



Processamento
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Probability of
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Other
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Processamento e Recuperação de Informação

Information Extraction : Hidden Markov Models

Departamento de Engenharia Informática
Instituto Superior Técnico

1º Semestre
2018/2019



Bibliography - Articles

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- Rakesh Dugad, U.B. Desai, *A Tutorial on Hidden Markov Models*, Technical Report, Department of Electrical Engineering, Indian Institute of Technology, 1996.
- Lawrence R. Rabiner, *A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition*, Proceedings of the IEEE, 77(2), February, 1989.



Outline

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An Example Generative Story

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- Suppose a person, inside a room, has three coins (possibly biased)
- The person chooses a coin, randomly, and throws it, chooses another, throws it, and so on...
- The choice of a coin depends on the previously chosen coin
- We are outside the room, looking through a window
- We can only see the outcome of the coin (heads or tails)

- Suppose we observe the sequence:

HHTTTTHHTHTTTHHTTHT

- What probabilities can influence this outcome?



Hidden Markov Model

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The outcome is influenced by three factors:

- 1 The probability of choosing a given coin first
- 2 The probability of choosing a given coin, after another
- 3 The probability of getting heads or tails

These three sets of probabilities characterize a **Hidden Markov Model** for the coin tossing experiment



Finite State Machine Representation

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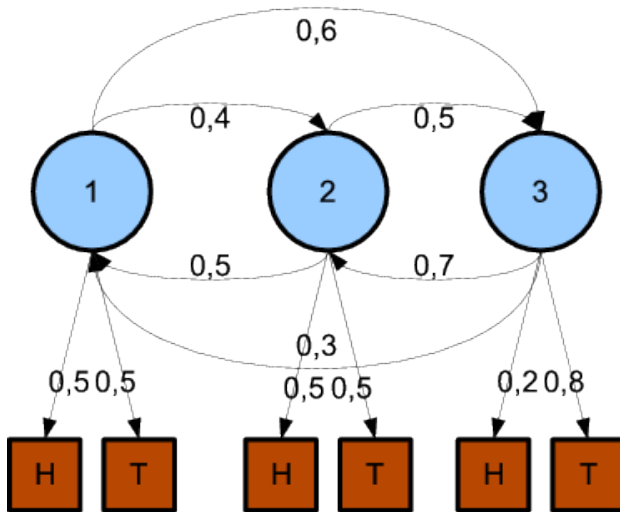
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Definitions

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We will use the following notation:

- N — the number of **states** in the model
- M — the number of distinct **observation symbols**
- T — the length of the **observation sequence**
- i_t — the state in which we are at time t
- $V = \{V_1, \dots, V_M\}$ — the set of observation symbols
- $\pi = \{\pi_i\}$ — the probability of being in state i at the beginning of the experiment, i.e. $\pi_i = P(i_1 = i)$
- $A = \{a_{ij}\}$ — the probability of being in state j at time $t + 1$ given that we were in state i at time t , i.e. $P(i_{t+1} = j | i_t = i)$
- $B = \{b_j(k)\}$ — the probability of observing symbol v_k given that we are in state j , i.e., $P(v_k \text{ at } t | i_t = j)$
- O_t the observation symbol observed at time t
- $\lambda = (A, B, \pi)$ — the **Hidden Markov Model**



An example

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Consider a set of n urns, each containing marbles of m different colors. We are randomly choosing an urn and randomly picking a marble from it. How do we model this as an HMM?

- What are the states?
- What are the observation symbols?



An example (cont.)

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- The model would represent the following sequence of events:



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- The model would represent the following sequence of events:
 - ① We choose one of the urns, according to probability distribution π



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- The model would represent the following sequence of events:
 - 1 We choose one of the urns, according to probability distribution π
 - 2 We choose a marble from that urn, according to probability distribution B



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- The model would represent the following sequence of events:
 - 1 We choose one of the urns, according to probability distribution π
 - 2 We choose a marble from that urn, according to probability distribution B
 - At this moment we are at time t_1 , state i_1 , and observed symbol O_1



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- The model would represent the following sequence of events:
 - 1 We choose one of the urns, according to probability distribution π
 - 2 We choose a marble from that urn, according to probability distribution B
 - At this moment we are at time t_1 , state i_1 , and observed symbol O_1
 - After the next step we will be at time t_2



