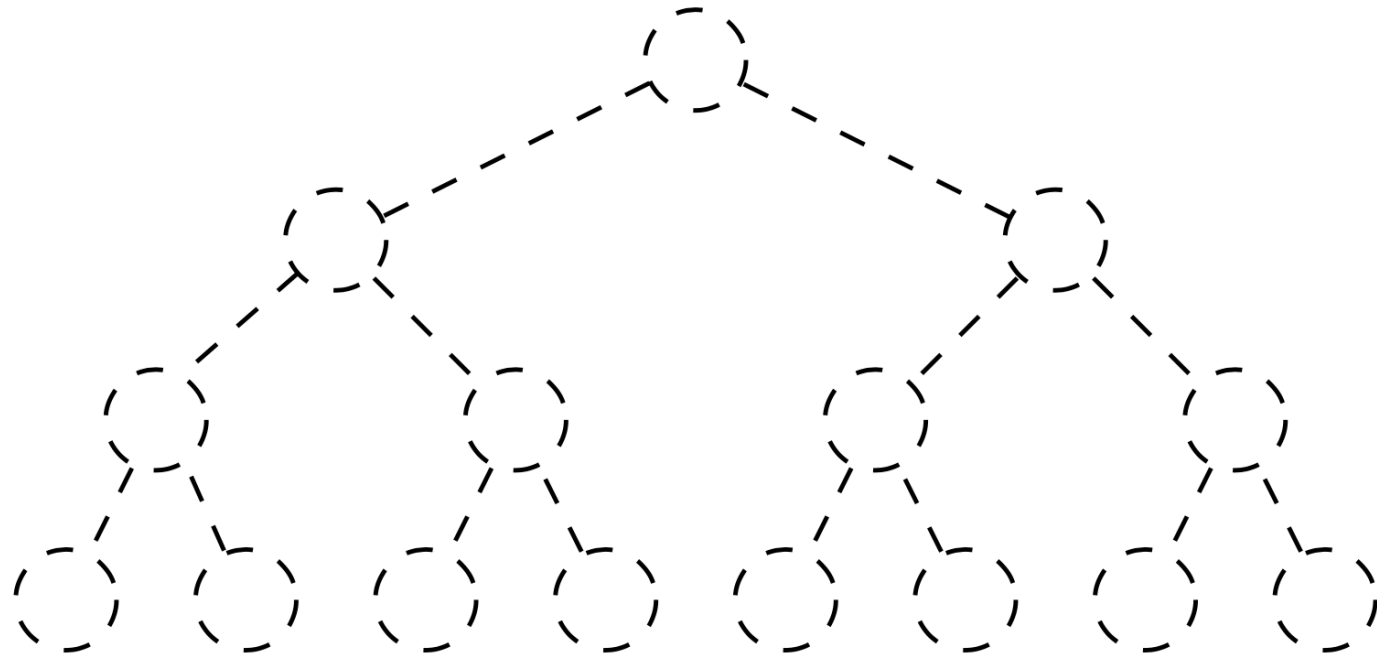
Each item in the first half, **from the right-most towards the left-most** will be subject to a down-heap bubbling.

No need to do anything for the second half, $(n+1)/2$ elements since these are the leaves.

