Logical Pluralism

Meaning, Rules and Counterexamples

Greg Restall

POSTERA CRESCAM LAUDE

THE UNIVERSITY OF
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Ingredients

Proofs and Syntax

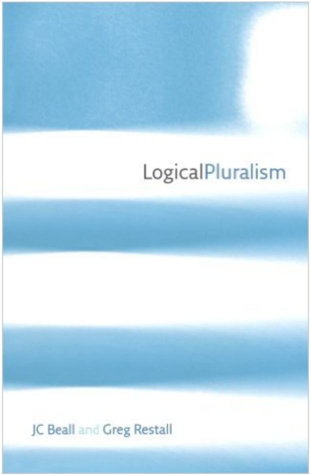
Abstract Proofs

Warrant and Choice

Where we've got, and where to from here

INGREDIENTS

Beall and Restall Pluralism



Beall and Restall Pluralism

- Pluralism about deductive logical consequence relations.
- Not pluralism about *truth*, or *languages*.
- Defined *representationally* or *model theoretically*.
- Different consequence relations correspond to different classes of *cases*.
- Including, at least, classical, constructive and paraconsistent logics.

Multiple Conclusions

Greg Restall*

Philosophy Department
The University of Melbourne
restall@unimelb.edu.au

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Abstract: I argue for the following four theses. (1) Denial is not to be analysed as the assertion of a negation. (2) Given the concepts of assertion and denial, we have the resources to analyse logical consequence as relating arguments with *multiple* premises and *multiple* conclusions. Gentzen's multiple conclusion calculus can be understood in a straightforward, motivated, non-question-begging way. (3) If a broadly anti-realist or inferentialist justification of a logical system works, it works just as well for *classical* logic as it does for *intuitionistic* logic. The special case for an anti-realist justification of intuitionistic logic over and above a justification of classical logic relies on an unjustified assumption about the shape of proofs. Finally, (4) this picture of logical consequence provides a relatively neutral shared vocabulary which can help us understand and adjudicate debates between proponents of classical and non-classical logics.

* * *

Multiple Conclusions

- A semantically antirealist defence of the classical sequent calculus.
- Bilateralism with respect to *assertion* and *denial*.
- Derivations of $X \succ Y$ related to the coherence of *positions* $[X : Y]$ consisting of assertions and denials.
- A defence of the *structural rules* of the classical sequent calculus in terms of norms governing assertion and denial.

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ORIGINAL ARTICLE

Pluralism and Proofs

Greg Restall

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Abstract Beall and Restall's *Logical Pluralism* (2006) characterises pluralism about logical consequence in terms of the different ways *cases* can be selected in the analysis of logical consequence as preservation of truth over a class of cases. This is not the only way to understand or to motivate pluralism about logical consequence. Here, I will examine pluralism about logical consequence in terms of different standards of

Pluralism and Proofs

- An attempt at a proof-first account of logical pluralism.
- Based on the sequent calculus: classical derivations for $X \succ Y$.
- Derivations with sequents where $|Y| \leqslant 1$ are *constructive*. Those with sequents where $|X| \leqslant 1$ are *paraconsistent*.
- Underivable positions are *counterexamples* (of each kind).

PROOF TERMS FOR CLASSICAL DERIVATIONS

Greg Restall*

Philosophy Department

The University of Melbourne

restall@unimelb.edu.au

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Abstract: I give an account of *proof terms* for derivations in a sequent calculus for classical propositional logic. The term for a derivation δ of a sequent $\Sigma \succ \Delta$ encodes *how* the premises Σ and conclusions Δ are related in δ. This encoding is many–to–one in the sense that different derivations can have the same proof term, since different derivations may be different ways of representing the same underlying connection between premises and conclusions. However, not all proof terms for a sequent $\Sigma \succ \Delta$ are the same. There may be *different* ways to connect those premises and conclusions.

