# Automating Code Generation for Higher Inductive Types

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## Abstract

This paper provides a sample of a LATEX document which conforms, somewhat loosely, to the formatting guidelines for ACM SIG Proceedings.

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## 1 Introduction

Type theory, a foundational language of mathematics, is a functional programming language and a theorem prover. We can view a type in type theory as a topological space in homotopy theory or a groupoid in a category theory. Elements of a type corresponds to points in a topological space and elements of identity type corresponds to paths in a topological space. Homotopy type theory extends Martin-Löf's intensional type theory by adding the notion of higher inductive type and univalence axiom. A higher inductive type is a general schema for defining new types using constructors for points and paths. The paths specified by a higher inductive type are realized by equivalences in the universe, and the mapping of paths to equivalences is made possible by univalence axiom [7].

Proof assistants like Coq [6] and Agda [5] does not have built-in mechanisms to support the specification of higher inductive types. However, we can work with higher inductive types in Coq and Agda by postulating them using axioms. In this approach, the elimination rule acting on the higher inductive type does not compute. Dan Licata suggested an alternative method to work with higher inductive types using type abstraction [4]. It involves defining a higher inductive type using an abstract type inside a module. The module consists of a boiler-plate code segment which defines the higher inductive type using a private base type. Inside the module, the recursion and the induction principles acts on the constructors of the private base type. The abstract type is then exported allowing the reduction rules for point constructors to hold definitionally.

Agda is a proof-assistant based on Martin-Löf's intensional type theory. In Agda, a metaprogram, which are programs used to construct other programs, run in a built-in type checking monad. Inspired by Idris's elaborator [3], Agda's reflection library expose an interface to the type checker using the type checking monad which allows for the construction of more interesting metaprograms. The type checking monad provides the interface to the Agda type checker through a set of primitive operations which can be used to retrieve static type information regarding various code segments. The primitive operations can also be used to build code fragments using constructs of abstract syntax tree, and to convert an abstract syntax tree to its concrete syntax.

In this paper, we discuss a tool used to automate the generation of bolier-plate code segments of the higher inductive type and its reduction rules. This tool extensively uses the Agda reflection library and it builds the abstract syntax tree of the code segments with static type informations obtained using the reflection primitives. The generated code is then brought into scope by another top-level reflection primitive. More specifically, we discuss the following contributions.

- We introduce a tool to automate construction of the recursion and the induction principles for inductive types with constructors taking zero arguments, one or more arguments, and the type being defined itself as an argument.
- We discuss the automation of code generation of boiler-plate code for a higher inductive type defined inside a module. The boiler-plate code depends on a base type defined as private inside the module. The constructors of the private base type are not accessible outside the module.
- We discuss the automation of code generation of the elimination and the computation rules for a higher inductive type with point and path constructors. We demonstrate the code generation of the reduction rules, specified as postulates, for the path constructors of the higher inductive type.
- We discuss the automation of code generation for patch theory [2] implementation enriched with patches of encryption.

We automated the code generation of approximately 1500 lines of code of the patch theory implementation with just

70 lines of automation code. The automation code extensively uses the static type information queried from the type checker using the reflection primitives.

## 2 Background

### 2.1 Higher Inductive Type

In homotopy type theory, an element of an *identity type* is used to model the notion of a path in a topological space or a morphism in a groupoid. An element of an identity type $a =_A b$ states that $a$ and $b$ are propositionally equal. In type theory, *propositional equality* is a proof-relevant notion of equality, internal to the theory, expressed by identity types. We also have a proof-irrelevant notion of equality, external to the theory, known as *judgmental equality* or *definitional equality*. Definitional equality is used to express equality by definition. For instance, when we have a function $f : Nat \rightarrow Nat$ defined as $f(x) = x^2$ then the expression $f(3)$ is definitionally equal to $3^2$.

A higher inductive type extends normal inductive type by providing constructors for generating paths in addition to providing constructors for generating points. For example, consider the following higher inductive type definition of circle.

```
1
2  data Circle : Set where
3    base : Circle          -- point constructor
4    loop : base ≡ base     -- path constructor
5
```

In Agda and Coq, we don't have built-in primitives to support the definition of `Circle`. However, the higher inductive types including `Circle` can be specified using Dan Licata's method [4]. According to this method, a higher inductive type is defined using type abstraction. For example, `Circle` is defined using Dan Licata's method as follows.

```
1
2  module Circle where
3    private
4      data S* : Set where
5        base* : S*
6
7    S : Set
8    S = S*
9
10   base : S
11   base = base*
12
13   postulate
14     loop : base ≡ base
15
```

Inside the module `Circle`, the type `S` is defined using a private type `S*`. The constructor `base` is defined using `base*` and the path `loop` is given as a propositional equality. The recursion and the induction principles are defined to pattern match on the constructor `base*` of the type `S*`. The clients of `Circle` will not have access to the constructor `base*` of the private type `S*` as it is not visible outside the module.

The clients only way of access to the constructor is through the non-dependent and dependent elimination rules. The following code gives the non-dependent eliminator or the recursion rule `recS`.

```
1
2  recS : {C : Set} →
3    (cbase : C) →
4    (cloop : cbase ≡ cbase) →
5    S → C
6  recS cbase cloop base* = cbase
7
8  postulate
9    βrecS : {C : Set} →
10     (cbase : C) →
11     (cloop : cbase ≡ cbase) →
12     ap (recS cbase cloop) loop ≡ cloop
13
```

`recS` ignores the path argument and simply computes to the appropriate answer for the point constructor. The computational behavior for the path constructor loop is postulated using reduction rule `βrecS`. In the above code, the function `ap` gives the action of a function $f$ on paths. The following equation gives the type of `ap`.

$$ap_f : (x =_A y) \rightarrow (f(x) =_A f(y)) \tag{1}$$

The following code gives the dependent eliminator or the induction rule `indS` and its computational rules.

```
1
2  indS : {C : S → Set} →
3    (cbase : C base) →
4    (cloop : transport C loop cbase ≡ cbase) →
5    (circle : S) → C circle
6  indS cbase cloop base* = cbase
7
8  postulate
9    βindS : {C : S → Set} →
10     (cbase : C base) →
11     (cloop : transport C loop cbase ≡ cbase) →
12     apd (indS {C} cbase cloop) loop ≡ cloop
13
```

In the above code, the function `apd` gives the action of dependent functions of type $f : \prod_{(x:A)} B(x)$ on paths. The following equation gives the type of `apd`.

$$apd_f : \prod_{p:x=y} (p_*(f(x)) =_{B(y)} f(y)) \tag{2}$$

In (2), the point $p_*(f(x))$ lying in space $B(y)$ can be thought of as an endpoint of a path obtained by lifting the path p from f(x) to a path in the total space $\sum_{(x:A)} B(x) \rightarrow A$. The following equation gives the type of $p*$ which is also known as *transport* [7].

$$transport_p^B : B(x) \rightarrow B(y) \tag{3}$$

where $p : x = y$ for $x, y : A$.

