

# CHAOS IN SIMPLE DYNAMICAL SYSTEMS – THE SITNIKOV PROBLEM

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**Abstract** It is well known that in low degrees of freedom dynamical systems chaotic behaviour appears. To examine this phenomenon the Sitnikov problem is a very good example which is a special case of the restricted three-body problem. In this paper we investigate the changing of the phase space structure due to the variation of the Surfaces of Sections.

**Keywords:** dynamical systems – chaos – resonances

## 1. Introduction

The investigation of dynamical systems in the past 50 years shows that chaotic behaviour appears not only in difficult, many degrees of freedom systems but in simple configurations as well. Therefore one of the most relevant tasks is to study these simple dynamical systems to understand chaos, and on the other hand it is a good starting point to investigate more difficult problems.

One of the simplest and most interesting systems in celestial mechanics is the Sitnikov problem. Essentially, it is a special case of the restricted three-body problem. Namely there are two equal masses  $m_1$  and  $m_2$  revolving in Keplerian orbits around each other, and a third massless body  $m_3$  moves on an axis perpendicular to the plane of the primaries through their barycenter.

Mac Millan [6] showed that in the circular problem, when the primaries revolve on circular orbit, the problem is integrable and the solution is expressed by elliptic integrals. The motion of the massless body is more various when we allow the two primaries to move in eccentric orbits. In this case quasi-periodic and chaotic orbits appear beside the periodic ones. The solution of the problem was first given by Sitnikov in 1960 [9], after that many authors examined the existence of periodic orbits in this configuration. The first mapping model was

derived by Liu and Sun [5], who showed that for small eccentricities the phase space becomes very complex. Perdios and Markellos [8] studied the stability and the bifurcations in the Sitnikov problem. Dvorak (1993) [1] investigated numerically the problem by using Poincaré's surfaces of section. Martínez Alfaro and Chilart [7] found that for certain eccentricities the fixed point  $z = \dot{z} = 0$  in the center becomes unstable. Kallrath et al. [4] explored the phase space in detail laying emphasis on resonances. For small eccentricities Hagel [3] and Faruque [2] applied perturbation methods and gave an analytical approximation to the problem.

In this study we investigate the phase space structure for different initial conditions, eminently for different positions of surfaces of sections (SOS). For the visualization of the results we use Poincaré's surfaces of section.

## 2. Equation of motion

As mentioned above, we investigate the motion of a massless body which moves along a line perpendicular to the plane of the primaries through their barycenter (Fig. 1). By introducing suitable units we can write the equation of motion. We choose the total mass of the primaries ( $m_1$  and  $m_2$ ) as mass unit, the rotating period equal to  $2\pi$ , the semi-major axis of the orbit of the primaries as distance unit, so the Gaussian constant becomes 1. Then the equation of motion of the massless body is

$$\ddot{z} = -\frac{1}{r^3}z, \quad (1)$$

where

$$r = \sqrt{R^2 + z^2}, \quad R = 1 - e \cos E. \quad (2)$$

$R$  is the distance between the primaries,  $z$  is the distance of the massless body from the plane of the primaries,  $e$  is the eccentricity, and  $E$  is the eccentric anomaly, which depends on the time according to Kepler's equation:

$$t - \tau = E - e \sin E. \quad (3)$$

The  $\tau = 0$  phase constant corresponds to the pericenter passage at  $t = 0$ . Since the problem is only one degree of freedom, we can introduce the true anomaly  $v$  as for independent variable instead of the time. (See [4].)

## 3. Structure of the phase space

We studied the Sitnikov problem for different initial conditions. On the phase portraits we plotted many trajectories corresponding to different initial condi-

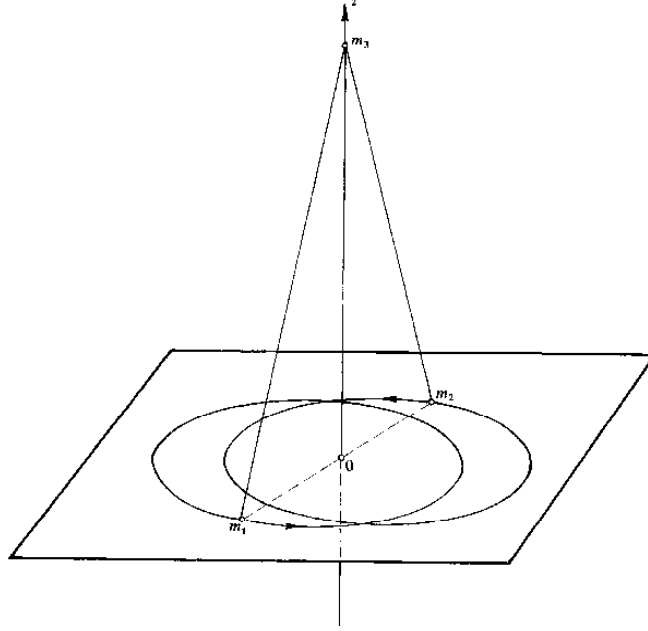


Figure 1. The Sitnikov problem.

tions. The set of initial conditions was  $z_0 = 0.15 - 1.8$  with  $\Delta z = 0.05$ , and the initial velocities were  $\dot{z} = 0$  in all cases. We chose the integration time to be 10000 periods of the primaries.

