# Problem 1

See attached PDF file problem1a.pdf and problem1b.pdf.

Problem 1b: It would probably be easier to use non-angular states.

# Problem 2

A)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\{\tilde{F}_1 e^{i\omega t}\} \\ \mathrm{Re}\{\tilde{F}_2 e^{i\omega t}\} \end{bmatrix}$$

B)

Parameters used:

```
m1=10
m2=19
k1=12
k2=3
k3=m2/2
c1=0.2
c2=0.2
c3=0.2
F1_abs=15
F2_abs=10
F1_phase=0
F2_phase=0
omega=1
```

To make it easier to work with mass-spring, we will substitute in complex terms for x:

$$x = \mathrm{Re}\left\{ \begin{bmatrix} \tilde{x}_1 e^{i\omega t} \\ \check{x}_2 e^{i\omega t} \end{bmatrix} \right\}$$

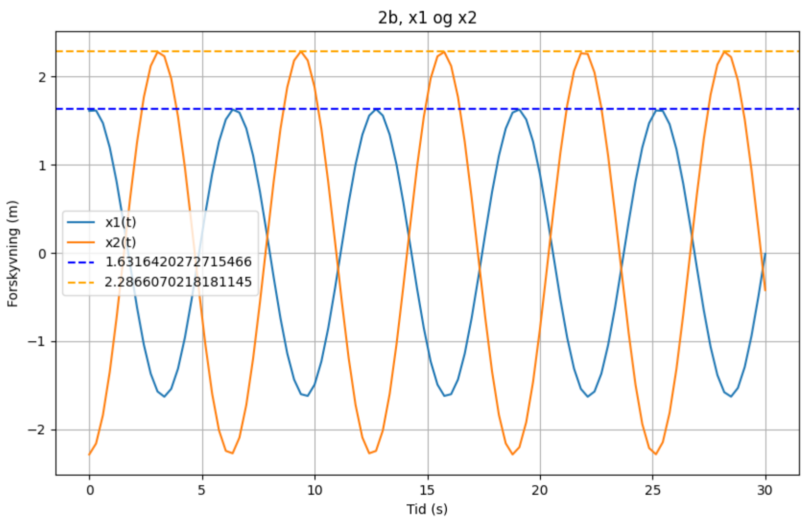
where the first element in the vector is the displacement for mass 1 and the second element is the displacement for mass 2. When substituting the complex x- vector in to the equation of motion, we will get this generalized equation:

$$\left( -\omega^2 \underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M} + i\omega \underbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}_{C} + \underbrace{\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}}_{K} \right) \begin{pmatrix} \check{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{bmatrix} \tilde{F}_1 \\ \check{F}_2 \end{bmatrix}$$

where M is mass matrix, C is damping matrix and K the stiffness matrix. By adding all the matrixes, we will get a 3 term equation, and by applying linear algebra rules we can solve for x, which is the real part of the displacement term of the equation.

$$\underbrace{\begin{bmatrix} -\omega^2 m_1 + i\omega(c_1 + c_2) + k_1 + k_2 & -i\omega c_2 - k_2 \\ -i\omega c_2 - k_2 & -\omega^2 m_2 + i\omega(c_2 + c_3) + k_2 + k_3 \end{bmatrix}}_{\text{Known}} \underbrace{\begin{bmatrix} \hat{x}_1 \\ \check{x}_2 \end{bmatrix}}_{\text{solving for}} = \underbrace{\begin{bmatrix} \check{f}_1 \\ \check{f}_2 \end{bmatrix}}_{\text{Known}}$$

The plot of the displacement of the two masses, varying with $e^{iwt}$, looks like this:




C)

The natural frequency is dependent on the stiffness and the mass of the mass-spring system. Writing the Newton's 2. law for this problem:

$$M\ddot{x} + Kx = 0$$

By substituting in the complex term for x we get this expression:

$$(-w^2 M + K)\tilde{x} = 0$$

$$\xrightarrow{\text{when}} \quad \det\left(\begin{bmatrix} -w^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & w^2 m_2 + k_2 + k_3 \end{bmatrix}\right) = 0$$

