$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{1}{x-3y} dx - \frac{3}{x-3y} dy$$

$$\nabla f(7,2) = (1,-3)$$

$$H_f = \begin{pmatrix} -\frac{1}{(x-3y)^2} & \frac{3}{(x-3y)^2} \\ \frac{3}{(x-3y)^2} & -\frac{9}{(x-3y)^2} \end{pmatrix} \qquad H_f(7,2) = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}$$

$$d^2 f = -dx^2 + 6dx\,dy - 9dy^2$$

$$T_2(x,y) = 0 + (1,-3)\cdot\begin{pmatrix} x-7 \\ y-2 \end{pmatrix} + \frac{1}{2}(x-7, y-2)\begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x-7 \\ y-2 \end{pmatrix}$$

$$T_2(6{,}9; 2{,}02) = (1;-3)\cdot\begin{pmatrix} -0{,}1 \\ 0{,}02 \end{pmatrix} + \frac{1}{2}(-0{,}1; 0{,}02)\begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} -0{,}1 \\ 0{,}02 \end{pmatrix}$$

---

a stacionární $\Leftrightarrow \nabla f(a) = \vec{0}$

$Hf(a)$

↳ poz. definitní ⇒ minimum

↳ neg. definitní ⇒ maximum

↳ indefinitní ⇒ sedlo

sylvester

+ + + poz. definitní

− + − neg. definitní

+ subdeterminant

+ subdeterminant

1 a) $f(x,y) = x^4 - 2x^2 + y^3 - 3y$

$\nabla f(x,y) = (4x^3 - 4x,\ 3y^2 - 3)$

řešme $\nabla f(x,y) = 0$

$4x^3 - 4x = 0$	$3y^2 - 3 = 0$
$x(x^2 - 1) = 0$	$y^2 - 1 = 0$
	$y_1 = 1$
$x_1 = 0$	$y_2 = -1$
$x_2 = -1$	

Stacionární body:

$[0,1]$ $[-1,1]$ $[1,1]$

$[0,-1]$ $[-1,-1]$ $[1,-1]$

$$H_f(x,y) = \begin{pmatrix} 12x^2 - 4 & 0 \\ 0 & 6y \end{pmatrix}$$

$$H_f(0,1) = \begin{pmatrix} -4 & 0 \\ 0 & 6 \end{pmatrix}$$ indefinitní

$$H_f(0,-1) = \begin{pmatrix} -4 & 0 \\ 0 & -6 \end{pmatrix}$$ neg. definitní

$$H_f(1,1) = H_f(-1,1) = \begin{pmatrix} 8 & 0 \\ 0 & 6 \end{pmatrix}$$ poz. definitní

$$H_f(1,-1) = H_f(-1,-1) = \begin{pmatrix} 8 & 0 \\ 0 & -6 \end{pmatrix}$$ indefinitní

závěr:

Lok. minima : $[-1,1]$, $[1,1]$

Lok. maxima : $[0,-1]$

sedlové body : $[0,1]$, $[-1,-1]$, $[1,-1]$

c) $f(x,y) = 1 - \sqrt{x^2+y^2}$

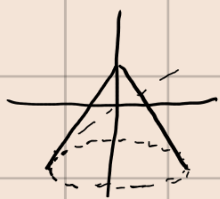
$$\nabla f(x,y) = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}} \right)$$

$\nabla f(0,0)$ ... neexistuje

f je $C^1(\mathbb{R}^2 \setminus \{0,0\})$

$\nabla f(x,y) = o$ ... stac. body neexistují

$r^2 = x^2+y^2$. $1-|r|$



v $(0,0)$

lok. maximum

$f(x,y) = x^4 - 2xy + y^2$

$\nabla f(x,y) = (4x^3 - 2y, -2x + 2y)$

$4x^3 - 2y = 0$

$2y - 2x = 0 \Rightarrow x = y$

$x(2x^2 - 1) = 0$

$\boxed{x_1 = 0}$ $\qquad y_1 = 0$

$2x^2 - 1 = 0$ $\qquad y_2 = \pm \frac{1}{\sqrt{2}}$

$x^2 = \frac{1}{2}$

$$\boxed{x = \pm \frac{1}{\sqrt{2}}}$$

Stac. body:

$[0,0]$, $\left[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$, $\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

$$H_f(x,y) = \begin{pmatrix} 12x^2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$H_f(0,0) = \begin{pmatrix} 0 & -2 \\ -2 & 2 \end{pmatrix} \qquad H_f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix}$$

$$\left.\begin{matrix} H_f(0,0)_{\{1\}} = 0 \\ H_f(0,0)_{\{1,2\}} = -4 \end{matrix}\right\} \text{indef.} \qquad \left.\begin{matrix} H_f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)_{\{1\}} \\ H_f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)_{\{1,2\}} \end{matrix}\right\} \text{poz. def.}$$

Závěr:

- Lok. minima: $\left[\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right]$
- sedlový bod $[0,0]$

2) $f(x,y,z) = x^2 + \frac{1}{2}y^2 + z^2 + xy + zy$

$\nabla f(x,y,z) = (2x+y,\ y+x+z,\ 2z+y)$

$$H_f(x,y,z) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$H_f(x,y,z)_{\{1\}} = 2$ $\qquad H_f(x,y,z) = 4 - 2 - 2 = 0$

$H_f(x,y,z)_{\{1,2\}} = 1$ $\qquad \Rightarrow$ poz. semidefinitní $\Rightarrow f$ konvexní na $\mathbb{R}^3$

3) $f(x,y) = \frac{1}{3}x^3 + y^2 - x$

$\nabla f(x,y) = \left(x^2 - 1,\ 2y\right)$

$H_f(x,y) = \begin{pmatrix} 2x & 0 \\ 0 & 2 \end{pmatrix}$

$x^2 - 1 = 0$

$2y = 0$

stac. body:

$[-1,0],\ [1,0]$

a) $C = \{(x,y) \in \mathbb{R}^2,\ x > 0\}$

b) $H_f(-1,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$ – indefinitní => sedlo

$H_f(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ – poz. def. => lok. minimum

c) $(1,0)$ je lok. minimum konvexní f na konvexní množině C

=> $(1,0)$ je globální minimum

5 a) $f(x) = ax + b$

$a(-1) + b = 0$

$a\left(-\frac{1}{2}\right) + b = 3$

$a(0) + b = 2$

$a\left(\frac{1}{2}\right) + b = 1$

$$\underbrace{\begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & 1 \\ 0 & 1 \\ \frac{1}{2} & 1 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 3 \\ 2 \\ 4 \end{pmatrix}}_{b}$$

$A^T \cdot A x = A^T b$

$A^T = \begin{pmatrix} -1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$A^T \cdot A = \begin{pmatrix} \frac{3}{2} & -1 \\ -1 & 4 \end{pmatrix}$

$A^T \cdot b = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

$$\left(\begin{array}{cc|c} \frac{3}{2} & -1 & -1 \\ -1 & 4 & 6 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 4 & 6 \\ \frac{3}{2} & -1 & -1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cc|c} -1 & 4 & 6 \\ 0 & 5 & 8 \end{array}\right)$$

$5b = 8$

$b = \frac{8}{5}$

$-a + \frac{4 \cdot 8}{5} = 6$

$-a = 6 - \frac{32}{5}$

$a = 4b - 6 = \frac{2}{5}$

$\Rightarrow f(x) = \frac{2}{5}x + \frac{8}{5}$