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Three-Dimensional Models for the Distribution of Mass in Galaxies

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Abstract

From TOOMRE's (1963) disk models of galaxies, we have generalized other convenient pairs of functions to represent the axisymmetric potential and density of models in which the mass is no longer confined to a single plane. These new pairs of functions include some that vary continuously from KUZMIN's (1956) thin-disk model to PLUMMER's (1911) spherical model for globular clusters. The models are free from singularities and involve functions that are quite explicit and elementary. Even without superposing any of them, the present density laws seem to mimic remarkably well the stratifications of mass both in the central bulges and in the disk parts of real galaxies.

Key words: Galaxies; Potential theory; Three-dimensional models.

1. *Introduction*

Several authors have attempted to construct simple three-dimensional models for the mass distribution in flat galaxies as well as in our own Galaxy. Collective descriptions of the older published models are found in PEREK's (1962) and SCHMIDT's (1965) articles. Their authors usually derived spheroidal stratifications of matter, assuming centrifugal balance in the galactic plane only.

To reproduce the linear variation with radius of the centrifugal force in our Galaxy, SCHMIDT (1956, 1965) assumed a special form of the density distribution, which diverges, however, at the galactic center. To estimate the total mass of flat galaxies, BURBIDGE et al. (1959) expanded the density distribution into a Taylor series, which itself diverges at large distances from the galactic center when the series is truncated. These singularities were considered to be not serious, since the total mass of flat galaxies was of primary interest.

BRANDT (1960), BRANDT and BELTON (1962), and BELTON and BRANDT (1963) considered the rotational velocity together with other dynamical quantities and proposed a model of flat galaxies which has been extensively investigated by TAKASE and KINOSHITA (1967) and TAKASE (1967). The applicability of their approach is however limited to highly flattened galaxies. Moreover, according to the methods developed by BURBIDGE et al. (1959) [as well as BRANDT (1960), BRANDT and BELTON (1962), and BELTON and BRANDT (1963)], no dynamical quantities can be easily estimated outside the galactic plane, except for the density distribution.

The above shortcomings in many ways limit the theoretical works associated with the dynamics of flat galaxies. In the present paper we derive quite elementary expressions for the most basic quantities of interest, i.e. the gravitational potential and the density distribution. These expressions are not only free from singularities everywhere in space, but are also differentiable an unlimited number of times with respect to the space coordinates. The present expressions involve very few parameters: the dimension of the galaxy a, the galactic thickness b, and the galactic mass M. With these three parameters, it will be seen that our expressions indeed cover a wide range of axisymmetric configurations for galaxies. More importantly, these should also serve as the first-order approximations to be employed in constructing fairly realistic three-dimensional models for galaxies with non-circular (i.e., random) motions of stars.

2. *Generalization of KUZMIN's (1956) and PLUMMER's (1911) Models*

In investigating the luminosity distribution of the globular clusters, PLUMMER (1911) has used an elementary solution of the Emden equation. Such a solution for the gravitational potential $\Phi(r)$ and the mass distribution $\rho(r)$ in the spherical coordinate r can be written as

$$\left.\begin{aligned} \Phi(r) &= \frac{GM}{[r^2+b^2]^{1/2}}\,, \\ \rho(r) &= \frac{3b^2 M}{4\pi}\frac{1}{[r^2+b^2]^{5/2}}\,, \end{aligned}\right\} \tag{1}$$

where M is the total mass of the spherical system and b is a non-zero constant with the dimension of length. The above stellar-dynamic model is one of the simplest known to be free from singularities, and it corresponds to Schuster's gas sphere with the polytropic index $n=5$. LYNDEN-BELL (1962) attempted a generalization of this configuration to a flattened system; however, he found it to yield not a disk-like mass distribution, but a toroidal one.

