Úloha 1. Nalezněte vlastní čísla a vlastní vektory matic

$$\boxed{\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}}, \quad \mathbf{Z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Poté dané matice diagonalisujte.

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\boxed{P \cdot \vec{x} = \lambda \cdot \vec{x} \qquad \boxed{\vec{x} \neq \vec{o}}}$$

$$\Updownarrow$$

$$P \cdot \vec{x} - \lambda \cdot \vec{x} = \vec{o}$$

$$\Updownarrow$$

$$P \cdot \vec{x} - \lambda \cdot E_2 \cdot \vec{x} = \vec{o}$$

$$\Updownarrow$$

$$(\vec{x} \neq \vec{o}) \quad \boxed{(P - \lambda \cdot E_2) \cdot \vec{x} = \vec{o}}$$

$$P - \lambda E_2 = \begin{pmatrix} 1/2-\lambda & 1/2 \\ 1/2 & 1/2-\lambda \end{pmatrix}$$

$$0 = \begin{vmatrix} 1/2-\lambda & 1/2 \\ 1/2 & 1/2-\lambda \end{vmatrix} = (1/2-\lambda)^2 - \frac{1}{4} =$$

$$= \frac{1}{4} - \lambda + \lambda^2 - \frac{1}{4} =$$

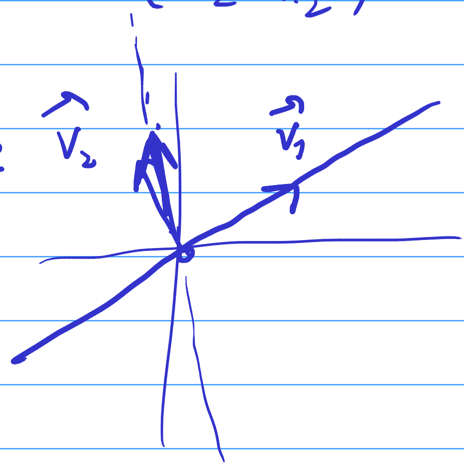
$$= \lambda^2 - \lambda = \underbrace{\lambda(\lambda - 1)}$$

$$\boxed{\lambda_1 = 1}, \quad \boxed{\lambda_2 = 0}$$

$$\Updownarrow$$

$$P \cdot \vec{v}_1 = 1 \cdot \vec{v}_1 \qquad P \cdot \vec{v}_2 = 0 \cdot \vec{v}_2 = \vec{o}$$

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$



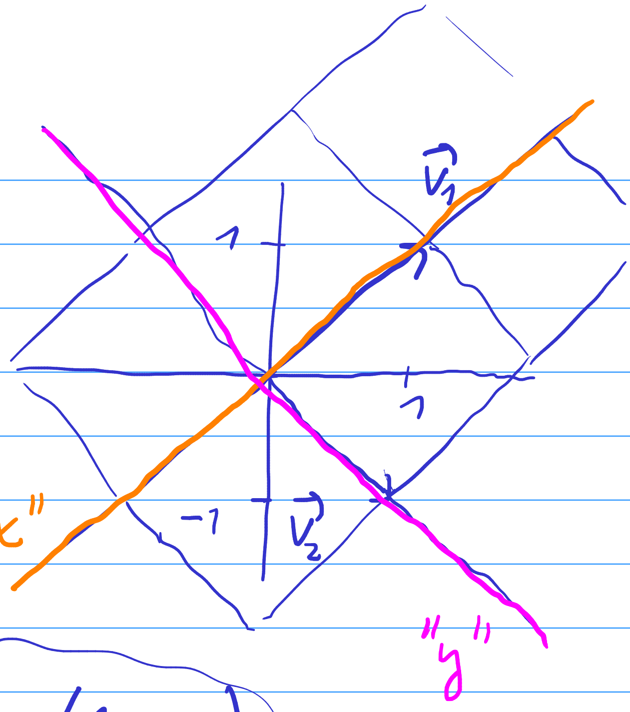
Pro $\boxed{\lambda_1 = 1}$:

$$\left(\begin{array}{cc|c} -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 + R_1 \end{array}$$

Řešení: $\mathrm{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Ať $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Pro $\boxed{\lambda_2=0}$:

$$\left(\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$



Řešení : $\operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

Atž $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$B = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = (\vec{v}_1, \vec{v}_2) \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Diagonalisuje matici P.