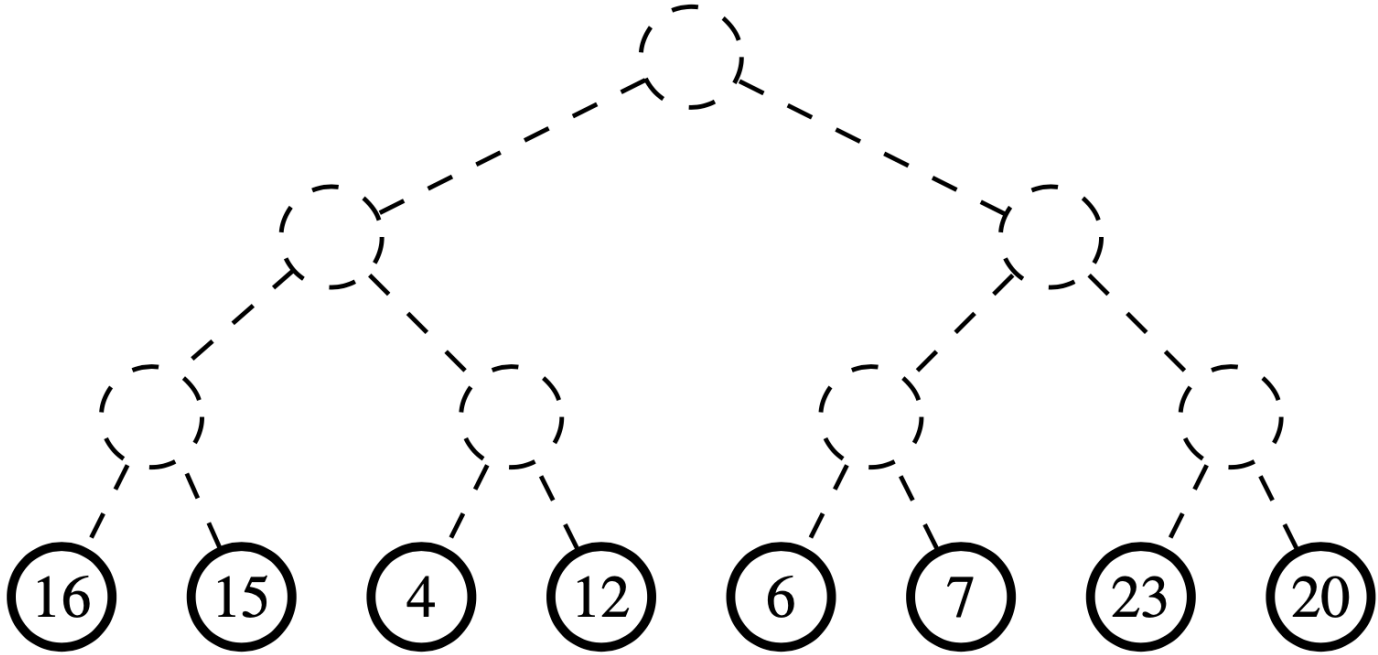
Heap Construction

Input Array:

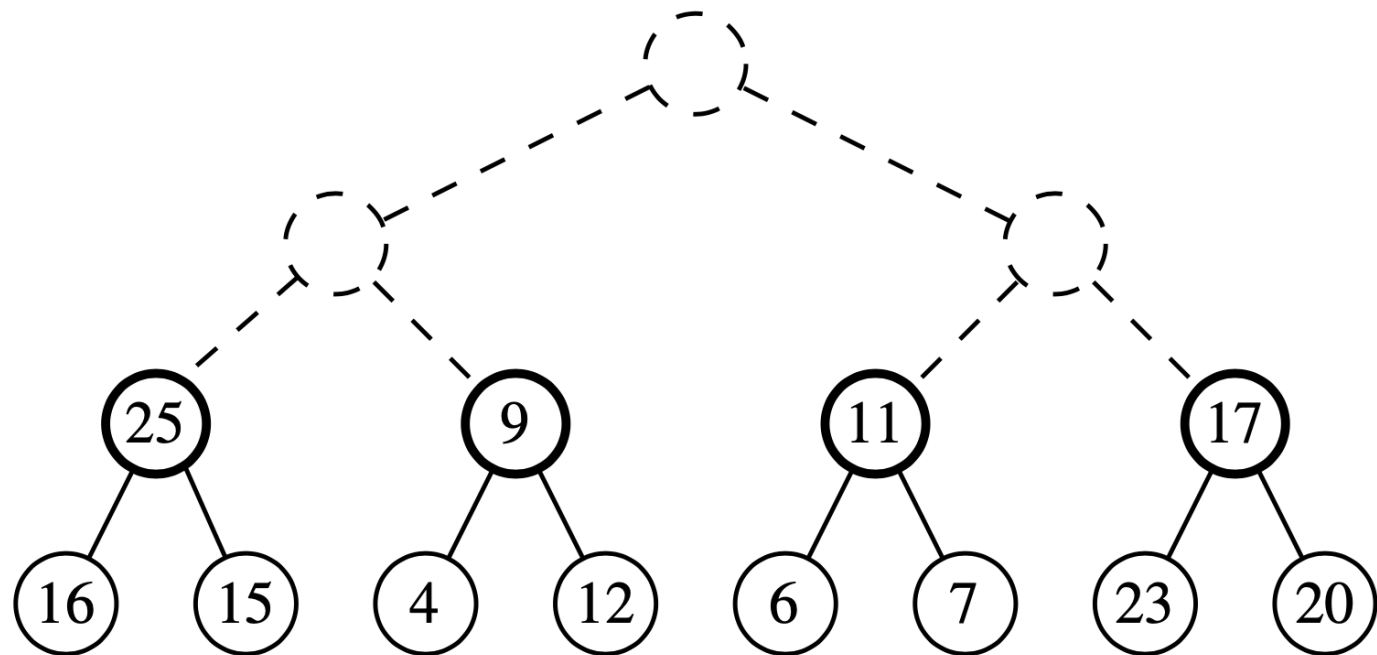
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|----