



Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

# Processamento e Recuperação de Informação

## Information Extraction : Hidden Markov Models

Departamento de Engenharia Informática  
Instituto Superior Técnico

1º Semestre  
2018/2019



# Bibliography - Articles

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Rakesh Dugad, U.B. Desai, *A Tutorial on Hidden Markov Models*, Technical Report, Department of Electrical Engineering, Indian Institute of Technology, 1996.
- Lawrence R. Rabiner, *A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition*, Proceedings of the IEEE, 77(2), February, 1989.



# Outline

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- 1 Hidden Markov Models
- 2 Probability of an Observation Sequence
- 3 Probability of a Sequence of States
- 4 Learning the Model
- 5 Other Sequential Classification Models



# An Example Generative Story

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Suppose a person, inside a room, has three coins (possibly biased)
- The person chooses a coin, randomly, and throws it, chooses another, throws it, and so on...
- The choice of a coin depends on the previously chosen coin
- We are outside the room, looking through a window
- We can only see the outcome of the coin (heads or tails)

- Suppose we observe the sequence:

HHTTTTHHTHTTTHHTTHT

- What probabilities can influence this outcome?



# Hidden Markov Model

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

The outcome is influenced by three factors:

- 1 The probability of choosing a given coin first
- 2 The probability of choosing a given coin, after another
- 3 The probability of getting heads or tails

These three sets of probabilities characterize a **Hidden Markov Model** for the coin tossing experiment



# Finite State Machine Representation

Processamento  
e Recuperação  
de Informação

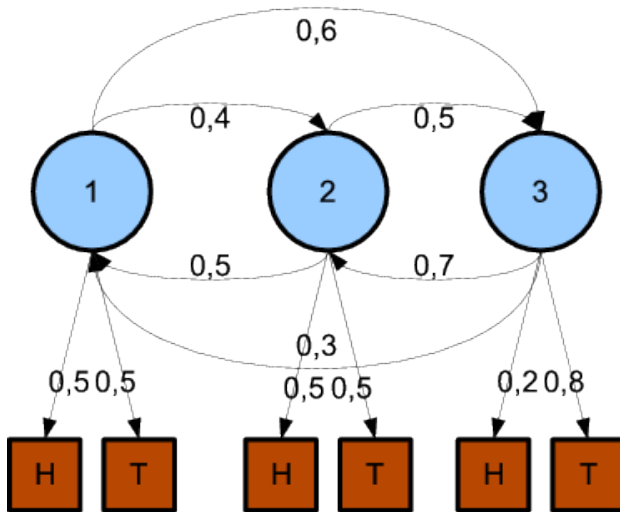
Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models





# Definitions

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

We will use the following notation:

- $N$  — the number of **states** in the model
- $M$  — the number of distinct **observation symbols**
- $T$  — the length of the **observation sequence**
- $i_t$  — the state in which we are at time  $t$
- $V = \{V_1, \dots, V_M\}$  — the set of observation symbols
- $\pi = \{\pi_i\}$  — the probability of being in state  $i$  at the beginning of the experiment, i.e.  $\pi_i = P(i_1 = i)$
- $A = \{a_{ij}\}$  — the probability of being in state  $j$  at time  $t + 1$  given that we were in state  $i$  at time  $t$ , i.e.  $P(i_{t+1} = j | i_t = i)$
- $B = \{b_j(k)\}$  — the probability of observing symbol  $v_k$  given that we are in state  $j$ , i.e.,  $P(v_k \text{ at } t | i_t = j)$
- $O_t$  the observation symbol observed at time  $t$
- $\lambda = (A, B, \pi)$  — the **Hidden Markov Model**



# An example

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

Consider a set of  $n$  urns, each containing marbles of  $m$  different colors. We are randomly choosing an urn and randomly picking a marble from it. How do we model this as an HMM?

- What are the states?
- What are the observation symbols?





