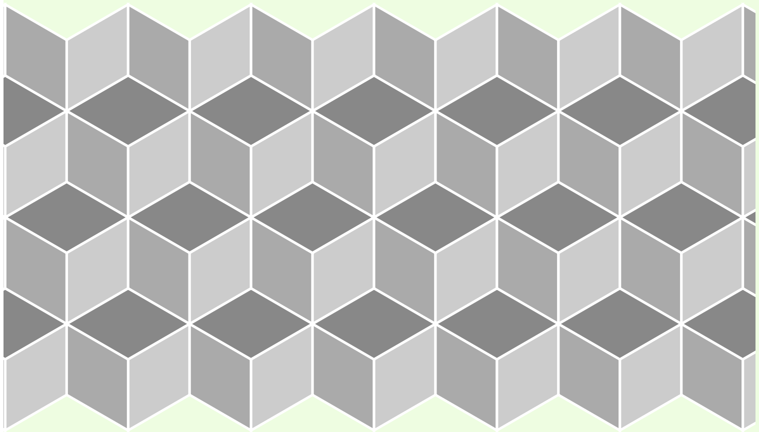
Geodesics on the Necker Cube Surface

Pat Hooper and Pavel Javornik

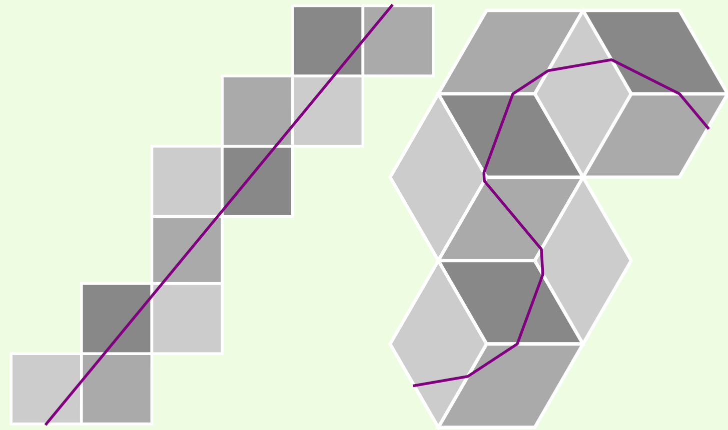
The Necker Cube Surface

The *Necker cube surface* is obtained by the edge to edge pairing of infinitely many unit cubes:



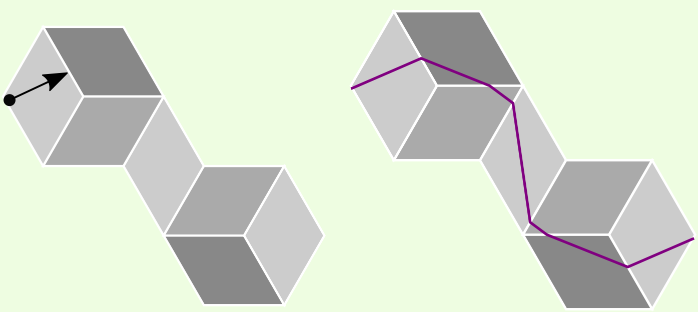
Geodesics

A *geodesic* is a locally, distance-minimizing curve or local "straight line path":



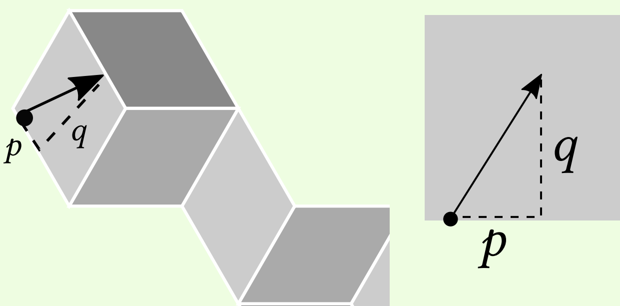
Geodesics

A geodesic is uniquely determined by an initial point and direction:



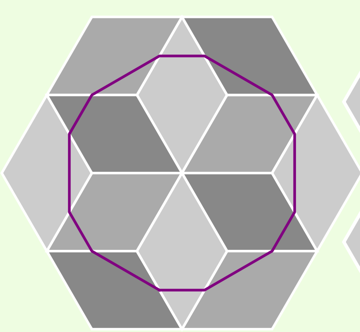
What is a direction?