|---|---|----|---|----|----|----|----|----|----|----|----|----|----|
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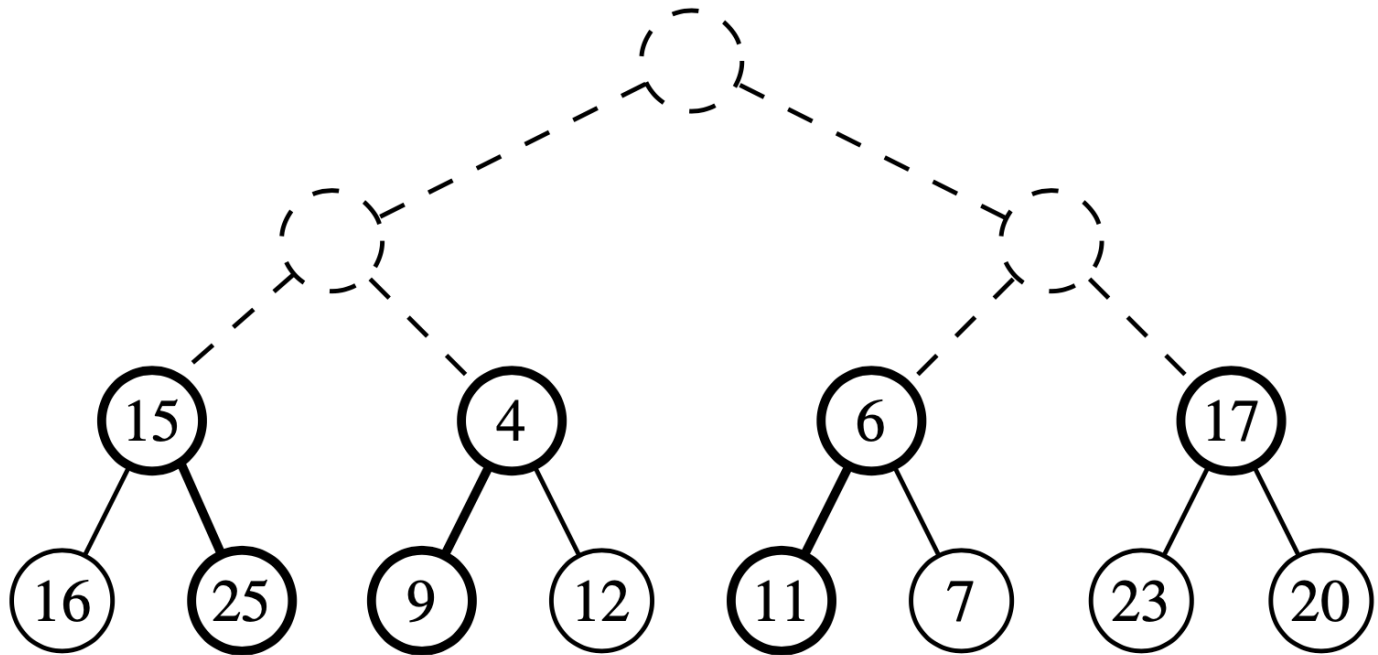
(a)



