ML Homework 02

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1. Answer to Exercise 1:
   a)

$$
\begin{aligned}
f(x) &= \int_0^{+\infty} \rho(x,y)dy = \lambda e^{-\lambda x} \\
f(y) &= \int_0^{+\infty} \rho(x,y)dx = \eta e^{-\eta y} \\
\rho(x,y) &= f(x)f(y)
\end{aligned}
$$

   so X and Y are independent
   b)

$$
\begin{aligned}
L(\lambda,\eta) &= \prod_i^N \rho(x,y) = \lambda^N \eta^N e^{-\lambda \sum_i^N x_i - \eta \sum_i^N y_i} \\
f(\lambda,\eta) &= Ln(L(\lambda,\eta)) = Nln\lambda + Nln\eta - \lambda \sum_i^N x_i - \eta \sum_i^N y_i \\
\frac{\partial f(\lambda,\eta)}{\partial \lambda} &= \frac{N}{\lambda} - \sum_i^N x_i = 0 \\
\lambda &= \frac{N}{\sum_i^N x_i}
\end{aligned}
$$

c)

$$f(\lambda,\eta) = Nln\lambda + Nln\eta - \lambda\sum_i^N x_i - \eta\sum_i^N y_i$$

$$\frac{\partial f(\lambda,\eta)}{\partial\lambda} = \frac{N}{\lambda} - \sum_i^N x_i = 0$$

$$constrain: \eta = \frac{1}{\lambda}$$

$$supposeg(\lambda,\eta) = \eta - \frac{1}{\lambda}$$

$$F(\lambda,\eta,\omega) = f(\lambda,\eta) + \omega g(\lambda,\eta)$$

$$\frac{\partial F}{\partial\lambda} = \frac{N}{\lambda} - \sum_i^N x_i + \frac{\omega}{\lambda^2} = 0$$

$$\frac{\partial F}{\partial\eta} = \frac{N}{\eta} - \sum_i^N y_i + \omega = 0$$

$$\frac{\partial F}{\partial\omega} = \eta - \frac{1}{\lambda} = 0$$

$$\lambda = \sqrt{\frac{\sum_i^N y_i}{\sum_i^N x_i}}$$

d)

$$constrain: \eta = 1 - \lambda$$

$$g(\lambda,\eta) = \eta + \lambda - 1$$

$$F(\lambda,\eta,\omega) = f(\lambda,\eta) + \omega g(\lambda,\eta)$$

$$\frac{\partial F}{\partial\lambda} = \frac{N}{\lambda} - \sum_i^N x_i + \omega = 0$$

$$\frac{\partial F}{\partial\eta} = \frac{N}{\eta} - \sum_i^N y_i + \omega = 0$$

$$\frac{\partial F}{\partial\omega} = \eta + \lambda - 1 = 0$$

$$\lambda = \frac{1}{2} + \frac{N}{\sum_i^N x_i - \sum_i^N y_i} - \sqrt{\frac{1}{4} + \frac{N^2}{(\sum_i^N x_i - \sum_i^N y_i)^2}}$$

$$or\lambda = \frac{1}{2} + \frac{N}{\sum_i^N x_i - \sum_i^N y_i} + \sqrt{\frac{1}{4} + \frac{N^2}{(\sum_i^N x_i - \sum_i^N y_i)^2}}$$

2. Answer to Exercise 2:

a)

$$P(D|\theta) = \prod_{i}^{N} p(x_k|\theta) = \theta^5(1-\theta^2)$$

b)

$$\begin{aligned}
Ln(P(D|\theta)) &= 5ln\theta + 2ln(1-\theta) \\
\frac{\partial Ln(P(D|\theta))}{\partial\theta} &= \frac{5}{\theta} - \frac{2}{1-\theta} = 0 \\
\hat{\theta} &= \frac{5}{7} \\
P(x_8 = head, x_9 = head|\hat{\theta}) &= \theta^7(1-\theta^2) \\
&= 0.007744
\end{aligned}$$

c)

$$\begin{aligned}
p(\theta|D) &= \frac{p(D|\theta)p(\theta)}{\int p(D|\theta)p(\theta)d\theta} \\
&= \frac{\prod_i^N p(x_k|\theta)p(\theta)}{\int_0^1 \prod_i^N p(x_k|\theta)p(\theta)d\theta} \\
&= \frac{\theta^5(1-\theta^2)}{\int_0^1 \theta^5(1-\theta^2)d\theta} \\
&= 24\theta^5(1-\theta^2) \\
\int P(x_8, x_9|\theta)p(\theta|D)d\theta &= \int \theta^2 * 24\theta^5(1-\theta^2)d\theta \\
&= 3\theta^8 - 2.4\theta^{10}(0 \le \theta \le 1)
\end{aligned}$$

3. Answer to Exercise 3

a) suppose

$$\begin{aligned}
a = \sigma_n^2, b &= \frac{\sigma^2}{n}, c = \sigma_0^2 \\
\frac{1}{a} &= \frac{1}{b} + \frac{1}{c} \\
a &= \frac{b * c}{b+c} \\
a - b &= \frac{-c^2}{b+c} \le 0 \\
a - c &= \frac{-b^2}{b+c} \le 0 \\
a &\le min(b, c) \\
\sigma_n^2 &\le min(\frac{\sigma^2}{n}, \sigma_0^2)
\end{aligned}$$

b)suppose

$$
\begin{aligned}
a = \sigma_n^2, b = \frac{\sigma^2}{n}, c = \sigma_0^2 \\
\frac{\mu_n}{\sigma_n^2} = \frac{n}{\sigma^2}\hat{\mu}_n + \frac{\mu_0}{\sigma^2} \\
\mu_n = \frac{a}{b}\hat{\mu}_n + \frac{a}{c}\mu_0 \\
\frac{a}{b} + \frac{a}{c} = 1 \\
min(\hat{\mu}_n, \mu_0) \le \mu_n \le max(\hat{\mu}_n, \mu_0)
\end{aligned}
$$