Since the face of each cube is flat, we consider a direction as a vector pair $\boldsymbol{v}=(p,q)$ as it is projected onto the plane:

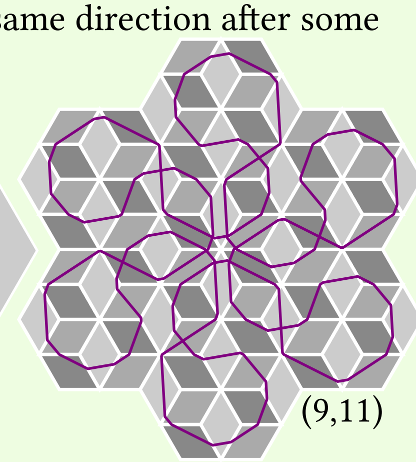


Periodic Geodesics

A geodesic is *periodic* if it returns to the initial point traveling the same direction after some finite time.



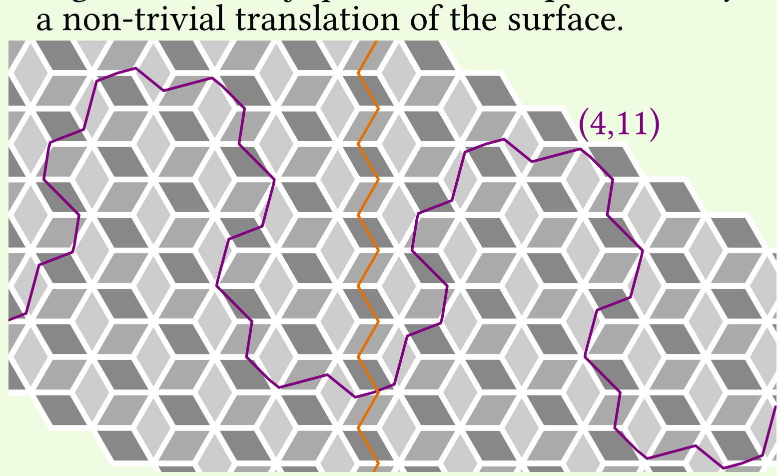
(1,1)



(9,11)

Drift-periodic Geodesics

A geodesic is *drift-periodic* if it is preserved by a non-trivial translation of the surface.



(1,0)

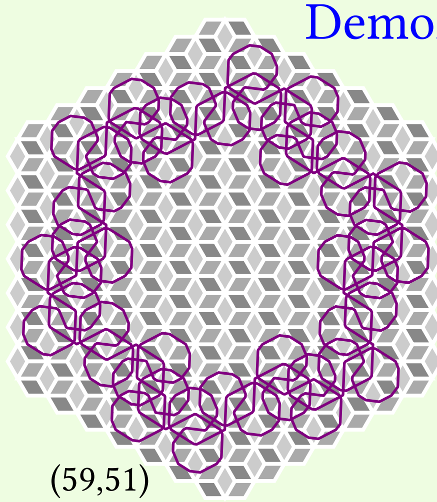
The Periodicity Theorem

A geodesic is called *singular* if it hits a vertex of a square.

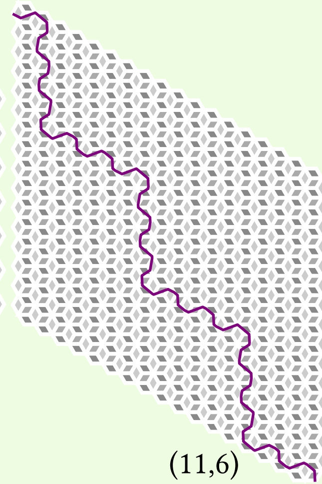
Theorem. A geodesic is periodic or drift-periodic if and only if it travels in an integral direction $\boldsymbol{v}=(p,q)$ with p and q having no common factor. Moreover:

- If p and q are both odd, then the geodesic is periodic.
- If one is odd and the other is even, then the geodesic is drift-periodic.

Demonstration



(59,51)



(11,6)

Irrational directions

The direction of a vector $\boldsymbol{v}=(x,y)$ is *rational* if it is parallel to a non-zero integer vector *(p,q)*, and is *irrational* otherwise.

An irrational direction can be approximated by integer vectors.

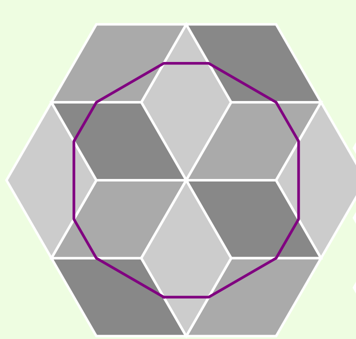
Example: The Fibonacci sequence is

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

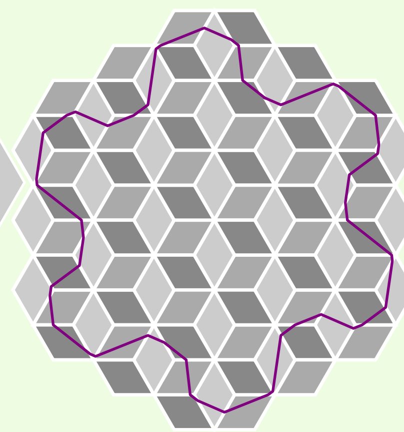
Vectors made from adjacent terms approach the golden mean direction, $(1, \frac{1+\sqrt{5}}{2})$,

(1,1), (3,5), (13,21), (55,89), ...

