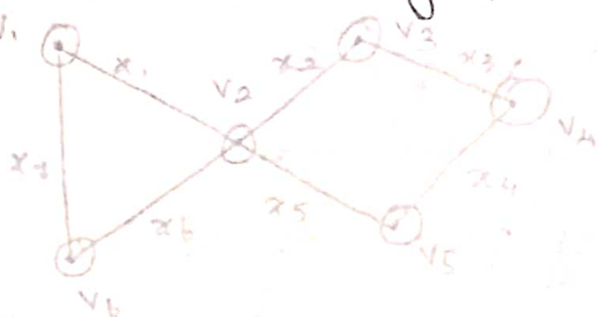
## Eulerian graph:

A closed trial containing all the points and the line is called a Eulerian graph.

A graph having an Eulerian trial is called an Eulerian graph.

Example:




## Lemma:

If G is a graph in which the degree of every vertex is atleast two then G is a cycle.

Proof:

Construct a sequence $v_1, v_2 \ldots$ of vertices as follows

Choose any vertex $v$.

Let $v_1$ be any vertex adjacent to $v$.

Let $v_2$ be any vertex adjacent to $v_1$ other than $v$.

At any stage if vertex $v_i$, $i \geq 2$ already chosen then choose $v_{i+1}$ to be any vertex adjacent to $v_i$ other than $v_{i-1}$.

Since the degree of each vertex is atleast 2 the existence of $v_{i+1}$ is always guaranteed.

Since $G$ has a finite number of vertices
have to choose a vertex which has been chosen before.

Let $v_k$ be such vertex and $v_k = v_i$ where $i < k$, then $v_i, v_{i+1}, \ldots v_k$ is a cycle.

Theorem:

The following statements are equivalent for a connected graph $G$.

i) $G$ is Eulerian

ii) Every point of $G$ has even degree.

iii) The set of edges of $G$ can be partitioned into cycles.

Proof:

(i) $\Rightarrow$ (ii)

Let $T$ be an Eulerian trail with $u$ as the initial and terminal point.

For every vertex $v$ in $T$ other than $u$, there are two edges adjacent with it.

Since an Eulerian trial contains all the edges of $G$. $d(v)$ is even for all $v \neq u$.

$d(u)$ is also even by considering the initial and terminal edge.

Hence every point of $G$ is even.

(ii) $\Rightarrow$ (iii)

Since $G$ is connected. Every vertex of $G$ has degree atleast 2.

Then by lemma,

$G$ Contains a cycle $z$.

Removal of lines of $z$ results in a Spanning subgraph of $G_1$.

If $G_1$ has no lines, then, all the lines of $G$ form one cycle and hence (iii) holds

Otherwise $G_1$ has a cycle $z_1$.

Removal of the lines of $z_1$ from $G_1$ results in a Spanning subgraph of $G_2$ in which every vertex has even degree

Continuing the above process, when a graph $G_n$ with no edges is obtained. we obtain a partitions of $G$ into $n$ cycles.

(iii) $\Rightarrow$ (i).

If the partition has only one cycle, then $G$ is obviously Eulerian.

Since it is Connected.

Otherwise let $z_1, z_2 \ldots z_n$ be the cycles forming a partition of the lines of $G$.

Since $G$ is connected, $\exists$ a cycle $z_i \neq z_1$ having a common point $v_1$ with $z_1$.

Without loss of generality.

Let it be $z_2$

The walk begining at $v_1$ and Consisting the cycles of $z_1$ and $z_2$ in Succession is a closed trial containing the edges of these two cycles.

Continuing this process, we can construct a closed trial containing all the edges of G.

Hence G is Eulerian.

Fleury's Algorithm to construct an Eulerian trail:

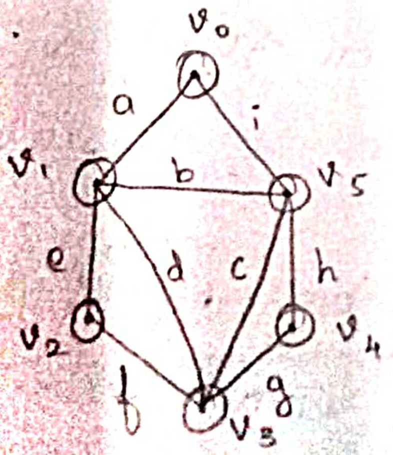
i) Choose an arbitrary vertex $v_0$ and set $W_0 = v_0$.

ii) Suppose that the trial $W_i = v_0$, $W_i = v_0 e_1 v_1 e_2 \ldots e_i v_i$ has been choosen then choose an edge $e_{i+1}$ from $X(G) - \{e_1, e_2 \ldots e_i\}$ in such a way that

* $e_{i+1}$ is incident with $v_i$
* unless there is no alternative, $e_{i+1}$ is not a bridge of $G - \{e_1, e_2 \ldots e_i\}$

iii) Stop when step (ii) can no longer be implemented.

For example, apply Fleury's algorithm to find a closed Eulerian trial in the following graph G.

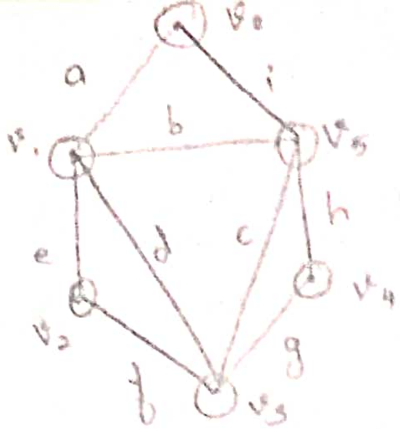



$G_0$

Step - 1 :

$w_0 = v_0$

$G = G_0$

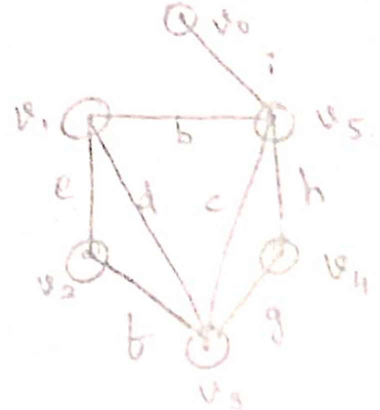



Step - 2 :

$w_1 = v_0 \, a \, v_1$

$= G_0 - \{a\}$

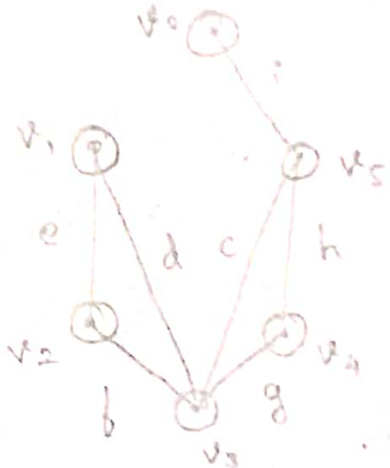
$= G - \{a\}$




Step - 3 :

$w_2 = v_0 \, a \, v_1 \, b \, v_5$

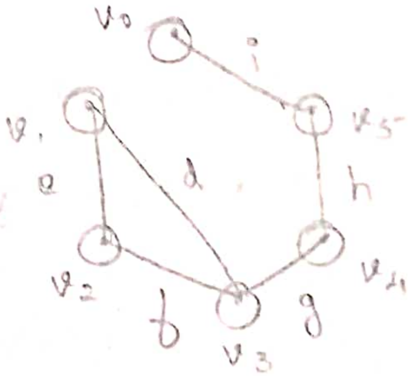
$= G_0 - \{a, b\}$




Step - 4 :

$w_3 = v_0 \, a \, v_1 \, b \, v_5 \, c \, v_3$

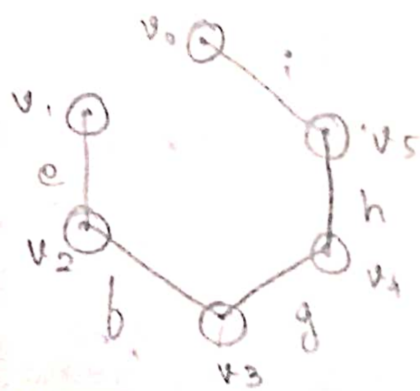
$= G_0 - \{a, b, c\}$




Step - 5 :

$w_4 = v_0 \, a \, v_1 \, b \, v_5 \, c \, v_3 \, d \, v_1$

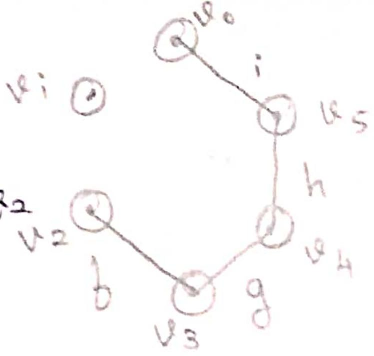
$= G - \{a, b, c, d\}$.



