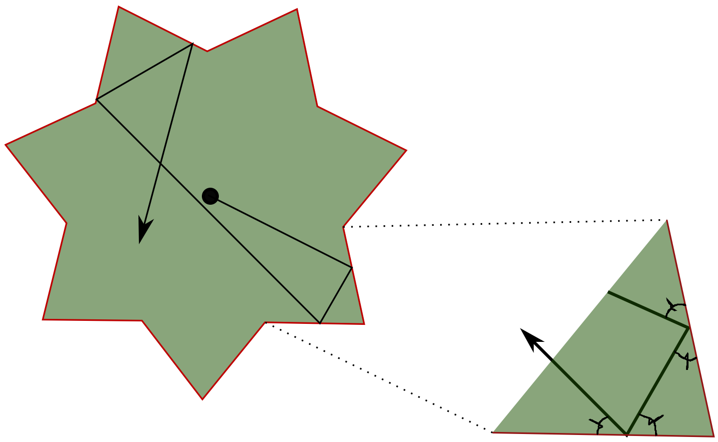
**Rational Billiards**
CCNY Math Club Date: 10/22/17

# 1 What is Billiards and Why Should I Care?

Billiards in math is the study of a single point mass moving in a billiards table at constant velocity. We look at how this billiard ball moves through an obstacle course, or billiards table.



Let $v_1, v_2$ be the initial and final velocity vectors in $\mathbb{R}^2$ at the moment of collision, and $m$ be the mass of the ball. We say that these collisions are perfectly elastic, i.e. $mv_1 = mv_2$. Energy is conserved, and nothing is lost to sound, friction, or heat.
We have an initial point, $x_0$, and a direction $\theta$ that the ball is shot in. A billiard ball bounces all over the board until one of two things happen:

1. We hit a sharp corner.
2. We come back to $x_0$.

We consider $\theta$ to be a *rational angle*, that is we can think of this as a vector $v_\theta \in \mathbb{Z}^2$ where the components $v_\theta = (a, b)$ are relatively prime. This is just so we have a *unique vector* for every rational direction. It's unique because there are infinitely many vectors in directions $kv_\theta$ for $k \in \mathbb{R}$, but only one where both components are coprime integers. In rational billiards we want to understand the dynamics of a particular trajectory, and see what patterns we can discern from our choice of point and direction, $(x_0, \theta)$.

## 1.1 Okay, so why should I care?

Well there are a few reasons.

1. There are very important tie-ins to Riemannian (usually hyperbolic) geometry, topology, and complex (as in $\mathbb{C}$) dynamics.
2. Numerous applications in physics/computer science such as photon/ray tracing. Opens doors to creative solutions to these problems.
3. In the material sciences, people use these methods to simulate how electrons flow through colloidal solutions, materials, and metal alloys. These complex topological structures that are used to study conductivity are known as *Fermi Surfaces*.

## 2 The Strategy

Say you're given a rational polygon, $P$. All lengths are rational, and all interior angles are rational multiples of $\pi$. On this polygon we have a billiard trajectory as a function of time.

$$\phi_t : \mathbb{R} \to P$$

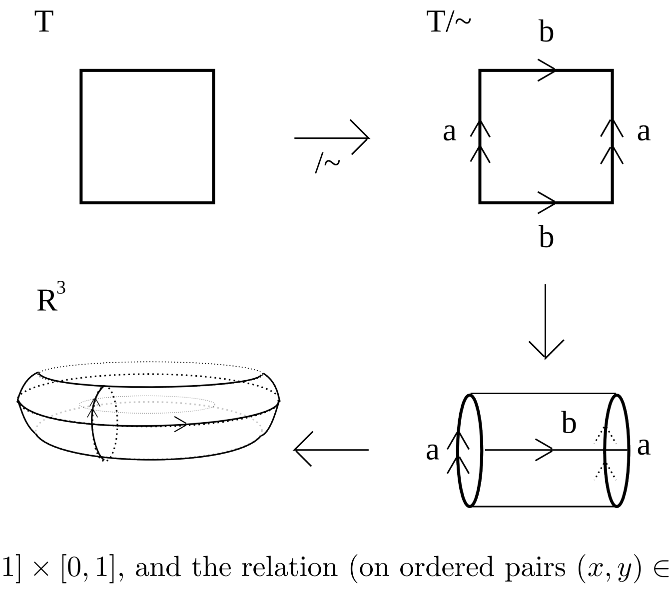
At time zero, $\phi(0) = x_0$; some initial point on the polygon. We also have an initial direction as a unit vector $u \in \mathbb{R}^2$. We might call this the derivative of $\phi$, but it's not entirely clear what this means right now.

