# Periodicity of Geodesic Flows on the Necker Cube Surface

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**Abstract**

The Necker cube surface is an infinite topological surface obtained by the periodic tiling of unit cubes in $\mathbb{R}^3$ along a 2-dimensional subspace realized as a Euclidean cone surface with a discrete, countable set of singularities of cone angles $3\pi$ and $\frac{3\pi}{2}$. When punctured at these points it is an infinite genus flat surface equipped with a flat metric with nontrivial holonomy. We employ techniques most commonly used in studying geodesic flow on finite flat surfaces and adapt them to this one.

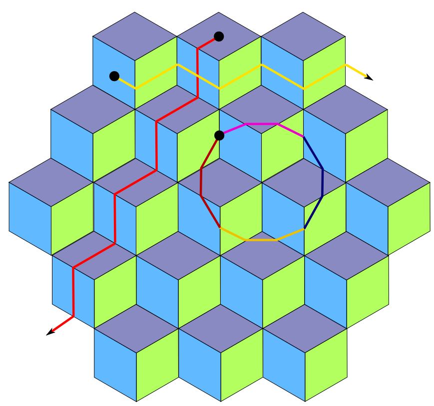


Figure 1: Periodic and drift-periodic flows on the Necker cube surface

# 1 Introduction

Put your Problem statement here! Example of a Citation[1, p.219]. Here's Another Citation [2]

The Necker cube[citation] has made numerous appearances throughout history. For example, M.C. Escher, the artist most famously known for his use of optical illusions in his artwork, has occasionally used tilings of the Necker cube such as in the lithographs "Metamorphosis I," and "Convex and Concave." (Pictured below.) The crystallographer Louis Albert Necker was credited for having extensively studied the lone Necker cube's geometry[citation] and remarked on this simple, optical illusion. Some might even recognize it as the same surface in which "Q*bert" hops around on in the 1982 arcade game published by Gottlieb.

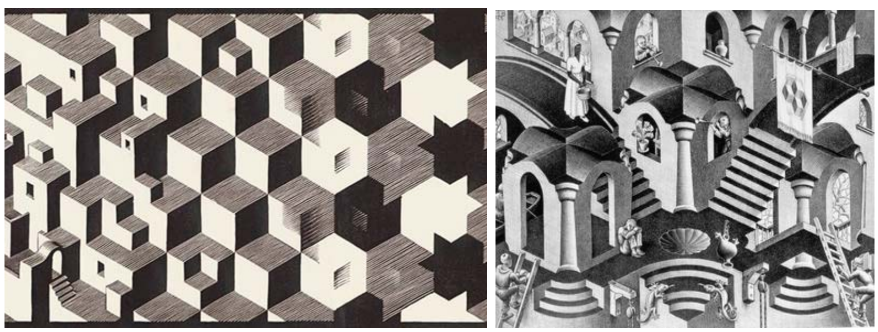
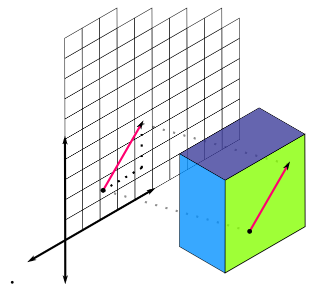


Figure 2: The Necker cube tiling as it appears briefly in "Metamorphosis I," and hung on a banner in "Convex and Concave."[citation]

**Remark.** *Familiarity is assumed on the part of the reader with covering space theory, translation (Veech) surfaces, and their associated Veech groups. For a general take on these topics check [cite] and [cite].*

## 1.1 Discussion of Results

Initial experiments suggested there was a correlation between choice of trajectory angle and periodicity. On a surface composed of infinitely many cubes, there is certainly a flat metric endowed on smaller neighborhoods of the surface that are locally isometric to the plane. Via parallel transports of tangent vectors over the edges of the cubes connecting one face to the next, it is possible to describe geodesic behavior on the Necker cube surface as a series of straight line segments that traverse faces and continuously flow in a fixed direction (see figure **??**).



We consider classes of directions obtained from unit vectors in $\mathbb{R}^3$ by projecting one of these line segments onto a plane parallel to some face. These vectors retain all necessary information about their directions and let us consider rational and irrational angles relative to a choice of basis in $\mathbb{R}^2$. This is what we refer to as an *initial trajectory angle*.

Rational angle trajectories are categorized according to the parities of their components.

