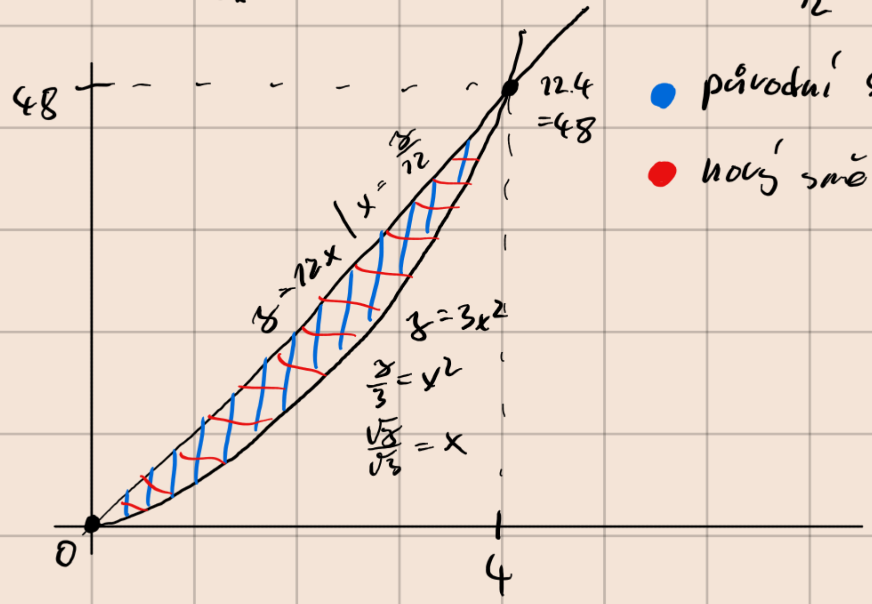
Lebesgueův integrál

1a) $$\int_0^4 \int_{3x^2}^{12x} f(x,y)\,dy\,dx = \int_0^{48} \int_{\frac{y}{12}}^{\sqrt{\frac{y}{3}}} f(x,y)\,dx\,dy$$



● původní směr integrace

● nový směr integrace

$3x^2 = 12x$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x_1 = 0$

$x_2 = 4$

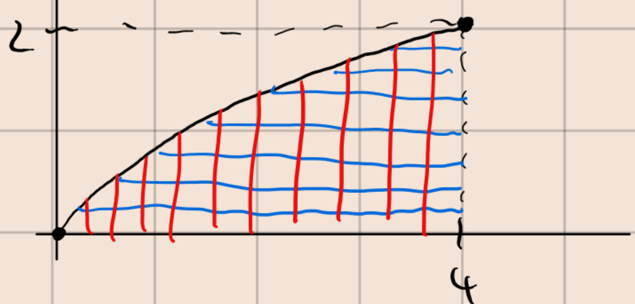
2b) $$\int_0^2 \int_{y^2}^4 \cos\sqrt{x^3}\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} \cos\sqrt{x^3}\,dy\,dx$$