On the other hand, to describe the mass distribution within highly flattened axisymmetric galaxies, TOOMRE (1963) has found a family of exact solutions for the Poisson equation

$$\nabla^2\Phi(R,z) = -4\pi G\rho(R,z) = -4\pi G\mu(R)\delta(z)\,, \tag{2}$$

where R and z denote two of the cylindrical coordinates and $\mu(R)$ and $\delta(z)$ denote the surface density and Dirac's δ-function respectively. The simplest solution of the family is given by

$$\left.\begin{aligned} \Phi(R,z) &= \frac{GM}{[R^2+(a+|z|)^2]^{1/2}}\,, \\ \rho(R,z) &= \mu(R)\delta(z) = \frac{aM}{2\pi}\frac{1}{[R^2+a^2]^{3/2}}\delta(z)\,, \end{aligned}\right\} \tag{3}$$

where M is the total mass of the disk-like system and a is a non-zero constant with the dimension of length. It was pointed out to us by TOOMRE (1975) that the so-called "TOOMRE's (1963) model 1" for the above simplest density-potential relationship should really be called "KUZMIN's (1956) model." Henceforth, we will indeed refer to it as KUZMIN's (1956) model, when meaning specifically the pair of functions in equations (3). In any case, it is for this mass distribution that MIYAMOTO (1971) has already given an exact solution of the Liouville equation.

One now finds that, except for $|z|$ and $\delta(z)$, there appear many formal similarities between the solutions (1) and (3) for the spherical and disk-like systems respectively. The occurrence of these similarities suggests that there may be a way to derive some intermediate states between the above two extreme states of the mass distribution. This can be achieved by replacing z^2+b^2 in $\Phi(r)=\Phi(R^2+z^2)$ of the formulae (1), or $(a+|z|)^2$ in $\Phi(R,z)$ of the formulae (3), with the expression $[a+(z^2+b^2)^{1/2}]^2$. Then we obtain the following gravitational potential generalizing both expressions (1) and (3):

$$\Phi(R,z)=\frac{GM}{\{R^2+[a+(z^2+b^2)^{1/2}]^2\}^{1/2}}\,. \tag{4}$$

It is free from singularities everywhere in space and tends to the Newtonian potential when R and z become large. Corresponding to this gravitational potential, the three-dimensional density distribution $\rho(R,z)$ can be easily derived from the Poisson equation (2) as

$$\rho(R,z)=\frac{b^2M}{4\pi}\,\frac{aR^2+[a+3(z^2+b^2)^{1/2}][a+(z^2+b^2)^{1/2}]^2}{\{R^2+[a+(z^2+b^2)^{1/2}]^2\}^{5/2}(z^2+b^2)^{3/2}}\,, \tag{5}$$

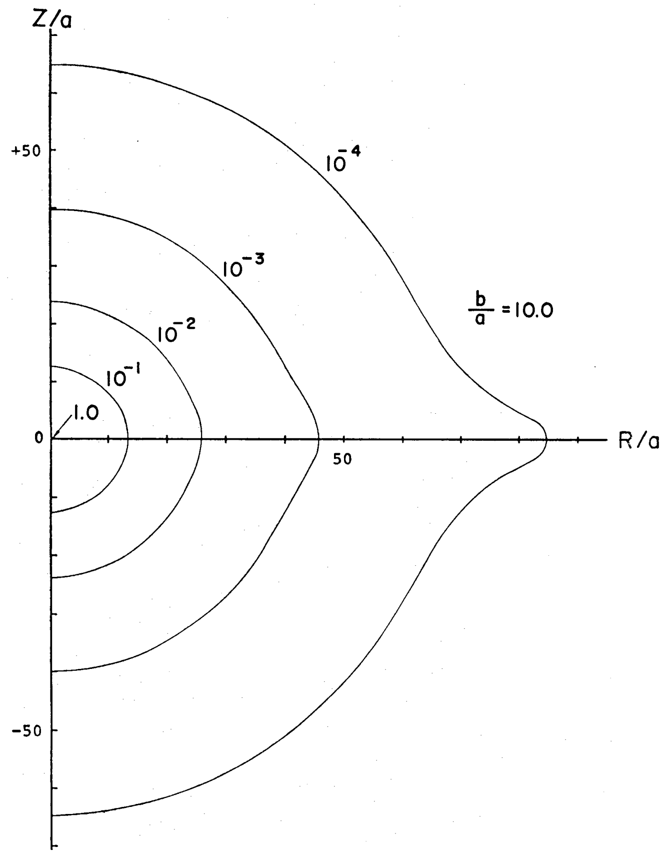
which is non-negative everywhere in space.

