Terms for Classical Sequents

Proof Invariants & Strong Normalisation

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My Aim

To introduce a new *invariant*
for classical propositional proofs
to help address questions
about proof identity.

Today's Plan

Background

Preterms

Derivations

Terms

Eliminating Cuts

Strong Normalisation

Further Work


BACKGROUND

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,\vee R}{p \wedge q \succ p \vee q}\,\wedge L \qquad \dfrac{\dfrac{p \wedge q}{p}\,\wedge E}{p \vee q}\,\vee I \qquad \dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,\wedge L}{p \wedge q \succ p \vee q}\,\vee R$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \lor q}\,{\scriptstyle \lor R}}{p \land q \succ p \lor q}\,{\scriptstyle \land L} \qquad \dfrac{\dfrac{p \land q}{p}\,{\scriptstyle \land E}}{p \lor q}\,{\scriptstyle \lor I} \qquad \dfrac{\dfrac{p \succ p}{p \land q \succ p}\,{\scriptstyle \land L}}{p \land q \succ p \lor q}\,{\scriptstyle \lor R}$$

$$\dfrac{\dfrac{q \succ q}{q \succ p \lor q}\,{\scriptstyle \lor R}}{p \land q \succ p \lor q}\,{\scriptstyle \land L} \qquad \dfrac{\dfrac{p \land q}{q}\,{\scriptstyle \land E}}{p \lor q}\,{\scriptstyle \lor I} \qquad \dfrac{\dfrac{q \succ q}{p \land q \succ q}\,{\scriptstyle \land L}}{p \land q \succ p \lor q}\,{\scriptstyle \lor R}$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \lor q \qquad \dfrac{[p]^1}{q \lor p}\lor I \qquad \dfrac{[q]^1}{q \lor p}\lor I}{q \lor p}\lor E^1}{(q \lor p) \lor r}\lor I \qquad\qquad \dfrac{p \lor q \qquad \dfrac{\dfrac{[p]^1}{q \lor p}\lor I}{(q \lor p) \lor r}\lor I \qquad \dfrac{\dfrac{[q]^1}{q \lor p}\lor I}{(q \lor p) \lor r}\lor I}{(q \lor p) \lor r}\lor E^1$$

Are these *different proofs*, or *different ways of presenting the same proof*?

Girard, Lafont and Taylor: *Proofs and Types*, Chapter 2

Natural deduction is a slightly paradoxical system: it is limited to the intuitionistic case (in the classical case it has no particularly good properties) but it is only satisfactory for the $(\wedge, \Rightarrow, \forall)$ fragment of the language: we shall defer consideration of $\vee$ and $\exists$ until chapter 10. Yet disjunction and existence are the two most *typically* intuitionistic connectors!

The basic idea of natural deduction is an asymmetry: a proof is a vaguely tree-like structure (this view is more a graphical illusion than a mathematical reality, but it is a pleasant illusion) with one or more hypotheses (possibly none) but a single conclusion. The deep symmetry of the calculus is shown by the *introduction* and *elimination* rules which match each other exactly. Observe, incidentally, that with a tree-like structure, one can always decide uniquely what was the *last* rule used, which is something we could not say if there were several conclusions.

$$
\begin{prooftree}
\AxiomC{$[x : p \supset (q \supset r)]$}
\AxiomC{$[z : p]$}
\RightLabel{$\supset E$}
\BinaryInfC{$xz : q \supset r$}
\AxiomC{$[y : p \supset q]$}
\AxiomC{$[z : p]$}
\RightLabel{$\supset E$}
\BinaryInfC{$yz : q$}
\RightLabel{$\supset E$}
\BinaryInfC{$(xz)(yz) : r$}
\RightLabel{$\supset I$}
\UnaryInfC{$\lambda z\,(xz)(yz) : p \supset r$}
\RightLabel{$\supset I$}
\UnaryInfC{$\lambda y \lambda z\,(xz)(yz) : (p \supset q) \supset (p \supset r)$}
\RightLabel{$\supset I$}
\UnaryInfC{$\lambda x \lambda y \lambda z\,(xz)(yz) : (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$}
\end{prooftree}
$$

Contraction and *weakening* are managed by variables

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\supset I$$

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$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\,{\supset}I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\,{\supset}I$$

$$\dfrac{\dfrac{\dfrac{x : p \supset (p \supset q) \quad [y : p]}{xy : p \supset q}\,{\supset}E \quad [y : p]}{(xy)y : q}\,{\supset}E}{\lambda y\, (xy)y : p \supset q}\,{\supset}I$$

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \lor \neg p}\,\lor R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \land \neg p \succ}\,\land L$$

