

# A Concrete Category of Classical Proofs

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THE UNIVERSITY OF  
MELBOURNE

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*Download slides here:* <http://consequently.org/presentation/2017/a-category-of-classical-proofs-tacl>

## My Aim

To show how *proof terms* for classical propositional logic form a *category*, and to examine some of its properties.

## Today's Plan

Proof Terms

The Proof Term Category

It's not *Cartesian*

It is *Monoidal*, and more...

Isomorphisms

Further Work



# PROOF TERMS

There can be different ways to prove the same thing

$$p \wedge q \succ p \vee q$$

## Four different derivations,

$$\frac{\frac{p \succ p}{p \wedge q \succ p} \wedge L}{p \wedge q \succ p \vee q} \vee R$$

$$\frac{\frac{p \succ p}{p \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

$$\frac{\frac{q \succ q}{p \wedge q \succ q} \wedge L}{p \wedge q \succ p \vee q} \vee R$$

$$\frac{\frac{q \succ q}{q \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

## Four different derivations, two *proofs*

$$\frac{\frac{p \succ p}{p \wedge q \succ p} \wedge L}{p \wedge q \succ p \vee q} \vee R$$

$$\frac{p \wedge q}{\frac{p}{p \vee q}}$$

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$$\frac{\frac{q \succ q}{q \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

## Motivating Idea

*Proof terms* are an *invariant* for derivations under rule permutation.

$\delta_1$  and  $\delta_2$  have the same *term* iff some permutation sends  $\delta_1$  to  $\delta_2$ .

# Four different derivations, two *proofterms*

$$\frac{\frac{x \rightsquigarrow y}{x : p \succ y : p} \wedge L \quad x : p \wedge q \succ y : p}{\frac{\lambda x \rightsquigarrow \forall y}{x : p \wedge q \succ y : p \vee q}} \vee R$$

$\lambda x \rightsquigarrow \forall y$

$$\frac{\frac{x \rightsquigarrow x}{x : p \succ y : p} \vee R \quad x : p \succ y : p \vee q}{\frac{\lambda x \rightsquigarrow \forall y}{x : p \wedge q \succ y : p \vee q}} \wedge L$$

$$\frac{\frac{x \rightsquigarrow y}{x : q \succ y : q} \wedge L \quad x : p \wedge q \succ y : q}{\frac{\lambda x \rightsquigarrow \forall y}{x : p \wedge q \succ y : p \vee q}} \vee R$$

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$$\frac{\frac{x \rightsquigarrow x}{x : q \succ y : q} \vee R \quad x : q \succ y : p \vee q}{\frac{\lambda x \rightsquigarrow \forall y}{x : p \wedge q \succ y : p \vee q}} \wedge L$$

# Ingredients

$\lambda$  terms

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$\lambda$  terms    ♦    flow graphs

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$\lambda$  terms    ♦    flow graphs    ♦    proof nets

# Slogan

A *proofterm* for  $\Sigma \succ \Delta$   
encodes the flow of information  
in a proof of  $\Sigma \succ \Delta$ .

# Results

- Cut elimination is *confluent* and *terminating*.

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- Cut elimination for proof terms is *local*.

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- Cut elimination is *confluent* and *terminating*.  
[So it can be understood as a kind of *evaluation*.]
- Cut elimination for proof terms is *local*.  
[So it is easily made parallel.]

# Proof Terms

See <http://consequently.org/writing/>

## PROOF TERMS FOR CLASSICAL DERIVATIONS

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*Abstract:* I give an account of *proof terms* for derivations in a sequent calculus for classical propositional logic. The term for a derivation  $\delta$  of a sequent  $\Sigma \succ \Delta$  encodes *how* the premises  $\Sigma$  and conclusions  $\Delta$  are related in  $\delta$ . This encoding is many-to-one in the sense that different derivations can have the same proof term, since different derivations may be different ways of representing the same underlying connection between premises and conclusions. However, not all proof terms for a sequent  $\Sigma \succ \Delta$  are the same. There may be *different* ways to connect those premises and conclusions.

Proof terms can be simplified in a process corresponding to the elimination of cut inferences in sequent derivations. However, unlike cut elimination in the sequent calculus, each proof term has a *unique normal form* (from which all cuts have been eliminated) and it is straightforward to show that term reduction is strongly normalising—every reduction process terminates in that unique normal form. Furthermore, proof terms are *invariants* for sequent derivations in a strong sense—two derivations  $\delta_1$  and  $\delta_2$  have the same proof term if and only if some permutation of derivation steps sends  $\delta_1$  to  $\delta_2$  (given a relatively natural class of permutations of derivations in the sequent calculus). Since not every derivation of a sequent can be permuted into every other derivation of that sequent, proof terms provide a non-trivial account of the identity of proofs, independent of the syntactic representation of those proofs.

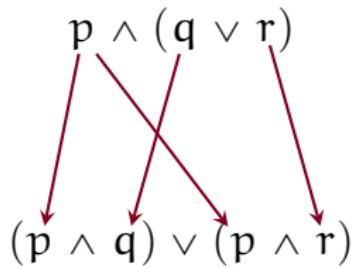
### OUTLINE

# Proof Terms

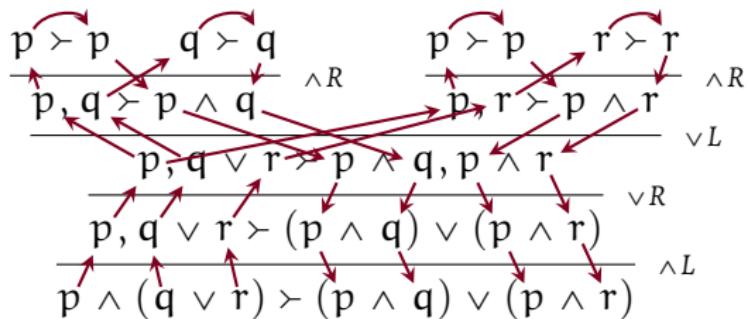
$$\lambda x \rightarrow \lambda \vee y \quad \lambda x \rightarrow \lambda \wedge y \quad \vee \lambda x \rightarrow \lambda \vee y \quad \wedge \lambda x \rightarrow \lambda \wedge y$$
$$x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$$

# Proof Terms as Graphs on Sequents

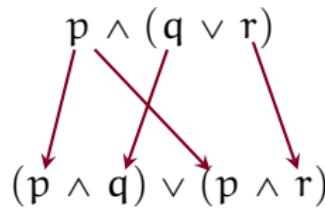
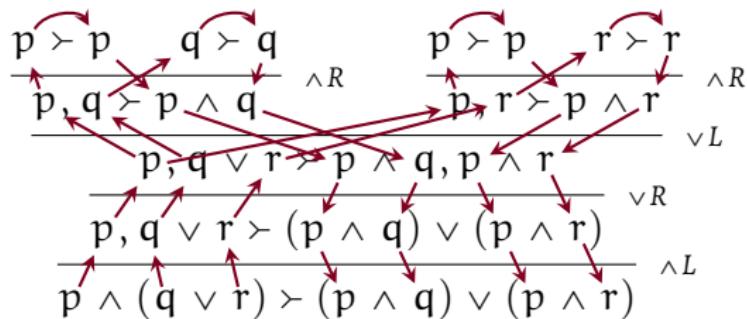
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# Finding a Proof Term from a Derivation



# Finding a Proof Term from a Derivation



# Finding a Proof Term from a Derivation

$$\frac{\begin{array}{c} p \succ p \\ q \succ q \\ \hline p, q \succ p \wedge q \end{array} \wedge R \quad \begin{array}{c} p \succ p \\ r \succ r \\ \hline p, r \succ p \wedge r \end{array} \wedge R}{\begin{array}{c} p, q \vee r \succ p \wedge q, p \wedge r \\ \hline p, q \vee r \succ (p \wedge q) \vee (p \wedge r) \end{array} \vee L} \vee R$$
$$\frac{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)} \wedge L$$

$$\begin{array}{c} p \wedge (q \vee r) \\ \swarrow \quad \searrow \\ (p \wedge q) \vee (p \wedge r) \end{array}$$