It can be easily seen that the gravitational potential and the density given by equations (4) and (5) tend to the solutions (1) and (3) as $a\to 0$ and $b\to 0$ respectively. Furthermore, the function (5) shows that the stratification of matter is oblate rather than prolate, since $\rho(x,0)>\rho(0,x)$ for $a\neq 0$. But that mass distribution is generally not spheroidal, except near the galactic center, where the density field (5) tends to an oblate spheroidal stratification. It is particularly interesting that all these intermediate models have the same radial force law, and hence the same rotation curve, in the plane $z=0$ provided $a+b=\text{const.}$ Of course this statement applies even to the pure KUZMIN (1956) and PLUMMER (1911) models at each extreme.

From the above pair of potential and density functions it is possible to derive a multitude of other exact pairs of solutions for the Poisson equation (2). Since the Poisson equation (2) is linear with respect to $\Phi(R,z)$ and $\rho(R,z)$, the matching equations (4) or (5) can be summed over any set of values for the parameters a, b, and M. However, such a summation seems rarely necessary, since even the single density function (5) can already reproduce the stratifications in the central bulge as well as in the disk part of galaxies, as is shown in the next section.

3. *Density Distribution of the Generalized* KUZMIN *(1956) Model*

Hereafter we shall refer to the pair of functions (4) and (5) as the generalized KUZMIN (1956) model. We demonstrate here in detail the stratification of mass which it implies. Figures 1a–e show meridional sections of the density field for various values of the ratio b/a. In each of figures 1a–e, the unit of length is a and the density has been normalized to unity at the center. As b/a decreases, these mass distributions obviously become flatter and flatter, so that b/a seems a


Z/a
+50
-50
0
50
R/a
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10^{-3}
10^{-2}
10^{-1}
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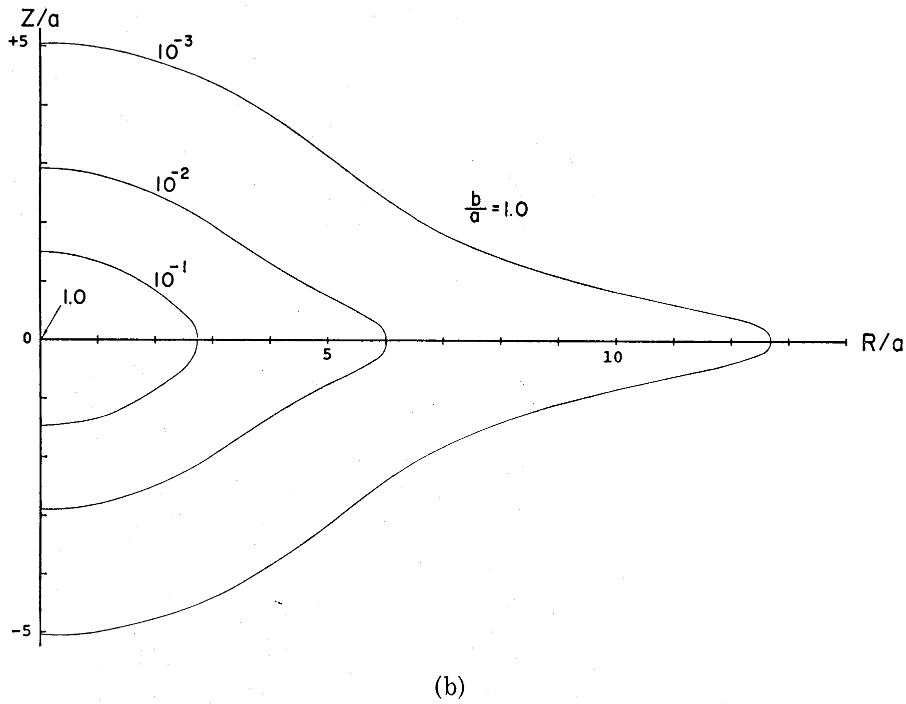
$\frac{b}{a}=10.0$