Step - 6 :

$w_5 = v_0 a v_1 b v_5 c v_3 d v_1 e v_2$

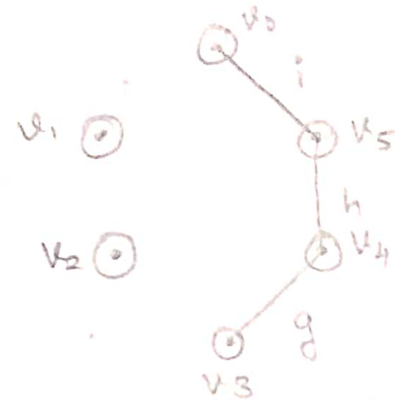
$= G_0 - \{a, b, c, d, e\}$




Step - 7 :

$w_6 = v_0 a v_1 b v_5 c v_3 d v_1 e$
$v_2 f v_3$

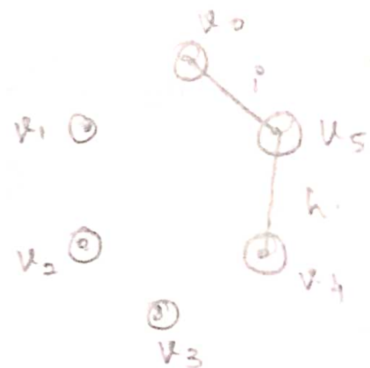
$= G_0 - \{a, b, c, d, e, f\}$




Step - 8 :

$w_7 = v_0 a v_1 b v_5 c v_3 d v_1 e$
$v_2 f v_3 g v_4$

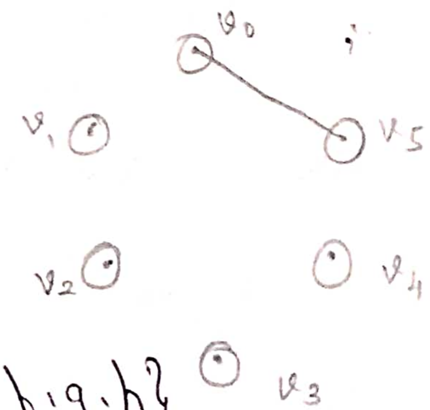
$= G_0 - \{a, b, c, d, e, f, g\}$




Step - 9 :

$w_8 = v_0 a v_1 b v_5 c v_3 d$
$v_1 e v_2 f v_3 g v_4 h v_5$

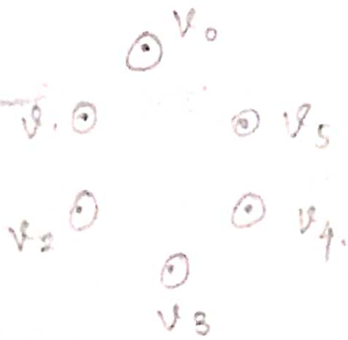
$= G_0 - \{a, b, c, d, e, f, g, h\}$




Step - 10 :

$w_9 = v_0 a v_1 b v_5 c v_3 d v_1 e$
$v_2 f v_3 g v_4 h v_5 i v_0$

$= G_0 - \{a, b, c, d, e, f, g, h, i\}$



Hamitonian graph:

A path in a graph G is called a Hamiltonian Path of G if it contains every vertex in G.

A cycle in a graph G that contains every vertex of G is called a Hamiltonian cycle of G.

A graph is said to be Hamiltonian graph if it contains a Hamiltonian cycle.

A block with two non-adjacent vertices of degree 3 and all other vertices of degree 2 is called a theta graph.

Note:

A theta graph is non-Hamiltonian graph.

A non-trivial graph is 2-connected if and only if it is a block.

Theorem:

Every Hamiltonian graph is 2-connected.

Proof:

Let G be a Hamiltonian graph.

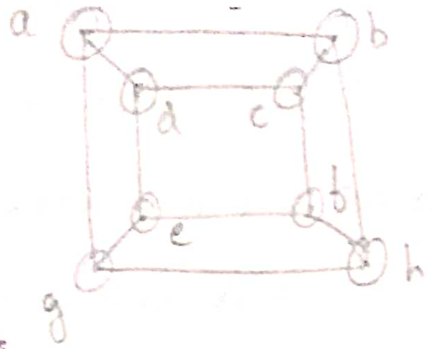
Let z be a Hamiltonian cycle of G.

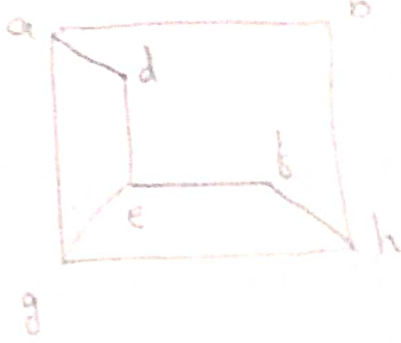
For any vertex v of G, z - v is connected

Hence G - v is also connected

$\Rightarrow$ G has not cut-points

$\Rightarrow$ G is a block. $\Rightarrow$ G is 2-connected




Theorem:

If G is Hamiltonian then for every non-em[pty] proper subset of $V(G)$, $\omega(G-S) \leq |S|$, where $\omega($[H]$)$ denotes the number of components in any graph H.

Proof: G is Hamiltonian

$\Rightarrow$ G has a Hamiltonian cycle. Let it be $z$

Let S be a non-empty proper subset of $V(G)$. Now,

$$\omega(z-S) \leq |S|.$$

$z-S$ is a spanning subgraph of $G-S$

Hence. $\omega(G-S) \leq \omega(z-S)$

$\leq |S|.$

Dirac theorem:

Statement:

If G is a graph with $p \geq 3$ vertices an[d] $\delta \geq p/2$ then G is Hamiltonian

Proof:

We shall prove the thm by the method of contradiction

Let G be a Non-Hamiltonian graph satisfying the hypothesis —— ①

Further let G be a maximal with num[ber] of edges.

Let $u$ and $v$ be two non-adjacent vertices of the graph $G$.

Now, $G+uv$ is Hamiltonian

Hence, there is a Hamiltonian path.

$$u = v_1, v_2 \ldots v_p = v.$$

Let

$$S = \{v_i \mid u\, v_{i+1} \in E(G)\}$$

$$T = \{v_i \mid v\, v_i \in E(G)\}$$

Now,

$$\left.\begin{array}{l} |S| = \deg u \\ |T| = \deg v \end{array}\right\} \text{——} (2)$$

We now assert that

i) $S \cap T = \phi$

ii) $|S \cup T| \leq p.$

i) Let $v_i \in S \cap T$, then

$$v_i \in S \Rightarrow u v_{i+1} \in E(G)$$

and $v_i \in T \Rightarrow v v_i \in E(G).$

Then $u = v_1, v_2 \ldots v_i, v_p, v_{p-1} \ldots v_{i+1}, v_1 = u$

is a Hamiltonian cycle in $G$.