$$= \int_0^4 \cos\sqrt{x^3} \left(\int_0^{\sqrt{x}} 1\,dy \right) dx =$$

$$= \int_0^4 \sqrt{x} \cos\sqrt{x^3}\,dx$$

$x = y^2$

$y = \sqrt{x}$



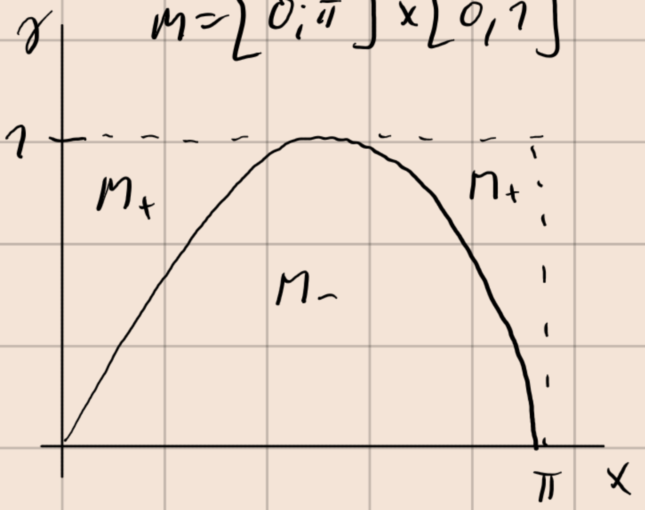
$$[\sin(x^{\frac{3}{2}})]' = \cos(x^{\frac{3}{2}}) \cdot \frac{3}{2} \cdot x^{\frac{1}{2}}$$

$$\int \cos(x^{\frac{3}{2}}) \cdot x^{\frac{1}{2}}\,dx = \frac{2}{3} \sin x^{\frac{3}{2}} + C$$

$$= \left[\frac{2}{3}\sin\left(\sqrt{x^3}\right)\right]_0^4 = \frac{2}{3}\sin(8)$$

3a) $$\int_M |y - \sin x| \, d\lambda^2 = \int_{M_+} (y - \sin x)\, d\lambda^2 + \int_{M_-} (\sin x - y)\, d\lambda^2$$

$M = [0;\pi] \times [0,1]$



$$= \int_0^\pi \int_{\sin x}^1 (y - \sin x)\, dy\, dx +$$

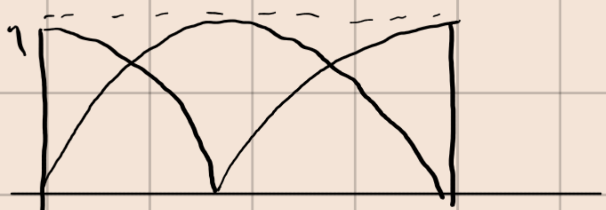
$$+ \int_0^\pi \int_0^{\sin x} (\sin x - y)\, d\lambda^2 =$$

$$= \int_0^\pi \left[\frac{y^2}{2} - y \sin x\right]_{\sin x}^1 dx +$$

$$+ \int_0^\pi \left[y \cdot \sin x - \frac{y^2}{2}\right]_0^{\sin x} dx =$$

$$= \int_0^\pi \left(\frac{1}{2} - \sin x - \frac{\sin^2 x}{2} + \sin^2 x\right) dx + \int_0^\pi \left(\sin^2 x - \frac{\sin^2 x}{2}\right) dx =$$

$$= \int_0^\pi \left(\sin^2 x - \sin x + \frac{1}{2}\right) dx =$$



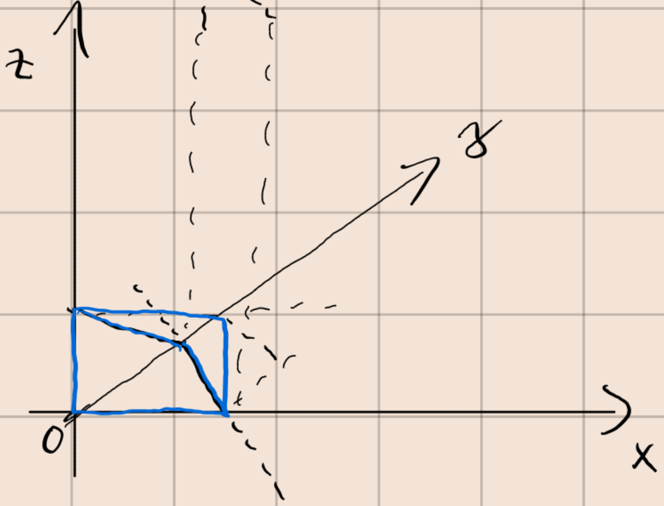
$$\pi = \int_0^\pi 1 = \int_0^\pi \sin^2 + \cos^2 dx = 2 \int_0^\pi \sin^2 dx$$