## An example (cont.)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- The model would represent the following sequence of events:



# An example (cont.)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- The model would represent the following sequence of events:
  - ① We choose one of the urns, according to probability distribution  $\pi$



# An example (cont.)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- The model would represent the following sequence of events:
  - 1 We choose one of the urns, according to probability distribution  $\pi$
  - 2 We choose a marble from that urn, according to probability distribution  $B$



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Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- The model would represent the following sequence of events:
  - 1 We choose one of the urns, according to probability distribution  $\pi$
  - 2 We choose a marble from that urn, according to probability distribution  $B$ 
    - At this moment we are at time  $t_1$ , state  $i_1$ , and observed symbol  $O_1$



# An example (cont.)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

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  - 1 We choose one of the urns, according to probability distribution  $\pi$
  - 2 We choose a marble from that urn, according to probability distribution  $B$ 
    - At this moment we are at time  $t_1$ , state  $i_1$ , and observed symbol  $O_1$
    - After the next step we will be at time  $t_2$



# An example (cont.)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

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  - 1 We choose one of the urns, according to probability distribution  $\pi$
  - 2 We choose a marble from that urn, according to probability distribution  $B$ 
    - At this moment we are at time  $t_1$ , state  $i_1$ , and observed symbol  $O_1$
    - After the next step we will be at time  $t_2$
  - 3 We choose another urn, according to probability distribution  $A$



# An example (cont.)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

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    - At this moment we are at time  $t_1$ , state  $i_1$ , and observed symbol  $O_1$
    - After the next step we will be at time  $t_2$
  - 3 We choose another urn, according to probability distribution  $A$
  - 4 Repeat from step 2, until we have made  $T$  observations (i.e.,  $t=T$ )



# An example (cont.)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

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  - 1 We choose one of the urns, according to probability distribution  $\pi$
  - 2 We choose a marble from that urn, according to probability distribution  $B$ 
    - At this moment we are at time  $t_1$ , state  $i_1$ , and observed symbol  $O_1$
    - After the next step we will be at time  $t_2$
  - 3 We choose another urn, according to probability distribution  $A$
  - 4 Repeat from step 2, until we have made  $T$  observations (i.e.,  $t=T$ )
- The generated observation sequence will be  $O_1, O_2, \dots, O_T$ .





# Three Problems for HMMs

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- ① Given the model  $\lambda = (A, B, \pi)$ , compute  $P(O|\lambda)$ 
  - I.e., compute the probability of observing a given sequence
  - Applications in [language modeling](#), [spelling correction](#), ...
- ② Given the model  $\lambda = (A, B, \pi)$ , choose a state sequence  $I = i_1, i_2, \dots, i_T$  such that  $P(O, I|\lambda)$  is maximized, for a given observation sequence  $O = O_1, O_2, \dots, O_T$ 
  - I.e., compute the most likely sequence of states to have generated an observation sequence (i.e., [decoding](#))
  - Applications in [information extraction](#) (e.g., chunking, named entity recognition, ...)
- ③ Adjust the model parameters  $\lambda = (A, B, \pi)$  such that  $P(O|\lambda)$  or  $P(O, I|\lambda)$ , is maximized
  - I.e., based on a series of observations and/or state sequences, compute the HMM
  - Learning model parameters from annotated data



# Outline

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- 1 Hidden Markov Models
- 2 Probability of an Observation Sequence**
- 3 Probability of a Sequence of States
- 4 Learning the Model
- 5 Other Sequential Classification Models



# Computing the Probabilities

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- We know that

$$P(O|\lambda) = \sum_I P(O|I, \lambda) P(I|\lambda)$$

- and since

$$P(O|I, \lambda) = b_{i_1}(O_1)b_{i_2}(O_2)\cdots b_{i_T}(O_T)$$

$$P(I|\lambda) = \pi_{i_1} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{T-1} i_T}$$

- we have that

$$P(O|\lambda) = \sum_I \pi_{i_1} b_{i_1}(O_1) a_{i_1 i_2} b_{i_2}(O_2) \cdots a_{i_{T-1} i_T} b_{i_T}(O_T)$$



# The Problem

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

$$P(O|\lambda) = \sum_I \pi_{i_1} b_{i_1}(O_1) a_{i_1 i_2} b_{i_2}(O_2) \cdots a_{i_{T-1} i_T} b_{i_T}(O_T)$$

- Computing each summand requires  $2T - 1$  multiplications
- There are  $N^T$  possible state sequences
- Thus, the complexity is  $O(2TN^T)$ : **unfeasible**
- However, there is a more efficient way of computing  $P(O|\lambda)$ : the **forward/backward procedure**



