Multi-period Order Decision Problem

Need to choose order quantity Q_t for each time period $t = 1, \ldots T$

$\Pi_t = pmin(Q_t + I_{t-1}, X) - Q_t c - FC$

i.e. find closed form solution for Qt^* *that maximizes $|sum*{t=1}^{T}E(\Pi_t)$

where the demand for each day is i.i.d $X \sim N(\mu, \sigma^2)$,

r is discount rate, we earn daily interest on profit accumulated to date

c = marginal cost of producing a single unit

FC = fixed cost

p = price per unit, prices are same each period

c, FC and p are constant over time

Q_t is the order quantity that we choose at the end of the business day of the prior day

Derivation

Working Backwards from hypothetical last period T

$E(\Pi_T) = E(pmin(Q_T + I_{T-1}, X) - Q_T c - FC) = p\int_{x=-\infty}^{Q_T+I_{T-1}} x f_X dx + p\int_{x=Q_T+I_{T-1}}^{\infty} (Q+I) f_X dx - Q_T C - FC$

$\frac{dE(\Pi_T)}{dQ} = p(Q+I)f(Q+I) + p(1-F(Q+I)) + p(Q+I)(-f(Q+I)) - C = p(1-F(Q+I)) - C$

$p(1-F(Q+I)) - C$ = 0

$\frac{C}{p} = 1 - F \Rightarrow F(Q+I) = \frac{p-C}{p} \Rightarrow Q^* = F^{-1}(\frac{p-C}{p}) - I_{T-1}$

$\frac{d^2E(\Pi_T)}{dQ^2} = -pf(Q+I) < 0$ so $E(\Pi_T)$ is concanve and Q^* maximizes $E(\Pi_T)$

$E[\Pi_T^*(I_{T-1})] = E(pmin(F^{-1}(\ldots), X) - [F^{-1}(\ldots) - I_{T-1}]C - FC = constant - I_{T-1}C$

$I_{T-1} = max(I_{T-2} + Q_{T-1} - X_{T-1}, 0)$

$E[\Pi_T^*(I_{T-1})] = constant + Cmax(I_{T-2} + Q_{T-1} - X_{T-1}, 0)$

Π_{T-1} = profit this period + discounted profit next period - cost of holding inventory for next period =

$\Pi_{T-1} = pmin(Q_{T-1} + I_{T-2}, X) - Q_{T-1}C - FC + \frac{1}{1+r}E[\Pi_T^*(I_{T-1})] - hmax(0, Q_{T-1} + I_{T-2} - X_{T-1})$

$\Pi_{T-1} = pmin(Q_{T-1} + I_{T-2}, X) - Q_{T-1}C - FC + \frac{1}{1+r}[constant + Cmax(I_{T-2} + Q_{T-1} - X_{T-1}, 0)] - hmax(0, Q_{T-1} + I_{T-2} - X_{T-1})$

$\Pi_{T-1} = pmin(Q_{T-1} + I_{T-2}, X) - Q_{T-1}C + constant + \frac{C}{1+r}[max(I_{T-2} + Q_{T-1} - X_{T-1}, 0)] - hmax(0, Q_{T-1} + I_{T-2} - X_{T-1})$

$\Pi_{T-1} = pmin(Q_{T-1} + I_{T-2}, X) - Q_{T-1}C + constant + [\frac{C}{1+r} - h][max(I_{T-2} + Q_{T-1} - X_{T-1}, 0)]$

the first part is same as at T when differentiating

$\frac{dE(\Pi_{T-1})}{dQ_{T-1}} = p(1-F(Q+I)) - C + [\frac{C}{1+r} - h]\frac{d}{dQ_{T-1}}\int max(I_{T-2} + Q_{T-1} - X_{T-1}, 0) f_X dx$

$\frac{dE(\Pi_{T-1})}{dQ_{T-1}} = p(1-F(Q+I)) - C + [\frac{C}{1+r} - h]\frac{d}{dQ_{T-1}}\int_{x=I_{T-2}+Q_{T-1}}^{\infty} (I_{T-2} + Q_{T-1} - x) f_X dx$

$\frac{dE(\Pi_{T-1})}{dQ_{T-1}} = p(1-F(Q+I)) - C + I(\frac{c}{1+r} - h)(-f(Q+I)) + (\frac{c}{1+r} - h)(1-F(Q+I)) + (\frac{c}{1+r} - h)Q(-f(Q+I)) + (\frac{c}{1+r} - h)(Q+I)f_X(Q+I)$

$\frac{dE(\Pi_{T-1})}{dQ_{T-1}} = p(1-F(Q+I)) - C + (\frac{c}{1+r} - h)(1-F(Q+I))$

$\frac{dE(\Pi_{T-1})}{dQ_{T-1}} = (p + \frac{c}{1+r} - h)(1-F(Q+I) - C$

reasonable to assume that $p + \frac{c}{1+r} - h > 0$ so we have concavity

$Q_{T-1}^* = F^{-1}(1 - \frac{c}{p+\frac{c}{1+r}-h}) - I_{T-2}$

$\frac{dQ}{dC} < 0$ and $\frac{dQ}{dp} > 0$ and $\frac{dQ}{dh} < 0$ which makes sense

