On equilibrium stability in the Sitnikov problem

Article *in* Cosmic Research · December 2011

DOI: 10.1134/S0010952511060049

P. S. Krasil'nikov
Moscow Aviation Institute

ISSN 0010-9525, Cosmic Research, 2011, Vol. 49, No. 6, pp. 534–537. © Pleiades Publishing, Ltd., 2011.
Original Russian Text © V.O. Kalas, P.S. Krasil'nikov, 2011, published in Kosmicheskie Issledovaniya, 2011, Vol. 49, No. 6, pp. 551–554.

On Equilibrium Stability in the Sitnikov Problem

V. O. Kalas and P. S. Krasil'nikov
Moscow Aviation Institute, Volokolamskoe shosse 4, Moscow 125993, Russia
Received March 25, 2010

Abstract—The problem of stability of the trivial equilibrium position in the Sitnikov problem is considered in the first approximation. The first approximation is shown to have the form of a linear second-order equation with time-periodic coefficient (the Hill-type equation). The equilibrium stability was studied on the basis of equation regularization in the vicinity of a singular point with subsequent calculation of the trace a of the monodromy matrix. The equilibrium stability is shown to be stable for almost all values of eccentricity e from the [0, 1] interval. The instability takes place on the discrete set of e values, when the mutipliers are multiple (with non-simple elementary divisors), $e = 1$ being a point of crowding of this set.

DOI: 10.1134/S0010952511060049

Let us consider the problem of stability of equilibrium of a passively gravitating point, which is located in the field of attraction of two heavy bodies of equal mass. The point of mass m is supposed to move along the Oz axis passing through the center of mass of attracting bodies perpendicular to the plane of their motion. The one-dimensional motion along the Oz axis is possible by virtue of the problem symmetry. The mass of a gravitating point is considered to be much less than the mass of attracting bodies; therefore, its effect on the motion of basic bodies can be neglected (the restricted three-body problem). We investigate the case, when the relative trajectory of motion of basic bodies represents a Keplerian ellipse with eccentricity e.

The equations of motion of a passively gravitating point along the z axis are well-known:

$$\ddot{z} + \frac{z}{\left(z^2 + r^2(t)\right)^{3/2}} = 0, \quad 2r = a\left(1 - e\cos E\right). \tag{1}$$

Here, r is a half of the distance between the bodies m_1, m_2; E is the eccentric anomaly, e is eccentricity of the orbit, and a is its semimajor axis. The measurement units are chosen so that $m_1 = m_2 = 1$, $a = 1$; in this case, period T of revolution of the basic bodies over the orbit equals 2π.

The anomaly E depends on time t on the strength of the Kepler equation

$$E - e\sin E = n\left(t - \tau\right), \tag{2}$$

where τ is one of Keplerian elements of an elliptical orbit, which represents the time of passing through the pericenter, $n = 2\pi/T$ is the mean motion of basic bodies. Without loss of generality we suppose that $\tau = 0$; in this case $n = 1$ by virtue of choice of measurement units.

It is known that, historically, the Sitnikov problem is associated with the problem of classifying the final motions in the three-body problem. The complete classification of types of final motions was given by Chazy [1] who postulated the existence of oscillating solutions, to which corresponded the unlimited oscillations of coordinate z, provided that z did not tend to infinity with time. In 1954 A.N. Kolmogorov proposed to study a particular case of the three-body problem for studying the topology of some phase space's subsets generating various types of final motions. K.A. Sitnikov [2] has proven, for this case, the existence of oscillating solutions, for which the coordinate z undergoes infinite number of overshoots for arbitrarily long distances; however, it always returns to the origin of coordinates. Later, V.M. Alexeev [3] has studied chaotic motions in the Sitnikov problem by using methods of symbolic dynamics and has shown that, under certain conditions, all possible combinations of Chazy final motions are realized in this problem. He has also proven the existence of oscillating and hyperbolic-elliptical solutions, which change their final type due to the "full capture" phenomenon. J. Mozer has acquainted the western researchers with the Sitnikov problem when delivering lectures on celestial mechanics at the Princeton University in early 1970s and publishing the results of his investigations into this problem in monograph [4]. Subsequently, a lot of works devoted to the Sitnikov problem has appeared, predominantly written by foreign authors. For example, the regular (periodic) orbits were studied in papers [5–9]; papers [10–13] were devoted to the issues of chaotic dynamics. Paper [16] was dedicated to stability of equilibrium of an elliptical restricted many-body problem of the "Sitnikov's problem" type for the case of small eccentricity values.