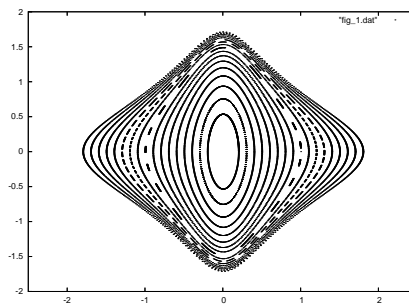


Figure 2. Invariant curves in the circular case of the Sitnikov problem. There are 17 initial conditions,  $z_0 = 0.2 - 1.8$ ,  $\Delta z = 0.1$ .

The circular case is equivalent to the two center problem, which was solved already by Euler in 1764. In this case, when the third mass has bounded motion,

the solutions are periodic or quasi-periodic depending on the initial conditions. The trajectories corresponding to the latter give close curves on a convenient surface of section in the phase space. Such curves are shown in Figure 2.

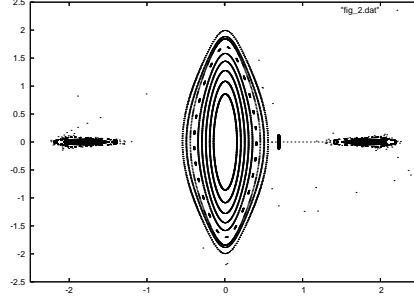


Figure 3.  $e = 0.4$  and  $v_0 = 0^\circ$  (pericenter passage). The islands outside the invariant curves correspond to the 1:1 and 2:1 mean motion resonances. Between the islands there are escaping trajectories.

In the eccentric case we have more various phenomena in the phase space. It is well known that increasing the parameter  $e$  the structure of the  $z - \dot{z}$  space is also changing (Kallrath et al., [4]). For initial conditions close to the plane of the primaries the solutions are quasi-periodic motions represented by invariant curves on the surface of section (Fig. 3). However, small islands appear for particular initial distances outside these invariant curves. These formations correspond to resonances with the primaries. The massless body escapes from the system in the region between the islands (Fig. 3).

In this paper we investigated the changes of the phase portraits when the SOS are not in the pericenter. We calculated the motions for four positions of the Surfaces of Sections  $v_0 = 45^\circ, 90^\circ, 135^\circ, 180^\circ$ . Figures 4–6 show the results. In Fig. 4 the eccentricity of the binary was 0.1. It can be seen that the 2:1 mean motion resonance (the two islands in the Fig. 3) remains in all cases, only the shape of the islands has changed. For example in the case  $v_0 = 45^\circ$  the islands moved from their initial positions (Fig. 4 top left). In addition, chaotic motion appears close to the separatrices as well. In Figure 4 the bottom right panel shows a quite distinct picture, but the islands are there. On the fourth panel the structure of the phase portrait is again symmetrical.

Fig. 5 shows the islands that correspond to 2:1 resonance ( $e=0.6$ ). The inner region of the phase plane has changed, because of the higher eccentricity. The invariant rotational curves disappeared, and the size of the tori decreased due to the growing of the perturbation parameter.

The four panels in Figure 6 where  $e = 0.6$  are similar to those of Figure 4. We can see the same behaviour (the rotation of the islands) after the bifurcation.