We can solve for $w^2$ , eigenfrequencies, by using linear algebra, and by substitution the eigenfrequencies into the complex Newton 2.law we get the mode shapes. The results will be:

```
Eigenfrequencies w_1^2 and w_2^2: [1.55292389 0.60497085]
w_1: 1.2461636675885155
w_2: 0.777798721022634
Mode shapes:
 [[ 0.98479327  0.3178071 ]
 [-0.17373029  0.94815539]]
```

The complex x-vector doesn't have a physical meaning, but can be used to compare the displacement of the two at a given eigenfrequency. If the complex displacements is 1:-1, it means that they have the same displacement amplitude, but different direction of motion. In this case we can see that, referring to the picture below, the displacement of mass 1 at eigenfrequency 1 is 0.322 times the displacement of mass 2, but they will move in the same direction. At eigenfrequency 2 the displacement of mass 2 is 0.184 the displacement of mass 1, and they move in opposite directions.

$$\begin{bmatrix} [0{,}985 & 0{,}318], \\ [-0{,}174 & 0{,}948] \end{bmatrix} \quad \begin{matrix} \text{mode shape of } \omega_1^2 \\ \text{mode shape of } \omega_2^2 \end{matrix}$$

$$\tilde{x}_{11} \cdot 0{,}985 + \tilde{x}_{12} \cdot 0{,}318 = 0$$

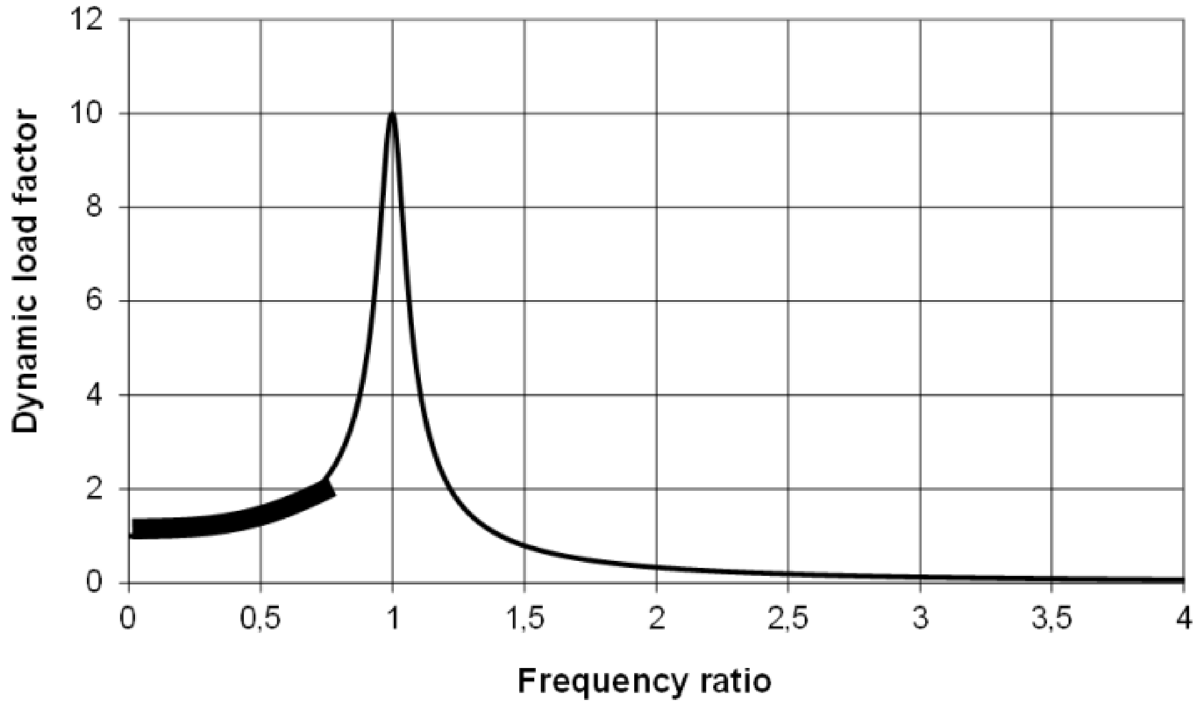
$$\tilde{x}_{11} = -\tilde{x}_{12} \cdot 0{,}322$$