Consider now the problem of stability of the trivial solution $z = 0$ of Eq. (1), (2) in the first approximation. For this purpose we linearize Eq. (1) in the vicinity of zero and, representing $r(t)$ in the form of a series in terms of eccentricity e, we make use of methods of parametric resonance theory. For this purpose we represent the solutions $E(t)$ to the Kepler equation in the form of a Lagrange series converging for $e \le 0.662...$:

$$E(e,\varsigma) = \varsigma + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^{n-1}\left(\sin^n \varsigma\right)}{d\varsigma^{n-1}} e^n, \quad \varsigma = n(t - \tau). \quad (3)$$

The first-approximation equation has the following form

$$\ddot{z} + \left(1/r^3\right) z = 0. \quad (4)$$

Using formula (3), let us represent the coefficient of z in the form of a series in terms of eccentricity and, retaining terms to the third order of smallness in e inclusive, we reduce Eq. (4) to the Hill-type equation:

$$\ddot{z} + \left(\omega^2 + p(t,e)\right) z = 0. \quad (5)$$

Here

$$\omega = \sqrt{8 + 12e^2},$$
$$p(t,e) = 24e\cos t + (36\cos 2t)e^2 + (27\cos t + 53\cos 3t)e^3.$$

The study of stability of the equilibrium of Eq. (5) on the basis of concepts of the classical theory of parametric resonance, where the disturbances are represented by a series in terms of a small parameter, is trusrworthy only for sufficiently small eccentricity values, since any rejected term of the order of e^k has essential influence the behavior of a stability boundary.

To study the stability problem for any e values from the [0, 1) interval, Eq. (4) should be reduced to the form

$$\ddot{z} + \frac{8}{(1 - e\cos E)^3} z = 0, \qquad \dot{E} = \frac{1}{1 - e\cos E}. \quad (6)$$

As was shown by Lyapunov [15], the characteristic equation of system (6) (we consider E as a periodic function of t on the strength of the second equation) has the form $\rho^2 + a(e)\rho + 1 = 0$; in this case, the trivial equilibrium $z = 0$ is unstable, if the coefficient $a(e)$, representing the trace of the monodromy matrix $Z(2\pi)$, satisfies the inequality $|a(e)| > 2$, and it is stable for $|a(e)| < 2$. If $|a(e)| = 2$, then the system's multipliers are real, multiple, and equal to unity in absolute value. In this case, the equilibrium position is unstable according to the power law, if the elementary divisors are non-simple, and it is stable in case of simple divisors.

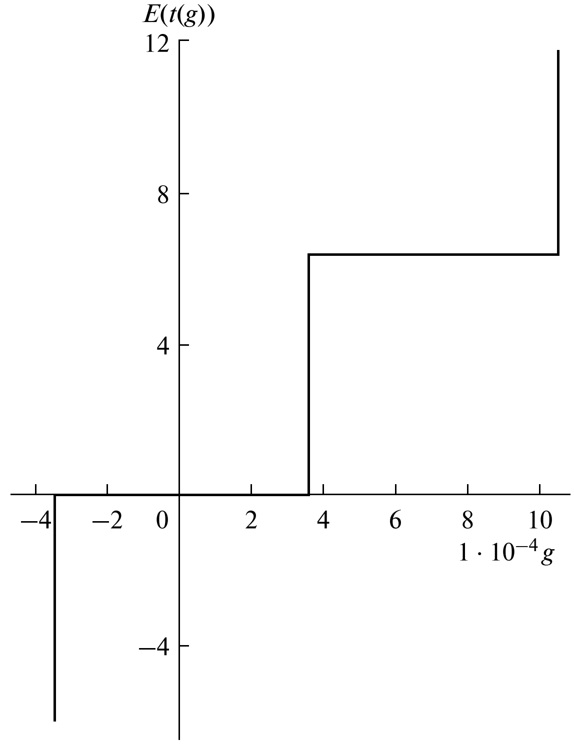

Fig. 1

Since the limiting value $e = 1$ is a singular parameter of system (6) (its right-hand side has a discontinuity at $E = 2\pi k$), it is necessary to perform regularization of Eqs. (6) in the vicinity of $e = 1$, $E = 2\pi k$). For this purpose we introduce (instead of t) dummy time g, so that the phase curve $x_1(g) = z(t(g))$, $x_2(g) = \dot{z}(t(g))$ would be a smooth function of parameter g on the strength of smoothness of right-hand sides of transformed equations. We let

$$g = \int_0^t \frac{dt}{(1 - e\cos E)^3}. \quad (7)$$

Equations (6) will take on the form

$$\frac{dx_1}{dg} = (1 - e\cos\Sigma)^3 x_2, \quad \frac{dx_2}{dg} = -8x_1, \qquad \frac{d\Sigma}{dg} = (1 - e\cos\Sigma)^2, \quad (8)$$

where $\Sigma(g) = E(t(g))$.