# Finding a Proof Term from a Derivation

$$\frac{\frac{\frac{p \succ p \quad q \succ q}{p, q \succ p \wedge q} \wedge R \quad \frac{p \succ p \quad r \succ r}{p, r \succ p \wedge r} \wedge R}{p, q \vee r \succ p \wedge q, p \wedge r} \vee L}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)} \vee R \quad \frac{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}{\wedge L}$$

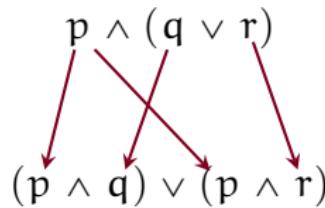
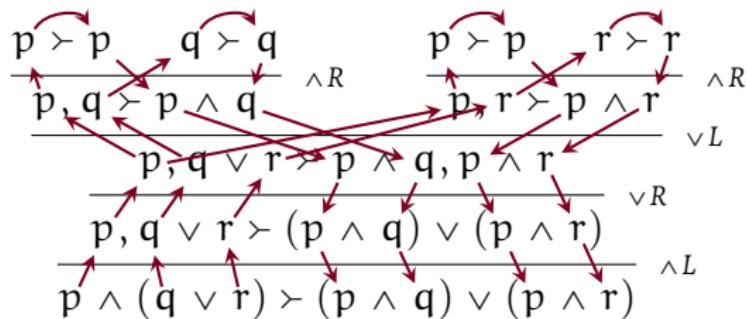
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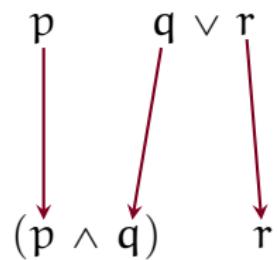
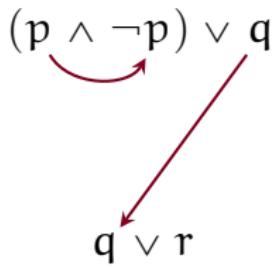
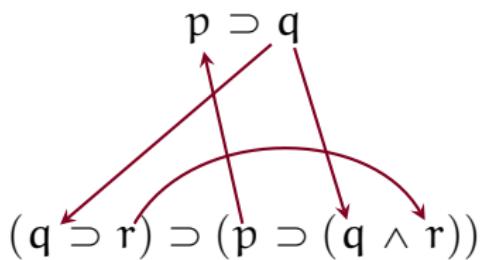
$$\frac{\begin{array}{c} p \succ p \quad q \succ q \\ \hline p, q \succ p \wedge q \end{array}}{\frac{}{p, q \vee r \succ p \wedge q, p \wedge r}} \wedge R$$
$$\frac{\begin{array}{c} p \succ p \quad r \succ r \\ \hline p, r \succ p \wedge r \end{array}}{\frac{}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}} \wedge R$$
$$\frac{\begin{array}{c} p, q \vee r \succ (p \wedge q) \vee (p \wedge r) \\ \hline p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r) \end{array}}{\vee L}$$
$$\frac{\begin{array}{c} p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r) \\ \hline \end{array}}{\vee R}$$
$$\frac{\begin{array}{c} p \wedge (q \vee r) \\ \hline (p \wedge q) \vee (p \wedge r) \end{array}}{\wedge L}$$

$$\frac{\begin{array}{c} p \wedge (q \vee r) \\ \hline (p \wedge q) \vee (p \wedge r) \end{array}}{(p \wedge q) \vee (p \wedge r)}$$

# Finding a Proof Term from a Derivation



# More Flow Graphs



## Proof Term Facts

Not every directed graph on occurrences of atoms in a sequent is a proof term.

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- ▶ They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]
- ▶ They must satisfy an “enough connections” condition, amounting to a non-emptiness under every *switching*. [e.g. the obvious linking between premise  $p \vee q$  and conclusion  $p \wedge q$  is not connected enough to be a proof term.]

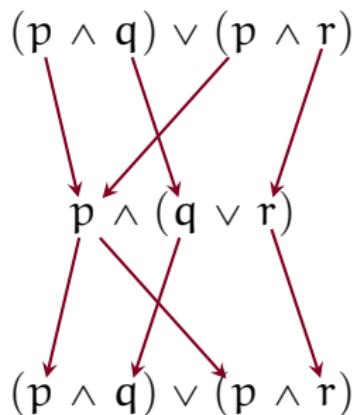
## Cut is chaining of proof terms

$$\begin{array}{c} (p \wedge q) \vee (p \wedge r) \\ \diagdown \quad \diagup \quad \diagup \\ p \wedge (q \vee r) \end{array}$$

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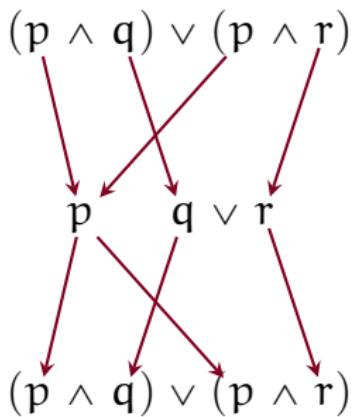


The *cut formula* is no longer a premise or a conclusion in the proof term.

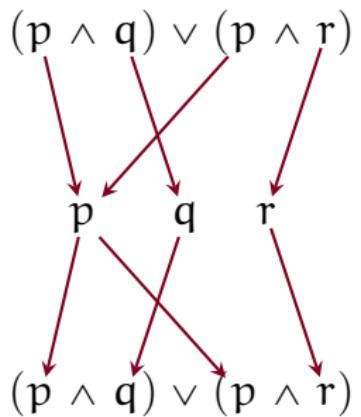
# Eliminating Cuts is Local

$$\begin{array}{c} ((p \wedge q) \vee (p \wedge r)) \\ \swarrow \quad \searrow \\ p \wedge (q \vee r) \\ \swarrow \quad \searrow \\ ((p \wedge q) \vee (p \wedge r)) \end{array}$$

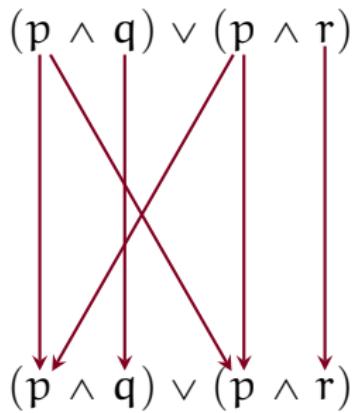
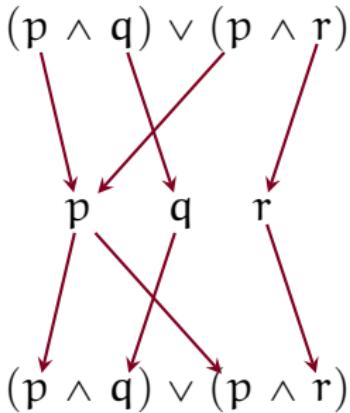
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# The Conjunction Reduction Case, for Derivations

$$\frac{\frac{\delta_1 \quad \delta_2}{\Sigma_1 \succ A, \Delta_1 \quad \Sigma_2 \succ B, \Delta_1} \wedge R \quad \frac{\delta_3}{\Sigma_3, A, B \succ \Delta_3} \wedge L}{\Sigma_{1,2} \succ A \wedge B, \Delta_{1,2}} \quad \frac{\Sigma_3, A \wedge B \succ \Delta_3}{\Sigma_{1-3} \succ \Delta_{1-3}} Cut_{A \wedge B}$$

reduces to

$$\frac{\frac{\delta_1 \quad \frac{\delta_2 \quad \delta_3}{\Sigma_2 \succ B, \Delta_1 \quad \Sigma_3, A, B \succ \Delta_3} Cut_B}{\Sigma_{2,3}, A \succ \Delta_{2,3}} Cut_A}{\Sigma_{1-3} \succ \Delta_{1-3}} Cut_A$$

## Two Different Proofs from $(p \wedge q) \vee (p \wedge r)$ to itself

$$(p \wedge q) \vee (p \wedge r)$$

$$(p \wedge q) \vee (p \wedge r)$$

The *second* proof term is the *identity* proof.