From $P$ we want to construct a *translation surface*, where we can understand trajectories easier. What is a translation surface? Say we have a polygon $T$.

First we make identifications on its edges. Every edge of the polygon needs an identification with exactly one other. This is an equivalence relation on a polygon in the plane, $\sim$.
An equivalence relation on a set or space satisfies the following:
Let $x, y, z \in P$.

1. $x \sim x$ (Reflexive)
2. $x \sim y \implies y \sim x$ (Symmetric)
3. $x \sim y$ and $y \sim z \implies x \sim z$ (Transitive)

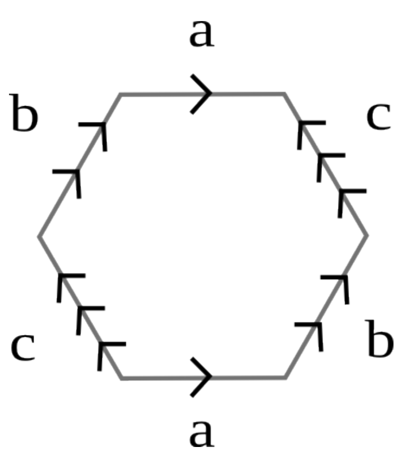
Our surface is then given as a quotient $T/\sim$. This just means everything looks the same except that when you encounter an edge, you are in two places at the polygon at once. Or you "glue" the edges. When you quotient out by a relation, you're creating a new space or set in this abstract sense.
For example we construct a torus, $\mathbb{T}$, out of a square like so:




Formally, $T = [0, 1] \times [0, 1]$, and the relation (on ordered pairs $(x, y) \in T$) is

1. $(0, y) \sim (1, y)$ for $0 \leq y \leq 1$
2. $(x, 0) \sim (x, 1)$ for $0 \leq x \leq 1$

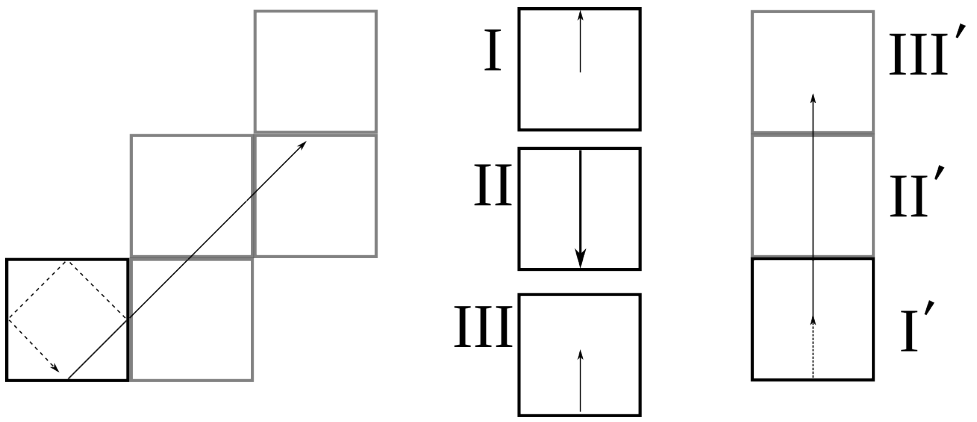
This is the usual construction of a compact surface (no boundaries). But what makes it a translation surface? Well the idea is basically this:
Every edge identified on $P$ MUST be by translations. That is if I want to take an edge to the other with the same label, I cannot reflect, rotate, distort the edge in any way. It has to be a smooth translation. For example:




This is actually a torus too. As far as regular polygonal constructions of the torus go, these are the only two polygons that can create the donut surface. See if you can fold this hexagon into a donut at home.

## 2.1 The Laundry Line

We want a translation surface of a polygonal billiards table because we can use a *geodesic* on that surface to model a billiard flow. To see how, we construct a *laundry line*. A laundry line is a billiard trajectory that doesnt reflect upon collision, but instead reflects the entire surface over the edge. This is unfolding over an edge.

