Separace proměnných

$$y' = -3\,\frac{y-3}{x}$$

$$\frac{y'}{y-3} = \frac{-3}{x}$$

$$\int \frac{1}{y-3}\,dy = \int \frac{-3}{x}\,dx$$

$$\ln|y-3| = -3\ln|x|$$

$$\ln|y-3| = \ln\left|\frac{1}{x^3}\right| + c$$

$$y-3 = \pm e^{\ln\left(\frac{1}{x^3}\right)+c}$$

$$y = \pm\frac{1}{x^3}\cdot e^{c} + 3$$

$$y = \frac{c}{x^3} + 3$$

$x \neq 0$

$y \neq 3$ → platí vždy

obecné řešení:

$$y(x) = \frac{c}{x^3} + 3 \quad x \neq 0$$

$$x' = \frac{x^2 - x}{y}$$

$$\frac{x'}{x^2 - x} = \frac{1}{y}$$

$$\int \frac{1}{x^2-x} = \int \frac{1}{y}$$

$$1 = \int \frac{1}{x(x-1)} = \int \frac{1}{x} + \frac{1}{x-1}$$

$$-\ln|x| + \ln|x-1| = \ln|y|$$

$$\ln\left|\frac{x-1}{x}\right| = \ln|y|$$

$$\frac{x-1}{x} = \pm e^{\ln|y|} \cdot e^{C}$$

$$\frac{x-1}{x} = \pm y \cdot C$$

$$1 - \frac{1}{x} = \pm yC$$

$$x(y) = \frac{1}{1-yC}$$

$t \neq 0$

$t \neq \frac{1}{C}$

$x \neq 0$

$x \neq 1$

$$\frac{y'}{y+1} = -4x^3 \qquad y(x) \neq -1$$

$$\int \frac{1}{y+1} = -4 \int x^3$$

$$\ln|y+1| = \underbrace{-4\frac{x^4}{4}}_{-x^4}$$

$$y+1 = e^{-x^4} \cdot C$$

$$y = Ce^{-x^4} - 1 \qquad \underbrace{C \neq 0 \Rightarrow y \neq -1}_{\Rightarrow x \in \mathbb{R}}$$

$x \to \infty$

$\hookrightarrow \quad e^{-\infty} - 1 = -1$

$$y' - 2xy = e^{x^2}$$

homogenní řešení:

$$y' - 2xy = 0$$

$$y' = 2xy$$

$$\frac{y'}{y} = 2x$$

$$\int \frac{1}{y} = \int 2x$$

$$\ln|y| = \cancel{2}\frac{x^2}{\cancel{2}}$$

$$\ln|y| = x^2 + C$$

$$y_h = e^{x^2} \cdot C \quad , x \in \mathbb{R}, y \in \mathbb{R}$$

$$y_p = C(x) \cdot e^{x^2}$$

$$y_p' = C'(x) \cdot e^{x^2} + C(x) \cdot e^{x^2} \cdot 2x$$

$$y' - 2xy = e^{x^2}$$

$$\rightarrow \quad y_p' - 2x y_p = e^{x^2}$$

$$C'(x) \cdot e^{x^2} + \boxed{C(x) \cdot e^{x^2} \cdot 2x} - \boxed{2x\, C(x) e^{x^2}} = e^{x^2}$$

$$C'(x) \cdot e^{x^2} = e^{x^2}$$

$$C(x) = \int \frac{e^{x^2}}{e^{x^2}}\, dx$$

$$C(x) = x$$

$$y_p = x \cdot e^{x}$$

$$y(x) = y_p + y_h$$

$y(x) = x \cdot e^x + e^x \cdot C \ , \ D \in \mathbb{R} \ , \ x \in \mathbb{R}$

$y'' + 2y' = 0$

$\lambda^2 + 2\lambda = 0$

$\lambda(\lambda + 2) = 0$

$\lambda_1 = 0 \quad \rightarrow y_1(x) = e^{0x} = 1$

$\lambda_2 = -2 \quad \rightarrow y_2(x) = e^{-2x}$

$y(x) = c_1 \cdot 1 + c_2 \cdot e^{-2x}$

$y(x) = c_1 + c_2 \cdot e^{-2x}$

$y'' + y' + 2y = 0$

$\lambda^2 + \lambda + 2 = 0$

$\lambda_1 = 1$

$\lambda_2 = -2$

$y(x) = c_1 \cdot e^x + c_1 \cdot e^{-2x}$

$y'(x) = c_1 e^x - 2c_1 e^{-2x}$

$y(0) = 1$

$y'(0) = 1$

$c_1 + c_2 = 1$

$c_1 - 2c_2 = 1$

$c_1 = 1 - c_2$

$c_1 = 1$

$c_2 = 0$

$\rightarrow y(x) = e^x$

y''