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P \cdot \vec{v}_1 = 1 \cdot \vec{v}_1 \qquad P \cdot \vec{v}_2 = 0\, \vec{v}_2 \qquad T = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}$$

$$P \cdot T = P \cdot (\vec{v}_1 \ \vec{v}_2) = (1 \cdot \vec{v}_1 \ \ 0 \cdot \vec{v}_2) = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = T \cdot D.$$

$$P \cdot T = T \cdot D$$

$\cdot T^{-1}$ zprava

$$P = T \cdot D \cdot T^{-1}$$

$$\boxed{T = T_{B \mapsto K_2}}$$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = P = T_{B \mapsto K_2} \cdot D \cdot T_{K_2 \mapsto B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

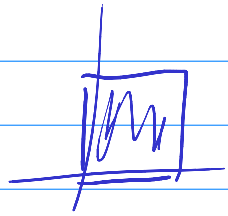
$$\boxed{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$$

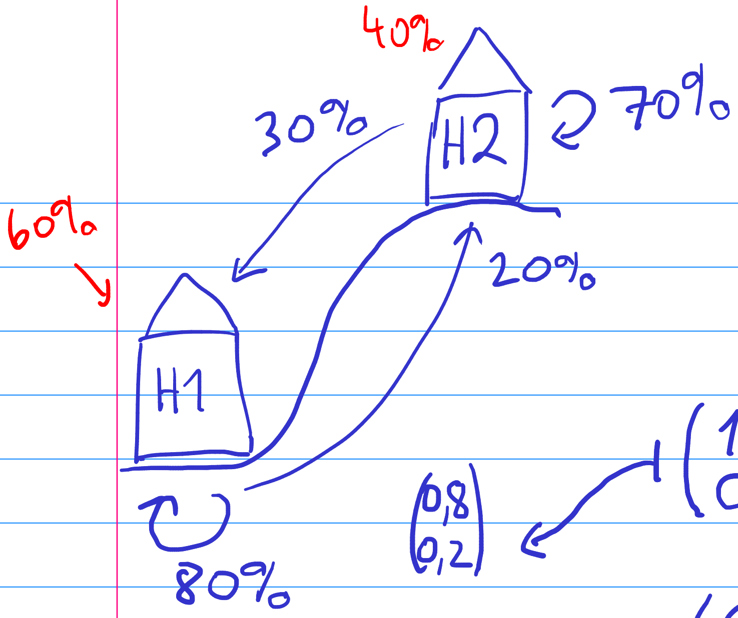
$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - \cancel{0}$$

$\lambda_1 = 1$ dvojnásobný

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Řešení: $\operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$





Počáteční stav: určitě v H1.

? Kde bude po 100 pivech?

$$\begin{pmatrix}0{,}8\\0{,}2\end{pmatrix} \longleftarrow\!\shortmid \begin{pmatrix}1\\0\end{pmatrix} = \vec{p} \quad \ldots \text{ na } 100\% \text{ v } H1$$

$$\begin{pmatrix}0{,}3\\0{,}7\end{pmatrix} \longleftarrow\!\shortmid \begin{pmatrix}0\\1\end{pmatrix} \quad \ldots \text{ na } 100\% \text{ v } H2$$

$$A = \begin{pmatrix}0{,}8 & 0{,}3\\0{,}2 & 0{,}7\end{pmatrix} \qquad \begin{pmatrix}0{,}9\\0{,}1\end{pmatrix} \ldots \text{ na } 90\% \text{ v } H1.$$

$$A \cdot A \cdot \vec{p} \qquad\qquad \underline{A^{100} \cdot \vec{p}}$$

Nalezneme vlastní čísla a vlastní vektory matice A.

$$\begin{vmatrix}0{,}8-\lambda & 0{,}3\\0{,}2 & 0{,}7-\lambda\end{vmatrix} = (0{,}8-\lambda)(0{,}7-\lambda) - 0{,}3\cdot 0{,}2 =$$

$$= 0{,}56 - 1{,}5\lambda + \lambda^2 - 0{,}06 =$$

$$= \lambda^2 - 1{,}5\lambda + 0{,}5$$

$$= (\lambda-1)(\lambda-0{,}5)$$

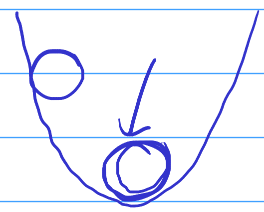
$\boxed{\lambda_1 = 1}$, $\boxed{\lambda_2 = 0{,}5}$.