Proof terms can be simplified in a process corresponding to the elimination of cut inferences in sequent derivations. However, unlike cut elimination in the sequent calculus, each proof term has a *unique normal form* (from which all cuts have been eliminated) and it is straightforward to show that term reduction is strongly normalising—*every* reduction process terminates in that unique normal form. Furthermore, proof terms are *invariants* for sequent derivations in a strong sense—two derivations δ_1 and δ_2 have the same proof term *if and only if* some permutation of derivation steps sends δ_1 to δ_2 (given a relatively natural class of permutations of derivations in the sequent calculus). Since not every derivation of a sequent can be permuted into every other derivation of that sequent, proof terms provide a non-trivial account of the identity of proofs, independent of the syntactic representation of those proofs.

Abstract Proofs

- A *syntax-independent* account of classical (propositional) proof.
- It abstracts away from irrelevant features of a derivations.
- An abstract proof for $X \succ Y$ shows how X and Y are connected in a derivation.

I aim to synthesise these ingredients in order to

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1. give a *pluralist* and *syntax-independent* account of classical and constructive *proof*,

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My Aim

I aim to synthesise these ingredients in order to

1. give a *pluralist* and *syntax-independent* account of classical and constructive *proof*, which is
2. grounded in a single underlying univocal theory of *meaning*, and which
3. explains the significance of classical and constructive counterexamples.


PROOFS & SYNTAX

Positions

$$[X : Y]$$

A position in which
each member of X has been asserted
and each member of Y has been denied.

$$X \succ Y$$

The position $[X : Y]$ is *out of bounds*.

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \wedge \neg p \succ}\ \wedge L$$

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\neg L}{p \land \neg p \succ}\land L \qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\neg R}{\succ p \lor \neg p}\lor R$$

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \wedge \neg p \succ}\,\wedge L \qquad\qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \vee \neg p}\,\vee R$$

$$X, A \succ A, Y \quad Id$$

Classical Sequent Derivations

$$\cfrac{\cfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \land \neg p \succ}\ \land L \qquad \cfrac{\cfrac{p \succ p}{\succ p, \neg p}\ \neg R}{\succ p \lor \neg p}\ \lor R$$

$$X, A \succ A, Y \ \ \mathit{Id} \qquad \cfrac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}\ \mathit{Cut}$$

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \wedge \neg p \succ}\,\wedge L \qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \vee \neg p}\,\vee R$$

$$X, A \succ A, Y \;\; Id \qquad \dfrac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}\,Cut$$

$$\dfrac{X, A, B \succ Y}{X, A, B \succ Y}\,\wedge L \qquad \dfrac{X \succ A, Y \quad X \succ B, Y}{X \succ A \wedge B, Y}\,\wedge R$$

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \wedge \neg p \succ}\ \wedge L \qquad\qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\ \neg R}{\succ p \vee \neg p}\ \vee R$$

$$X, A \succ A, Y \ \ \mathit{Id} \qquad\qquad \dfrac{X \succ A, Y \qquad X, A \succ Y}{X \succ Y}\ \mathit{Cut}$$

$$\dfrac{X, A, B \succ Y}{X, A, B \succ Y}\ \wedge L \qquad\qquad \dfrac{X \succ A, Y \qquad X \succ B, Y}{X \succ A \wedge B, Y}\ \wedge R$$

$$\dfrac{X \succ A, Y}{X, \neg A \succ Y}\ \neg L \qquad\qquad \dfrac{X, A \succ Y}{X \succ \neg A, Y}\ \neg R$$

Constructive Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \wedge \neg p \succ}\ \wedge L$$

Constructive Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \wedge \neg p \succ}\ \wedge L \qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\ \neg R}{\succ p \vee \neg p}\ \vee R$$

Constructive Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \wedge \neg p \succ}\,\wedge L \qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \vee \neg p}\,\vee R$$