(b)



(c)

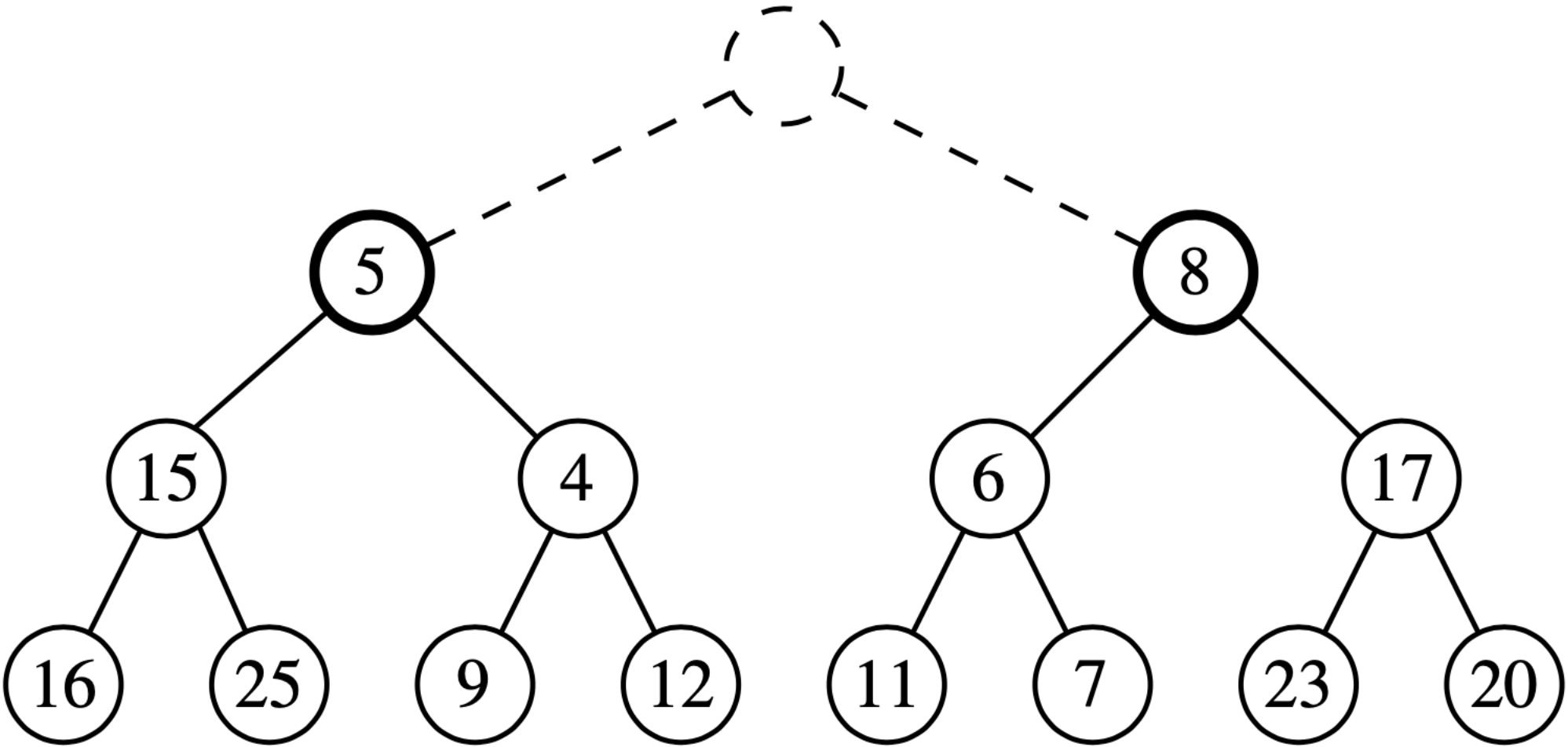


(d)

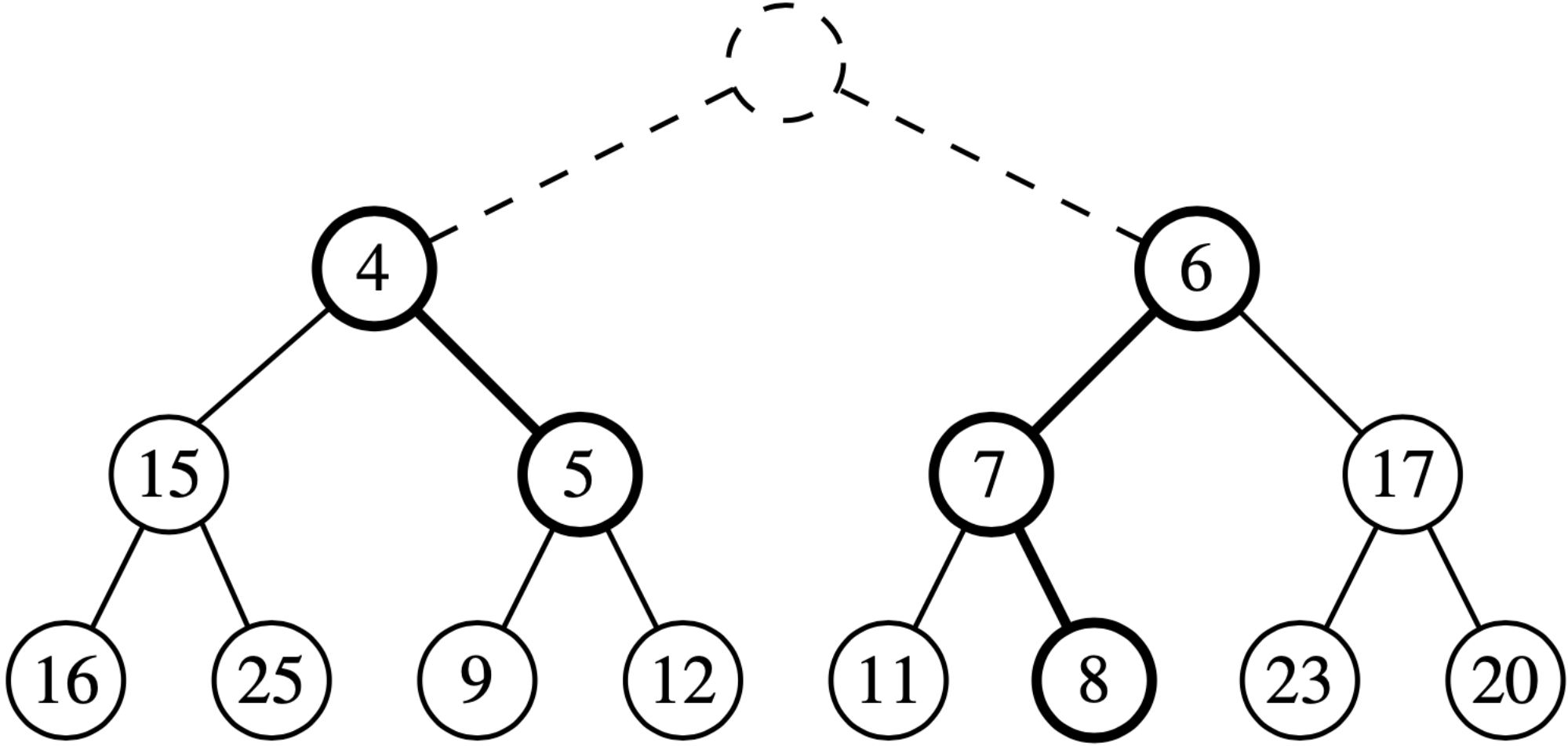
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|----|---|---|----|---|---|----|----|----|----|----|----|----|----|----|
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Heap Construction