### 2.2 Agda Reflection

Agda has a reflection library that enables compile time metaprogramming. The reflection library provides an interface

to the Agda type checker through a built-in type check-ing TC monad. Using the reflection library and its meta-programming interface, we can convert a code fragment to its corresponding abstract syntax tree, and we can also convert the abstract representation back to its concrete agda code. Internally, agda translates the reflection code to a unique reflection syntax. The core library of Agda is enriched with functions supporting translations from reflection syntax to abstract syntax and from abstract syntax to concrete syntax and vice-versa. In this section, we will see usage examples for some of the primitive operations specified in the Agda reflection library which are used by the tool described in this paper. More information on the reflection library can be found in the Agda documentation [1].

In Agda, macros are used to construct a metaprogram that executes in a `Term` position. Macros are functions of type `t1 → t2 → ... → Term → TC ⊤` defined inside a macro block. During a macro invocation, the last argument of type `Term` is provided by the type checker, and it represents the metavariable instantiated with the result of the macro invocation. Also, the arguments of type `Term` and `Name` are automatically quoted. For example, consider the following macro invocation on an identity function.

```
1
2  macro
3    mc1 : Term → Term → TC ⊤
4    mc1 exp hole = bindTC (quoteTC exp)
5                        (λ exp' → unify hole exp')
6
7  sampleTerm : Term
8  sampleTerm = mc1 (λ (n : Nat) → n)
9
```

The value of `sampleTerm` is as follows:

$$lam\ visible\ (abs\ ”n”\ (var\ 0\ [\,]))\qquad(4)$$

In (4), the constructor *lam* encodes the abstract representa-tion of the identity function. The argument n is represented as explicit and the body of the lambda function refers to n using de-bruijn index.

`Term` is the central type that defines an abstract syntax tree representation for the Agda terms. It is mapped to the AGDATERM built-in. The surface syntax for reflection prim-itives is not fixed, and so they are mapped to their corre-sponding built-ins, using BUILTIN pragma, to inform the type checker about the reflection concept being defined. In the above macro, the `quoteTC` primitive converts a concrete agda syntax into its `Term` representation. `unify` performs the unification of two terms and attempts to solve any meta variables present during the unification process. `bindTC` un-wraps the result of `(quoteTC exp)`, which is captured in `exp'`, and passes it to `unify` for the unification process.

The `unquoteTC` primitive converts the abstract syntax tree representation of `Term` into its concrete Agda syntax. The following code gives an example usage for the `unquoteTC` primitive.

```
1
2  macro
3    mc2 : Term → Term → TC ⊤
4    mc2 exp hole = bindTC (unquoteTC exp)
5                        (λ exp' → unify hole exp')
6
7  sampleSyntax : Nat → Nat
8  sampleSyntax = mc2 (lam visible (abs "n" (var 0 [])))
9
```

Macro `mc2` converts the abstract representation of the identity function given by (1) to its concrete syntax. We also have a type `Name`, an internal identifier mapped to a built-in type QName. In Agda, `Name` literals are created using `quote` primitive.

Metaprograms to create top-level definitions are brought into scope and executed using `unquoteDecl` or `unquoteDef` primitive. A function of a given type is declared using `declareDef`, and is later defined by `defineFun`. A postulate of a given type is defined using `declarePostulate`. Newly defined functions and postulates are brought into scope and executed by `unquoteDecl`. The usage of `declareDef`, `defineFun` and `unquoteDecl` is best illustrated with the fol-lowing example:

```
1
2  plus : Nat → Nat → Nat
3  plus zero b = b
4  plus (suc n) b = suc (plus n b)
5
```

The above function computes addition of two natural num-bers. It can be defined using Agda reflection primitives as follows:

```
1
2   pattern vArg x = arg (arg-info visible relevant) x
3   pattern _`⇒_ a b = pi (vArg a) (abs "_" b)
4   pattern `Nat = def (quote Nat) []
5
6   unquoteDecl plus =
7     bindTC (declareDef (vArg plus) (`Nat `⇒ `Nat `⇒ `Nat)) λ _ →
8       defineFun plus
9         (clause (vArg (con (quote zero) []) ::
10                    vArg (var "y") :: [])
11              (var 0 []) ::
12          clause (vArg (con (quote suc)
13                      (vArg (var "x") :: [])) ::
14             vArg (var "y") :: [])
15          (con (quote suc)
16              (vArg
17                  (def plus
18                      (vArg (var 1 []) ::
19                      vArg (var 0 []) :: [])) :: [])) :: [])
20
```

In the above code, `declareDef` declares the type of `plus` using the constructor `pi`, and `defineFun` uses the declared type information to define the clauses for `plus` using con-structors `con` and `def`. Finally, the function `plus` is brought into scope by `unquoteDecl`. Terms use the de-bruijn index to refer to the declared variables. For example, the variable y declared at line 10 is referenced using a de-bruijn index 0

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at line 11. vArg encodes the visibility and relevance informations of the arguments to the reflection constructs.

In the next section, we will see automation of code generation for normal inductive types. In section 4, we will see automation of code generation for higher inductive types, and in section 5, we will revisit patch theory [2] application and discuss automation of code generation for the higher inductive types defined to model Darc's version control system.

## 3 Code Generation for Inductive Type

An inductive type X is a type that is freely generated by a finite collection of constructors. The constructors of X are functions with zero or more arguments and codomain X. The constructors can also take an element of type X itself as an argument, but only strictly positively. In Agda reflection library, data-type of type Definition stores the constructors of an inductive type as a list of Name. The type of a constructor can be retrieved by giving its Name as an input to the getType primitive. In the following subsections, we will discuss how to use the constructor informations to generate code for the elimination rules of an inductive type.

### 3.1 Non-dependent Eliminator

For an inductive type $W$, with a constructor $g$, the recursion principle says that to define a mapping $f : W \rightarrow P$, it suffices to define the action of $f$ on input $g$. For example, the recursion principle for Nat says that a mapping f : Nat → P can be given by defining the action of f on the constructors zero and (suc x) for x : Nat.