(a)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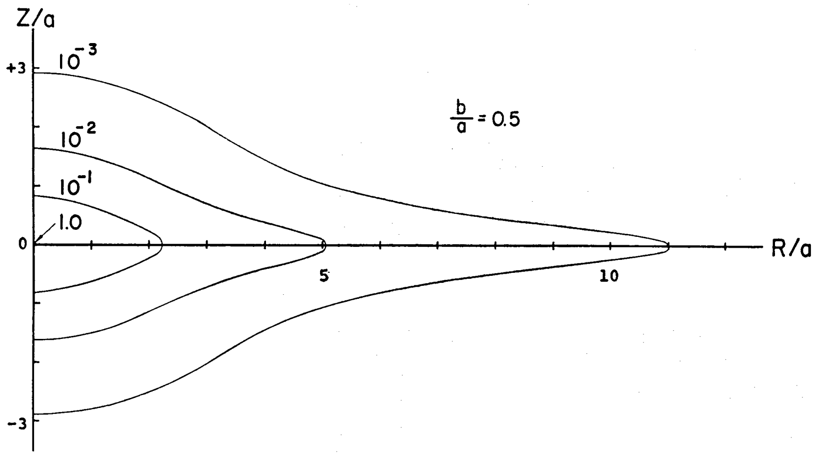
(b)



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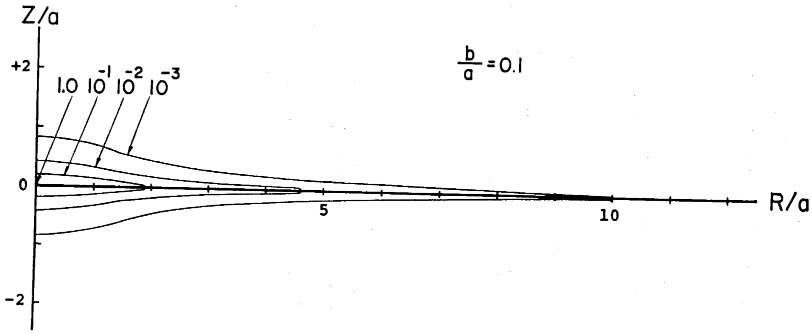

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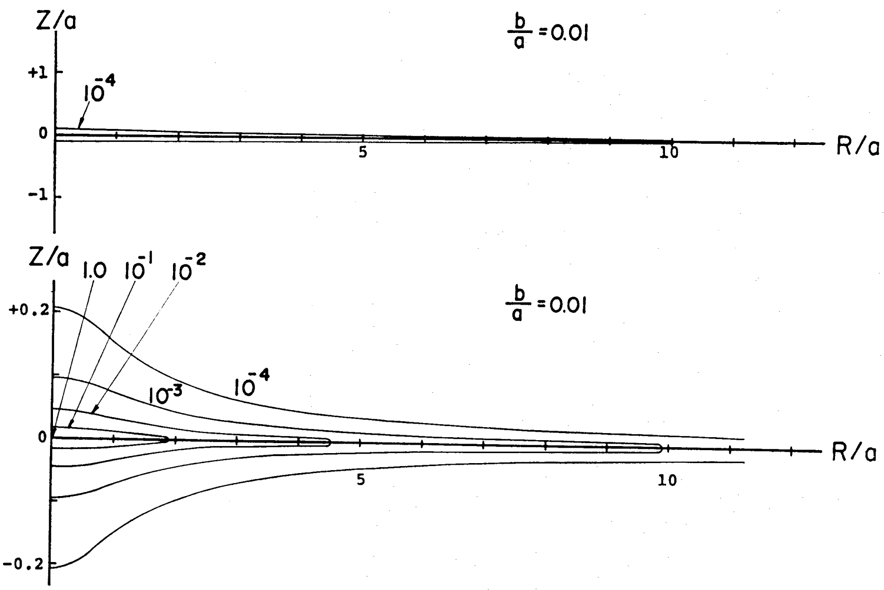

(e)

Fig. 1. Meridional sections of the stratifications of mass for the generalized KUZMIN (1956) model. Various values of the ratio b/a, which is a measure of the flatness in the present model, are indicated in the respective figures. The density is expressed as multiples of the central density. In figure 1e, the z-scale has been magnified by a factor of 10 in the lower meridional section to show the stratification.

suitable measure of flatness in the present models. More importantly, the central bulge plus surrounding disk character of observed edge-on galaxies seems to be imitated surprisingly well by the density function (5).

It is of interest to compare the vertically integrated density $\rho_*(R)$ of these generalized KUZMIN's (1956) models with the corresponding surface density $\mu(R)$ of KUZMIN's (1956) extremely thin model. The comparison is made in figure 2, where the units of length and density are a and M/a^2 respectively. In figures 1a-e it is seen that the radial dimension of the generalized KUZMIN (1956) models increases as the ratio b/a increases. Therefore, this character of the models reflects on the vertically integrated density, in such a manner that for large radial distances R, $\rho_*(R)$ is larger than $\mu(R)$ and for small R the reverse is true, provided M and a are fixed.