Constructive Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \wedge \neg p \succ}\ \wedge L \qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\ \neg R}{\succ p \vee \neg p}\ \vee R$$

$$X, A \succ A \quad Id$$

Constructive Sequent Derivations

$$\cfrac{\cfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \land \neg p \succ}\ \land L \qquad \cfrac{\cfrac{p \succ p}{\succ p, \neg p}\ \neg R}{\succ p \lor \neg p}\ \lor R$$

$$X, A \succ A \ \ \mathit{Id} \qquad \cfrac{X \succ A \qquad X, A \succ C}{X \succ C}\ \mathit{Cut}$$

Constructive Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \land \neg p \succ}\ \land L \qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\ \neg R}{\succ p \lor \neg p}\ \lor R$$

$$X, A \succ A \quad \mathit{Id} \qquad \dfrac{X \succ A \qquad X, A \succ C}{X \succ C}\ \mathit{Cut}$$

$$\dfrac{X, A, B \succ C}{X, A, B \succ C}\ \land L \qquad \dfrac{X \succ A \qquad X \succ B}{X \succ A \land B}\ \land R$$

Constructive Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \wedge \neg p \succ}\ \wedge L \qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\ \neg R}{\succ p \vee \neg p}\ \vee R$$

$$X, A \succ A \ \ Id \qquad \dfrac{X \succ A \qquad X, A \succ C}{X \succ C}\ Cut$$

$$\dfrac{X, A, B \succ C}{X, A, B \succ C}\ \wedge L \qquad \dfrac{X \succ A \qquad X \succ B}{X \succ A \wedge B}\ \wedge R$$

$$\dfrac{X \succ A}{X, \neg A \succ}\ \neg L \qquad \dfrac{X, A \succ}{X \succ \neg A}\ \neg R$$

But what *is* a proof?

A proof from A to B *shows how* B follows from A.

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When I possess a proof from A to B,
I can transform a warrant for A
into a warrant for B
(by following the proof) . . .

But what *is* a proof?

A proof from A to B *shows how* B follows from A.

When I possess a proof from A to B,
I can transform a warrant for A
into a warrant for B
(by following the proof) . . .

. . . and equally, I can transform
a warrant *against* B
into a warrant *against* A.

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \wedge q}{p}\ {\scriptstyle \wedge E}}{p \vee q}\ {\scriptstyle \vee I}$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \vee q}\ {\scriptstyle \wedge L}}{p \wedge q \succ p \vee q}\ {\scriptstyle \wedge L}$$

$$\dfrac{\dfrac{p \wedge q}{p}\ {\scriptstyle \wedge E}}{p \vee q}\ {\scriptstyle \vee I}$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \vee q}\ {\scriptstyle \wedge L}}{p \wedge q \succ p \vee q}\ {\scriptstyle \wedge L} \qquad \dfrac{\dfrac{p \wedge q}{p}\ {\scriptstyle \wedge E}}{p \vee q}\ {\scriptstyle \vee I} \qquad \dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\ {\scriptstyle \wedge L}}{p \wedge q \succ p \vee q}\ {\scriptstyle \vee R}$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,{\wedge L}}{p \wedge q \succ p \vee q}\,{\wedge L}$$

$$\dfrac{\dfrac{p \wedge q}{p}\,{\wedge E}}{p \vee q}\,{\vee I}$$

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,{\wedge L}}{p \wedge q \succ p \vee q}\,{\vee R}$$

$$\dfrac{\dfrac{p \wedge q}{q}\,{\wedge E}}{p \vee q}\,{\vee I}$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \lor q}\;\land L}{p \land q \succ p \lor q}\;\land L \qquad \dfrac{\dfrac{p \land q}{p}\;\land E}{p \lor q}\;\lor I \qquad \dfrac{\dfrac{p \succ p}{p \land q \succ p}\;\land L}{p \land q \succ p \lor q}\;\lor R$$