$$\tilde{x}_{21} \cdot (-0{,}174) + \tilde{x}_{22} \cdot 0{,}948 = 0$$

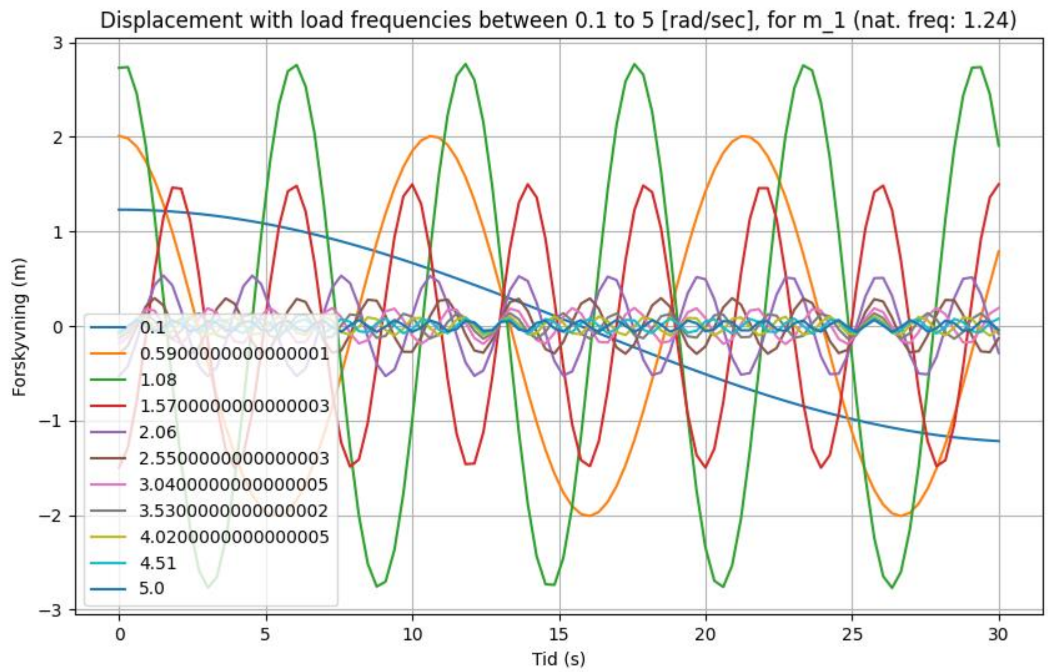
$$\tilde{x}_{22} = \tilde{x}_{21} \cdot 0{,}184$$