# Bounds

$\top$  and  $\perp$  are interesting.

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$$p \wedge \neg p$$

$$q$$

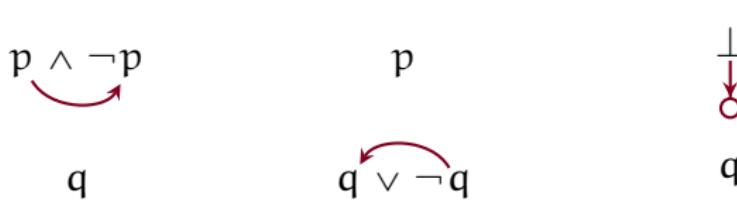
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$$p$$
  
$$q \vee \neg q$$


$$\perp$$


$$q$$

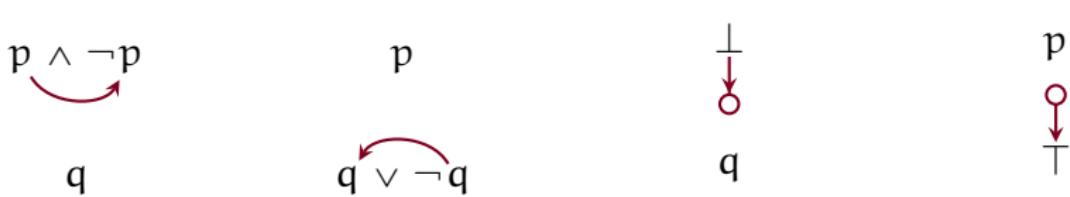
$$p$$


$$\top$$

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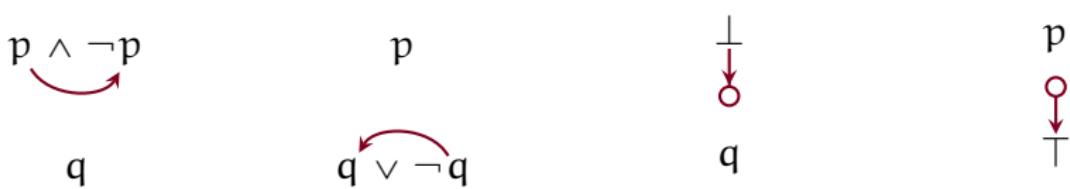
A  $\perp$  link has an input but no output.

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A  $\perp$  link has an input but no output.

A  $\top$  link has an output but no input.

No links have  $\top$  as an input.

No links have  $\perp$  as an output.

# Identity Proofs

$$\begin{array}{c} A \\ \Downarrow \\ A \end{array}$$

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$$\begin{array}{ccc} p & \Downarrow & T \\ \downarrow & & \downarrow \\ p & \circ & T \end{array}$$

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$$p \downarrow p$$

$$T \downarrow T$$

$$\perp \downarrow \perp$$

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$$\begin{array}{c} A \wedge B \\ \Downarrow \\ A \wedge B \end{array}$$

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$$\begin{array}{c} \neg A \\ \Updownarrow \\ \neg A \end{array}$$

# Identity Proofs

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$$\begin{array}{c} p \\ \Downarrow \\ T \end{array}$$

$$\begin{array}{c} T \\ \Downarrow \\ \top \end{array}$$

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$$\begin{array}{c} A \wedge B \\ \Downarrow \\ A \wedge B \end{array}$$

$$\begin{array}{c} A \vee B \\ \Downarrow \\ A \vee B \end{array}$$

$$\begin{array}{c} \neg A \\ \Updownarrow \\ \neg A \end{array}$$

$$\begin{array}{c} A \supset B \\ \Updownarrow \\ A \supset B \end{array}$$

A photograph of a vast, open landscape. In the foreground, a dark asphalt road curves from the bottom left towards the center. To the left of the road, there's a small puddle of water reflecting the sky. The middle ground is a flat, dry plain with sparse, yellowish-brown vegetation. In the background, there are several large, rugged mountains with distinct layered rock faces. The sky above is a clear blue with scattered white and grey clouds.

# THE PROOF TERM CATEGORY

# OBJECTS: *Formulas* — ARROWS: *Cut-Free Proof Terms*

- $\pi : A \rightarrow B$  iff  $\pi(x)[y]$  is a *cut-free* proof for  $x : A \succ y : B$ .

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- ▶ Composition is chaining proofs & elimination of cuts.
  - If  $\pi : A \rightarrow B$  and  $\tau : B \rightarrow C$  then  $\tau \circ \pi : A \rightarrow C$  is  $(\pi(x)[\bullet] \tau(\bullet)[y])^*$ .

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- ▶ Composition is chaining proofs & elimination of cuts.
  - If  $\pi : A \rightarrow B$  and  $\tau : B \rightarrow C$  then  $\tau \circ \pi : A \rightarrow C$  is  $(\pi(x)[\bullet] \tau(\bullet)[y])^*$ .
- ▶ Composition is associative.
- ▶ Identity proofs are indeed identities in the category:
  - $(\pi(x)[\bullet] \bullet \rightleftarrows y)^* = \pi(x)[y]$ , and  $(x \rightleftarrows \bullet \pi(\bullet)[y])^* = \pi(x)[y]$ , when  $\pi$  is cut-free.

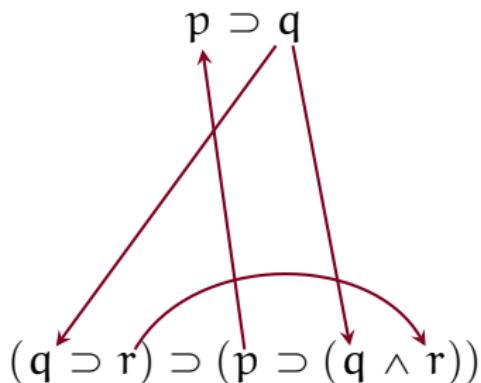
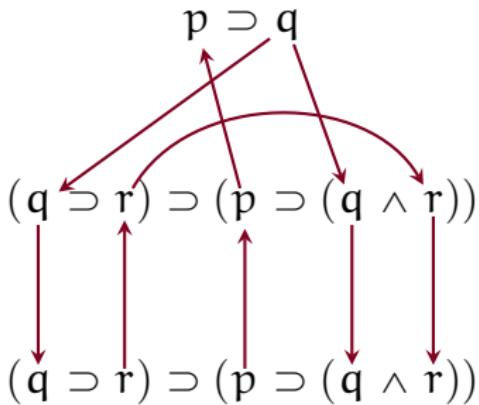
# How Identity Proofs Compose

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$$\begin{array}{c} p \supset q \\ \swarrow \quad \searrow \\ ((q \supset r) \supset (p \supset (q \wedge r))) \\ \downarrow \quad \downarrow \\ ((q \supset r) \supset (p \supset (q \wedge r))) \end{array}$$

# How Identity Proofs Compose



# We have a Category

## Proof Terms

- ▶  $\pi$  has type  $\Sigma \succ \Delta$ .
- ▶ Proofs are SET–SET.
- ▶ Proofs include *Cuts*.

$$\frac{x : A \succ y : A \quad x : A \succ y : B}{\frac{x \not\sim \bullet \quad \pi(\bullet)[y]}{x : A \succ y : B}} \text{Cut}$$

## The Category $\mathcal{T}$ of Cut-Free Terms

- ▶  $\pi : A \rightarrow B$ .
- ▶ Proofs are FMLA–FMLA.
- ▶ Proofs have no *Cuts*.

$$\frac{id_A \quad \pi}{\frac{\pi \circ id_A = \pi}{A \rightarrow B}}$$

What is the proof term category like?