$\Rightarrow G$ is a Hamiltonian

which is a Contradiction

$$\Rightarrow v_i \notin S \cap T$$

$$\Rightarrow S \cap T = \phi$$

$$\Rightarrow |S \cap T| = 0$$

ii) Let $S \cup T \subseteq \{v_1, v_2 \ldots v_p\}.$

Now, $v_p = v$ is neither in $S$ nor in $T$

$\Rightarrow |S \cup T| < p$.

By eqn (2)

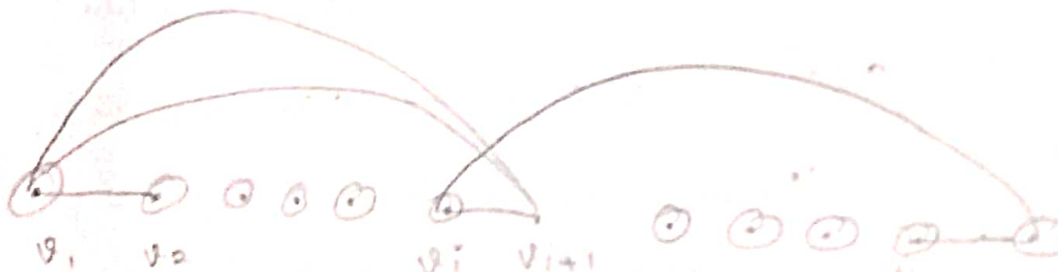
$$d(u) + d(v) = |S| + |T|$$
$$= |S \cup T|$$
$$< p \quad \text{——— (3)}$$

But given that $\delta \geq p/2$.

$$\therefore d(u) + d(v) \geq \frac{p}{2} + \frac{p}{2}$$
$$\geq p \quad \text{——— (4)}$$

Which is contradiction of eqn (3)

Hence $G$ is Hamiltonian.




Lemma:

Let $G$ be a graph with $p$ points & let $u$ & $v$ be non-adjacent points in $G$ $\ni$ $d(u) + d(v) \geq p$, then $G$ is Hamiltonian iff $G + uv$ is Hamiltonian.

Proof:

If $G$ is Hamiltonian, then $G + uv$ is also Hamiltonian.

Conversely, Suppose that $G + uv$ is hamiltoni but G is not.

By Previous thm,

$$d(u) + d(v) < p.$$

Contradicting hypothesis that

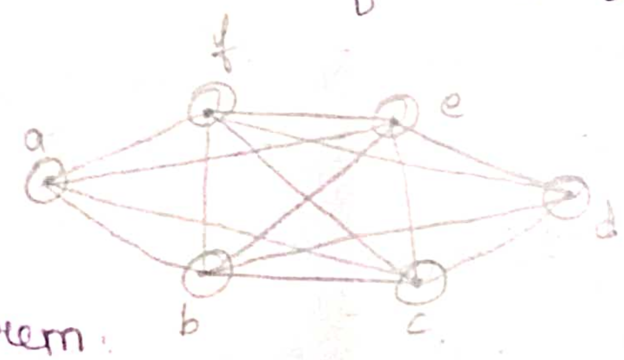
$$d(u) + d(v) \geq p.$$

G is Hamiltonian.

Defn: Closure of G.

The closure of G with p Points is the graph obtained from G by repeatedly joining pairs of non-adjacent vertices whose degree sum is atleast p unit no such pairs remains.

The closure of G is denoted by $c(G)$




Theorem:

A graph G is Hamiltonian iff its closure is Hamiltonian.

Proof:

Let $x_1, x_2 \ldots x_n$ be the sequence of edges added to G in obtaining $c(G)$.

Let $G_1, G_2 \ldots G_n = c(G)$ be the successive graphs obtained.

By previous lemma, G is hamiltonian.

$\Longleftrightarrow G_1$ is Hamiltonian.

$\Leftrightarrow$ $G_2$ is Hamiltonian

. . . . . . . .

$\Leftrightarrow$ $G_n$ is Hamiltonian

$\Leftrightarrow$ $c(G)$ is Hamiltonian

Theorem:

$c(G)$ is well defined.

Proof:

Let $G$ have $p$ vertices.

Suppose that $G$ has two closures as $G_1$ &

Let $G_1$ & $G_2$ be two graphs obtained from $G$ repeatedly joining the pairs of non-adjacent vertices, whose degree sum is atleast $p$ until no such pair remains.

Let $x_1, x_2 \ldots x_m$ & $y_1, y_2 \ldots y_n$ be the Sequence of edges added to $G$ in obtaining $G_1$ & $G_2$ respectively.

We claim that $\{x_1, x_2 \ldots x_m\} = \{y_1, y_2 \ldots$

If $\{x_1, x_2 \ldots x_m\} \neq \{y_1, y_2 \ldots y_n\}$

Let $x_{i+1} = uv$ be the first edge in $G_1$ but not in $G_2$.

Let $H$ be the graph obtained by adding the edges $\{x_1, x_2 \ldots x_i\}$ to $G$.

i.e) $H = G + \{x_1, x_2 \ldots x_i\}$

Clearly, $H$ is a subgraph of $G_1$ & $G_2$.

Since $x_{i+1} = uv$ is the next edge.
i.e) to be added to $H$, in the Construction of $G_1$..

We have, $d_H(u) + d_H(v) \geq p$.

Since $H$ is a subgraph of $G_2$.

$dG_2(u) \geq d_H(u)$

$dG_2(v) \geq d_H(v)$

Now, $dG_2(u) + dG_2(v) \geq d_H(u) + d_H(v)$
$\geq p$.

By the definition of $G_2$, $u$ & $v$ must be adjacent in $G_2$, which is a contradiction

Hence each $x_i$ is an edge of $G_2$.

|||ly we can prove that each $y_i$ is an edge of $G_1$

Hence $G_1 = G_2$.

Thus $C(G)$ is unique.

$\therefore C(G)$ is well defined.

Corollary:

Let $G$ be a graph with atleast 3 points If $C(G)$ is Complete, then $G$ is Hamiltonian

Proof: Given, $C(G)$ is Complete

Closure of $G$ is Hamiltonian.

Since a Complete graph is hamiltonian.

$\therefore G$ is Hamiltonian.

Chavatal theorem:

Statement:

Let $G$ be a graph with degree sequence $(d_1, d_2 \ldots d_p)$, where $d_1 \leq d_2 \leq d_3 \leq \ldots \leq d_p$, $p \geq 3$. Suppose that for each value of $m <$ either $d_m > m$ (or) $d_{p-m} \geq p-m$, then $G$ is Hamiltonian.

Proof:

We know that,

If $c(G)$ is Complete then $G$ is Hamiltonian

It is enough to show that $c(G)$ is Complete

Suppose that $c(G)$ is not Complete

$\Rightarrow \exists$ non-adjacent vertices in $c(G)$.

Let us denote the degree of a vertex $v$ in $c(G)$ by $d'(v)$.

Let $u$ and $v$ be any two non-adjacent vertices in $c(G)$ $\ni$

$$d'(u) \leq d'(v) \quad \text{—①} \quad \&$$

$$d'(u) + d'(v) = \text{maximum}$$

Let $d'(u) = m$.

Since no two non-adjacent vertices in $c(G)$ have degree sum $p$ or more

Now,

$$d'(u) + d'(v) < p.$$

$$m + d'(v) < p$$

$$d'(v) < p - m.$$

from ①

$$d'(u) < d'(v)$$
$$m < p-m$$
$$m < p/2.$$

Let $S = \{v_i \in V - \{v\} \,/\, v_i$ is not adjacent to $v$ in $C(G)\}$

Let $T = \{v_i \in V - \{u\} \,/\, v_i$ is not adjacent to $u$ in $C(G)\}$.