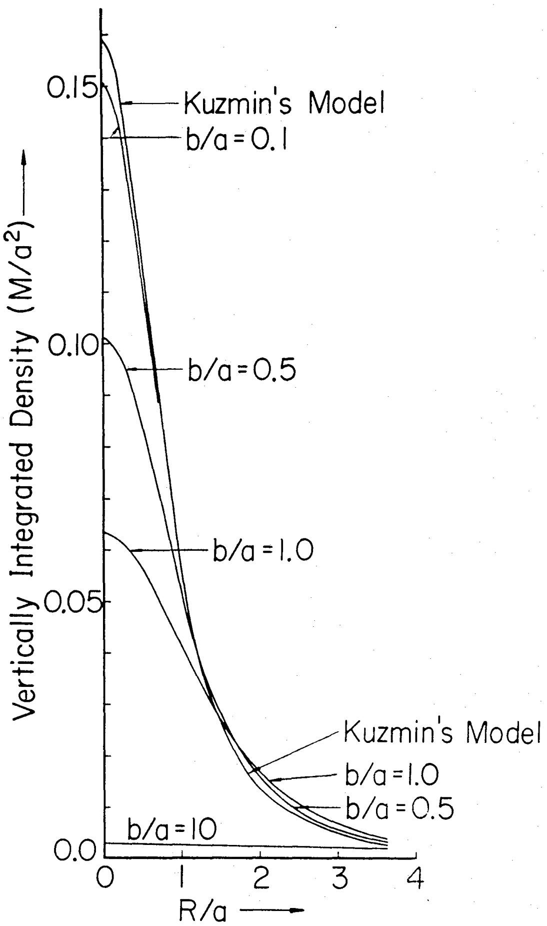

Fig. 2. Vertically integrated densities of generalized KUZMIN (1956) models and the corresponding surface density of KUZMIN's (1956) extremely thin model.

Of course, all the dynamical quantities in these models can be deduced quite easily from the basic formulae (4) and (5). What we wish to stress here is that just the single density function (5), which is quite simple, already simulates the stratifications of mass both in the central bulges and in the disk parts of real galaxies. By contrast, according to previously proposed methods summarized by PEREK (1962) and SCHMIDT (1965), at least two non-homogeneous spheroids with different oblateness would have had to be superposed.

4. *Generalization of TOOMRE's (1963) Other Models*

Similarly, by replacing $(a+|z|)$ in the potential function $\Phi(R,z)=\Phi[R,(a+|z|)]$ of TOOMRE's (1963) other disk-like models with $[a+(z^2+b^2)^{1/2}]$, it is possible to inflate them. For example, TOOMRE's (1963) models 2 and 3 are described by

$$\left.\begin{aligned}\Phi(R,z)&=\frac{GM}{[R^2+(a+|z|)^2]^{1/2}}\left\{1+\frac{a(a+|z|)}{R^2+(a+|z|)^2}\right\},\\ \rho(R,z)&=\frac{3a^3M}{2\pi}\frac{1}{(R^2+a^2)^{5/2}}\delta(z),\end{aligned}\right\}\qquad(6)$$

and

$$\left.\begin{aligned}\Phi(R,z)&=\frac{GM}{[R^2+(a+|z|)^2]^{1/2}}\left[1+\frac{a(a+|z|)}{R^2+(a+|z|)^2}-\frac{1}{3}\frac{a^2[R^2-2(a+|z|)^2]}{[R^2+(a+|z|)^2]^2}\right],\\ \rho(R,z)&=\frac{5a^5M}{2\pi}\frac{1}{(R^2+a^2)^{7/2}}\delta(z)\ ,\end{aligned}\right\}\qquad (7)$$

respectively. Corresponding to the above pairs of functions we have the following three-dimensional density functions, which are again non-negative everywhere in space and free from singularities:

$$\rho(R,z)=\frac{3b^2M}{4\pi}\frac{1}{[R^2+(\zeta+a)^2]^{7/2}\zeta^3}[R^2(\zeta+a)(\zeta^2-a\zeta+a^2)+(\zeta+a)^3(\zeta^2+4a\zeta+a^2)] \qquad (8)$$

