1) a) $y'' + y' = 2\cos(x) + 2$

$y(0) = -2$

$y'(0) = 4$

Char. funkce

$\lambda^2 + \lambda = 0$

$\lambda(\lambda+1) = 0$

$\lambda_1 = 0$ $\quad \mathcal{R} = \{1, e^{-x}\}$

$\lambda_2 = -1$

$y_h = c_1 + c_2 \cdot e^{-x}$ z char. funkce

$y_p = A\cdot\cos(x) + B\cdot\sin(x) + Cx$ odhad z pravé strany

$y_p' = -A\cdot\sin(x) + B\cdot\cos(x) + C$

$y_p'' = -A\cdot\cos(x) - B\cdot\sin(x)$

$\underline{-A\cdot\cos(x)}\ \underline{-B\cdot\sin(x)}\ \underline{-A\cdot\sin(x)} + \underline{B\cdot\cos(x)} + C = 2\cos(x) + 2$

$(B-A)\cos(x) + (-B-A)\sin(x) + C = 2\cos(x) + 0\sin(x) + 2$

$C = 2$

$B - A = 2 \qquad B + B = 2$

$-B - A = 0 \qquad 2B = 2$

$-B = A \qquad B = 1$

$$\boxed{\begin{aligned} C &= 2 \\ B &= 1 \\ A &= -1 \end{aligned}}$$

$y = y_p + y_h$

$y = \sin(x) - \cos(x) + 2x + c_1 + c_2 \cdot e^{-x}$

$y' = \cos(x) + \sin(x) + 2 - c_2 \cdot e^{-x}$

$$\sin(0) - \cos(0) + C_1 + C_2 = 2$$

$$\cos(0) + \sin(0) + 2 - C_2 = 4$$

$$-1 + C_1 + C_2 = -2 \qquad C_1 + C_2 = -1$$

$$1 + 0 + 2 - C_2 = 4$$

$$3 - C_2 = 4$$

$$-C_2 = 1$$

$$C_2 = -1$$

$$C_1 - 1 = -1$$

$$C_1 = 0$$

$$y = \sin(x) - \cos(x) + 2x - e^{-x}, \ \underline{x \in \mathbb{R}}$$

b) $\{x, e^x\}$ je fund. systém pro rovnici:

$$(x-1)y'' - xy' + y = 0$$

- jsou LN
- jsou 2 = dim prostoru řešení

$$y = x \qquad y = e^x$$
$$y' = 1 \qquad y' = e^x$$
$$y'' = 0 \qquad y'' = e^x$$

$$(x-1)0 - x\cdot 1 + x = 0$$
$$-x + x = 0$$

$$(x-1)\cdot e^x - x\cdot e^x + e^x = 0$$
$$\underbrace{x\cdot e^x - x\cdot e^x}_{0} \underbrace{- e^x + e^x}_{0} = 0$$

- jsou řešením

(2) $$y' = \frac{e^{-y}}{2\sqrt{x}}$$

$$y(1) = 0$$

$$\int e^{y}\,dy = \int \frac{1}{2\sqrt{x}}\,dx$$

$$e^{y} = \frac{1}{2} \cdot x^{-1/2}$$

$$e^{y} = \sqrt{x} + C$$

$$y = \ln(\sqrt{x} + C) \quad , x > 0$$
$$\sqrt{x} + C > 0$$

$$0 = \ln(\sqrt{1} + C)$$
$$0 = \ln(1 + C) \Rightarrow C = 0$$

$$y_P = y = \ln\sqrt{x} \quad , x \in (0; \infty)$$

b) $\left.\begin{aligned} y_1' &= 2y_1 - y_2 \\ y_2' &= 8y_1 - 4y_2 \end{aligned}\right\} A = \begin{pmatrix} 2 & -1 \\ 8 & -4 \end{pmatrix}$

$$A - E\cdot\lambda = \begin{pmatrix} 2-\lambda & -1 \\ 8 & -4-\lambda \end{pmatrix}$$

vlastní čísla:

