

Proof Terms as *Invariants*

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My Aim

To introduce an *invariant*
for classical propositional proofs
to help address questions
about proof identity.

Project Outline

1. Why proof identity is important, but difficult.

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2. Proof terms for classical sequent derivations.

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6. Cut elimination for derivations.

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1. Why proof identity is important, but difficult.
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5. Cut reduction for proof terms.
6. Cut elimination for derivations.

Today's Plan

Background

Derivations and Terms

Permutations

Further Work

BACKGROUND

On Proofs of THE IRRATIONALITY OF $\sqrt{2}$

By V. C. HARRIS

San Diego State College
San Diego, California

IN THIS paper there are given thirteen proofs that $\sqrt{2}$ is irrational. Indications are given as to whether the method employed extends to proving the irrationality of other square roots or of roots of higher order. In addition, a reference is provided to an incorrect proof recently published and its criticism.

Proof 1: Terminal-digit proof. Assume that $\sqrt{2}$ is rational, so that we can write $\sqrt{2} = a/b$ and $a^2 = 2b^2$, where a and b are

terminal digit 0, which implies they are not relatively prime—a contradiction.

Proof 2: First prime-divisor proof. Assume $\sqrt{2} = a/b$ and $a^2 = 2b^2$, where a and b are relatively prime positive integers. We see that $b > 1$, since otherwise $b = 1$ and $\sqrt{2} = a$, an integer, which is incorrect. By division, $b^2 = (a/2) \cdot a$. Now (1) any positive integer > 1 is either a prime or a product of primes, and (2) if p is a prime dividing the product rs of two integers, then p divides r or p divides s . There is a prime p dividing b , since $b > 1$. Also, p divides $a/2$ or p divides a . In either case, p divides a , so that p divides both a and b —a contradiction to the assumption that

When is π_1 the *same proof* as π_2 ?

Can you see how *these* two proofs are different?

$$\frac{\frac{\frac{(p)^1 \quad (p)^1}{p \wedge p} \wedge I}{p \supset (p \wedge p)} \supset I^1}{p \supset (p \supset (p \wedge p))} \supset I^2 \qquad \frac{\frac{(p)^1 \quad (p)^2}{p \wedge p} \wedge I}{p \supset (p \wedge p)} \supset I^1}{p \supset (p \supset (p \wedge p))} \supset I^2$$

Annotate with λ -terms

$$\frac{\frac{\frac{(x:p)^1 \quad (x:p)^1}{\langle x, x \rangle : p \wedge p} \wedge I}{\lambda x. \langle x, x \rangle : p \supset (p \wedge p)} \supset I^1}{\lambda y. \lambda x. \langle x, x \rangle : p \supset (p \supset (p \wedge p))} \supset I^2$$

$$\frac{\frac{\frac{(x:p)^1 \quad (y:p)^2}{\langle x, y \rangle : p \wedge p} \wedge I}{\lambda x. \langle x, y \rangle : p \supset (p \wedge p)} \supset I^1}{\lambda y. \lambda x. \langle x, y \rangle : p \supset (p \supset (p \wedge p))} \supset I^2$$

Sequent Derivations can be Annotated, Too

$$\frac{\frac{x : p \succ x : p \quad x : p \succ x : p}{x : p \succ \langle x, x \rangle : p \wedge p} \wedge R}{\succ \lambda x. \langle x, x \rangle : p \supset (p \wedge p)} \supset R$$
$$\frac{}{\succ \lambda y. \lambda x. \langle x, x \rangle : p \supset (p \supset (p \wedge p))} \supset R$$

$$\frac{x : p \succ x : p \quad y : p \succ y : p}{x : p, y : p \succ \langle x, y \rangle : p \wedge p} \wedge R$$
$$\frac{\frac{y : p \succ \lambda x. \langle x, y \rangle : p \supset (p \wedge p)}{y : p \succ \lambda x. \langle x, y \rangle : p \supset (p \supset (p \wedge p))} \supset R}{\succ \lambda y. \lambda x. \langle x, y \rangle : p \supset (p \supset (p \wedge p))} \supset R$$

Different Sequent Derivations have the same term

$$\frac{\frac{\frac{x:p \succ x:p}{y:p \wedge q \succ \text{fst } y:p} \wedge L}{x:p, y:p \wedge q \succ \langle x, \text{fst } y \rangle : p \wedge p} \wedge R}{y:p \wedge q \succ \lambda x. \langle x, \text{fst } y \rangle : p \supset (p \wedge p)} \supset R \\ \frac{}{\succ \lambda y. \lambda x. \langle x, \text{fst } y \rangle : (p \wedge q) \supset (p \supset (p \wedge p))} \supset R$$

$$\frac{\frac{\frac{x:p \succ x:p \quad z:p \succ z:p}{x:p, z:p \succ \langle x, z \rangle : p \wedge p} \wedge R}{z:p \succ \lambda x. \langle x, z \rangle : p \supset (p \wedge p)} \supset R}{y:p \wedge q \succ \lambda x. \langle x, \text{fst } y \rangle : p \supset (p \wedge p)} \wedge L \\ \frac{}{\succ \lambda y. \lambda x. \langle x, \text{fst } y \rangle : (p \wedge q) \supset (p \supset (p \wedge p))} \supset R$$

Different Sequent Derivations have the same term

$$\frac{\frac{\frac{x:p \succ x:p}{y:p \wedge q \succ \text{fst } y:p} \wedge L}{x:p, y:p \wedge q \succ \langle x, \text{fst } y \rangle : p \wedge p} \wedge R}{y:p \wedge q \succ \lambda x. \langle x, \text{fst } y \rangle : p \supset (p \wedge p)} \supset R \\ \succ \lambda y. \lambda x. \langle x, \text{fst } y \rangle : (p \wedge q) \supset (p \supset (p \wedge p)) \supset R$$

$$\frac{\frac{\frac{x:p \succ x:p \quad z:p \succ z:p}{x:p, z:p \succ \langle x, z \rangle : p \wedge p} \wedge R}{z:p \succ \lambda x. \langle x, z \rangle : p \supset (p \wedge p)} \supset R}{y:p \wedge q \succ \lambda x. \langle x, \text{fst } y \rangle : p \supset (p \wedge p)} \wedge L \\ \succ \lambda y. \lambda x. \langle x, \text{fst } y \rangle : (p \wedge q) \supset (p \supset (p \wedge p)) \supset R$$

An *excursus* on eliminating *Cut*: Contraction and Permutation

$$\frac{\frac{p \succ p \quad p \succ p}{p \vee p \succ p, p} \vee L \quad \frac{p \succ p \quad p \succ p}{p, p \succ p \wedge p} \wedge R}{\frac{p \vee p \succ p}{p \succ p \wedge p}} W \quad Cut$$

An *excursus* on eliminating *Cut*: Contraction and Permutation

$$\frac{\frac{\frac{p \succ p \quad p \succ p}{p \vee p \succ p, p} \textcolor{red}{\vee L}}{p \vee p \succ p} \textcolor{brown}{W}}{p \vee p \succ p \wedge p} \quad \frac{\frac{p \succ p \quad p \succ p}{p, p \succ p \wedge p} \textcolor{red}{\wedge R}}{p \succ p \wedge p} \textcolor{brown}{W}$$

Cut

$$p \vee p \succ p \wedge p$$

$$\frac{\frac{\frac{p \succ p \quad p \succ p}{p, p \succ p \wedge p} \textcolor{red}{\wedge R}}{p \succ p \wedge p} \textcolor{brown}{W} \quad \frac{\frac{p \succ p \quad p \succ p}{p, p \succ p \wedge p} \textcolor{red}{\wedge R}}{p \succ p \wedge p} \textcolor{brown}{W}}{\frac{p \vee p \succ p \wedge p, p \wedge p}{p \vee p \succ p \wedge p} \textcolor{brown}{W}} \textcolor{brown}{\vee L}$$
$$\frac{\frac{\frac{p \succ p \quad p \succ p}{p \vee p \succ p, p} \textcolor{red}{\vee L}}{p \vee p \succ p} \textcolor{brown}{W} \quad \frac{\frac{p \succ p \quad p \succ p}{p \vee p \succ p, p} \textcolor{red}{\vee L}}{p \vee p \succ p} \textcolor{brown}{W}}{\frac{p \vee p, p \vee p \succ p \wedge p}{p \vee p \succ p \wedge p} \textcolor{brown}{W}} \textcolor{brown}{\wedge R}$$
$$p \vee p \succ p \wedge p$$