and

$$\begin{aligned}\rho(R,z)=\frac{b^2M}{4\pi}&\frac{1}{[R^2+(\zeta+a)^2]^{9/2}\zeta^3}[3R^4\zeta^3+R^2(\zeta+a)^2(6\zeta^3+15a\zeta^2-10a^2\zeta+5a^3)\\ &+(\zeta+a)^4(3\zeta^3+15a\zeta^2+25a^2\zeta+5a^3)]\ ,\end{aligned} \qquad (9)$$

where $\zeta=(z^2+b^2)^{1/2}$. The above density functions (8) and (9) again yield the oblate stratifications of matter, since $\rho(x,0)>\rho(0,x)$ for $a\neq 0$. But these are again not spheroidal except near the galactic center. The stratifications which these generalized TOOMRE (1963) models 2 and 3 imply are not illustrated here, since they show roughly similar density fields (the central bulge plus the disk part) as the generalized KUZMIN (1956) model. It can be shown that the vertically integrated densities of these generalized TOOMRE's (1963) models are larger than the corresponding surface densities of TOOMRE's (1963) extremely thin models for large radial distances R, and for small R the reverse is true, just as with the generalized KUZMIN (1956) model.

In a similar fashion, it is possible to deduce easily all the higher order pairs of three-dimensional potential and density functions from the higher order disk-like models given by TOOMRE (1963). However, no mathematical proof is given here that all the higher order three-dimensional density functions $\rho(R,z)$'s are always non-negative. The proof will be given in a forthcoming paper. Anyhow, the above three pairs of functions, deduced from TOOMRE's (1963) models 1-3, seem to be sufficient for practical purposes, since suitable summations of the above density functions can cover a wide range of the stratification in real galaxies. It should be noticed that any pair of functions, which should be deduced from these TOOMRE's (1963) models, tends to PLUMMER's (1911) pair of functions (1) as $a\rightarrow 0$.

5. *Discussion*

By replacing $(a+|z|)$ in TOOMRE's (1963) potential functions for completely flattened mass distributions with $[a+(z^2+b^2)^{1/2}]$, we have generalized other convenient functions to represent the axisymmetric potential and density of three-dimensional models. This replacement to inflate TOOMRE's (1963) disk models seems to be the simplest one. However, another class of replacements is supposedly also obtainable to bridge the gap between TOOMRE's (1963) disk models and the class of spherical systems, for example, described by the Emden equation. By such a generalization it seems to be possible to reproduce more general axisymmetric mass distributions of astrophysical interest as well.

In the forthcoming paper it will be shown that in particular the simplest pair of functions (4) and (5) is very powerful to describe three-dimensional axisymmetric stellar system with random motions of stars. As an example of the utility of the pair of functions (4) and (5) we have examined a simple model for the mass distribution of our Galaxy in the Appendix.

We would like to acknowledge Professor A. Toomre for several valuable suggestions and for very constructive criticisms of this paper. We are also very grateful to Professors S. Aoki and B. Takase, who have encouraged us constantly on this work.

Appendix. *A Model for the Mass Distribution of Our Galaxy*

As an example, we apply here the pair of functions (4) and (5) to our Galaxy and examine a simple model for the mass distribution of the Galaxy. We assume the centrifugal balance only in the galactic plane; outside the galactic plane no kinds of equilibria are considered (a "pressure-supported" equilibrium in the z-direction will be described in our forthcoming papers). As is well known, the galactic rotation curve has two peaks, at $R \sim 0.5$ kpc and $R \sim 9$ kpc (see SCHMIDT 1965). Therefore, to reproduce this feature (not to reproduce the central bulge plus the disk part!) we need two pairs of functions (4) and (5) with different sets of parameters (a_1, b_1, M_1) and (a_2, b_2, M_2), as has been attempted by MIYAMOTO (1974, 1975) in constructing an extremely thin galactic model with random motions of stars.

