směrová derivace

$$\nabla_v f(a) = \lim_{t\to 0} \frac{f(a+tv)-f(a)}{t} = \lim_{t\to 0}\frac{\varphi(t)-\varphi(0)}{t} = \varphi'(0)$$

$\varphi(t) = f(a+t.v)$

Parciální derivace:

$$\frac{\partial f(a)}{\partial x} = \nabla_{(1,0)} f(a) = \lim_{t\to 0}\frac{f(a_1 t, a_2) - f(a_1, a_2)}{t}$$

$\frac{\partial f}{\partial y}(a) = \nabla_{(0,1)} f(a)$ ... obvykle počítáme jako derivaci jedné proměnné, ostatní prom. jako konstanty

$$\frac{\partial^2 f}{\partial y\,\partial x}(a) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(a)$$

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b) $f(x,y) = \frac{xy}{x+y}$, $a=(2,-1)$, $v=(4,1)$, $D = \mathbb{R}^2 \setminus \{(x,-x), x\in\mathbb{R}\}$

$a+tv = (2+4t, -1+t.1) = (2+4t, -1+t)$

$$\nabla_v f(a) = \lim_{t\to 0}\frac{f(2+4t,-1+t) - f(2,-1)}{t} =$$

$$= \lim_{t\to 0}\frac{\frac{(2+4t)(-1+t)}{2+4t-1+t} - \frac{2(-1)}{2-1}}{t} = \lim_{t\to 0}\frac{\frac{-2+2t-4t+4t^2}{1+5t}+2}{t} =$$

$$= \lim_{t\to 0}\frac{-2+2t+4t^2+2+10t}{t(1+5t)} = \lim_{t\to 0}\frac{t(4t+8)}{t(1+5t)} = \lim_{t\to 0}\frac{4t+8}{1+5t} = 8$$

$$\varphi(t) = \frac{(2+4t)(-1+t)}{1+5t}$$

$$\varphi'(0) = \left(\frac{4t^2-2t-2}{1+5t}\right)'\Bigg|_{t=0} = \frac{(8t-2)(1+5t)-5(4t^2-2t-2)}{(1+5t)^2}\Bigg|_{t=0} =$$

$$= \frac{-2\cdot 1 - 5(-2)}{1^2} = \frac{8}{1} = 8$$

(2)

$f(x,y) = \sqrt{|xy|}$, $D = \mathbb{R}^2$, $a = (0,0)$, $v = (v_1, v_2)$, $a + tv = (tv_1, tv_2)$

$$\nabla_v f(a) = \lim_{t \to 0} \frac{\sqrt{|tv_1 \cdot tv_2|} - \sqrt{|0 \cdot 0|}}{t} = \lim_{t \to 0} \frac{\sqrt{|v_1 \cdot v_2|} \cdot \sqrt{t^2}}{t} =$$

$$= \lim_{t \to 0} \sqrt{v_1 v_2} \cdot \frac{|t|}{t} = \lim_{t \to 0} \sqrt{v_1 v_2} \cdot \operatorname{sgn}(t) =$$

$$= \begin{cases} v_1 \cdot v_2 \neq 0: & \left.\begin{array}{l} \lim_{t \to 0^+} \sqrt{|v_1 \cdot v_2|} \cdot \operatorname{sgn}(t) = \sqrt{|v_1 \cdot v_2|} \\ \lim_{t \to 0^-} \sqrt{|v_1 \cdot v_2|} \cdot \operatorname{sgn}(t) = -\sqrt{|v_1 \cdot v_2|} \end{array}\right\} \Rightarrow \text{limita \underline{zde} neexistuje} \\ v_1 \cdot v_2 = 0: & \lim_{t \to 0} \sqrt{|v_1 \cdot v_2|} \cdot \operatorname{sgn}(t) = \lim_{t \to 0} 0 = 0 \Rightarrow \text{směrová derivace zde existuje a je rovna } 0 \end{cases}$$

(4) a) $f(x,y) = x^2 - 3x^2 y + 5y^3$, $a = (-2, 1)$

- $$\frac{\partial f}{\partial x}(-2,1) = \frac{\partial (x^2 - 3x^2 y + 5y^3)}{\partial x}(-2,1) = 2x - 3y \cdot 2x + 0 \Big|_{(-2,1)} = -4 - 3 \cdot 1 \cdot 2(-2)$$

$$= 8$$

- $$\frac{\partial f}{\partial y}(-2,1) = \frac{\partial (x^2 - 3x^2 y + 5y^3)}{\partial y}(-2,1) = -3x^2 + 15y^2 \Big|_{(-2,1)} = -3 \cdot 4 + 15 \cdot 1 = -12 + 15$$

$$= 3$$

(5) b) $f(x, y, z) = \sqrt{x^2 + y^2 \cos^2 z + 1}$, $D = \mathbb{R}^3$

- $$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{1}{\sqrt{x^2+y^2\cdot\cos^2 z+1}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2\cdot\cos^2 z+1}},\quad (x,y,z)\in\mathbb{R}^3$$