An *excursus* on eliminating *Cut*: Contraction and Permutation

$$\frac{\frac{\frac{p \succ p \quad p \succ p}{p \vee p \succ p, p} \textcolor{red}{\vee L}}{p \vee p \succ p} \textcolor{brown}{W}}{p \vee p \succ p \wedge p} \quad \frac{\frac{p \succ p \quad p \succ p}{p, p \succ p \wedge p} \textcolor{red}{\wedge R}}{p \succ p \wedge p} \textcolor{brown}{W}}
 {p \succ p \wedge p} \textcolor{brown}{Cut}$$

$$\frac{\frac{\frac{p \succ p \quad p \succ p}{p, p \succ p \wedge p} \textcolor{red}{\wedge R}}{p \succ p \wedge p} \textcolor{brown}{W} \quad \frac{\frac{p \succ p \quad p \succ p}{p, p \succ p \wedge p} \textcolor{red}{\wedge R}}{p \succ p \wedge p} \textcolor{brown}{W}}{\frac{p \succ p \wedge p}{p \vee p \succ p \wedge p, p \wedge p} \textcolor{red}{\vee L}} \textcolor{brown}{W}}{p \vee p \succ p \wedge p} \quad \frac{\frac{\frac{p \succ p \quad p \succ p}{p \vee p \succ p, p} \textcolor{red}{\vee L}}{p \vee p \succ p} \textcolor{brown}{W} \quad \frac{\frac{p \succ p \quad p \succ p}{p \vee p \succ p, p} \textcolor{red}{\vee L}}{p \vee p \succ p} \textcolor{brown}{W}}{\frac{p \vee p, p \vee p \succ p \wedge p}{p \vee p \succ p \wedge p} \textcolor{red}{\wedge R}} \textcolor{brown}{W}}$$

An *excursus* on eliminating *Cut*: Contraction and Permutation

$$\frac{\frac{\frac{p \succ p \quad p \succ p}{p \vee p \succ p, p} \textcolor{red}{\vee L}}{p \vee p \succ p} \textcolor{brown}{W}}{p \vee p \succ p \wedge p} \quad \frac{\frac{p \succ p \quad p \succ p}{p, p \succ p \wedge p} \textcolor{red}{\wedge R}}{p \succ p \wedge p} \textcolor{brown}{W}}$$

$\frac{}{p \vee p \succ p \wedge p} \textcolor{brown}{Cut}$

$$\frac{\frac{\frac{p \succ p \quad p \succ p}{p, p \succ p \wedge p} \textcolor{red}{\wedge R}}{p \succ p \wedge p} \textcolor{brown}{W} \quad \frac{\frac{p \succ p \quad p \succ p}{p, p \succ p \wedge p} \textcolor{red}{\wedge R}}{p \succ p \wedge p} \textcolor{brown}{W}}{\frac{}{p \vee p \succ p \wedge p} \textcolor{red}{\vee L}} \textcolor{brown}{W}}$$
$$\frac{\frac{\frac{p \succ p \quad p \succ p}{p \vee p \succ p, p} \textcolor{red}{\vee L}}{p \vee p \succ p} \textcolor{brown}{W} \quad \frac{\frac{p \succ p \quad p \succ p}{p \vee p \succ p, p} \textcolor{red}{\vee L}}{p \vee p \succ p} \textcolor{brown}{W}}{\frac{}{p \vee p, p \vee p \succ p \wedge p} \textcolor{red}{\wedge R}} \textcolor{brown}{W}}$$
$$\frac{}{p \vee p \succ p \wedge p} \textcolor{brown}{W}$$

An excursus on eliminating Cut: Weakening and Blend

$$\frac{\begin{array}{c} \delta_1 \\ \vdots \\ \Sigma_1 \succ \Delta_1 \end{array} \quad \begin{array}{c} \delta_2 \\ \vdots \\ \Sigma_2 \succ \Delta_2 \end{array}}{\frac{\Sigma_1 \succ C, \Delta_1 \quad \Sigma_2, C \succ \Delta_2}{\Sigma_{1,2} \succ \Delta_{1,2}}} K \quad Cut$$

An excursus on eliminating Cut: Weakening and Blend

$$\frac{\begin{array}{c} \delta_1 \\ \vdots \\ \Sigma_1 \succ \Delta_1 \\ \hline \Sigma_1 \succ C, \Delta_1 \end{array} \quad \begin{array}{c} \delta_2 \\ \vdots \\ \Sigma_2 \succ \Delta_2 \\ \hline \Sigma_2, C \succ \Delta_2 \end{array}}{\Sigma_{1,2} \succ \Delta_{1,2}} \frac{K}{Cut}$$

$$\frac{\Sigma_1 \succ \Delta_1 \quad \Sigma_2 \succ \Delta_2}{\Sigma_{1,2} \succ \Delta_{1,2}} \frac{}{Blend}$$



DERIVATIONS AND TERMS

Symmetry in Classical Logic

$$\frac{\begin{array}{c} p \succ p \\ p, \neg p \succ \end{array}}{p \wedge \neg p \succ} \wedge L \qquad \frac{p \succ p}{\succ p, \neg p} \neg R$$

Symmetry in Classical Logic

$$\frac{p \succ p}{p, \neg p \succ} \neg L \qquad \frac{p \succ p}{\succ p, \neg p} \neg R$$
$$\frac{}{p \wedge \neg p \succ} \wedge L \qquad \frac{}{\succ p \vee \neg p} \vee R$$

These sequents aren't $x_1 : A_1, \dots, x_n : A_n \succ t(x_1, \dots, x_n) : B$.

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$$\frac{p \succ p}{p, \neg p \succ} \neg L \qquad \frac{p \succ p}{\succ p, \neg p} \neg R$$
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These sequents aren't $x_1 : A_1, \dots, x_n : A_n \succ t(x_1, \dots, x_n) : B$.

Instead, the shape for terms will be

$$x_1 : A_1, \dots, x_n : A_n \succ y_1 : B_1, \dots, y_m : B_m$$
$$\pi(x_1, \dots, x_n)[y_1, \dots, y_m]$$

Example Derivations with Terms

$$\frac{\begin{array}{c} x \rightsquigarrow x \\ x : p \succ x : p \end{array}}{x \rightsquigarrow \neg y} \neg L$$
$$\frac{x : p, y : \neg p \succ}{\lambda z \rightsquigarrow \lambda z} \wedge L$$
$$z : p \wedge \neg p \succ$$

$$\frac{\begin{array}{c} x \rightsquigarrow x \\ x : p \succ x : p \end{array}}{\neg y \rightsquigarrow x} \neg R$$
$$\frac{\succ x : p, y : \neg p}{\neg \vee z \rightsquigarrow \vee z} \vee R$$
$$\succ z : p \vee \neg p$$

Labelled Formulas and Sequents

$x : A, y : A, v : B \succ z : C, w : D$

Labelled Formulas and Sequents

$$x:A, y:A, v:B \succ z:C, w:D$$

$$\frac{x:A, y:A \succ z:C, w:D \quad v:B \succ z:C, w':D}{x:A, u:A \vee B \succ z:C, w:D, w':D} \vee L$$

Labelled Formulas and Sequents

$$x:A, y:A, v:B \succ z:C, w:D$$

$$\frac{x:A, y:A \succ z:C, w:D \quad v:B \succ z:C, w':D}{x:A, u:A \vee B \succ z:C, w:D, w':D} \vee L$$

The Σ and Δ are *sets* of labelled formulas in $\Sigma \succ \Delta$.

Basic Axioms

$$\Sigma, \textcolor{red}{x} : \perp \succ \Delta$$

$$\Sigma, \textcolor{red}{x} : p \succ \textcolor{red}{y} : p, \Delta$$

$$\Sigma \succ \textcolor{red}{y} : \top, \Delta$$

Blending Sequents

$$\Sigma_i \succ \Delta_i^{\pi_i}$$

Blending Sequents

$$\Sigma_i \succ \Delta_i$$

$$\bigcup_i \Sigma_i \succ \bigcup_i \Delta_i$$

Axioms

$$\Sigma, \textcolor{red}{x} : \perp \succ \Delta$$

$$\Sigma, \textcolor{red}{x} : p \succ \textcolor{red}{y} : p, \Delta$$

$$\Sigma \succ \textcolor{red}{y} : \top, \Delta$$

The *blend* of any finite collection of basic axioms is also an axiom.

Axioms

$$\Sigma, \textcolor{red}{x} : \perp \succ \Delta$$

$$\Sigma, \textcolor{red}{x} : p \succ \textcolor{red}{y} : p, \Delta$$

$$\Sigma \succ \textcolor{red}{y} : \top, \Delta$$

The *blend* of any finite collection of basic axioms is also an axiom.

$$\textcolor{red}{x} : p, \textcolor{red}{y} : q, \textcolor{red}{z} : r \wedge \neg s \succ \textcolor{red}{x} : p, \textcolor{red}{y} : q, \textcolor{red}{w} : \top$$

Conjunction

$$\frac{\pi(x, y)}{\Sigma, \mathbf{x}: A, \mathbf{y}: B \succ \Delta} \wedge L \quad \frac{\pi_1[x] \quad \pi_2[y]}{\Sigma_1 \succ \mathbf{x}: A, \Delta_1 \quad \Sigma_2 \succ \mathbf{y}: B, \Delta_2} \wedge R$$
$$\frac{}{\Sigma, \mathbf{z}: A \wedge B \succ \Delta} \quad \frac{\pi[\lambda z] \quad \pi'[\lambda z]}{\Sigma_{1,2} \succ \mathbf{z}: A \wedge B, \Delta_{1,2}}$$

Disjunction

$$\frac{\begin{array}{c} \pi_1(x) \\ \Sigma_1, \textcolor{red}{x}: A \succ \Delta_1 \end{array} \quad \begin{array}{c} \pi_2(y) \\ \Sigma_2, \textcolor{red}{y}: B \succ \Delta_2 \end{array}}{\pi_1(\dot{\vee} z) \ \pi_2(\dot{\vee} z)} \quad \Sigma \succ \textcolor{red}{x}: A, \textcolor{red}{y}: B, \Delta \quad \frac{\pi[x, y]}{\pi[\dot{\vee} z, \dot{\vee} z]} \quad \Sigma \succ \textcolor{red}{z}: A \vee B, \Delta$$