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \lor \neg p}\,\lor R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \land \neg p \succ}\,\land L$$

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{q \succ q \quad r \succ r}{q \lor r \succ q, r}\,\lor L}{p, q \lor r \succ p \land q, r}\,\land R}{p \land (q \lor r) \succ p \land q, r}\,\land L}{p \land (q \lor r) \succ (p \land q) \lor r}\,\lor R$$

Sequents and Terms

$$X \succ Y \qquad X \succ A, Y \qquad X, A \succ Y$$

Where do you put the *variables*,
and where do you put the *terms*?

$$x_1 : A_1, \ldots, x_n : A_n \succ y_1 : B_1, \ldots, y_m : B_m$$

Our Choice

$$x_1 : A_1, \ldots, x_n : A_n \succ y_1 : B_1, \ldots, y_m : B_m$$

Each premise and conclusion is decorated with variables.

Our Choice

$$x_1 : A_1, \ldots, x_n : A_n \overset{\pi(x_1,\ldots,x_n)[y_1,\ldots,y_m]}{\succ} y_1 : B_1, \ldots, y_m : B_m$$

Each premise and conclusion is decorated with variables.

The *sequent* gets the term, showing how inputs & outputs are connected, with as much parallelism as possible.

The Core Idea

$$x_1 : A_1, \ldots, x_n : A_n \overset{\pi(x_1, \ldots, x_n)[y_1, \ldots, y_m]}{\succ} y_1 : B_1, \ldots, y_m : B_m$$

The term shows *how* a proof
connects premises and conclusions.

The Core Idea

$$\overset{\pi(x_1,\ldots,x_n)[y_1,\ldots,y_m]}{x_1 : A_1, \ldots, x_n : A_n \succ y_1 : B_1, \ldots, y_m : B_m}$$

The term shows *how* a proof
connects premises and conclusions.

If π_1 and π_2 connect premises and conclusions
in the same way, they represent the same proof.

Example 1

$$
\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \dfrac{\overset{y\frown y}{y : q \succ y : q} \qquad \overset{z\frown z}{z : r \succ z : r}}{\overset{Lw\frown y \quad Rw\frown z}{w : q \vee r \succ y : q, z : r}}\,{\vee L}}{\overset{x\frown Fv \quad Lw\frown Sv \quad Rw\frown z}{x : p, w : q \vee r \succ v : p \wedge q, z : r}}\,{\wedge R}
$$

$$
\dfrac{\dfrac{x : p, w : q \vee r \succ v : p \wedge q, z : r}{\overset{Fu\frown Fv \quad LSu\frown Sv \quad RSu\frown z}{u : p \wedge (q \vee r) \succ v : p \wedge q, z : r}}\,{\wedge L}}{\overset{Fu\frown FLt \quad LSu\frown SLt \quad RSu\frown Rt}{u : p \wedge (q \vee r) \succ t : (p \wedge q) \vee r}}\,{\vee R}
$$

Example 2

$$
\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown \mathsf{F}y \quad x\frown \mathsf{S}y}{x : p \succ y : p \land p}}\,{\land}R \qquad \dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{\mathsf{F}w\frown z}{w : p \land p \succ z : p}}\,{\land}L}{\overset{x\frown \mathsf{F}\bullet \quad x\frown \mathsf{S}\bullet \quad \mathsf{F}\bullet\frown z}{x : p \succ z : p}}\,\mathit{Cut}
$$


PRETERMS

Variables and Cut Points

- For each formula A, $x_1^A, x_2^A, \ldots$ are VARIABLES of type A.

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- For each formula A, $\bullet_1^A, \bullet_2^A, \ldots$ are CUT POINTS of type A.
  - We use $x, y, z, u, v, w, \ldots; \bullet, \star, *, \sharp, \flat$ as schematic letters for variables and cut points, ommitting type superscripts where possible.

Nodes and Subnodes

- A variable x of type A and a cut point $\bullet$ of type A are both A NODES.

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- If n is an $A \wedge B$ node, then Ln is an A node and Rn is a B node.
- If n is an $A \vee B$ node, then Fn is an A node and Sn is a B node.
- If n is an $A \supset B$ node, then An is an A node and Cn is a B node.
- If n is a $\neg A$ node, then Nn is an A node.