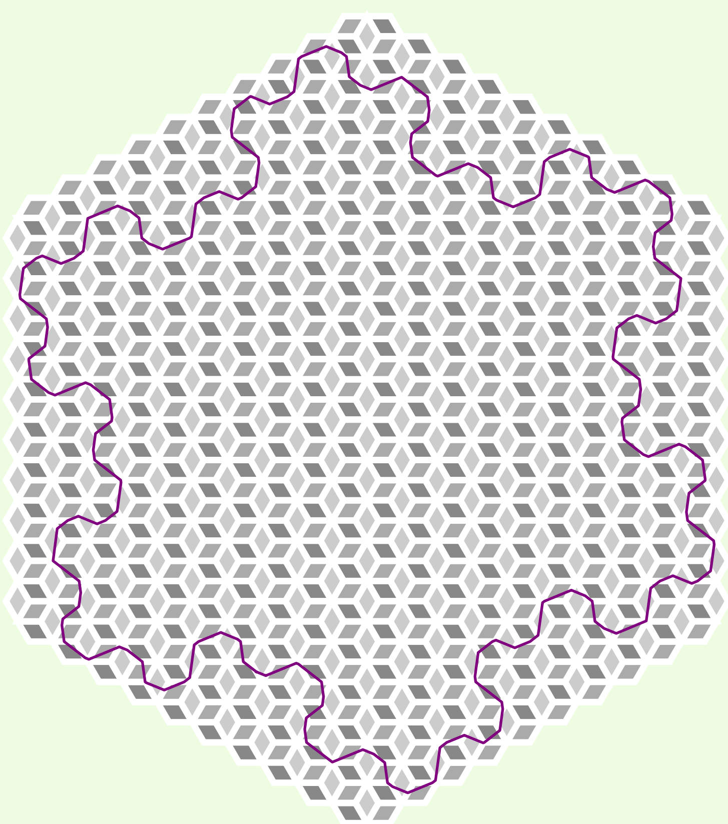
The Golden Mean Snowflake



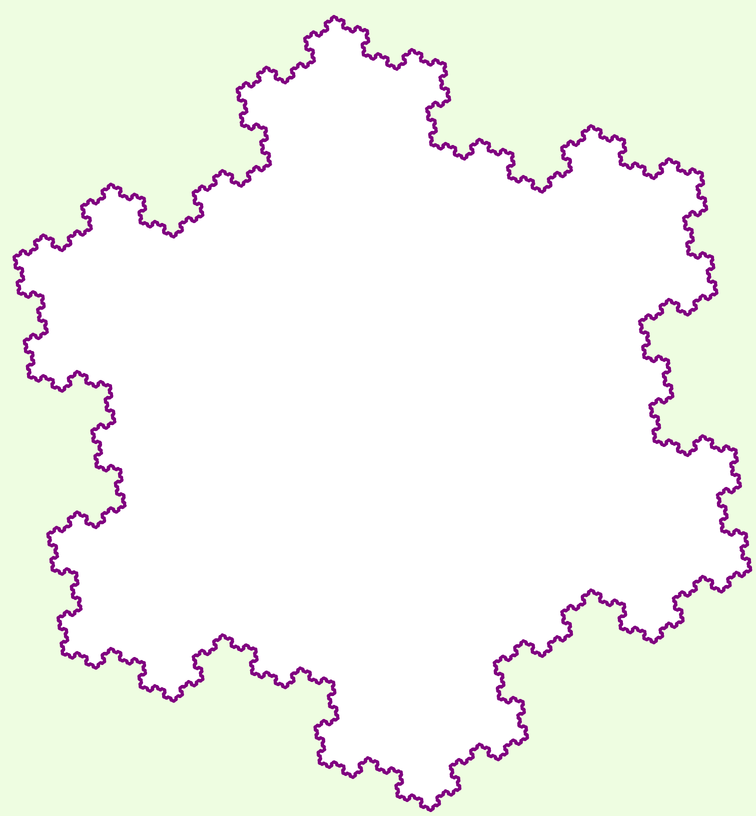
(1,1)



(3,5)



(13,21)



(4181,6765)