$$\begin{vmatrix} 2-\lambda & -1 \\ 8 & -4-\lambda \end{vmatrix} = (2-\lambda)\cdot(-4-\lambda) + 8 =$$

$$= \cancel{-8} - 2\lambda + 4\lambda + \lambda^2 \cancel{+8}$$

$$= 2\lambda + \lambda^2 = \lambda(\lambda+2)$$

$$\boxed{\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -2 \end{aligned}}$$

z $\lambda = 0$:

$$\ker\begin{pmatrix} 2 & -1 \\ 8 & -4 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$

z $\lambda = -2$:

$$\ker\begin{pmatrix} 4 & -1 \\ 8 & -2 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right)$$

$$y = A\begin{pmatrix}1\\2\end{pmatrix} + B\begin{pmatrix}1\\4\end{pmatrix}\cdot e^{-2x}$$

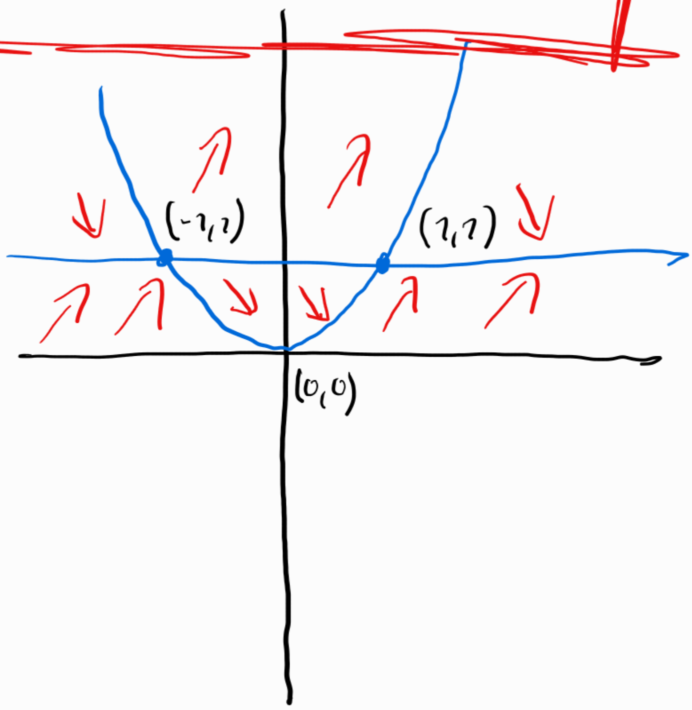
$$\boxed{\begin{aligned} y_1 &= Be^{-2x} + A \\ y_2 &= 4B\cdot e^{-2x} + 2A \end{aligned}}$$

$x \in \mathbb{R}$

Stabilita řešení:

- záporné λ_1, λ_2 – stabilní
- alespoň jedno kladné – nestabilní
- záporné a alespoň jedno 0 – spíše nestabilní
  - vlastní vektory LN $\Rightarrow$ diagonalizovatelná $\Rightarrow$ nestabilní nebo matice má defekt

3 a) $y' = (y - x^2)(y - 1)$



Stac. body:

$(y - x^2)(y - 1) = 0$

$y - x^2 = 0 \Rightarrow y = x^2$

$y - 1 = 0 \Rightarrow y = 1$

$y' = \left(\frac{1}{2} - \frac{1}{4}\right)\left(\frac{1}{2} - 1\right) = -$

$(-4)(-1)$

b) $y' = e^{x-y}$

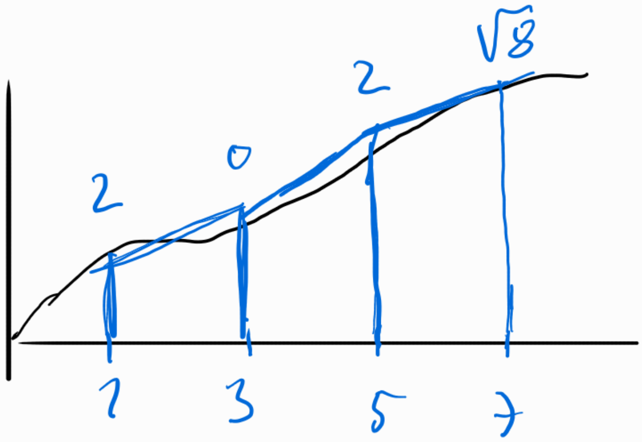
- separace :
- lineární + odhad : ne
- lineární + variace C : ne