$$\dfrac{\dfrac{q \succ q}{q \succ p \lor q}\;\lor R}{p \land q \succ p \lor q}\;\land L \qquad \dfrac{\dfrac{p \land q}{q}\;\land E}{p \lor q}\;\lor I$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \lor q}\;{\land L}}{p \land q \succ p \lor q}\;{\land L}
\qquad
\dfrac{\dfrac{p \land q}{p}\;{\land E}}{p \lor q}\;{\lor I}
\qquad
\dfrac{\dfrac{p \succ p}{p \land q \succ p}\;{\land L}}{p \land q \succ p \lor q}\;{\lor R}$$

$$\dfrac{\dfrac{q \succ q}{q \succ p \lor q}\;{\lor R}}{p \land q \succ p \lor q}\;{\land L}
\qquad
\dfrac{\dfrac{p \land q}{q}\;{\land E}}{p \lor q}\;{\lor I}
\qquad
\dfrac{\dfrac{q \succ q}{p \land q \succ q}\;{\land L}}{p \land q \succ p \lor q}\;{\lor R}$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \lor q \qquad \dfrac{[p]^1}{q \lor p}\,{\lor I} \qquad \dfrac{[q]^1}{q \lor p}\,{\lor I}}{q \lor p}\,{\lor E^1}}{(q \lor p) \lor r}\,{\lor I}
\qquad
\dfrac{p \lor q \qquad \dfrac{\dfrac{[p]^1}{q \lor p}\,{\lor I}}{(q \lor p) \lor r}\,{\lor I} \qquad \dfrac{\dfrac{[q]^1}{q \lor p}\,{\lor I}}{(q \lor p) \lor r}\,{\lor I}}{(q \lor p) \lor r}\,{\lor E^1}$$

Are these *different proofs*, or *different ways of presenting the same proof*?


ABSTRACT PROOFS

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \land q}{p}\ \land E}{p \lor q}\ \lor I \qquad\qquad \dfrac{\dfrac{p \succ p}{p \succ p \lor q}\ \land L}{p \land q \succ p \lor q}\ \land L$$

When is π_1 the same proof as π_2?

$$\cfrac{\cfrac{p \land q}{p}\ {\land}E}{p \lor q}\ {\lor}I \qquad\qquad \cfrac{\cfrac{p \succ p}{p \succ p \lor q}\ {\land}L}{p \land q \succ p \lor q}\ {\land}L$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \wedge q}{p}\,\wedge E}{p \vee q}\,\vee I \qquad\qquad \dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,\wedge L}{p \wedge q \succ p \vee q}\,\wedge L$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \land q}{p}\,\land E}{p \lor q}\,\lor I \qquad\qquad \dfrac{\dfrac{p \succ p}{p \succ p \lor q}\,\land L}{p \land q \succ p \lor q}\,\land L$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \wedge q}{p}\ {\scriptstyle \wedge E}}{p \vee q}\ {\scriptstyle \vee I} \qquad \dfrac{\dfrac{p \succ p}{p \succ p \vee q}\ {\scriptstyle \wedge L}}{p \wedge q \succ p \vee q}\ {\scriptstyle \wedge L}$$

$$\dfrac{\dfrac{p \wedge q}{q}\ {\scriptstyle \wedge E}}{p \vee q}\ {\scriptstyle \vee I} \qquad \dfrac{\dfrac{q \succ q}{q \succ p \vee q}\ {\scriptstyle \vee R}}{p \wedge q \succ p \vee q}\ {\scriptstyle \wedge L}$$

When is π_1 the same proof as π_2?

$$\cfrac{\cfrac{p \wedge q}{p}\ {\wedge E}}{p \vee q}\ {\vee I}
\qquad\qquad
\cfrac{\cfrac{p \succ p}{p \succ p \vee q}\ {\wedge L}}{p \wedge q \succ p \vee q}\ {\wedge L}$$

$$\cfrac{\cfrac{p \wedge q}{q}\ {\wedge E}}{p \vee q}\ {\vee I}
\qquad\qquad
\cfrac{\cfrac{q \succ q}{q \succ p \vee q}\ {\vee R}}{p \wedge q \succ p \vee q}\ {\wedge L}$$