Consider $S$.

If $v_i \in S$, then $v_i$ is not adjacent to $v$ in $C(G)$

$\Rightarrow d'(v_i) + d'(v) < p.$

$\Rightarrow d'(v_i) + d'(v) \leq d'(u) + d'(v)$

$\Rightarrow d'(v_i) \leq d'(u)$

$\Rightarrow d'(v_i) \leq m.$

$\Rightarrow$ Degree of every vertex in $S$ is almost $m$

Now,

$$|S| = p-1-d'(v)$$
$$> p-1-(p-m)$$
$$> p-1-p+m$$
$$> m-1$$
$$|S| \geq m.$$

Hence $C(G)$ has atleast $m$ vertices of degree almost $m$.

i.e) $d'(m) \leq m$ —— ②

Consider $T$,

If $v_i \in T$, then $v_i$ is not adjacent to $u$ in $C(G)$.

$\Rightarrow d'(v_i) + d'(u) < p$

$\Rightarrow d'(v_i) + d'(u) \leq d'(u) + d'(v)$

$\Rightarrow d'(v_i) \leq d'(v)$

$\Rightarrow d'(v_i) < p-m$

$\Rightarrow$ Every vertex in T has degree almost p-m

Now, $|T| = p-1-d'(u)$

$= p-1-m$

$< p-m$

Hence $c(G)$ has atmost p-m vertices of degree atmost p-m.

i.e) $d'_{p-m} < p-m$ —— (II)

Now, G is a spanning subgraph of $c(G)$ + degree of every vertex in G cannot exceed th in $c(G)$.

Hence the statements (I) & (II) holds for G also.

i.e) $d_m \leq m$

$d_{p-m} < p-m$

Contradicting the hypothesis.

Hence $c(G)$ is Complete

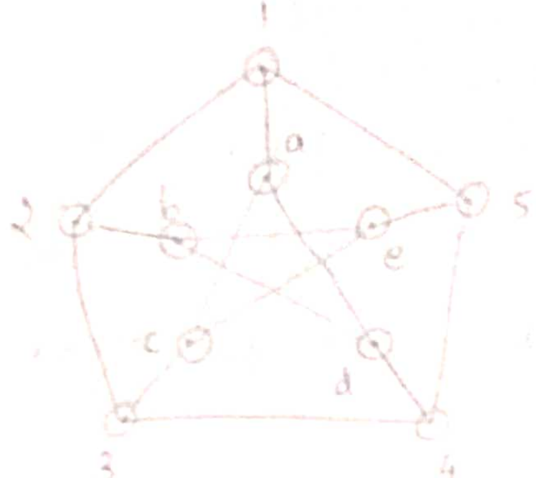
$\Rightarrow$ G is Hamiltonian.

Problems:



1. Show that the peterson graph is non-hamiltonian

Proof:




Let us suppose that the peterson graph G has a Hamiltonian cycle c.

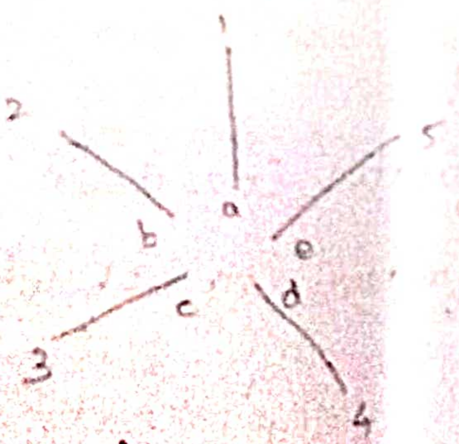
Now G - E(C) must be a regular spanning Subgraph of degree 1.

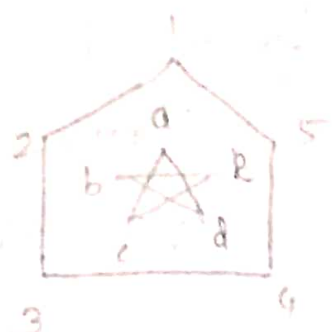
A regular subgraph of degree 1 is called a one function

Case (i)

Consider the subset

A = {1a, 2b, 3c, 4d, 5e}




Clearly, A is a one factor of G

Now, G - A is a union of two distinct cycles $C_1$ hence it is not a any hamiltonian cycle of G.

Case (ii):

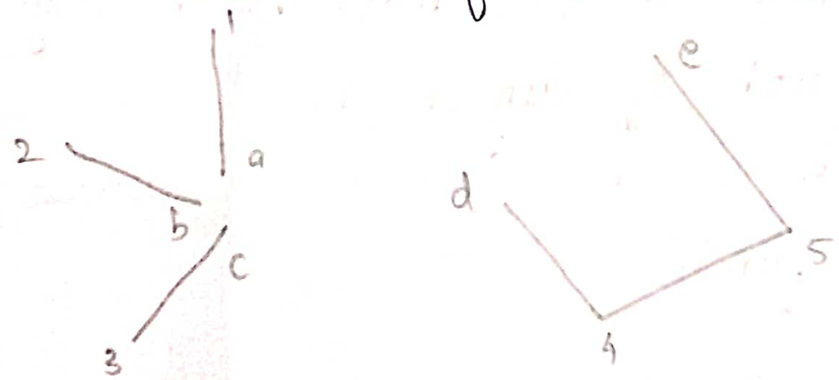
Consider the one factor that Contains only four edges.

Now, the line passing through the remaining two points must also be included the one factor, so we again get A.

Case (iii):

If the one factor contain 3 edges then the following cases arises.

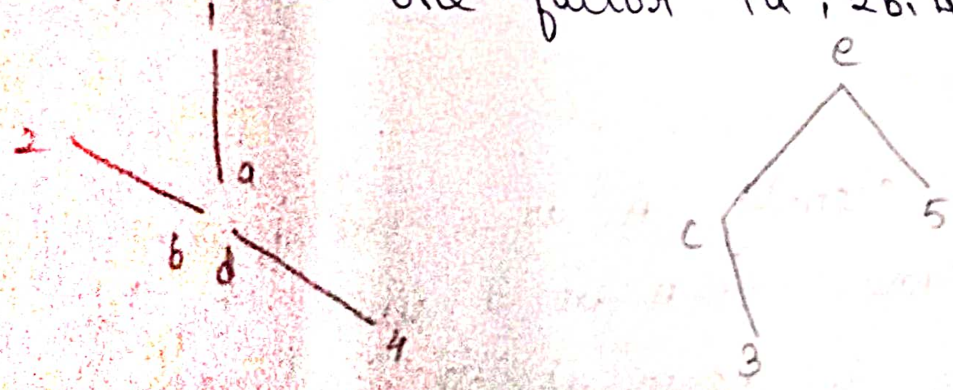
i) Let the one factor Contain 1a, 2b, 3c




The Subgraph induced by the remaining four points is a path whose unique one factor is {4d, 5e}.

Thus the one factor Considered becomes A.

ii) Let, the one factor 1a, 2b, 4d



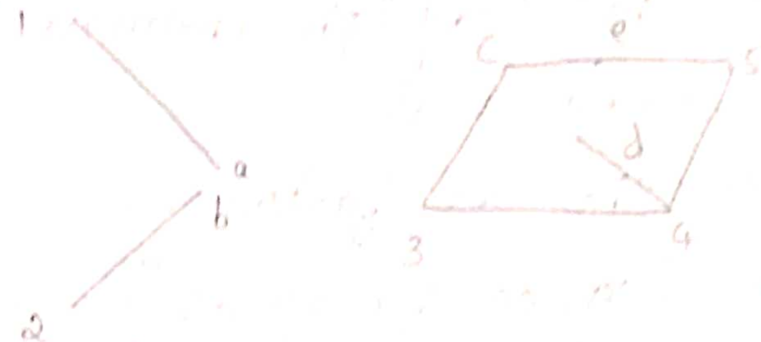
The subgraph induced by the remaining four points in a path whose unique one factor is $\{3c, 5e\}$.

