

A Category of Classical Proofs

Greg Restall



THE UNIVERSITY OF

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My Aim

To show how *proof terms* for classical propositional logic form a *category*, and to examine some of its properties.

Today's Plan

Proof Terms

Categories

The Proof Term Category

It's not *Cartesian*

It is *Monoidal*, and more...

Further Work

A wide-angle photograph of a mountainous landscape. In the foreground, dark green coniferous trees are silhouetted against a bright sky. A wide, light-colored river or valley floor stretches across the middle ground, framed by dark, rugged mountains. The sky above is filled with scattered white and grey clouds.

PROOF TERMS

There can be different ways to prove the same thing

$$p \wedge q \succ p \vee q$$

There can be different ways to prove the same thing

$$p \wedge q \succ p \vee q$$

$$p \wedge \neg p \succ p \vee \neg p$$

Proof Terms

PROOF TERMS FOR CLASSICAL DERIVATIONS

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Abstract: I give an account of *proof terms* for derivations in a sequent calculus for classical propositional logic. The term for a derivation δ of a sequent $\Sigma \succ \Delta$ encodes *how* the premises Σ and conclusions Δ are related in δ . This encoding is many-to-one in the sense that different derivations can have the same proof term, since different derivations may be different ways of representing the same underlying connection between premises and conclusions. However, not all proof terms for a sequent $\Sigma \succ \Delta$ are the same. There may be *different* ways to connect those premises and conclusions.

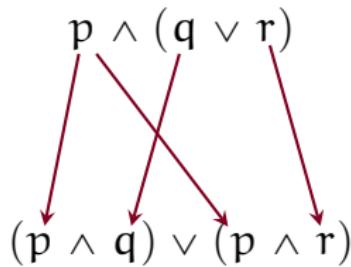
Proof terms can be simplified in a process corresponding to the elimination of cut inferences in sequent derivations. However, unlike cut elimination in the sequent calculus, each proof term has a *unique normal form* (from which all cuts have been eliminated) and it is straightforward to show that term reduction is strongly normalising—every reduction process terminates in that unique normal form. Furthermore, proof terms are *invariants* for sequent derivations in a strong sense—two derivations δ_1 and δ_2 have the same proof term if and only if some permutation of derivation steps sends δ_1 to δ_2 (given a rela-

Proof Terms

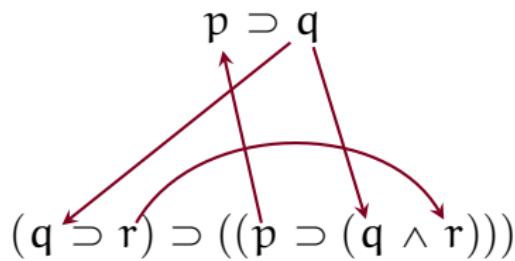
$$\lambda x \rightarrow \lambda \neg y \quad \lambda x \rightarrow \lambda \neg y \quad \vee \lambda x \rightarrow \lambda \neg y \quad \vee \lambda x \rightarrow \lambda \neg y$$
$$x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$$

Proof Terms as Graphs on Sequents

$$\lambda x \rightarrow \lambda \neg y \quad \lambda x \rightarrow \lambda \neg y \quad \vee \lambda x \rightarrow \lambda \neg y \quad \vee \lambda x \rightarrow \lambda \neg y \\ \textcolor{red}{x} : p \wedge (q \vee r) \succ \textcolor{red}{y} : (p \wedge q) \vee (p \wedge r)$$



Proof Terms as Graphs on Sequents



Composing Sequents and Eliminating Cuts

$$\begin{array}{c} (p \wedge q) \vee (p \wedge r) \\ \swarrow \quad \searrow \\ p \wedge (q \vee r) \\ \downarrow \quad \downarrow \\ (p \wedge q) \vee (p \wedge r) \end{array}$$

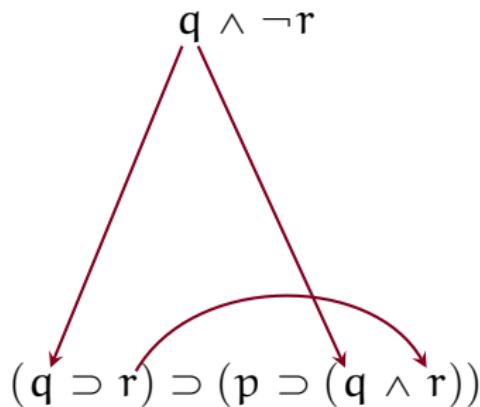
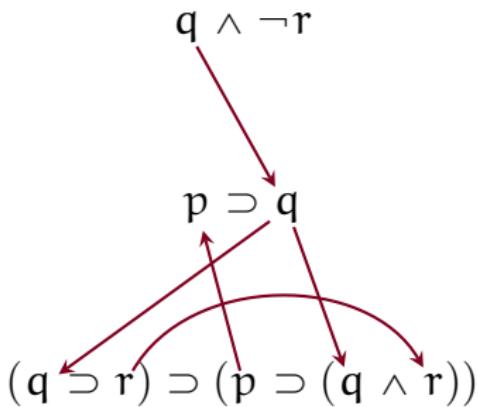
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That's not the Identity Proof

$$(p \wedge q) \vee (p \wedge r)$$
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Composing Sequents and Eliminating Cuts



Bounds

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$$\begin{array}{ccc} p \wedge \neg p & p & \perp \\ \text{---} \curvearrowright & \text{---} & \downarrow \\ q & q \vee \neg q & q \end{array}$$