The constructor $g$ can take zero or more arguments including an element of the type $W$. Lets define the type of $g$ as follows.

$$g : (A \rightarrow W) \rightarrow B \rightarrow W \rightarrow W$$

To define the action of $f$ on input $g$, we need a function $d$ of the following type.

$$d : (A \rightarrow W) \rightarrow (A \rightarrow P) \rightarrow B \rightarrow W \rightarrow P \rightarrow P$$

The function $f$ will map the constructor $g$ to $d$. To construct the type of the recursion rule for $W$, we need to build the type of $d$. We can retrieve the static type information of $g$ using reflection primitives, and use that to construct the type of $d$. The constructor pi of type Term encodes the abstract syntax tree (AST) representation of $g$ (fig. 1). We can retrieve and traverse the AST of $g$, and add new type information into it to build a new type representing $d$.

During the traversal of the abstract syntax tree of $g$, when we identify a function f1 with codomain $W$, we add a new function f2, with the same arguments as f1 and codomain $P$, to the tree. For example, in figure 1, a new function is built from the first argument $A \rightarrow W$ by modifying the codomain $W$ to $P$. The new function of type $A \rightarrow P$ is added after the function $A \rightarrow W$ as in figure 2. Constant types require

```
(pi (vArg
   (pi (vArg `A) (abs "_" `W)))      -- f1 : A → W
   (abs "_" (pi (vArg `B)            -- B
   (abs "_" (pi (vArg `W)            -- W
   (abs "_" `W))))))                 -- W
```

**Figure 1.** Abstract syntax tree for constructor $g$

```
(pi (vArg
   (pi (vArg `A) (abs "_" `W)))      -- f1 : A → W
   (abs "_" (pi (vArg
   (pi (vArg `A) (abs "_" `P)))      -- f2 : A → P
   (abs "_" (pi (vArg `B)            -- B
   (abs "_" (pi (vArg `W)            -- W
   (abs "_" (pi (vArg `P)            -- P
   (abs "_" `P)))))))))))            -- P
```

**Figure 2.** Abstract syntax tree for constructor $d$

no modifications. Therefore, we copy $B$ into the new type without any changes. When the type $W$ occurs directly in a non-codomain position, we add the type $P$ next to it. Finally, we change the codomain $W$ of $g$ to $P$ resulting in an abstract syntax tree representation of $d$ (fig. 2). We repeat this process for all the constructors of a given type. In the case of Nat, we construct the output types for zero and (suc x) for x : Nat using the above method.

The parameter and the index of $W$, if present, should be encoded as part of the type of function $f$. We can retrieve information about the parameter and the index of $W$ from its type. The constructors refer to the parameter and the index using de-bruijn indices. During the construction of the output type $d$, we should update the de-bruijn indices accordingly. The constructor data-type contains the count $cp$ of parameters occurring in a defined type. It also encodes the constructors of the type as a list of Name. We can retrieve the index count by finding the difference between $cp$ and length of the constructor list. The parameters are common to all the constructors of a type. But the index values are different for each constructor. So we have to encode unique indices for each constructor. Also, some constructors might not take the same number of indices as the parent type. For example, in the case of Vec, the constructor [] excludes the index Nat from its type. We do not have any reflection primitive to retrieve the index count from a constructor name. A workaround is to pass the index count of each constructor explicitly to the automation tool.

Once we have the type of $d$, we can build the type of the recursion rule for $W$. To encode the mapping $W \rightarrow P$ in the recursion type we need to declare $P$. We can use the constructor agda-sort to introduce the type *(P : Set)*. The

```
(clause
  (vArg (var "P") ::                        -- P
   vArg (var "d") ::                        -- d
   vArg (con (quote g)                      -- g
         (vArg (var "alpha") ::             -- α
          vArg (var "beta") ::              -- β
          vArg (var "omega") :: [])) :: [])  -- ω
  (var 3                                    -- d
   (vArg (var 2 []) ::                      -- α
    vArg (lam visible (abs "a"             -- a
     (def f                                 -- f ∘ α
       (vArg (var 3
        (vArg (var 0 []) :: [])) :: []))) ::
    vArg (var 1 []) ::                      -- β
    vArg (var 0 []) ::                      -- ω
    vArg (def f (vArg (var 0 []) :: []))))))  -- f(ω)
```

**Figure 3.** Clause definition for the computation rule of $W$

type of the recursion rule $f$ is given as follows.

$$f : (P : Set) \rightarrow$$
$$(d : (A \rightarrow W) \rightarrow (A \rightarrow P) \rightarrow B \rightarrow W \rightarrow P \rightarrow P) \rightarrow$$
$$W \rightarrow P$$

The above type is declared using declareDef. We can build the computation rule representing the action of function $f$ on $d$ using clause (fig. 3). The first argument to clause encodes variables corresponding to the above type, and it also includes the abstract representation of $g$ on which the pattern matching should occur. The second argument to clause, which is of type Term, refers to the variables in the first argument using de-bruijn indices, and it encodes the output of the action of function $f$ on $d$. The constructor var in Pattern is used to introduce new variables in the clause definition. The type Pattern also has another constructor con used to represent the pattern matching term $g$. The type Term has similar constructors var and con, but with different types, used to encode the output of the recursion rule. Given $\alpha : A \rightarrow W$, $\beta : B$ and $\omega : W$, the computation rule corresponding to the above type is given as follows.

$$f(P, d, g(\alpha, \beta, \omega)) \equiv d(\alpha, f \circ \alpha, \beta, \omega, f(\omega))$$

The second argument $(f \circ \alpha)$ to $d$ is the composite of function $f$ and $\alpha$. We can build a composite function inside a lambda using the constructor lam. The arguments to lam are referenced using de-bruijn indices inside the lambda body. So, the de-bruijn indices for referring variables outside the lambda body are updated accordingly. The clause definition, which evaluates to the above computation rule of $W$, is given in figure 3. The de-bruijn index reference increments right to left starting from the last argument $\omega$. Inside the lambda body, the reference 0 refers to the lambda argument a, and the index references to the variables outside the lambda body starts from 1 and increments towards the left. The above

clause definition is defined using defineFun primitive, and the function $f$ is brought into scope by unquotedecl.