$$= \frac{\pi}{2} - \left[-\cos x\right]_0^\pi + \frac{\pi}{2} =$$

$$= \frac{\pi}{2} - 1 - 1 + \frac{\pi}{2} = \boxed{\pi - 2}$$

3e) $$\int_M y \, d\lambda^3 = \int_0^1 \int_0^{1-x} \int_0^{1-y} y \, dz\, dy\, dx =$$

$$\int^1 \int^{1-x}$$



$$= \int_0 \int_0 (1-y)\, y \, dy\, dx =$$

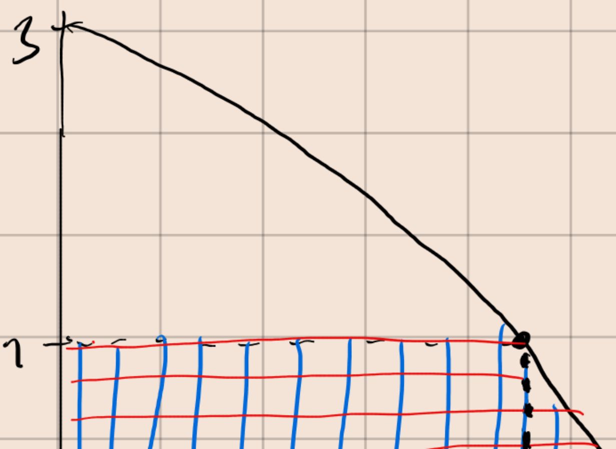
$$= \int_0^1 \int_0^{1-x} (y - y^2)\, dy\, dx =$$

$$= \int_0^1 \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^{1-x} dx = \int_0^1 \left(\frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} \right) dx =$$

$$= \left| \begin{array}{l} \text{sub.} \\ 1-x = t \\ -dx = dt \end{array} \right| = \int_0^1 \left(\frac{t^2}{2} - \frac{t^3}{3} \right) dt = \left[\frac{t^3}{6} - \frac{t^4}{12} \right]_0^1 =$$

$$= \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$$

4) $$\int_0^1 \int_0^{\sqrt{9-y^2}} f(x,y)\, dx\, dy = \circledast$$



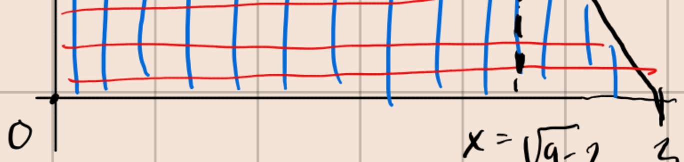
$x = \sqrt{9-y^2}$

$x^2 = 9 - y^2$

$x^2 + y^2 = 9$

$y = \sqrt{9 - \dots}$

$y = \sqrt{9-x^2}$



$x = \sqrt{9-1}$

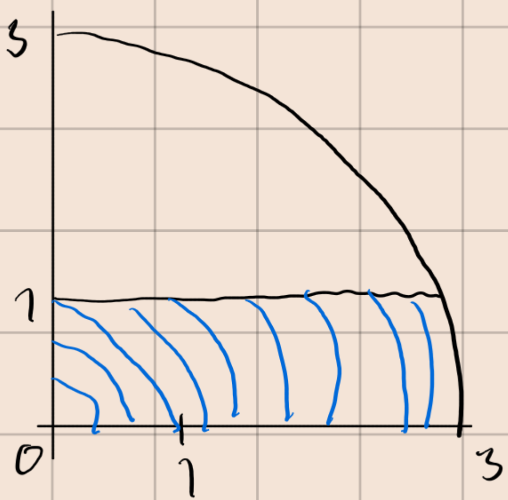
$= \sqrt{8} = 2\sqrt{2}$

$$\circledast = \int_0^{2\sqrt{2}} \int_0^1 f(x,y)\, dy\, dx + \int_{2\sqrt{2}}^{3} \int_0^{\sqrt{9-x^2}} f(x,y)\, dy\, dx$$

převod do polárních souřadnic:

$x = r \cdot \cos\varphi$

$y = r \cdot \sin\varphi$; $\phi(r,\varphi) = (r \cdot \cos\varphi, r \cdot \sin\varphi)$



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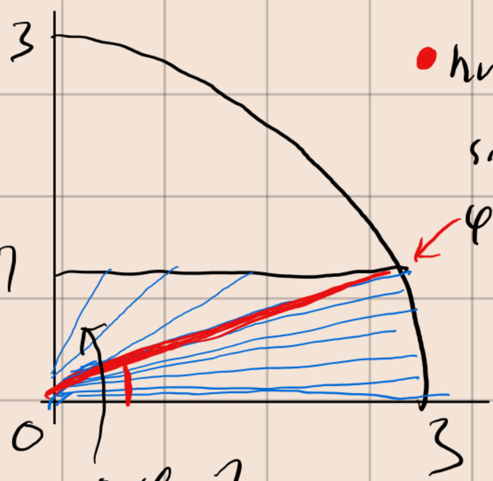


hran. úhel:

$\sin\varphi = \frac{1}{3}$

$\varphi = \arcsin\frac{1}{3}$

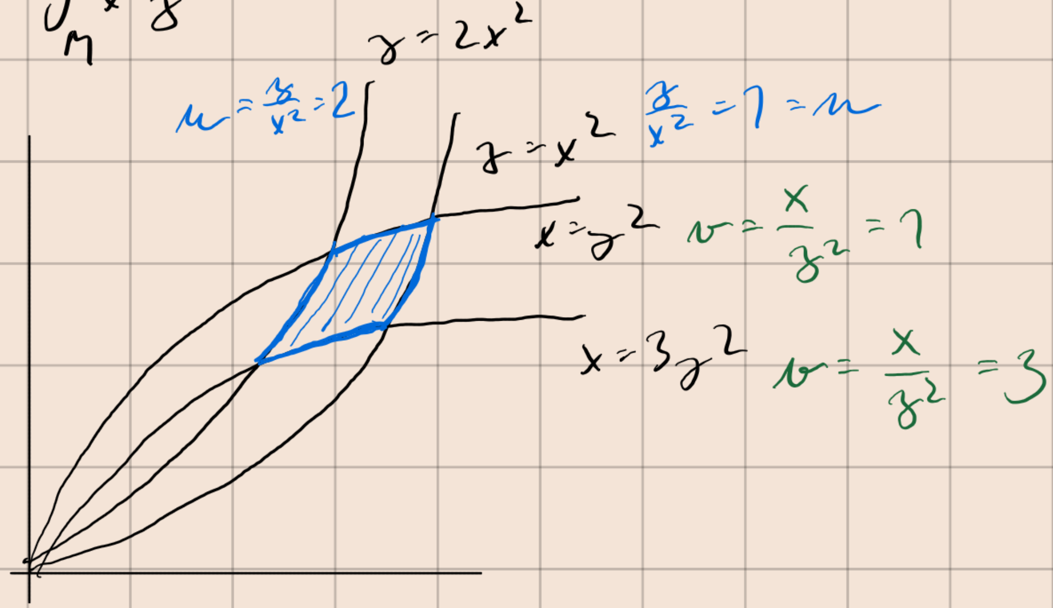
$\sin\varphi = \frac{1}{r}$

$r = \frac{1}{\sin\varphi}$

$$\int_0^{\arcsin\frac{1}{3}} \int_0^3 f(r\cdot\cos\varphi, r\cdot\sin\varphi)\cdot r\, dr\, d\varphi +$$

$$+ \int_{\arcsin\frac{1}{3}}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\varphi}} f(r\cdot\cos\varphi, r\cdot\sin\varphi)\cdot r\, dr\, d\varphi$$