**Definition 1.** *Let $v$ be a unit vector of the form $\frac{1}{k}(x,y) \simeq \mathbb{R}/2\pi\mathbb{Z}$ for $x, y \in \mathbb{Z}$ and $k = \sqrt{x^2+y^2} \in \mathbb{R}$. We call $v$ an* ***odd-odd*** *vector if $x$ and $y$ are relatively prime integers and both odd. We denote the* ***set of all odd-odd directions*** *$(x,y) \in \mathcal{O}$. We say that $v$ is an* ***even-odd*** *vector if $x$ and $y$ are relatively prime integers, and $x$ is even if and only if $y$ is odd. We denote the* ***set of all even-odd directions*** *$(x,y) \in \mathcal{E}$.*

If $v$ does not fall into one of these two categories, it is an *irrational* direction. We prove the following is true of all rational trajectory angles:

**Theorem 1.** *Let $\Phi_t$ be the unit tangent flow on the Necker cube surface, and $\Phi_0 = (x_0, v')$ be its initial point in direction $v' \in \mathbb{R}^3$. Let $v$ be a unit tangent vector in $\mathbb{R}^2$ identified with $v'$. Then the following is true:*

*(i) $\Phi$ is periodic with period $T \in \mathbb{R}$ if and only if $v \in \mathcal{O}$.*

*(ii) $\Phi$ is drift-periodic with period $T \in \mathbb{R}$ if and only if $v \in \mathcal{E}$.*

The proof of this theorem is in section X.
**theorem about most directions being recurrent?**

## 1.2 Acknowledgements


-Pat Hooper
- Vincent Delecroix, Ferrán Valdez, pascal hubert
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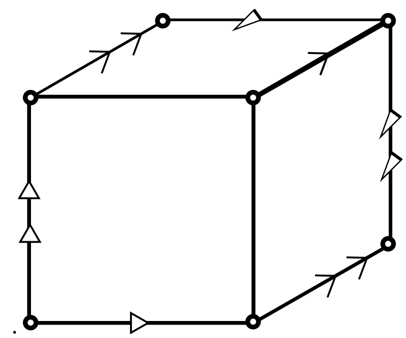
# 2 Construction

This section details the construction of the Necker cube surface and how two different interpretations of the surface are one and the same.

## 2.1 Geometric Interpretation

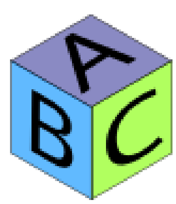
Consider the following surface embedded in $\mathbb{R}^3$:



We call this surface $\mathbf{G}$, the *Necker cube*. $\mathbf{G}$ is obtained by taking three copies of the unit square and identifying the edges so as to resemble a cube when viewed from the front. $\mathbf{G}$ has a total of three vertices. Every vertex has a cone angle of either $3\pi$ or $\frac{3\pi}{2}$ and is therefore a conical point/singularity. It is apparent that $\mathbf{G}$ has an induced flat metric and non-trivial linear holonomy, so geodesic flows on its unit tangent bundle are well-defined via parallel transports over the edges. We denote the surface without singularities by $\mathbf{G}^\circ$. The periodic tiling of $\mathbf{G}$ will take infinitely many copies of $\mathbf{G}$ and glue them translationally along the solid triangles and create our Necker cube surface, denoted $\mathbf{S}$.

Take the following unit squares in $\mathbb{R}^3$:

$$\begin{aligned}\mathbf{A}_{m,n,p} &= [m, m+1] \times [n, n+1] \times \{p\},\\ \mathbf{B}_{m,n,p} &= \{m+1\} \times [n, n+1] \times [p-1, p],\\ \mathbf{C}_{m,n,p} &= [m, m+1] \times \{n+1\} \times [p-1, p].\end{aligned}$$




If we take the integer triple $(m, n, p)$ and require that the individual faces based around them has the triple lie on the plane $x + y + z = 0$, $\mathbf{S}$ is broken down into the following sets:

$$\mathbf{A} = \bigcup \{\mathbf{A}_{m,n,p} : m + n + p = 0\},$$
$$\mathbf{B} = \bigcup \{\mathbf{B}_{m,n,p} : m + n + p = 0\},$$
$$\mathbf{C} = \bigcup \{\mathbf{C}_{m,n,p} : m + n + p = 0\}.$$

The sets $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are contained in $\mathbf{S}$ their union is the Necker cube surface.

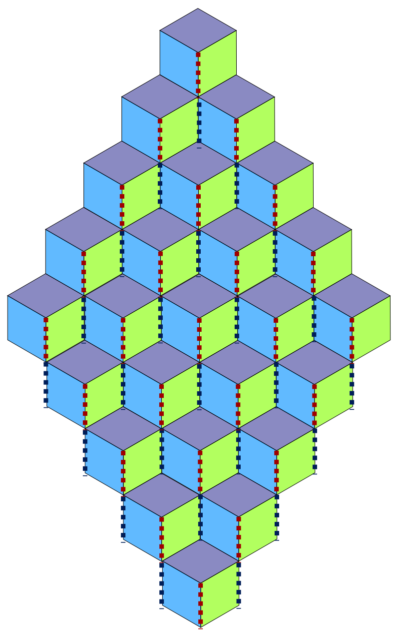
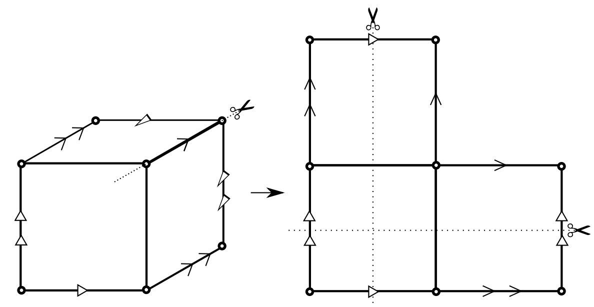


Figure 3: The tiling of $\mathbf{G}$ where the $\mathbf{A}$ faces are copies of copies of the front-facing square of $\mathbf{G}$.(Edges that were not used in the tiling are labeled).