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$$p$$

$$q \vee \neg q$$

$$\perp$$

$$q$$

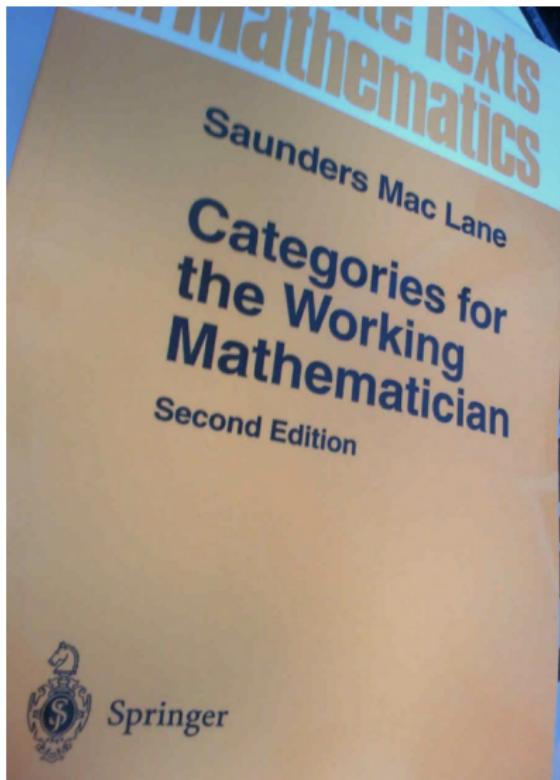
$$p$$

$$\top$$

A nighttime satellite view of Earth from space, showing city lights and clouds.

CATEGORIES

Categories as a Unifying Mathematical Vocabulary



Objects, Arrows, Composition and Identities

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- ▶ For each object A , there is an *identity arrow* $id_A : A \rightarrow A$
 - such that $f \circ id_A = f$, when $f : A \rightarrow B$ and $id_A \circ g = g$ when $g : C \rightarrow A$.

Many properties are definable at this level of generality

$f : A \rightarrow B$ is an *isomorphism* if
there is some arrow $g : B \rightarrow A$
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We write ' $f : A \xrightarrow{\sim} B$ ' to show that f is an isomorphism.

Examples of Categories

- ▶ Sets & *functions*.

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 - *Objects*: one object. *Arrows*: an arrow for each element in the monoid, with the identity arrow as the monoid identity, and composition the monoid operation.
- ▶ Formulas & *proofs*.

A photograph of a vast, open landscape. In the foreground, a dark asphalt road curves from the bottom left towards the center. To the left of the road, there's a small puddle of water reflecting the sky. The middle ground features a wide, flat plain with sparse, dry vegetation. In the background, there are several large, rugged mountains with distinct layered rock faces. The sky above is a clear blue with scattered white and grey clouds.

THE PROOF TERM CATEGORY

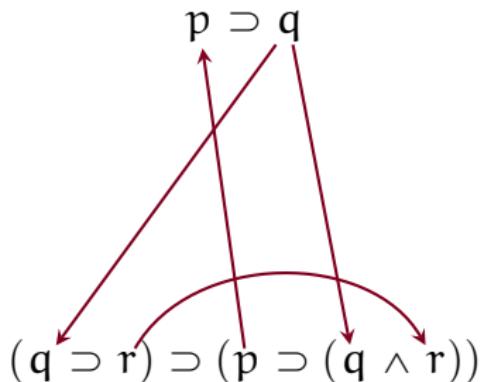
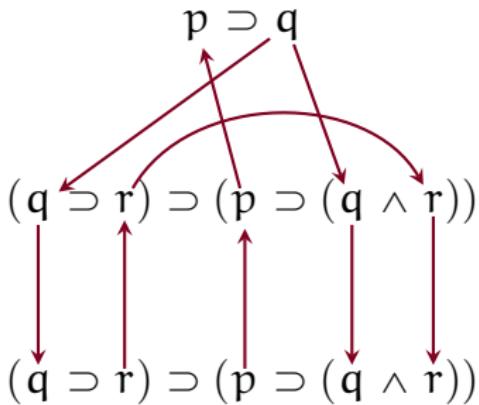
- ▶ $\pi : A \rightarrow B$ iff $\pi(x)[y]$ is a *cut-free* proof for $x : A \succ y : B$.
- ▶ $id_A : A \rightarrow A$ is the identity proof term $x \rightleftarrows y$ of type A .
- ▶ Composition is chaining proofs & elimination of cuts.
 - If $\pi : A \rightarrow B$ and $\tau : B \rightarrow C$ then $\tau \circ \pi : A \rightarrow C$ is $(\pi(x)[\bullet] \tau(\bullet)[y])^*$.
- ▶ Composition is associative.
- ▶ Identity proofs are identities in the category:
 - $(\pi(x)[\bullet] \bullet \rightleftarrows y)^* = \pi(x)[y]$, and $(x \rightleftarrows \bullet \pi(\bullet)[y])^* = \pi(x)[y]$, when π is cut-free.

How Identity Proofs Compose

$$(q \supset r) \supset (p \supset (q \wedge r))$$

$$(q \supset r) \supset (p \supset (q \wedge r))$$

How Identity Proofs Compose



We have a Category

Proof Terms—in general

- ▶ π has type $\Sigma \succ \Delta$.
- ▶ Proofs are SET–SET.
- ▶ Proofs include *Cuts*.

Categories of Cut-Free Terms

- ▶ $\pi : A \rightarrow B$.
- ▶ Proofs are FMLA–FMLA.
- ▶ Proofs have no *Cuts*.

$$\frac{x:\mathbf{A} \succ y:\mathbf{A} \quad x:\mathbf{A} \succ y:\mathbf{B}}{x \not\sim \bullet \quad \pi(\bullet)[y]} \quad \frac{x:\mathbf{A} \succ y:\mathbf{A} \quad \pi(x)[y]}{x:\mathbf{A} \succ y:\mathbf{B}}$$

Cut

$$\frac{id_A \quad \pi}{\pi \circ id_A = \pi} \quad \frac{A \rightarrow A \quad A \rightarrow B}{A \rightarrow B}$$

What is the proof term category like?