6a) $\int_M \frac{1}{x^2 y^2} \, d\lambda^2$



$$\varphi(x,y) = \left(\frac{y}{x^2} ; \frac{x}{y^2} \right)$$

$$J_\varphi(x,y) = \begin{pmatrix} -2\frac{y}{x^3} & \frac{1}{x^2} \\ \frac{1}{y^2} & -2\frac{x}{y^3} \end{pmatrix}$$

$$|J_\varphi(x,y)| = 4 \cdot \frac{xy}{x^3 y^3} - \frac{1}{x^2 y^2} = \frac{3}{x^2 y^2}$$

$$\int_M \frac{1}{3} \frac{3}{x^2 y^2} \, d\lambda^2 = \int_{\varphi(M)} \frac{1}{3} \, d\lambda^2 =$$

$$\varphi(M) = [1,2] \times [1,3]$$

$$= \int_1^2 \int_1^3 \frac{1}{3} \, dv \, du = \boxed{\frac{2}{3}}$$

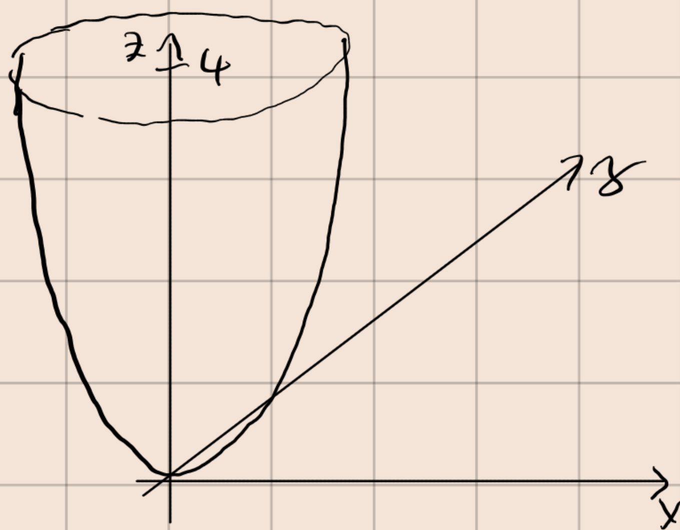
6f) $\int_M z \, d\lambda^3 =$

válcové souřadnice

$x = r \cdot \cos\varphi$

$y = r \cdot \sin\varphi$

$z = z$



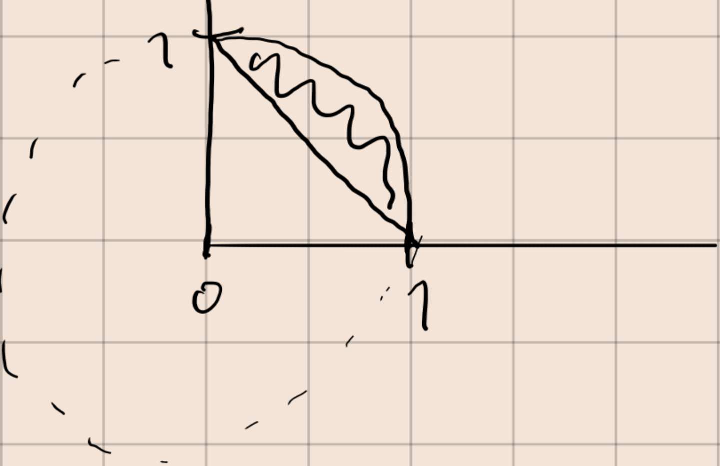
$\varphi(r, \varphi)$

7) $\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} \frac{x+y}{x^2+y^2} \, dy \, dx$