$Q_{T-1}^* + I_{T-2} = F^{-1}(1 - \frac{c}{p+\frac{c}{1+r}-h})$

$\Pi_{T-1}(I_{T-2}) = pmin(F^{-1}(1 - \frac{c}{p+\frac{c}{1+r}-h}), X) - [F^{-1}(1 - \frac{c}{p+\frac{c}{1+r}-h}) - I_{T-2}]C + constant + [\frac{C}{1+r} - h][max(F^{-1}(1 - \frac{c}{p+\frac{c}{1+r}-h}) - X_{T-1}, 0)]$

$E[\Pi_{T-1}(I_{T-2})] = constant - [F^{-1}(1 - \frac{c}{p+\frac{c}{1+r}-h}) - I_{T-2}]C$

$\Pi_{T-2} = pmin(Q_{T-2} + I_{T-3}, X_{T-2}) - Q_{T-2}C - FC + \frac{1}{1+r}E[\Pi_{T-1}^*(I_{T-2})] - hmax(0, Q_{T-2} + I_{T-3} - X_{T-2})$

$\Pi_{T-2} = pmin(Q_{T-2} + I_{T-3}, X_{T-2}) - Q_{T-2}C - FC + \frac{1}{1+r}[constant - [F^{-1}(1 - \frac{c}{p+\frac{c}{1+r}-h}) - I_{T-2}]C] - hmax(0, Q_{T-2} + I_{T-3} - X_{T-2})$

$\Pi_{T-2} = pmin(Q_{T-2} + I_{T-3}, X_{T-2}) - Q_{T-2}C + \frac{C}{1+r}I_{T-2} - hmax(0, Q_{T-2} + I_{T-3} - X_{T-2}) + constant$

note that $I_{T-2} = max(Q_{T-2} + I_{T-3} - X, 0)$ so substitute this in

$\Pi_{T-2} = pmin(Q_{T-2} + I_{T-3}, X_{T-2}) - Q_{T-2}C + [\frac{C}{1+r} - h]max(Q_{T-2} + I_{T-3} - X, 0) + constant$

but this is exactly the same form as we had for Π_{T-1} hence the solution will also be in the same form

$Q_{T-i}^* + I_{T-1-i} = F^{-1}(1 - \frac{c}{p+\frac{c}{1+r}-h})$ for $i = 1, \ldots, n$

Solution:

$Q_{T-i}^* + I_{T-1-i} = F^{-1}(1 - \frac{c}{p+ce^{-r}-h})$

Economic Logic of Result:

a) A high holding cost reduces optimal order quantity $\frac{dQ}{dh} < 0$

b) Higher interest rates reduce order quatity as we discount future profit value and it costs to tie up capital in inventory

c) Higher prices mean mean stockouts are costlier and value of future profit on excess stock is also higher.

d) Since it is reasonable that p-h>0 and r is very small, higher costs reduce order quantity which makes sense given lower profitability means it is less attractive to order more and pay extra holding costs.

note:

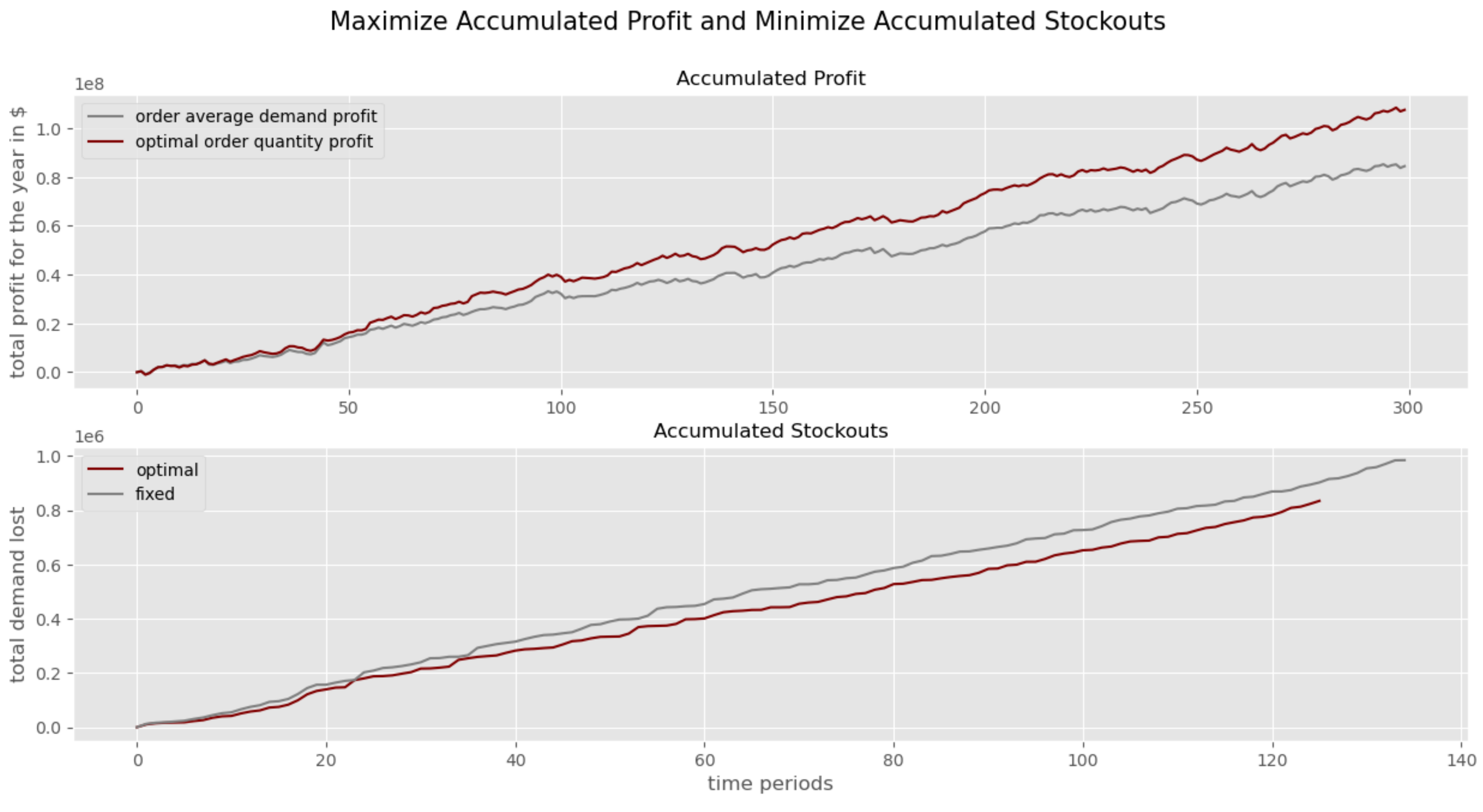
1) Stock-outs possible but can be incorporated