Nodes and Subnodes

- A variable x of type A and a cut point $\bullet$ of type A are both A NODES.
- If n is an $A \wedge B$ node, then $\mathsf{L}n$ is an A node and $\mathsf{R}n$ is a B node.
- If n is an $A \vee B$ node, then $\mathsf{F}n$ is an A node and $\mathsf{S}n$ is a B node.
- If n is an $A \supset B$ node, then $\mathsf{A}n$ is an A node and $\mathsf{C}n$ is a B node.
- If n is a $\neg A$ node, then $\mathsf{N}n$ is an A node.
- For each complex node $\mathsf{L}n$, $\mathsf{R}n$, $\mathsf{F}n$, $\mathsf{S}n$, $\mathsf{A}n$, $\mathsf{C}n$ and $\mathsf{N}n$, n is its IMMEDIATE subnode, and the subnodes of n are also subnodes of the original node.

Linkings, Inputs and Outputs

- A LINKING is a pair $\mathfrak{n}\frown\mathfrak{m}$ of nodes of the same type.

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- In $n \frown m$, n is in INPUT POSITION, and m is in OUTPUT POSITION.
- Positions generalise to subnodes as follows:
  - If $\mathsf{L}n$, $\mathsf{R}n$, $\mathsf{F}n$, $\mathsf{S}n$ or $\mathsf{C}n$ are in *input position*, n is also in *input position*.
  - If $\mathsf{L}n$, $\mathsf{R}n$, $\mathsf{F}n$, $\mathsf{S}n$ or $\mathsf{C}n$ are in *output position*, n is also in *output position*.
  - – L, R, F, S and C each *preserve position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *input position*, n is in *output position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *output position*, n is in *input position*.
  - – A and N *reverse position*.

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- In $n \frown m$, n is in INPUT POSITION, and m is in OUTPUT POSITION.
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  - – L, R, F, S and C each *preserve position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *input position*, n is in *output position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *output position*, n is in *input position*.
  - – A and N *reverse position*.
- The INPUTS (OUTPUTS) of a linking are the *variables* in INPUT (OUTPUT) position of that linking.

Example Linkings

x of type $((p \supset q) \supset p) \supset p$

$\mathsf{AAA}x \frown \mathsf{C}x$

$\mathsf{CA}x \frown \mathsf{C}x$

Preterms

- DEFINITION: A PRETERM is a finite set of linkings.

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  - The INPUTS of a preterm are the inputs of its linkings.
  - Its OUTPUTS are the outputs of its linkings.

DERIVATIONS

Annotating Derivations: Identity

$$\Sigma, x : A \overset{x \frown y}{\succ} y : A, \Delta$$

Annotating Derivations: Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz, Sz)}{\Sigma, z : A \land B \succ \Delta}} \land L$$

Annotating Derivations: Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(\mathsf{F}z,\mathsf{S}z)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L \qquad \dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[\mathsf{F}z] \quad \pi'[\mathsf{S}z]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta'}} \wedge R$$

Excursus on *Weakening* and *Variables*

$$\cfrac{\cfrac{[x : p]}{\lambda y\, x : q \supset p}\ \supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\ \supset I$$

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\,{\supset}I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\,{\supset}I$$

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz, Sz)}{\Sigma, z : A \wedge B \succ \Delta}}\,{\wedge}L \quad \text{can be} \quad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Fz)}{\Sigma, z : A \wedge B \succ \Delta}}\,{\wedge}L$$

Excursus on *Weakening* and *Variables*

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\supset I$$

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \land B \succ \Delta}}\land L \quad \text{can be} \quad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Fz)}{\Sigma, z : A \land B \succ \Delta}}\land L$$

In a premise $\pi(x, y)$ the indicated x and y display all of the x and y inputs to the proof term.

There might be *none*.

Annotating Derivations: Negation

$$\dfrac{\begin{array}{c}\pi[x] \\ \Sigma \succ x : A, \Delta\end{array}}{\begin{array}{c}\pi[\mathsf{N}z] \\ \Sigma, z : \neg A \succ \Delta\end{array}}\,\neg L \qquad \dfrac{\begin{array}{c}\pi(x) \\ \Sigma, x : A \succ \Delta\end{array}}{\begin{array}{c}\pi(\mathsf{N}z) \\ \Sigma \succ z : \neg A, \Delta\end{array}}\,\neg R$$

Annotating Derivations: Disjunction

$$\dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(Lz)\quad \pi'(Rz)}{\Sigma, \Sigma', z : A \vee B \succ \Delta, \Delta'}}\ \vee L \qquad\qquad \dfrac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[Lz, Rz]}{\Sigma \succ z : A \vee B, \Delta}}\ \vee R$$

Annotating Derivations: Conditional

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az] \quad \pi'(Lz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}} \supset L \qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(Az)[Cz]}{\Sigma \succ z : A \supset B, \Delta}} \supset R$$