Gem a věci k tomu:

- spektrální poloměr: největší vlastní číslo
- řádková/sloupcová norma: největší součet čísel v řádcích/sloupcích

4a)

$$\int_1^7 \sqrt{|2x-6|}\,dx$$

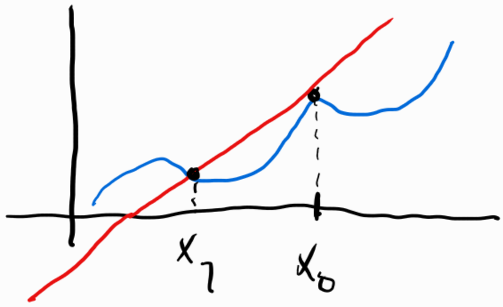


$$\int \dots \sim \frac{2+0}{2}\cdot 2 + \frac{0+2}{2}\cdot 2 + \frac{2+\sqrt{8}}{2}\cdot 2$$

$$\sim 2+2+2+\sqrt{8} = 6+\sqrt{8}$$

$$= 8{,}82$$

b)



$$\boxed{f'(x_0) = \frac{f(x_0)}{x_0 - x_1}}$$

$$f'(x_0)(x_0 - x_1) = f(x_0)$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$-x_1 = \frac{f(x_0)}{f'(x_0)} - x_0$$

$$x_1 = -\frac{f(x_0)}{f'(x_0)} + x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Lineární diferenciální rovnice:

$$y'' - 4y' + 3y = (x+1)e^{3x}$$

$$\lambda^2 - 4\lambda + 3 = 0 \qquad \hookrightarrow (Ax+B)e^{3x}$$

$$1$$
$$1 \cdot 3$$

$$\lambda_1 = 1 \rightarrow e^{x}$$

$$\lambda_2 = 3 \rightarrow e^{3x}$$

$$\Downarrow$$

$$(Ax^2+Bx)e^{3x}$$

$$x^2+1 \rightarrow Ax^2+Bx+C$$

$$12\sin(2x) = A\cdot\sin(3x) + B\cdot\cos(3x)$$

$$(x^2-3)e^{2x} = (Ax^2+Bx+C)e^{2x}$$

$$2e^{x}\sin(2x) + (x-1)e^{x}\cdot\cos(2x)$$

$$e^{x}\big(2\sin(2x) + (x-1)\cdot\cos(2x)\big)$$

$$e^{x}\big((Ax+B)\sin(2x) + (Cx+D)\cdot\cos(2x)\big)$$

$y''' - 2y'' = 2e^x - 1$

$\lambda^3 - 2\lambda^2$

$\lambda^2(\lambda - 2)$

$\lambda_1 = 0$

$\lambda_2 = 0$

$\lambda_3 = 2$

$R = \{1, x, e^{2x}\}$ ✓

$y_h = A + Bx + Ce^{2x}$

$y_p = A \cdot e^x + Bx^2$

$y_p' = A \cdot e^x + 2Bx$

$y_p'' = Ae^x + 2B$

$y_p''' = Ae^x$

$Ae^x - 2Ae^x + 4B = 2e^x - 1$

$-Ae^x + 4B = 2e^x - 1$

$A = -2$

$B = \frac{1}{4}$

$$y_p = -2e^x + \frac{1}{4}x^2 + A + Bx + Ce^{2x}, \; x \in \mathbb{R}$$

$$y'' - 2y' = 6e^{3x} - 4x$$

$$\lambda^2 - 2\lambda = 0 \qquad \lambda_1 = 0 \qquad \tilde{R} = \{1, e^{2x}\}$$
$$\lambda(\lambda - 2) \qquad \lambda_2 = 2$$