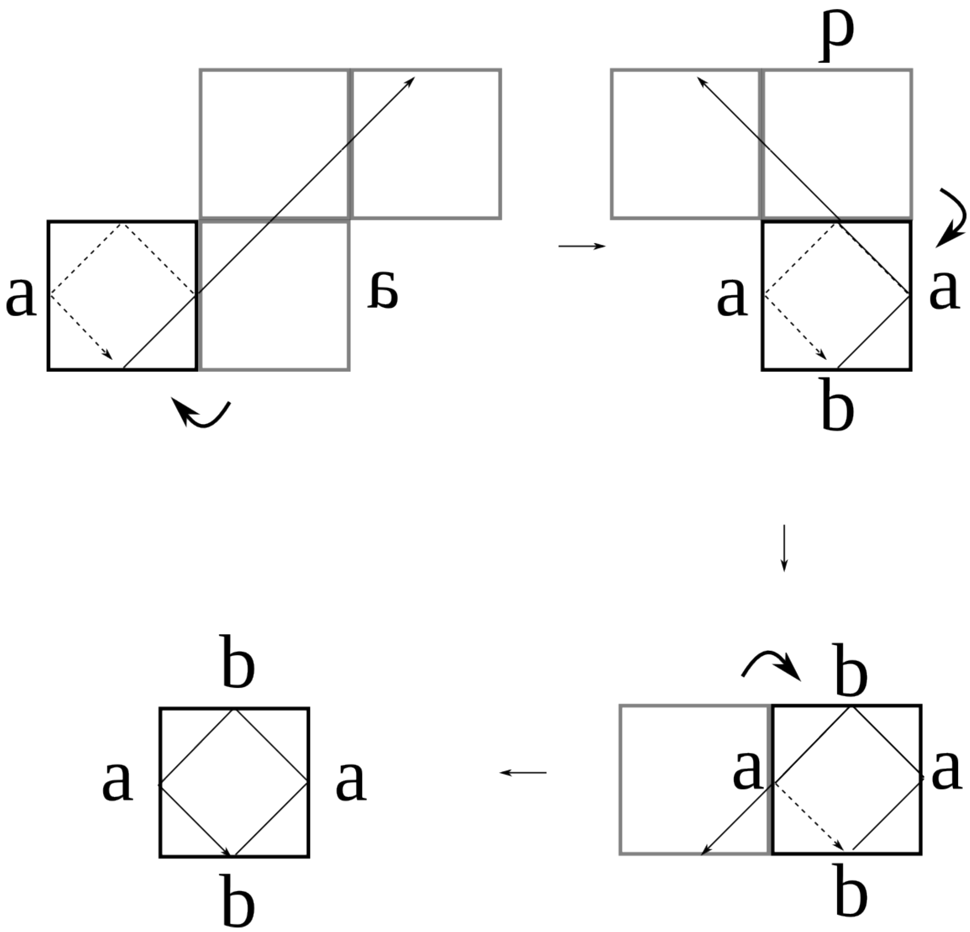



Left: There is a diamond shaped billiard trajectory (dashed line) on a square, and its laundry line.

Right: There is a billiard ball shot straight up the square. The roman numerals $I, II, III$ are the different phases of the trajectory, and the other roman numerals show us how that translates to the laundry line.

### 2.2 Back to surfaces

Let's fold our unfolding, and keep track of the edges that we reflect over:




From the laundry line construction, the square torus is actually a model of a square billiard table. You just have to play around with laundry lines to determine a proper translation surface for P.