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(e)

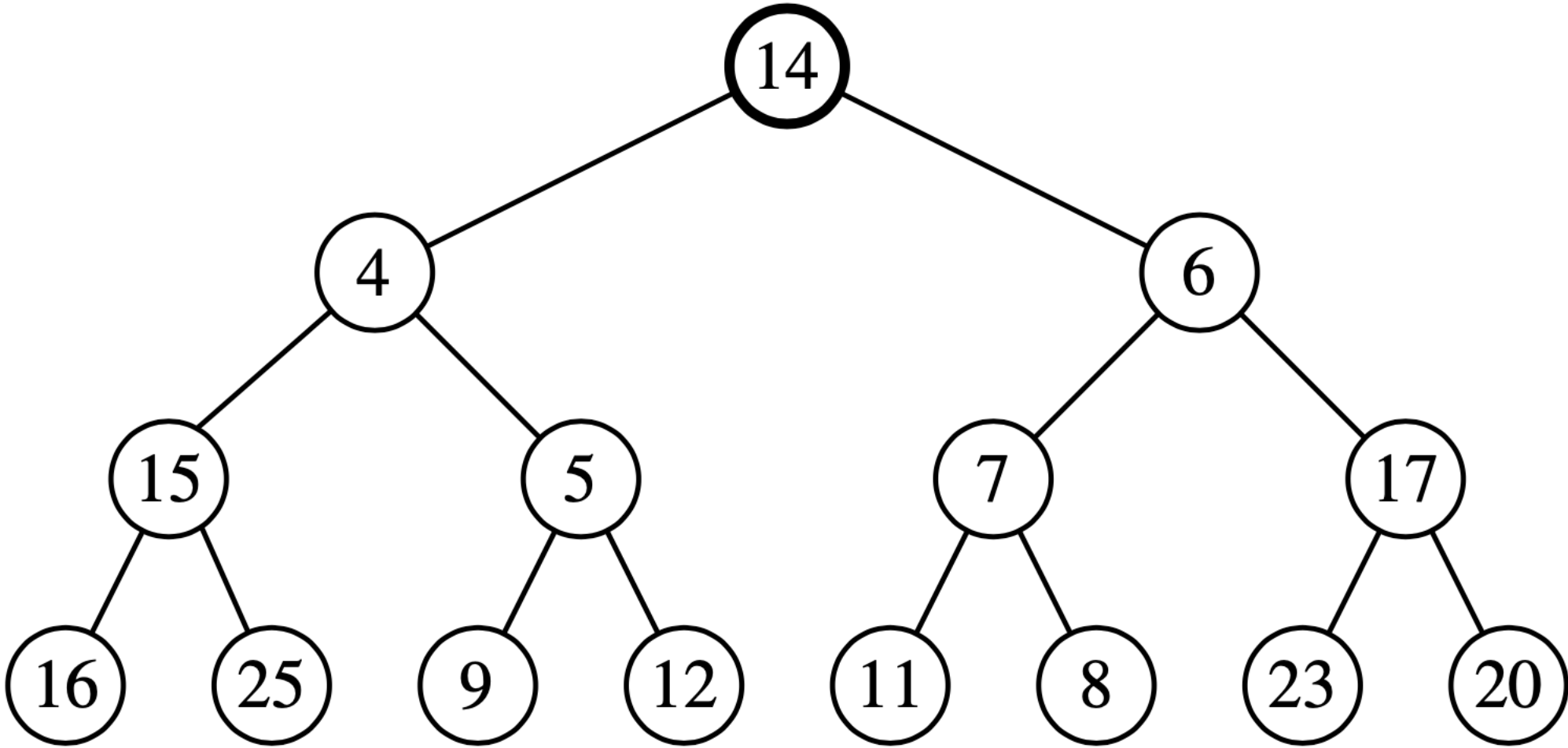


(f)