When is π_1 the same proof as π_2?

$$\frac{\dfrac{p \land q}{p}\ \land E}{p \lor q}\ \lor I \qquad\qquad \frac{\dfrac{p \succ p}{p \succ p \lor q}\ \land L}{p \land q \succ p \lor q}\ \land L$$

$$\frac{\dfrac{p \land q}{q}\ \land E}{p \lor q}\ \lor I \qquad\qquad \frac{\dfrac{q \succ q}{q \succ p \lor q}\ \lor R}{p \land q \succ p \lor q}\ \land L$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \wedge q}{p}\,{\wedge}E}{p \vee q}\,{\vee}I \qquad\qquad \dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,{\wedge}L}{p \wedge q \succ p \vee q}\,{\wedge}L$$

$$\dfrac{\dfrac{p \wedge q}{q}\,{\wedge}E}{p \vee q}\,{\vee}I \qquad\qquad \dfrac{\dfrac{q \succ q}{q \succ p \vee q}\,{\vee}R}{p \wedge q \succ p \vee q}\,{\wedge}L$$

Term Representation

$$\acute{\wedge}x \curvearrowright \acute{\vee}y \qquad\qquad \grave{\wedge}x \curvearrowright \grave{\vee}y$$

$$x : p \wedge q \succ y : p \vee q \qquad\qquad x : p \wedge q \succ y : p \vee q$$

Term Representation

$$\acute{\wedge}x \curvearrowright \acute{\vee}y$$
$$x : p \wedge q \succ y : p \vee q$$

$$\grave{\wedge}x \curvearrowright \grave{\vee}y$$
$$x : p \wedge q \succ y : p \vee q$$

$$\pi(x_1, \ldots, x_m)[y_1, \ldots, y_n]$$
$$x_1 : A_1, \ldots, x_m : A_m \succ y_1 : B_1, \ldots, y_n : B_n$$

Sequent Rules: Axioms

$$\overset{x \curvearrowright}{\Sigma, x : \bot \succ \Delta} \qquad \overset{x \curvearrowright y}{\Sigma, x : p \succ y : p, \Delta} \qquad \overset{\curvearrowright y}{\Sigma \succ y : \top, \Delta}$$

Sequent Rules: Axioms, Conjunction

$$\overset{x \curvearrowright}{\Sigma, x : \bot \succ \Delta} \qquad \overset{x \curvearrowright y}{\Sigma, x : p \succ y : p, \Delta} \qquad \overset{\curvearrowright y}{\Sigma \succ y : \top, \Delta}$$

$$\frac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(\acute{\wedge} z, \grave{\wedge} z)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L \qquad \frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[\acute{\wedge} z] \;\; \pi'[\grave{\wedge} z]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta'}} \wedge R$$

Sequent Rules: Axioms, Conjunction and Disjunction

$$\overset{x \curvearrowright}{\Sigma, x : \bot \succ \Delta} \qquad \overset{x \curvearrowright y}{\Sigma, x : p \succ y : p, \Delta} \qquad \overset{\curvearrowright y}{\Sigma \succ y : \top, \Delta}$$

$$\frac{\overset{\pi(x, y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(\acute{\wedge} z, \grave{\wedge} z)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L \qquad \frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[\acute{\wedge} z] \;\; \pi'[\grave{\wedge} z]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta'}} \wedge R$$

$$\frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(\acute{\vee} z) \;\; \pi'(\grave{\vee} z)}{\Sigma, \Sigma', z : A \vee B \succ \Delta, \Delta'}} \vee L \qquad \frac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[\acute{\vee} z, \grave{\vee} z]}{\Sigma \succ z : A \vee B, \Delta}} \vee R$$