$\vee L$ $\vee R$

Conditional

$$\frac{\begin{array}{c} \pi_1[x] & \pi_2(y) \\ \Sigma_1 \succ x : A, \Delta_1 & \Sigma_2, y : B \succ \Delta_2 \end{array}}{\pi_1[\dot{\exists}z] \quad \pi_2(\dot{\exists}z)} \supset L \qquad \frac{\begin{array}{c} \pi(x)[y] \\ \Sigma, x : A \succ y : B, \Delta \end{array}}{\pi(\dot{\exists}z)[\dot{\exists}z]} \supset R$$
$$\Sigma_{1,2}, z : A \supset B \succ \Delta_{1,2} \qquad \qquad \Sigma \succ z : A \supset B, \Delta$$

Negation

$$\frac{\pi[x]}{\Sigma \succ \mathbf{x} : A, \Delta} \neg^L \quad \frac{\pi(x)}{\Sigma, \mathbf{x} : A \succ \Delta} \neg^R$$
$$\Sigma, \mathbf{z} : \neg A \succ \Delta \qquad \Sigma \succ \mathbf{z} : \neg A, \Delta$$

Cut

$$\frac{\begin{array}{c} \pi_1[x] \\ \Sigma_1 \succ \textcolor{red}{x} : A, \Delta_1 \end{array} \qquad \begin{array}{c} \pi_2(y) \\ \Sigma_2, \textcolor{red}{y} : A \succ \Delta_2 \end{array}}{\begin{array}{c} \pi_1[\bullet] \quad \pi_2(\bullet) \\ \Sigma_{1,2} \succ \Delta_{1,2} \end{array}} \text{Cut}$$

α Rules

$$\frac{\pi(x, y)}{\Sigma, \textcolor{red}{x}: A, \textcolor{red}{y}: B \succ \Delta} \wedge L \quad \frac{\pi[x, y]}{\Sigma \succ \textcolor{red}{x}: A, \textcolor{red}{y}: B, \Delta} \vee R$$

$$\Sigma, \textcolor{red}{z}: A \wedge B \succ \Delta \quad \Sigma \succ \textcolor{red}{z}: A \vee B, \Delta$$

$$\frac{\pi(x)[y]}{\Sigma, \textcolor{red}{x}: A \succ \textcolor{red}{y}: B, \Delta} \supset R \quad \frac{\pi[x]}{\Sigma \succ \textcolor{red}{x}: A, \Delta} \neg L \quad \frac{\pi(x)}{\Sigma, \textcolor{red}{x}: A \succ \Delta} \neg R$$

$$\Sigma \succ \textcolor{red}{z}: A \supset B, \Delta \quad \Sigma, \textcolor{red}{z}: \neg A \succ \Delta \quad \Sigma \succ \textcolor{red}{z}: \neg A, \Delta$$

$$\frac{\pi\{\alpha_1\}\{\alpha_2\}}{\mathfrak{S}\{\textcolor{red}{a}_1: \alpha_1\}\{\textcolor{red}{a}_2: \alpha_2\}}$$

$$\frac{\pi\{\dot{\alpha}\alpha\}\{\dot{\alpha}\alpha\}}{\mathfrak{S}\{\textcolor{red}{a}: \alpha\}}$$

α Rule Rubric

RULE	$\{a_1 : \alpha_1\}$	$\{a_2 : \alpha_2\}$	$\{a : \alpha\}$	$\check{\alpha}$	$\check{\alpha}$
$\wedge L$	input, A	input, B	input, $A \wedge B$	$\check{\wedge}$	λ
$\vee R$	output, A	output, B	output, $A \vee B$	$\check{\vee}$	$\check{\vee}$
$\supset R$	input, A	output, B	output, $A \supset B$	$\check{\supset}$	$\check{\supset}$
$\neg L$	output, A	output, A	input, $\neg A$	$\check{\neg}$	$\check{\neg}$
$\neg R$	input, A	input, A	output, $\neg A$	$\check{\neg}$	$\check{\neg}$

β Rules

$$\frac{\begin{array}{c} \pi_1[x] \\ \Sigma_1 \succ \textcolor{red}{x}: A, \Delta_1 \end{array} \quad \begin{array}{c} \pi_2[y] \\ \Sigma_2 \succ \textcolor{red}{y}: B, \Delta_2 \end{array}}{\begin{array}{c} \pi_1[\lambda z] \quad \pi_2[\lambda z] \\ \Sigma_{1,2} \succ \textcolor{red}{z}: A \wedge B, \Delta_{1,2} \end{array}} \wedge R \quad \frac{\begin{array}{c} \pi_1(x) \\ \Sigma_1, \textcolor{red}{x}: A \succ \Delta_1 \end{array} \quad \begin{array}{c} \pi_2(y) \\ \Sigma_2, \textcolor{red}{y}: B \succ \Delta_2 \end{array}}{\begin{array}{c} \pi_1(\vee z) \quad \pi_2(\vee z) \\ \Sigma_{1,2}, \textcolor{red}{z}: A \vee B \succ \Delta_{1,2} \end{array}} \vee L$$

$$\frac{\begin{array}{c} \pi_1[x] \\ \Sigma_1 \succ \textcolor{red}{x}: A, \Delta_1 \end{array} \quad \begin{array}{c} \pi_2(y) \\ \Sigma_2, \textcolor{red}{y}: B \succ \Delta_2 \end{array}}{\begin{array}{c} \pi_1[\dot{\cup} z] \quad \pi_2(\dot{\cup} z) \\ \Sigma_{1,2}, \textcolor{red}{z}: A \supset B \succ \Delta_{1,2} \end{array}} \supset L \quad \frac{\begin{array}{c} \pi_1[x] \\ \Sigma_1 \succ \textcolor{red}{x}: A, \Delta_1 \end{array} \quad \begin{array}{c} \pi_2(y) \\ \Sigma_2, \textcolor{red}{y}: A \succ \Delta_2 \end{array}}{\begin{array}{c} \pi_1[\bullet] \quad \pi_2(\bullet) \\ \Sigma_{1,2} \succ \Delta_{1,2} \end{array}} Cut$$

$$\frac{\begin{array}{c} \pi_1\{b_1\} \\ \mathfrak{S}_1\{b_1: \beta_1\} \end{array} \quad \begin{array}{c} \pi_2\{b_2\} \\ \mathfrak{S}_2\{b_2: \beta_2\} \end{array}}{\begin{array}{c} \pi_1\{\dot{\beta} b\} \quad \pi_2\{\dot{\beta} b\} \\ \mathfrak{S}_{1,2}\{b: \beta\} \end{array}}$$

β Rule Rubric

RULE	$\{b_1 : \beta_1\}$	$\{b_2 : \beta_2\}$	$\{b : \beta\}$	β	$\dot{\beta}$
$\wedge R$	output, A	output, B	output, $A \wedge B$	\wedge	λ
$\vee L$	input, A	input, B	input, $A \vee B$	\vee	$\dot{\vee}$
$\supset L$	output, A	input, B	input, $A \supset B$	\supset	$\dot{\supset}$
<i>Cut</i>	output, A	input, A	—	•	•

Example 1—derivations with different terms

$$\frac{\frac{x \rightsquigarrow x}{x : p \succ x : p} \quad \frac{x \rightsquigarrow x}{x : p \succ x : p}}{\frac{\frac{\frac{\frac{\forall y \rightsquigarrow x \quad \forall y \rightsquigarrow x}{y : p \vee p \succ x : p}}{\forall y \rightsquigarrow \bullet \quad \forall y \rightsquigarrow \bullet} \quad \frac{\bullet \rightsquigarrow \lambda z \quad \bullet \rightsquigarrow \lambda z}{x : p \succ z : p \wedge p}}{x : p \succ y : p \wedge p}}{\forall y \rightsquigarrow \bullet \quad \forall y \rightsquigarrow \bullet \quad \frac{\bullet \rightsquigarrow \lambda z \quad \bullet \rightsquigarrow \lambda z}{x : p \succ z : p \wedge p}} \quad \text{Cut}$$

$\forall y \rightsquigarrow \bullet \quad \forall y \rightsquigarrow \bullet \quad \bullet \rightsquigarrow \lambda z \quad \bullet \rightsquigarrow \lambda z$

$y : p \vee p \succ y : p \wedge p$

Example 1—derivations with different terms

$$\frac{\frac{x : p \succ x : p \quad x : p \succ x : p}{\frac{\frac{\frac{\frac{v y \rightsquigarrow x \quad v y \rightsquigarrow x}{y : p \vee p \succ x : p}}{\frac{\frac{v y \rightsquigarrow \bullet \quad v y \rightsquigarrow \bullet}{y : p \vee p \succ y : p \wedge p}}{\frac{x : p \succ x : p \quad x : p \succ x : p}{\frac{\frac{\frac{x \rightsquigarrow \lambda z \quad x \rightsquigarrow \lambda z}{x : p \succ z : p \wedge p}}{\frac{x \rightsquigarrow \lambda z \quad x \rightsquigarrow \lambda z}{x : p \succ z : p \wedge p}}}}}}{\wedge R} \quad Cut$$

$$\frac{\frac{x : p \succ x : p \quad x : p \succ x : p}{\frac{\frac{\frac{x \rightsquigarrow \lambda z \quad x \rightsquigarrow \lambda z}{x : p \succ z : p \wedge p}}{\frac{\frac{\frac{v y \rightsquigarrow \lambda z \quad v y \rightsquigarrow \lambda z}{v y \rightsquigarrow \lambda z \quad v y \rightsquigarrow \lambda z}}{\frac{\frac{x : p \succ x : p \quad x : p \succ x : p}{\frac{\frac{\frac{x \rightsquigarrow \lambda z \quad x \rightsquigarrow \lambda z}{x : p \succ z : p \wedge p}}{\frac{x \rightsquigarrow \lambda z \quad x \rightsquigarrow \lambda z}{x : p \succ z : p \wedge p}}}}{\wedge R} \quad \wedge L$$