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- The model would represent the following sequence of events:
 - 1 We choose one of the urns, according to probability distribution π
 - 2 We choose a marble from that urn, according to probability distribution B
 - At this moment we are at time t_1 , state i_1 , and observed symbol O_1
 - After the next step we will be at time t_2
 - 3 We choose another urn, according to probability distribution A



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- The model would represent the following sequence of events:
 - 1 We choose one of the urns, according to probability distribution π
 - 2 We choose a marble from that urn, according to probability distribution B
 - At this moment we are at time t_1 , state i_1 , and observed symbol O_1
 - After the next step we will be at time t_2
 - 3 We choose another urn, according to probability distribution A
 - 4 Repeat from step 2, until we have made T observations (i.e., $t=T$)



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- The model would represent the following sequence of events:
 - 1 We choose one of the urns, according to probability distribution π
 - 2 We choose a marble from that urn, according to probability distribution B
 - At this moment we are at time t_1 , state i_1 , and observed symbol O_1
 - After the next step we will be at time t_2
 - 3 We choose another urn, according to probability distribution A
 - 4 Repeat from step 2, until we have made T observations (i.e., $t=T$)
- The generated observation sequence will be O_1, O_2, \dots, O_T .



Three Problems for HMMs

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- ① Given the model $\lambda = (A, B, \pi)$, compute $P(O|\lambda)$
 - I.e., compute the probability of observing a given sequence
 - Applications in [language modeling](#), [spelling correction](#), ...
- ② Given the model $\lambda = (A, B, \pi)$, choose a state sequence $I = i_1, i_2, \dots, i_T$ such that $P(O, I|\lambda)$ is maximized, for a given observation sequence $O = O_1, O_2, \dots, O_T$
 - I.e., compute the most likely sequence of states to have generated an observation sequence (i.e., [decoding](#))
 - Applications in [information extraction](#) (e.g., chunking, named entity recognition, ...)
- ③ Adjust the model parameters $\lambda = (A, B, \pi)$ such that $P(O|\lambda)$ or $P(O, I|\lambda)$, is maximized
 - I.e., based on a series of observations and/or state sequences, compute the HMM
 - Learning model parameters from annotated data



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Computing the Probabilities

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- We know that

$$P(O|\lambda) = \sum_I P(O|I, \lambda) P(I|\lambda)$$

- and since

$$\begin{aligned} P(O|I, \lambda) &= b_{i_1}(O_1)b_{i_2}(O_2)\cdots b_{i_T}(O_T) \\ P(I|\lambda) &= \pi_{i_1} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{T-1} i_T} \end{aligned}$$

- we have that

$$P(O|\lambda) = \sum_I \pi_{i_1} b_{i_1}(O_1) a_{i_1 i_2} b_{i_2}(O_2) \cdots a_{i_{T-1} i_T} b_{i_T}(O_T)$$



The Problem

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$$P(O|\lambda) = \sum_I \pi_{i_1} b_{i_1}(O_1) a_{i_1 i_2} b_{i_2}(O_2) \cdots a_{i_{T-1} i_T} b_{i_T}(O_T)$$

- Computing each summand requires $2T - 1$ multiplications
- There are N^T possible state sequences
- Thus, the complexity is $O(2TN^T)$: **unfeasible**
- However, there is a more efficient way of computing $P(O|\lambda)$: the **forward/backward procedure**



The Forward Procedure

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- Consider the **forward variable** $\alpha_t(i)$, defined as:

$$\alpha_t(i) = P(O_1, O_2, \dots, O_t, i_t = i | \lambda)$$

i.e., the probability of a partial observation sequence (until time t) that ends in state i

- $\alpha_t(i)$ can be computed as follows:
 - 1 Compute the probability of starting in state i and observing O_1 : $\alpha_1(i)$
 - 2 For time $t + 1$ compute the probability of reaching a state j and observing O_{t+1} , knowing that we already computed all probabilities for all times $\leq t$
 - 3 The final probability (at time T) will be the sum of all probabilities for each possible ending state i : $\alpha_T(i)$