Cartesian Products

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

Cartesian Products

$$\begin{array}{ccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ & \nwarrow f & & \nearrow g & \\ & & C & & \end{array}$$

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This looks a *lot* like conjunction.

Many interesting categories have cartesian products

Final Objects — the empty product

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & \nearrow \langle f, g \rangle & \uparrow & \searrow g & \\ C & & & & \end{array}$$

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Coproducts and Terminal Objects

$$\begin{array}{ccccc} A & \xrightarrow{\iota_1} & A + B & \xleftarrow{\iota_2} & B \\ f \searrow & & \downarrow [f,g] & & \swarrow g \\ & & C & & \end{array} \quad \perp \quad \begin{array}{c} \downarrow \\ C \end{array}$$

Residuating Products — internalising arrows

$$f : A \times B \rightarrow C \quad \tilde{f} : A \rightarrow B \supset C \quad ev : (B \supset C) \times B \rightarrow C$$

$$\begin{array}{ccc} (B \supset C) \times B & \xrightarrow{ev} & C \\ \tilde{f} \times id \uparrow & & \nearrow f \\ A \times B & & \end{array}$$

Cartesian Closed Categories...

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...model intuitionistic logic.

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...model intuitionistic logic.

They *collapse* when made classical.

So what *is* the proof term category?

A dark, moody landscape featuring snow-covered mountain peaks in the background and a rocky, brownish-orange terrain with patches of snow in the foreground.

IT'S NOT CARTESIAN

\top is not Terminal, \perp is not Initial

\top is *terminal* iff for each C there's a *unique* arrow $C \rightarrow \top$.

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$$p \wedge \neg p \quad \downarrow \quad \top$$

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\top is *terminal* iff for each C there's a *unique* arrow $C \rightarrow \top$. \top is not terminal.

$$p \wedge \neg p \downarrow \top$$

$$p \wedge \neg p \xrightarrow{\text{red curved arrow}} \top$$

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Conjunction isn't Cartesian Product

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & \nearrow \langle f, g \rangle & \uparrow & & g \searrow \\ C & & & & \end{array}$$

Conjunction isn't Cartesian Product

We *do* have projection arrows.

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\ f \swarrow & \uparrow \langle f, g \rangle & \downarrow & \searrow g & \\ C & & & & \end{array}$$

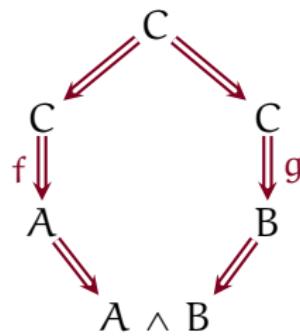
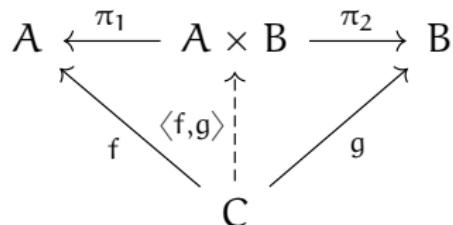
$$A \wedge B \quad \begin{matrix} \downarrow \\ A \end{matrix}$$

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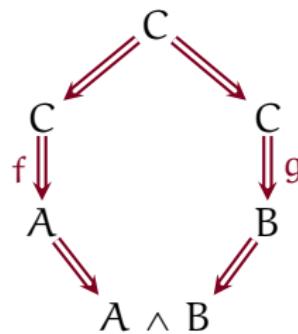
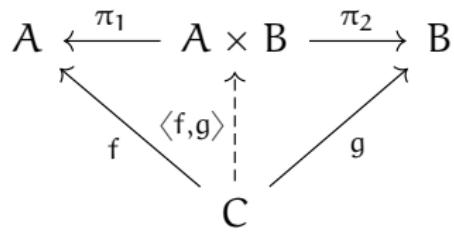
And we have a conjunctive arrow.



Conjunction isn't Cartesian Product

We *do* have projection arrows.

And we have a conjunctive arrow.



But their composition need not restore the original arrows f and g .

An Example

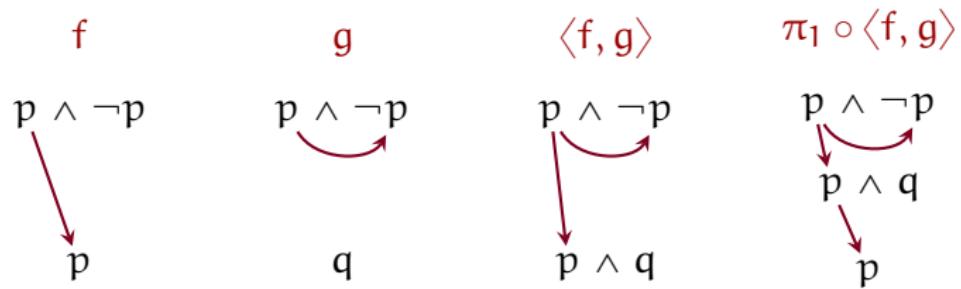
$$\begin{array}{ccc} f & & g \\ p \wedge \neg p & \swarrow & p \wedge \neg p \\ p & & q \end{array}$$

The diagram illustrates a proof in a sequent calculus system. It shows two formulas, f and g , at the top. Below f is the formula $p \wedge \neg p$. A red arrow points from this formula down to the variable p . Below g is also the formula $p \wedge \neg p$. A red curved arrow points from this formula up to the variable q .