# Cartesian Products

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

# Cartesian Products

$$\begin{array}{ccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ & \nwarrow f & & \nearrow g & \\ & & C & & \end{array}$$

# Cartesian Products

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & & \uparrow \langle f, g \rangle & & \searrow g \\ & & C & & \end{array}$$

# Cartesian Products

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & & \uparrow \langle f, g \rangle & & \searrow g \\ C & & & & \end{array}$$

$$\pi_1 \circ \langle f, g \rangle = f \quad \pi_2 \circ \langle f, g \rangle = g$$

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$$\pi_1 \circ \langle f, g \rangle = f \quad \pi_2 \circ \langle f, g \rangle = g$$

This looks a lot like conjunction.

Many interesting categories have cartesian products.

# The Empty Product

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & \nearrow \langle f, g \rangle & \uparrow & \searrow g & \\ C & & & & \end{array}$$

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# The Empty Product

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & \uparrow \langle f, g \rangle & & \searrow g & \\ C & & & & C \end{array}$$

A commutative diagram showing the empty product construction. At the top, there is a horizontal sequence of objects:  $A$ ,  $A \times B$ , and  $B$ . Above this sequence, there are two arrows:  $\pi_1$  from  $A$  to  $A \times B$ , and  $\pi_2$  from  $A \times B$  to  $B$ . Below the sequence, there is a single object labeled  $C$ . From  $A$ , there is a diagonal arrow  $f$  pointing down and to the right towards  $C$ . From  $B$ , there is a diagonal arrow  $g$  pointing up and to the left towards  $C$ . Between  $A$  and  $B$ , there is a vertical dashed arrow labeled  $\langle f, g \rangle$  pointing upwards. To the right of the diagram, there is a vertical dashed arrow labeled  $T$  pointing upwards, with a label  $C$  at its bottom.

# Coproducts and Initial Objects

$$\begin{array}{ccccc} A & \xrightarrow{\iota_1} & A + B & \xleftarrow{\iota_2} & B \\ f \searrow & & \downarrow [f,g] & & \swarrow g \\ & & C & & \end{array} \quad \perp \quad \begin{array}{c} \downarrow \\ C \end{array}$$

## Residuating Products — internalising arrows

$$f : A \times B \rightarrow C \quad \tilde{f} : A \rightarrow B \supset C \quad ev : (B \supset C) \times B \rightarrow C$$

$$\begin{array}{ccc} (B \supset C) \times B & \xrightarrow{ev} & C \\ \tilde{f} \times id \uparrow & & \nearrow f \\ A \times B & & \end{array}$$

# Cartesian Closed Categories...

## Cartesian Closed Categories...

...model intuitionistic logic.

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...model intuitionistic logic.

They collapse into preorders when made classical.

So what is the proof term category?

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Since it isn't a preorder, and it is classical...

A dark, moody landscape featuring snow-covered mountain peaks in the background and a rocky, brownish-orange terrain with patches of snow in the foreground.

IT'S NOT CARTESIAN

$\top$  is not Terminal,  $\perp$  is not Initial

$\top$  is not Terminal,  $\perp$  is not Initial

$$p \wedge \neg p$$

```
graph TD; A((p ∧ ¬p)) --> T1[T]; T1 --> B((p ∧ ¬p))
```

$$p \wedge \neg p$$

```
graph TD; A((p ∧ ¬p)) --> T2[T]; T2 --> B((p ∧ ¬p))
```

$\top$  is not Terminal,  $\perp$  is not Initial

$$p \wedge \neg p$$

A red arrow points downwards from the formula  $p \wedge \neg p$  to the terminal object  $\top$ .

$$p \wedge \neg p$$

A red curved arrow points back to the formula  $p \wedge \neg p$  from the terminal object  $\top$ .

$$\perp$$

A red arrow points downwards from the initial object  $\perp$  to the formula  $q \vee \neg q$ .

$$\perp$$

A red curved arrow points back to the initial object  $\perp$  from the formula  $q \vee \neg q$ .

... and nothing else is initial or terminal either

If  $T$  is a candidate terminal object,  
then there is some arrow  $\top \rightarrow T$ .

In this arrow all links are internal to  $T$   
(since  $\top$  is never a source in a link).

These links generate a proof term for  $\perp \rightarrow T$ ,  
and this proof ignores  $\perp$ .

There is a different proof term for  $\perp \rightarrow T$   
using  $\perp$  and ignoring  $T$ .

(This dualises for any candidate initial object  $I$ .)

## Conjunction isn't Cartesian Product

$$\begin{array}{ccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & \nearrow \langle f, g \rangle & \uparrow & \searrow & g \\ C & & & & \end{array}$$

# Conjunction isn't Cartesian Product

We have candidate projection arrows.

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & \nearrow \langle f, g \rangle & \uparrow & \searrow g & \\ C & & & & \end{array}$$

$$\begin{array}{c} A \wedge B \\ \downarrow \\ A \end{array}$$

$$\begin{array}{c} A \wedge B \\ \downarrow \\ B \end{array}$$

# Conjunction isn't Cartesian Product

We have candidate projection arrows.

And a candidate pairing arrow.

$$\begin{array}{ccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & \nearrow \langle f, g \rangle & \uparrow & \searrow g & \\ C & & & & \end{array}$$

$$\begin{array}{c} C \\ f \swarrow \quad \searrow g \\ A \wedge B \end{array}$$

# Conjunction isn't Cartesian Product

We have candidate projection arrows.

And a candidate pairing arrow.

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & \nearrow \langle f, g \rangle & \uparrow & \searrow g & \\ C & & & & \end{array}$$

$$\begin{array}{c} C \\ f \swarrow \quad \searrow g \\ A \wedge B \end{array}$$

But its composition with “projection”  
need not restore  $f$  and  $g$ .

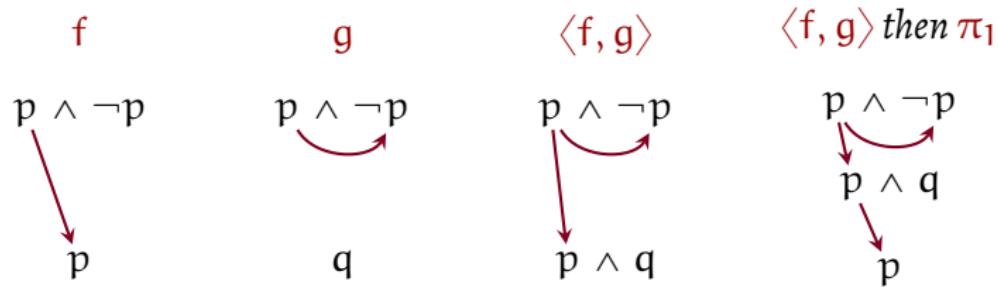
# An Example

$$\begin{array}{ccc} f & & g \\ p \wedge \neg p & \xrightarrow{\hspace{1cm}} & p \wedge \neg p \\ p & & q \end{array}$$

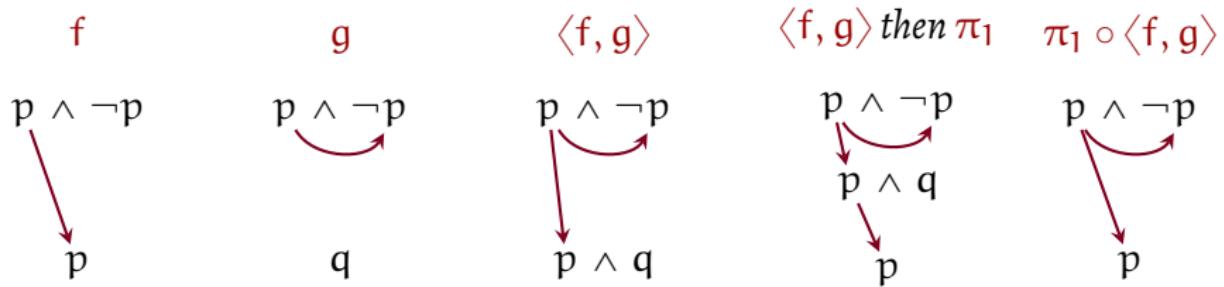
## An Example

$$\begin{array}{ccc} f & g & \langle f, g \rangle \\ p \wedge \neg p & p \wedge \neg p & p \wedge \neg p \\ \downarrow & \curvearrowright & \downarrow \\ p & q & p \wedge q \end{array}$$

# An Example



# An Example



## An Example



Notice:  $\pi_1 \circ \langle f, g \rangle$  is not  $f$ .