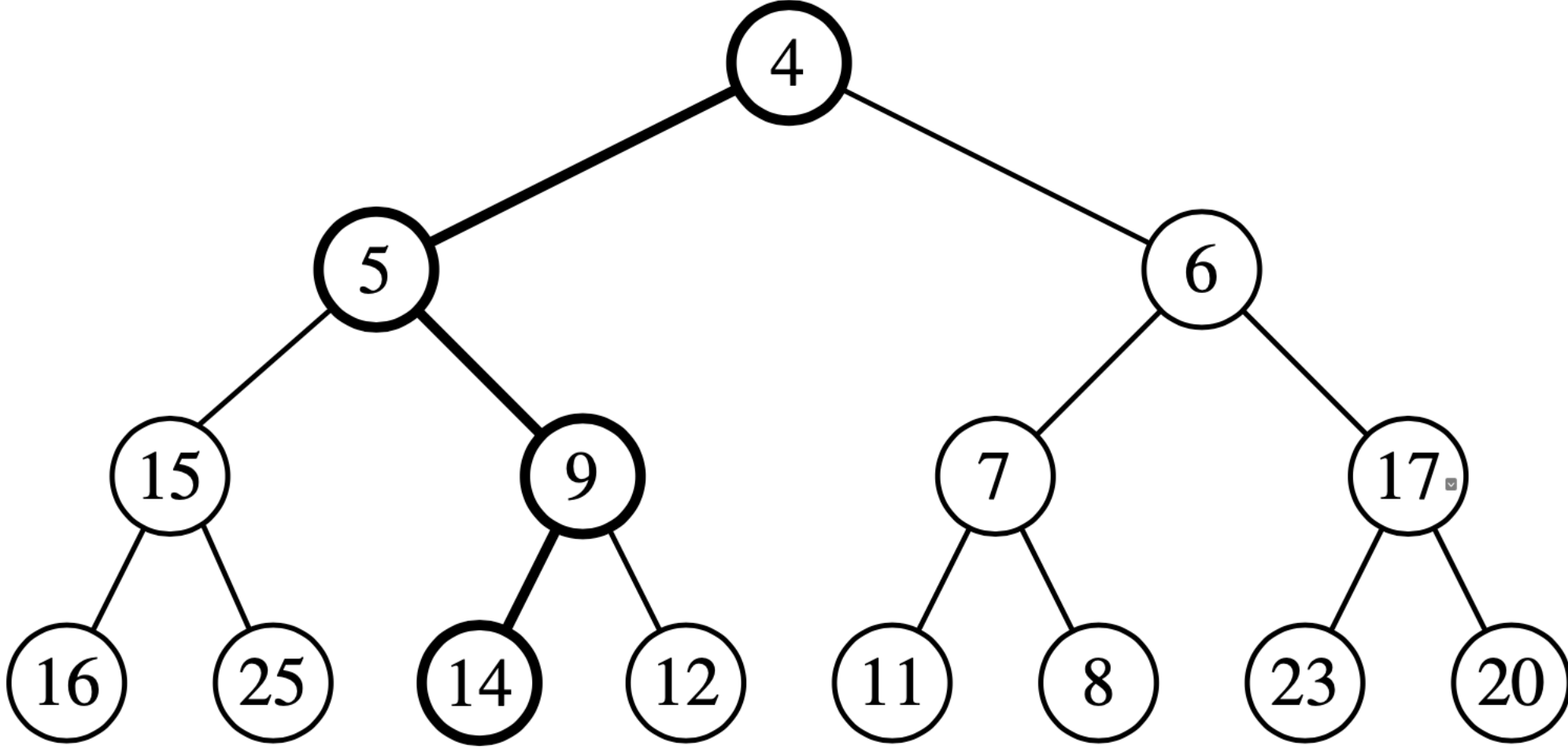
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Heap Construction

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
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(g)