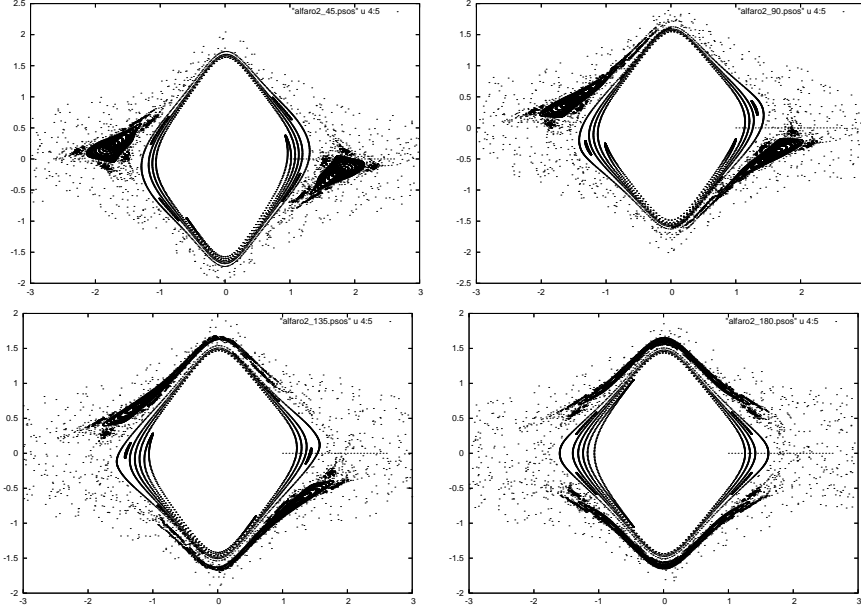


Figure 4.  $e = 0.1$ . The figures show that the periodic orbits and the tori around them turn round the centre while the position of the surface of section is changing. The panels correspond to different positions of SOS: Top left panel:  $v_0 = 45^\circ$ , top right panel:  $v_0 = 90^\circ$ , bottom left panel:  $v_0 = 135^\circ$ , bottom right panel:  $v_0 = 180^\circ$ . We can see that the islands do not disappear, only their form has changed.

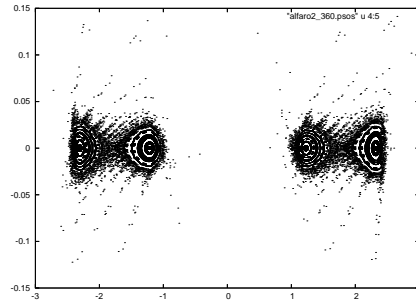


Figure 5.  $e = 0.6$ . After a critical value of the eccentricity  $e_c = 0.54325$  the stable fixed point becomes unstable. In this moment two stable island appears next to the original position of the periodic orbit.

#### 4. Concluding remarks

We investigated the phase space of the Sitnikov problem for different initial conditions. Four initial positions of the primaries were studied beside different

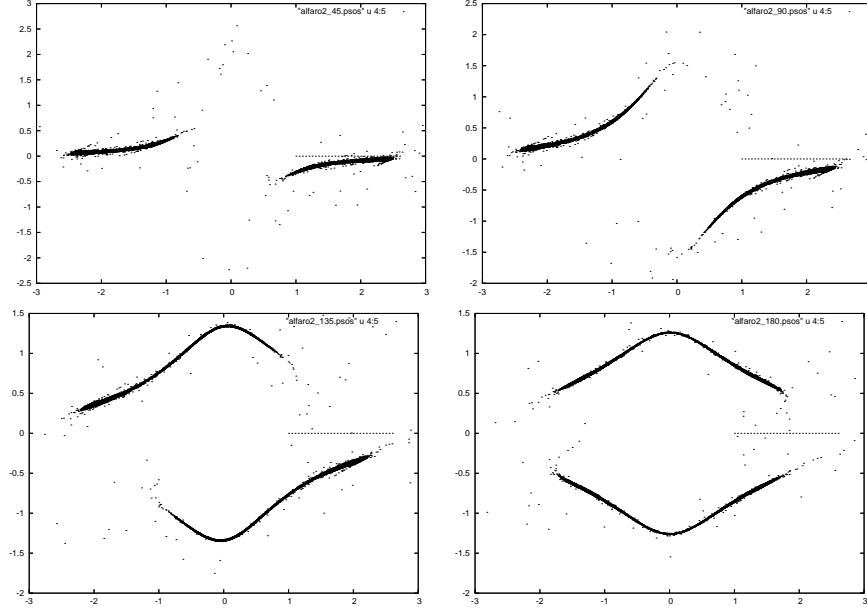


Figure 6. The eccentricity of the primaries is 0.6. In this case the invariant curves are confined to a smaller part of the phase portrait as in the case  $e = 0.15$ . The panels show the moving of the islands that correspond to the 2:1 resonance. Top left panel:  $v_0 = 45^\circ$ , top right panel:  $v_0 = 90^\circ$ , bottom left panel:  $v_0 = 135^\circ$ , bottom right panel:  $v_0 = 180^\circ$ .

positions of the SOS. There are closed curves on the surfaces of section corresponding to quasi-periodic orbits and small islands which mean resonances. These small islands break up with varying the eccentricity, or higher order resonances appear. It is important to note that varying the positions of the surfaces of sections the topology of the phase portrait do not change. The islands remain they only turn round the centre, their area is constant in time.

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