# The Forward Procedure

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Consider the **forward variable**  $\alpha_t(i)$ , defined as:

$$\alpha_t(i) = P(O_1, O_2, \dots, O_t, i_t = i | \lambda)$$

i.e., the probability of a partial observation sequence (until time  $t$ ) that ends in state  $i$

- $\alpha_t(i)$  can be computed as follows:
  - 1 Compute the probability of starting in state  $i$  and observing  $O_1$ :  $\alpha_1(i)$
  - 2 For time  $t + 1$  compute the probability of reaching a state  $j$  and observing  $O_{t+1}$ , knowing that we already computed all probabilities for all times  $\leq t$
  - 3 The final probability (at time  $T$ ) will be the sum of all probabilities for each possible ending state  $i$ :  $\alpha_T(i)$



# Computing The Forward Procedure

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

① Initial step:

$$\alpha_1(i) = \pi_i b_i(O_1) , 1 \leq i \leq N$$



# Computing The Forward Procedure

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

① Initial step:

$$\alpha_1(i) = \pi_i b_i(O_1), \quad 1 \leq i \leq N$$

② For  $t = 1, 2, \dots, T - 1, 1 \leq j \leq N$

$$\alpha_{t+1}(j) = \left[ \sum_{i=1}^N \alpha_t(i) a_{ij} \right] b_j(O_{t+1})$$



# Computing The Forward Procedure

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- 1 Initial step:

$$\alpha_1(i) = \pi_i b_i(O_1) , 1 \leq i \leq N$$

- 2 For  $t = 1, 2, \dots, T - 1, 1 \leq j \leq N$

$$\alpha_{t+1}(j) = \left[ \sum_{i=1}^N \alpha_t(i) a_{ij} \right] b_j(O_{t+1})$$

- 3 Thus, we have that:

$$P(O|\lambda) = \sum_{i=1}^N \alpha_T(i)$$





# Time Complexity

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Step 1 requires  $N$  multiplications
- Step 2 requires  $N + 1$  multiplications. This is performed for all  $N$  states and  $T - 1$  times, yielding  $(N + 1)N(T - 1)$  multiplications
- Step 3 requires only to sum the computed values
- Thus, the time complexity is  $O(N^2 T)$



# Backward Procedure

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- A similar procedure can be applied *moving backwards*
- Consider the backward variable  $\beta_t(i)$ , defined as:

$$\beta_t(i) = P(O_{t+1}, O_{t+2}, \dots, O_T | i_t = i, \lambda)$$

i.e., the probability of observing a partial sequence starting at time  $t + 1$  and state  $i$

- $\beta_t(i)$  can also be computed as follows:

①

$$\beta_T(i) = 1, 1 \leq i \leq N$$

②

For  $t = T - 1, T - 2, \dots, 1, 1 \leq i \leq N$ , we have

$$\beta_t(i) = \sum_{j=1}^N a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)$$

③

Thus, we have that:

$$P(O|\lambda) = \sum_{i=1}^N \pi_i b_i(O_1) \beta_1(i)$$



# Outline

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- 1 Hidden Markov Models
- 2 Probability of an Observation Sequence
- 3 Probability of a Sequence of States**
- 4 Learning the Model
- 5 Other Sequential Classification Models



# The Decoding Problem

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- We want to find a sequence of states  $I = i_1, i_2, \dots, i_T$  such that the probability of observing a sequence  $O = O_1, O_2, \dots, O_T$  is greater than for any other sequence
- I.e., Find  $I$  that maximizes  $P(O, I|\lambda)$

$$\arg \max_{\{i_t\}_{t=1}^T} P(O, i_1, i_2, \dots, i_T|\lambda)$$

- This can be computed using the **Viterbi Algorithm**



# The Viterbi Algorithm

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- We know that

$$\begin{aligned} P(O, I|\lambda) &= P(O|I, \lambda)P(I|\lambda) \\ &= \pi_{i_1} b_{i_1}(O_1) a_{i_1 i_2} b_{i_2}(O_2) \cdots a_{i_{T-1} i_T} b_{i_T}(O_T) \end{aligned}$$

- Thus, we can define

$$U(i_1, i_2, \dots, i_T) = - \left[ \ln(\pi_{i_1} b_{i_1}(O_1)) + \sum_{t=2}^T \ln(a_{i_{t-1} i_t} b_{i_t}(O_t)) \right]$$