Thus a one factor considered becomes A.

Case (ii)

If the one factor contains two edges, then the following cases arise

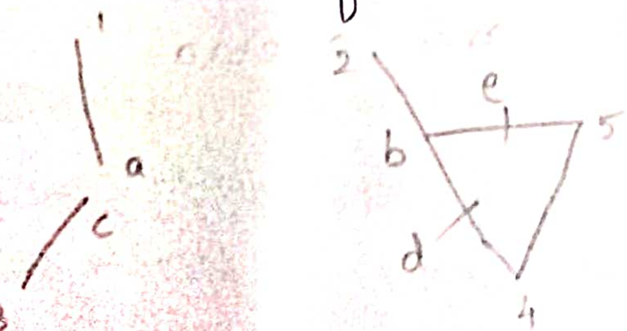
i) Let the one factor contain 1a & 2b.




In the subgraph induced by the remaining points, point d has degree 1 & hence any one factor of the subgraph must contain the edge 4d.

Hence case III is repeated.

ii) Let the one factor contain 1a and 3c.



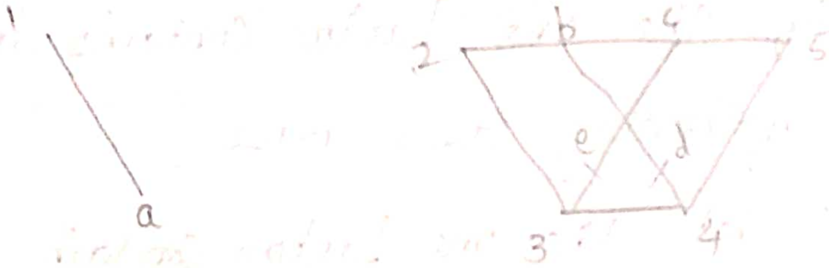

In the subgraph induced by the remaining points, point 2 has degree 1 and hence any

one factor of the subgraph must contain th
edge 2b

Hence case (iii) is repeated.

Case (iv):

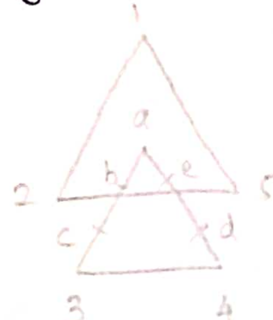
Let the one factor contain just one edge




In the subgraph induced by the remain points cebd, 2345.

Hence the one factor is

$B = \{1a, ce, bd, 23, 45\}$

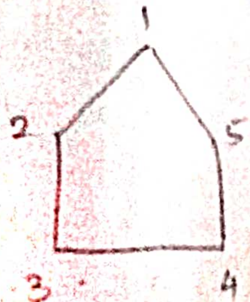


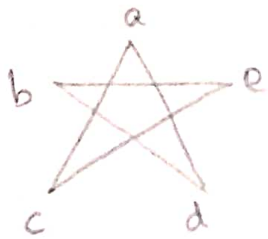

Now $G - B$ is the union of two disjoint cycle and hence it is not a Hamiltonian.

Case (iv):

Suppose that $\exists$ a one factor does not contain any edge from A.

Now, the graph can contain atmost 2 edg from 123451 and acebda.




Hence it contain atmost 4 edges.

$\Rightarrow \nexists$ no such one factor.

From the above cases it can be concluded that there is no such one factor.

$\Rightarrow$ G has no Hamiltonian cycle.

$\Rightarrow$ G is not Hamiltonian.

2. For what values of n is $K_n$ Eulerian.

Soln:

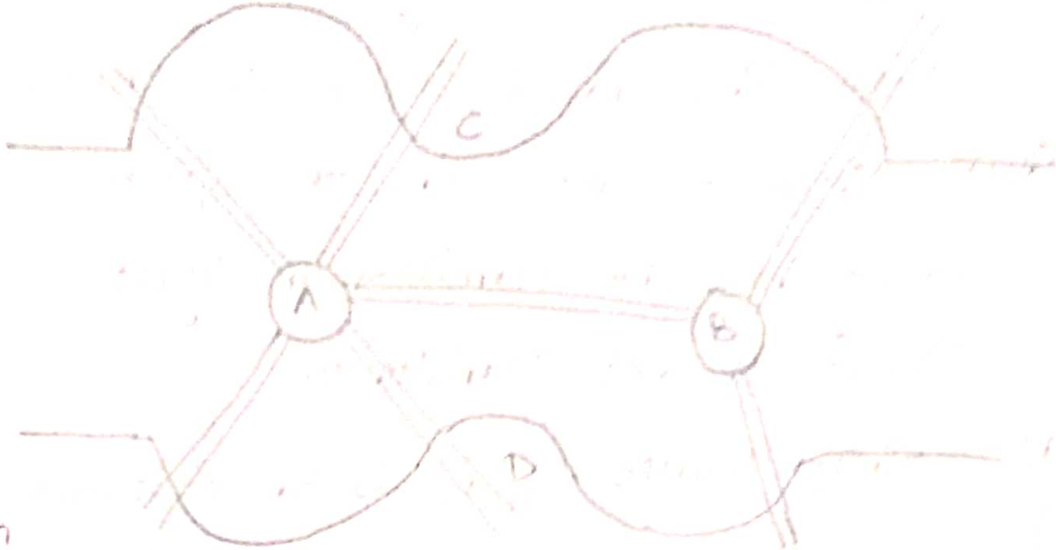
The degree of any vertex in a complete graph with n vertices is $n-1$.

$\therefore$ The graph is Eulerian iff n is odd.

Konigsberg Bridge Problem.

Two islands A & B formed by the Pregel river in Konigsberg were connected to each other & to the banks C & D by 7 bridges as described above the problem was to start at the any of the four lands areas A, B, C or D walk over each of the seven bridges exactly once and return to the starting point.

This pbm remain unsolved for more than two centuries Euler getted this pbm in 1736, He viewed this pbm as a graph with four vertices & 7 edge in which each edge represent a bridge & each vertex represent the land areas.




Graph of Konigsberg bridge Problem:

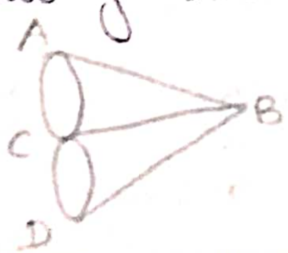
The Konigsberg bridge Problem is the sa as the Problem of drawing the above figure without lifting the pen from the paper and without retracing a line & Coming back to the starting point.

Soln:

Euler proved that the Konigsberg bridg Pbm is equivalent to that of finding a closed walk that traverses all the edges of the above graph.

He Proved that the pbm had a soln on if the vertices were all of even degree But in the above graph not all its vertices are o Even degree Hence it is not an Euler graph

That is not possible to walk over each of the 7 bridges exactly once & return to the starting point



One - Marks:

1. A closed trail containing all the points & the line is called ___

   a) cycle  b) closed cycle  c) Eulerian  d) none

2. If G is a graph in which the degree of every vertex is atleast two then G is a ___

   a) path  b) Trail  c) cycle  d) both a & b

3. A path in a graph G is called ___

   a) Hamiltonian cycle  b) Hamiltonian path

   c) Hamiltonian graph  d) none

4. A cycle in a graph G that contains every vertex of G is called ___

   a) Hamiltonian cycle  b) Hamiltonian path

   c) Hamiltonian graph  d) none

5. A block with two non-adjacent vertices of degree 3 & all other vertices of degree 2 is called ___

   a) Gamma graph  b) theta graph  c) none  d) both a & b.