## 2.2 Algebraic Interpretation

Rigid motions of the faces of $\mathbf{G}^\circ$ allow us to use a series of cutting and gluing operations to obtain an L-shaped surface homeomorphic to the two-torus with three points removed,



which is then cut along the dotted lines to obtain a surface that resembles a torus with a unit square shaped hole cut out:

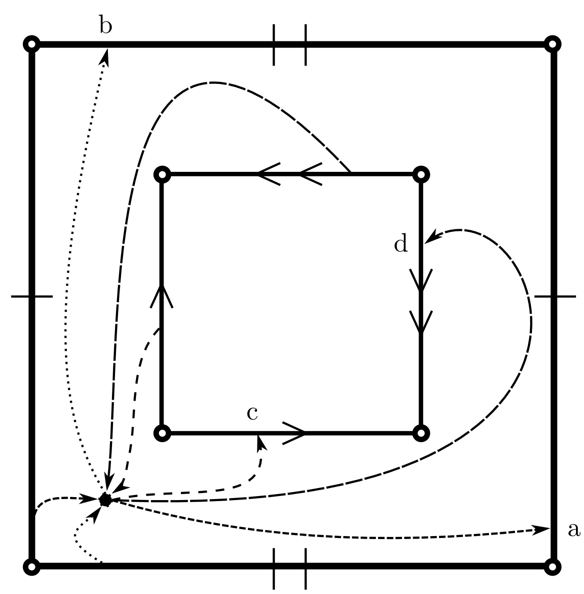



Figure 4: $\mathbf{G}^\circ$ with 4 closed paths labeled identified

We assert that $\pi_1(\mathbf{G}^\circ) \cong \langle a, b, c, d\rangle$ since it is homeomorphic to the thrice-branched torus. We tile this surface so that the associated deck group, $\Delta_{\mathbf{G}^\circ}$, of the cover is isomorphic to $\mathbb{Z}^2$.

**Definition 2.** *Let $\varphi : \pi_1(\mathbf{G}^\circ) \to \mathbb{Z}^2$ be the map between groups such that*
$\varphi(1) = \varphi(c) = \varphi(d) = (0, 0)$
$\varphi(a) = (1, 0)$
$\varphi(b) = (0, 1)$
$\varphi(ab) = (1, 1) = \varphi(ba)$.

**Lemma 1.** *$\varphi$ is a group homomorphism.*

*Proof.* $\varphi$ preserves identities and is well defined on the generators of $\pi_1(\mathbf{G}^\circ)$, and is therefore a homomorphism. $\square$

**Definition 3.** $\mathbf{U}^\circ$ *is the cover of* $\mathbf{G}^\circ$ *such that* $\pi_1(\mathbf{U}^\circ) = \ker\varphi$.

The kernel of the homomorphism is the free product of the commutator subgroup $[a,b]$ and free group $\langle c,d\rangle$, so $\Delta_{\mathbf{G}^\circ} = \pi_1(\mathbf{G}^\circ)/\pi_1(\mathbf{U}^\circ) = \langle a,b,c,d\rangle/\langle c,d\rangle \otimes [a,b] \cong \mathbb{Z}^2$. The covering map that induces the inclusion $k_{\varphi *} : \pi_1(\mathbf{U}^\circ) \hookrightarrow \pi_1(\mathbf{G}^\circ)$ is the function $k_\varphi : \mathbf{U}^\circ \hookrightarrow \mathbf{G}^\circ$ that takes every point on $\mathbf{U}^\circ$ to its modular equivalent under these translational symmetries. $\mathbf{U}^\circ$ is realized as an infinite $\mathbb{Z}^2$-tiling of $\mathbf{G}^\circ$ in $\mathbb{C}$:

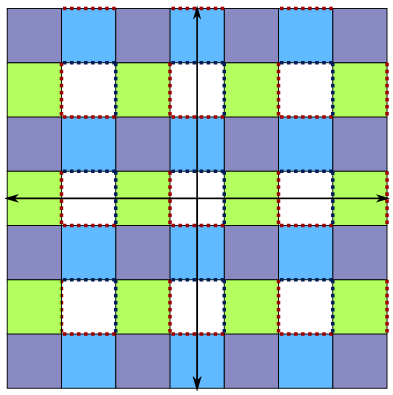


Figure 5: The infinite surface $\mathbf{U}^\circ$ embedded in $\mathbb{C}$ with edges identified.