2) Can use empirical CDF for demand instead Normal

In [84]:
```python
#import libraries
from scipy.stats import norm
import numpy as np
import matplotlib.pyplot as plt
plt.style.use('ggplot')
```

In [51]:
```python
#Assumptions
np.random.seed(100)
I= 0 # starting inventory
c= 70 # cost per unit for each period
p = 100 # price per unit for each period
r= (1 + 0.25)**(1/365) - 1 # daily interest rate
FC = 100000 #fixed cost..... rough expected profit before inventory holding cost = mu*(p-c)-FC = 25k units (100-70) -100k = 650k$
mu = 25000
sigma = 10000
t=0 #starting time period
T=300 #number of periods
h=5 # nightly holding period
fixed_order_quantity =mu # assume we order mean demand thinking it demand will even out over time
```

In [85]:
```python
#Run Simulation
total_profit = [0]
total_profit_fixed = [0] # profit if we choose to just order average demand
stock_outs=[0] # total missed demand
stock_outs2=[0] # total missed demand

fig,ax=plt.subplots(nrows=2,ncols=1,figsize=(15,7))
for t in range(1,T):
    #calculate demand for the period
    D=np.random.normal(mu,sigma,1)[0] #generate r.v. demand

    #calcualte optimal order quantity as per derived quantity
    order_quantity = norm.ppf(1-c/(p+c/(1+r)-h))*sigma + mu-I # optimal

    #calculate profit
    current_profit = p*min(D,I+order_quantity)-order_quantity*c - FC - h*max(0,D-I-order_quantity)
    current_profit_fixed = p*min(D,I+fixed_order_quantity)-fixed_order_quantity*c - FC - h*max(0,D-I-fixed_order_quantity)

    #record stock outs
    if D>order_quantity+I:
        stock_outs.append(stock_outs[-1]+D-order_quantity-I)
    if D>fixed_order_quantity+I_fixed:
        stock_outs2.append(stock_outs2[-1]+D-fixed_order_quantity-I_fixed)

    #update inventory
    I = max(0,D-I-order_quantity)
    I_fixed = max(0,D-I-fixed_order_quantity)
    total_profit.append(total_profit[-1]*(1+r)+current_profit)
    total_profit_fixed.append(total_profit_fixed[-1]*(1+r)+current_profit_fixed)

# Draw Results of the Simulation
ax[0].plot(range(0,T),total_profit_fixed,c='grey',label='order average demand profit')
ax[0].plot(range(0,T),total_profit,c='maroon',label='optimal order quantity profit')
plt.suptitle('Maximize Accumulated Profit and Minimize Accumulated Stockouts',fontsize=15)
ax[0].set_title('Accumulated Profit',fontsize=12)
ax[1].set_title('Accumulated Stockouts',fontsize=12)
ax[1].set_xlabel('time periods');ax[0].set_ylabel('total profit for the year in $')
ax[1].set_ylabel('total demand lost')
ax[1].plot(range(len(stock_outs)),stock_outs,c='maroon',label='optimal')
ax[1].plot(range(len(stock_outs2)),stock_outs2,c='grey',label='fixed')
[ax[i].legend() for i in range(2)]
plt.show()
```



The above is what we would expect: above average accumulated profit and below average accumulated stockouts