Computing The Forward Procedure

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① Initial step:

$$\alpha_1(i) = \pi_i b_i(O_1) , 1 \leq i \leq N$$



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① Initial step:

$$\alpha_1(i) = \pi_i b_i(O_1), \quad 1 \leq i \leq N$$

② For $t = 1, 2, \dots, T - 1, 1 \leq j \leq N$

$$\alpha_{t+1}(j) = \left[\sum_{i=1}^N \alpha_t(i) a_{ij} \right] b_j(O_{t+1})$$



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- 1 Initial step:

$$\alpha_1(i) = \pi_i b_i(O_1) , 1 \leq i \leq N$$

- 2 For $t = 1, 2, \dots, T - 1, 1 \leq j \leq N$

$$\alpha_{t+1}(j) = \left[\sum_{i=1}^N \alpha_t(i) a_{ij} \right] b_j(O_{t+1})$$

- 3 Thus, we have that:

$$P(O|\lambda) = \sum_{i=1}^N \alpha_T(i)$$



Time Complexity

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- Step 1 requires N multiplications
- Step 2 requires $N + 1$ multiplications. This is performed for all N states and $T - 1$ times, yielding $(N + 1)N(T - 1)$ multiplications
- Step 3 requires only to sum the computed values
- Thus, the time complexity is $O(N^2 T)$



Backward Procedure

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- A similar procedure can be applied *moving backwards*
- Consider the backward variable $\beta_t(i)$, defined as:

$$\beta_t(i) = P(O_{t+1}, O_{t+2}, \dots, O_T | i_t = i, \lambda)$$

i.e., the probability of observing a partial sequence starting at time $t + 1$ and state i

- $\beta_t(i)$ can also be computed as follows:

①

$$\beta_T(i) = 1, 1 \leq i \leq N$$

②

For $t = T - 1, T - 2, \dots, 1, 1 \leq i \leq N$, we have

$$\beta_t(i) = \sum_{j=1}^N a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)$$

③

Thus, we have that:

$$P(O|\lambda) = \sum_{i=1}^N \pi_i b_i(O_1) \beta_1(i)$$



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The Decoding Problem

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- We want to find a sequence of states $I = i_1, i_2, \dots, i_T$ such that the probability of observing a sequence $O = O_1, O_2, \dots, O_T$ is greater than for any other sequence
- I.e., Find I that maximizes $P(O, I|\lambda)$

$$\arg \max_{\{i_t\}_{t=1}^T} P(O, i_1, i_2, \dots, i_T|\lambda)$$

- This can be computed using the **Viterbi Algorithm**



The Viterbi Algorithm

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- We know that

$$\begin{aligned}P(O, I|\lambda) &= P(O|I, \lambda)P(I|\lambda) \\ &= \pi_{i_1} b_{i_1}(O_1) a_{i_1 i_2} b_{i_2}(O_2) \cdots a_{i_{T-1} i_T} b_{i_T}(O_T)\end{aligned}$$

- Thus, we can define

$$U(i_1, i_2, \dots, i_T) = - \left[\ln(\pi_{i_1} b_{i_1}(O_1)) + \sum_{t=2}^T \ln(a_{i_{t-1} i_t} b_{i_t}(O_t)) \right]$$

- so that

$$P(O, I|\lambda) = \exp(-U(i_1, i_2, \dots, i_T))$$

- and our problem becomes

$$\arg \min_{\{i_t\}_{t=1}^T} U(i_1, i_2, \dots, i_T)$$



The Viterbi Algorithm (cont.)

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- We can view the term $-\ln(a_{ij_i_k} b_{i_k}(O_t))$ as the cost of going from state i_j to state i_k at time t
- The Viterbi Algorithm is a **dynamic programming** approach to compute the path of least cost
- The total cost of a path is the sum of the weights on the edges we cross
 - Note that this is equivalent to multiplying the probabilities



Computing the Viterbi Algorithm (1)

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- Let $\delta_t(i)$ be the accumulated weight at state i and time t
- Let $\psi_t(j)$ be the state at time $t - 1$ with the lowest cost transition to state j at time t

- 1 Initialization, for $1 \leq i \leq N$:

$$\delta_1(i) = -\ln(\pi_i) - \ln(b_i(O_1))$$

$$\psi_1(i) = 0$$

- 2 Recursive computation, for $2 \leq t \leq T$, $1 \leq j \leq N$:

$$\delta_t(j) = \min_{1 \leq i \leq N} [\delta_{t-1}(i) - \ln(a_{ij})] - \ln(b_j(O_t))$$

$$\psi_t(j) = \arg \min_{1 \leq i \leq N} [\delta_{t-1}(i) - \ln(a_{ij})]$$



Computing the Viterbi Algorithm (2)

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3 Termination:

$$P^* = \min_{1 \leq i \leq N} [\delta_T(i)]$$
$$q_T^* = \arg \min_{1 \leq i \leq N} [\delta_T(i)]$$