6. Every hamiltonian graph is ___

   a) 2-connected  b) 3 connected  c) not connected  d) none

7. A graph G is Hamiltonian iff its closure is ___

   a) Non hamiltonian  b) Hamiltonian  c) none

   d) both a & b.

8. C(G) is ___

a) not unique b) unique c) well defined d) both a & c

9. Let G be a graph with atleast 3 pts. If C(G) is ___

a) not Complete b) Complete c) both a & b d) none

10 peterson graph is ___

a) not Complete b) Complete c) Non-hamiltonian

d) both b & c

11 The closure of G is denoted by ___

a) C(G) b) L(G) c) d(G) d) G(C)

12 The theta graph is ___

a) Hamiltonian b) Non-hamiltonian c) both a &

d) none

13. A non-trivial graph is 2-Connected iff

a) Dense b) block c) cut-edge d) none

14. Every hamiltonian graph is 2-Connected then it is ___

a) not cut points b) block c) 2-Connected d) all the above

15. If G is a graph with $p \geq 3$ vertices & ___ then G is hamiltonian.

a) $\delta \geq p/2$ b) $\delta \leq p/2$ c) $\delta = 0$ d) $\delta = p/2$

5- Marks

1. If G is a graph in which the degree of every vertex is atleast two then G is a cycle
2. Fleury's Algorithm to Construct an Eulerian trail
3. Every hamiltonian graph is 2-connected
4. If G is hamiltonain then for every non-empty Proper subsets of $V(G)$ $w(G) - S \leq |S|$, where $w(H)$ denotes the number of components in any graph H
5. State & Prove Dirac thm.
6. A graph G is hamiltonian $\Leftrightarrow$ its closure is hamiltonian
7. Let G be a graph with atleast 3 points. If $c(G)$ is complete then G is Hamiltonian

10- Marks

1. State & Prove chavatal thm
2. State & Prove Peterson graph is non-hamiltonian
3. State & Prove Konigsberg thm
4. State & Prove Dirac thm

# UNIT - IV

# UNIT - IV

**Defn: Tree**

i) A graph that contains no cycle is called a acyclic graph.

ii) A connected acyclic graph is called a tree.

iii) Any graph without cycles is also called as forest. So that the components of a forest is a trees.

**Thm:**

Let G be a (p, q) graph the following statements are equivalent

i) G is tree

ii) Every two points of G are joined by a unique path.

iii) G is connected and $p = q + 1$

iv) G is acyclic and $p = q + 1$

**Proof**

(i) ⇒ (ii)

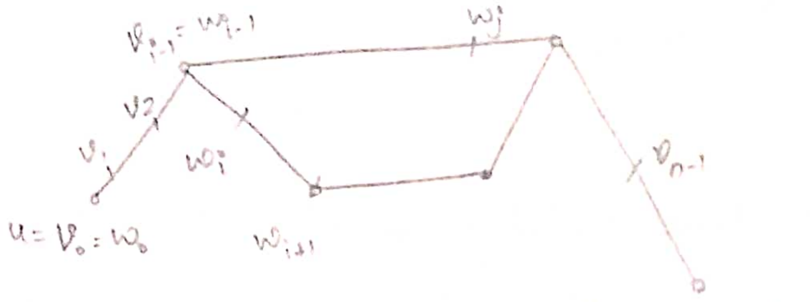
Let u, v be any two points of G.

Given that G is tree.

G is connected.

∃ a u-v path in G.

Now, suppose that ∃ two distinct u-v path.




$\therefore P_1 : u = v_0, v_1, v_2 \dots v_n = v.$

$P_2 : u = w_0, w_1, w_2 \dots w_n = v$

Let $i$ be the least +ve int $\ni$ $1 \le i < m$, $w_i \notin P_1$

then $w_{i-1} \in P_1 \cap P_2$

Let $j$ be the least +ve int $\ni$ $1 < i \le m$ and $w_j \in$ $(w_j \in P_2)$

Now, $w_{i-1} - w_j$ along $P_2$ and $w_j - w_{i-1}$ along $P_1$ form cycle,

which is a contradiction.

Hence $\exists$ a unique $u-v$ path in $G$.

(ii) $\Rightarrow$ (iii)

Since every two points of $G$ have a path.

$\Rightarrow G$ is connected.

To prove : $p = q + 1$

This is trivial for a connected graph with one (or) two path.

Assume that the result is true for a graph with less than $p$ points.

Let $G$ be a graph with $p$ points.

Let $x = uv$ be any line of $G$.

Since $\exists$ a unique $u-v$ path in $G$.

$G - x$ is disconnected with exactly two components $G_1$ and $G_2$.

Let $G_1$ be a $(p_1, q_1)$ graph and $G_2$ be a $(p_2, q_2)$ graph.

Then, $p_1 + p_2 = p$ and $q_1 + q_2 = q - 1$

Hence,

$$p = p_1 + p_2$$
$$= q_1 + 1 + q_2 + 1$$
$$= q_1 + q_2 + 2$$
$$= q + 2 - 1$$
$$p = q + 1$$

(iii) $\Rightarrow$ (iv)

To prove: G is acyclic

Suppose that G has a cycle of length n.

$\Rightarrow$ G has n points and n lines on this cycle.

Fix a point u on the cycle c.

Consider any one of the remaining p-n points not on the cycle c.

Since G is connected.

$\Rightarrow \exists$ a shortest u-v path in G.

The p-n lines these obtained are distinct.

Hence,

$$q \geq p - n + n$$
$$q \geq p$$

Which is $\Rightarrow\Leftarrow$ to $p = q + 1$

(iv) $\Rightarrow$ (i) $\qquad \Rightarrow$ G is acyclic.

To prove,

G is connected.

Suppose that, G is not connected.

Let $G_1, G_2 \cdots G_K$ $(K \geq 2)$ be the components of $G$.

Since $G$ is acyclic. each of these components is a tree.

Hence $P_i = q_i + 1$. where $G_i$ is a $(P_i, q_i)$ graph.

$$\sum_{i=1}^{K} P_i = \sum_{i=1}^{K} (q_i + 1)$$

$$\Rightarrow P = q + K \quad (K \geq 2)$$

$$\Rightarrow q = q + K \quad (K \geq 2)$$

Which is $\Rightarrow\Leftarrow$

$\Rightarrow G$ is connected.

**Corollary:**

Every non-trivial tree $G$ has atleast two vertices of degree 1.

**Proof**

Since $G$ is a non-trivial tree.

$d(v) \geq 1$ for all points $v(G)$.

W.K.T,

The sum of the degrees of the pt of a grap $G$ is twice the no. of lines.

Now,

$$\Sigma d(v) = 2q$$

$= 2(P-1)$ $(P = q+1)$ [Since $G$ is a tre

$\Rightarrow G$ has atleast two vertices of degree.

**Thm:**

Every connected graph has a spanning tree.

**Proof**

Let $G$ be a connected graph.

Let $T$ be a minimum connected spanning subg of $G$.

Now for any line $x$ of $T$.

$T-x$ is disconnected.

$\Rightarrow$ $x$ is a bridge of $T$.

$\Rightarrow$ $x$ is not an cycle of $T$.

$\Rightarrow$ $T$ is acyclic

$\Rightarrow$ $T$ is connected.

Hence It is a spanning tree.

**Corollary:**

Let $G$ be a $(p,q)$ connected graph then $q \geq p-1$.

**Proof**

We know that, Every connected graph has a spanning tree.

Let $T$ be a spanning tree of $G$.

Then no. of vertices in $T$ is $p$ and no. of edge in $T$ is $p-1$.

Then $q \geq p-1$ ($T$ is a subgraph of $G$).