Figure 1 presents the plot of the dependence $\Sigma(g)$ for $e = 0.999$, $E(0) = 0$. The plot of function $t = t(g)$ determined by equality (2) has similar appearance. It follows from this fact that, during the long interval of variation of dummy time g, the eccentric anomaly $\Sigma(g)$ retains the values close to zero or to $2\pi k$, and the transition from one "step" to another occurs in a very short time interval. Calculations have shown that during the time of "jump" of function $\Sigma(g)$ quantities $x_1(g)$ and

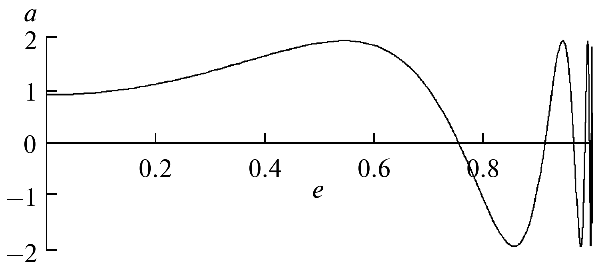

Fig. 2

$x_2(g)$ quickly oscillate, while during the remaining time they execute slow oscillations. Such a behavior of phase variables is explained by the effect of time slowdown in the vicinity of singular points $E = 2\pi k$.

As parameter e approaches unity, the value of duration of steps $\Sigma = 0, \pm 2\pi, \pm 4\pi\ldots$ sharply grows, and in the limit, when $e = 1$, function $\Sigma = \Sigma(g)$ (accordingly, $t = t(g)$) becomes many-valued one, every branch of which looks like an infinite "step" contained between two horizontal asymptotes $\Sigma = 2\pi(k-1)$ and $\Sigma = 2\pi k$.

We have $a(e) = z^{(1)}(2\pi) + \dot{z}^{(2)}(2\pi)$, where $\left(z^{(k)}, \dot{z}^{(k)}\right)$, $k = 1,2$ is the fundamental normal system of solutions to Eqs. (6). It follows from this relation, that

$$a(e) = x_1^{(1)}(g^*) + x_2^{(2)}(g^*) \tag{9}$$

under the condition that $t(g^*) = 2\pi$. It is seen from formula (7), that

$$g^*(e) = \int_0^{2\pi} \frac{dt}{(1 - e\cos E)^3} = \int_0^{2\pi} \frac{dE}{(1 - e\cos E)^2}.$$

By virtue of the obvious equality $t(g + g^*) = t(g) + 2\pi$ we have $\Sigma(g + g^*) = \Sigma(g) + 2\pi$, therefore, $g^*(e)$ is the width of a "step". Letting the initial conditions to be equal to

$$x_1^{(1)}(0) = 1,\quad x_2^{(1)}(0) = 0,\quad E(0) = 0,$$
$$x_1^{(2)}(0) = 0,\quad x_2^{(2)}(0) = 1,\quad E(0) = 0,$$

we find the normal fundamental system of solutions $x^{(1)}(g)$, $x^{(2)}(g)$ to Eqs. (8), after which we calculate the trace $a(e)$ of the monodromy matrix

$$X(g^*) = \|x_{ij}(g^*)\|_{i,j=1}^2,\quad x_{ij}(g^*) = x_i^{(j)}(g^*)$$

on the basis of formula (9). The results of calculations of $a(e)$ are given in Fig. 2.

It follows from this figure that the equilibrium $z = 0$ is stable for almost all e values excluding the discrete set of zeroes of the equation

$$|a(e)| = 2. \tag{10}$$

The value $e = 1$ is a limiting point of this set, function $a(e)$ has singularity at $e = 1$, it behaves itself like the function $2\sin(1/(1-e))$.

Now let us demonstrate that for the first nine terms of the infinite sequence of roots of Eq. (10) the equilibrium is unstable. Indeed, the approximate values of these roots are presented in the first line of the table.

The second and third lines of the table are filled with off-diagonal elements of the monodromy matrix. They are nonzero; as a result, on the strength of continuity of solutions to the Cauchy problem in parameter e, the off-diagonal elements of the monodromy matrix, corresponding to strict values of the roots of Eq. (10), will be nonzero as well. But this implies that matrix $X(g^*)$ has non-simple elementary divisors (simple elementary divisors exist only in the case, when $X(g^*) == \pm E$); therefore, the equilibrium is unstable.

Note that the equilibrium instability in the first approximation for "large values" of e was studied also in paper [16] on the basis of approximate analysis. It has been shown that for $e \approx 0.876551$ the instability of trivial equilibrium takes place, which represents an approximation for a more rigorous value $e = 0.855860$.

Calculations of the fundamental matrix of solutions were performed on the basis of the rosenbrock technique to an accuracy of $1 \cdot 10^{-17}$, quantity $a(e)$ was calculated to an accuracy of the order of $1 \cdot 10^{-7}$. The sign of quantity $a(e)$ alternates sequentially beginning with a positive one.