4 Trace back, for $t = T, T - 1, T - 2, \dots, 1$:

$$q_t^* = \psi_{t+1}(q_{t+1}^*)$$

- $Q^* = \{q_1^*, q_2^*, \dots, q_T^*\}$ is the optimal state sequence
- $\exp(-P^*)$ is the optimized probability for the state sequence
- Complexity: $O(N^2 T)$



An Example Computation

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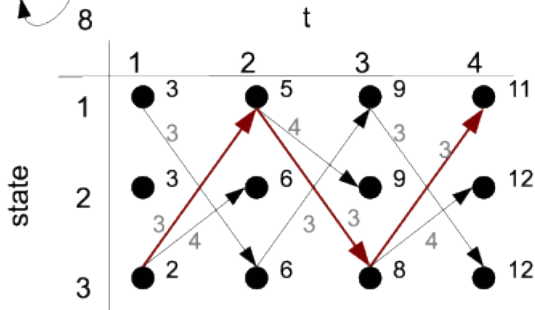
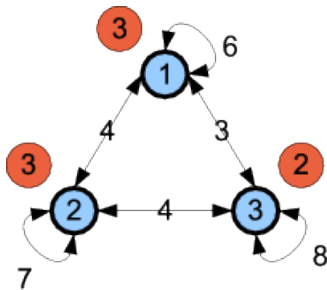
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- The Viterbi algorithm can be used with HMMs, and also with other sequential classification models (e.g., structured Perceptrons, CRFs, neural network approaches, ...)
- Other *decoding* approaches are also frequently used in practice, one example being **posterior decoding**
 - Determine, independently for every symbol O_t , the most probable state using the forward/backward procedure
 - Often more effective when several concurring paths have similar probabilities
- Some practical implementations of Information Extraction tools, leveraging sequential classification models, rely on methods such as **beam search** to find an approximate solution to the problem of finding state sequences



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Learning HMMs

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- In a **supervised setting**, we use the training data to estimate the probabilities
- Transition probabilities

$$\hat{P}(i \rightarrow i') = \frac{c(i \rightarrow i')}{\sum_{s \in I} c(i \rightarrow s)}$$

- Emission probabilities

$$\hat{P}(i \uparrow o) = \frac{c(i \uparrow o)}{\sum_{\rho \in O} c(i \uparrow \rho)}$$

- $c(i \rightarrow i')$ is the number of times there is a transition from state i to state i' (in a training set)
- $c(i \uparrow o)$ counts the number of times symbol o is observed in state i (in a training set)
- The **estimation of beginning probabilities** is similar to that of transition probabilities, but we count the number of times there is a transition from the start (i.e., the beginning of a training sequence) to a state i



Improving Probability Estimates

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- Problem: sparse training data causes poor probability estimates
 - E.g., unseen symbols have emission probabilities of zero
- Solution: use probability smoothing techniques
 - Laplace smoothing
 - Absolute discounting
 - ...



Laplace Smoothing

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- Adds 1 to every count of occurrences
- Moves all estimates towards the uniform distribution
- All unseen words will have equal probability

An example:

$$\hat{P}(i \uparrow o) = \frac{c(i \uparrow o) + 1}{\sum_{\rho \in O} c(i \uparrow \rho) + |O|}$$



Absolute Discounting

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- Localized (per state) smoothing
- Appropriate if zero probabilities vary from state to state
- Subtracts a fixed discount $0 < d < 1$ from all symbols with count > 0
- The total discounted value is distributed by the remaining symbols

An example:

$$\hat{P}(i \uparrow o) = \begin{cases} \frac{c(i \uparrow o) - d}{\sum_{\rho \in O} c(i \uparrow \rho)} & \text{if } c(i \uparrow o) > 0 \\ \frac{d(|O| - |Z_q|)}{|Z_q| * \sum_{\rho \in O} c(i \uparrow \rho)} & \text{if } c(i \uparrow o) = 0 \end{cases}$$

where $|Z_q|$ is the number of symbols with zero count in state i .