Example 2—two *different* derivations with the *same* term

$$\begin{array}{c}
 \frac{x \rightsquigarrow x \quad z \rightsquigarrow z}{x : p \succ x : p \quad z : p \succ z : p} \wedge L \\
 \frac{x : p \succ x : p \quad y : p \wedge q \succ z : p}{x \rightsquigarrow \lambda w \quad y \rightsquigarrow \lambda w} \wedge R \\
 \frac{x : p, y : p \wedge q \succ w : p \wedge p}{\dot{\varsigma} v \rightsquigarrow \dot{\varsigma} v \quad \dot{\varsigma} y \rightsquigarrow \dot{\varsigma} v} \supset R \\
 \frac{y : p \wedge q \succ v : p \supset (p \wedge p)}{\dot{\varsigma} u \rightsquigarrow \dot{\varsigma} u \quad \dot{\varsigma} v \rightsquigarrow \dot{\varsigma} u} \supset R \\
 \succ u : (p \wedge q) \supset (p \supset (p \wedge p))
 \end{array}$$

$$\begin{array}{c}
 \frac{x \rightsquigarrow x \quad z \rightsquigarrow z}{x : p \succ x : p \quad z : p \succ z : p} \wedge R \\
 \frac{x : p \succ x : p \quad x \rightsquigarrow \lambda w \quad z \rightsquigarrow \lambda w}{x : p, z : p \succ w : p \wedge p} \supset R \\
 \frac{x : p, z : p \succ w : p \wedge p}{\dot{\varsigma} v \rightsquigarrow \dot{\varsigma} v \quad z \rightsquigarrow \dot{\varsigma} v} \supset R \\
 \frac{z : p \succ v : p \supset (p \wedge p)}{\dot{\varsigma} v \rightsquigarrow \dot{\varsigma} v \quad \dot{\varsigma} y \rightsquigarrow \dot{\varsigma} v} \wedge L \\
 \frac{y : p \wedge q \succ v : p \supset (p \wedge p)}{\dot{\varsigma} u \rightsquigarrow \dot{\varsigma} u \quad \dot{\varsigma} v \rightsquigarrow \dot{\varsigma} u} \supset R \\
 \succ u : (p \wedge q) \supset (p \supset (p \wedge p))
 \end{array}$$

Example 2—two *different* derivations with the *same* term

$$\begin{array}{c}
 \frac{\begin{array}{c} z \rightsquigarrow z \\ z : p \succ z : p \end{array}}{\begin{array}{c} \lambda y \rightsquigarrow z \\ y : p \wedge q \succ z : p \end{array}} \wedge L \\
 \frac{\begin{array}{c} x \rightsquigarrow x \\ x : p \succ x : p \end{array} \quad y : p \wedge q \succ z : p}{\begin{array}{c} x \rightsquigarrow \lambda w \quad \lambda y \rightsquigarrow \lambda w \\ x : p, y : p \wedge q \succ w : p \wedge p \end{array}} \wedge R \\
 \frac{\begin{array}{c} \lambda v \rightsquigarrow \lambda v \\ \lambda v \rightsquigarrow \lambda v \quad \lambda y \rightsquigarrow \lambda v \end{array}}{\begin{array}{c} \lambda v \rightsquigarrow \lambda v \\ y : p \wedge q \succ v : p \supset (p \wedge p) \end{array}} \supset R \\
 \frac{\begin{array}{c} \lambda u \rightsquigarrow \lambda u \\ \lambda u \rightsquigarrow \lambda u \quad \lambda v \rightsquigarrow \lambda u \end{array}}{\begin{array}{c} \lambda u \rightsquigarrow \lambda u \\ y : p \wedge q \succ v : p \supset (p \wedge p) \end{array}} \supset R \\
 \succ u : (p \wedge q) \supset (p \supset (p \wedge p))
 \end{array}$$

$$\begin{array}{c}
 \frac{\begin{array}{c} x \rightsquigarrow x \\ x : p \succ x : p \end{array} \quad \begin{array}{c} z \rightsquigarrow z \\ z : p \succ z : p \end{array}}{\begin{array}{c} \lambda w \rightsquigarrow \lambda w \\ x : p, z : p \succ w : p \wedge p \end{array}} \wedge R \\
 \frac{\begin{array}{c} \lambda v \rightsquigarrow \lambda v \\ \lambda v \rightsquigarrow \lambda v \end{array}}{\begin{array}{c} \lambda v \rightsquigarrow \lambda v \\ z : p \succ v : p \supset (p \wedge p) \end{array}} \supset R \\
 \frac{\begin{array}{c} \lambda u \rightsquigarrow \lambda u \\ \lambda u \rightsquigarrow \lambda u \end{array}}{\begin{array}{c} \lambda u \rightsquigarrow \lambda u \\ y : p \wedge q \succ v : p \supset (p \wedge p) \end{array}} \supset R \\
 \succ u : (p \wedge q) \supset (p \supset (p \wedge p))
 \end{array}$$

Example 3—two *different* derivations with the *same* term

$$\frac{\frac{x : p \succ x : p \quad x : p \succ x : p}{\frac{x \curvearrowleft z \quad x \curvearrowleft z}{x : p \succ z : p \wedge p}} \wedge R \quad \frac{x : p \succ x : p \quad x : p \succ x : p}{\frac{x \curvearrowleft z \quad x \curvearrowleft z}{x : p \succ z : p \wedge p}} \wedge R}{\frac{\frac{\frac{\frac{\frac{\forall y \curvearrowleft z \quad \forall y \curvearrowleft z}{\forall y \curvearrowleft z} \quad \frac{\frac{\forall y \curvearrowleft z \quad \forall y \curvearrowleft z}{\forall y \curvearrowleft z}}{\forall y \curvearrowleft z}}{\forall y \curvearrowleft z} \quad \forall y \curvearrowleft z}{\forall y \curvearrowleft z} \quad \forall y \curvearrowleft z}{\forall y \curvearrowleft z} \quad \forall y \curvearrowleft z}{\forall y \curvearrowleft z} \quad \forall y \curvearrowleft z}}{y : p \vee p \succ z : p \wedge p} \vee L$$

$$\frac{\frac{x : p \succ x : p \quad x : p \succ x : p}{\frac{\forall y \curvearrowleft x \quad \forall y \curvearrowleft x}{\forall y \curvearrowleft x \quad \forall y \curvearrowleft x}} \vee L \quad \frac{x : p \succ x : p \quad x : p \succ x : p}{\frac{\forall y \curvearrowleft x \quad \forall y \curvearrowleft x}{\forall y \curvearrowleft x \quad \forall y \curvearrowleft x}} \vee L}{\frac{\frac{\frac{\frac{\forall y \curvearrowleft z \quad \forall y \curvearrowleft z}{\forall y \curvearrowleft z} \quad \frac{\frac{\forall y \curvearrowleft z \quad \forall y \curvearrowleft z}{\forall y \curvearrowleft z}}{\forall y \curvearrowleft z}}{\forall y \curvearrowleft z} \quad \forall y \curvearrowleft z}{\forall y \curvearrowleft z} \quad \forall y \curvearrowleft z}{\forall y \curvearrowleft z} \quad \forall y \curvearrowleft z}}{y : p \vee p \succ z : p \wedge p} \wedge R$$

Example 3—two *different* derivations with the *same* term

$$\frac{\frac{x:p \succ x:p \quad x:p \succ x:p}{\frac{x:\tilde{z} \quad x:\tilde{z}}{x:p \succ z:p \wedge p}} \wedge R \quad \frac{x:p \succ x:p \quad x:p \succ x:p}{\frac{x:\tilde{z} \quad x:\tilde{z}}{x:p \succ z:p \wedge p}} \wedge R}{\frac{\frac{\frac{\tilde{y}:\tilde{z} \quad \tilde{y}:\tilde{z}}{\frac{\tilde{y}:\tilde{z} \quad \tilde{y}:\tilde{z}}{y:p \vee p \succ z:p \wedge p}} \vee L}{y:p \vee p \succ z:p \wedge p}}{\vee L}}$$

$$\frac{\frac{x:p \succ x:p \quad x:p \succ x:p}{\frac{\tilde{y}:x \quad \tilde{y}:x}{y:p \vee p \succ x:p}} \vee L \quad \frac{x:p \succ x:p \quad x:p \succ x:p}{\frac{\tilde{y}:x \quad \tilde{y}:x}{y:p \vee p \succ x:p}} \vee L}{\frac{\frac{\frac{\tilde{y}:\tilde{z} \quad \tilde{y}:\tilde{z}}{\frac{\tilde{y}:\tilde{z} \quad \tilde{y}:\tilde{z}}{y:p \vee p \succ z:p \wedge p}} \wedge R}{y:p \vee p \succ z:p \wedge p}}{\wedge R}}$$

Height Preserving Admissibility—Contraction and Weakening

For the single premise rules of *weakening* and *contraction*

$$\frac{\pi}{\Sigma \succ \Delta} \text{KL} \quad \frac{\pi}{\Sigma \succ \Delta, \Delta} \text{KR} \quad \frac{\pi(x, y)}{\Sigma, x:A, y:A \succ \Delta} \text{WL} \quad \frac{\pi[x, y]}{\Sigma \succ x:A, y:A, \Delta} \text{WR}$$
$$\Sigma, x:A \succ \Delta \quad \Sigma \succ x:A, \Delta \quad \Sigma, x:A \succ \Delta \quad \Sigma \succ x:A, \Delta$$

if there is a derivation δ of the *premise* of the rule, of height $h(\delta)$, then there is a derivation δ' of the *conclusion* of the rule, of the same height, $h(\delta)$.