Example Annotation

$$
\dfrac{
  \dfrac{\overset{x\frown x}{x : p \succ x : p}
  \qquad
  \dfrac{\overset{y\frown y}{y : q \succ y : q} \qquad \overset{z\frown z}{z : r \succ z : r}}{\overset{Lw\frown y \;\; Rw\frown z}{w : q \lor r \succ y : q, z : r}}\,{\lor}L}{\overset{x\frown Fv \;\; Lw\frown Sv \;\; Rw\frown z}{x : p, w : q \lor r \succ v : p \land q, z : r}}\,{\land}R
}{
  \dfrac{\overset{Fu\frown Fv \;\; LSu\frown Sv \;\; RSu\frown z}{u : p \land (q \lor r) \succ v : p \land q, z : r}}{\overset{Fu\frown FLt \;\; LSu\frown SLt \;\; RSu\frown Rt}{u : p \land (q \lor r) \succ t : (p \land q) \lor r}}\,{\lor}R
}\,{\land}L
$$

Annotating Derivations: Cut

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet] \quad \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Cut}$$

Identify Terms up to α equivalence

If π can be transformed into π' by relabelling cut points we treat them as identical (they are α equivalent).

Example Annotation, with *Cut*

$$
\dfrac{
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,{\lor}L
\qquad
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,{\land}R
}{\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \lor p \succ z : p \land p}}\,\mathit{Cut}
$$

$$\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{z\frown Ly}{z : p \succ y : p \lor q}}\,{\lor}R}{\overset{Fx\frown Ly}{x : p \land q \succ y : p \lor q}}\,{\land}L \qquad \dfrac{\dfrac{p \land q}{p}\,{\land}E}{p \lor q}\,{\lor}I \qquad \dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fx\frown z}{x : p \land q \succ z : p}}\,{\lor}R}{\overset{Fx\frown Ly}{x : p \land q \succ y : p \lor q}}\,{\land}L$$

When is π_1 the same proof as π_2 (revisited)?

$$\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{z\frown Ly}{z : p \succ y : p \vee q}}{\vee}R}{\overset{Fx\frown Ly}{x : p \wedge q \succ y : p \vee q}}{\wedge}L \qquad \dfrac{\dfrac{p \wedge q}{p}{\wedge}E}{p \vee q}{\vee}I \qquad \dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fx\frown z}{x : p \wedge q \succ z : p}}{\wedge}L}{\overset{Fx\frown Ly}{x : p \wedge q \succ y : p \vee q}}{\vee}R$$

$$\dfrac{\dfrac{\overset{w\frown w}{w : q \succ w : q}}{\overset{w\frown Ry}{w : q \succ y : p \vee q}}{\vee}R}{\overset{Sx\frown Ry}{x : p \wedge q \succ y : p \vee q}}{\wedge}L \qquad \dfrac{\dfrac{p \wedge q}{q}{\wedge}E}{p \vee q}{\vee}I \qquad \dfrac{\dfrac{\overset{w\frown w}{w : q \succ w : q}}{\overset{Sx\frown w}{x : p \wedge q \succ w : q}}{\wedge}L}{\overset{Sx\frown Ry}{x : p \wedge q \succ y : p \vee q}}{\vee}R$$

When is π_1 the same proof as π_2 (revisited)?

$$\dfrac{\dfrac{p \lor q \qquad \dfrac{[p]^1}{q \lor p}\,{\lor}I \qquad \dfrac{[q]^1}{q \lor p}\,{\lor}I}{q \lor p}\,{\lor}E^1}{(q \lor p) \lor r}\,{\lor}I$$

$$\dfrac{\dfrac{\dfrac{\overset{x \frown x}{x : p \succ x : p}}{\overset{x \frown Rz}{x : p \succ z : q \lor p}}\,{\lor}R \qquad \dfrac{\overset{y \frown y}{y : q \succ y : q}}{\overset{y \frown Lz}{y : q \succ z : q \lor p}}\,{\lor}R}{\overset{Lw \frown Rz \quad Rw \frown Lz}{w : p \lor q \succ z : q \lor p}}\,{\lor}L}{\overset{Lw \frown RLu \quad Rw \frown LLu}{w : p \lor q \succ u : (q \lor p) \lor r}}\,{\lor}R$$

$$\dfrac{p \lor q \qquad \dfrac{\dfrac{[p]^1}{q \lor p}\,\lor I}{(q \lor p) \lor r}\,\lor I \qquad \dfrac{\dfrac{[q]^1}{q \lor p}\,\lor I}{(q \lor p) \lor r}\,\lor I}{(q \lor p) \lor r}\,\lor E^1$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} x \frown x \\ x : p \succ x : p \end{array}}{\begin{array}{c} x \frown Rz \\ x : p \succ z : q \lor p \end{array}}\,\lor R}{\begin{array}{c} x \frown RLu \\ x : p \succ u : (q \lor p) \lor r \end{array}}\,\lor R \qquad \dfrac{\dfrac{\begin{array}{c} y \frown y \\ y : q \succ y : q \end{array}}{\begin{array}{c} x \frown LLu \\ y : q \succ z : q \lor p \end{array}}\,\lor R}{\begin{array}{c} y \frown LLu \\ y : q \succ u : (q \lor p) \lor r \end{array}}\,\lor R}{\begin{array}{c} Lw \frown RLu \quad Rw \frown LLu \\ w : p \lor q \succ u : (q \lor p) \lor r \end{array}}\,\lor L$$