It has some of  $g$  (in this case, *all* of the links of  $g$ ) left behind.

## An Example



Notice:  $\pi_1 \circ \langle f, g \rangle$  is not  $f$ .

It has some of  $g$  (in this case, *all* of the links of  $g$ ) left behind.

However, in general,  $f \subseteq \pi_1 \circ \langle f, g \rangle$  and  $g \subseteq \pi_2 \circ \langle f, g \rangle$ .

# Diagnosis

This arises from the *locality* of cut reduction.

$$\frac{\frac{\frac{p, \neg p \succ p}{p \wedge \neg p \succ p} \quad \frac{\frac{p \succ p, q}{p, \neg p \succ q}}{p \wedge \neg p \succ q} \wedge R \quad \frac{p, q \succ p}{p \wedge q \succ p} \wedge L}{p \wedge \neg p \succ p \wedge q} Cut_{p \wedge q}}$$

~~~>

$$\frac{\frac{\frac{p, \neg p \succ p}{p \wedge \neg p \succ p} \quad \frac{\frac{p \succ p, q}{p, \neg p \succ q}}{p \wedge \neg p \succ q} \quad p, q \succ p}{p, p \wedge \neg p \succ p} Cut_q}{p \wedge \neg p \succ p} Cut_p$$

## In fact, there are *no* Cartesian Products

A *slightly* more general argument shows that there is *no* object  $p \times q$

- equipped with projection arrows  $\pi_1 : p \times q \rightarrow p$  and  $\pi_2 : p \times q \rightarrow q$ ,
- where there is some proof  $h : p \wedge \neg p \rightarrow p \times q$ , such that
- $\pi_1 \circ h = f$  and  $\pi_2 \circ h = g$ .

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- where there is some proof  $h : p \wedge \neg p \rightarrow p \times q$ , such that
- $\pi_1 \circ h = f$  and  $\pi_2 \circ h = g$ .

(The argument is a dilemma: does  $h$  contain a link between the instances of  $p$  in the premise  $p \wedge \neg p$ ? If it *does*, then composition with  $\pi_1$  preserves that link, and  $\pi_1 \circ h$  isn't  $f$ . If it *doesn't*, there is no way for  $\pi_2 \circ h$  to contain that link.)

So, if it isn't Cartesian, what *is* the category like?

A wide-angle landscape photograph of a vast, winding river valley in a rugged mountain range under a cloudy sky. The river flows from the background towards the foreground, its banks lined with green vegetation and rocky terrain. The mountains on either side are steep and brownish-yellow, with patches of snow and ice clinging to their peaks. The sky is filled with heavy, grey clouds.

IT IS MONOIDAL,  
& MORE...

# Monoidal Categories

Many categories have something *like* cartesian product, but different.

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Tensor product —  $\otimes$  — in vector spaces is an important example.

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Many categories have something *like* cartesian product, but different.

Tensor product —  $\otimes$  — in vector spaces is an important example.

This motivates the definition of a *monoidal* category.

# Symmetric Monoidal Categories

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \quad 1 \in Ob(\mathcal{C})$$

# Symmetric Monoidal Categories

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \quad 1 \in Ob(\mathcal{C})$$

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$$

$$\sigma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A \quad \iota_A : 1 \otimes A \xrightarrow{\sim} A$$

where *associativity* ( $\alpha$ ), *symmetry* ( $\sigma$ ) and *unit* ( $\iota$ ) behave sensibly.

# Associativity

$$\begin{array}{ccccc} & & (A \otimes B) \otimes (C \otimes D) & & \\ & \nearrow \alpha_{A,B,C \otimes D} & & \searrow \alpha_{A \otimes B,C,D} & \\ A \otimes (B \otimes (C \otimes D)) & & & & ((A \otimes B) \otimes C) \otimes D \\ id_A \otimes \alpha_{B,C,D} \downarrow & & & & \uparrow \alpha_{A,B,C} \otimes id_D \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha_{A,B \otimes C,D}} & & & (A \otimes (B \otimes C)) \otimes D \end{array}$$

# Associativity

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(The ‘Pentagon’)

# Symmetry

$$\begin{array}{ccc} & \sigma_{A,B} & \\ A \otimes B & \swarrow \quad \searrow & B \otimes A \\ & \sigma_{B,A} & \end{array}$$

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$$\begin{array}{ccc} & \sigma_{A,B} & \\ A \otimes B & \swarrow & \searrow \\ & \sigma_{B,A} & \end{array}$$

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\sigma_{A,B \otimes C}} & (B \otimes C) \otimes A \\ \downarrow \sigma_{A,B} \otimes id_C & & & & \downarrow \alpha_{B,C,A} \\ (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{id_B \otimes \sigma_{A,C}} & B \otimes (C \otimes A) \end{array}$$

(The ‘Hexagon’)

(Let's drop the subscripts on  $\alpha$ ,  $\sigma$ ,  $\iota$ ,  $id$  where there's no ambiguity.)

# Unit

$$\begin{array}{ccc} (A \otimes 1) \otimes B & \xrightarrow{\alpha} & A \otimes (1 \otimes B) \\ \sigma \otimes id \downarrow & & \downarrow id \otimes \iota \\ (1 \otimes A) \otimes B & \xrightarrow{\iota \otimes id} & A \otimes B \end{array}$$

(The ‘Square’)

# Proof Terms are a Symmetric Monoidal Category under $\wedge/\top$

$$\wedge : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T} \quad \top \in Ob(\mathcal{T})$$

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$$\wedge : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T} \quad \top \in Ob(\mathcal{T})$$

$$\hat{\alpha} : A \wedge (B \wedge C) \xrightarrow{\sim} (A \wedge B) \wedge C$$

$$\hat{\sigma} : A \wedge B \xrightarrow{\sim} B \wedge A \quad \hat{\iota} : \top \wedge A \xrightarrow{\sim} A$$

and indeed, *associativity* ( $\hat{\alpha}$ ), *symmetry* ( $\hat{\sigma}$ ) and *unit* ( $\hat{\iota}$ ) behave sensibly.

# $\hat{\alpha}$ , $\hat{\sigma}$ and $\hat{\iota}$

$$\hat{\alpha}_{A,B,C} : A \wedge (B \wedge C) \rightarrow (A \wedge B) \wedge C$$