In the automation tool, generateRec interface is used to generate the resursion rule $f$ for the type $W$. The implementation of generateRec is given as follows.

```
1
2  generateRec : Arg Name → Name → List Nat → TC ⊤
3  generateRec (arg i f) t indLs =
4    bindTC (getConstructors t) λ cns →
5    bindTC (getLength cns) λ lcons →
6    bindTC (getClause lcons zero t f indLs cns) λ cls →
7    bindTC (getType t) λ RTy →
8    bindTC (getRtype t indLs zero RTy) λ funType →
9    bindTC (declareDef (arg i f) funType) λ _ →
10   (defineFun f cls)
11
```

generateRec uses getClause and getRtype to build the computation and elimination rules respectively. It takes three arguments which consists of the function name to be defined as an element of type Arg Name, the quoted Name of the type $W$ and a list containing the index count of the individual constructors. generateRec can be used to automate the generation of recursion rules for the inductive types with point constructors. It can be used with the constructors of inductive types that takes no arguments (eg. Bool), one or multiple arguments (eg. coproduct, cartesian product) and the constructors that takes argument from the inductive type being defined (eg. Nat, List, Vec). The recursion rule generated by generateRec is brought into scope using unquoteDecl as follows.

```
unquoteDecl f = generateRec (vArg f) (quote W) (0 :: [])
```

The third argument to generateRec is a list consisting of the index count for the constructors. Since the constructor $g$ does not have any index, the count is given as 0. It is required to pass the index count for each constructor explicitly as the Agda reflection library does not have built-in primitives to retrieve the index value.

### 3.2 Dependent Eliminator

The dependent eliminator or the induction principle is used to define the behavior of a mapping $f'$ when the output type is dependent on the input element to $f'$. For the inductive type $W$, the induction principle says that to define a mapping $f' : (w : W) \rightarrow P(w)$, it suffices to define the action of $f'$ on $g$. To define the action of $f'$ on $g$, we need an element of the following type.

$$d' : \prod_{\alpha : A \rightarrow W} (\prod_{a : A} P(\alpha(a))) \rightarrow \prod_{\beta : B} \prod_{\omega : W} P(\omega) \rightarrow P(g(\alpha, \beta, \omega))$$

In the first argument of $d'$, the type $P$ depends on the action of the function $\alpha$ on input $a : A$. In the second argument, it depends on the input element $\omega : W$. The output of function $d'$ depends on the action of constructor $g$ on inputs $\alpha$, $\beta$, and $\omega$.

```
 (pi (vArg
    (pi (vArg `A) (abs "_" `W)))        -- α : A → W
    (abs "_" (pi (vArg                  -- A → P(α(a))
    (pi (vArg `A)                       -- a : A
    (abs "_" (var 2)                    -- P(α(a))
    (vArg (var 1                        -- α ref
    (vArg (var 0 []) :: [])) :: []))))) -- a ref
    (abs "_" (pi (vArg `B)              -- β : B
    (abs "_" (pi (vArg `W)              -- ω : W
    (abs "_" (pi (vArg (var 4)          -- P(ω)
    (vArg (var 0 []) :: [])))           -- ω ref
    (abs "_" (var 5)                    -- P(g i β ω)
    (vArg (con (quote g)                -- g ref
    (vArg (var 4 []) ::                 -- α ref
     vArg (var 2 []) ::                 -- β ref
     vArg (var 1 []) :: [])) :: [])))))))))))))) -- ω ref
```

**Figure 4.** Abstract syntax tree for constructor $d'$

To build the dependent eliminator, we need the type of the function $d'$. We can construct the abstract syntax tree of $d'$ using the static type information obtained from $g$. To construct $d'$, during the traversal of the abstract syntax tree of $g$, when we identify a function $\alpha$ with codomain $W$, we add a new function with the same argument A as in $\alpha$ and codomain $P$, which depends on the action of $\alpha$ on (a : A). We copy the constant $B$ directly without any changes as in the case of the non-dependant eliminator. When we see $(\omega : W)$ in a non-codomain position, we add the type $P$, which depends on the element $\omega$, next to it. Finally, the output type $W$ of the constructor $g$ is changed to $P$, which depends on the action of the constructor $g$ on inputs $\alpha$, $\beta$ and $\omega$ (fig. 4).

We can construct the type of the induction principle $f'$ using $d'$. The type $P$ in the mapping $f'$ depends on the element of the input type $W$. The following equation gives the type of the induction principle $f'$.

$$
\begin{aligned}
f' : (P : W &\to Set) \to \\
(d' : \prod_{\alpha:A\to W}(\prod_{a:A} P(\alpha(a))) &\to \prod_{\beta:B}\prod_{\omega:W} P(\omega) \to \\
P(g(\alpha,\beta,\omega))) &\to \\
(w : W) &\to P(w)
\end{aligned}
$$

The computation rule corresponding to the above type is the same as the computation rule of the recursion principle of $W$ except that the function $f$ and $d$ are changed to $f'$ and $d'$ respectively. It is constructed using clause definitions following the same approach as the recursion principle. The computation rule corresponding to $f'$ is given as follows.

$$f'(P, d', g(\alpha, \beta, \omega)) \equiv d'(\alpha, f' \circ \alpha, \beta, \omega, f'(\omega))$$

We can automate the generation of the induction rule $f'$ and its corresponding computation rules for the type $W$ using generateInd interface. The following code gives the implementation of generateInd.

```
1
2  generateInd : Arg Name → Name → List Nat → TC ⊤
3  generateInd (arg i f) t indLs =
4    bindTC (getIndex t indLs) λ indLs' →
5    bindTC (getConstructors t) λ cns →
6    bindTC (getLength cns) λ lcons →
7    bindTC (getClauseDep lcons zero t f indLs' cns) λ cls →
8    bindTC (getType t) λ RTy →
9    bindTC (getRtypeInd t zero indLs' RTy) λ funType →
10   bindTC (declareDef (arg i f) funType) λ _ →
11   (defineFun f cls)
12
```

generateInd uses getClauseDep to generate the clause definitions representing the computation rule of the type $W$. The abstract representation of the type of $W$ is provided by getRtypeInd. $f'$ generated by generateInd is brought into scope by unquoteDecl as follows.

```
unquoteDecl f' = generateInd (vArg f') (quote W) []
```

We pass an empty list to generateInd as the type $W$ has no index. We can also pass an empty list if all the constructors of a type has the same number of index as the parent type. But if any one constructor has an index count different from the index count of the parent type, then we have to explicitly pass the index count of all the constructors.

## 4 Code Generation for Higher Inductive Type

In Agda, there are no built-in primitives to support the definition of higher inductive types. However, we can still define a higher inductive type with a base type using Dan Licata's [4] method as discussed in section 2.1. In this section, we discuss the automation of code generation for the boiler-plate code segments defining the higher inductive type. We also describe how to automate the code generation for the elimination and the computation rules of the higher inductive type using static type information obtained from the base type.