$y'' - 5y' - 6y = e^{3x} + 12x - 4$

$y'' - 5y' + 6y = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$\lambda_1 = 3$

$\lambda_2 = 2$

$y_h = c_1 \cdot e^{2x} + c_2 \cdot e^{3x}$

$y_p = A \cdot x \cdot e^{3x} + Bx + C$

$$y_p' = Ae^{3x} + 3Axe^{3x} + B$$

$$y_p'' = 6Ae^{3x} + 9Axe^{3x}$$

$$\left.\begin{aligned} y_2' &= 2y_1 - 3y_2 \\ y_2' &= 4y_1 - 5y_2 \end{aligned}\right\} \quad A = \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} \qquad A - E\lambda = \begin{pmatrix} 2-\lambda & -3 \\ 4 & -5-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & -3 \\ 4 & -5-\lambda \end{vmatrix} = (2-\lambda)(-5-\lambda) - (-3)\cdot(4)$$

$$= -2\lambda + 5\lambda + \lambda^2 + 2 =$$

$$= \boxed{3\lambda + \lambda^2 + 2}$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\frac{-3 \pm \sqrt{9 - 4\cdot 2}}{2} = \frac{-3 \pm 1}{2} = \frac{-2}{2} = -1$$

$$= \frac{-4}{2} = -2$$

$$\left.\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= -2 \end{aligned}\right\} \breve{R} = \{e^{-x}, e^{-2x}\}$$

$\lambda := -1$

$$\ker\begin{pmatrix} 3 & -3 \\ 4 & -4 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$\lambda := -2$

$$\ker\begin{pmatrix} 4 & -3 \\ 4 & -3 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right)$$

$$y_a(x) = \begin{pmatrix} e^{-x} \\ e^{-x} \end{pmatrix}$$

$$y_b(x) = \begin{pmatrix} 3e^{-2x} \\ 4e^{-2x} \end{pmatrix}$$

$$y = A \cdot \begin{pmatrix} e^{-x} \\ e^{-x} \end{pmatrix} + B \cdot \begin{pmatrix} 3e^{-2x} \\ 4e^{-2x} \end{pmatrix}$$

$$\boxed{\begin{aligned} y_1 &= A \cdot e^{-x} + B \cdot 3e^{-2x} \\ y_2 &= A \cdot e^{-x} + B \cdot 4e^{-2x} \end{aligned}}$$

$$\left.\begin{aligned} y_1' &= -y_1 + y_2 \\ & \end{aligned}\right\} \quad A = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \quad A - E\lambda = \begin{pmatrix} -1-\lambda & 1 \\ 2 & -2-\lambda \end{pmatrix}$$

$y_2' = 2y_1 - 2y_2$

$$\begin{vmatrix} -1-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = (-1-\lambda)(-2-\lambda) - 2 =$$

$$\lambda + 2\lambda + \lambda^2 = \lambda^2 + 3\lambda = \lambda(\lambda+3)$$

$$\lambda(\lambda+3) = 0 \rightarrow \lambda_1 = 0$$
$$\lambda_2 = -3$$

vl. vektory:

$\lambda := 0$

$$\ker \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$\lambda := -3$

$$\ker \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$$

$$y_a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y_b = \begin{pmatrix} e^{-3x} \\ -2e^{-3x} \end{pmatrix}$$

$$y(x) = A \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B \begin{pmatrix} e^{-3x} \\ -2e^{-3x} \end{pmatrix}$$

$$3y' = \frac{2x}{y^2} \qquad y(1) = 1$$

$\longrightarrow y \neq 0$

$$3y' \cdot y^2 = 2x$$

$$\int 3y^2 = \int 2x$$

$$\cancel{3}\frac{y^3}{\cancel{3}} = 2\frac{x^2}{2} + C$$

$$y^3 = x^2 + C$$

$$y = \sqrt[3]{x^2 + C} \qquad C > x^2$$

2) $$\frac{y'}{y+1} = -4x^3 \qquad \longrightarrow y \neq -1$$

$$\int \frac{1}{y+1} = \int -4x^3$$

$$\ln|y+1| = -\cancel{4}\frac{x^4}{\cancel{4}}$$

$$\ln|y+1| = -x^4$$

$$y+1 = \pm e^{-x^4+C}$$

$$\ldots = \pm e^{-x^4} \ldots D$$

$y+1 = \pm e \cdot D \rightarrow D \neq 0$

$y = \pm e^{-x^4} \cdot D - 1 \quad x \in \mathbb{R}$

$0 = D - 1$

$D = 1$

$y' = \frac{2xy}{x^2-4}$

$y = 0 \rightarrow$ stac.