$$y_h = A + Be^{2x}$$

$$y_p = A \cdot e^{3x} + Bx + C$$

$$y_p' = 3Ae^{3x} + B$$

$$y_p'' = 9Ae^{3x}$$

$$9Ae^{3x} - 2(3Ae^{3x} + B) = 6e^{3x} - 4x$$

$$9Ae^{3x} - 6Ae^{3x} - 2B = 6e^{3x} - 4x$$

$$3Ae^{3x} - 2B = 6e^{3x} - 4x$$

$$3A = 6 \qquad -2B = -4$$
$$A = 2 \qquad B = 2$$

$$y_p = 2e^{3x} + 2x$$

$$y = A + Be^{2x} + 2e^{3x} + 2x$$

$$y' = 2Be^{2x} + 6e^{3x} + 2$$

$$A + B = 1$$

$$7 = 2B + 6 + 2$$

$$-1 = 2B \quad \rightarrow B = -\frac{1}{2}$$

$$A = \frac{3}{2}$$

$$y = \frac{3}{2} - \frac{1}{2}e^{2x} + 2e^{3x} + 2x$$

$$y'' + \frac{1}{x}y' - \frac{16}{x^2}y = 0$$

$$\lambda^2 + \frac{1}{x}\lambda - \frac{16}{x^2} = 0$$

$$y' = \pi e^x . \cos(\pi e^x) y$$

$$y(0) = 2$$

$$\int \frac{1}{y} = \int \pi e^{x} \cdot \cos(\pi e^{x})^{dx} = \left| \overset{\text{sub.}}{u \cdot \cos(u)} \right|$$

$$u = \pi e^{x}$$

$$du = \pi e^{x} dx$$

$$dx = \frac{du}{\pi e^{x}}$$

$$\ln|y| = \int \cos(u)\, du$$

$$\ln|y| = \sin(\pi e^{x})$$

$$y = e^{\sin \pi e^{x}} \cdot C$$

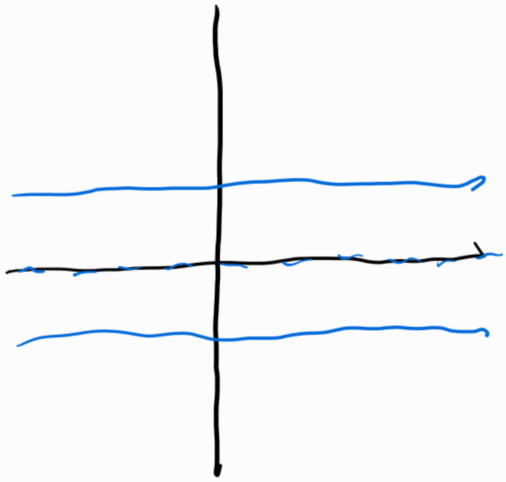
$$y(0) = 2$$

$$2 = e^{\sin \pi} \cdot C$$

$$\frac{2}{e^{\sin \pi}} = C$$

$$\frac{2}{1} = C \to C = 2$$

$$y = 2e^{\sin(\pi e^{x})}$$



$$y = \frac{y^{2}-1}{y} \to y \neq 0$$

$$y^{2} - 1 = 0$$

$$y = \pm 1$$

$$\begin{pmatrix} 3 & -3 \\ 2 & -4 \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & -3 \\ 2 & -4-\lambda \end{vmatrix} = (3-\lambda)(-4-\lambda)+6 = -12-3\lambda+4\lambda+\lambda^2$$

$$= -6+\lambda+\lambda^2$$

$$= \lambda^2+\lambda-6$$

$\lambda_1 = -3$

$\lambda_2 = 2$

$\lambda_1 = -3$

$$\ker\begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$

$\lambda_2 = 2$

$$\ker\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right)$$

$$y = c_1 \cdot e^{2x}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \cdot e^{-3x}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Řády metod:

Integrály:

- obdélníková: 1
- lichoběžníková: 2
- simpsonova: 4

Derivace:

- dopředná diference: 1

Kořeny rovnic

- Newtonova: 2
- Bisekce: 1
- Pevný bod

Dif. rovnice:

- Eulerova: 1
- Runge-kutta: 4

Soustavy rovnic:

- GEM: $\left(\frac{2}{3}n^3\right)$, dosazení: (n^2)
- Gauss-Seidel: 1 (kn^2)
- Jacobiho: 1 (kn^2)

time complexity $O(\dots)$

Taylorův polynom:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} \dots$$

Řád metody v integraci:

$$E_n \approx \frac{C}{n^p}$$

C → konstanta

p → řád metody

n → počet dělení

Řád metody při hledání kořene

$$\underbrace{|x_{n+1} - r|}_{E_{n+1}} \approx C\,\underbrace{|x_n - r|}_{E_n}^{\,p}$$

Relaxace:

→ $x_{n+1} = (1-\lambda)x_n + \lambda g(x_n)$

$x^2 = x+1 \;\rightarrow\; x = x^2 - 1 \;\rightarrow\; x_0 = 3$

$g(x) = x^2 - 1 \qquad x_1 = g(3)$

...

$|g'(r)| < 1$ ... mohlo by konvergovat

Řád metod dif. rovnic:

$$E_n = C h^p$$

p → řád

h → krok

E_n → glob. chyba

Eulerova metoda

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

$y' = x + y$ (1) $x_0 = 1$ (2) $x_1 = 2$

$y(1) = 13$ $y_0 = 13$ $y_1 = 27$

$h = 1$ $y_1 = 13 + 1 \cdot 14 = 27$ $\vdots$

$\Downarrow$

$f(x_n, y_n) = x + y$

Newtonova metoda (kořeny)

$$f'(x_0) = \frac{f(x_0)}{x_0 - x}$$

Gem a věci k tomu:

- spektrální poloměr: největší vlastní číslo
- řádková / sloupcová norma: největší součet čísel v řádcích/sloupcích
- změna znaménka při záměně řádků!

Gauss-Seidel metoda

zadání

$$x - z = 1$$
$$2x + y - z = 1$$
$$x + 2y - z = -1$$

výchozí vektor:

$$x_0 = (0, 0, 0)$$

$$x = 1 + z$$
$$y = 1 + z - 2x$$
$$z = x + 2y + 1$$

$$\boxed{\begin{aligned} x_{k+1} &= 1 + z_k \\ y_{k+1} &= 1 + z_k - 2x_{k+1} \\ z_{k+1} &= x_{k+1} + 2y_{k+1} + 1 \end{aligned}}$$

$$x_1 = 1 + 0 = 1$$
$$y_1 = 1 + 0 - 2 \cdot 1 = -1$$
$$z_1 = 1 + 2(-1) + 1 = 0$$

...

Jacobiho metoda

Stejná, jen používá pouze předchozí iterace.

$$\boxed{\begin{aligned} x_{k+1} &= 1 + z_k \\ y_{k+1} &= 1 + z_k - 2x_k \\ z_{k+1} &= x_k + 2y_k + 1 \end{aligned}}$$