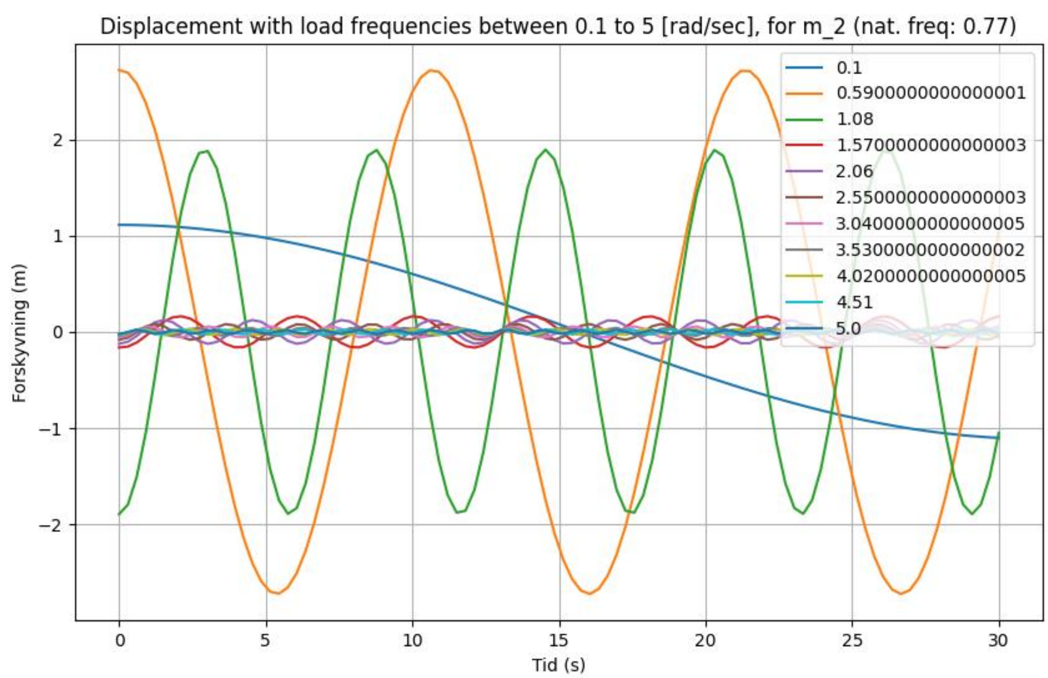
D)

From 2c) we have solved for the eigenfrequencies for the two masses. From earlier in this lecture, we have learned about resonance and how a mass-spring system will displace when it is stiffness dominated (w_force<w_0), inertia dominated (w_force>w_0) or at resonance (w_force≈w_0). Referring to this picture from MarineDynamics2024.pdf:

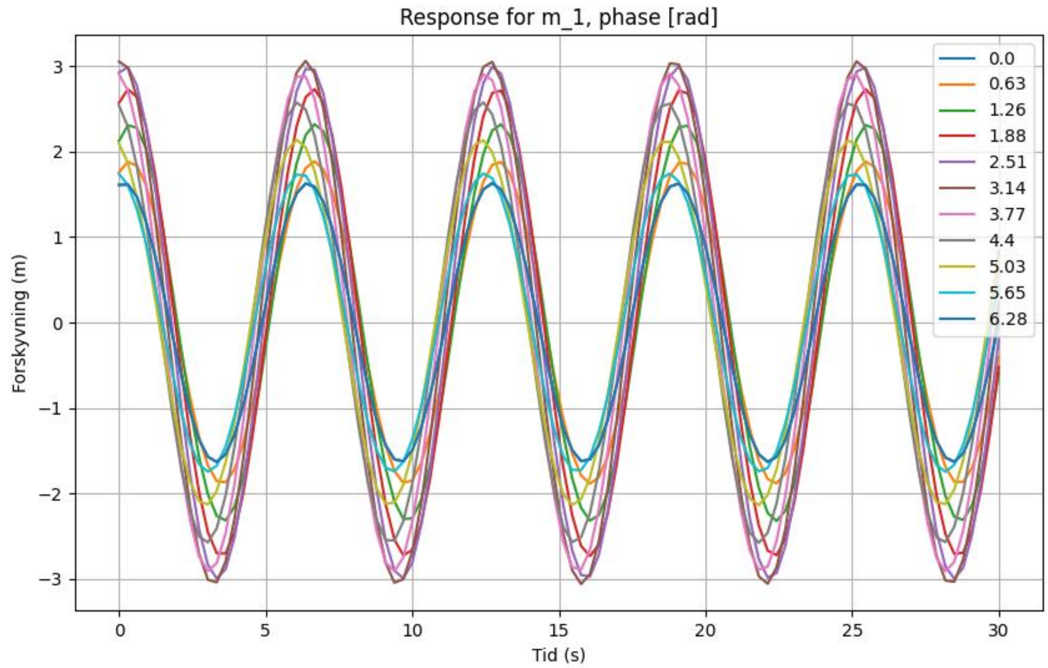


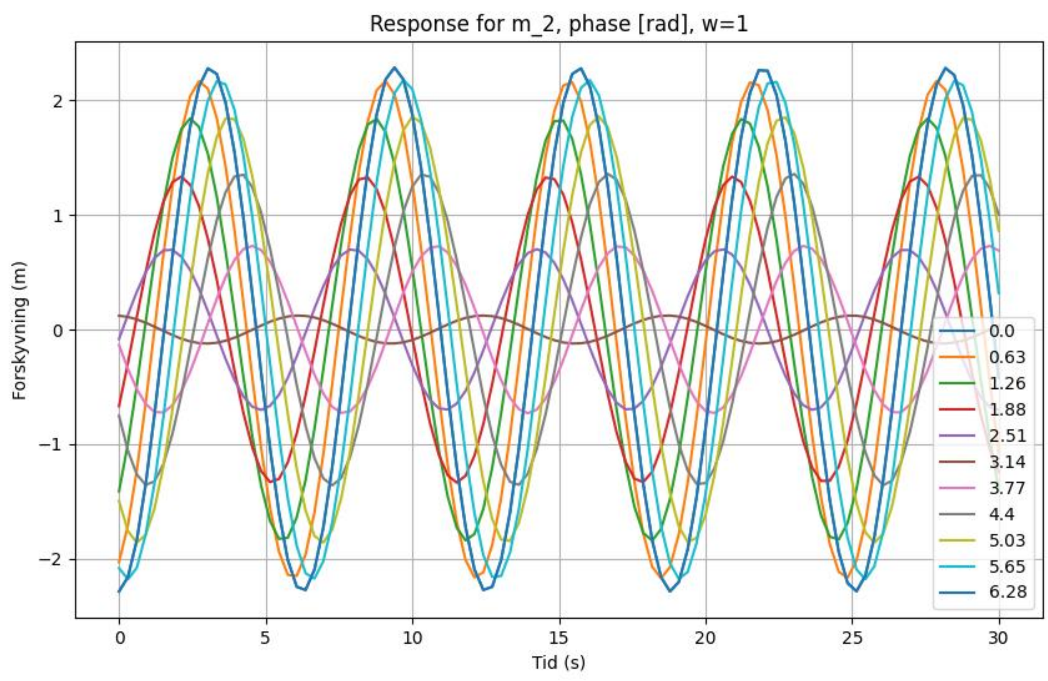

This is also the case in this exercise. When studying the response for m_1, we can see the graphs closest to the eigenfrequency to m_1 is the largest, and those with frequency ratio <0.5 and >1.5 is small in comparison. This is also the case for mass 2.


Displacement with load frequencies between 0.1 to 5 [rad/sec], for m_1 (nat. freq: 1.24)
Forskyvning (m)
Tid (s)
0.1
0.5900000000000001
1.08
1.5700000000000003
2.06
2.5500000000000003
3.0400000000000005
3.5300000000000002
4.0200000000000005
4.51
5.0


Displacement with load frequencies between 0.1 to 5 [rad/sec], for m_2 (nat. freq: 0.77)
Forskyvning (m)
Tid (s)
0.1
0.5900000000000001
1.08
1.5700000000000003
2.06
2.5500000000000003
3.0400000000000005
3.5300000000000002
4.0200000000000005
4.51
5.0

E)




For a set of phases between 0 and 2 π, calculate the displacement for both masses.

When forces are in phase then the resulting motion is greater because they act in the same direction at the same time. The opposite is true when forces are out of phase. This is what we see in plots for mass 2 as 0 and 2π phases are maximized but for mass 1 it is the opposite and not expected.

F)

The response was not as expected, we got overlap between x1 and x1_var and between x2 and x2_var. X1 and x2 were given a constant phase of 0, while x1_var and x2_var were given an array from –180 to 180.