- so that

$$P(O, I|\lambda) = \exp(-U(i_1, i_2, \dots, i_T))$$

- and our problem becomes

$$\arg \min_{\{i_t\}_{t=1}^T} U(i_1, i_2, \dots, i_T)$$



# The Viterbi Algorithm (cont.)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- We can view the term  $-\ln(a_{ij}b_{i_k}(O_t))$  as the cost of going from state  $i_j$  to state  $i_k$  at time  $t$
- The Viterbi Algorithm is a **dynamic programming** approach to compute the path of least cost
- The total cost of a path is the sum of the weights on the edges we cross
  - Note that this is equivalent to multiplying the probabilities



# Computing the Viterbi Algorithm (1)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Let  $\delta_t(i)$  be the accumulated weight at state  $i$  and time  $t$
- Let  $\psi_t(j)$  be the state at time  $t - 1$  with the lowest cost transition to state  $j$  at time  $t$

- 1 Initialization, for  $1 \leq i \leq N$ :

$$\delta_1(i) = -\ln(\pi_i) - \ln(b_i(O_1))$$

$$\psi_1(i) = 0$$

- 2 Recursive computation, for  $2 \leq t \leq T$ ,  $1 \leq j \leq N$ :

$$\delta_t(j) = \min_{1 \leq i \leq N} [\delta_{t-1}(i) - \ln(a_{ij})] - \ln(b_j(O_t))$$

$$\psi_t(j) = \arg \min_{1 \leq i \leq N} [\delta_{t-1}(i) - \ln(a_{ij})]$$



# Computing the Viterbi Algorithm (2)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

## 3 Termination:

$$P^* = \min_{1 \leq i \leq N} [\delta_T(i)]$$
$$q_T^* = \arg \min_{1 \leq i \leq N} [\delta_T(i)]$$

## 4 Trace back, for $t = T, T - 1, T - 2, \dots, 1$ :

$$q_t^* = \psi_{t+1}(q_{t+1}^*)$$

- $Q^* = \{q_1^*, q_2^*, \dots, q_T^*\}$  is the optimal state sequence
- $\exp(-P^*)$  is the optimized probability for the state sequence
- Complexity:  $O(N^2 T)$





# An Example Computation

Processamento  
e Recuperação  
de Informação

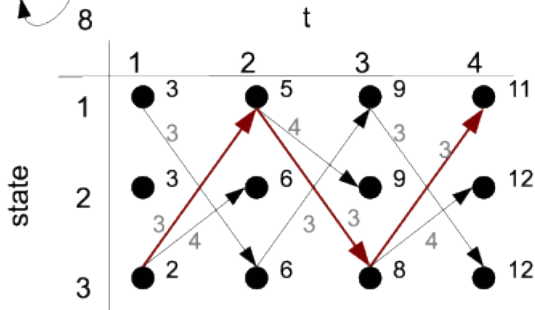
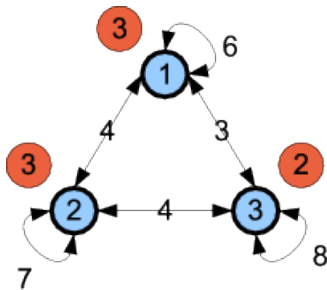
Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models





# Notes

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- The Viterbi algorithm can be used with HMMs, and also with other sequential classification models (e.g., structured Perceptrons, CRFs, neural network approaches, ...)
- Other *decoding* approaches are also frequently used in practice, one example being **posterior decoding**
  - Determine, independently for every symbol  $O_t$ , the most probable state using the forward/backward procedure
  - Often more effective when several concurring paths have similar probabilities
- Some practical implementations of Information Extraction tools, leveraging sequential classification models, rely on methods such as **beam search** to find an approximate solution to the problem of finding state sequences



# Outline

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- 1 Hidden Markov Models
- 2 Probability of an Observation Sequence
- 3 Probability of a Sequence of States
- 4 Learning the Model**
- 5 Other Sequential Classification Models



# Learning HMMs

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- In a **supervised setting**, we use the training data to estimate the probabilities
- Transition probabilities

$$\hat{P}(i \rightarrow i') = \frac{c(i \rightarrow i')}{\sum_{s \in I} c(i \rightarrow s)}$$