Unsupervised Learning of HMMs

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- We want to train an HMM model with a set of **example observation sequences** such that, when a similar sequence is discovered later the model is able to identify it.
- Most well known method
 - **Baum-Welch** algorithm
- Other methods exist
 - E.g. **Segmental K-means**



The Baum-Welch Method

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an
Observation
Sequence

Probability of
a Sequence of
States

Learning the
Model

Other
Sequential
Classification
Models

- Assume an initial model λ
 - Can be constructed in any way (e.g. randomly)
- Maximizes $P(O|\lambda)$ by adjusting λ
 - Called the **maximum likelihood criterion**



Probability of Visiting a State

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- Let

$$\gamma_t(i) = P(i_t = i | O, \lambda)$$

i.e. the probability of being in state i at time t given the observation sequence O and the model λ

- Applying Bayes rule:

$$\gamma_t(i) = \frac{P(i_t = i, O)}{P(O|\lambda)} = \frac{\alpha_t(i)\beta_t(i)}{P(O|\lambda)}$$

where $\alpha_t(i)$ is computed as in the Forward procedure and $\beta_t(i)$ is computed as in the Backward procedure



Probability of Transitioning

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- Let

$$\xi_t(i) = P(i_t = i, i_{t+1} = j | O, \lambda)$$

i.e. the probability of being in state i at time t and making a transition to state j at time $t + 1$, given the observation sequence O and the model λ

- Applying Bayes rule:

$$\xi_t(i) =$$

$$\frac{P(i_t = i, i_{t+1} = j, O | \lambda)}{P(O | \lambda)} = \frac{\alpha_t(i) a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)}{P(O | \lambda)}$$



Expected Number of Transitions

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Expected number of visits to state i :

$$\sum_{t=1}^T \gamma_t(i)$$

Expected number of transitions from state i :

$$\sum_{t=1}^{T-1} \gamma_t(i)$$

Expected number of transitions from state i to state j :

$$\sum_{t=1}^{T-1} \xi_t(i, j)$$



Baum-Welch Re-Estimation Formulas

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The new model parameters $\hat{\lambda} = (\hat{A}, \hat{B}, \hat{\pi})$ can be computed as:

$$\hat{\pi}_i = \gamma_1(i)$$

$$\hat{a}_{ij} = \sum_{t=1}^{T-1} \xi_t(i, j) / \sum_{t=1}^{T-1} \gamma_t(i)$$

$$\hat{b}_i(k) = \sum_{t=1|O_t=k}^T \gamma_t(i) / \sum_{t=1}^T \gamma_t(i)$$



Multiple Observation Sequences

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For multiple observation sequences, sum $\xi_t(i, j)$ and $\gamma_t(i)$ and over all sequences:

$$\hat{\pi}_i = \sum_O \gamma_1(i)$$

$$\hat{a}_{ij} = \sum_O \sum_{t=1}^{T-1} \xi_t(i, j) / \sum_O \sum_{t=1}^{T-1} \gamma_t(i)$$

$$\hat{b}_i(k) = \sum_O \sum_{t=1|O_t=k}^T \gamma_t(i) / \sum_O \sum_{t=1}^T \gamma_t(i)$$

The final values will then have to be normalized.



Outline

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- 1 Hidden Markov Models
- 2 Probability of an Observation Sequence
- 3 Probability of a Sequence of States
- 4 Learning the Model
- 5 Other Sequential Classification Models



Other Models

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Other
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Models

- Structured Perceptron
- Conditional Random Fields
- Recurrent or Convolutional Deep Neural Networks
- ...



Restructuring HMMs With Features

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- In a regular HMM, we have that:

$$\begin{aligned}P(O, I|\lambda) &= P(O|I, \lambda)P(I|\lambda) \\ &= \pi_{i_1} b_{i_1}(O_1) a_{i_1, i_2} b_{i_2}(O_2) \cdots a_{i_{T-1} i_T} b_{i_T}(O_T)\end{aligned}$$

- Considering the log likelihood:

$$\begin{aligned}\log(P(O, I|\lambda)) &= \log(\pi_{i_1} + \log(b_{i_1}(O_1))) + \\ &\quad \log(a_{i_1 i_2}) + \log(b_{i_2}(O_2))) + \cdots + \\ &\quad \log(a_{i_{T-1} i_T}) + \log(b_{i_T}(O_T)))\end{aligned}$$

- Considering scores, and assuming that $W(t, O_t) = \log(a_{i_{t-1} i_t}) + \log(b_{i_t}(O_t))$, we have:

$$S(O, I|\lambda) = \sum_{t=1}^T W(t, O_t)$$



Structured Perceptron (just a hint)

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- Simple discriminative model that enables exploring features representing symbols (e.g., *capital letters denote nouns?*)
- Viterbi algorithm now considers feature weights for computing costs
- Features represent each possible transition/emission
- Update feature weights incrementally, so as to increase/decrease score of correct/incorrect labellings

