Metoda substituce

$$1)\quad \int f(x)\,dx = \overset{\text{SUB.}}{\left| \begin{matrix} x=\varphi(t) \\ dx=\varphi'(t)\,dt \end{matrix} \right|} = \int f(\varphi(t))\cdot\varphi'(t)\,dt$$

φ - ryze monotónní na otevřeném intervalu I.

$$\int\sqrt{1-x^2}\,dx = \overset{\text{SUB.}}{\left| \begin{matrix} x=\sin(t) \\ dx=\cos(t)\,dt \\ t\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)=I \\ \sin(t)\text{ rostoucí} \\ \text{na } I \\ \arcsin(x)=t \end{matrix} \right|} = \int\underbrace{\sqrt{1-\sin^2 t}}_{\substack{=\sqrt{\cos^2 t} \\ =|\cos t|=\cos t \\ \text{pro } t\in I}}\cdot\cos t\,dt = \int\cos t\cdot\cos t\,dt$$

$$=\int\cos^2 t\,dt = \int\frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)+C$$

$$=\frac{1}{2}t+\frac{1}{4}\underbrace{\sin 2t}_{2\cdot\sin t\cdot\cos t}+C = \frac{1}{2}t+\frac{1}{2}\sin t\cdot\sqrt{1-\sin^2 t}+C$$

$$=\underline{\frac{1}{2}\arcsin x+\frac{1}{2}x\cdot\sqrt{1-x^2}+C} \qquad x\in(-1;1)$$

ze spojitosti integrandu a PF $x\in\langle-1;1\rangle$

$$2)\quad \int f(\varphi(x))\,\varphi'(x)\,dx = \overset{\text{SUB.}}{\left| \begin{matrix} \varphi(x)=z \\ \varphi'(x)\,dx=dz \end{matrix} \right|} = \int f(z)\,dz$$

$$\int(5x-2)^5\,dx = \left| \begin{matrix} 5x-2=z \\ 5\,dx=dz \end{matrix} \right| = \frac{1}{5}\int(5x-2)^5\cdot 5\,dx = \frac{1}{5}\int z^5\,dz =$$

$$= \frac{1}{5}\cdot\frac{z^6}{6} = \frac{z^6}{30} + C = \frac{(5x-2)^6}{30} + C;\ x\in\mathbb{R}$$

$$\int \frac{1}{(2x+3)}\,dx = \left| \begin{matrix} 2x+3=z \\ 2\,dx = dz \\ dx=\frac{1}{2}\,dz\end{matrix}\right| = \int \frac{1}{z}\cdot\frac{1}{2}\,dz = \frac{1}{2}\int\frac{1}{z}\,dz =$$

$$= \frac{1}{2}\cdot\ln|z| + C = \frac{1}{2}\ln|2x+3|+C;\ \begin{matrix} x\in(-\infty;-\frac{3}{2}) \\ x\in(-\frac{3}{2};\infty)\end{matrix}$$

$$\int\frac{x^2}{\sqrt{x^3+1}}\,dx = \int\frac{1}{\sqrt{x^3+1}}\cdot x^2\,dx = \left|\begin{matrix} x^3+1=z \\ 3x^2\,dx=dz \\ x^2\,dx=\frac{1}{3}\,dz\end{matrix}\right| =$$

$x^3>-1$

$x>-1$

$$=\frac{1}{3}\int\frac{1}{\sqrt{z}}\,dz = \frac{1}{3}\int z^{-1/2}\,dz = \frac{1}{3}\cdot\frac{z^{\frac{1}{2}}}{\frac{1}{2}} = \frac{1}{3}\cdot\frac{2\sqrt{z}}{3}+C=$$

$$\frac{1}{9}\sqrt{z} + C = \frac{1}{9}\sqrt{x^3+1}+C;\ x\in(-1;\infty)$$

$$\int\frac{1}{x}\sqrt{\ln x}\,dx = \left|\begin{matrix}\ln x = z \\ \frac{1}{x}\,dx = dz\end{matrix}\right| = \int\sqrt{z}\,dz =$$

$x\geq 1$

$$= \frac{z^{3/2}}{3/2} + C = \frac{2z^{3/2}}{3} + C = \frac{2}{3}\sqrt{z^3} + C = \frac{2}{3}\sqrt{\ln^3 x} + C;$$