$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$



$$\varphi(x,y) = (r\cdot\cos\varphi,\ r\cdot\sin\varphi)$$

$$J_\varphi = \begin{pmatrix} \cos\varphi , & -r\cdot\sin\varphi \\ \sin\varphi , & r\cdot\cos\varphi \end{pmatrix}$$

$$\det J_\varphi = \cos\varphi\cdot r\cdot\cos\varphi + \sin\varphi\cdot r\cdot\sin\varphi$$

$$= r(\cos^2\varphi + \sin^2\varphi) = r\cdot 1 = \underline{r}$$

$$y = 1 - x$$

$$r\cdot\sin\varphi = 1 - (r\cdot\cos\varphi)$$

$$r\cdot\sin\varphi + r\cdot\cos\varphi = 1$$

$$r(\sin\varphi + \cos\varphi) = 1$$

$$r = \frac{1}{\sin\varphi + \cos\varphi}$$

$$\int_0^{\frac{\pi}{2}} \int_{\frac{1}{\sin\varphi + \cos\varphi}}^{1} \frac{r\cdot\cos\varphi + r\cdot\sin\varphi}{r^2\cdot\cos^2\varphi + r^2\cdot\sin^2\varphi}\cdot r \, dr \, d\varphi =$$

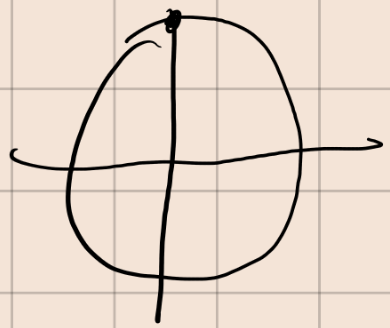
$$\left\| \frac{r^2(\cos\varphi + \sin\varphi)}{r^2(\cos^2\varphi + \sin^2\varphi)} = \frac{\cos\varphi + \sin\varphi}{1} = \cos\varphi + \sin\varphi \right\|$$

$$\int_0^{\frac{\pi}{2}} \quad 1 \quad \frac{\pi}{2}$$

$$= \int_0^{\frac{\pi}{2}} \left[\cos\varphi + \sin\varphi\right] \frac{1}{\sin\varphi + \cos\varphi} = \int_0^{\frac{\pi}{2}} \cos\varphi + \sin\varphi - 1 \, d\varphi =$$

$$\left[\sin\varphi - \cos\varphi - \varphi\right]_0^{\frac{\pi}{2}} = \left(1 - 0 - \frac{\pi}{2}\right) - \left(0 - 1 - 0\right) =$$

$$= 1 - \frac{\pi}{2} + 1 = 1 + \frac{\pi}{2}$$



$$= 2 - \frac{\pi}{2}$$

$$\int_0^1 \int_{\sqrt{y}-2}^{-1} \frac{x}{(x+2)^2} \, dx \, dy =$$

$x = \sqrt{y} - 2 \quad x + 2 = \sqrt{y}$

$(x+2)^2 = y$

$$\int_{-2}^{-1} \int_0^{(x+2)^2} \frac{x}{(x+2)^2} \, dy \, dx =$$

$$= \int_{-2}^{-1} \frac{x\cancel{(x+2)^2}}{\cancel{(x+2)^2}} = \int_{-2}^{-1} x \, dx = \left[\frac{x^2}{2}\right]_{-2}^{-1} =$$

$$= \frac{1}{2} - \frac{4}{2} = -\frac{3}{2}$$

$$f(x,y) = x - 2y^2$$

$$[2,0],\ [0,1]$$

$$r(t) = (1-t)A + t \cdot B$$

$$x(t) = (1-t)2 = 2-2t$$

$$y(t) = t$$

$$f(x(t), y(t)) = x(t) - 2(y(t)^2)$$

$$= 2(1-t) - 2t^2 = 2-2t-2t^2$$

$$\int_0^1 2-2t-2t^2 \cdot \|r'(t)\| \, dt = \int_0^1 2-2t-2t^2 \cdot \sqrt{5} \, dt =$$