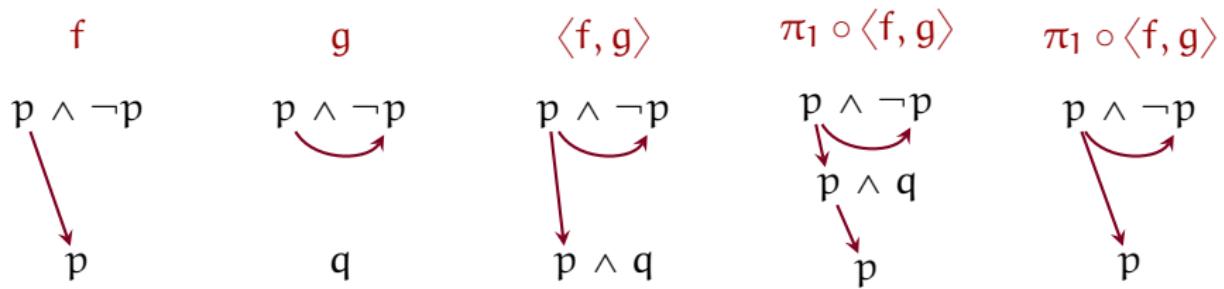
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$$\begin{array}{ccc} f & g & \langle f, g \rangle \\ p \wedge \neg p & p \wedge \neg p & p \wedge \neg p \\ \searrow & \curvearrowright & \downarrow \\ p & q & p \wedge q \end{array}$$

An Example



An Example



So, what is the category *like*?

A wide-angle landscape photograph of a vast, winding river valley in a rugged mountain range under a cloudy sky. The river flows from the background towards the foreground, its banks lined with green vegetation and rocky terrain. The mountains on either side are steep and brownish-grey, with patches of snow and ice clinging to their peaks. The sky is filled with heavy, grey clouds, creating a dramatic and moody atmosphere.

IT IS MONOIDAL,
& MORE...

Monoidal Categories

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Tensor product — \otimes — in vector spaces is an important example.

This motivates the definition of a *monoidal* category.

Symmetric Monoidal Categories

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \quad 1 \in Ob(\mathcal{C})$$

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$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \quad 1 \in Ob(\mathcal{C})$$

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$$

$$\sigma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A \quad \iota_A : 1 \otimes A \xrightarrow{\sim} A$$

where *associativity* (α), *symmetry* (σ) and *unit* (ι) behave sensibly.

Associativity

$$\begin{array}{ccccc} & & (A \otimes B) \otimes (C \otimes D) & & \\ & \nearrow \alpha_{A,B,C \otimes D} & & \searrow \alpha_{A \otimes B,C,D} & \\ A \otimes (B \otimes (C \otimes D)) & & & & ((A \otimes B) \otimes C) \otimes D \\ id_A \otimes \alpha_{B,C,D} \downarrow & & & & \uparrow \alpha_{A,B,C} \otimes id_D \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha_{A,B \otimes C,D}} & & & (A \otimes (B \otimes C)) \otimes D \end{array}$$

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(The 'Pentagon')

Symmetry

$$\begin{array}{ccc} & \sigma_{A,B} & \\ A \otimes B & \swarrow \quad \searrow & B \otimes A \\ & \sigma_{B,A} & \end{array}$$

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$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\sigma_{A,B \otimes C}} & (B \otimes C) \otimes A \\ \sigma_{A,B} \otimes id_C \downarrow & & & & \downarrow \alpha_{B,C,A} \\ (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{id_B \otimes \sigma_{A,C}} & B \otimes (C \otimes A) \end{array}$$

(The ‘Hexagon’)

Symmetry

$$\begin{array}{ccc} & \sigma & \\ A \otimes B & \swarrow & \searrow \\ & \sigma & \end{array}$$

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \xrightarrow{\alpha} & A \otimes (B \otimes C) & \xrightarrow{\sigma} & (B \otimes C) \otimes A \\ \downarrow \sigma \otimes id & & & & \downarrow \alpha \\ (B \otimes A) \otimes C & \xrightarrow{\alpha} & B \otimes (A \otimes C) & \xrightarrow{id \otimes \sigma} & B \otimes (C \otimes A) \end{array}$$

(The ‘Hexagon’)

Let's drop subscripts when there is no ambiguity.

Unit

$$\begin{array}{ccc} (A \otimes 1) \otimes B & \xrightarrow{\alpha} & A \otimes (1 \otimes B) \\ \sigma \otimes id \downarrow & & \downarrow id \otimes \iota \\ (1 \otimes A) \otimes B & \xrightarrow{\iota \otimes id} & A \otimes B \end{array}$$

(The ‘Square’)

Proof Terms are a Symmetric Monoidal Category under \wedge/\top

$$\wedge : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T} \quad \top \in Ob(\mathcal{T})$$

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$$\wedge : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T} \quad \top \in Ob(\mathcal{T})$$

$$\hat{\alpha} : A \wedge (B \wedge C) \xrightarrow{\sim} (A \wedge B) \wedge C$$

$$\hat{\sigma} : A \wedge B \xrightarrow{\sim} B \wedge A \quad \hat{\iota} : \top \wedge A \xrightarrow{\sim} A$$

and indeed, *associativity* ($\hat{\alpha}$), *symmetry* ($\hat{\sigma}$) and *unit* ($\hat{\iota}$) behave sensibly.