Sequent Rules: Conditional

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad\qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[\acute{\supset} z]\ \ \pi'(\grave{\supset} z)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}}\ {\supset}L \qquad\qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(\acute{\supset} z)[\grave{\supset} z]}{\Sigma \succ z : A \supset B, \Delta}}\ {\supset}R$$

$$\frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[\acute{\supset} z]\ \ \pi'(\grave{\supset} z)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}}\ {\supset}L \qquad \frac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(\acute{\supset} z)[\grave{\supset} z]}{\Sigma \succ z : A \supset B, \Delta}}\ {\supset}R$$

$$\frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\overset{\pi[\grave{\neg} z]}{\Sigma, z : \neg A \succ \Delta}}\ \neg L \qquad \frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(\grave{\neg} z)}{\Sigma \succ z : \neg A, \Delta}}\ \neg R$$

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[\acute{\supset} z]\ \ \pi'(\grave{\supset} z)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}}\ {\supset}L \qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(\acute{\supset} z)[\grave{\supset} z]}{\Sigma \succ z : A \supset B, \Delta}}\ {\supset}R$$

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\overset{\pi[\grave{\neg} z]}{\Sigma, z : \neg A \succ \Delta}}\ \neg L \qquad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(\grave{\neg} z)}{\Sigma \succ z : \neg A, \Delta}}\ \neg R$$

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet]\ \ \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Cut}$$

Classical Derivations

$$\dfrac{\dfrac{\overset{x \curvearrowright y}{x : p \succ y : p}}{\overset{x \curvearrowright \grave{\neg} z}{x : p, z : \neg p \succ}}\ \neg L}{\overset{\acute{\wedge} w \curvearrowright \grave{\neg} \grave{\wedge} w}{w : p \wedge \neg p \succ}}\ \wedge L
\qquad\qquad
\dfrac{\dfrac{\overset{x \curvearrowright y}{x : p \succ y : p}}{\overset{\grave{\neg} z \curvearrowright y}{\succ y : p, z : \neg p}}\ \neg R}{\overset{\grave{\neg} \grave{\vee} v \curvearrowright \acute{\vee} v}{\succ v : p \vee \neg p}}\ \vee R$$

Classical Derivations

$$\dfrac{\dfrac{\overset{x\curvearrowright y}{x : p \succ y : p}}{\overset{x\curvearrowright \grave{\neg}z}{x : p, z : \neg p \succ}}\;\neg L}{\overset{\acute{\wedge}w\curvearrowright \grave{\neg}\grave{\wedge}w}{w : p \wedge \neg p \succ}}\;\wedge L
\qquad\qquad
\dfrac{\dfrac{\overset{x\curvearrowright y}{x : p \succ y : p}}{\overset{\grave{\neg}z\curvearrowright y}{\succ y : p, z : \neg p}}\;\neg R}{\overset{\grave{\neg}\grave{\vee}v\curvearrowright \acute{\vee}v}{\succ v : p \vee \neg p}}\;\vee R$$

$$\dfrac{\dfrac{\overset{x\curvearrowright y}{x : p \succ y : p} \qquad \dfrac{\overset{x\curvearrowright y}{x : p \succ y : p}}{\overset{\acute{\supset}z\curvearrowright y}{\succ y : p, z : p \supset q}}\;\supset R}{\overset{\grave{\supset}v\curvearrowright y \quad \acute{\supset}\acute{\supset}v\curvearrowright y}{v : (p \supset q) \supset p \succ y : p}}\;\supset L}{\overset{\grave{\supset}\acute{\supset}u\curvearrowright \grave{\supset}u \quad \acute{\supset}\acute{\supset}\acute{\supset}u\curvearrowright \grave{\supset}u}{\succ u : ((p \supset q) \supset p) \supset p}}\;\supset R$$