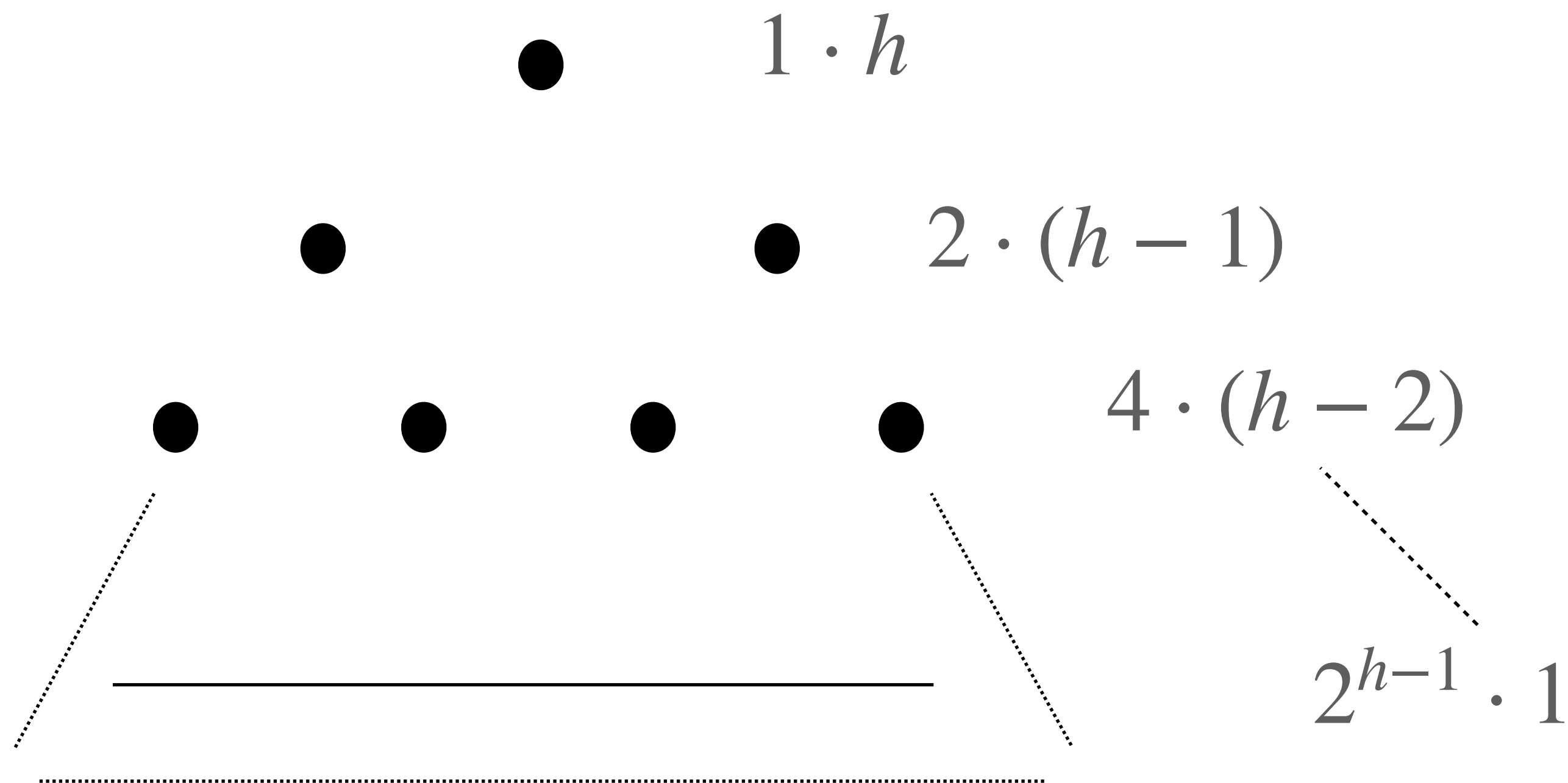
(h)