Pro $\boxed{\lambda_1 = 1}$: $\left(\begin{array}{cc|c} 0{,}8-1 & 0{,}3 & 0 \\ 0{,}2 & 0{,}7-1 & 0 \end{array}\right)$

$$= \left(\begin{array}{cc|c} -0{,}2 & 0{,}3 & 0 \\ 0{,}2 & -0{,}3 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -0{,}2 & 0{,}3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

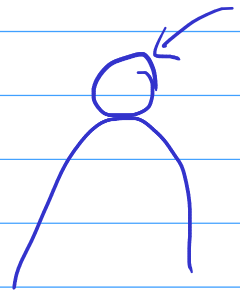
Řešení: $\mathrm{span}\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right)$

Až $\vec{v}_1 = \begin{pmatrix} 0{,}6 \\ 0{,}4 \end{pmatrix}$. $\quad A \cdot \vec{v}_1 = \vec{v}_1 \quad$ $\vec{v}_1$ pevný bod A. ekvilibrium.



$\boxed{\lambda_2 = 0{,}5}$

$$\left(\begin{array}{cc|c} 0{,}8-0{,}5 & 0{,}3 & 0 \\ 0{,}2 & 0{,}7-0{,}5 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 0{,}3 & 0{,}3 & 0 \\ 0{,}2 & 0{,}2 & 0 \end{array}\right) \sim$$



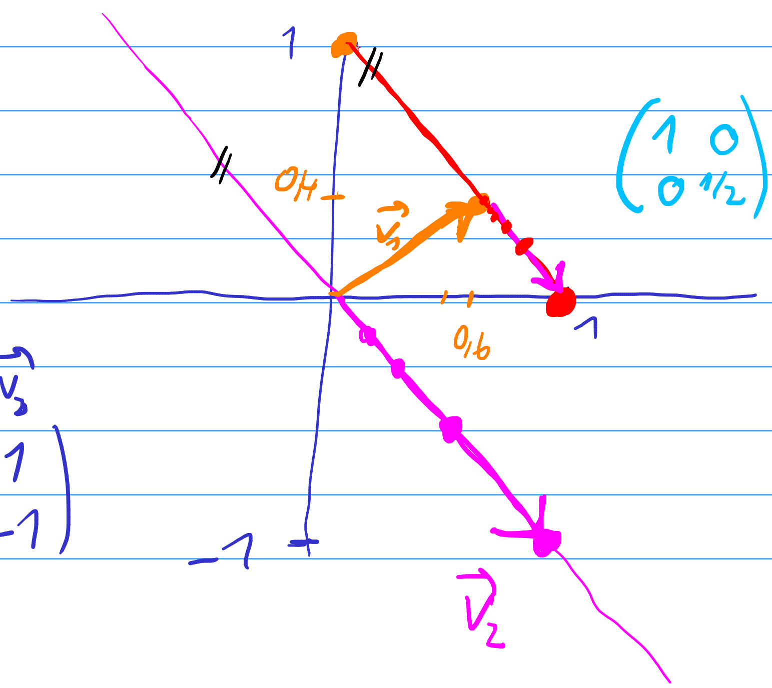
$$\sim \left(\begin{array}{cc|c} 0{,}3 & 0{,}3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Řešení: $\mathrm{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$ $\quad$ Až $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\begin{pmatrix} x \\ y \end{pmatrix}$

$x + y = 1$

$A^{100} \cdot \vec{p}$



$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underset{=}{1} \overset{\vec{v}_1}{\begin{pmatrix} 0{,}6 \\ 0{,}4 \end{pmatrix}} + \underset{=}{0{,}4} \overset{\vec{v}_2}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$A \quad A$

$$A^{100}\cdot\begin{pmatrix}1\\0\end{pmatrix} = A^{100}\left(\begin{pmatrix}0{,}6\\0{,}4\end{pmatrix} + 0{,}4\begin{pmatrix}1\\-1\end{pmatrix}\right) =$$

$$= A^{100}\underbrace{\begin{pmatrix}0{,}6\\0{,}4\end{pmatrix}}_{\vec{v}_1} + 0{,}4\cdot A^{100}\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{\vec{v}_2} =$$

$$= 1\begin{pmatrix}0{,}6\\0{,}4\end{pmatrix} + 0{,}4\cdot(0{,}5)^{100}\begin{pmatrix}1\\-1\end{pmatrix} =$$

$$\approx \begin{pmatrix}0{,}6\\0{,}4\end{pmatrix}.$$

$$\mathbb{D} = \begin{pmatrix}1 & 0\\0 & 1/2\end{pmatrix} \qquad \mathbb{D}^n = \begin{pmatrix}1^n & 0\\0 & (1/2)^n\end{pmatrix}$$

Úloha 4. Spočtěte $\mathbf{A}^2$ a $\mathbf{A}^6$ pro matici

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$