- Emission probabilities

$$\hat{P}(i \uparrow o) = \frac{c(i \uparrow o)}{\sum_{\rho \in O} c(i \uparrow \rho)}$$

- $c(i \rightarrow i')$  is the number of times there is a transition from state  $i$  to state  $i'$  (in a training set)
- $c(i \uparrow o)$  counts the number of times symbol  $o$  is observed in state  $i$  (in a training set)
- The **estimation of beginning probabilities** is similar to that of transition probabilities, but we count the number of times there is a transition from the start (i.e., the beginning of a training sequence) to a state  $i$



# Improving Probability Estimates

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Problem: sparse training data causes poor probability estimates
  - E.g., unseen symbols have emission probabilities of zero
- Solution: use probability smoothing techniques
  - Laplace smoothing
  - Absolute discounting
  - ...



# Laplace Smoothing

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Adds 1 to every count of occurrences
- Moves all estimates towards the uniform distribution
- All unseen words will have equal probability

An example:

$$\hat{P}(i \uparrow o) = \frac{c(i \uparrow o) + 1}{\sum_{\rho \in O} c(i \uparrow \rho) + |O|}$$



# Absolute Discounting

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Localized (per state) smoothing
- Appropriate if zero probabilities vary from state to state
- Subtracts a fixed discount  $0 < d < 1$  from all symbols with count  $> 0$
- The total discounted value is distributed by the remaining symbols

An example:

$$\hat{P}(i \uparrow o) = \begin{cases} \frac{c(i \uparrow o) - d}{\sum_{\rho \in O} c(i \uparrow \rho)} & \text{if } c(i \uparrow o) > 0 \\ \frac{d(|O| - |Z_q|)}{|Z_q| * \sum_{\rho \in O} c(i \uparrow \rho)} & \text{if } c(i \uparrow o) = 0 \end{cases}$$

where  $|Z_q|$  is the number of symbols with zero count in state  $i$ .



# Unsupervised Learning of HMMs

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- We want to train an HMM model with a set of **example observation sequences** such that, when a similar sequence is discovered later the model is able to identify it.
- Most well known method
  - **Baum-Welch** algorithm
- Other methods exist
  - E.g. **Segmental K-means**





# The Baum-Welch Method

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Assume an initial model  $\lambda$ 
  - Can be constructed in any way (e.g. randomly)
- Maximizes  $P(O|\lambda)$  by adjusting  $\lambda$ 
  - Called the **maximum likelihood criterion**



# Probability of Visiting a State

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Let

$$\gamma_t(i) = P(i_t = i | O, \lambda)$$

i.e. the probability of being in state  $i$  at time  $t$  given the observation sequence  $O$  and the model  $\lambda$

- Applying Bayes rule:

$$\gamma_t(i) = \frac{P(i_t = i, O)}{P(O|\lambda)} = \frac{\alpha_t(i)\beta_t(i)}{P(O|\lambda)}$$

where  $\alpha_t(i)$  is computed as in the Forward procedure and  $\beta_t(i)$  is computed as in the Backward procedure



# Probability of Transitioning

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Let

$$\xi_t(i) = P(i_t = i, i_{t+1} = j | O, \lambda)$$

i.e. the probability of being in state  $i$  at time  $t$  and making a transition to state  $j$  at time  $t + 1$ , given the observation sequence  $O$  and the model  $\lambda$

- Applying Bayes rule:

$$\xi_t(i) =$$

$$\frac{P(i_t = i, i_{t+1} = j, O | \lambda)}{P(O | \lambda)} = \frac{\alpha_t(i) a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)}{P(O | \lambda)}$$



# Expected Number of Transitions

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

Expected number of visits to state  $i$ :

$$\sum_{t=1}^T \gamma_t(i)$$

Expected number of transitions from state  $i$ :

$$\sum_{t=1}^{T-1} \gamma_t(i)$$

Expected number of transitions from state  $i$  to state  $j$ :

$$\sum_{t=1}^{T-1} \xi_t(i, j)$$



# Baum-Welch Re-Estimation Formulas

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

The new model parameters  $\hat{\lambda} = (\hat{A}, \hat{B}, \hat{\pi})$  can be computed as:

$$\hat{\pi}_i = \gamma_1(i)$$

$$\hat{a}_{ij} = \sum_{t=1}^{T-1} \xi_t(i, j) / \sum_{t=1}^{T-1} \gamma_t(i)$$

$$\hat{b}_i(k) = \sum_{t=1|O_t=k}^T \gamma_t(i) / \sum_{t=1}^T \gamma_t(i)$$



# Multiple Observation Sequences

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

For multiple observation sequences, sum  $\xi_t(i, j)$  and  $\gamma_t(i)$  and over all sequences:

$$\hat{\pi}_i = \sum_O \gamma_1(i)$$

$$\hat{a}_{ij} = \sum_O \sum_{t=1}^{T-1} \xi_t(i, j) / \sum_O \sum_{t=1}^{T-1} \gamma_t(i)$$

$$\hat{b}_i(k) = \sum_O \sum_{t=1|O_t=k}^T \gamma_t(i) / \sum_O \sum_{t=1}^T \gamma_t(i)$$

The final values will then have to be normalized.



# Outline

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- 1 Hidden Markov Models
- 2 Probability of an Observation Sequence
- 3 Probability of a Sequence of States
- 4 Learning the Model
- 5 Other Sequential Classification Models



# Other Models

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Structured Perceptron
- Conditional Random Fields
- Recurrent or Convolutional Deep Neural Networks
- ...





# Restructuring HMMs With Features (1)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- In a regular HMM, we have that:

$$\begin{aligned}P(O, I|\lambda) &= P(O|I, \lambda)P(I|\lambda) \\ &= \pi_{i_1} b_{i_1}(O_1) a_{i_1, i_2} b_{i_2}(O_2) \cdots a_{i_{T-1} i_T} b_{i_T}(O_T)\end{aligned}$$

- Considering the log likelihood:

$$\begin{aligned}\log(P(O, I|\lambda)) &= \log(\pi_{i_1} + \log(b_{i_1}(O_1))) + \\ &\quad \log(a_{i_1 i_2}) + \log(b_{i_2}(O_2))) + \cdots + \\ &\quad \log(a_{i_{T-1} i_T}) + \log(b_{i_T}(O_T)))\end{aligned}$$

- Considering scores, and assuming that  $W(t, O_t) = \log(a_{i_{t-1} i_t}) + \log(b_{i_t}(O_t))$ , we have:

$$S(O, I|\lambda) = \sum_{t=1}^T W(t, O_t)$$



# Restructuring HMMs With Features (2)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- In the previous example, we saw how to express a model equivalent to an HMM through scoring functions  $W(t, O_t)$  that leverage state transitions between adjacent positions in the sequence (i.e., from  $t - 1$  to  $t$ ), and symbol emissions for position  $t$  of the input sequence.
- Scoring functions can also be written as a linear combination of  $K$  different features, again describing state transitions between adjacent positions in the sequence, and symbol emissions for position  $t$  of the input sequence.
- In information extraction applications, we want to find the state sequence  $I$  that satisfies:

$$\hat{I} = \arg \max_I S(O, I | \lambda) = \arg \max_I \sum_{t=1}^T \sum_{k=1}^K \lambda_k f_k(y_t, y_{t-1}, x_t)$$



# Structured Perceptron (just a hint)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Simple discriminative model that enables exploring features representing symbols (e.g., *capital letters denote nouns?*)
- Viterbi algorithm now considers feature weights for computing costs
- Features represent each possible transition/emission
- Update feature weights incrementally, so as to increase/decrease score of correct/incorrect labellings

- 1 Create feature map and set initial feature weights  $w$
- 2 For  $\epsilon$  iterations, and for each labeled pair  $\{O, I\}$  in the training data
  - 1 Compute  $\hat{I}$  for the observation sequence  $O$ , using the Viterbi algorithm and the feature vector  $w$
  - 2 If  $I = \hat{I}$  do not update the model, else
    - 1 Compute the feature vector  $f$  for the pair  $\{O, I\}$
    - 2 Compute the feature vector  $\hat{f}$  for the pair  $\{O, \hat{I}\}$
    - 3 Update the feature vector:  $w = w + f - \hat{f}$