$$\hat{\sigma}_{A,B} : A \wedge B \rightarrow B \wedge A$$


$$\hat{\iota}_A : \top \wedge A \rightarrow A$$


$\hat{\alpha}$ ,  $\hat{\sigma}$  and  $\hat{\iota}$  are isomorphisms

$$\begin{array}{c} \hat{\alpha}^{-1} \circ \hat{\alpha} \\ A \wedge (B \wedge C) \\ \Downarrow \\ (A \wedge B) \wedge C \\ \Downarrow \\ A \wedge (B \wedge C) \end{array}$$

$$\begin{array}{c} id \\ A \wedge (B \wedge C) \\ \Downarrow \\ A \wedge (B \wedge C) \end{array}$$

$$\begin{array}{c} \hat{\sigma}^{-1} \circ \hat{\sigma} \\ A \wedge B \\ \times \\ B \wedge A \\ \times \\ A \wedge B \\ \Downarrow \\ A \wedge B \end{array}$$

$$\begin{array}{c} id \\ A \wedge B \\ \Downarrow \\ A \wedge B \end{array}$$

$\hat{\alpha}$ ,  $\hat{\sigma}$  and  $\hat{\iota}$  are isomorphisms

$$\hat{\iota}^{-1} \circ \hat{\iota}$$
$$\begin{array}{ccc} T \wedge A & & \\ \searrow & \swarrow & \\ A & & \\ \circ \downarrow & \searrow & \\ T \wedge A & & \end{array}$$

$$id$$
$$\begin{array}{ccc} T \wedge A & & \\ \downarrow & \downarrow & \\ T \wedge A & & \end{array}$$

$$\hat{\iota} \circ \hat{\iota}^{-1}$$
$$\begin{array}{ccc} & A & \\ & \searrow & \swarrow \\ \circ \downarrow & & \\ T \wedge A & & \\ \searrow & & \swarrow \\ A & & \end{array}$$

$$id$$
$$\begin{array}{ccc} A & & \\ \downarrow & \downarrow & \\ A & & \end{array}$$

# The Pentagon, Hexagon, Square, etc., commute

$$\begin{array}{ccc}
 (A \wedge \top) \wedge B & \xrightarrow{\hat{\alpha}} & A \wedge (\top \wedge B) \\
 \hat{\sigma} \wedge id \downarrow & & \downarrow id \wedge \hat{\iota} \\
 (\top \wedge A) \wedge B & \xrightarrow{\hat{\iota} \wedge id} & A \wedge B
 \end{array}$$

$$(\hat{\iota} \wedge id) \circ (\hat{\sigma} \wedge id)$$

$$\begin{array}{c}
 (A \wedge \top) \wedge B \\
 \searrow \text{○} \downarrow \quad \swarrow \downarrow \\
 (\top \wedge A) \wedge B \\
 \searrow \downarrow \quad \swarrow \downarrow \\
 A \wedge B
 \end{array}$$

=

$$(id \wedge \hat{\iota}) \circ \hat{\alpha}$$

$$\begin{array}{c}
 (A \wedge \top) \wedge B \\
 \searrow \downarrow \quad \swarrow \downarrow \\
 A \wedge B
 \end{array}$$

$$\begin{array}{c}
 (A \wedge \top) \wedge B \\
 \searrow \downarrow \quad \swarrow \text{○} \downarrow \\
 A \wedge (\top \wedge B) \\
 \searrow \downarrow \quad \swarrow \downarrow \\
 A \wedge B
 \end{array}$$

## The Pentagon, Hexagon, Square, etc., commute

$$\begin{array}{ccccc} & & (A \wedge B) \wedge (C \wedge D) & & \\ & \nearrow \hat{\alpha} & & \searrow \hat{\alpha} & \\ A \wedge (B \wedge (C \wedge D)) & & & & ((A \wedge B) \wedge C) \wedge D \\ id \wedge \hat{\alpha} \downarrow & & & & \uparrow \hat{\alpha} \wedge id \\ A \wedge ((B \wedge C) \wedge D) & \xrightarrow{\hat{\alpha}} & & & (A \wedge (B \wedge C)) \wedge D \end{array}$$

## The Pentagon, Hexagon, Square, etc., commute

$$\begin{array}{ccccc} (A \wedge B) \wedge C & \xrightarrow{\hat{\alpha}} & A \wedge (B \wedge C) & \xrightarrow{\hat{\sigma}} & (B \wedge C) \wedge A \\ \hat{\sigma} \wedge id \downarrow & & & & \downarrow \hat{\alpha} \\ (B \wedge A) \wedge C & \xrightarrow{\hat{\alpha}} & B \wedge (A \wedge C) & \xrightarrow{id \wedge \hat{\sigma}} & B \wedge (C \wedge A) \end{array}$$

# Proof Terms are a Symmetric Monoidal Category under $\vee/\perp$

$$\vee : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T} \quad \perp \in Ob(\mathcal{T})$$

$$\stackrel{\vee}{\alpha} : A \vee (B \vee C) \xrightarrow{\sim} (A \vee B) \vee C$$

$$\stackrel{\vee}{\sigma} : A \vee B \xrightarrow{\sim} B \vee A \quad \stackrel{\vee}{\iota} : \perp \vee A \xrightarrow{\sim} A$$

and *associativity* ( $\stackrel{\vee}{\alpha}$ ), *symmetry* ( $\stackrel{\vee}{\sigma}$ ) and *unit* ( $\stackrel{\vee}{\iota}$ ) behave just as sensibly.

# Linear Distributive Categories

The operators  $\wedge$  and  $\vee$  are connected by  $\delta$  and  $\delta'$

$$\delta : A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C \quad \delta' : (A \vee B) \wedge C \rightarrow A \vee (B \wedge C)$$

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$$\delta : A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C \quad \delta' : (A \vee B) \wedge C \rightarrow A \vee (B \wedge C)$$

If the operators are *symmetric*, then we need only one.

$$\begin{array}{ccc} (A \vee B) \wedge C & \xrightarrow{\delta'} & A \vee (B \wedge C) \\ \hat{\sigma} \downarrow & & \uparrow \check{\sigma} \\ C \wedge (A \vee B) & & (B \wedge C) \vee A \\ id \wedge \check{\sigma} \downarrow & & \uparrow \hat{\sigma} \vee id \\ C \wedge (B \vee A) & \xrightarrow{\delta} & (C \wedge B) \vee A \end{array}$$

$\delta$  and  $\delta'$  are *obvious* proof terms

$$\begin{array}{ccc} \delta & & \delta' \\ A \wedge (B \vee C) & \downarrow & (A \vee B) \wedge C \\ (A \wedge B) \vee C & & A \vee (B \wedge C) \end{array}$$

# Linear Distributivity Conditions

$$\begin{array}{ccc}
 & (A \wedge B) \wedge (C \vee D) & \xrightarrow{\hat{\alpha}} A \wedge (B \wedge (C \vee D)) \\
 \begin{array}{c} T \wedge (A \vee B) \\ \downarrow \delta \\ ((T \wedge A) \vee B) \xrightarrow{\hat{\iota} \vee id} A \vee B \end{array} & \downarrow \delta & \begin{array}{c} A \wedge ((B \wedge C) \vee D) \\ \downarrow id \wedge \delta \\ ((A \wedge B) \wedge C) \vee D \xrightarrow{\hat{\alpha} \vee id} (A \wedge (B \wedge C)) \vee D \end{array} \\
 & & \downarrow \delta
 \end{array}$$
  

$$\begin{array}{ccc}
 ((A \vee B) \wedge C) \vee D & \xleftarrow{\delta} & (A \vee B) \wedge (C \vee D) & \xrightarrow{\delta'} & A \vee (B \wedge (C \vee D)) \\
 & \downarrow \delta' \vee id & & & \downarrow id \vee \delta \\