Height Preserving Admissibility—Blend

For the two premise rule of *blend*,

$$\frac{\begin{array}{c} \pi_1 \\ \Sigma_1 \succ \Delta_1 \end{array} \quad \begin{array}{c} \pi_2 \\ \Sigma_2 \succ \Delta_2 \end{array}}{\begin{array}{c} \pi_1 \ \pi_2 \\ \Sigma_{1,2} \succ \Delta_{1,2} \end{array}} \text{ Blend}$$

if there is a derivation δ_1 of the premise $\Gamma_1 \succ \Delta_1$ (with term π) and a derivation δ_2 of the premise $\Gamma_2 \succ \Delta_2$ (with term π') then there is a derivation of height less than or equal to $h(\delta) + h(\delta')$ of the blend $\Gamma_{1,2} \succ \Delta_{1,2}$ with term $\pi_1 \ \pi_2$.

An aerial photograph of a lush green valley. The landscape is characterized by numerous winding white lines, likely representing roads or paths, which follow the contours of the terrain. Small clusters of buildings, possibly farmhouses or settlements, are scattered throughout the valley floor. The overall scene is one of a rural, agricultural area.

PERMUTATIONS

$\wedge L/\neg R$ Permutation

$$\frac{\pi(x, y, z)}{\Sigma, x : A, y : B, z : C \succ \Delta} \quad \wedge L$$
$$\frac{\pi(\lambda v, \lambda v, z)}{\Sigma, v : A \wedge B, z : C \succ \Delta} \quad \neg R$$
$$\Sigma, v : A \wedge B \succ w : \neg C, \Delta$$

$$\frac{\pi(x, y, z)}{\Sigma, x : A, y : B, z : C \succ \Delta} \quad \neg R$$
$$\frac{\pi(x, y, \dot{\neg} w)}{\Sigma, x : A, y : B \succ w : \neg C, \Delta} \quad \wedge L$$
$$\Sigma, v : A \wedge B \succ w : \neg C, \Delta$$

$\wedge L / \neg R$ Permutation

$$\frac{\pi(x, y, z)}{\Sigma, x : A, y : B, z : C \succ \Delta} \quad \wedge L$$

$$\frac{\pi(\lambda v, \lambda v, z)}{\Sigma, v : A \wedge B, z : C \succ \Delta} \quad \neg R$$

$$\Sigma, v : A \wedge B \succ w : \neg C, \Delta$$

$$\frac{\pi(x, y, z)}{\Sigma, x : A, y : B, z : C \succ \Delta} \quad \neg R$$

$$\frac{\pi(x, y, \dot{\neg} w)}{\Sigma, x : A, y : B \succ w : \neg C, \Delta} \quad \wedge L$$

$$\Sigma, v : A \wedge B \succ w : \neg C, \Delta$$

α/α Permutations

$$\frac{\begin{array}{c} \pi\{a_1\}\{a_2\}\{a'_1\}\{a'_2\} \\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\} \end{array}}{\begin{array}{c} \pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{a'_1\}\{a'_2\} \\ \mathfrak{S}\{a : \alpha\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\} \end{array}} \begin{array}{l} \alpha \\ \alpha' \end{array}$$
$$\frac{\pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}'a'\}\{\dot{\alpha}'a'\}}{\mathfrak{S}\{a : \alpha\}\{a' : \alpha'\}}$$

α/α Permutations

$$\frac{\begin{array}{c} \pi\{a_1\}\{a_2\}\{a'_1\}\{a'_2\} \\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\} \end{array}}{\begin{array}{c} \pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{a'_1\}\{a'_2\} \\ \mathfrak{S}\{a : \alpha\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\} \end{array}}_{\alpha}$$
$$\frac{\begin{array}{c} \pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}'a'\}\{\dot{\alpha}'a'\} \\ \mathfrak{S}\{a : \alpha\}\{a' : \alpha'\} \end{array}}{\begin{array}{c} \pi\{a_1\}\{a_2\}\{a'_1\}\{a'_2\} \\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\} \end{array}}_{\alpha'}$$

$$\frac{\begin{array}{c} \pi\{a_1\}\{a_2\}\{a'_1\}\{a'_2\} \\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\} \end{array}}{\begin{array}{c} \pi\{a_1\}\{a_2\}\{\dot{\alpha}'a'\}\{\dot{\alpha}'a'\} \\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a' : \alpha'\} \end{array}}_{\alpha'}$$
$$\frac{\begin{array}{c} \pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}'a'\}\{\dot{\alpha}'a'\} \\ \mathfrak{S}\{a : \alpha\}\{a' : \alpha'\} \end{array}}{\begin{array}{c} \pi\{a_1\}\{a_2\}\{a'_1\}\{a'_2\} \\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\} \end{array}}_{\alpha}$$

$\wedge R/\neg R$ Permutations

$$\frac{\Sigma_1, z : C \succ x : A, \Delta_1 \quad \Sigma_2, z : C \succ y : B, \Delta_2}{\frac{\pi_1(z)[x] \quad \pi_2(z)[y]}{\frac{\pi_1(z)[\lambda v] \quad \pi_2(z)[\lambda v]}{\frac{\Sigma_{1,2}, z : C \succ v : A \wedge B, \Delta_{1,2}}{\frac{\pi_1(\dot{v})[\lambda v] \quad \pi_2(\dot{v})[\lambda v]}{\Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2}}}}}}_{\wedge R}$$
$$\frac{\pi_1(z)[\lambda v] \quad \pi_2(z)[\lambda v]}{\frac{\Sigma_{1,2}, z : C \succ v : A \wedge B, \Delta_{1,2}}{\frac{\pi_1(\dot{v})[\lambda v] \quad \pi_2(\dot{v})[\lambda v]}{\Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2}}}}}_{\neg R}$$

$\wedge R/\neg R$ Permutations

$$\frac{\begin{array}{c} \pi_1(z)[x] \\ \Sigma_1, z : C \succ x : A, \Delta_1 \end{array} \quad \begin{array}{c} \pi_2(z)[y] \\ \Sigma_2, z : C \succ y : B, \Delta_2 \end{array}}{\begin{array}{c} \pi_1(z)[\lambda v] \quad \pi_2(z)[\lambda v] \\ \Sigma_{1,2}, z : C \succ v : A \wedge B, \Delta_{1,2} \end{array}} \wedge R$$

$$\frac{\begin{array}{c} \pi_1(\neg w)[\lambda v] \quad \pi_2(\neg w)[\lambda v] \\ \Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2} \end{array}}{\pi_1(\neg w)[\lambda v] \quad \pi_2(\neg w)[\lambda v]} \neg R$$

$$\frac{\begin{array}{c} \pi_1(z)[x] \\ \Sigma_1, z : C \succ x : A, \Delta_1 \end{array} \quad \begin{array}{c} \pi_2(z)[y] \\ \Sigma_2, z : C \succ y : B, \Delta_2 \end{array}}{\begin{array}{c} \pi_1(\neg w)[x] \\ \pi_2(\neg w)[y] \end{array}} \neg R$$

$$\frac{\Sigma_1 \succ w : \neg C, x : A, \Delta_1 \quad \Sigma_2 \succ w : \neg C, y : B, \Delta_2}{\begin{array}{c} \pi_1(\neg w)[\lambda v] \quad \pi_2(\neg w)[\lambda v] \\ \Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2} \end{array}} \wedge R$$

α/β Permutations

$$\frac{\frac{\frac{\mathfrak{S}_1\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{b_1 : \beta_1\}}{\pi_1\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{b_1\}}}{\mathfrak{S}_1\{a : \alpha\}\{b_1 : \beta_1\}}}{\pi_1\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\beta}b\}} \quad \frac{\frac{\mathfrak{S}_2\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{b_2 : \beta_2\}}{\pi_2\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{b_2\}}}{\mathfrak{S}_2\{a : \alpha\}\{b_2 : \beta_2\}}}{\pi_2\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\beta}b\}}}_{\alpha\beta} \mathfrak{S}_{1,2}\{a : \alpha\}\{b : \beta\}$$