Since the presence of non-simple elementary divisors represents the case of a general situation, one should expect that the trivial equilibrium instability takes place also for the remaining infinite sequence of zeroes of Eq. (10) (the rigorous substantiation of this statement requires to prove the multiplicity of multipliers $\rho_1 = \rho_2 = \pm 1$ with respect to elementary divisors for the Hill-type equation). Note also that the stability analysis in the non-linear approximation demands separate consideration.

Finally, we note that the conclusion of paper [17] about the trivial equilibrium stability in the linear approximation for any $e \in [0, 1)$ is erroneous.

Table

e	0.544880	0.85860	0.944770	0.977520	0.990605	0.996021	0.998305	0.999276	0.999690
$x_{12}(g^*)$	0.000210	$0.720807 \cdot 10^{-6}$	$0.514462 \cdot 10^{-7}$	$0.270881 \cdot 10^{-6}$	$0.123914 \cdot 10^{-6}$	$0.556056 \cdot 10^{-7}$	$-0.57712 \cdot 10^{-9}$	$-0.13619 \cdot 10^{-7}$	$0.12155 \cdot 10^{-8}$
$x_{21}(g^*)$	−0.01878	−0.00384	−0.00267	−0.26457	−1.31505	−8.58158	1.044799	305.5066	−359.381

REFERENCES

1. Chazy, J., Sur l'Allure Final du Mouvement dans le Probleme des Trios Corps Quant le Temps Croit Indefiniment, *Annales de l'Ecole Norm. Sup. 3, ser.* 1922, vol. 39, pp. 29–130.
2. Sitnikov, K.A., Existence of Oscillating Motions in the Three-Body Problem, *Dokl. Akad. Nauk*, 1960, vol. 133, no. 2, pp. 303–306.
3. Alekseev, V.M., Quasi-Random Dynamical Systems I, II, III, *Mat. Sb.*, 1968, vol. 76, no. 1, pp. 72–134; 1968, vol. 77, no. 4, pp. 545–601; 1969, vol. 78, no. 1, pp. 3–50.
4. Moser, J., *Stable and Random Motions in Dynamical Systems*, NJ: Princeton University Press, 1973.
5. Belbruno, E., Llibre, J., and Olle, M., On the Families of Periodic Orbits which Bifurcate from the Circular Sitnikov Motions, *Celestial Mech. and Dynam. Astronom.*, 1994, no. 60, pp. 99–129.
6. Corbera, M. and Llibre, J., Periodic Orbits of the Sitnikov Problem via a Poincare Map, *Celestial Mech. Dynam. Astronom.*, 2000, no. 77, pp. 273–303.
7. Jimenez-Lara, L. and Escalona-Buendia, A., Symmetries and Bifurcations in the Sitnikov Problem, *Celestial Mech. Dynam. Astronom.*, 2001, no. 79, pp. 97–117.
8. Llibre, J. and Ortega, R., On the Families of Periodic Orbits of the Sitnikov Problem, *SIAM J. Applied Dynamical Systems,* 2008, no. 7, pp. 561–576.
9. Hagel, J., A New Analytic Approach to the Sitnikov Problem, *Celes. Mech.,* 1992, no. 53, pp. 267–292.
10. Kovacs, T. and Erdi, B., The Structure of the Extended Phase Space of the Sitnikov Problem, *Astron. Nachr.*, 2007, AN 328, no. 8, pp. 801–804.
11. Liu, Jie and Sun, Yi-Sui., On the Sitnikov Problem, *Celes. Mech.*, 1990, no. 49, pp. 285–302.
12. Jalali, M.A. and Pourtakdoust, S.H., Regular and Chaotic Solutions of the Sitnikov Problem near the 3/2 Commensurability, *Celestial Mechanics and Dynamical Astronomy*, 1997, no. 68, pp. 151–162.
13. Robinson, Clark, Uniform Subharmonic Orbits for Sitnikov Problem, *Discrete and Continuous Dynamical Systems*, Series S, 2008, vol. 1, no. 4, pp. 647–652.
14. Merkin, D.R., *Vvedenie v teoriyu ustoichivosti dvizheniya* (Introduction to the Motion Stability Theory), Moscow: Nauka, 1976.
15. Lyapunov, A.M., *Obshchaya zadacha ob ustoichivosti dvizheniya* (General Problem of Motion Stability), Moscow–Leningrad: GITTL, 1950.
16. Prokopenya, A.N., Investigation of Stability of Equilibrium Solutions to Elliptical Restricted Many-Body Problem by Methods of Computer Algebra, *Math. Model.*, 2006, vol. 18, no. 10, pp. 102–112.
17. Tkhai, V.N., Periodic Motions in Reversible Mechanical System of the Second Order: Application to the Sitnikov Problem, *Prikl. Math. Mekh.*, 2006, vol. 70, no. 5, pp. 813–834.