### 2.3 Your turn!

### 2.4 Step it up a notch.

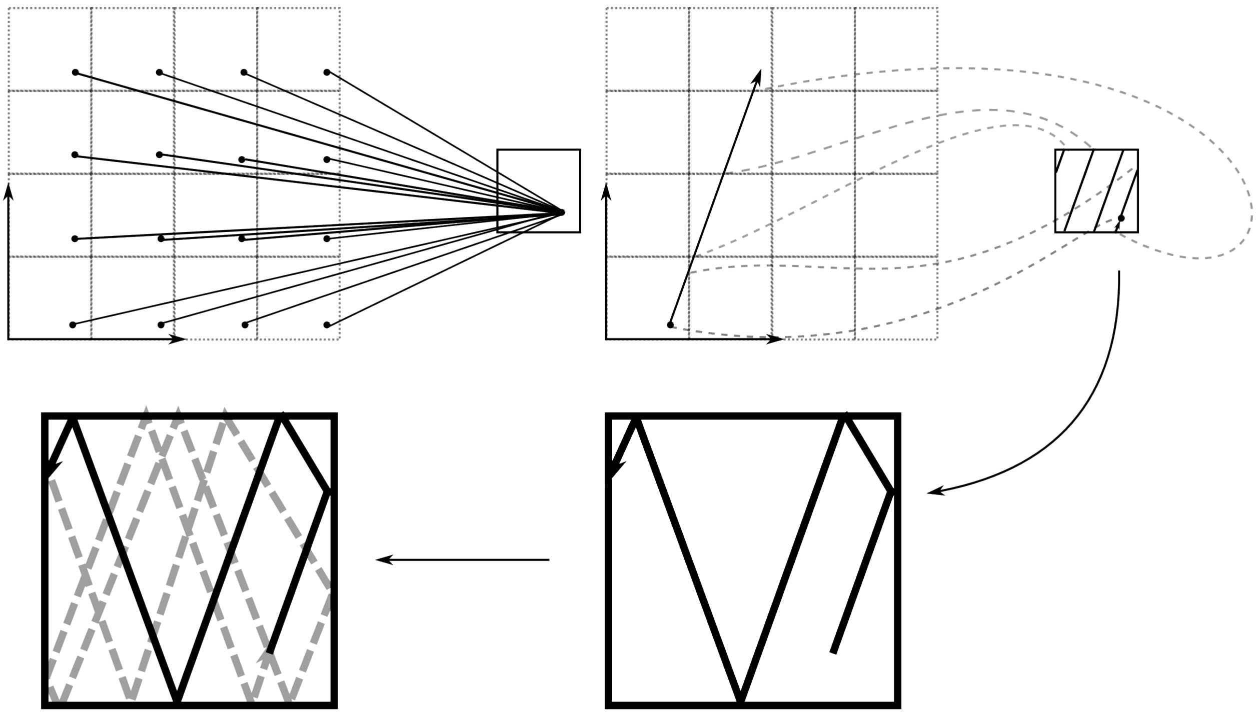
### 2.5 Challenge?

Try drawing some trajectories on the 3-4-5 triangle. Can you determine a good translation surface? Hint: Keep winding around the $30°$ vertex. This is a little more difficult because this translation surface is *hyperbolic*. It's canonical polygon construction wont tile the plane like the square and hexagon do.

# 3 Symmetries and Covering Spaces

We mentioned tiling a plane earlier. This is a very important concept in algebraic topology, and it is what lets us categorize surfaces according to their global geometries (Euclidean, Spherical, Hyperbolic). Can they cover $\mathbb{R}^2$?
But how does this help determine properties of a billiard flow? Well lets say you had a *very surjective map*, $f : \mathbb{R}^2 \to (P/\sim)$. For example, the map from $\mathbb{R}^2$ to the (unit) square torus is given by:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\ (mod\ 1) \\ y\ (mod\ 1) \end{pmatrix}$$



Top left: All the equivalent points that are mapped to our base point on the torus.
Top right: A line segment that closes on the torus.
Bottom right: How the billiard trajectory looks with that length.
Bottom left: Continuing that billiard trajectory.

This is known as a *cover* of a surface. A covering space of the torus is $\mathbb{R}^2$. Covering spaces preserve local geometric properties like angles, distances, curvature, etc. This lets us talk about *lifted paths*. Let $I = [0, 1]$. A closed geodesic on the torus is a continuous loop $\gamma : I \to \mathbb{T}$.

Fix a base point in $p \in \mathbb{R}^2$ such that $p \in f^{-1}(\gamma(0))$. A lift to $\mathbb{R}^2$ is a unique map $\tilde{\gamma} : I \to \mathbb{R}^2$ such that $f \circ \tilde{\gamma} = \gamma$. It's represented by this diagram:

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "f"] \\ I \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"] & \mathbb{T} \end{tikzcd}$$

In this case, the lengths of both paths are the same.

### 3.1 But then how long does it take for a trajectory to close up?

We've seen that it takes longer to close on the square table than the square torus. In fact its actually twice as long. But how do we know, given a "slope" how long the billiard trajectory is? This is where symmetries come to play.