# Conditional Random Fields (just a hint)

Processamento  
e Recuperação  
de Informação

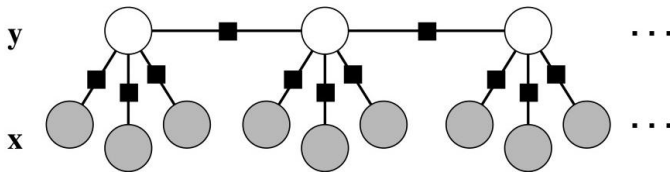
Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models



$$P(y|x) = \frac{1}{Z(x)} \exp \left\{ \sum_{t=1}^T \sum_{k=1}^K \lambda_k f_k(y_t, y_{t-1}, x_t) \right\}$$



Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

# Questions?



Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

## Extra Credits



# The Segmental K-means Algorithm

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- Segmental K-means adjusts the parameters  $\lambda = (A, B, \pi)$  to maximize  $P(O, I|\lambda)$ , where  $I$  is the optimal sequence of states for observation sequence  $O$
- Idea: evolve from  $\lambda^k$  to  $\lambda^{k+1}$  such that  $P(O, I_k^*|\lambda^k) \leq P(O, I_{k+1}^*|\lambda^{k+1})$
- $I_k^*$  is the optimal state sequence for  $O = O_1, O_2, \dots, O_T$  and  $\lambda_k$
- Function  $P(O, I^*|\lambda) = \max_I P(O, I|\lambda)$  is called the **state optimized likelihood function**
- This optimization criterion is called **maximum state optimized likelihood criterion**



# The Segmental K-means Algorithm (cont.)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

Basic assumptions:

- We have a set of  $w$  observation sequences available (**training sequences**)
- Each training sequence  $O = O_1, O_2, \dots, O_T$  consists of  $T$  observation symbols
- Each observation symbol  $O_i$  is a **vector of  $D$  ( $\geq 1$ ) dimensions**





# Computing the Algorithm (1)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- 1 Randomly choose  $N$  observation symbols; assign each of the  $wT$  training symbols to the closest chosen symbol (e.g., using Euclidean distance)
- 2 Calculate the initial probabilities and transition probabilities:

- For  $1 \leq i \leq N$ :

$$\hat{\pi}_i = \frac{\text{Number of occurrences of } \{O_1 \in i\}}{\text{Total number of occurrences of } O_1 \text{ (i.e., } w\text{)}}$$

- For  $1 \leq i \leq N, 1 \leq j \leq N$ :

$$\hat{a}_{ij} = \frac{\text{Number of occurrences of } \{O_t \in i \text{ and } O_{t+1} \in j\}, \forall t}{\text{Total number of occurrences of } O_t \in i, \forall t}$$



## Computing the Algorithm (2)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- ③ Compute mean and covariance matrix of each state. For  $1 \leq i \leq N$ :

$$\hat{\mu}_i = \frac{1}{N_i} \sum_{O_t \in i} O_t$$

$$\hat{V}_i = \frac{1}{N_i} \sum_{O_t \in i} (O_t - \hat{\mu}_i)^T (O_t - \hat{\mu}_i)$$

- ④ Calculate the probability distribution of each symbol in each state:

$$\hat{b}_i(O_t) = \frac{1}{((2\pi)^{D/2} |\hat{V}_i|^{1/2})} \exp\left[-\frac{1}{2} (O_t - \hat{\mu}_i)^T \hat{V}_i^{-1} (O_t - \hat{\mu}_i)\right]$$

- We are assuming a Gaussian distribution. Others could be used.



# Computing the Algorithm (3)

Processamento  
e Recuperação  
de Informação

Hidden  
Markov  
Models

Probability of  
an  
Observation  
Sequence

Probability of  
a Sequence of  
States

Learning the  
Model

Other  
Sequential  
Classification  
Models

- 5 Find the optimal state sequence  $I^*$  for each training sequence, using  $\hat{\lambda}_i = (\hat{A}_i, \hat{B}_i, \hat{\pi}_i)$ ; reassign  $O_t$  (of the  $k$ -th training sequence) to state  $i$  iff  $i_t^*$  (of the  $k$ -th training sequence) is  $i$ 
    - For instance: if  $O_2$  of the 5th sequence was in state 3, and in  $I^*$  (for the 5th training sequence) we have that  $i_2^*$  is 4, we assign  $O_2$  of the 5th sequence to state 4.
  - 6 If any symbol was reassigned, repeat from step ??, otherwise stop.
- It can be proved that the algorithm converges to the state-optimized likelihood function for many different observation probability distributions