$\hat{\alpha}$, $\hat{\sigma}$ and $\hat{\iota}$

$$\hat{\alpha}_{A,B,C} : A \wedge (B \wedge C) \rightarrow ((A \wedge B) \wedge C)$$


$$\hat{\sigma}_{A,B} : A \wedge B \rightarrow B \wedge A$$


$$\hat{\iota} : \top \wedge A \rightarrow A$$


$\hat{\alpha}$, $\hat{\sigma}$ and $\hat{\iota}$ are isomorphisms

$$\begin{array}{c} \hat{\alpha}^{-1} \circ \hat{\alpha} \\ A \wedge (B \wedge C) \\ \Downarrow \\ (A \wedge B) \wedge C \\ \Downarrow \\ A \wedge (B \wedge C) \end{array}$$

$$\begin{array}{c} id \\ A \wedge (B \wedge C) \\ \Downarrow \\ A \wedge (B \wedge C) \end{array}$$

$$\begin{array}{c} \hat{\sigma}^{-1} \circ \hat{\sigma} \\ A \wedge B \\ \times \\ B \wedge A \\ \times \\ A \wedge B \\ \Downarrow \\ A \wedge B \end{array}$$

$$\begin{array}{c} id \\ A \wedge B \\ \Downarrow \\ A \wedge B \end{array}$$

$\hat{\alpha}$, $\hat{\sigma}$ and $\hat{\iota}$ are isomorphisms

$$\begin{array}{c} \hat{\iota}^{-1} \circ \hat{\iota} \\ \text{---} \\ T \wedge A \\ \swarrow \quad \searrow \\ A \\ \downarrow \quad \downarrow \\ T \wedge A \end{array}$$

$$\begin{array}{c} id \\ \text{---} \\ T \wedge A \\ \downarrow \quad \downarrow \\ T \wedge A \end{array}$$

$$\begin{array}{c} \hat{\iota} \circ \hat{\iota}^{-1} \\ \text{---} \\ A \\ \downarrow \quad \searrow \\ T \wedge A \\ \swarrow \\ A \end{array}$$

$$\begin{array}{c} id \\ \text{---} \\ A \\ \downarrow \quad \downarrow \\ A \end{array}$$

The Pentagon, Hexagon, Square, etc., commute

$$\begin{array}{ccc}
 (A \wedge \top) \wedge B & \xrightarrow{\hat{\alpha}} & A \wedge (\top \wedge B) \\
 \hat{\sigma} \wedge id \downarrow & & \downarrow id \wedge \hat{\iota} \\
 (\top \wedge A) \wedge B & \xrightarrow{\hat{\iota} \wedge id} & A \wedge B
 \end{array}$$

$$(\hat{\iota} \wedge id) \circ (\hat{\sigma} \wedge id)$$

$$\begin{array}{ccc}
 (A \wedge \top) \wedge B & & \\
 \searrow \downarrow \quad \swarrow \downarrow & & \\
 (\top \wedge A) \wedge B & & \\
 \searrow \downarrow \quad \swarrow \downarrow & & \\
 A \wedge B & &
 \end{array}$$

=

$$(id \wedge \hat{\iota}) \circ \hat{\alpha}$$

$$\begin{array}{ccc}
 (A \wedge \top) \wedge B & & \\
 \searrow \downarrow \quad \swarrow \downarrow & & \\
 A \wedge B & &
 \end{array}$$

$$\begin{array}{ccc}
 (A \wedge \top) \wedge B & & \\
 \searrow \downarrow \quad \swarrow \downarrow & & \\
 A \wedge (\top \wedge B) & & \\
 \searrow \downarrow \quad \swarrow \downarrow & & \\
 A \wedge B & &
 \end{array}$$

The Pentagon, Hexagon, Square, etc., commute

$$\begin{array}{ccccc} & & (A \wedge B) \wedge (C \wedge D) & & \\ & \nearrow \hat{\alpha} & & \searrow \hat{\alpha} & \\ A \wedge (B \wedge (C \wedge D)) & & & & ((A \wedge B) \wedge C) \wedge D \\ id \wedge \hat{\alpha} \downarrow & & & & \uparrow \hat{\alpha} \wedge id \\ A \wedge ((B \wedge C) \wedge D) & \xrightarrow{\hat{\alpha}} & & & (A \wedge (B \wedge C)) \wedge D \end{array}$$

The Pentagon, Hexagon, Square, etc., commute

$$\begin{array}{ccccc} (A \wedge B) \wedge C & \xrightarrow{\hat{\alpha}} & A \wedge (B \wedge C) & \xrightarrow{\hat{\sigma}} & (B \wedge C) \wedge A \\ \hat{\sigma} \wedge id \downarrow & & & & \downarrow \hat{\alpha} \\ (B \wedge A) \wedge C & \xrightarrow{\hat{\alpha}} & B \wedge (A \wedge C) & \xrightarrow{id \wedge \hat{\sigma}} & B \wedge (C \wedge A) \end{array}$$

Proof Terms are a Symmetric Monoidal Category under \vee/\perp

$$\vee : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T} \quad \perp \in Ob(\mathcal{T})$$

$$\stackrel{\vee}{\alpha} : A \vee (B \vee C) \xrightarrow{\sim} (A \vee B) \vee C$$

$$\stackrel{\vee}{\sigma} : A \vee B \xrightarrow{\sim} B \vee A \quad \stackrel{\vee}{\iota} : \perp \vee A \xrightarrow{\sim} A$$

and *associativity* ($\stackrel{\vee}{\alpha}$), *symmetry* ($\stackrel{\vee}{\sigma}$) and *unit* ($\stackrel{\vee}{\iota}$) behave just as sensibly.