Classical Derivations (outputs in blue)

$$\dfrac{\dfrac{\overset{x \frown y}{x : p \succ y : p}}{\overset{x \frown \dot{\neg} z}{x : p,\, z : \neg p \succ}}\;\neg L}{\overset{\acute{\land} w \frown \dot{\neg} \grave{\land} w}{w : p \land \neg p \succ}}\;\land L
\qquad\qquad
\dfrac{\dfrac{\overset{x \frown y}{x : p \succ y : p}}{\overset{\dot{\neg} z \frown y}{\succ y : p,\, z : \neg p}}\;\neg R}{\overset{\dot{\neg} \grave{\lor} v \frown \acute{\lor} v}{\succ v : p \lor \neg p}}\;\lor R$$

$$\dfrac{\dfrac{\overset{x \frown y}{x : p \succ y : p} \qquad \dfrac{\overset{x \frown y}{x : p \succ y : p}}{\overset{\acute{\supset} z \frown y}{\succ y : p,\, z : p \supset q}}\;\supset R}{\overset{\grave{\supset} v \frown y \quad \acute{\supset}\acute{\supset} v \frown y}{v : (p \supset q) \supset p \succ y : p}}\;\supset L}{\overset{\grave{\supset}\acute{\supset} u \frown \grave{\supset} u \quad \acute{\supset}\acute{\supset}\acute{\supset} u \frown \grave{\supset} u}{\succ u : ((p \supset q) \supset p) \supset p}}\;\supset R$$

Constructive Proof Terms

A *constructive* proof term is *hereditarily* single (or zero) output.

$$\pi(x_1, \ldots, x_n)[y] \qquad \pi'(x_1, \ldots, x_n)$$

Constructive Proof Terms and Lambda Terms

$$\pi(x_1, \ldots, x_m)[y]$$
$$x_1 : A_1, \ldots, x_m : A_m \succ y : B$$

Constructive Proof Terms and Lambda Terms

$$\pi(x_1, \ldots, x_m)[y]$$
$$x_1 : A_1, \ldots, x_m : A_m \succ y : B$$

$$x_1 : A_1, \ldots, x_m : A_m \succ M(x_1, \ldots, x_m) : B$$

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$$x_1 : A_1, \ldots, x_m : A_m \succ M'(x_1, \ldots, x_m) : \bot$$


WARRANT & CHOICE

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Constructs a warrant for B out of warrants for $A_1, \ldots, A_m$.

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What Proof Terms *Describe*

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$$x_1 : A_1, \ldots, x_m : A_m \succ y_1 : B_1, \ldots, y_n : B_n$$

Describes a clash between warrants *for* $A_1, \ldots, A_n$
and warrants *against* $B_1, \ldots, B_n$.

Example Proof Terms

$$\overset{x \frown \acute{\wedge} z \quad y \frown \grave{\wedge} z}{x : p, y : q \succ z : p \wedge q}$$

A clash between warrants for p and for q and a warrant against $p \wedge q$.

Example Proof Terms

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A clash between warrants for p and for q
and a warrant against $p \wedge q$.

$$\acute{\vee} x \curvearrowright y \qquad \grave{\vee} x \curvearrowright z$$

$$x : p \vee q \succ y : p, z : q$$

A clash between a warrant for $p \vee q$
and warrants against p and against q.

On Warrants for and Against

$$\Sigma, x : \bot \succ \Delta \qquad \Sigma, x : p \succ y : p, \Delta \qquad \Sigma \succ y : \top, \Delta$$

$$\Sigma, \overset{x\curvearrowright}{x : \bot} \succ \Delta \qquad \Sigma, \overset{x \curvearrowright y}{x : p \succ y} : p, \Delta \qquad \Sigma \succ \overset{\curvearrowright y}{y} : \top, \Delta$$

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet]\ \ \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Cut}$$

This *doesn't* mean that we always possess a warrant *for* or *against* A.

It *does* mean that the only way a context has some clash with a warrant *for* A and a warrant *against* A is if it contains a clash within itself.