 (A \vee (B \wedge C)) \vee D & \xrightarrow{\alpha} & & & A \vee ((B \wedge C) \vee D)
 \end{array}$$

# Linear Distributivity Conditions

$$\begin{array}{ccc}
 & (A \wedge B) \wedge (C \vee D) & \xrightarrow{\hat{\alpha}} A \wedge (B \wedge (C \vee D)) \\
 \begin{array}{c} T \wedge (A \vee B) \\ \downarrow \delta \\ ((T \wedge A) \vee B) \xrightarrow{\hat{\iota} \vee id} A \vee B \end{array} & \downarrow \delta & \begin{array}{c} A \wedge ((B \wedge C) \vee D) \\ \downarrow id \wedge \delta \\ ((A \wedge B) \wedge C) \vee D \xrightarrow{\hat{\alpha} \vee id} (A \wedge (B \wedge C)) \vee D \end{array} \\
 & & \downarrow \delta
 \end{array}$$
  

$$\begin{array}{ccc}
 ((A \vee B) \wedge C) \vee D & \xleftarrow{\delta} & (A \vee B) \wedge (C \vee D) & \xrightarrow{\delta'} & A \vee (B \wedge (C \vee D)) \\
 & \downarrow \delta' \vee id & & & \downarrow id \vee \delta \\
 (A \vee (B \wedge C)) \vee D & \xrightarrow{\alpha} & & & A \vee ((B \wedge C) \vee D)
 \end{array}$$

(These diagrams *clearly* commute in the proof term category.)

## Star-Autonomous Categories

There are a number of ways to define *Star-Autonomous Categories*.

We have a  $\neg A$  for each object  $A$ , and two sets of arrows.

$$\gamma_A : A \wedge \neg A \rightarrow \perp \quad \tau_A : \top \rightarrow \neg A \vee A$$

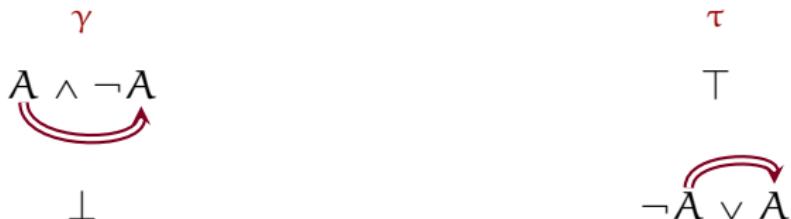
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We have a  $\neg A$  for each object  $A$ , and two sets of arrows.

$$\gamma_A : A \wedge \neg A \rightarrow \perp \quad \tau_A : \top \rightarrow \neg A \vee A$$

These arrows have natural proof terms.



# These Diagrams Must Commute

$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \vee \\ A \wedge \top & \xrightarrow[\wedge]{\iota} & & & \end{array}$$

$$\begin{array}{ccccc} (\neg A \vee A) \wedge \neg A & \xrightarrow{\delta'} & \neg A \vee (A \wedge \neg A) & \xrightarrow{id \vee \gamma} & \neg A \vee \perp \\ \tau \wedge id \uparrow & & & & \downarrow \vee \\ \top \wedge \neg A & \xrightarrow[\wedge]{\iota} & & & \end{array}$$

# These Diagrams Must Commute

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$$\begin{array}{ccc} (\neg A \vee A) \wedge \neg A & \xrightarrow{\delta'} & \neg A \vee (A \wedge \neg A) \xrightarrow{id \vee \gamma} \neg A \vee \perp \\ \tau \wedge id \uparrow & & \downarrow \vee \\ \top \wedge \neg A & \xrightarrow[\wedge]{\iota} & \neg A \end{array}$$

These aren't so obviously commutative as proof terms.

# The negation diagrams commute in the proof term category

$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \vdash \\ A \wedge \top & \xrightarrow[\wedge \mathfrak{t}]{} & & & A \end{array}$$

# The negation diagrams commute in the proof term category

$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \vdash \\ A \wedge \top & \xrightarrow[\wedge]{} & & & A \end{array}$$

$$\begin{array}{c} (id \wedge \tau) \\ \swarrow \quad \searrow \\ A \wedge \top \\ \swarrow \quad \searrow \\ A \wedge (\neg A \vee A) \end{array}$$

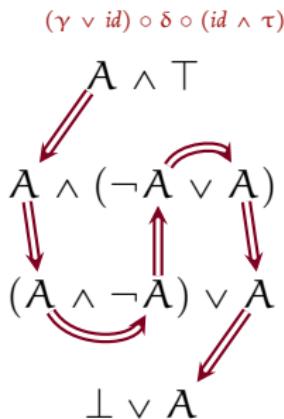
# The negation diagrams commute in the proof term category

$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \vee \\ A \wedge \top & \xrightarrow[\wedge]{} & & & A \end{array}$$

$$\begin{array}{c} \delta \circ (id \wedge \tau) \\ \swarrow \quad \searrow \\ A \wedge \top \\ \downarrow \quad \downarrow \\ A \wedge (\neg A \vee A) \\ \uparrow \quad \downarrow \\ (A \wedge \neg A) \vee A \end{array}$$

# The negation diagrams commute in the proof term category

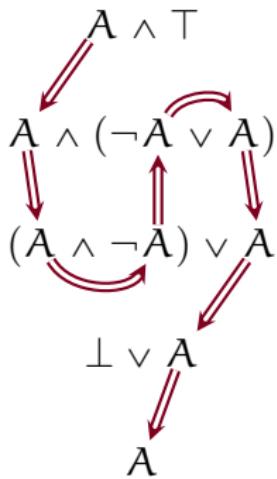
$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \vdash \\ A \wedge \top & \xrightarrow[\wedge]{} & & & A \end{array}$$



# The negation diagrams commute in the proof term category

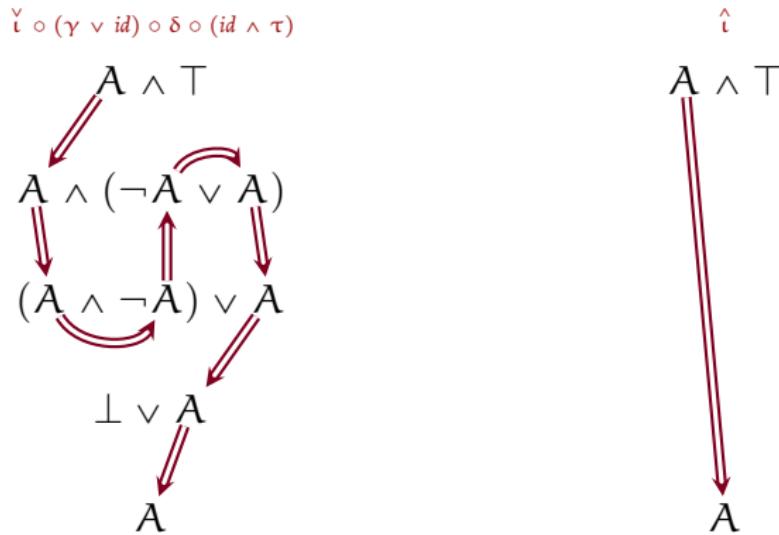
$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \textcolor{red}{\text{v}} \iota \\ A \wedge \top & \xrightarrow[\wedge \iota]{} & & & A \end{array}$$

$$\textcolor{red}{\text{v}} \iota \circ (\gamma \vee id) \circ \delta \circ (id \wedge \tau)$$



# The negation diagrams commute in the proof term category

$$\begin{array}{ccccc}
 A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\
 \uparrow id \wedge \tau & & & & \downarrow \check{\iota} \\
 A \wedge \top & \xrightarrow[\textcolor{red}{\wedge}]{} & & & A
 \end{array}$$



# Star-Autonomous Categories and Linear Logic

These categories model the multiplicative fragment of linear logic.

## Linear Implication

I won't pause now to explain how  $A \supset B$ , definable as  $\neg A \vee B$  (or as  $\neg(A \wedge \neg B)$ , to which it's isomorphic) is a right adjoint to  $\wedge$ .

## We can do more

Our proof terms allow *contraction* and *weakening*.

# Weakening and Contraction Monoids and Comonoids

$$\nabla_A : A \vee A \rightarrow A$$

$$\stackrel{\perp}{\beta}_A : \perp \rightarrow A$$

$$\begin{array}{ccc} \nabla_A & & \stackrel{\perp}{\beta}_A \\ A \vee A & \Downarrow & \perp \\ A & \Downarrow & A \end{array}$$

# Weakening and Contraction Monoids and Comonoids

$$\nabla_A : A \vee A \rightarrow A$$

$$\beta_A^\perp : \perp \rightarrow A$$

$$\Delta_A : A \rightarrow A \wedge A$$

$$\beta_A^T : A \rightarrow T$$



# What Makes $\beta^\perp$ and $\beta^T$ weakening?

$$A \xrightarrow{\vee\ i} A \vee \perp \xrightarrow{id \vee \beta_B^\perp} A \vee B$$

$$A \wedge B \xrightarrow{id \wedge \beta_B^T} A \wedge \top \xrightarrow{\wedge\ i} A$$