Linear Distributive Categories

The operators \wedge and \vee are connected by δ and δ'

$$\delta : A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C \quad \delta' : (A \vee B) \wedge C \rightarrow A \vee (B \wedge C)$$

Linear Distributive Categories

The operators \wedge and \vee are connected by δ and δ'

$$\delta : A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C \quad \delta' : (A \vee B) \wedge C \rightarrow A \vee (B \wedge C)$$

If the operators are *symmetric*, then we need only one.

$$\begin{array}{ccc} (A \vee B) \wedge C & \xrightarrow{\delta'} & A \vee (B \wedge C) \\ \hat{\sigma} \downarrow & & \uparrow \check{\sigma} \\ C \wedge (A \vee B) & & (B \wedge C) \vee A \\ id \wedge \check{\sigma} \downarrow & & \uparrow \hat{\sigma} \vee id \\ C \wedge (B \vee A) & \xrightarrow{\delta} & (C \wedge B) \vee A \end{array}$$

δ and δ' are *obvious* proof terms

$$\begin{array}{c} \delta \\ A \wedge (B \vee C) \\ \Downarrow \\ (A \wedge B) \vee C \end{array}$$

$$\begin{array}{c} \delta' \\ (A \vee B) \wedge C \\ \Downarrow \\ A \vee (B \wedge C) \end{array}$$

Linear Distributivity Conditions

$$\begin{array}{ccc} & (A \wedge B) \wedge (C \vee D) & \xrightarrow{\hat{\alpha}} A \wedge (B \wedge (C \vee D)) \\ \begin{array}{c} T \wedge (A \vee B) \\ \downarrow \delta \\ ((T \wedge A) \vee B) \xrightarrow{\hat{\iota} \vee id} A \vee B \end{array} & \downarrow \delta & \begin{array}{c} \downarrow id \wedge \delta \\ A \wedge ((B \wedge C) \vee D) \\ \downarrow \delta \\ ((A \wedge B) \wedge C) \vee D \xrightarrow{\hat{\alpha} \vee id} (A \wedge (B \wedge C)) \vee D \end{array} \\ & & \end{array}$$
$$\begin{array}{ccc} ((A \vee B) \wedge C) \vee D & \xleftarrow{\delta} & (A \vee B) \wedge (C \vee D) & \xrightarrow{\delta'} & A \vee (B \wedge (C \vee D)) \\ & \downarrow \delta' \vee id & & & \downarrow id \vee \delta \\ (A \vee (B \wedge C)) \vee D & \xrightarrow{\alpha} & & & A \vee ((B \wedge C) \vee D) \end{array}$$

Linear Distributivity Conditions

$$\begin{array}{ccc}
 & (A \wedge B) \wedge (C \vee D) & \xrightarrow{\hat{\alpha}} A \wedge (B \wedge (C \vee D)) \\
 \begin{array}{c} T \wedge (A \vee B) \\ \downarrow \delta \\ ((T \wedge A) \vee B) \xrightarrow{\hat{\iota} \vee id} A \vee B \end{array} & \downarrow \delta & \begin{array}{c} A \wedge ((B \wedge C) \vee D) \\ \downarrow id \wedge \delta \\ ((A \wedge B) \wedge C) \vee D \xrightarrow{\hat{\alpha} \vee id} (A \wedge (B \wedge C)) \vee D \end{array} \\
 & & \downarrow \delta
 \end{array}$$

$$\begin{array}{ccc}
 ((A \vee B) \wedge C) \vee D & \xleftarrow{\delta} & (A \vee B) \wedge (C \vee D) & \xrightarrow{\delta'} A \vee (B \wedge (C \vee D)) \\
 & \downarrow \delta' \vee id & & \downarrow id \vee \delta \\
 (A \vee (B \wedge C)) \vee D & \xrightarrow{\alpha} & & A \vee ((B \wedge C) \vee D)
 \end{array}$$

(These diagrams *clearly* commute in the proof term category.)

Star-Autonomous Categories

There are a number of ways to define *Star-Autonomous Categories*.

We have a $\neg A$ for each object A , and two sets of arrows.

$$\gamma_A : A \wedge \neg A \rightarrow \perp \quad \tau_A : \top \rightarrow \neg A \vee A$$

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We have a $\neg A$ for each object A , and two sets of arrows.

$$\gamma_A : A \wedge \neg A \rightarrow \perp \quad \tau_A : \top \rightarrow \neg A \vee A$$

These arrows have natural proof terms.



These Diagrams Must Commute

$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \vee \\ A \wedge \top & \xrightarrow[\wedge]{\iota} & & & \end{array}$$

$$\begin{array}{ccccc} (\neg A \vee A) \wedge \neg A & \xrightarrow{\delta'} & \neg A \vee (A \wedge \neg A) & \xrightarrow{id \vee \gamma} & \neg A \vee \perp \\ \tau \wedge id \uparrow & & & & \downarrow \vee \\ \top \wedge \neg A & \xrightarrow[\wedge]{\iota} & & & \end{array}$$

These Diagrams Must Commute

$$\begin{array}{ccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A \xrightarrow{\gamma \vee id} \perp \vee A \\ id \wedge \tau \uparrow & & \downarrow \vee \\ A \wedge \top & \xrightarrow[\wedge]{\iota} & A \end{array}$$

$$\begin{array}{ccc} (\neg A \vee A) \wedge \neg A & \xrightarrow{\delta'} & \neg A \vee (A \wedge \neg A) \xrightarrow{id \vee \gamma} \neg A \vee \perp \\ \tau \wedge id \uparrow & & \downarrow \vee \\ \top \wedge \neg A & \xrightarrow[\wedge]{\iota} & \neg A \end{array}$$

These aren't so obviously commutative as proof terms.