**Thm:**

Let $T$ be a spanning tree of a connected graph $G$. Let $x=uv$ be an edge of $G$ has not in $T$. Then $T+x$ contains a unique cycle.

**Proof.**

Given $T$ is a spanning tree of $G$.

$\Rightarrow$ $T$ is acyclic.

Let $x=uv$ be added of $T$.

Then every cycle in $T+x$ must contain $x$.

Now there is a 1-1 correspondence b/w the cycle in $T+x$ & $u$-$v$ paths in $T$. $\Rightarrow$ $T+x$ contains a unique cycle.

## Defn:

**Centre of a tree**

Let $v$ be a point in a connected graph $G$.

i) The eccentricity $e(v)$ of $v$ is defined by

$$e(v) = \max\{d(u,v) / u \in V(G)\}$$


$e(a) = 2$

$e(b) = 1$

$e(c) = 2$

**NOTE:**

A vertex with minimum eccentricity is called the centre of $G$.

ii) The radius $r(G)$ is defined by

$$r(G) = \min\{e(v) / v \in V(G)\}$$

$V$ is called a central point if $e(v) = r(G)$ and t set of all central points is called the centre of $G$.

**Thm:**

Every tree has a centre consisting of either or points or two adjacent points.

**Proof**

The result is obvious for the trees $K_1$ and $K_2$

Let $T$ be any tree with $v \geq 2$ points then $T$ h atleast two end points.

$\Rightarrow$ The maximum distance b/w any two points occurs only if the points are end points.

Now, delete all the end points from $T$.

The resulting graph $T'$ is also a tree and t accentricity of each point of $T'$ is exactly one less th the eccentricity of the same point in $T$.

Hence T and T' have the same centre.

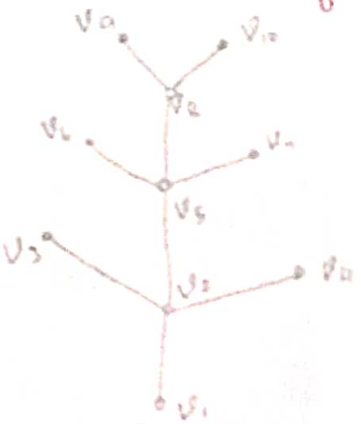
On the repeating above process we obtain a tree which is either $K_1$ or $K_2$.

Hence the centre of T consist of either one point or two adjacent points.

## Spanning tree:

A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all the vertices of G.

Find the central point for given tree.




Sol

| | |
|---|---|
| $d(v_1) = 4$ | $d(v_6) = 3$ |
| $d(v_2) = 3$ | $d(v_7) = 3$ |
| $d(v_3) = 4$ | $d(v_8) = 3$ |
| $d(v_4) = 4$ | $d(v_9) = 4$ |
| $d(v_5) = 2$ | $d(v_{10}) = 4$ |

Minimum deg of $(v_2)$ is $2 = r(v_2)$

maximum deg of $(v_2)$ is $2 = e(v_2)$

$\therefore r(v_2) = e(v_2)$

$\therefore$ centre point is $v_2$.

incognito mode

# UNIT - V

# UNIT-V

# Some Applications

## Introduction:

In this chapter, we discuss some application of graph theory. Some of the problems discussed here, like the travelling salesman problem, job sequencing problem etc. have no efficient solutions.

## Connector problem:

**Defn: Weighted graph:**

A Graph $G$ is called a weighted graph if there is a real number associated with each edge of $G$. The real number associated with each edge is called its weight. The weight of a subgraph $H$ of a weighted graph is defined as the sum of the weights of all the edges of $H$.

## Kruskal's Algorithm:

Kruskal has given an efficient algorithm for finding a shortest spanning tree in a weighted graph $G$. Let $w(e)$ denote the weight of an edge $e$.

**Algorithm:**

* Choose an edge $e_1$ such that $w(e_1)$ is as small as possible.

* If edges $e_1, e_2 \dots e_i$ have been chosen, then choose an edge $e_{i+1}$ from $E(G) - \{e_1, e_2 \dots e_i\}$ s.t.

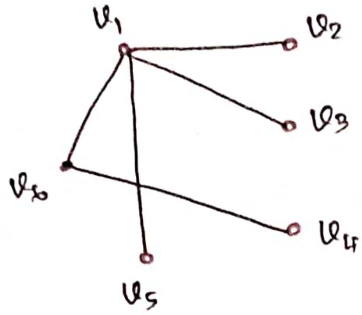
  (i) The subgraph induced by $e_1, e_2 \dots e_{i+1}$ is acyclic

  (ii) $w(e_{i+1})$ is as small as possible subject to (i)

* Stop when step 2 cannot be implemented further.

Example:

Consider the connected weighted graph G on 6 vertices the weights of whose edges are given in th table.




$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 3 & 13 & 2 & 8 \\ 1 & 0 & 8 & 9 & 2 & \infty \\ 3 & 8 & 0 & \infty & 12 & \infty \\ 13 & 9 & \infty & 0 & 10 & 7 \\ 2 & 2 & 12 & 10 & 0 & 12 \\ 8 & \infty & \infty & 7 & 12 & 0 \end{bmatrix} \end{array}$$

The entries $\infty$ correspond to edges that are n in G.

The first edge chosen is $v_1 v_2$.

The Second edge chosen is $v_1 v_5$.

The third edge chosen is $v_1 v_3$ $(w(v_1, v_3) = 3)$.

[Even though $w(v_2 v_5) = 2$, it cannot be chosen as forms a cycle with the edges already chosen].

The next edge chosen is $v_4 v_6$.

(Note that the edges already chosen, namely $v_1 v_2$, $v_1 v_3$, $v_1 v_5$ and $v_4 v_6$ do not induce a connecte subgraph).

The next edge chosen is $v_1 v_6$. Since we have g a spanning tree.

Now, we stop the process, the weight of the spanning tree obtained is $1+2+3+7+8 = 21$.

**NOTE:**

The array of weights will be symmetric in gene However if it is not symmetric, then for every i a j, replace the $(i,j)^{th}$ entry by min $\{(i,j)^{th}$ entry, $(j,i$ entry$\}$ and proceed with the resultant array.

## Shortest Path Problem

Given a railway network connecting various stations, determine a shortest route b/w two specified stations in the network. In graph theoretic terms, the problem is that of finding a path of minimum weight connecting two specified vertices $u_0$ and $v_0$ in a weighted graph.

The weight of each edge corresponds to the distance b/w the points joined by that edge. Dijkstra's algorithm is an efficient method of finding such a path.

## Dijkstra's algorithm for finding a shortest $u_0$-$v_0$ path.

Let $S_0 = \{u_0\}$

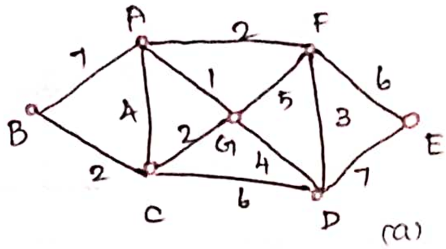
Let $u_1$ be a point of $V - S_0$ that is at minimum distance from $u_0$. Let $S_1 = \{u_0, u_1\}$. clearly $u_0 u_1$ is a shortest $u_0 - u_1$ path.

Let $u_2$ be a point of $V - S_1$ that is at minimum distance from $u_0$. Let $S_2 = \{u_0, u_1, u_2\}$.

Now find a point $u_3$ in $V - S_2$ that is minimum distance from $u_0$. (The procedure used for selecting $u_2$ may be used here also). Proceeding like this, when $v_0$ gets selected, we get a shortest $u_0 - v_0$ path. The shortest distance b/w $u_0$ and $v_0$ is given by the weight of this path.

