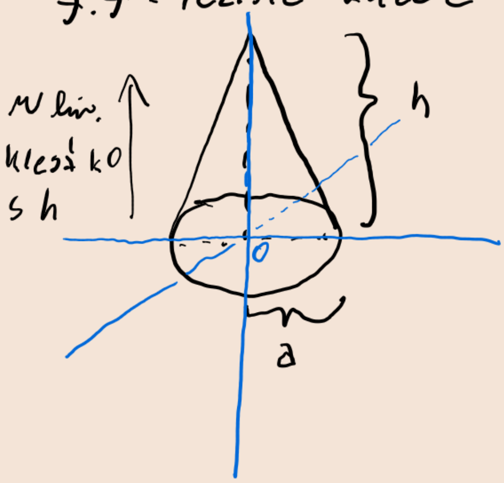
7.7 - Težiště kužele



1. výpočet objemu

- do válcových souřadnic:

$$\left.\begin{array}{l} x = r.\cos\varphi \\ y = r.\sin\varphi \\ z = z \end{array}\right\} \quad J_\varphi = \begin{pmatrix} \cos\varphi & -r.\sin\varphi & 0 \\ \sin\varphi & r.\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(J_\varphi) = 1(\cos\varphi \cdot r.\cos\varphi - (-r.\sin\varphi.\sin\varphi = r.(\cos^2\varphi + \sin^2\varphi) = r$$

- rozsah integrace:

$\varphi \in [0, 2\pi]$

$z \in [0, h]$

$r \in [0, \frac{a}{h}.z]$

- integrace:

$$V = \int_0^{2\pi}\int_0^h\int_0^{\frac{a}{h}z} \det(J_\varphi)\, dr\, dz\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^h\int_0^{\frac{a}{h}z} r\, dr\, dz\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^h \left[\frac{1}{2}r^2\right]_0^{\frac{a}{h}z} \underset{dz\,d\varphi}{=} \int_0^{2\pi}\int_0^h \frac{1}{2}\left(\frac{a^2}{h^2}z^2\right) dz\, d\varphi$$

$$= \int_0^{2\pi} \frac{1}{2}\frac{a^2}{h^2} \cdot \left[\frac{1}{3}z^3\right]_0^h = \int_0^{2\pi} \frac{1}{6}a^2h\, d\varphi =$$

$$= \left[\frac{1}{6}a^2h.\varphi\right]_0^{2\pi} = \frac{1}{6}a^2h.2\pi = \frac{1}{3}\pi a^2 h$$

2. těžiště:

$$z_T = \frac{1}{V}\iiint_K z\cdot r \, dr\, dz\, d\varphi = \frac{1}{V}\int_0^{2\pi}\int_0^h\int_0^{\frac{a}{h}\cdot z} z r \, dr\, dz\, d\varphi =$$

$$= \frac{1}{V}\int_0^{2\pi}\int_0^h z\left[\frac{1}{2}r^2\right]_0^{\frac{a}{h}z} dz\, d\varphi = \frac{1}{V}\int_0^{2\pi}\int_0^h z\cdot\frac{1}{2}\left(\frac{a^2}{h^2}z^2\right) dz\, d\varphi =$$

$$= \frac{1}{V}\int_0^{2\pi}\int_0^h \frac{1}{2}\frac{a^2}{h^2}z^3 \, dz\, d\varphi = \frac{1}{V}\int_0^{2\pi}\frac{1}{2}\frac{a^2}{h^2}\left[\frac{z^4}{4}\right]_0^h d\varphi = \frac{1}{V}\int_0^{2\pi}\frac{1}{2}\frac{a^2}{h^2}\cdot\frac{1}{4}h^4 \, d\varphi =$$

$$= \frac{1}{V}\int_0^{2\pi}\frac{1}{8}a^2h^2 \, d\varphi = \frac{1}{V}\cdot\left[\frac{1}{8}a^2h^2\cdot\varphi\right]_0^{2\pi} = \frac{1}{V}\cdot\frac{1}{8}a^2h^2\, 2\pi = \frac{1}{V}\cdot\frac{1}{4}\pi\cdot a^2h^2 =$$

$$= \frac{1}{\frac{1}{3}\cancel{\pi}\cdot\cancel{a^2}h}\cdot\frac{1}{4}\cancel{\pi}\cancel{a^2}h^{\cancel{2}} = \underline{\underline{\frac{3}{4}h \text{ (od vrchu)}}} \quad \vee \quad \underline{\underline{h-\frac{3}{4}h = \frac{1}{4}h \text{ (od spodu)}}}$$