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Heap Construction

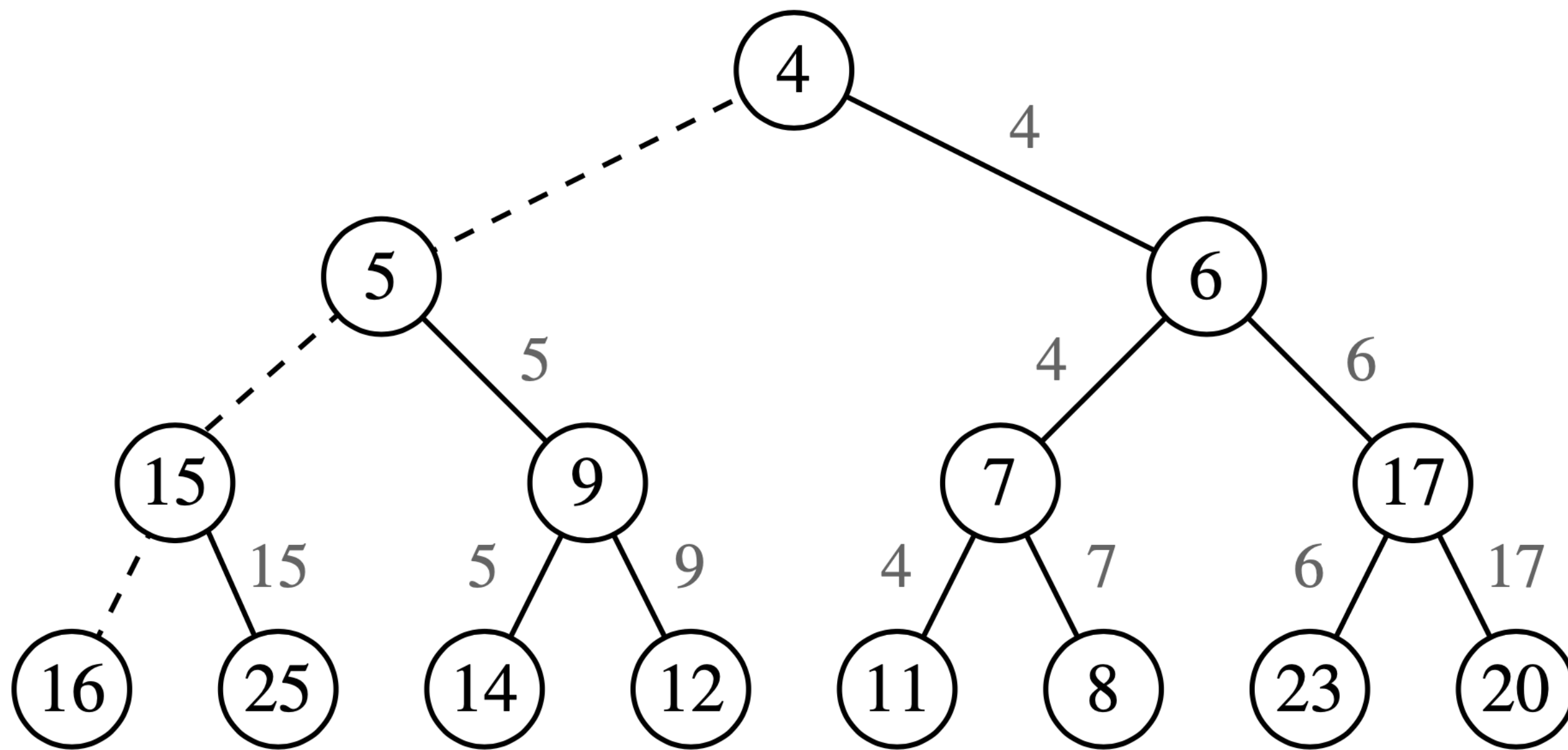
- How many *bubbling* steps are used per item ?
- In the worst case, each item will down-bubble to the lowest level, which depends on the level of the item in the tree.

$$1 \cdot h + 2 \cdot (h - 1) + 4 \cdot (h - 2) + \dots + 2^r \cdot (h - r) + \dots 2^{h-1} \cdot 1 = ?$$



Heap Construction

- A clever way to get this count
 - Associate each edge with an update operation
 - Observe that no edge is shared with multiple nodes
 - The number of edges in the tree, which is less than n , is upper-bound for the construction steps.
 - Hence, the construction is $O(n)$ -time



- The path for each node is go right child, then always follow left-child until a leaf node.
- Successor of each node according to in-order traversal
- Each node have its path to a leaf and no edge is shared between any nodes !
- Maybe the exact path during the construction is different, but the path length does not change and we are trying to count the steps.

Heap Sort

- So, we can modify an input array to become a heap in $O(n)$ -time
- We can use this heap for sorting
 - Extract the root for n times
- Building the heap plus n times root removal is $O(n) + O(n \log n) \rightarrow O(n \log n)$
- This is the heap-sort algorithm!
Notice that it is an in-place sorting algorithm, no need for auxiliary space

Top-k Queries

- Return the k largest elements of a given sequence.
- In static case, just sort everything in $O(n \log n)$ -time and return the largest k elements.
- In streaming case, it needs different handling.
 - Maintain the set of k largest elements observed so far, and then compare each newcomer with this set to replace the minimum element when the newcomer is larger than that.
 - Different ways can be modeled, but naively it yields $O(n \cdot k)$ -time.

Top-k Queries

- A better solution is with the min-heap data structure:
 - Construct a min-heap with the first k elements in $O(k)$ time.
 - For every position after k , compare the candidate's value with the root of min-heap, and perform root-removal with new item insertion, if candidate is larger than the root. This takes $O(\log k)$ time.
- Thus, top-k queries can be answered in $O(n \log k)$ time with the heap.

Another example

- Given a string (or a stream !), find the first non-repeating k symbols.
- For instance, if $S = \text{abracadabraXYXZ}$ and $k = 3$, then the output is c, d, Y
- Naive solution: Maintain a table holding the individual symbols with occurrence count 1 and the first position of their appearance. $O(n)$ -time and -space.
- Better solution with the heap ?

Yet another one...

- Given n ropes, we want to connect them to make a single one. The cost of connecting two ropes is the sum of their lengths. What should be a good solution for that.

Example: If rope lengths are $\{3,5,1,8\}$, then

$$1 + 3 = 4 \quad \{4,5,8\}$$

$$4 + 5 = 9 \quad \{8,9\}$$

$$8 + 9 = 17$$

Total cost: $4 + 9 + 17 = 30$

Reading assignment

- Read the chapter 9 from Goodrich, chapter 6 from Cormen.