- 1 Create feature map and set initial feature weights w
- 2 For ϵ iterations, and for each labeled pair $\{O, I\}$ in the training data
 - 1 Compute \hat{I} for the observation sequence O , using the Viterbi algorithm and the feature vector w
 - 2 If $I = \hat{I}$ do not update the model, else
 - 1 Compute the feature vector f for the pair $\{O, I\}$
 - 2 Compute the feature vector \hat{f} for the pair $\{O, \hat{I}\}$
 - 3 Update the feature vector: $w = w + f - \hat{f}$



Conditional Random Fields (just a hint)

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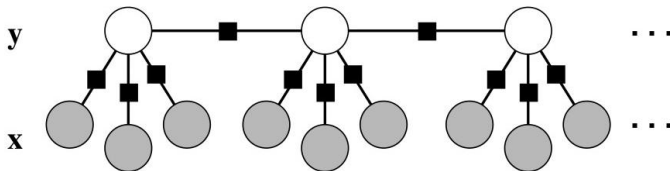
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$$P(y|x) = \frac{1}{Z(x)} \exp \left\{ \sum_{k=1}^K \lambda_k f_k(y_t, y_{t-1}, x_t) \right\}$$



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Questions?



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Extra Credits



The Segmental K-means Algorithm

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- Segmental K-means adjusts the parameters $\lambda = (A, B, \pi)$ to maximize $P(O, I|\lambda)$, where I is the optimal sequence of states for observation sequence O
- Idea: evolve from λ^k to λ^{k+1} such that $P(O, I_k^*|\lambda^k) \leq P(O, I_{k+1}^*|\lambda^{k+1})$
- I_k^* is the optimal state sequence for $O = O_1, O_2, \dots, O_T$ and λ_k
- Function $P(O, I^*|\lambda) = \max_I P(O, I|\lambda)$ is called the **state optimized likelihood function**
- This optimization criterion is called **maximum state optimized likelihood criterion**



The Segmental K-means Algorithm (cont.)

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Basic assumptions:

- We have a set of w observation sequences available (**training sequences**)
- Each training sequence $O = O_1, O_2, \dots, O_T$ consists of T observation symbols
- Each observation symbol O_i is a **vector of D (≥ 1) dimensions**



Computing the Algorithm (1)

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- ① Randomly choose N observation symbols; assign each of the wT training symbols to the closest chosen symbol (e.g., using Euclidean distance)
- ② Calculate the initial probabilities and transition probabilities:

- For $1 \leq i \leq N$:

$$\hat{\pi}_i = \frac{\text{Number of occurrences of } \{O_1 \in i\}}{\text{Total number of occurrences of } O_1 \text{ (i.e., } w)}$$

- For $1 \leq i \leq N, 1 \leq j \leq N$:

$$\hat{a}_{ij} = \frac{\text{Number of occurrences of } \{O_t \in i \text{ and } O_{t+1} \in j\}, \forall t}{\text{Total number of occurrences of } O_t \in i, \forall t}$$



Computing the Algorithm (2)

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- ③ Compute mean and covariance matrix of each state. For $1 \leq i \leq N$:

$$\hat{\mu}_i = \frac{1}{N_i} \sum_{O_t \in i} O_t$$

$$\hat{V}_i = \frac{1}{N_i} \sum_{O_t \in i} (O_t - \hat{\mu}_i)^T (O_t - \hat{\mu}_i)$$

- ④ Calculate the probability distribution of each symbol in each state:

$$\hat{b}_i(O_t) = \frac{1}{((2\pi)^{D/2} |\hat{V}_i|^{1/2})} \exp\left[-\frac{1}{2}(O_t - \hat{\mu}_i)^T \hat{V}_i^{-1} (O_t - \hat{\mu}_i)\right]$$

- We are assuming a Gaussian distribution. Others could be used.



Computing the Algorithm (3)

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- 5 Find the optimal state sequence I^* for each training sequence, using $\hat{\lambda}_i = (\hat{A}_i, \hat{B}_i, \hat{\pi}_i)$; reassign O_t (of the k -th training sequence) to state i iff i_t^* (of the k -th training sequence) is i
 - For instance: if O_2 of the 5th sequence was in state 3, and in I^* (for the 5th training sequence) we have that i_2^* is 4, we assign O_2 of the 5th sequence to state 4.
 - 6 If any symbol was reassigned, repeat from step 2, otherwise stop.
- It can be proved that the algorithm converges to the state-optimized likelihood function for many different observation probability distributions