$$x(t) = 2-2t \to x'(t) = -2$$

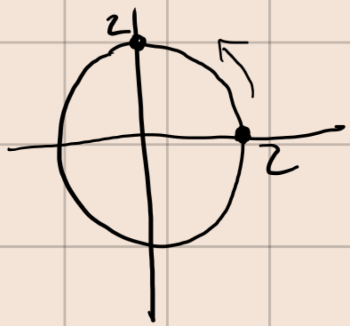
$$y(t) = t \to y'(t) = 1$$

$$\|r'(t)\| = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}$$

$$\sqrt{5} \cdot \int_0^1 2-2t-2t^2 \, dt = \sqrt{5} \cdot \left[2t - \frac{2t^2}{2} - \frac{2t^3}{3}\right]_0^1 =$$

$$= \sqrt{5} \left[2t - t^2 - \frac{2}{3}t^3\right]_0^1 = \sqrt{5} \cdot \left(2-1-\frac{2}{3}\right) = \sqrt{5} \cdot \frac{3}{3} - \frac{2}{3} =$$

$= \frac{\sqrt{5}}{3}$



$x = r \cdot \cos\varphi \qquad \rightarrow \qquad x = 2 \cdot \cos(t)$

$y = r \cdot \sin\varphi \qquad\qquad y = 2 \cdot \sin(t)$

$t \in [0; \frac{\pi}{2}]$

$\dot{r}(t) = \left(-2\sin(t), 2\cos(t)\right)$

$F(x, y) = (-xy, x^2)$

$f(x(t), y(t)) = \left(-2 \cdot \cos(t) \cdot 2 \cdot \sin(t), r^2 \cdot \cos^2(t)\right)$

$= \left(-4\cos(t) \cdot \sin(t), 4\cos^2(t)\right)$

$\int_0^{\frac{\pi}{2}} \left(-4\cos(t) \cdot \sin(t), 4\cos^2(t)\right) \cdot \left(-2\sin(t), 2\cos(t)\right)$

$= \int_0^{\frac{\pi}{2}} -4\cos(t) \cdot \sin(t) \cdot \left(-2\sin(t)\right) + 4\cos^2(t) \cdot 2\cos(t))$

$= \int_0^{\frac{\pi}{2}} 8\cos(t) \cdot \sin^2(t) + 8\cos^3(t) =$

$= \int_0^{\frac{\pi}{2}} 8\cos(t) \cdot \left(\sin^2(t) + \cos^2(t)\right) = \int_0^{\frac{\pi}{2}} 8\cos(t)$

$= 8\int_0^{\frac{\pi}{2}} \cos(t) = 8\left[\sin(t)\right]_0^{\frac{\pi}{2}} = 8$

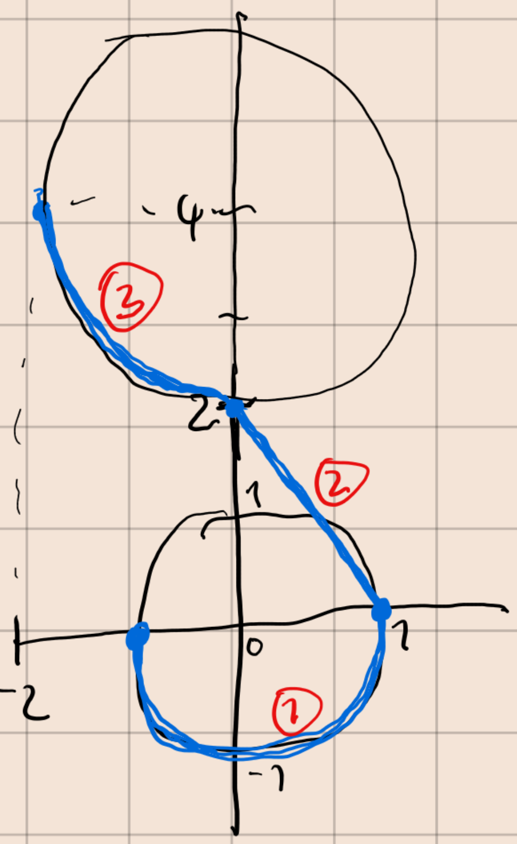
① $\varphi(t) = (t^2, 9t, 4t^{3/2}) \qquad t \in [1,4]$