### 2.3 Isometry From S to U

The Necker cube surface can be *flattened* onto the plane by piecewise isometric maps onto a subspace of $\mathbb{R}^3$ with (open) unit squares removed at every even integer pair in the plane, what we claim to be $\mathbf{U}$.

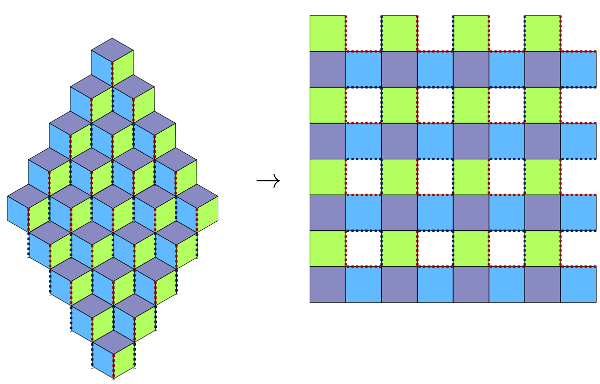


Figure 6: A piecewise isometry from $\mathbf{S}$ to $\mathbf{U}$.

The red/blue dotted lines represent the edges that are split on the plane. The map from one surface to the other is composed of piecewise isometries, $\Psi : \mathbf{S} \to \mathbb{R}^3$.

$$\Psi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ z - \lfloor z \rfloor \end{bmatrix} & \text{if } (x,y,z) \in \mathbf{A} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ x - \lfloor x \rfloor \end{bmatrix} & \text{if } (x,y,z) \in \mathbf{B} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ y - \lfloor y \rfloor \end{bmatrix} & \text{if } (x,y,z) \in \mathbf{C} \end{cases} \tag{1}$$

The flattened surface is contained entirely in $(x, y, 0) \in \mathbb{R}^3$, which is isometric to $\mathbb{C}$. $\mathbf{U}$ is recovered as a topological quotient on the domain

$$\mathbf{P} = \mathbb{C} \setminus \bigcup_{m,n \in \mathbb{Z}} \{u + vi : u \in (2m - \frac{1}{2}, 2m + \frac{1}{2}),\ v \in (2n - \frac{1}{2}, 2n + \frac{1}{2})\}. \tag{2}$$

**Definition 4.** $\mathbf{U}$ *is the surface obtained as the topological quotient* $\mathbf{P}/\sim_{\mathbf{P}}$*, where* $\mathbb{C}$ *is identified with* $\mathbb{R}^2$ *in the usual way and* $\sim_{\mathbf{P}}$ *is a minimal relation on* $\mathbf{P}$ *defined as follows:*

$$\textit{Let } x_0 = (u_0, v_0), x_1 = (u_1, v_1) \in \mathbf{P}.\ \sim_{\mathbf{P}} \textit{ is given as the relation}$$
$$x_0 \sim_{\mathbf{P}} x_1 \textit{ iff } x_0 = x_1$$
$$\textit{or, for some } m, n \in \mathbb{Z}\ x_0, x_1 \in \partial\left(\left[2m - \frac{1}{2}, 2m + \frac{1}{2}\right] \times \left[2n - \frac{1}{2}, 2n + \frac{1}{2}\right]\right)$$
$$\begin{bmatrix} u_1 - 2m \\ v_1 - 2n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_0 - 2m \\ v_0 - 2n \end{bmatrix}.$$

**Remark.** *While* $\Psi$ *does send edges and vertices to different line segments and points* $(x, y, 0) \sim \mathbf{P}$ *depending on the choice of subset of* $\mathbf{S}$*, the surface* $\mathbf{U}$ *is recovered as a topological quotient. This being an isometry follows from* $\Psi$ *being composed of Euclidean matrices*

## 2.4 Four-Fold Cover

We denote the *unit tangent bundle* on the surface $\mathbf{U}^\circ$ by $UT(\mathbf{U}^\circ) = \mathbf{U}^\circ \times \mathbb{R}^2$, and describe a lift by $SO(2, \mathbb{Z})$ action on the fibers of $\mathbf{U}^\circ$. Initial experiments have shown us that geodesics viewed as discontinuous line segments on $\mathbf{P}$ flow in any one of four directions at a time.