### 4.1 Higher Inductive Type Definition

In Agda, we define an inductive type using data keyword. The data literal characterize a data type by declaring its type and specifying its constructors. Consider the following generic form for the definition of an inductive type $W$ with constructors $g_1 \ldots g_n$.

$$
\begin{aligned}
data \;\; W^* \;\; &(x_1 : P_1) \ldots (x_n : P_n) : Q_1 \to \ldots \to Q_n \to Set\; \ell \; where \\
g_1^* : &\{i_1 : Q_1\} \ldots \{i_n : Q_n\} \to Type_1 \to W^* x_1 \ldots x_n\, i_1 \ldots i_n \\
&\vdots \\
g_n^* : &\{j_1 : Q_1\} \ldots \{j_n : Q_n\} \to Type_n \to W^* x_1 \ldots x_n\, j_1 \ldots j_n
\end{aligned}
$$

The parameters $(x_1 : P_1) \ldots (x_n : P_n)$ are common to all the constructors, and the type of each constructor implicitly encodes the parameter references. However, the indices

are different for each constructor. So, the type of each constructor explicitly exhibits the index declaration. During the construction of a higher inductive type, we have to represent the parameters and the indices as explicit arguments in the type of the constructors. We define a higher inductive type $W$ as a top-level definition using a base type $W^*$ similar to the module `Circle` in section 2.1. The reflection type `Definition` provides us the type and the constructors of the base type $W^*$. We copy the type of $W^*$ to $W$ and for the constructors $g_1 \ldots g_n$ of $W$, we traverse the abstract representation of the type of $g_1^* \ldots g_n^*$ respectively replacing the occurrence of $W^*$ to $W$ in every strict positive position. Consider a constructor $g_i^*$ having the following type.

$$g_i^* : (A \rightarrow W^*) \rightarrow (B \rightarrow W^*) \rightarrow C \rightarrow W^* \rightarrow W^*$$

We built the type of $g_i$ by traversing the abstract syntax tree of $g_i^*$ and replacing the base type $W^*$ with the higher inductive type $W$. The abstract syntax tree of $g_i^*$ incorporates the type of the parameters and the indices if present. We have to retain the parameters and the indices explicitly during the construction of $g_i$. The following equation represents the type of the constructor $g_i$.

$$g_i : (A \rightarrow W) \rightarrow (B \rightarrow W) \rightarrow C \rightarrow W \rightarrow W$$

We explicitly pass the type of the path constructors to the automation tool. The higher inductive type definition of `Circle` in section 2.1 represents the path constructors as propositional equalities. The automation tool takes the path types as input and declares them as propositional equalities using the reflection primitive `declarePostulate`. We introduce a new data type `ArgPath` to input the path types to the automation tool.

$$data\ ArgPath\ \{\ell_1\} : Set\ (lsuc\ \ell_1)\ where$$
$$argPath : Set\ \ell_1 \rightarrow ArgPath$$

The constructor $argPath$ takes the type of a path constructor as input. We define the generic form of a higher inductive type as follows.

$$data\text{-}hit\ (quote\ W^*)\ W$$
$$\quad Wpoints\quad (g_1 :: \ldots :: g_n :: [])$$
$$\quad Wpaths\quad (p_1 :: \ldots :: p_n :: [])$$
$$\quad (argPath$$
$$\quad\quad (\{x_1 : P_1\} \rightarrow \ldots \rightarrow \{x_n : P_n\} \rightarrow$$
$$\quad\quad \{i_1 : Q_1\} \rightarrow \ldots \rightarrow \{i_n : Q_n\} \rightarrow Type_1 \rightarrow$$
$$\quad\quad (g_i\{x_1\}\ldots\{x_n\}\{i_1\}\ldots\{i_n\}\ldots) \equiv (g_j \ldots)) ::$$
$$\vdots$$
$$\quad argPath$$
$$\quad\quad (\{x_1 : P_1\} \rightarrow \ldots \rightarrow \{x_n : P_n\} \rightarrow$$
$$\quad\quad \{j_1 : Q_1\} \rightarrow \ldots \rightarrow \{j_n : Q_n\} \rightarrow Type_n \rightarrow$$
$$\quad\quad (g_i\{x_1\}\ldots\{x_n\}\{j_1\}\ldots\{j_n\}\ldots) \equiv (g_j \ldots)) :: [])$$

We define holders $Wpoints$ for point constructors and $Wpaths$ for path constructors as part of the higher inductive type definition of $W$. We cannot retrieve the constructors of

the higher inductive type $W$ using `Definition`. Therefore, $Wpoints$ and $Wpaths$ act as the only references for the constructors of $W$. The elements of the $argPath$ list represent the type of the path constructors $p_1 \ldots p_n$ respectively. We explicitly include the parameter references $\{x_1 : P_1\} \ldots \{x_n : P_n\}$ and the index references $\{k_1 : Q_1\} \ldots \{k_n : Q_n\}$ in the type of the arguments to $argPath$. The points $g_1 \ldots g_n$ are not in scope when used in the identity type passed to $argPath$. The automation tool uses the base type constructors $g_1^* \ldots g_n^*$ as dummy arguments in the place of $g_1 \ldots g_n$ respectively. The automation tool implements the interface `data-hit` as follows.

```
1
2   data-hit : ∀{ℓ₁} (baseType : Name) → (indType : Name) →
3     (pointHolder : Name) → (lcons : List Name) →
4     (pathHolder : Name) → (lpaths : List Name) →
5     (lpathTypes : (List (ArgPath {ℓ₁}))) → TC ⊤
6   data-hit base ind pointH lcons pathH lpaths lpathTypes =
7     bindTC (defineHindType base ind) λ _ →
8     bindTC (getConstructors base) λ lcons' →
9     bindTC (defineHitCons base ind lcons' lcons) λ _ →
10    bindTC (getPathTypes base ind lcons' lcons lpathTypes) λ lp →
11    bindTC (defineHitPathCons lpaths lp) λ _ →
12    bindTC (definePointHolder pointHolder lcons) λ _ →
13    definePathHolder pathHolder lpaths
14
```
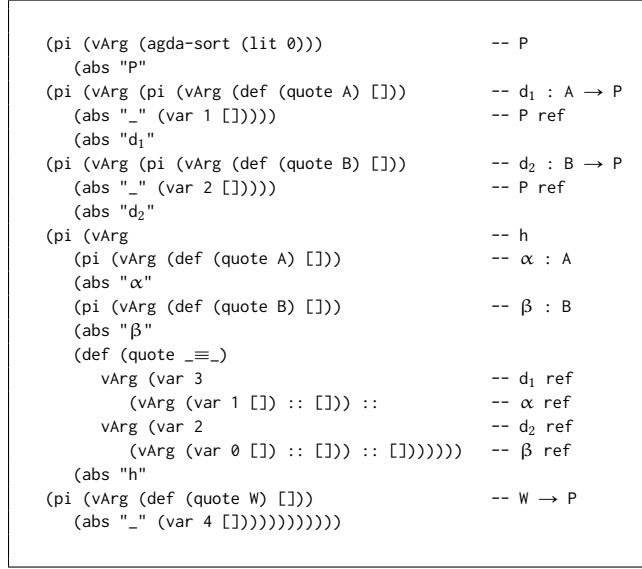
The higher inductive type $W$, the points $g_1 \ldots g_n$, the paths $p_1 \ldots p_n$, and the holders $Wpoints$ and $Wpaths$ are brought into scope by `unquoteDecl`. In the above implementation of `data-hit`, `defineHindType` defines the higher inductive type as a top-level definition using the base type. `defineHitCons` specifies the point constructors of the higher inductive type using the type information obtained from the constructors of the base type, and `defineHitPathCons` builds the paths constructors of the higher inductive type using the `argPath` list. The following code automates the generation of the higher inductive type definition for `Circle` given in section 2.1.

```
unquoteDecl S Spoints base Spaths loop =
  data-hit (quote S*) S
    Spoints (base :: []) -- point constructors
    Spaths (loop :: [])  -- path constructors
    (argPath (base* ≡ base*) :: []) -- base replaces base*
```