DEFINITION: A preterm is SEQUENTIALISABLE iff it is the conclusion of some derivation.


TERMS

Nonsequentialisable Preterms

$$\begin{array}{c} \mathsf{L}x \frown \mathsf{F}y \quad \mathsf{R}x \frown \mathsf{S}y \\ x : p \lor q \succ y : p \land q \end{array}$$

This is connected, but it is not connected *enough*.

Switching Example

$$\overset{Lx\frown Fy}{} \quad \overset{Rx\frown Sy}{}$$

$$x : p \lor q \succ y : p \land q$$

Switching Example

$$\overset{Lx^\frown Fy \quad Rx^\frown Sy}{x : p \vee q \succ y : p \wedge q}$$

$$\overset{Lx^\frown Fy \quad \cancel{Rx}^\frown \cancel{Sy}}{x : p \vee - \succ y : p \wedge -}$$

$$\overset{Lx^\frown \cancel{Fy} \quad \cancel{Rx}^\frown Sy}{x : p \vee - \succ y : - \wedge q}$$

$$\overset{\cancel{Lx}^\frown Fy \quad Rx^\frown \cancel{Sy}}{x : - \vee q \succ y : p \wedge -}$$

$$\overset{\cancel{Lx}^\frown \cancel{Fy} \quad Rx^\frown Sy}{x : - \vee q \succ y : - \wedge q}$$

Switchings

- The SWITCHINGS of a preterm π are found by selecting for each pair of subterms Ln and Rn in *input* position; Fn and Sn in *output position*, An in *output* position and Cn in *input position*; or the cut point • (in both *input* and *output position*), one item of the pair to keep, and the other to DELETE.
- A LINKING in a switching of a preterm π SURVIVES if and only if neither side of the link involves a deletion.
- A preterm is SPANNED if every switching has at least one surviving linking.

Example

$$\mathsf{Fu}^\frown\mathsf{FLt} \quad \mathsf{LSu}^\frown\mathsf{SLt} \quad \mathsf{RSu}^\frown\mathsf{Rt}$$

This has two pairs for switching:

$\mathsf{LSu}/\mathsf{RSu}$ in *input position*. $\mathsf{FLt}/\mathsf{SLt}$ in *output position*.

Example

$$\mathsf{Fu}\frown\mathsf{FLt} \quad \mathsf{LSu}\frown\mathsf{SLt} \quad \mathsf{RSu}\frown\mathsf{Rt}$$

This has two pairs for switching:

$\mathsf{LSu}/\mathsf{RSu}$ in *input position*. $\mathsf{FLt}/\mathsf{SLt}$ in *output position*.

$$\mathsf{Fu}\frown\cancel{\mathsf{FLt}} \quad \cancel{\mathsf{LSu}}\frown\mathsf{SLt} \quad \mathsf{RSu}\frown\mathsf{Rt}$$

$$\mathsf{Fu}\frown\mathsf{FLt} \quad \cancel{\mathsf{LSu}}\frown\cancel{\mathsf{SLt}} \quad \mathsf{RSu}\frown\mathsf{Rt}$$

$$\mathsf{Fu}\frown\cancel{\mathsf{FLt}} \quad \mathsf{LSu}\frown\mathsf{SLt} \quad \cancel{\mathsf{RSu}}\frown\mathsf{Rt}$$

$$\mathsf{Fu}\frown\mathsf{FLt} \quad \mathsf{LSu}\frown\cancel{\mathsf{SLt}} \quad \cancel{\mathsf{RSu}}\frown\mathsf{Rt}$$

DEFINITION: A preterm π is a TERM when it is SPANNED.

By induction on the derivation sequentialising π.