② $\frac{y'}{y+1} = -4x^3 \quad \rightarrow y \neq -1$

$\int \frac{1}{y+1} dy = -\int 4x^3 dx$

$\ln|y+1| = -\cancel{4}\frac{x^4}{\cancel{4}} + C$

$\ln|y+1| = -x^4 + C$

$y+1 = e^{-x^4} \cdot e^C$

$y = De^{-x^4} - 1 \quad , D \neq 0$

$x \in \mathbb{R}$

$x \rightarrow \infty : -1$

$y(0) = 0$

$0 = D - 1 \quad \rightarrow y = e^{-x^4} - 1$

$D = 1$

③ $3y' = \frac{1}{y^2} \qquad y^2 \neq 0$

$y \neq 0$

$3y' \cdot y^2 = 1$

$\int 3y^2 = \int 1$

$\cancel{3}\frac{y^3}{\cancel{3}} = x + C$

$y^3 = x + C$

$y = \sqrt[3]{x+C} \qquad x + C \neq 0$

$x \neq -C$

④

$$y' = e^{x-y}$$

$$y' = e^x \cdot e^{-y}$$

$$\frac{y'}{e^{-y}} = e^x$$

$$\int e^y \, dy = \int e^x \, dx$$

$$e^y = e^x + C$$

$$y = \ln|e^x + C| \rightarrow e^x + C > 0$$

⑤

$$x' = \frac{x^2}{t^2}$$

$$\frac{x'}{x^2} = t^{-2}$$

$$\int \frac{1}{x^2} \, dx = \int t^{-2} \, dt$$

$$-\frac{1}{x} = \frac{t^{-1}}{-1} + C$$

$$-\frac{1}{x} = -\frac{1}{t} + C$$

$$\frac{1}{x} = \frac{1}{t} - C$$

⑥ $y' = \frac{2xy}{x^2-4}$

$$\frac{y'}{y} = \frac{2x}{x^2-4}$$

$$\int \frac{1}{y}\,dy = \int \frac{2x}{x^2-4}\,dx$$

$$\ln|y| = \int \frac{A}{x-2}\,dx + \frac{B}{x+2}\,dx$$

$$\frac{2x}{\underset{(x-2)(x+2)}{x^2-4}} = \frac{A}{x-2} + \frac{B}{x+2}$$

$y = 0$

$x \neq \pm 2$

$$2x = A(x+2) + B(x-2)$$

$$2x = Ax + 2A - 2B + Bx$$

$$2x = (A+B)x + 2A - 2B$$

$A + B = 2$

$2A - 2B = 0$ $\quad A = 1$

$2A = 2B$ $\quad B = 1$

$A = B$

$$\ln|y| = \int \frac{1}{x-2}\,dx + \frac{1}{x+2}\,dx$$

$$\ln|y| = \ln|x-2| + C + \ln|x+2| + D$$

$$\ln|y| = \ln|x^2-4| + D$$

$$y = e^{\ln|x^2-4|} \cdot e^{D}$$

$$y = E(x^2-4), \; \cancel{E \neq 0}, \; x \neq \pm 2$$

$$y = 0$$

(7) $y \cdot y' = -x$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$y^2 = -x^2 + C$$

$$y = \sqrt{C - x^2} \quad |x| < C$$

(8) $$\frac{2y'}{1-y^2} = \frac{2}{x}$$

$$2\int \frac{1}{1-y^2}\,dy = 2\int \frac{1}{x}\,dx$$

$$\ln|1-y^2| = \ln|x|\,dx$$

$$1-y^2 = x + C$$

$$\cancel{-y^2 = x - 1 + C}$$

$$\cancel{y^2 = 1 - x - C}$$

$$\cancel{y = \sqrt{-x + C}}$$

$$y'' - 4y' + 4y = e^{3x} + 8x$$

homogenní:

$$\lambda^2 - 4\lambda + 4 = 0$$

$$2 \quad 2$$

$$\left.\begin{array}{l}\lambda_1 = 2 \\ \lambda_2 = 2\end{array}\right\} R = \{e^{2x}, xe^{2x}\}$$

$$y_h = a \cdot e^{2x} + b \cdot x \cdot e^{2x}$$

$$y_p = A \cdot e^{3x} + Bx + C$$

$$y_p' = 3Ae^{3x} + B$$

$$y_p'' = 9Ae^{3x}$$

$$9Ae^{3x} - 4(3Ae^{3x} + B) + 4(Ae^{3x} + Bx + C) = e^{3x} + 8x$$

$$9Ae^{3x} - 12Ae^{3x} - 4B + 4Ae^{3x} + 4Bx + 4C = e^{3x} + 8x$$

$$Ae^{3x} + (-4 + 4x)B + 4C = 1e^{3x}$$

$$Ae^{3x} + 4Bx + (C - B)4 \rightarrow A = 1,\ B = 2,\ C = 2$$

$$4B = 8$$

$$B = 2$$

$$C - B = 0$$

$$C = B$$

$$y_p = e^{3x} + 2x + 2$$

$$y = y_p + y_h = e^{3x} + 2x + 2 + a \cdot e^{2x} \cdot bx \cdot e^{2x}$$

2 d)