The identity type input (base* ≡ base*) to argPath represents the type of the path `loop`, and it uses the inductive type constructor `base*` as a dummy argument in the place of the higher inductive type constructor `base`. The constructor `base` comes into scope only during the execution of `unquoteDecl`, and so cannot be used in the identity type reference in `argPath`. We use the constructor `base*` of type `S*` as dummy argument because the type of `base*` is similar to `base`, and has the same references for the common arguments. The automation tool traverses the abstract syntax tree of `loop` and replaces the occurrences of `base*` with `base`.

```
  (pi (vArg (agda-sort (lit 0)))          -- P
    (abs "P"
  (pi (vArg (pi (vArg (def (quote A) []))  -- d₁ : A → P
    (abs "_" (var 1 []))))                 -- P ref
    (abs "d₁"
  (pi (vArg (pi (vArg (def (quote B) []))  -- d₂ : B → P
    (abs "_" (var 2 []))))                 -- P ref
    (abs "d₂"
  (pi (vArg                               -- h
    (pi (vArg (def (quote A) []))         -- α : A
    (abs "α"
    (pi (vArg (def (quote B) []))         -- β : B
    (abs "β"
    (def (quote _≡_)
      vArg (var 3                         -- d₁ ref
        (vArg (var 1 []) :: [])) ::       -- α ref
      vArg (var 2                         -- d₂ ref
        (vArg (var 0 []) :: [])) :: []))))))) -- β ref
    (abs "h"
  (pi (vArg (def (quote W) []))           -- W → P
    (abs "_" (var 4 []))))))))))))))
```

**Figure 5.** Abstract syntax tree for constructor $f$

### 4.2  Non-dependent Eliminator

Non-dependent eliminator or the recursion principle of a higher inductive type $W$ maps the points and paths of $W$ to an output type $P$. Let $g_1 : A \to W$ and $g_2 : B \to W$ be the point constructors, and $l : (\alpha : A) \to (\beta : B) \to (g_1\,\alpha) \equiv (g_2\,\beta)$ be the path constructor of $W$. To define a mapping $f : W \to P$, we need points $d_1 : A \to P$ and $d_2 : B \to P$ and a path $h : (\alpha : A) \to (\beta : B) \to (d_1\,\alpha) \equiv (d_2\,\beta)$ in the space $P$. The path constructor $l$ can take arguments of any type including the higher inductive type $W$ but for simplicity we will consider only arguments of constant type.

To construct the recursion principle $f$, we need to build the type of points $d1, d2$ and path $h$. The type of $d1$ and $d2$ are built from the abstract syntax tree of points $g1$ and $g2$ respectively using the approach described in section (3.1). We build the type of $h$ by traversing the abstract syntax tree of $l$. During the traversal, we replace the points $g_1$ and $g_2$, which are arguments to the identity type, in the co-domain of the path $l$ by the points $d_1$ and $d_2$ respectively. In figure (5), the arguments of the identity type in the abstract syntax tree of $h$ references the points $d_1$ and $d_2$ using de-bruijn index 3 and 2 respectively. We copy the other two arguments of $l$, which are of constant type $A$ and $B$, without any changes into the type of $h$.

The recursion rule $f$ corresponding to figure (5) is given by the following equation.

$$f : (P : Set) \to (d_1 : A \to P) \to (d_2 : B \to P) \to$$
$$(h : (\alpha : A) \to (\beta : B) \to (d_1\,\alpha) \equiv (d_2\,\beta)) \to$$
$$W \to P$$

The computation rules for constructors $g_1$ and $g_2$ give the action of eliminator $f$ on input $g_1(\alpha)$ and $g_2(\beta)$ respectively.

The computation rules say that the function $f$ maps the points $g_1(\alpha)$ and $g_2(\beta)$ to the points $d_1(\alpha)$ and $d_2(\beta)$ respectively in the output type $P$. We specify the clause definition to build the computation rules for $g_1$ and $g_2$ in the same way as for a regular inductive type (sec. 3.1) except that the clause definition for the higher inductive type $W$ has an additional reference variable to the path $h$. The following definitional equalities give the computation rules for $g_1$ and $g_2$.

$$f(P, d_1, d_2, h, g_1(\alpha)) \equiv d_1(\alpha)$$
$$f(P, d_1, d_2, h, g_2(\beta)) \equiv d_2(\beta)$$

The computation rule for the path constructor $l$ defines the action of function $f$ on the path $(l\,\alpha\,\beta)$. In type theory, the functions are functorial, and they preserve the path structures in their mapping to the output type [7]. The function $ap_f$ (1) defines the action of the non-dependent eliminator $f$ on the path $l$. Using (1), we can build the computation rule for the path $l$ as follows.

$$\beta_f : (P : Set) \to (d_1 : A \to P) \to (d_2 : B \to P) \to$$
$$(h : (\alpha : A) \to (\beta : B) \to (d_1\,\alpha) \equiv (d_2\,\beta)) \to$$
$$\{\alpha : A\} \to \{\beta : B\} \to$$
$$ap_{(\lambda x \to f(P, d_1, d_2, h, x))}(l\,\alpha\,\beta) \equiv (h\,\alpha\,\beta)$$