Sequentialisable Preterms are Terms: Identity

$$\Sigma, x : A \overset{x \frown y}{\succ} y : A, \Delta$$

Sequentialisable Preterms are Terms: Conjunction

$$\frac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \land B \succ \Delta}}\land L \qquad \frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \quad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \land B, \Delta, \Delta'}}\land R$$

Sequentialisable Preterms are Terms: Negation

$$\frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\underset{}{\overset{\pi[Nz]}{\Sigma, z : \neg A \succ \Delta}}}\ \neg L \qquad \frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Nz)}{\Sigma \succ z : \neg A, \Delta}}\ \neg R$$

Sequentialisable Preterms are Terms: Disjunction

$$\dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(Lz) \quad \pi'(Rz)}{\Sigma, \Sigma', z : A \lor B \succ \Delta, \Delta'}} \lor L \qquad \dfrac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[Lz, Rz]}{\Sigma \succ z : A \lor B, \Delta}} \lor R$$

Sequentialisable Preterms are Terms: Conditional

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az] \quad \pi'(Cz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}} \supset L \qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(Az)[Cz]}{\Sigma \succ z : A \supset B, \Delta}} \supset R$$

Sequentialisable Preterms are Terms: Cut

$$\dfrac{\Sigma \succ \overset{\pi[x]}{x} : A, \Delta \qquad \Sigma', \overset{\pi'(y)}{y} : A \succ \Delta'}{\Sigma, \Sigma' \succ \overset{\pi[\bullet] \quad \pi'(\bullet)}{\Delta, \Delta'}}\ Cut$$

By induction on the number of pairs for switching in π.

Except …

$$x \frown y \quad u \frown v$$


ELIMINATING CUTS

$$
\dfrac{
\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta}} \wedge R
\qquad
\dfrac{\overset{\pi''(u, v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi''(Fw, Sw)}{\Sigma'', w : A \wedge B \succ \Delta''}} \wedge L
}{
\overset{\pi[F\bullet] \quad \pi'[S\bullet] \quad \pi''(F\bullet, S\bullet)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}
} \mathit{Cut}
$$

Conjunction Cut Reduction

$$
\dfrac{
\dfrac{
\overset{\pi[x]}{\Sigma \succ x : A, \Delta}
\qquad
\overset{\pi'[y]}{\Sigma' \succ y : B, \Delta}
}{
\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta}
}{\scriptstyle \wedge R}
\qquad
\dfrac{
\overset{\pi''(u,v)}{\Sigma'', u : A, v : B \succ \Delta''}
}{
\overset{\pi''(Fw, Sw)}{\Sigma'', w : A \wedge B \succ \Delta''}
}{\scriptstyle \wedge L}
}{
\overset{\pi[F\bullet] \quad \pi'[S\bullet] \quad \pi''(F\bullet, S\bullet)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}
}\;\mathit{Cut}
$$

reduces to

$$
\dfrac{
\overset{\pi[x]}{\Sigma \succ x : A, \Delta}
\qquad
\dfrac{
\overset{\pi'[y]}{\Sigma' \succ y : B, \Delta}
\qquad
\overset{\pi''(u,v)}{\Sigma'', u : A, v : B \succ \Delta''}
}{
\overset{\pi'[\star] \quad \pi''(u, \star)}{\Sigma', \Sigma'', u : A \succ \Delta', \Delta''}
}\;\mathit{Cut}
}{
\overset{\pi[*] \quad \pi'[\star] \quad \pi''(*, \star)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}
}\;\mathit{Cut}
$$

Identity Cut Reduction

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{y\frown z}{\Sigma', y : A \succ z : A, \Delta'}}{\overset{\pi[\bullet] \quad \bullet\frown z}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}}\ Cut$$

Identity Cut Reduction

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{y^\frown z}{\Sigma', y : A \succ z : A, \Delta'}}{\overset{\pi[\bullet] \quad \bullet^\frown z}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}}\ \mathit{Cut}$$

reduces to

$$\overset{\pi[z]}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}$$

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{\mathsf{L}y\frown x \quad \mathsf{R}y\frown x}{y : p \vee p \succ x : p}}\,{\vee L} \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown \mathsf{F}z \quad x\frown \mathsf{S}z}{x : p \succ z : p \wedge p}}\,{\wedge R}}{\overset{\mathsf{L}y\frown\bullet \quad \mathsf{R}y\frown\bullet \quad \bullet\frown\mathsf{F}z \quad \bullet\frown\mathsf{S}z}{y : p \vee p \succ z : p \wedge p}}\,\mathit{Cut}$$

Difficult Cases: Contraction

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \vee p \succ x : p}}\,\vee L \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \wedge p}}\,\wedge R}{\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \vee p \succ z : p \wedge p}}\,\mathit{Cut}$$