$$y' = \frac{\cos(x)\, y}{\sin(x)}$$

$$\frac{y'}{y} = \frac{\cos(x)}{\sin(x)}$$

$$\frac{y'}{y} = \cot g(x)$$

$$\int \frac{1}{y}\, dy = \int \cot g(x)\, dx \quad \to y = 0 \text{ part.}$$

$$\int \frac{1}{y}\, dy = \int \frac{\cos(x)}{\sin(x)}\, dx \quad \Bigg|\ \begin{array}{l} \text{Sub.} \\ u = \sin(x) \\ du = \cos(x)\, dx \\ dx = \frac{du}{\cos(x)} \end{array}$$

$$\ln|y| = \int \frac{\cancel{\cos(x)}}{\sin(x)} \frac{du}{\cancel{\cos(x)}}$$

$$\ln|y| = \ln|\sin(x)| + C$$

$$y = C \cdot \sin(x), \quad C \neq 0$$

$$y = 0 \to C = 0$$

$$y = C \cdot \sin(x), \quad x, C \in \mathbb{R}$$

$y\left(\frac{\pi}{2}\right) = 13$

$13 = C \cdot 1$

$C = 13$

$y = 13 \cdot \sin(x)$

2b) $y' - 2y = \frac{1}{x} e^{2x}$

$\lambda - 2 = 0 \rightarrow \lambda = 2$

$y_h = C \cdot e^{2x}$

$y_p = C(x) \cdot e^{2x}$

$y_p' = C'(x) \cdot e^{2x} + C(x) \cdot 2e^{2x}$

$C'(x) \cdot e^{2x} + C(x) \cdot 2e^{2x} - 2\left(C(x) \cdot e^{2x}\right) = \frac{1}{x} e^{2x}$

$C'(x) \cdot e^{2x} + \underline{C(x) \cdot 2e^{2x}} \; \underline{- 2C(x) \cdot e^{2x}} = \frac{1}{x} e^{2x}$

$C'(x) \cdot e^{2x} = \frac{1}{x} e^{2x}$

$C'(x) = \frac{1}{x}$

$C(x) = \ln|x|$

$y_p = \ln|x| \cdot e^{2x}$

$y = C \cdot e^{2x} + \ln|x| \cdot e^{2x} \quad x \neq 0$

3b) $y' = x + xy^2$ $\qquad$ $y' = x(1+y^2)$

- separace: ano
- lin. + odhad: ne
- lin. + variace: ne

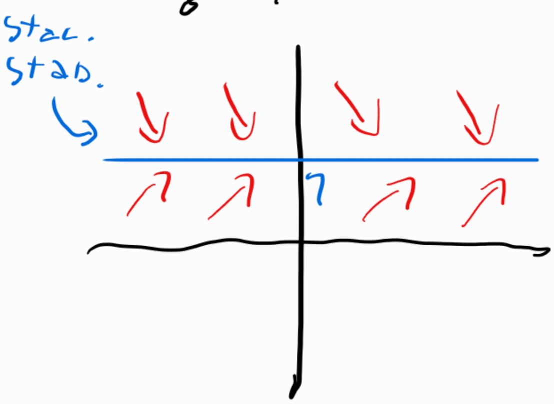
$$y' = \frac{1-y}{y^2+1}$$

$$\frac{1-y}{y^2+1} = 0$$

$y^2+1 \neq 0$

$1-y=0$ -> stac.

$y=1$ -> stac.



$$\frac{1-0}{0+1} = +$$

$$\frac{1-2}{4+1} = -$$

$$\frac{x}{\sqrt{x^2+1}} = 1$$

$$x+2y-2z=-5$$
$$2x+6y-2z=-8$$
$$x+4y-z=-5$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & -5 \\ 2 & 6 & -2 & -8 \\ 1 & 4 & -1 & -5 \end{array}\right) \sim -1 \left(\begin{array}{ccc|c} 1 & 3 & -1 & -4 \\ 1 & 2 & -2 & -5 \\ 1 & 4 & -1 & -5 \end{array}\right) \sim$$