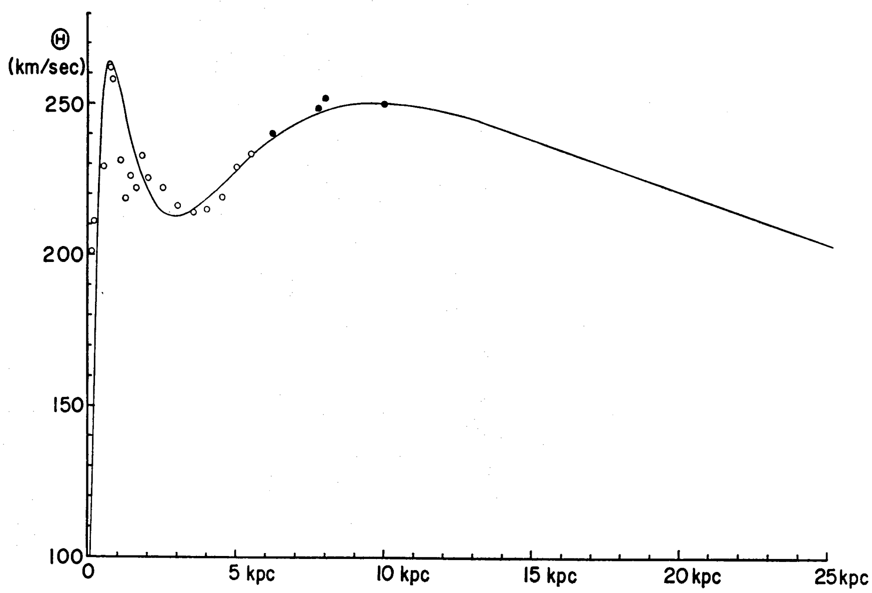

Fig. A1. The variation of the circular velocity in our Galaxy with respect to R, which is represented by equation (A1) with the set of parameters (A3). The filled circles show observational data given by SCHMIDT (1965) and the open circles are suitably chosen from figure 1 of SIMONSON and MADER (1973).

Even in the present three-dimensional case, the circular velocity $\Theta(R,0)$ in the galactic plane, which is deduced from function (4), is quite simple (the same as in the case of disk models), and we have

$$\Theta(R,0)=R\left\{\frac{GM_1}{[R^2+(a_1+b_1)^2]^{3/2}}+\frac{GM_2}{[R^2+(a_2+b_2)^2]^{3/2}}\right\}^{1/2}. \tag{A1}$$

We first fix four parameters (a_1+b_1), M_1, (a_2+b_2), and M_2 so as to approximate the observational data given by SCHMIDT (1965) for $6\,\mathrm{kpc}<R<10\,\mathrm{kpc}$ and by SIMONSON and MADER (1973) for $R<6\,\mathrm{kpc}$, as closely as possible. The circular velocity $\Theta(R,0)$ thus obtained is illustrated in figure A1. The circular velocity $\Theta(R,0)$ takes the value of $250.0\,\mathrm{km\,s^{-1}}$ at $R=10\,\mathrm{kpc}$.

Next, we tentatively put $a_1=0$, since we have no reliable observation about the three-dimensional configuration of the galactic central part. Then, based on the formula

$$\rho(R,z)=\frac{1}{4\pi}\sum_{i=1}^{2}b_i^2M_i\,\frac{a_iR^2+[a_i+3(z^2+b_i^2)^{1/2}][a_i+(z^2+b_i^2)^{1/2}]^2}{\{R^2+[a_i+(z^2+b_i^2)^{1/2}]^2\}^{5/2}(z^2+b_i^2)^{3/2}}, \tag{A2}$$

it is possible to determine a_2 and b_2 separately so as to yield the value of $0.150M_\odot\,\mathrm{pc^{-3}}$ (OORT 1965) for the total mass density at the sun. All the parameters thus deduced are:

$$\left.\begin{aligned}a_1&=0.0\,\mathrm{kpc}\,,\\ b_1&=0.495\,\mathrm{kpc}\,,\\ M_1&=2.05\times10^{10}M_\odot\,,\\ a_2&=7.258\,\mathrm{kpc}\,,\\ b_2&=0.520\,\mathrm{kpc}\,,\\ M_2&=25.47\times10^{10}\,M_\odot\,.\end{aligned}\right\} \tag{A3}$$

For this set of parameters, a meridional section of the possible stratification of matter in our Galaxy is illustrated in figure A2. Based on this model, we