First, let's define the *special linear group.* With integer components,

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ ad - bc = 1 \text{ and } a,b,c,d \in \mathbb{Z} \right\}.$$

These are determinant one integer matrices. On $\mathbb{R}^2$, these matrices preserve *area* and *orientation.*

A *symmetry* is an isometry of a surface. Specific kinds of isometries are *orientation-preserving affine diffeomorphisms.* These are some maps on the square torus that have derivatives in $\mathrm{SL}(2,\mathbb{Z})$ that affect vector directions:

1. Rotations

$$\psi_r = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2. Dehn-twists

$$\psi_h = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} (mod\ 1)$$
$$\psi_v = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} (mod\ 1)$$

Their derivatives form the *Veech group*, denoted $V(\mathbb{T})$, of the surface. It's generated by the matrices $\langle \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \rangle$. This matrices actually generate all of $\mathrm{SL}(2,\mathbb{Z})$. Let $\mathcal{S} = \{(a,b) \ : \ a,b \in \mathbb{Z}\ a,b \text{ relatively prime}\}$. What this means is that

For any $(a,b) \in \mathcal{S}$ there exists an $M \in V(\mathbb{T})$ such that $M \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. The Veech group acts transitively on the set of rational directions $\mathcal{S}$. This is not always the case. Sometimes Veech groups of translation surfaces split up or partition $\mathcal{S}$ into classes of attainable vectors.

### 3.2 Quick Look at Homology

Homology is a hard thing to define in a few minutes, but the gist is this.
If you have some curves, like $\alpha, \beta$ on a surface like $\mathbb{T}$, you can find an integer linear combination of the form $k_1\alpha + k_2\beta$ that describes every kind of possible path on the surface. The $k_1, k_2$ count the number of times a path winds around each of $\alpha$ and $\beta$. If these curves are enough to generate all of homology, then we say $\alpha$ and $\beta$ spans the homology group, $H_1(\mathbb{T},\mathbb{Z})$.

image of alpha beta curves on torus

Homology is useful because they give us an algebraic structure to study the path components of a surface. Moreover, we can treat it like a module, or *almost*-vector space, and define linear transformations $L : H_1(\mathbb{T},\mathbb{Z}) \to G$ to other groups. For example, we saw that a billiard path on the square has to traverse twice the length of a geodesic on a torus, so the map $H_1(\mathbb{T},\mathbb{Z}) \to 2H_1(\mathbb{T},\mathbb{Z})$ reliably takes a closed geodesic on $\mathbb{T}$ to a closed billiard on $T$, and preserves properties like directions.

Let $\Omega : H_1(\mathbb{T},\mathbb{Z}) \to \mathbb{Z}^2$ be the map that takes $k_1\alpha + k_2\beta$ to $(k_1, k_2) \in \mathbb{Z}^2$. It seems like a boring map, but it actually tells you where a geodesic lift terminates on the plane.

include lift image

The affine diffeomorphisms induce automorphisms of $H_1(\mathbb{T},\mathbb{Z})$ as $2 \times 2$ invertible matrices. For example, a horizontal dehn-twist induces a map $\psi_h^* : H_1(\mathbb{T},\mathbb{Z}) \to H_1(\mathbb{T},\mathbb{Z})$ that sends $k_1\alpha + k_2\beta$ to $k_1\alpha + (k_1 + k_2)\beta$:

include dehntwist on homology

Or as a matrix: $\psi_h^* = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{bmatrix} \alpha & \beta \end{bmatrix}$. The matrix of the automorphism is the same as the derivative of the dehn-twist and leads us to the following theorem:

**Theorem 3.1.** *Let $\phi_t : \mathbb{R} \to T$ be a unit-speed billiard on the unit square in rational direction $(a, b) \in \mathcal{S}$. Then $\phi$ is periodic with period $m$, i.e. $\phi(t + m) = \phi(t)$, given as $m = 2\sqrt{a^2 + b^2}$.*

*Proof.* A geodesic on its translation cover is twice the length of a geodesic on the square torus. There exists an element $M$ of $\mathrm{SL}(2,\mathbb{Z})$ and induced automorphism $M^*$ of $H_1(\mathbb{T},\mathbb{Z})$ such that $M(1, 1) = (a, b)$, and $M^*(\alpha+\beta) = a\alpha+b\beta$. Under $\Omega$, $\Omega \cdot M^*(\alpha+\beta) = (a, b)$. The lift of the path $\gamma : I \to \mathbb{T}$ with this homology class to $\mathbb{R}^2$ would satisfy $\tilde{\gamma(1)} - \tilde{\gamma}(0) = (a, b)$. This total length is $\sqrt{a^2 + b^2}$, so the length of the closed billiard would be $2\sqrt{a^2 + b^2}$. Since $\phi$ is unit-speed, the period of $\phi$ would be its length after a single period. Therefore $m = 2\sqrt{a^2 + b^2}$. $\square$