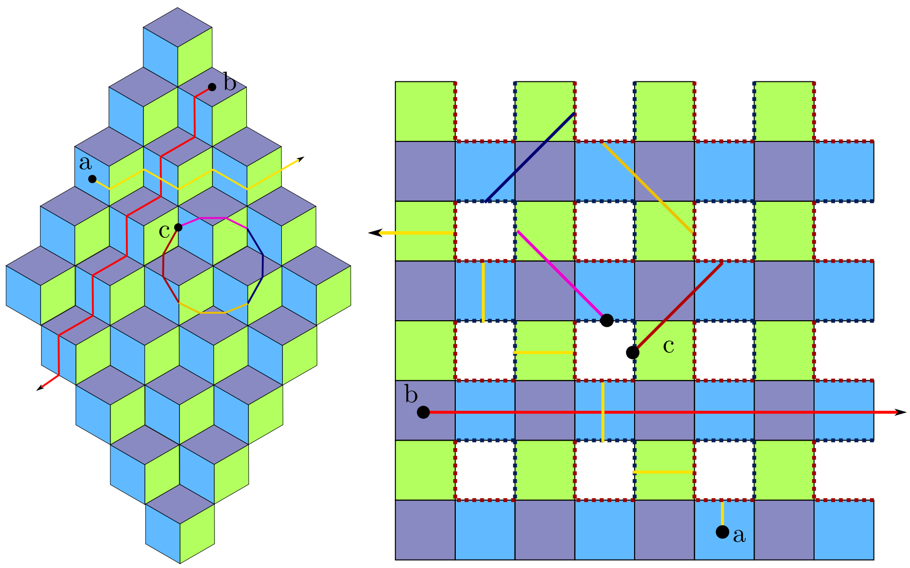



Figure 7: Simple periodic (c) and drift-periodic (a,b) trajectories represented as line segments on $\mathbf{S}$ and $\mathbf{U}$.

Observe that a closed loop on $\mathbf{U}^\circ$ will act on the vectors by a rotation of $\pm\frac{\pi}{2}$ radians. Let $\eta : UT(\mathbf{U}^\circ) \to \mathbf{U}^\circ$ be the projection map onto $\mathbf{U}^\circ$, and $\gamma : [0,1] \to \mathbf{U}^\circ$ be a closed path based at $x_0 \in \mathbf{U}^\circ$. We lift $\gamma$ to $\tilde{\gamma}$ under $\eta$ such that $\tilde{\gamma}(0) = p = (x_0, v_0)$, where $v_0$ is an arbitrary vector in $\mathbb{R}^2$. A holonomy map $\tau : \pi_1(\mathbf{U}^\circ) \to \mathrm{Hol}_p$ takes $c \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = R$, $d \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = R^3$, and maps $a, b$ trivially. Then $\tau(\pi_1(\mathbf{U}^\circ)) = \tau([a,b] \otimes \langle c, d\rangle) = [I_2, I_2] \otimes \langle R, R^3 \rangle \cong SO(2,\mathbb{Z})$. $A \in SO(2,\mathbb{Z})$ acts on $(x,v) \in UT(\mathbf{U}^\circ)$ by the left action: $A \cdot (x,v) = (x, Av)$ that corresponds to the effect that looping around these cone singularities has on a vector. The corresponding lifted path thus returns as $\tilde{\gamma}(1) = \tau([\gamma]) \cdot (x_0, v_0) = (x_0, \tau([\gamma])v_0)$.

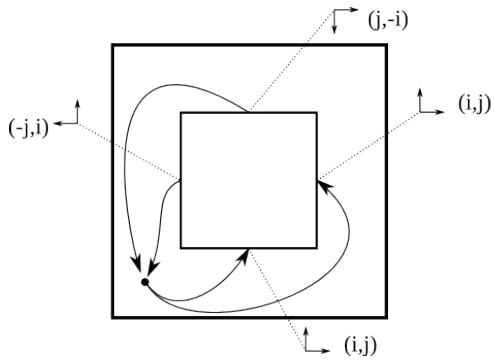



Figure 8: Effect a nontrivial loop has on arbitrary basis (i,j).

Thus, $\tilde{\gamma}$ is closed if and only if $\gamma$ has trivial linear holonomy.

**Definition 5.** *The four-fold cover* $\tilde{\mathbf{U}}_0^\circ$ *is the surface with fundamental group* $\pi_1(\tilde{\mathbf{U}}_0^\circ) = \ker\tau$.

We consider this to be a four-fold cover, as this is a cyclic cover whose sole purpose is to have a trivial holonomy on all *closed* paths in order to lift closed geodesics to closed geodesics. Arbitrary paths do *not* have trivial holonomy by parallel transport over the edges:

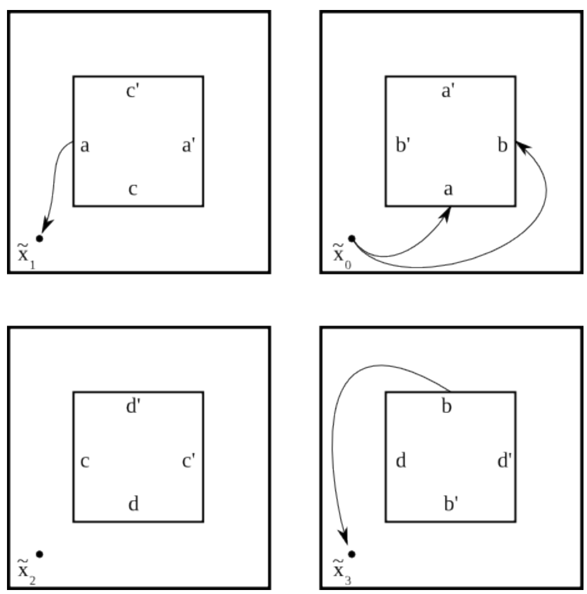



Figure 9: Arbitrary lifts to fibers

The inclusion map $k_{\tau*} : \pi_1(\tilde{\mathbf{U}}_0^\circ) \hookrightarrow \pi(\mathbf{U}^\circ)$ is induced by the projection $k_\tau : \tilde{\mathbf{U}}_0^\circ \hookrightarrow \mathbf{U}^\circ$. A simple calculation also shows us that $\ker\tau = [a,b] \otimes \langle c^4, d^4\rangle$.