- $$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{1}{\sqrt{x^2+y^2\cdot\cos^2 z+1}}\cdot 2y\cos^2 z = \frac{y\cdot\cos^2 z}{\sqrt{x^2\cdot y^2\cdot\cos^2 z+1}}$$

- $$\frac{\partial f}{\partial z} = \frac{1}{2}\cdot\frac{1}{\sqrt{x^2\cdot y^2\cdot\cos^2 z+1}}\cdot 2\cos z\,(-\sin z)\,y^2 = \frac{-y^2\cos z\,(-\sin z)}{\sqrt{x^2\cdot y^2\cdot\cos^2 z+1}}$$

(6) $$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

- pro $(x,y)\neq(0,0)$:

$$\frac{\partial f}{\partial x} = \frac{y(x^2+y^2)-xy\,2x}{(x^2+y^2)^2} = \frac{y^3-x^2y}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$ (ze simetrie x a y)

$\Rightarrow$ pro $(x,y)\neq(0,0)$ jsou parc. derivace spojité (ze spojitosti polynomů a podílů)

- pro $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(0+t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{\frac{t\cdot 0}{t^2+0^2}-0}{t} = \lim_{t\to 0}\frac{0-0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$ (ze symetrie)

spojitost v $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \lim_{(x,y)\to(0,0)} \frac{y(y^2-x^2)}{(x^2+y^2)^2} \stackrel{?}{=} \frac{\partial f}{\partial x}(0,0) = 0$$

$x = y$ $\quad \lim_{x\to 0} \frac{x(x^2-x^2)}{(x^2+x^2)^2} = 0$

$x = 0$ $\quad \lim_{y\to 0} \frac{y^3}{y^4} = \lim_{y\to 0} \frac{1}{y} =$ neexistuje

$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}$ neexistuje

$\Rightarrow \frac{\partial f}{\partial x}$ není v $(0,0)$ spojitá

(pro y ze symetrie)

(10)

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}, \quad D = \mathbb{R}^3 \setminus \{\vec{0}\}$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- $$\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} 2x = -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial x^2} = -x(x^2+y^2+z^2)^{-\frac{3}{2}} - x \cdot \left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}} \cdot 2x$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}^3}\left(-1 + \frac{3x^2}{x^2+y^2+z^2}\right)$$

- $$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\sqrt{x^2+y^2+z^2}^3}\left(-1 + \frac{3y^2}{x^2+y^2+z^2}\right)$$

- $$\frac{\partial^2 f}{\partial x} = \frac{1}{\sqrt{\quad}^3}\left(-1 + \frac{3y^2}{x^2+y^2+z^2}\right)$$

$$\sqrt{x^2+y^2+z^2}$$

$$\Delta f = \_ + \_ + \_ = \frac{1}{\sqrt{x^2+y^2+z^2}^3}\left(-3+\frac{3x^2+3y^2+3z^2}{x^2+y^2+z^2}\right) =$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\left(-3+3\cdot 1\right) = 0$$

(8) $f(x,y) = \dfrac{y}{2x+3y}$, $2x \neq -3y$, $x \neq -\frac{3}{2}y$, $D = \mathbb{R}^2 \setminus \left\{\left(x, -\frac{2}{3}x\right), x \in \mathbb{R}\right\}$

- $\dfrac{\partial f}{\partial x} = \dfrac{0 - y\cdot 2}{(2x+3y)^2} = \dfrac{-2y}{(2x+3y)^2}$
- $\dfrac{\partial f}{\partial y} = \dfrac{2x+3y - y\cdot 3}{(2x+3y)^2} = \dfrac{2x}{(2x+3y)^2}$

$$\frac{\partial f}{\partial y \partial x} = \frac{-2(2x+3y)^2 + 2y\cdot 2\cdot(2x+3y)\cdot 3}{(2x+3y)^4} = \frac{-4x-6y-12y}{(2x+3y)^3} = \frac{-4x+6y}{(2x+3y)^3}$$

$$\frac{\partial f}{\partial x \partial y} = \frac{2(2x+3y)^2 - 2x\cdot 2(2x+3y)2}{(2x+3y)^3} = \frac{4x-6y-8x}{(2x+3y)^3} = \frac{-4x+6y}{(2x+3y)^3}$$