The computation rule $\beta_f$ exists only as propositional equality. We build the type of $\beta_f$ using the same approach as for the recursion rule $f$. The type of $f$ and $\beta_f$ is identical except for the mapping $W \to P$ in $f$ which is replaced by the term representing the action of function $f$ on the path $(l\,\alpha\,\beta)$. The function $ap$ applies $f$, which is nested inside a lambda function, on the arguments $(g_1\,\alpha)$ and $(g_2\,\beta)$ of the path $(l\,\alpha\,\beta)$. We use the constructor lam of Term to introduce the lambda function with argument x (fig. 6). Inside the lambda body, the argument x gets a de-bruijn index reference of zero and we update the remaining de-bruijn indices accordingly. The application of function $f$ to the path $(l\,\alpha\,\beta)$ substitutes the points $(g_1\,\alpha)$ and $(g_2\,\beta)$ for the lambda argument x, and it evaluates to the path $(h\,\alpha\,\beta)$ in the output type $P$. We implement the generateRecHit interface as follows.

```
1
2  generateRecHit : Arg Name → List (Arg Name) →
3    (baseType : Name) → (indexList : List Nat) →
4    (baseRec : Name) → (indType : Name) →
5    (points : List Name) → (paths : List Name) → TC ⊤
6  generateRecHit (arg i f) argD base il rec ind points paths =
7    bindTC (getConstructors base) λ lcons →
8    bindTC (getLength points) λ lpoints →
9    bindTC (getLength paths) λ lpaths →
10   bindTC (getPathClause lpoints lpaths rec) λ clause →
11   bindTC (getType base) λ RTy →
12   bindTC (getRtypePath base ind rec il paths zero RTy) λ fty →
13   bindTC (declareDef (arg i f) fty) λ _ →
14   bindTC (defineFun f clause) λ _ →
15   (generateβRecHit argD base il rec ind f points paths)
16
```

generateRecHit takes the base type recursion rule as input and uses that to map the points $g_1(\alpha)$ and $g_2(\beta)$ to

```
(def (quote _≡_)
   (vArg (def (quote ap)                      -- ap
     (vArg (lam visible (abs "x"              -- x
       (def (quote f)                         -- f
         (vArg (var 6 []) ::                   -- P ref
          vArg (var 5 []) ::                   -- d₁ ref
          vArg (var 4 []) ::                   -- d₂ ref
          vArg (var 3 []) ::                   -- h ref
          vArg (var 0 []) :: [])))) ::         -- x ref
       vArg (def (quote l)                     -- l
         (vArg (var 1 []) ::                   -- α ref
          vArg (var 0 []) :: [])) :: [])) ::   -- β ref
       vArg (var 2                             -- h ref
         (vArg (var 1 []) ::                   -- α ref
          vArg (var 0 []) :: [])) :: []))       -- β ref
```

**Figure 6.** Abstract syntax tree for the term representing the action of function $f$ on path $l$

$d_1(\alpha)$ and $d_2(\beta)$ respectively in the abstract syntax tree of the path $(l\,\alpha\,\beta)$. The second argument argD is a list of terms representing the computation rules for the path constructors. The generateβRecHit interface takes argD as input and builds the computation rule for the path constructor $l$. Other inputs to generateRecHit are the point and path holders declared during the higher inductive type definition of $W$. The following automation code generates the recursion rule and the computation rules for Circle.

```
unquoteDecl recS₁* = generateRec (vArg recS₁*)
                               (quote S₁*) (0 :: [])

unquoteDecl recS₁ βrecS₁ = generateRecHit (vArg recS₁)
                            ((vArg βrecS₁) :: [])
                            (quote S₁*) (0 :: [])
                            (quote recS₁*)
                            (quote S₁) S₁points S₁paths
```

The term βrecS₁ represents the computation rule for the path constructor loop and is brought into scope by unquoteDecl.

### 4.3 Dependent Eliminator

Dependent eliminator or the induction principle of a higher inductive type $W$ is a dependent function that maps an element $\omega$ of $W$ to an output type $P\,\omega$. For the type $W$ with point constructors $g_1 : A \to W$ and $g_2 : B \to W$ and path constructor $l : (\alpha \to A) \to (\beta \to B) \to (g_1\,\alpha) \equiv (g_2\,\beta)$, to define a mapping $f' : (\omega : W) \to P\,\omega$, we need $d_1 : (\alpha : A) \to P\,\alpha$, $d_2 : (\beta : B) \to P\,\beta$, and $h : (\alpha \to A) \to (\beta : B) \to$ $transport\,P\,(l\,\alpha\,\beta)\,(d_1\,\alpha) \equiv (d_2\,\beta)$, where $(h\,\alpha\,\beta)$ is a heterogeneous path transported over $(l\,\alpha\,\beta)$ lying in the space $P\,\beta$. Equation (3) gives the type of $transport$.

In the abstract syntax tree of $f'$ (fig. 7), the type of constructors $d_1$ and $d_2$ are built from the type of $g_1$ and $g_2$ respectively using the same approach as in section (3.2). When building the type of $d_1$, we copy the constant argument

```
(pi (vArg (`W `⇒ agda-sort (lit 0)))
   (abs "P"
 (pi (vArg
    (pi (vArg `A)
    (abs "α"
    (var 1 (vArg (con (quote g₁)
            (vArg (var 0 []) :: [])) :: []))))))
    (abs "d₁"
 (pi (vArg
    (pi (vArg `B)
    (abs "β"
    (var 2 (vArg (con (quote g₂)
            (vArg (var 0 []) :: [])) :: []))))))
    (abs "d₂"
 (pi (vArg
    (pi (vArg `A)
    (abs "α"
    (pi (vArg `B)
    (abs "β"
    (def (quote _≡_)
      (vArg (def (quote transport)
        (vArg (var 4 []) ::
         vArg (def (quote l) (vArg (var 1 []) ::
         vArg (var 0 []) :: [])) ::
         vArg (var 3 (vArg (var 1 []) :: [])) :: [])) ::
      vArg (var 2 (vArg (var 0 []) :: [])) :: []))))))))
    (abs "h"
 (pi (vArg `W)
    (abs "ω" (var 4 (vArg (var 0 []) :: [])))))))))))))))
```