The negation diagrams commute in the proof term category

$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \vdash \\ A \wedge \top & \xrightarrow[\tau]{} & & & A \end{array}$$

The negation diagrams commute in the proof term category

$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \vdash \\ A \wedge \top & \xrightarrow[\wedge]{} & & & A \end{array}$$

$$\begin{array}{c} (id \wedge \tau) \\ \swarrow \quad \searrow \\ A \wedge \top \\ \swarrow \quad \searrow \\ A \wedge (\neg A \vee A) \end{array}$$

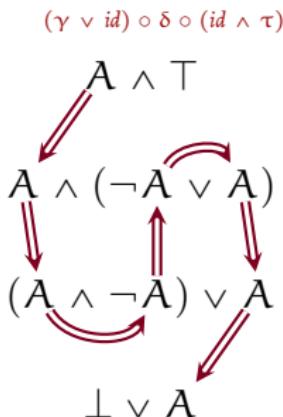
The negation diagrams commute in the proof term category

$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \vee \\ A \wedge \top & \xrightarrow[\wedge]{} & & & A \end{array}$$

$$\begin{array}{c} \delta \circ (id \wedge \tau) \\ \swarrow \quad \searrow \\ A \wedge \top \\ \downarrow \quad \downarrow \\ A \wedge (\neg A \vee A) \\ \uparrow \quad \downarrow \\ (A \wedge \neg A) \vee A \end{array}$$

The negation diagrams commute in the proof term category

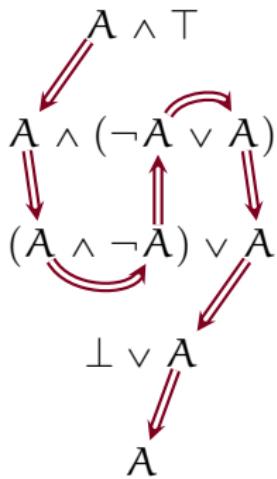
$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \vdash \\ A \wedge \top & \xrightarrow[\wedge]{} & & & A \end{array}$$



The negation diagrams commute in the proof term category

$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\ id \wedge \tau \uparrow & & & & \downarrow \check{\iota} \\ A \wedge \top & \xrightarrow[\wedge]{} & & & A \end{array}$$

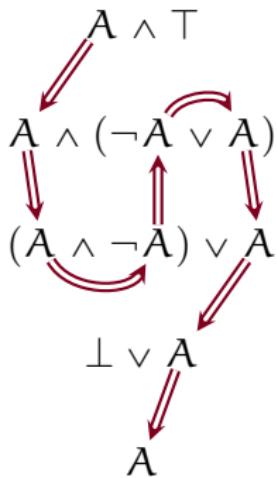
$$\check{\iota} \circ (\gamma \vee id) \circ \delta \circ (id \wedge \tau)$$



The negation diagrams commute in the proof term category

$$\begin{array}{ccccc}
 A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \perp \vee A \\
 \uparrow id \wedge \tau & & & & \downarrow \check{\iota} \\
 A \wedge \top & \xrightarrow[\textcolor{red}{\wedge}]{} & & & A
 \end{array}$$

$$\check{\iota} \circ (\gamma \vee id) \circ \delta \circ (id \wedge \tau)$$



Star-Autonomous Categories and Linear Logic

These categories model the multiplicative fragment of linear logic.

Linear Implication

I won't pause now to explain how $A \supset B$, definable as $\neg A \vee B$ (or as $\neg(A \wedge \neg B)$, to which it's isomorphic) is a right adjoint to \wedge .

We can do more

Our proof terms allow *contraction* and *weakening*.

Weakening and Contraction Monoids and Comonoids

$$\nabla_A : A \vee A \rightarrow A$$

$$\stackrel{\perp}{\beta}_A : \perp \rightarrow A$$

$$\begin{array}{ccc} \nabla_A & & \stackrel{\perp}{\beta}_A \\ A \vee A & \Downarrow & \perp \\ A & \Downarrow & A \end{array}$$

Weakening and Contraction Monoids and Comonoids

$$\nabla_A : A \vee A \rightarrow A$$

$$\beta_A^\perp : \perp \rightarrow A$$

$$\Delta_A : A \rightarrow A \wedge A$$

$$\beta_A^T : A \rightarrow T$$



$$\beta_A^\perp : \perp \rightarrow A$$

$$\begin{array}{c} \delta_A \\ A \\ \swarrow \quad \searrow \\ A \wedge A \\ \downarrow \\ T \end{array}$$

Contraction and Weakening Conditions

$$\begin{array}{ccc} (A \vee A) \vee A & \xrightarrow{\check{\alpha}} & A \vee (A \vee A) \\ \nabla \vee id \downarrow & & \downarrow id \vee \nabla \\ A \vee A & \xrightarrow{\nabla} & A \xleftarrow{\nabla} A \vee A \end{array}$$

$$\begin{array}{ccc} A \vee \perp & \xrightarrow{id \vee \frac{1}{\beta}} & A \vee A \xleftarrow{\frac{1}{\beta} \vee id} \perp \vee A \\ \check{\sigma} \downarrow & & \nabla \downarrow \\ \perp \vee A & \xrightarrow{\check{\iota}} & A \xleftarrow{\check{\iota}} \perp \vee A \end{array} \qquad \begin{array}{ccc} A \vee A & \xrightarrow{\check{\sigma}} & A \vee A \\ \nabla \searrow & & \swarrow \nabla \\ & A & \end{array}$$