What is special about constructive proof terms?

$$\pi(x_1, \ldots, x_n)[y]$$

- A *clash* between $x_1, \ldots, x_n$ (for $A_1, \ldots, A_n$) and y (against B) can *also* be understood as a construction of a warrant *for* B.

$$\overset{\grave{\neg}\acute{\vee}\nu \frown \grave{\vee}\nu}{\succ \nu : p \vee \neg p}$$

- This is describes a warrant for $p \vee \neg p$, but it is not *hereditarily* single output. (Both $\acute{\vee}\nu$ and $\grave{\vee}\nu$ are in output position.) It gives a warrant for $p \vee \neg p$ without giving a warrant for either disjunct.

Choice Free Warrants

- A hereditarily single (or zero) output proof term describes a *choice-free* clash between warrants.

- A warrant for $A \wedge B$ gives a warrant for A and a warrant for B (*no choice*).
- A warrant against $A \wedge B$ gives *either* a warrant against A or a warrant against B (*choice*).
- A warrant for $A \vee B$ gives *either* a warrant for A or a warrant for B (*choice*).
- A warrant against $A \vee B$ gives a warrant against A and a warrant against B (*no choice*).
- A warrant for $\neg A$ gives a warrant against A (*no choice*).
- A warrant against $\neg A$ gives a warrant for A (*no choice*).

Is $\neg\neg p$ effectively *the same* as p? *No*

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$$\dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{\grave{\neg} y \curvearrowright x}{\succ x : p,\ y : \neg p}}\,\neg R}{\overset{\grave{\neg}\grave{\neg} z \curvearrowright x}{z : \neg\neg p \succ x : p}}\,\neg L$$

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Univocal Rules

$$\overset{x\curvearrowright}{\Sigma, x : \bot \succ \Delta} \qquad \overset{x\curvearrowright y}{\Sigma, x : p \succ y : p, \Delta} \qquad \overset{\curvearrowright y}{\Sigma \succ y : \top, \Delta}$$

$$\frac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(\acute{\wedge} z,\ \grave{\wedge} z)}{\Sigma, z : A \wedge B \succ \Delta}}\ \wedge L \qquad \frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[\acute{\wedge} z]\ \ \pi'[\grave{\wedge} z]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta'}}\ \wedge R$$

$$\frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(\acute{\vee} z)\ \ \pi'(\grave{\vee} z)}{\Sigma, \Sigma', z : A \vee B \succ \Delta, \Delta'}}\ \vee L \qquad \frac{\overset{\pi[x,y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[\acute{\vee} z,\ \grave{\vee} z]}{\Sigma \succ z : A \vee B, \Delta}}\ \vee R$$

$$\frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[\acute{\supset} z]\ \ \pi'(\grave{\supset} z)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}}\ \supset L \qquad \frac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(\acute{\supset} z)[\grave{\supset} z]}{\Sigma \succ z : A \supset B, \Delta}}\ \supset R$$

$$\frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\overset{\pi[\acute{\neg} z]}{\Sigma, z : \neg A \succ \Delta}}\ \neg L \qquad \frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(\acute{\neg} z)}{\Sigma \succ z : \neg A, \Delta}}\ \neg R \qquad \frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet]\ \ \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Cut}$$

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  - The position $[\Sigma : \Delta]$ may be filled out into a partition of the language $[\Sigma' : \Delta']$ where no warrants for Σ' can be transformed in a choice-free way into a warrant for anything in Δ'.
  - This is a *construction*.


WHERE WE'VE GOT,
AND WHERE TO
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2. grounded in univocal rules for evaluating assertions and denials for the connectives,
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4. and which give us an interpretation of classical and constructive counterexamples.

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- Explore the structural relationship between constructive proof terms and λ terms, and normalisaiton and cut-elimination.
- Develop the dual, *paraconsistent* case of single premise, multiple conclusion proof.
- Extend the account to predicate logic and beyond.

THANK YOU!

http://consequently.org/presentation/2017/
logical-pluralism-meaning-counterexamples

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