α/β Permutations

$$\frac{\frac{\pi_1\{a_1\}\{a_2\}\{b_1\}}{\mathfrak{S}_1\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{b_1 : \beta_1\}}}{\pi_1\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{b_1\}} \quad \frac{\pi_2\{a_1\}\{a_2\}\{b_2\}}{\mathfrak{S}_2\{a_1 : \alpha_1\}\{a_2 : \alpha_1\}\{b_2 : \beta_2\}} \quad \alpha$$

$$\frac{\mathfrak{S}_1\{a : \alpha\}\{b_1 : \beta_1\}}{\pi_1\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\beta}b\}} \quad \frac{\mathfrak{S}_2\{a : \alpha\}\{b_2 : \beta_2\}}{\pi_2\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\beta}b\}} \quad \beta$$

$$\mathfrak{S}_{1,2}\{a : \alpha\}\{b : \beta\}$$

$$\frac{\pi_1\{a_1\}\{a_2\}\{b_1\}}{\mathfrak{S}_1\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{b_1 : \beta_1\}} \quad \frac{\pi_2\{a_1\}\{a_2\}\{b_2\}}{\mathfrak{S}_2\{a_1 : \alpha_1\}\{a_2 : \alpha_1\}\{b_2 : \beta_2\}} \quad \beta$$

$$\frac{\mathfrak{S}_{1,2}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{b : \beta\}}{\pi_1\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\beta}b\} \quad \pi_2\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\beta}b\}} \quad \alpha$$

$$\mathfrak{S}_{1,2}\{a : \alpha\}\{b : \beta\}$$

β/β Permutations

$$\frac{\begin{array}{c} \pi_1\{b_1\}\{b'_1\} \\ \mathfrak{S}_1\{b_1:\beta_1\}\{b'_1:\beta'_1\} \end{array} \quad \begin{array}{c} \pi_2\{b_2\}\{b'_2\} \\ \mathfrak{S}_2\{b_2:\beta_2\}\{b'_2:\beta'_2\} \end{array} \quad \begin{array}{c} \pi_3\{b_1\}\{b'_2\} \\ \mathfrak{S}_3\{b_1:\beta_1\}\{b'_2:\beta'_2\} \end{array} \quad \begin{array}{c} \pi_4\{b_2\}\{b'_2\} \\ \mathfrak{S}_4\{b_2:\beta_2\}\{b'_2:\beta'_2\} \end{array}}{\beta} \quad \frac{\begin{array}{c} \pi_1\{\dot{\beta} b\}\{b'_1\} \quad \pi_2\{\dot{\beta} b\}\{b'_1\} \\ \mathfrak{S}_{1,2}\{b:\beta\}\{b'_1:\beta'_1\} \end{array} \quad \begin{array}{c} \pi_3\{\dot{\beta} b\}\{b'_2\} \quad \pi_2\{\dot{\beta} b\}\{b'_2\} \\ \mathfrak{S}_{3,4}\{b:\beta\}\{b'_2:\beta'_2\} \end{array}}{\beta'} \quad \frac{\begin{array}{c} \pi_1\{\dot{\beta} b\}\{\dot{\beta}' b'\} \quad \pi_2\{\dot{\beta} b\}\{\dot{\beta}' b'\} \quad \pi_3\{\dot{\beta} b\}\{\dot{\beta}' b'\} \quad \pi_4\{\dot{\beta} b\}\{\dot{\beta}' b'\} \\ \mathfrak{S}_{1-4}\{b:\beta\}\{b':\beta'\} \end{array}}{\beta'}$$

β/β Permutations

$$\begin{array}{c}
 \frac{\pi_1\{b_1\}\{b'_1\} \quad \pi_2\{b_2\}\{b'_2\} \quad \pi_3\{b_1\}\{b'_2\} \quad \pi_4\{b_2\}\{b'_2\}}{\mathfrak{S}_1\{b_1:\beta_1\}\{b'_1:\beta'_1\} \quad \mathfrak{S}_2\{b_2:\beta_2\}\{b'_1:\beta'_1\} \quad \mathfrak{S}_3\{b_1:\beta_1\}\{b'_2:\beta'_2\} \quad \mathfrak{S}_4\{b_2:\beta_2\}\{b'_2:\beta'_2\}} \quad \beta \\
 \hline
 \pi_1\{\dot{\beta} b\}\{b'_1\} \quad \pi_2\{\dot{\beta} b\}\{b'_1\} \quad \pi_3\{\dot{\beta} b\}\{b'_2\} \quad \pi_2\{\dot{\beta} b\}\{b'_2\} \\
 \mathfrak{S}_{1,2}\{b:\beta\}\{b'_1:\beta'_1\} \quad \mathfrak{S}_{3,4}\{b:\beta\}\{b'_2:\beta'_2\} \\
 \hline
 \pi_1\{\dot{\beta} b\}\{\dot{\beta}' b'\} \quad \pi_2\{\dot{\beta} b\}\{\dot{\beta}' b'\} \quad \pi_3\{\dot{\beta} b\}\{\dot{\beta}' b'\} \quad \pi_4\{\dot{\beta} b\}\{\dot{\beta}' b'\} \\
 \mathfrak{S}_{1-4}\{b:\beta\}\{b':\beta'\}
 \end{array} \quad \beta'$$

$$\begin{array}{c}
 \frac{\pi_1\{b_1\}\{b'_1\} \quad \pi_3\{b_1\}\{b'_2\} \quad \pi_2\{b_2\}\{b'_1\} \quad \pi_4\{b_2\}\{b'_2\}}{\mathfrak{S}_1\{b_1:\beta_1\}\{b'_1:\beta'_1\} \quad \mathfrak{S}_3\{b_1:\beta_1\}\{b'_2:\beta'_2\} \quad \mathfrak{S}_2\{b_2:\beta_2\}\{b'_1:\beta'_1\} \quad \mathfrak{S}_4\{b_2:\beta_2\}\{b'_2:\beta'_2\}} \quad \beta' \\
 \hline
 \pi_1\{b_1\}\{\dot{\beta}' b'\} \quad \pi_3\{b_1\}\{\dot{\beta}' b'\} \quad \pi_3\{b_2\}\{\dot{\beta}' b'\} \quad \pi_2\{b_2\}\{\dot{\beta}' b'\} \\
 \mathfrak{S}_{1,3}\{b_1:\beta_1\}\{b':\beta'\} \quad \mathfrak{S}_{2,4}\{b_2:\beta_2\}\{b':\beta'\} \\
 \hline
 \pi_1\{\dot{\beta} b\}\{\dot{\beta}' b'\} \quad \pi_2\{\dot{\beta} b\}\{\dot{\beta}' b'\} \quad \pi_3\{\dot{\beta} b\}\{\dot{\beta}' b'\} \quad \pi_4\{\dot{\beta} b\}\{\dot{\beta}' b'\} \\
 \mathfrak{S}_{1-4}\{b:\beta\}\{b':\beta'\}
 \end{array} \quad \beta'$$

Duplicating Rule Instances

$$\frac{\begin{array}{c} \pi_1(z)[x] \\ \Sigma_1, z : C \succ x : A, \Delta_1 \end{array}}{\pi_1(\dot{\neg}w)[x]} \quad \frac{\begin{array}{c} \pi_2(z)[y] \\ \Sigma_2, z : C \succ y : B, \Delta_2 \end{array}}{\pi_2(\dot{\neg}w)[y]} \quad \frac{}{\pi_1(\dot{\neg}w)[\lambda v] \quad \pi_2(\dot{\neg}w)[\lambda v]} \quad \frac{\Sigma_1 \succ w : \neg C, x : A, \Delta_1 \quad \Sigma_2 \succ w : \neg C, y : B, \Delta_2}{\Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2}}$$

$\neg R$

$\wedge R$

$$\frac{\begin{array}{c} \pi_1(z)[x] \\ \Sigma_1, z : C \succ x : A, \Delta_1 \end{array}}{\pi_1(\dot{\neg}w)[x]} \quad \frac{\begin{array}{c} \pi_2(z)[y] \\ \Sigma_2, z : C \succ y : B, \Delta_2 \end{array}}{\pi_2(z)[\lambda v]} \quad \frac{}{\pi_1(\dot{\neg}w)[\lambda v] \quad \pi_2(z)[\lambda v]} \quad \frac{\Sigma_1 \succ w : \neg C, x : A, \Delta_1 \quad \Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2}}{\Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2}}$$

$\neg R$

$\wedge R$

$\wedge L/\wedge L$ Contraction/Expansion

$$\frac{\pi(x, y, x', y')}{\Sigma, x : A, y : B, x' : A, y' : B \succ \Delta} \wedge L$$
$$\frac{\pi(\lambda z, \lambda z, x', y')}{\Sigma, z : A \wedge B, x' : A, y' : B \succ \Delta} \wedge L$$
$$\Sigma, z : A \wedge B \succ \Delta$$

$\wedge L/\wedge L$ Contraction/Expansion

$$\frac{\pi(x, y, x', y')}{\Sigma, \mathbf{x}: A, \mathbf{y}: B, \mathbf{x}': A, \mathbf{y}': B \succ \Delta} \wedge L$$
$$\frac{\pi(\lambda z, \lambda z, x', y')}{\Sigma, \mathbf{z}: A \wedge B, \mathbf{x}': A, \mathbf{y}': B \succ \Delta} \wedge L$$
$$\Sigma, \mathbf{z}: A \wedge B \succ \Delta$$

$$\frac{\pi(x, y, x, y)}{\Sigma, \mathbf{x}: A, \mathbf{y}: B \succ \Delta} \wedge L$$
$$\frac{\pi(\lambda z, \lambda z, \lambda z, \lambda z)}{\Sigma, \mathbf{z}: A \wedge B \succ \Delta}$$