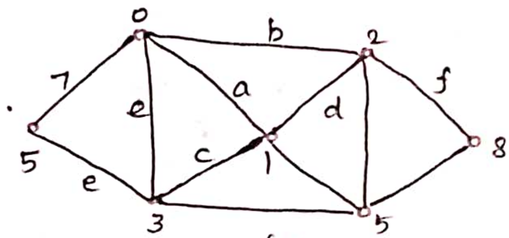
**NOTE:** When each point get selected in $S_0$ the point is labled with its distance from $u_0$ so that further computation becomes easier.

**Eg:** Find the shortest distance from vertex A to all vertices in the following weighted graph.




(a)

is an extension of




(b)

In fig (b) Each point is labled with its shortest distance from A and the edges chosen successively a labeled a, b, c, d, e and f.

$S_0 = \{A\}$

$S_1 = \{A, G\}$ since G is the point at minimum distance from A.

Now, lines from $S_1$ to $V-S_1$ are AB, AC, AF GC, GD and GF. The lengths of the paths from A to $V-S_1$ having these as the last edges are respectively 7, 4, 2, 1+2, 1+4 and 1+5. The minimum among these is 2. Hence we pick up F and $S_2 = \{A,$

The shortest distance from A to F is 2.

Now, Lines from $S_2$ to $V - S_2$ are AB, AC, GC, GD, FD and FE and the weights of the corresponding paths from A to $V - S_2$ are 7, 4, 1+2, 1+4, 2+3 and 2+6.

The minimum among these is 3. Hence we pick up C and now $S_3 = \{A, G, F, C\}$. The shortest distance from A to C is 3.

Lines from $S_3$ to $V - S_3$ are AB, GD, FD, FE, CB and CD. The corresponding paths have lengths 7, 1+4, 2+3, 2+6, (1+2)+2, (1+2)+6. The minimum among these is 5. We pick up D and thus $S_4 = \{A, G, F, C, D\}$

Lines from $S_4$ to $V - S_4$ are are AB, FE, CB and DE. The corresponding paths have lengths 7, 2+6, 3+2 and 5+7. The minimum is 5. and we pick up vertex B.

Thus $S_5 = \{A, G, F, C, D, B\}$

Now lines from $S_5$ to $V - S_5$ are FE and DE. The corresponding paths have lengths 2+6 and 5+7.

The minimum is 8. Hence E is picked up

Thus the shortest distance from A to B, C, D, E, F and G are respectively 5, 3, 5, 8, 2 and 1.

## Transformation and kinematic graph:

**Defn: transformation:**

Let X and Y be two non-empty sets. A transformation from X to Y is an operator which associates with each elts of X one or more elements of Y.

**operands:** The elts of $x$ are called operands.

**Image:** The elts of $y$ which are associated with $x \in X$ are called the images of $x$.

Eg:

let $x = \{a, b, c\}$ and $y = \{a', b', c', d', e'\}$, $\phi : x \to y$ defined by $\phi(a) = a'$ or $b'$, $\phi(b) = b'$ or $c'$ and $\phi(c) = d'$ is a transformation. This transformation can represented schematically as

$$\begin{array}{ccc} a & b & c \\ \downarrow & \downarrow & \downarrow \\ a' \text{ or } b' & b' \text{ or } c' & d' \end{array}$$

A transformation from $X$ to $X$ is called closed.

A transformation is called single valued if each operand has a unique image.

Eg: $\begin{array}{ccc} a & b & c \\ \downarrow & \downarrow & \downarrow \\ c & a & a \end{array}$ is closed and single valued

**Defn:** Each single valued transformation can be represented by means of a diagram called its kinematic graph:

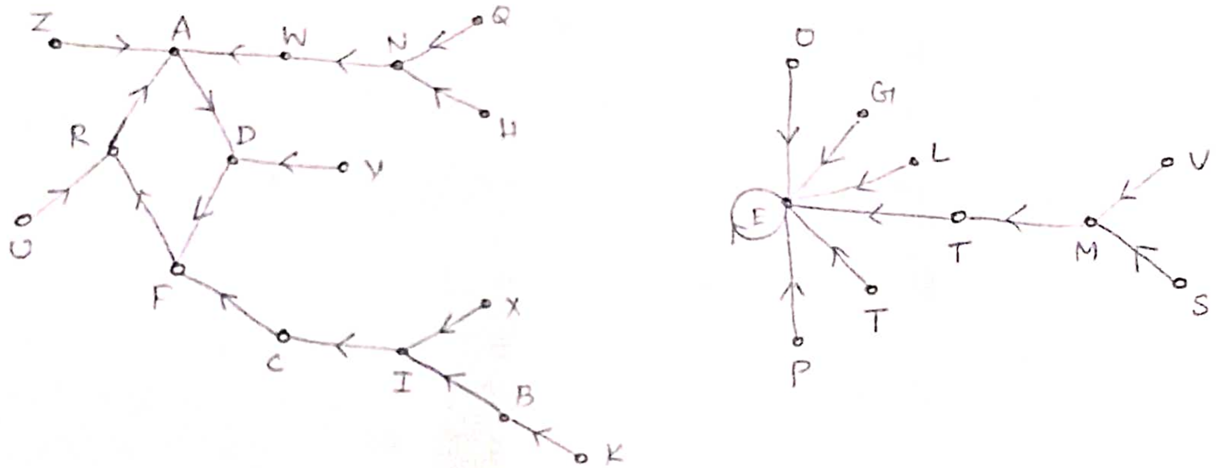
Denote each operand as well as its image by means of a point. Draw directed arc from each operand its image. If the transform is single valued and closed, then the kinematic graph will be functional.

Ex: 1

The kinematic graph of the following transformation is given in

| A | B | C | D | E | F | G | H | I | J | K | L | M | N |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
| D | I | F | F | E | R | E | N | C | E | B | E | T | W |

| O | P | Q | R | S | T | U | V | W | X | Y | Z |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
| E | E | N | A | M | E | R | M | A | I | D | A |




Eg: The kinematic graph of the identify transformation on n elements is the graph consisting of n isolated loops.

Thm:

The arcs of a kinematic graph can be partioned into vertex disjoint cycles iff the transformation is one to one.

Proof

If the transformation is 1-1 then every element has a Pre-Image. Also, since, there are only finite number of elements. Hence the indegree and outdegree of each vertex is unity. Hence the arcs can be partitioned into cycles, which are necessarily vertex disjoint.

Conversely,

If the arcs form a set of vertex disjoint cycles, the indegree and outdegree of each vertex is unity. He each elt has a unique image and unique preimage.

Hence the transformation is 1-1.

Eg: 3

In eg-1, find out the minimal set of elements to which the original set of 26 elements can be reduc by repeating the transformation. Also find the mini number of times the transformation must be applie to arrive at this minimal set.

Sol

Set of elements that lie on cycles or loops is A, D, F,

Hence this is the minimal set of elements to which t original set of 26 elements can be reduced.

The max. length of a directed path from any poi of indegree 0 to a point on a cycle is 4. Hence the minimum number of times the transformation m be applied to arrive at the minimal set is 4.

One interesting application of this concept is in politics. After each general election, a series of defections - the so called "realignment of political force takes place. If we take political parties as element and assume that the realignment pattern is same in all general elections, we can easily find out the political parties that will survive ultimately and the number of general elections needed to arrive at this stage.

Transformations can be represented by means of matrices also.

For eg. $\begin{matrix} a & b & c \\ \downarrow & \downarrow & \downarrow \\ c & a & a \end{matrix}$ can be represented by

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 0 & 1 \\ b & 1 & 0 & 0 \\ c & 0 & 0 & 0 \end{array}$$

The problem of which elements will be selected by the $m^{th}$ repetition of a transformation is equivalent to asking which columns of the matrix will contain +ve elements after raising the matrix to the $m^{th}$ power.