Structurality for ∇ and β^\perp

$$\begin{array}{ccccc}
 (A \vee B) \vee (A \vee B) & \xrightarrow{\alpha} & A \vee (B \vee (A \vee B)) & \xrightarrow{id \vee \alpha} & A \vee ((B \vee A) \vee B) \\
 \downarrow \nabla & & & & \downarrow id \vee (\sigma \vee id) \\
 & & & & A \vee ((A \vee B) \vee B) \\
 & & & & \downarrow id \vee \alpha \\
 & & & & A \vee (A \vee (B \vee B)) \\
 & & & & \downarrow \alpha \\
 A \vee B & \xleftarrow{\nabla \vee \nabla} & & & (A \vee A) \vee (B \vee B)
 \end{array}$$

$$\begin{array}{ccc}
 \perp & \xrightarrow{\nabla \iota} & \perp \vee \perp \\
 & \searrow \perp \beta & \swarrow \perp \beta \vee \perp \\
 & A \vee B &
 \end{array}$$

Structurality for ∇ and β^\perp

$$\begin{array}{ccc} \perp \vee \perp & \begin{array}{c} \xrightarrow{\nabla_\perp} \\ \xleftarrow{\vee_\perp} \end{array} & \perp \\ & \text{and} & \\ \perp & \begin{array}{c} \xrightarrow{\beta_\perp^\perp} \\ \xleftarrow{id_\perp} \end{array} & \perp \end{array}$$

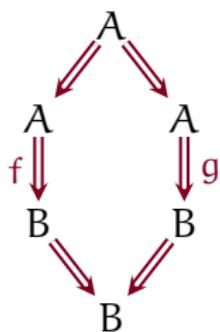
Structurality for ∇ and β^\perp

$$\begin{array}{ccc} \perp \vee \perp & \xrightarrow{\nabla_\perp} & \perp \\ & \xrightarrow{\vee_\perp} & \end{array} \quad \begin{array}{ccc} \perp & \xrightarrow{\beta_\perp^\perp} & \perp \\ & \xrightarrow{id_\perp} & \end{array}$$

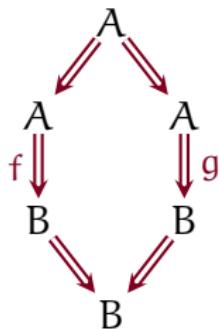
All these conditions are straightforward to verify for proof terms.

And dually for Δ and β .^T

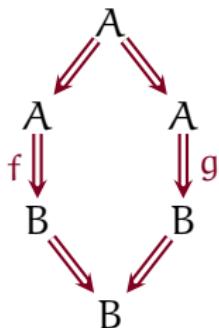
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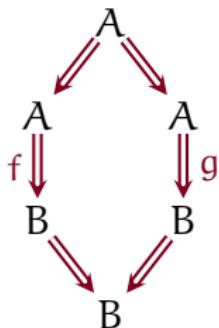
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\cup is a semilattice join on $\text{Hom}(A, B)$.

$$(f \cup f') \circ g = (f \circ g) \cup (f' \circ g) \quad f \circ (g \cup g') = (f \circ g) \cup (f \circ g')$$

Blend



\cup is a semilattice join on $\text{Hom}(A, B)$.

$$(f \cup f') \circ g = (f \circ g) \cup (f' \circ g) \quad f \circ (g \cup g') = (f \circ g) \cup (f \circ g')$$

The term category \mathcal{T} is *enriched in* SLat.

Classical Categories

Classical categories are
star autonomous categories
with *structural monoids and comonoids*,
enriched in SLat.

Isomorphisms

$$A \wedge T \cong A \quad A \wedge B \cong B \wedge A \quad A \wedge (B \wedge C) \cong (A \wedge B) \wedge C$$

$$A \vee \perp \cong A \quad A \vee B \cong B \vee A \quad A \vee (B \vee C) \cong (A \vee B) \vee C$$

$$\neg(A \wedge B) \cong (\neg A \vee \neg B) \quad \neg(A \vee B) \cong (\neg A \wedge \neg B) \quad \neg\neg A \cong A$$

$$\neg T \cong \perp \quad \neg \perp \cong T \quad T \vee T \cong T \quad \perp \wedge \perp \cong \perp$$

Isomorphisms

$$A \wedge \top \cong A \quad A \wedge B \cong B \wedge A \quad A \wedge (B \wedge C) \cong (A \wedge B) \wedge C$$

$$A \vee \perp \cong A \quad A \vee B \cong B \vee A \quad A \vee (B \vee C) \cong (A \vee B) \vee C$$

$$\neg(A \wedge B) \cong (\neg A \vee \neg B) \quad \neg(A \vee B) \cong (\neg A \wedge \neg B) \quad \neg\neg A \cong A$$

$$\neg\top \cong \perp \quad \neg\perp \cong \top \quad \top \vee \top \cong \top \quad \perp \wedge \perp \cong \perp$$

$$p \not\cong p \wedge p \quad p \not\cong p \vee p \quad p \wedge (q \vee r) \not\cong (p \wedge q) \vee (p \wedge r)$$

$$p \wedge (p \vee q) \not\cong p \vee (p \wedge q) \quad p \vee \neg p \not\cong \top \quad p \wedge \neg p \not\cong \perp$$

A scenic view of Bryce Canyon National Park, featuring a vast landscape of red rock hoodoos and green pine trees. A dirt trail winds its way through the canyon, with several people walking along it. The sky is clear and blue.

FURTHER WORK

To Do List

- ▶ Finish the completeness proof, to the effect that $\mathcal{T}_{\mathcal{L}}$ is the free classical category on \mathcal{L} .
- ▶ Explore other examples of classical categories.
- ▶ Extend to first order predicate logic.

THANK YOU!

[http://consequently.org/presentation/2017/
a-category-of-classical-proofs-logicmelm](http://consequently.org/presentation/2017/a-category-of-classical-proofs-logicmelm)

@consequently