α/α Contraction/Expansion

$$\frac{\pi\{a_1\}\{a_2\}\{a'_1\}\{a'_2\}}{\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\}} \alpha$$
$$\frac{\pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{a'_1\}\{a'_2\}}{\mathfrak{S}\{a : \alpha\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\}} \alpha$$
$$\frac{\pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}a\}}{\mathfrak{S}\{a : \alpha\}}$$

α/α Contraction/Expansion

$$\frac{\pi\{a_1\}\{a_2\}\{a'_1\}\{a'_2\} \\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\}}{\pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{a'_1\}\{a'_2\} \\ \mathfrak{S}\{a : \alpha\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\}} \quad \alpha$$
$$\frac{\pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}a\} \\ \mathfrak{S}\{a : \alpha\}}{\pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}a\}} \quad \alpha$$

$$\frac{\pi\{a_1\}\{a_2\}\{a_1\}\{a_2\} \\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}}{\pi\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}a\}\{\dot{\alpha}a\}} \quad \alpha$$
$$\mathfrak{S}\{a : \alpha\}$$

$\wedge R / \wedge R$ Contraction/Expansion?

$$\frac{\begin{array}{c} \pi_1[x, x'] \\ \Sigma_1 \succ \textcolor{red}{x}: A, \textcolor{red}{x}' : A, \Delta_1 \end{array} \quad \begin{array}{c} \pi_2[x, y'] \\ \Sigma_2 \succ x: A, \textcolor{red}{y}' : B, \Delta_2 \end{array} \quad \begin{array}{c} \pi_3[y, x'] \\ \Sigma_3 \succ \textcolor{red}{y}: B, \textcolor{red}{x}' : A, \Delta_3 \end{array} \quad \begin{array}{c} \pi_4[y, y'] \\ \Sigma_4 \succ y: B, \textcolor{red}{y}' : B, \Delta_4 \end{array}}{\begin{array}{c} \pi_1[x, \lambda z] \quad \pi_2[x, \lambda z] \\ \Sigma_{1,2} \succ x: A, \textcolor{red}{z}: A \wedge B, \Delta_{1,2} \end{array} \quad \begin{array}{c} \pi_3[y, \lambda z] \quad \pi_4[y, \lambda z] \\ \Sigma_{3,4} \succ \textcolor{red}{y}: B, \textcolor{red}{z}: A \wedge B, \Delta_{3,4} \end{array}}$$
$$\frac{\pi_1[\lambda z, \lambda z] \quad \pi_2[\lambda z, \lambda z] \quad \pi_3[\lambda z, \lambda z] \quad \pi_4[\lambda z, \lambda z]}{\Sigma_{1-4} \succ \textcolor{red}{z}: A \wedge B, \Delta_{1-4}}$$

$\wedge R / \wedge R$ Contraction/Expansion?

$$\begin{array}{c}
 \frac{\pi_1[x, x'] \quad \pi_2[x, y']}{\Sigma_1 \succ x : A, x' : A, \Delta_1 \quad \Sigma_2 \succ x : A, y' : B, \Delta_2} \quad \frac{\pi_3[y, x'] \quad \pi_4[y, y']}{\Sigma_3 \succ y : B, x' : A, \Delta_3 \quad \Sigma_4 \succ y : B, y' : B, \Delta_4} \\
 \hline
 \frac{\pi_1[x, \lambda z] \quad \pi_2[x, \lambda z]}{\Sigma_{1,2} \succ x : A, z : A \wedge B, \Delta_{1,2}} \quad \frac{\pi_3[y, \lambda z] \quad \pi_4[y, \lambda z]}{\Sigma_{3,4} \succ y : B, z : A \wedge B, \Delta_{3,4}} \\
 \hline
 \pi_1[\lambda z, \lambda z] \quad \pi_2[\lambda z, \lambda z] \quad \pi_3[\lambda z, \lambda z] \quad \pi_4[\lambda z, \lambda z] \\
 \Sigma_{1-4} \succ z : A \wedge B, \Delta_{1-4}
 \end{array}$$

Collapsing this into one step removes the π_2 and π_3 connections.

$$\frac{\pi_1[x, x] \quad \pi_4[y, y]}{\Sigma_1 \succ x : A, \Delta_1 \quad \Sigma_4 \succ y : B, \Delta_4} \\
 \hline
 \frac{\pi_1[\lambda z, \lambda z] \quad \pi_4[\lambda z, \lambda z]}{\Sigma_{1,4} \succ z : A \wedge B, \Delta_{1,4}}$$

β/β Contraction/Expansion

We can contract stacked occurrences of the same β rule into this triangle.

$$\frac{\begin{array}{c} \pi_1\{b_1\}\{b'_1\} \\ \mathfrak{S}_1\{b_1:\beta_1\}\{b'_1:\beta_1\} \end{array} \quad \begin{array}{c} \pi_2\{b_2\}\{b'_2\} \\ \mathfrak{S}_2\{b_2:\beta_2\}\{b'_2:\beta_2\} \end{array} \quad \begin{array}{c} \pi_3\{b_1\}\{b'_2\} \\ \mathfrak{S}_3\{b_1:\beta_1\}\{b'_2:\beta_2\} \end{array} \quad \begin{array}{c} \pi_4\{b_2\}\{b'_2\} \\ \mathfrak{S}_4\{b_2:\beta_2\}\{b'_2:\beta_2\} \end{array}}{\begin{array}{c} \pi_1\{\dot{\beta} b\}\{b'_1\} \quad \pi_2\{\dot{\beta} b\}\{b'_2\} \\ \mathfrak{S}_{1,2}\{b:\beta\}\{b'_1:\beta_1\} \end{array} \quad \begin{array}{c} \pi_3\{\dot{\beta} b\}\{b'_2\} \quad \pi_2\{\dot{\beta} b\}\{b'_2\} \\ \mathfrak{S}_{3,4}\{b:\beta\}\{b'_2:\beta_2\} \end{array}} \quad \beta \quad \beta$$
$$\frac{\pi_1\{\dot{\beta} b\}\{\dot{\beta} b\} \quad \pi_2\{\dot{\beta} b\}\{\dot{\beta} b\} \quad \pi_3\{\dot{\beta} b\}\{\dot{\beta} b\} \quad \pi_4\{\dot{\beta} b\}\{\dot{\beta} b\}}{\mathfrak{S}_{1-4}\{b:\beta\}} \quad \beta$$

There's one more permutation...

I'll introduce that later.

(It's more complicated, and I'm not satisfied that I've got the best formulation.)

Soundness of permutations

If $\delta_1 \approx \delta_2$ then $\pi(\delta_1) = \pi(\delta_2)$

Completeness of permutations

If $\pi(\delta_1) = \pi(\delta_2)$ then $\delta_1 \approx \delta_2$

Strategy for Completeness

- ▶ Start with δ_1 and δ_2 where $\pi(\delta_1) = \pi(\delta_2)$

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Strategy for Completeness

- ▶ Start with δ_1 and δ_2 where $\pi(\delta_1) = \pi(\delta_2)$
- ▶ If they're both axioms, we're done.
- ▶ If they're not, then look at the last rule application in δ_1 .
 - ▶ Permute the *other* instances of that rule application in δ_1 to the bottom, where they collapse into one instance (α)—or a triangle (β).
 - ▶ Do the same, with the corresponding instances in δ_2 .
 - ▶ Chop off those last rules in both derivations.
 - ▶ The premises are the same, with the same proof terms [*Caveat*].
- ▶ Rinse and repeat.

Same instances of rule applications?

In a derivation δ with term π , annotate steps with the *node* in π introduced.

$$\frac{\frac{\frac{x : p \succ x : p}{x : p, y : p \wedge q \succ z : p} \quad \frac{z : p \succ z : p}{\lambda y \rightarrow z}}{x : p, y : p \wedge q \succ w : p \wedge p} \quad \frac{x : p \succ x : p}{x : p, y : p \wedge q \succ v : p \supset (p \wedge p)}}{x : p, y : p \wedge q \succ u : (p \wedge q) \supset (p \supset (p \wedge p))} \quad \text{u}$$

$$\frac{\frac{\frac{x : p \succ x : p}{x : p, z : p \succ w : p \wedge p} \quad \frac{z : p \succ z : p}{\lambda w \rightarrow z}}{x : p, z : p \succ w : p \wedge p} \quad \frac{x : p, z : p \succ w : p \wedge p}{\lambda v \rightarrow \lambda w \rightarrow z}}{\frac{x : p, z : p \succ v : p \supset (p \wedge p)}{\frac{x : p, z : p \succ v : p \supset (p \wedge p)}{\frac{x : p, z : p \succ v : p \supset (p \wedge p)}{x : p, z : p \succ u : (p \wedge q) \supset (p \supset (p \wedge p))}}}} \quad \text{u}$$