**Figure 7.** Abstract syntax tree of function $f'$

$A$ directly into the abstract syntax tree of $d_1$, and for the codomain P, which depends on the action of constructor $g_1$ on the constant argument $\alpha : A$, we pass the de-bruijn index reference of $\alpha$ as an argument to the constructor $g_1$ built using the reflection construct con. Similarly, for $d_2$, we copy the type $B$ directly into the abstract syntax tree of $d_2$ and pass the de-bruijn index of $\beta : B$ to the constructor $g_2$ in the co-domain. We build the type of path $h$ by traversing the abstract syntax tree of $l$ and adding relevant type information into it. We copy the constant arguments of type $A$ and $B$ without modification. For the codomain of $l$, which is the identity type $(g_1\,\alpha) \equiv (g_2\,\beta)$, we insert the quoted name of function $transport$ with arguments $P$, the path $(l\,\alpha\,\beta)$ and $(d_1\,\alpha)$. We apply the base eliminator to the arguments of the path $(l\,\alpha\,\beta)$ to form the elements $(d_1\,\alpha)$ and $(d_2\,\beta)$. The following declaration gives the type of $f'$ corresponding to figure (7).

$$f' : (P : W \to Set) \to$$
$$(d_1 : (\alpha : A) \to P(g_1\,\alpha)) \to (d_2 : (\beta : B) \to P(g_2\,\beta)) \to$$
$$(h : (\alpha : A) \to (\beta : B) \to$$
$$transport\,P\,(l\,\alpha\,\beta)(d_1\,\alpha) \equiv (d_2\,\beta)) \to$$
$$(\omega : W) \to P(\omega)$$

We build the computation rules corresponding to the mapping $f'$ for the point constructors $g_1$ and $g_2$ using the same

```
(def (quote _≡_)
  (vArg (def (quote apd)                           -- apd
    (vArg (lam visible (abs "x"                     -- x
      (def (quote f')                               -- f
        (vArg (var 6 []) ::                         -- P ref
         vArg (var 5 []) ::                         -- d₁ ref
         vArg (var 4 []) ::                         -- d₂ ref
         vArg (var 3 []) ::                         -- h ref
         vArg (var 0 []) :: []))))) ::              -- x ref
     vArg (def (quote l)                            -- l
      (vArg (var 1 []) ::                           -- α ref
       vArg (var 0 []) :: [])) :: [])) ::           -- β ref
   vArg (var 2                                      -- h ref
    (vArg (var 1 []) ::                             -- α ref
     vArg (var 0 []) :: [])) :: []))                -- β ref
```
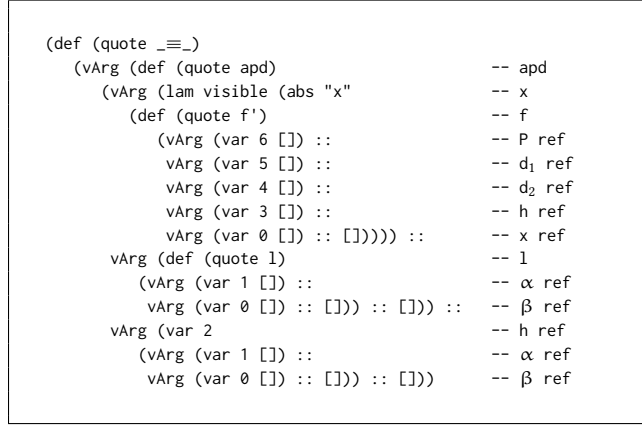
**Figure 8.** Abstract syntax tree for the term representing the action of dependent function $f'$ on path $l$

approach as for the non-dependent eliminator $f$. The following equations give the computation rules for the point constructors $g_1$ and $g_2$.

$$f'(P, d_1, d_2, h, g_1(\alpha)) \equiv d_1(\alpha)$$
$$f'(P, d_1, d_2, h, g_2(\beta)) \equiv d_2(\beta)$$

The computation rule $\beta_{f'}$ for the heterogeneous path $h$ depends on *apd* (2) which gives the action of dependent function $f'$ on the path $h$. We build the type of $\beta_{f'}$ using (2) as follows.

$$\beta_{f'} : (P : W \to Set) \to$$
$$(d_1 : (\alpha : A) \to P(g_1\,\alpha)) \to (d_2 : (\beta : B) \to P(g_2\,\beta)) \to$$
$$(h : (\alpha : A) \to (\beta : B) \to$$
$$\quad transport\, P\, (l\,\alpha\,\beta)(d_1\,\alpha) \equiv (d_2\,\beta)) \to$$
$$apd_{(\lambda x \to f'(P, d_1, d_2, h, x))}(l\,\alpha\,\beta) \equiv (h\,\alpha\,\beta)$$

We build the abstract syntax tree for the computation rule $\beta_{f'}$ in the same way as for induction principle $f'$. In the codomain of $\beta_{f'}$ (fig. 8), which is an identity type, we insert the abstract representation of the function *apd* with arguments $f'$, nested inside a lambda body, and the path $(l\,\alpha\,\beta)$. We pass the path $(h\,\alpha\,\beta)$, built using de-bruijn indices, as the second argument to the identity type. The following code gives the implementation of the generateIndHit interface in the automation tool.

```
1
2  generateIndHit : Arg Name → List (Arg Name) →
3    (baseType : Name) → (indLs : List Nat) →
4    (baseElm : Name) → (indType : Name) →
5    (points : List Name) → (paths : List Name) → TC ⊤
6  generateIndHit (arg i f) argD b indLs elm ind p1 p2 =
7    bindTC (getIndex base indLs) λ il →
8    bindTC (getConstructors base) λ lcons →
9    bindTC (getLength points) λ l1 →
10   bindTC (getLength paths) λ l2 →
11   bindTC (getPathClauseDep l1 l2 base elm il lcons) λ cl →
12   bindTC (getType baseType) λ RTy →
```

```
13   bindTC (getRtypePathDep base ind elm p1 p2 zero il RTy) λ fty →
14   bindTC (declareDef (arg i f) fty) λ _ →
15   bindTC (defineFun f cl) λ _ →
16   (generateβIndHit argD base indLs elm ind f points paths)
17
```

The interface generateIndHit takes the base eliminator as an argument and uses that to map the point constructors $g_1$ and $g_2$ to output $d_1$ and $d_2$ when building the codomain of the path constructor $h$. generateβIndHit builds the computation rule for the path constructor $h$. The following code builds the induction principle and the computation rules for Circle.

```
unquoteDecl indS₁⋆ = generateInd (vArg indS₁⋆)
                                 (quote S₁⋆) []

unquoteDecl indS₁ βindS₁ = generateIndHit (vArg indS₁)
                           ((vArg βindS₁) :: [])
                           (quote S₁⋆) []
                           (quote indS₁⋆)
                           (quote S₁) S₁points S₁paths
```

The primitive unquoteDecl declares $\beta indS_1$ as a postulate. $\beta indS_1$ gives the action of dependent function $indS_1$ on the path constructor loop.

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