Naturally, this also extends to a monodromy map from the fundamental group to the cyclic group of permutations of elements in the fiber $k_\tau^{-1}(x_0)$.

**Theorem 2.** *A periodic geodesic on $\mathbf{U}^\circ$ lifts to a periodic geodesic on $\tilde{\mathbf{U}}_0^\circ$.*

*Proof.* Let $\gamma : [0,1] \to \mathbf{U}^\circ$ be a closed, periodic geodeisc in unit direction $\gamma'$. Suppose that $\gamma$ does not lift to a closed path on $\tilde{\mathbf{U}}_0^\circ$. Then $[\gamma] \notin \ker\tau$, which implies that $\tau([\gamma]) \neq I_2$. Now lift $\gamma$ to $UT(\mathbf{U}^\circ)$ under $\eta$ as $\tilde{\gamma} : [0,1] \to UT(\mathbf{U}^\circ)$ with base point $(\gamma(0), \gamma')$. Since $\gamma$ is periodic it must be true that the parallel transport of a vector on the unit tangent bundle returns unaltered. But because $\mathbf{U}^\circ$ has nontrivial linear holonomy, $\tilde{\gamma}(1) = \tau([\gamma]) \cdot (\gamma(1), \gamma') = (\gamma(1), \tau([\gamma])\gamma') \neq (\gamma(1), \gamma')$ since $\tau([\gamma]) \neq I_2$. Hence $[\gamma] \in \ker\tau = \pi_1(\tilde{\mathbf{U}}_0^\circ)$ and $\gamma$ lifts to a closed geodesic on $\tilde{\mathbf{U}}_0^\circ$. $\square$

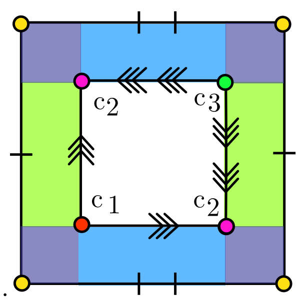



Figure 10: A $2 \times 2$ cut out section centered at each missing square. Edges and vertices identified.

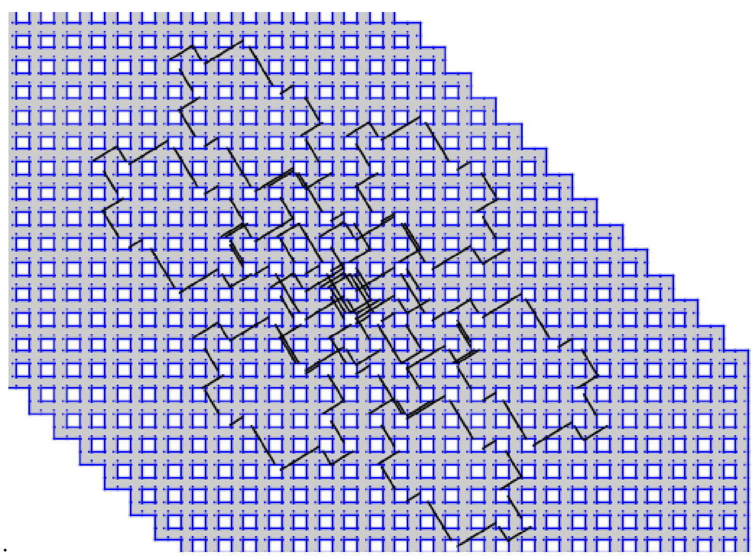


Figure 11: Geodesic flow on $\mathbf{U}$ modeled using sage-flatsurf.

## 2.5 Translation Surface

A better picture of $\tilde{\mathbf{U}}_0^\circ$ is obtained by making the following edge identifications on $\mathbf{P}' = \mathbf{P} \times \mathbb{Z}/4\mathbb{Z}$:

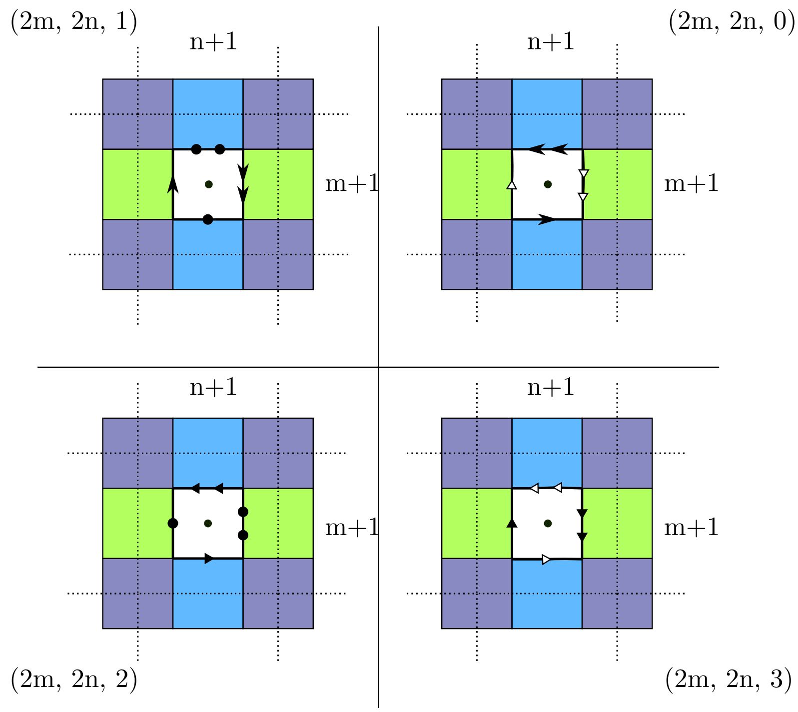
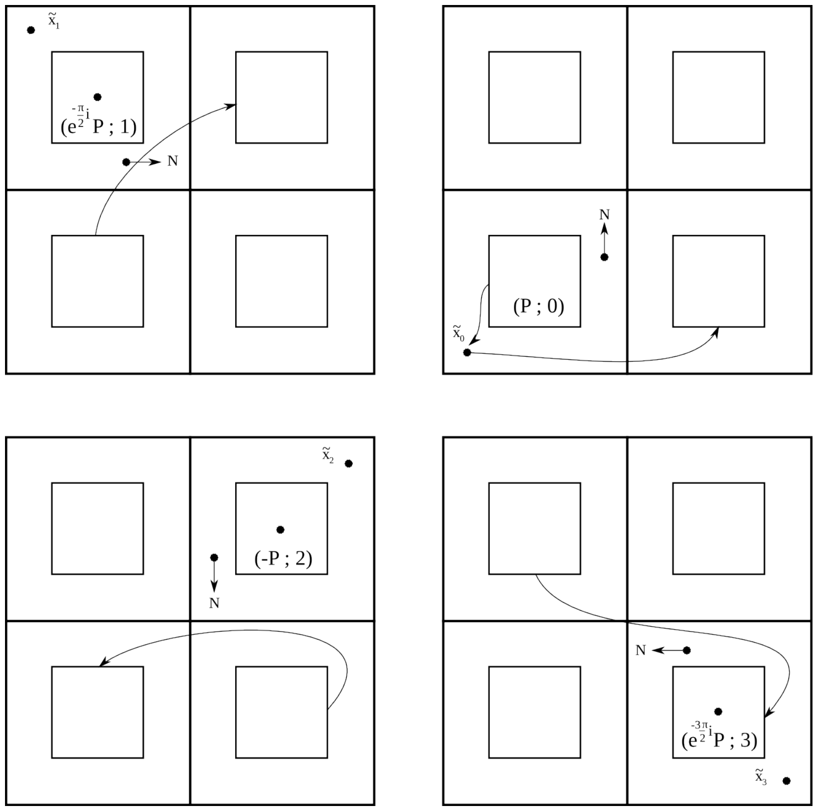



Figure 12: Branched cover of $\mathbf{U}$ of degree four.

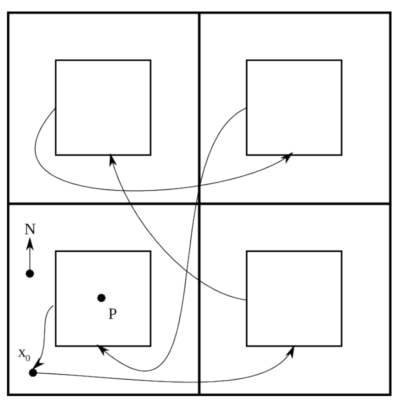
**Definition 6.** *$\tilde{\mathbf{U}}_0^\circ$ is the surface obtained as the quotient $\mathbf{P}'/\sim_{\mathbf{P}'}$. Denote paths $\zeta$ and $\eta$ on $\mathbf{P}$, parameterized by integers $m, n \in \mathbb{Z}$ and $t \in [-\frac{1}{4}, \frac{1}{4}]$, and defined:*
*$\zeta(t) = (2m + t) + i(2n - \frac{1}{2})$,*
*$\eta(t) = (2m + t) + i(2n + \frac{1}{2})$.*
*Let $\aleph = 2m + i2n \in \mathbb{C}$. The minimal relation $\sim_{\mathbf{P}'}$ is given as:*

$$\begin{aligned}(\zeta(t); j) &\sim_{\mathbf{P}'} (e^{i\frac{\pi}{2}}\ \overline{\zeta(t) - \aleph}; j+1)\\ (\eta(t); j) &\sim_{\mathbf{P}'} (e^{i\frac{\pi}{2}}\ \overline{\eta(t) - \aleph}; j+1)\end{aligned} \tag{3}$$