$\frac{y'}{y} = \frac{2x}{x^2-4}$

$x^2 - 4 \neq 0$

$x \neq \pm 2$

$\int \frac{1}{y} = \int \frac{2x}{x^2-4} \quad \Big| \quad \begin{matrix} x^2-4 = y \\ 2x\,dx = dy \\ dx = \frac{dy}{2x} \end{matrix} \quad = \int \frac{\cancel{2x}}{y} \frac{dy}{\cancel{2x}}$

$\ln|y| = \ln|x^2+4| \quad$ $= \int \frac{1}{y} dy = \ln|y| + c$

$\rightarrow \ln|x^2-4|$

$y = \pm(x^2+4) \cdot c, \; x \neq \pm 2$

$c \neq 0$ nebo

$c = 0$ jsou stac. řešením

$y' = \frac{2}{x^3} - \frac{3y}{x}$

hom. řešení!

$y' = -\frac{3y}{x}$

$\frac{y'}{\;} = \frac{-3}{\;}$

$y \qquad x$

$$\int \frac{1}{y} = -3 \int \frac{1}{x}$$

$$\ln|y| = -3 \ln|x| + C$$

$$y = \pm \ln|x^{-3}| + C$$

$$y = \pm \ln\left|\frac{1}{x^3}\right| + C$$

$$y = \pm \frac{1}{x^3} \cdot C$$

$$y_h = \pm \frac{C}{x^3} \quad , x \neq 0$$

variace:

$$y_p(x) = \frac{c(x)}{x^3}$$

$$y_p'(y) = c'(x) \cdot x^{-3} + c(x) \cdot 3x^2$$

$$c'(x) \cdot x^{-3} + c(x) \cdot 3x^2 = \frac{2}{x^3}$$

$$\frac{c'(x)}{x^3} = \frac{2}{x^3} \qquad c'(x) = 2$$

$$c(x) = 2x + C$$

$$y(x) = \frac{2x + C}{x^3}$$

$y' + y = 13x$

hom.

$y' + y = 0$

$\lambda + 1 = 0$

$\lambda_1 = -1$

$y_h = C e^{-x}, \ x \in \mathbb{R}$

$y_p(x) = C(x) \cdot e^{-x}$

$C'(x) \cdot e^{-x} = 13x$

$C'(x) = 13x e^{x}$

$C(x) = 13x e^{x} - 13 e^{x}$

$y(x) = (13x e^{x} - 13 e^{x}) \cdot e^{-x}$

$y(x) = 13x - 13 + C \cdot e^{-x}, \ x \in \mathbb{R}$

$y' = \frac{x^2}{y^2}$ $\quad -1 \ y \neq 0$

$y' \cdot y^2 = x^2$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

$$y = \sqrt[3]{x^3 + 3C} \quad \to x \neq 0$$

$$y'' + 4y' + 4y = 0$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$-2 \cdot -2$$

$$\left.\begin{aligned} \lambda_1 &= -2 \\ \lambda_2 &= -2 \end{aligned}\right\} \text{dvojný!}$$

$$y = C_1 \cdot e^{-2x} + C_2 \cdot x \cdot e^{-2x}$$

$$y' = \left((C_1 + xC_2)(e^{-2x})\right)'$$

$$y' = (C_2 + 0) \cdot e^{-2x} + 2(C_1 + xC_2)e^{-2x}$$

$$y' = -C_2 e^{-2x} + 2C_1 e^{-2x} - 2C_2 x e^{-2x}$$

$$y'' + 4y' + 4y = \underbrace{e^{-2x}} + 2e^x$$

$$\lambda_{1,2} = -2 \quad \to \quad \tilde{R} = \{e^{-2x}\}$$

$y_p = A \cdot x^2 \cdot e^{-2x} + B \cdot e^x$

① $y' = -e^x y^2 \qquad y(1) = \frac{1}{e-1}$

$\frac{y'}{y^2} = -e^x \quad \rightarrow y \neq 0$ ... pot. stac. řešení

$\int \frac{1}{y^2} = -\int e^x$

$-\frac{1}{y} = -e^x + C$

$\frac{1}{y} = e^x - C$

$y = \frac{1}{e^x - C} \qquad e^x - C \neq 0$

$e^x \neq C$

$x \neq \ln C$

$y(1) = \frac{1}{e-1}$

$$\frac{1}{e-1} = \frac{1}{e^0 - c}$$

$$\frac{1}{e-1} = \frac{1}{e-c} \quad \rightarrow c = 1$$

(2) $y'' + 3y' + 2y = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$-1 \cdot -2$