$x \in (0; \infty)$

$$\int \sin x \cdot \cos^3 x \, dx = -\int -\sin x \cdot \cos^3 x \, dx = \left| \begin{array}{l} \cos x = z \\ -\sin x \, dx = dz \end{array} \right|$$

$$= -\int z^3 \, dz = -\frac{z^4}{4} + C = -\frac{\cos^4 x}{4} + C; \; x \in R$$

Ai: $\int \frac{1}{x} \ln^2 x \, dx$ $\qquad \int \frac{4x-6}{x^2-3x+5} \, dx$

$\int \sin^6 x \cdot \cos x \, dx$

$$\int \frac{f'(x)}{f(x)} \, dx = \left| \begin{array}{l} f(x) = t \\ f'(x) \, dx = dt \end{array} \right| = \int \frac{1}{t} \, dt = \ln|t| + C$$

$$= \ln|f(x)| + C$$

$$\int \cotg 3x \, dx = \int \frac{\cos 3x}{\sin 3x} \, dx = \frac{1}{3} \int \frac{3\cos 3x}{\sin 3x} \, dx = \frac{1}{3} \ln|\sin x| + C$$

$\sin 3x \neq 0$

$3x \neq k.\pi$

$x \neq k \cdot \frac{\pi}{3} \; ; k \in \mathbb{Z}$

$x = \left(0; \frac{\pi}{3}\right) + k \cdot \frac{\pi}{3} \; ; k \in \mathbb{Z}$

$$\int \operatorname{arccotg} x \, dx = \int 1 \cdot \operatorname{arccotg} x \, dx = \left| \begin{matrix} PP \\ u = \operatorname{arccotg} x & v' = 1 \\ u' = \frac{-1}{x^2+1} & v = x \end{matrix} \right| =$$

$$= x \cdot \operatorname{arccotg} x - \int \frac{-1}{x^2+1} x \, dx = x \cdot \operatorname{arccotg} x + \int \frac{x}{x^2+1} dx =$$

$$= x \cdot \operatorname{arccotg} + \frac{1}{2} \int \frac{2x}{x^2+1} dx = x \cdot \operatorname{arccotg} + \frac{1}{2} \ln |x^2+1| + C$$

$x \in \mathbb{R}$

Integrace racionálních funkcí

$$\frac{P(x)}{Q(x)} = \text{polynom} + \text{fce ryze lomená} \quad \deg(P) < \deg(Q) \quad =$$

P, Q ... polynomy

= polynom + součet parciálních zlomků :

$$\frac{A}{(x-a)^n}, \; \frac{Bx+C}{(x^2+px+q)^n}$$

$$\frac{2x^2-9x+8}{x^2-5x+6} = 2 + \frac{x-4}{x^2-5x+6}$$

$$(2x^2-9x+8):(x^2-5x+6) = 2 + \frac{x-4}{\underset{(x-2)(x-3)}{x^2-5x+6}} = 2 + \underbrace{\frac{x-4}{(x-2)(x-3)}}_{R_1(x)} =$$

$$\begin{array}{l} -2x^2+10x-12 \\ \quad\quad x-4 \end{array}$$

$$\frac{x-4}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

$$x-4 = A(x-3) + B(x-2) = Ax-3A+Bx-2B =$$

$$= (A+B)x-3A-2B$$

porovnáme koeficienty:

$$\left.\begin{array}{l} x^1: \ 1=(A+B) \\ x^0: \ -4=-3A-2B \end{array}\right\} \begin{array}{l} A=2 \\ B=-1 \end{array}$$

$$= 2 + \frac{2}{x-2} - \frac{1}{x-3} \ ; \ x \in \{2;3\}$$

zakrývací pravidlo:

$x:=2$

$$A = \left.\frac{x-4}{x-3}\right|_{x:=2} = \frac{2-4}{2-3} = 2$$

$x:=3$

$$B = \frac{3-4}{3-2} = \frac{-1}{1} = -1$$

$$\int \frac{2x^2-9x+8}{x^2-5x+6}\,dx = \int 2 + \frac{2}{x-2} - \frac{1}{x-3}\,dx =$$

$$= 2x + 2\ln|x-2| - \ln|x-3| + C; \quad \begin{array}{l} x \in (-\infty; 2) \\ x \in (2; 3) \\ x \in (3; \infty) \end{array}$$