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \vee p \succ x : p}}\,\vee L \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \vee p \succ x : p}}\,\vee L}{\overset{Ly\frown Fz \quad Ry\frown Fz \quad Ly\frown Sz \quad Ry\frown Sz}{y : p \vee p \succ z : p \wedge p}}\,\wedge R$$

Difficult Cases: Contraction

$$
\dfrac{
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,{\lor}L
\qquad
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,{\land}R
}{
\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \lor p \succ z : p \land p}
}\,\mathit{Cut}
$$

$$
\dfrac{
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,{\land}R
\qquad
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,{\land}R
}{
\overset{Ly\frown Fz \quad Ry\frown Fz \quad Ly\frown Sz \quad Ry\frown Sz}{y : p \lor p \succ z : p \land p}
}\,{\lor}L
$$

$$
\dfrac{\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x : A, \Delta}} \qquad \dfrac{\overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi'}{\Sigma', y : A \succ \Delta'}}}{\overset{\pi \;\; \pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\,\mathit{Cut}
$$

Difficult Cases: Weakening

$$
\dfrac{
\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x : A, \Delta}}
\qquad
\dfrac{\overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi'}{\Sigma', y : A \succ \Delta'}}
}{
\overset{\pi\ \ \pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}
}\ \mathit{Cut}
$$

$$
\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Weak}
\qquad
\dfrac{\overset{\pi}{\Sigma \succ \Delta} \qquad \overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi\ \ \pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Mix}
\qquad
\dfrac{\overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Weak}
$$

Back to Sequentialisation

$$x \frown y \quad u \frown v$$

$$\dfrac{\overset{x \frown y}{x : A \succ y : A} \qquad \overset{u \frown v}{u : B \succ v : B}}{\overset{x \frown y \quad u \frown v}{x : A, u : B \succ y : B, v : A}}\ \mathit{Mix}$$

Sequentialisation: Terms with *No* Switchings

The term contains no Ln, Rn, Cn and • in input position or Fn, Sn, An and • in output position.

It has a derivation using the linear rules $\wedge L$, $\neg L$, $\neg R$, $\vee R$ and $\supset R$ and *mixes*.

The term contains no $\mathsf{L}n$, $\mathsf{R}n$, $\mathsf{C}n$ and $\bullet$ in input position or $\mathsf{F}n$, $\mathsf{S}n$, $\mathsf{A}n$ and $\bullet$ in output position.

It has a derivation using the linear rules $\wedge L$, $\neg L$, $\neg R$, $\vee R$ and $\supset R$ and *mixes*.

$$\mathsf{F}y\frown\mathsf{L}z \quad \mathsf{NR}z\frown\mathsf{L}z \quad \mathsf{S}y\frown\mathsf{R}z$$

$$y : p \wedge \neg p \succ z : p \vee \neg p$$

$$
\dfrac{
\dfrac{
\dfrac{\begin{matrix}x\frown x\\ x : p \succ x : p\end{matrix}}{\begin{matrix}x\frown Lz\\ x : p \succ z : p \lor \neg p\end{matrix}}\;{\lor}R
}{\begin{matrix}Fy\frown Lz\\ y : p \land \neg p \succ z : p \lor \neg p\end{matrix}}\;{\land}L
\qquad
\dfrac{
\dfrac{
\dfrac{\begin{matrix}u\frown u\\ u : p \succ u : p\end{matrix}}{\begin{matrix}Nv\frown u\\ \succ u : p, v : \neg p\end{matrix}}\;\neg R
}{\begin{matrix}NRz\frown Lz\\ \succ z : p \lor \neg p\end{matrix}}\;{\lor}R
\quad
\dfrac{
\dfrac{\begin{matrix}v\frown v\\ v : \neg p \succ v : \neg p\end{matrix}}{\begin{matrix}v\frown Rz\\ v : \neg p \succ z : p \lor \neg p\end{matrix}}\;{\lor}R
}{\begin{matrix}Sy\frown Rz\\ y : p \land \neg p \succ z : p \lor \neg p\end{matrix}}\;{\land}L
}{\begin{matrix}NRz\frown Lz \quad Sy\frown Rz\\ y : p \land \neg p \succ z : p \lor \neg p\end{matrix}}\;\mathit{Mix}
}{\begin{matrix}Fy\frown Lz \quad NRz\frown Lz \quad Sy\frown Rz\\ y : p \land \neg p \succ z : p \lor \neg p\end{matrix}}\;\mathit{Mix}
$$

By induction on the number of switched pairs.