$\lambda_1 = -1$

$\lambda_2 = -2$

$y(x) = c_1 \cdot e^{-x} + c_2 \cdot e^{-2x}$

$y(x) \xrightarrow[x \to \infty]{} e^{-\infty} = 0$

$y(0) = -1$

$y'(0) = 1$

$y'(x) = \left(c_1 \cdot e^{-x} + c_2 \cdot e^{-2x}\right)' =$

$-c_1 \cdot e^{-x} - 2c_2 \cdot e^{-2x}$

$$-1 = c_1 + c_2 \quad \rightarrow c_1 = -1 - c_2$$

$$1 = -c_1 - 2c_2$$

$$1 = -(-1 - c_2) - 2c_2 = 1 + c_2 - 2c_2$$

$$0 = c_2 - 2c_2$$

$$c_2 = 0$$

$$c_1 = -1 - 0 = -1$$

$$\boxed{y(x) = -e^{-x}}$$

(3) $y'' + 3y' + 2y = e^x + 13\cos(x)$

$$y_h = c_1 e^{-x} + c_2 \cdot e^{-2x}$$

$$y_p = A \cdot e^x + B \cdot \cos(x) + C \cdot \sin(x)$$

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(1)

$$3y' = \frac{2x}{y^2} \qquad y(1) = 1$$

$$3y' \cdot y^2 = 2x$$

$$3\int y^2 = \int 2x$$

$y \neq 0$ ...

$$3\frac{y^3}{3} = 2\frac{x^2}{2} + C$$

$$y^3 = x^2 + C$$

$$y = \sqrt[3]{x^2 + C}$$

$$1 = \sqrt[3]{1 + C}$$

$$1 = 1 + C$$

$$C = 0$$

$$y = \sqrt[3]{x^2}$$

$$\boxed{y = x^{\frac{2}{3}}}$$

② $y'' + 2y' + y = 0$

$$\lambda^2 + 2\lambda + 1 = 0$$

$-1, -1$

$$\lambda_{1,2} = -1$$

$$y(x) = C_1 \cdot e^{-x} + C_2 \cdot x \cdot e^{-x} \to 0$$

$$y'(x) = \left((C_1 + C_2 x) e^{x}\right)' =$$

$$= C_2 \cdot e^{x} + (C_1 + C_2 x) \cdot e^{x} =$$

$$y' = C_2 e^{x} - C_1 e^{x} - C_2 x e^{x}$$

$$y(0) = 0$$

$$y'(0) = 1$$

$$0 = C_1$$

$$-1 = C_2$$

$$y(x) = -x \cdot e^{-x}$$

(3) $$y'' + 2y' + y = e^{-x} - 23e^{2x}$$

$$y_h = C_1 \cdot e^{-x} + C_2 \cdot x \cdot e^{-x}$$

$$y_p = A \cdot x^2 \cdot e^{-x} - Be^{2x}$$

(4) $$y_1' = 0y_1 - 1y_2$$

$$y_2' = 3y_1 + 4y_2 \qquad A = \begin{pmatrix} 0 & -1 \\ 3 & 4 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & -1 \\ 3 & 4-\lambda \end{vmatrix} = (-\lambda)(4-\lambda)+3 = -4\lambda+\lambda^2+3$$

$$\lambda^2-4\lambda+3=0$$

1 3

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$

$\lambda := 1$

$$\ker\begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

$\lambda := 3$

$$\ker\begin{pmatrix} -3 & -1 \\ 3 & 1 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right)$$

$$y(x) = c_1 e^{x}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3x}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

① $y' = \dfrac{x^2}{y^2}$

$$y' \cdot y^2 = x^2$$

$$\int y^2 = \int x^2$$

$$\frac{y^3}{3} = \frac{x^3 + 3c}{3}$$

$$y^3 = x^3 + 3c$$

$$y = \sqrt[3]{x^3 + 3c} \qquad y \neq 0$$

$$x^3 + D \neq 0$$

$$y = \sqrt[3]{x^3 + D} \qquad D \neq -x^3$$

7. $$y' = -3\,\frac{y-3}{x}$$

$$y - 3 = 0$$

$$\frac{y'}{y-3} = \frac{-3}{x} \qquad y = 3 \text{ ... stac.}$$

$$y \neq 0$$

$$\int \frac{1}{y-3} = -3\int \frac{1}{x}$$

$$\ln|y-3| = -3\ln|x|$$

$$y - 3 = \frac{c}{x^3}$$

$$y = \frac{c}{x^3} + 3, \quad x \neq 0$$

$$\frac{c}{(-1)^3} + 3 = 5$$

$c = -2$

$$y = \frac{-2}{x^3} + 3$$

(2) $y'' + 2y' + 5y = 0$

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda_1 = -1 + 2i$

$\lambda_2 = -1 - 2i$

$y(x) = C_1 \cdot e^{(-1+2i)x} + C_2 \cdot e^{(-1-2i)x}$

$y(x) = C_1 \cdot e^{-x} \cdot \sin(2x) + C_2 \cdot e^{-x} \cdot \cos(2x)$