7.7 - Těžiště kužele
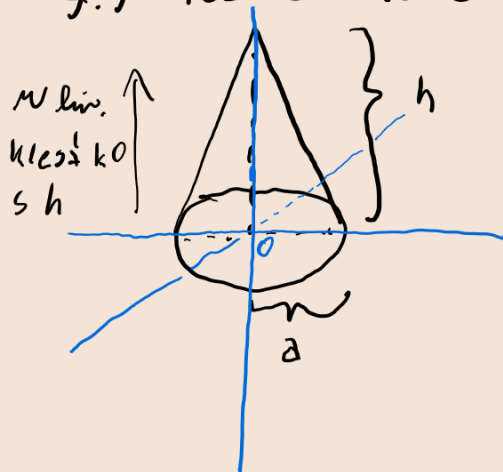

1. výpočet objemu

- do válcových souřadnic:

$$\left.\begin{array}{l} x = r \cdot \cos\varphi \\ y = r \cdot \sin\varphi \\ z = z \end{array}\right\} \quad J_\varphi = \begin{pmatrix} \cos\varphi & -r\cdot\sin\varphi & 0 \\ \sin\varphi & r\cdot\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(J_\varphi) = 1\left(\cos\varphi \cdot r \cdot \cos\varphi - (-r\cdot\sin\varphi \cdot \sin\varphi\right) = r\cdot(\cos^2\varphi + \sin^2\varphi) = r$$

- rozsah integrace:

$$\varphi \in [0, 2\pi]$$
$$z \in [0, h]$$
$$r \in \left[0, \frac{a}{h}\cdot z\right]$$

- integrace:

$$V = \int_0^{2\pi} \int_0^h \int_0^{\frac{a}{h}z} \det(J_\varphi)\, dr\, dz\, d\varphi =$$

$$= \int_0^{2\pi} \int_0^h \int_0^{\frac{a}{h}z} r\, dr\, dz\, d\varphi =$$

$$= \int_0^{2\pi} \int_0^h \left[\frac{1}{2} r^2\right]_0^{\frac{a}{h}z} dz\, d\varphi = \int_0^{2\pi} \int_0^h \frac{1}{2}\left(\frac{a^2}{h^2}z^2\right) dz\, d\varphi$$

$$= \int_0^{2\pi} \frac{1}{2}\frac{a^2}{h^2}\cdot\left[\frac{1}{3}z^3\right]_0^h = \int_0^{2\pi} \frac{1}{6}a^2 h\, d\varphi =$$

$$= \left[\frac{1}{6}a^2 h \cdot \varphi\right]_0^{2\pi} = \frac{1}{6}a^2 h \cdot 2\pi = \frac{1}{3}\pi a^2 h$$

2. těžiště :

$$z_T = \frac{1}{V} \iiint_k z \cdot r \; dr \; dz \; d\varphi = \frac{1}{V} \int_0^{2\pi} \int_0^h \int_0^{\frac{a}{h} z} z r \; dr \; dz \; d\varphi =$$

$$= \frac{1}{V} \int_0^{2\pi} \int_0^h z \left[\frac{1}{2} r^2 \right]_0^{\frac{a}{h} z} dz \; d\varphi = \frac{1}{V} \int_0^{2\pi} \int_0^h z \cdot \frac{1}{2} \left(\frac{a^2}{h^2} z^2 \right) dz \; d\varphi =$$

$$= \frac{1}{V} \int_0^{2\pi} \int_0^h \frac{1}{2} \frac{a^2}{h^2} z^3 \; dz \; d\varphi = \frac{1}{V} \int_0^{2\pi} \frac{1}{2} \frac{a^2}{h^2} \left[\frac{z^4}{4} \right]_0^h d\varphi = \frac{1}{V} \int_0^{2\pi} \frac{1}{2} \frac{a^2}{h^2} \cdot \frac{1}{4} h^4 \; d\varphi =$$

$$= \frac{1}{V} \int_0^{2\pi} \frac{1}{8} a^2 h^2 \; d\varphi = \frac{1}{V} \cdot \left[\frac{1}{8} a^2 h^2 \cdot \varphi \right]_0^{2\pi} = \frac{1}{V} \cdot \frac{1}{8} a^2 h^2 \; 2\pi = \frac{1}{V} \cdot \frac{1}{4} \pi \cdot a^2 h^2 =$$

$$= \frac{1}{\frac{1}{3} \pi \cdot a^2 h} \cdot \frac{1}{4} \pi a^2 h^2 = \frac{3}{4} h \quad (\text{od vrchu}) \quad \vee \quad h - \frac{3}{4} h = \frac{1}{4} h \; (\text{od spodu})$$