$\varphi'(t) = (2t, 9, 6\sqrt{t})$

$$\int_1^4 \|\varphi'(t)\|\, dt = \int_1^4 \sqrt{4t^2+81+36t}\, dt = \int_1^4 2t+9 =$$

$$= \Big[t^2\Big]_1^4 + 27 = 15+27 = 42$$

② $\int_C x+y\, ds$



① $x(t) = \cos\varphi$
$y(t) = \sin\varphi \qquad \varphi \in [\pi, 2\pi]$
$r'(t) = (-\sin\varphi, \cos\varphi)$

② $x(t) = (1-t)\cdot 1 + 0t$
$y(t) = (1-t)\cdot 0 + 2t$
$\Downarrow$
$x(t) = 1-t \qquad t \in [0,1]$
$y(t) = 2t$
$r'(t) = (-1, 2)$

$$\int_\pi^{2\pi} (\cos(t)+\sin(t))\cdot\sqrt{\sin^2 t+\cos^2 t}\, dt =$$

$$= \int_\pi^{2\pi} \cos(t)+\sin(t)\, dt =$$

③ $x(t) = 2\cos\varphi$
$y(t) = 2\sin\varphi \qquad \varphi \in [\frac{\pi}{2}, 2\pi]$

$= \left[\sin(t) - \cos(t)\right]_{\pi} = -1 - 1 = -2$ $\quad \varphi'(t) = (-2\sin\varphi, 2\cos\varphi)$

$$\int_0^1 (1 - t + 2t) \cdot \sqrt{5}\, dt = \int_0^1 \sqrt{5} - t\sqrt{5}\, dt = \sqrt{5} - \sqrt{5}\left[\frac{t^2}{2}\right]_0^1 =$$

$$\frac{2\sqrt{5}}{2} - \frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{2}$$

③ $\int (y, z^2, x)\, ds$

$x^2 + y^2 = 1$ $\quad x = \cos(t)$

$\quad y = \sin(t) \quad t \in [0, 2\pi]$

$x + z = 1$ $\quad z = 1 - \cos(t)$

$\cos(t) + z = 1$ $\quad \varphi(t) = (\cos(t), \sin(t), 1 - \cos(t))$

$z = 1 - \cos(t)$ $\quad \varphi'(t) = (-\sin(t), \cos(t), \sin(t))$

$$\int_0^{2\pi} (\sin(t), (1 - \cos(t))^2, \cos(t)) \cdot (-\sin(t), \cos(t), \sin(t))\, dt =$$

$$\int_0^{2\pi} -\sin^2(t) + \cos(t) - 2\cos^2(t) + \cos^3(t) + \sin(t) \cdot \cos(t)\, dt =$$

$$= \int_0^{2\pi} \left(-\frac{1}{2} + \frac{\cos 2t}{2}\right) dt + \int_0^{2\pi} -(1 + \cos 2t)\, dt + \ldots =$$

$= -3\pi$

④ $F(x, y, z) = (\ldots z, y \ldots)$

1 $F(x,y,z) = (y^2, z, x)$

$$x^2+y^2+z^2=1 \qquad x = \sin$$

$$x=y$$

$$2x^2+z^2=1$$

$$\varphi(t) = \left(\sqrt{\frac{1-t^2}{2}}, \sqrt{\frac{1-t^2}{2}}, t\right) \qquad t\in[0,1]$$

$$\left(\sqrt{\frac{1-t^2}{2}}\right)' = \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{1-t^2}{2}}}\cdot -t = \frac{-t}{\sqrt{2(1-t^2)}} =$$

$$\left(\frac{1-t^2}{2}\right)' = \frac{1}{2}\cdot(1-t^2) = -\frac{2t}{2} = -t$$

$$\varphi'(t) = \left(\frac{-t}{\sqrt{2(1-t^2)}}, \frac{-t}{\sqrt{2(1-t^2)}}, 1\right)$$