This allows identification of steps inside the *same* derivation

$$\frac{\begin{array}{c} x : p \succ x : p \quad x : p \succ x : p \\ \hline x : p \succ z : p \wedge p \end{array}}{z} \quad \frac{\begin{array}{c} x : p \succ x : p \quad x : p \succ x : p \\ \hline x : p \succ z : p \wedge p \end{array}}{z}$$

$$\frac{\begin{array}{c} \forall y \succ \lambda z \quad \forall y \succ \lambda z \\ \hline y : p \vee p \succ z : p \wedge p \end{array}}{y}$$

$$\frac{\begin{array}{c} x : p \succ x : p \quad x : p \succ x : p \\ \hline \forall y \succ x \quad \forall y \succ x \end{array}}{y} \quad \frac{\begin{array}{c} x : p \succ x : p \quad x : p \succ x : p \\ \hline \forall y \succ x \quad \forall y \succ x \end{array}}{y}$$

$$\frac{\begin{array}{c} \forall y \succ \lambda z \quad \forall y \succ \lambda z \quad \forall y \succ \lambda z \quad \forall y \succ \lambda z \\ \hline y : p \vee p \succ z : p \wedge p \end{array}}{z}$$

Deleting an α rule

$$\frac{\pi\{a_1\}\{a_2\} \\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}}{\pi\{\dot{\alpha}a\}\{\dot{\alpha}a\} \\ \mathfrak{S}\{a : \alpha\}}$$

Deleting a β rule — where do you split?

$$\frac{\begin{array}{c} \pi_1\{b_1\} & \pi_2\{b_2\} \\ \mathfrak{S}_1\{b_1 : \beta_1\} & \mathfrak{S}_2\{b_2 : \beta_2\} \end{array}}{\begin{array}{c} \pi_1\{\dot{\beta}b\} \quad \pi_2\{\dot{\beta}b\} \\ \mathfrak{S}_{1,2}\{b : \beta\} \end{array}}$$

Deleting a β rule — where do you split?

$$\frac{\begin{array}{c} \pi_1\{b_1\} \\ \mathfrak{S}_1\{b_1 : \beta_1\} \end{array} \qquad \begin{array}{c} \pi_2\{b_2\} \\ \mathfrak{S}_2\{b_2 : \beta_2\} \end{array}}{\begin{array}{c} \pi_1\{\dot{\beta}b\} \quad \pi_2\{\dot{\beta}b\} \\ \mathfrak{S}_{1,2}\{b : \beta\} \end{array}}$$

For example ...

$$\frac{\begin{array}{c} \pi_1[x] \\ \Sigma_1 \succ x : A, \Delta_1 \end{array} \qquad \begin{array}{c} \pi_2[y] \\ \Sigma_2 \succ y : B, \Delta_2 \end{array}}{\begin{array}{c} \pi_1[\lambda z] \quad \pi_2[\lambda z] \\ \Sigma_{1,2} \succ z : A \wedge B, \Delta_{1,2} \end{array}}$$

$$\frac{\begin{array}{c} \pi_3[x] \\ \Sigma_3 \succ x : A, \Delta_3 \end{array} \qquad \begin{array}{c} \pi_4[y] \\ \Sigma_4 \succ y : B, \Delta_4 \end{array}}{\begin{array}{c} \pi_3[\lambda z] \quad \pi_4[\lambda z] \\ \Sigma_{3,4} \succ z : A \wedge B, \Delta_{3,4} \end{array}}$$

Be Lazy

$$\frac{\pi_1[x] \quad \pi_2[y]}{\Sigma_1 \succ \textcolor{red}{x} : A, \Delta_1 \quad \Sigma_2 \succ \textcolor{red}{y} : B, \Delta_2}$$
$$\frac{}{\pi_1[\lambda z] \quad \pi_2[\lambda z]}$$
$$\Sigma_{1,2} \succ \textcolor{red}{z} : A \wedge B, \Delta_{1,2}$$

$$\frac{\pi[x, -] \quad \pi[-, y]}{\Sigma \succ \textcolor{red}{x} : A, \Delta \quad \Sigma \succ \textcolor{red}{y} : B, \Delta}$$
$$\frac{}{\pi[\lambda z, \lambda z]}$$
$$\Sigma \succ \textcolor{red}{z} : A \wedge B, \Delta$$

Be Lazy

$$\frac{\pi_1[x] \quad \pi_2[y]}{\Sigma_1 \succ \textcolor{red}{x} : A, \Delta_1 \quad \Sigma_2 \succ \textcolor{red}{y} : B, \Delta_2}$$

$$\frac{\pi[x, -] \quad \pi[-, y]}{\Sigma \succ \textcolor{red}{x} : A, \Delta \quad \Sigma \succ \textcolor{red}{y} : B, \Delta}$$

$$\frac{\pi_1[x] \ \pi_2[-] \quad \pi_1[-] \ \pi_2[y]}{\Sigma \succ \textcolor{red}{x} : A, \Delta \quad \Sigma \succ \textcolor{red}{y} : B, \Delta}$$

$$\frac{\pi_1[\lambda z] \ \pi_2[\lambda z]}{\Sigma \succ \textcolor{red}{z} : A \wedge B, \Delta}$$

The last permutation—horizontal copying of *prooffragments*

$$\frac{\begin{array}{c} \pi_1 \\ \Sigma_1 \succ \Delta_1 \end{array} \quad \begin{array}{c} \pi_2[y] \\ \Sigma_2 \succ \textcolor{red}{y} : A, \Delta_2 \end{array}}{\Sigma \succ \Delta}$$

$$\frac{\begin{array}{c} \pi_1 \quad \pi_2[-] \\ \Sigma_{1,2} \succ \Delta_{1,2} \end{array} \quad \begin{array}{c} \pi_2[y] \\ \Sigma_2 \succ \textcolor{red}{y} : A, \Delta_2 \end{array}}{\Sigma \succ \Delta}$$

Example of weakening in a proof fragment

$$\frac{\begin{array}{c} \textcolor{red}{y \rightsquigarrow y} \\ \textcolor{red}{y : A \succ y : A} \end{array} \quad \begin{array}{c} \textcolor{red}{z \rightsquigarrow z} \\ \textcolor{red}{z : B \succ z : B} \end{array}}{\textcolor{red}{\vee x \rightsquigarrow y \ \vee x \rightsquigarrow z}} \quad \textcolor{black}{\vee L}$$
$$\textcolor{red}{x : A \vee B \succ y : A, z : B}$$

Example of weakening in a proof fragment

$$\frac{\begin{array}{c} \textcolor{red}{y \rightsquigarrow y} \\ \textcolor{red}{y : A \succ y : A} \end{array} \quad \begin{array}{c} \textcolor{red}{z \rightsquigarrow z} \\ \textcolor{red}{z : B \succ z : B} \end{array}}{\textcolor{red}{\vee x \rightsquigarrow y \vee x \rightsquigarrow z}} \quad \textcolor{black}{\vee L}$$
$$\textcolor{red}{x : A \vee B \succ y : A, z : B}$$

Delete y to get $\textcolor{red}{\vee x \rightsquigarrow z}$, for the (invalid) sequent $x : A \vee B \succ z : B$.

Example of weakening in a proof fragment

$$\frac{\begin{array}{c} \textcolor{red}{y \rightsquigarrow y} \\ \textcolor{red}{y : A \succ y : A} \end{array} \quad \begin{array}{c} \textcolor{red}{z \rightsquigarrow z} \\ \textcolor{red}{z : B \succ z : B} \end{array}}{\textcolor{red}{\vee x \rightsquigarrow y \vee x \rightsquigarrow z}} \quad \textcolor{brown}{\vee L}$$
$$\textcolor{red}{x : A \vee B \succ y : A, z : B}$$

Delete y to get $\textcolor{red}{\vee x \rightsquigarrow z}$, for the (invalid) sequent $x : A \vee B \succ z : B$.

$$\Sigma \stackrel{\pi}{\succ} \Delta$$

Example of weakening in a proof fragment

$$\frac{\begin{array}{c} y \rightsquigarrow y \\ y : A \succ y : A \end{array} \quad \begin{array}{c} z \rightsquigarrow z \\ z : B \succ z : B \end{array}}{\begin{array}{c} \vee x \rightsquigarrow y \quad \vee x \rightsquigarrow z \\ x : A \vee B \succ y : A, z : B \end{array}} \vee L$$

Delete y to get $\vee x \rightsquigarrow z$, for the (invalid) sequent $x : A \vee B \succ z : B$.

$$\Sigma \succ \Delta \quad \text{becomes} \quad \frac{\begin{array}{c} \pi \\ \Sigma, y : A \succ \Delta \end{array} \quad \begin{array}{c} z \rightsquigarrow z \\ z : B \succ z : B \end{array}}{\begin{array}{c} \pi \quad \vee x \rightsquigarrow z \\ \Sigma, x : A \vee B \succ z : B, \Delta \end{array}} \vee L$$

A scenic view of Bryce Canyon National Park, featuring a vast landscape of red rock hoodoos and green pine trees. A dirt trail winds its way through the canyon, with several people walking along it. The sky is clear and blue.

FURTHER WORK

To Do List

- ▶ Find a clear formulation of horizontal copying, and establish whether it follows from the other permutations.
- ▶ Apply these terms to other kinds of proofs (Fitch, Lemmon, tableaux, Hilbert, resolution...)
- ▶ *Categories* (The class of *single input, single output* terms with composition by defined by *Cut + reduction* is a category. What are its properties?)
- ▶ Extend beyond propositional logic.

THANK YOU!

[http://consequently.org/presentation/2016/
proof-terms-invariants](http://consequently.org/presentation/2016/proof-terms-invariants)

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