Take a switched pair at the *adjacent to variables* or *cut points* (peel away unswitched steps if there aren't any).

$$\dfrac{\overset{\pi[x](-)}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi[-](y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az](Lz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}} \supset L$$

Back to Eliminating Cuts: Cuts can be Complicated

$$\dfrac{\dfrac{\overset{\pi[x, u]}{\succ x : A \wedge B, u : A} \qquad \overset{\pi'[x, v]}{\succ x : A \wedge B, v : B}}{\overset{\pi[x, Fx] \quad \pi'[x, Sx]}{\succ x : A \wedge B}}\,{\wedge R} \qquad \dfrac{\overset{\pi''(y, z, x)}{y : A, z : B, x : A \wedge B \succ}}{\overset{\pi''(Fx, Sx, x)}{x : A \wedge B \succ}}\,{\wedge L}}{\overset{\pi[\bullet, F\bullet] \quad \pi'[\bullet, S\bullet] \quad \pi''(F\bullet, S\bullet, \bullet)}{\Sigma \succ \Delta}}\,\mathit{Cut}$$

Cut Reductions

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

- *atomic*: replace each pair $\mathfrak{n}^\frown\bullet$ and $\bullet^\frown\mathfrak{m}$ by $\mathfrak{n}^\frown\mathfrak{m}$.

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- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.

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$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

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- *conjunction*: for each $F\bullet/S\bullet$, add new cut points $\star$ and $*$. For any $\bullet{}^\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n{}^\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.

$$Sz{}^\frown F\bullet \quad Fz{}^\frown S\bullet \quad F\bullet{}^\frown Sx \quad S\bullet{}^\frown Fx \quad Ny{}^\frown\bullet \quad \bullet{}^\frown\nu$$

$Sz{}^\frown\star \quad Fz{}^\frown * \quad \star{}^\frown Sx \quad *{}^\frown Fx$

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

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$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

$$\mathsf{S}z\frown\star \quad \mathsf{F}z\frown* \quad \star\frown\mathsf{S}x \quad *\frown\mathsf{F}x \quad \mathsf{FN}y\frown\mathsf{S}x \quad \mathsf{SN}y\frown\mathsf{F}x \quad \mathsf{N}y\frown\nu$$

Cut Reductions

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

- *atomic*: replace each pair $n\frown\bullet$ and $\bullet\frown m$ by $n\frown m$.
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$$Sz\frown F\bullet \quad Fz\frown S\bullet \quad F\bullet\frown Sx \quad S\bullet\frown Fx \quad Ny\frown\bullet \quad \bullet\frown\nu$$

$$Sz\frown\star \quad Fz\frown * \quad \star\frown Sx \quad *\frown Fx \quad FNy\frown Sx \quad SNy\frown Fx \quad Ny\frown\nu \quad Sz\frown F\nu \quad Fz\frown S\nu$$

Cut Reductions

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

- *atomic*: replace each pair $n \frown \bullet$ and $\bullet \frown m$ by $n \frown m$.
- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet \frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n \frown \bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.
- *negation*: for each $\mathsf{N}\bullet$, add a new cut point $\star$. For any $\bullet \frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n \frown \bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

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- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.
- *negation*: for each $\mathsf{N}\bullet$, add a new cut point $\star$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.
- *disjunction*: for each $\mathsf{L}\bullet/\mathsf{R}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

Cut Reductions

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

- *atomic*: replace each pair $n\frown\bullet$ and $\bullet\frown m$ by $n\frown m$.
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- *negation*: for each $\mathsf{N}\bullet$, add a new cut point $\star$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.
- *disjunction*: for each $\mathsf{L}\bullet/\mathsf{R}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.
- *conditional*: for each $\mathsf{A}\bullet/\mathsf{C}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.


STRONG
NORMALISATION

Any reduction for π terminates in a unique* term π^*

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- There is *some* terminating reduction process.

Any reduction for π terminates in a unique* term π^*

- There is *some* terminating reduction process.

- Proof reduction is confluent.

– If $\pi \rightsquigarrow_{\bullet} \pi'$ and $\pi \rightsquigarrow_{\star} \pi''$ then there is a π''' where $\pi' \rightsquigarrow_{\star} \pi'''$ and $\pi'' \rightsquigarrow_{\bullet} \pi'''$. (This is where α equivalence is required.)


FURTHER WORK

To Do List

- Are these genuine *invariants*? (Can we show that if two derivations have the same term, some set of permutations permute one to the other?)
- Apply these terms to other kinds of proofs (Fitch, Lemmon, tableaux, Hilbert, resolution…)
- *Categories* (The class of *single input*, *single output* terms with composition by defined by *Cut* + *reduction* is a category. What are its properties?)
- Apply terms to theories of warrants.
- Extend beyond propositional logic.

THANK YOU!

`http://consequently.org/presentation/2016/terms-for-classical-sequents-aal-2016`

@consequently on Twitter