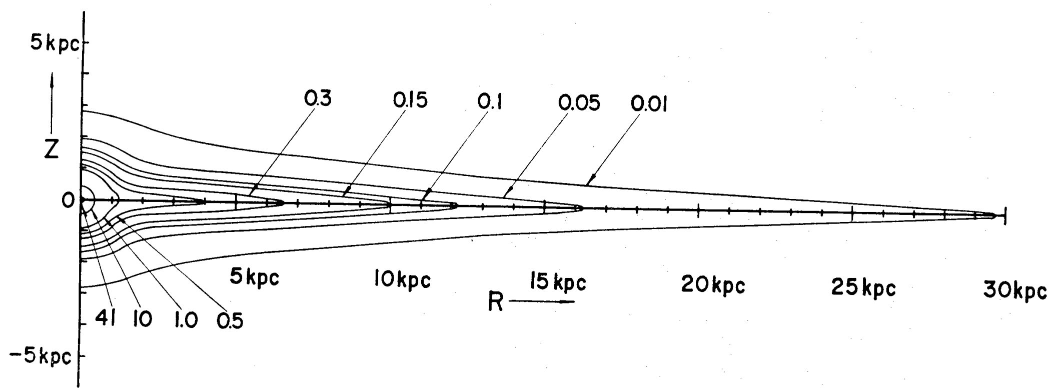

Fig. A2. A meridional section of a possible stratification of mass in our Galaxy. The stratification is drawn by using equation (A2) and parameters (A3). The unit of volume density is $M_\odot\,\mathrm{pc^{-3}}$. Values of the volume density at the sun and the galactic center are $0.150M_\odot\,\mathrm{pc^{-3}}$ and $41M_\odot\,\mathrm{pc^{-3}}$ respectively.

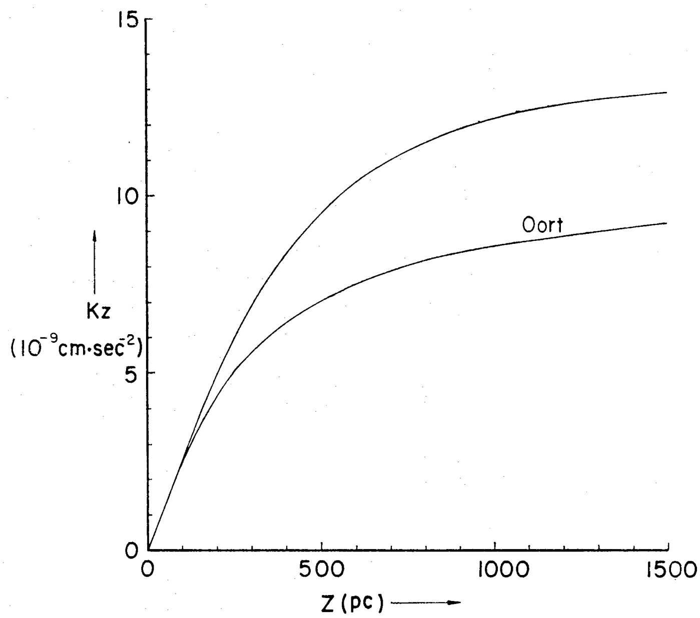

Fig. A3. The z-component of the gravitational force K_z given by equation (A4) and parameters (A3). The lower curve, derived by OORT (1965), has been added for comparison.

estimate the oblateness of the overall configuration to be about 1/10, which corresponds approximately to the value of b_2/a_2. Moreover, we can estimate the possible mass density at the galactic center to be $41M_\odot\,\mathrm{pc}^{-3}$, which is close to the value of $39M_\odot\,\mathrm{pc}^{-3}$ estimated by TAKASE (1967).

In the present galactic model, the z-component K_z of the gravitational force can be easily derived from the potential function (4), and we have

$$K_z = Gz\sum_{i=1}^{2}\frac{M_i[a_i+(b_i^2+z^2)^{1/2}]}{\{R^2+[a_i+(b_i^2+z^2)^{1/2}]^2\}^{3/2}(b_i^2+z^2)^{1/2}}\,. \tag{A4}$$

For the above set of parameters the variation of K_z at $R=10\,\mathrm{kpc}$ is compared with that given by OORT (1965) in figure A3. For $z<200\,\mathrm{pc}$ the agreement of the both curves is satisfactory, while for larger z the present K_z departs from OORT's (1965) K_z considerably. This latter discrepancy may be reconciled by adopting the higher order generalized TOOMRE (1963) models.

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