# (Co)monoidal Conditions for Contraction and Weakening

$$\begin{array}{ccc} (A \vee A) \vee A & \xrightarrow{\check{\alpha}} & A \vee (A \vee A) \\ \nabla \vee id \downarrow & & \downarrow id \vee \nabla \\ A \vee A & \xrightarrow{\nabla} & A \xleftarrow{\nabla} A \vee A \end{array}$$

$$\begin{array}{ccc} A \vee \perp & \xrightarrow{id \vee \frac{1}{\beta}} & A \vee A \xleftarrow{\frac{1}{\beta} \vee id} \perp \vee A \\ \check{\sigma} \downarrow & & \nabla \downarrow \\ \perp \vee A & \xrightarrow{\check{\iota}} & A \xleftarrow{\check{\iota}} \perp \vee A \end{array} \qquad \begin{array}{ccc} A \vee A & \xrightarrow{\check{\sigma}} & A \vee A \\ \nabla \searrow & & \swarrow \nabla \\ & A & \end{array}$$

# Structurality for $\nabla$ and $\beta$ : disjunctions

$$\begin{array}{c}
 (A \vee B) \vee (A \vee B) \xrightarrow{\alpha} A \vee (B \vee (A \vee B)) \xrightarrow{id \vee \alpha} A \vee ((B \vee A) \vee B) \\
 \downarrow id \vee (\sigma \vee id) \\
 A \vee ((A \vee B) \vee B) \\
 \downarrow id \vee \alpha \\
 A \vee (A \vee (B \vee B)) \\
 \downarrow \alpha \\
 A \vee B \leftarrow \nabla \vee \nabla
 \end{array}$$

$$\begin{array}{ccc}
 \perp & \xrightarrow{\text{!}^\vee} & \perp \vee \perp \\
 & \searrow \perp \beta & \swarrow \perp \beta \vee \perp \\
 & A \vee B &
 \end{array}$$

# Structurality for $\nabla$ and $\beta^\perp$ : bounds

$$\begin{array}{ccc} \perp \vee \perp & \begin{array}{c} \xrightarrow{\nabla_\perp} \\ \curvearrowright \\ \xleftarrow{\text{v}_\perp} \end{array} & \perp \\ & & \\ \perp & \begin{array}{c} \xrightarrow{\beta_\perp^\perp} \\ \curvearrowright \\ \xleftarrow{id_\perp} \end{array} & \perp \end{array}$$

## Structurality for $\nabla$ and $\beta^\perp$ : bounds

$$\begin{array}{ccc} \perp \vee \perp & \xrightarrow{\nabla_\perp} & \perp \\ & \xrightarrow{\vee_\perp} & \end{array} \quad \begin{array}{ccc} \perp & \xrightarrow{\beta_\perp^\perp} & \perp \\ & \xrightarrow{id_\perp} & \end{array}$$

All these conditions are straightforward to verify for proof terms.

And dually for  $\Delta$  and  $\beta$ .<sup>T</sup>

# Blend

$$\begin{array}{c} A \\ \Downarrow \\ f \\ \Downarrow \\ B \end{array} \qquad \begin{array}{c} A \\ \Downarrow \\ g \\ \Downarrow \\ B \end{array}$$

# Blend

$$\begin{array}{c} A \\ \Downarrow \\ f \\ \Downarrow \\ B \end{array}$$

$$\begin{array}{c} A \\ \Downarrow \\ g \\ \Downarrow \\ B \end{array}$$

$$f \left( \begin{array}{c} A \\ \Downarrow \\ B \end{array} \right) g$$

# Blend

$$A \xrightarrow{f} B$$

$$A \xrightarrow{g} B$$

$$f \left( \begin{array}{c} A \\[-1ex] B \end{array} \right) g$$

$$f \cup g \downarrow A \\[-1ex] B$$

# Blend

$$\begin{array}{c} A \\ \Downarrow \\ f \\ \Downarrow \\ B \end{array}$$

$$\begin{array}{c} A \\ \Downarrow \\ g \\ \Downarrow \\ B \end{array}$$

$$f \left( \begin{array}{c} A \\ \Downarrow \\ B \end{array} \right) g$$

$$\begin{array}{c} A \\ \Downarrow \\ f \cup g \\ \Downarrow \\ B \end{array}$$

$\cup$  is a semilattice join on  $\text{Hom}(A, B)$ .

$$(f \cup f') \circ g = (f \circ g) \cup (f' \circ g) \quad f \circ (g \cup g') = (f \circ g) \cup (f \circ g')$$

The term category  $\mathcal{T}$  is *enriched in*  $\text{SLat}$ .

# Classical Categories

Classical categories are  
*star autonomous categories*  
with *structural monoids and comonoids*,  
*enriched in SLat*.

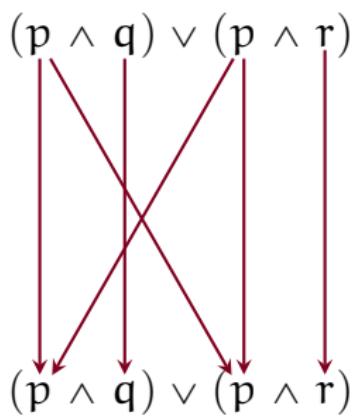
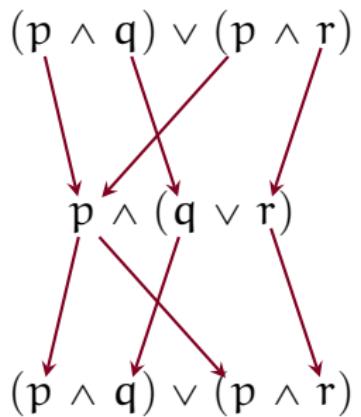
Cf. Führmann and Pym:

- ▶ “Order-enriched categorical models of the classical sequent calculus”  
JPAA (2006)
- ▶ “On categorical models of classical logic and the Geometry of Interaction”  
MSCS (2007).



# ISOMORPHISMS

$(p \wedge q) \vee (p \wedge r)$  is not isomorphic to  $p \wedge (q \vee r)$



## Also Not Isomorphisms

$$p \not\cong p \wedge p \quad p \not\cong p \vee p$$

$$p \wedge (p \vee q) \not\cong p \vee (p \wedge q) \quad p \vee \neg p \not\cong \top \quad q \wedge \neg q \not\cong \perp$$

## Isomorphisms

$$A \wedge T \cong A \quad A \wedge B \cong B \wedge A \quad A \wedge (B \wedge C) \cong (A \wedge B) \wedge C$$

$$A \vee \perp \cong A \quad A \vee B \cong B \vee A \quad A \vee (B \vee C) \cong (A \vee B) \vee C$$

$$\neg(A \wedge B) \cong (\neg A \vee \neg B) \quad \neg(A \vee B) \cong (\neg A \wedge \neg B) \quad \neg\neg A \cong A$$

$$\neg T \cong \perp \quad \neg \perp \cong T \quad T \vee T \cong T \quad \perp \wedge \perp \cong \perp$$

These isomorphisms (together with substitution into arbitrary contexts)  
*characterise* isomorphism in the term category  $\mathcal{T}$ .

# Hyperintensionality

*Inside classical logic,*  
there is a fine-grained,  
hyperintensional notion  
of sameness of content,  
tighter than logical equivalence  
but looser than syntactic identity.

A scenic view of Bryce Canyon National Park, featuring a vast landscape of red rock hoodoos and green pine trees under a clear blue sky. A paved trail winds through the foreground on the left, leading towards a group of people standing on a rocky outcrop. The terrain is rugged and layered, with deep canyons and high plateaus.

# FURTHER WORK

## To Do List

- ▶ Finish the completeness proof, to the effect that  $\mathcal{T}_{\mathcal{L}}$  is the free classical category on  $\mathcal{L}$ .
- ▶ Explore other examples of classical categories.
- ▶ Consider the restriction to terms for intuitionist derivations.  
(This still isn't Cartesian. What sort of category is it?)
- ▶ Extend all of this to *first order predicate logic*.

# THANK YOU!

[http://consequently.org/presentation/2017/  
a-category-of-classical-proofs-tacl](http://consequently.org/presentation/2017/a-category-of-classical-proofs-tacl)

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