This relation is similar to $\sim_{\mathbf{P}}$ in that it translates line segments of squares surrounding even integer pairs to the origin and relates points on one edge of a square to points on an adjacent edge. Where it differs is that these adjacent edges now belong to a different copy of $\mathbf{P}$. It is in this cyclic manner that edges are glued that allows for trivial linear holonomy on arbitrary paths by rotating each copy of $\mathbf{P}$ accordingly. For example, here is a path on a section of the surface in the neighborhood of $P = 2m + i2n \in \mathbb{C}$ $(m, n \in \mathbb{Z})$ after rotation:




and its subsequent projection onto $\mathbf{U}^{\circ}$ :




## 3 Construction

The surface itself has a variety of constructions. One interpretation is the periodic tiling of "*half-cubes*," or sets of three squares sharing a single vertex.

We will introduce some objects and definitions to be used later in the paper. We give the Necker cube surface a concrete structure as a subset of $\mathbb{R}^3$. Take the following unit squares in $\mathbb{R}^3$:
If we take the integer triple $(m, n, p)$ and require that the individual faces based around them has the triple lie on the plane $x + y + z = 0$, we obtain the sets

$$\mathbf{A} = \bigcup \{\mathbf{A}_{m,n,p} : m + n + p = 0\},$$
$$\mathbf{B} = \bigcup \{\mathbf{B}_{m,n,p} : m + n + p = 0\},$$
$$\mathbf{C} = \bigcup \{\mathbf{C}_{m,n,p} : m + n + p = 0\}.$$

**Definition 7.** *We denote the* ***Necker cube surface*** *by* $\mathbf{S}$*, and define it to be the surface of the union of unit squares of the form* $\mathbf{S} = \mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$*.*

## 3.1 Four-Fold Cover

This section describes how a branched translation cover of $\mathbf{U}$ is obtained by winding around cone singularities of the surface.

We denote the surface $\mathbf{U}$ with its countably many cone singularities removed by $\mathbf{U}^\circ$. $\mathbf{U}^\circ$ is an infinite-genus surface obtainable as a translation cover of a thrice-punctured torus (figure 10).

There is a nontrivial linear holonomy as can be seen in figure 7. The reader is encouraged to check that the total angle around each singularity is either $\frac{3\pi}{2}$ or $3\pi$ radians.

As to be expected, a parallel transport of a geodesic flow over an edge will act on the unit tangent bundle by a rotation of $\pm\frac{\pi}{2}$ radians. This is best exemplified by the periodic flow labeled $c$ in figure 7, where the geodesic flowing in the $\frac{1}{\sqrt{2}}(1, 1)$ can be seen as line segments on $\mathbf{P}$. When a unit direction is fixed, it is mapped to its equivalence class of directions in the orbit under $SO(2, \mathbb{Z})$ action.

If we begin flowing a unit vector in some direction there are only one of four possible directions in the ambient space of the surface (See figure 11 for further convincing). This is primarily what motivated taking four copies of the surface so as to lift a flow to a surface whose rotational symmetries could align the flow by associating every one of the four possible directions with its own "copy" of $\mathbf{P}$. $\tilde{\mathbf{U}}$ will become what is known as an infinite translation surface, equipped with a flat metric and trivial linear holonomy.

Surrounding each pair of even integers in $(2\mathbb{Z} + 2\mathbb{Z}i)^2 \subset \mathbb{C}$, there is an open unit square removed. Four copies of the domain $\mathbf{P}$ are indexed by elements in $\mathbb{Z}/4\mathbb{Z}$, and we consider the set of points $\mathbf{P}' = \mathbf{P} \times \mathbb{Z}/4\mathbb{Z}$. It is on $\mathbf{P}'$ that we define a minimal relation $\sim_{\mathbf{P}'}$ that is used to identify edges in the following manner:

We call this surface $\tilde{\mathbf{U}}_0$. It is another infinite surface constructed in a cyclic manner that complements $\mathbf{U}^\circ$'s holonomy. Winding around each cone singularity shows us that there are 4 distinct vertices for each removed square centered at $(2\mathbb{Z}+2\mathbb{Z}i)^2 \times \mathbb{Z}/4\mathbb{Z}$, each of cone angle $6\pi$. The surface with these singularities removed is denoted $\tilde{\mathbf{U}}_0^\circ$. The obvious covering map $p : \tilde{\mathbf{U}}_0^\circ \to \mathbf{U}^\circ$ is taken to be a projection map onto $\mathbf{U}^\circ$ induced by the map $\mathbf{P'} \hookrightarrow \mathbf{P}$, where $(z; j) \mapsto (z)$.

## 3.2 Monodromy Action on Fibers of $\mathbf{U}^\circ$

Let $x_0$ be a point on $\mathbf{U}^\circ$, and fix a point $\tilde{x}_0 \in \tilde{\mathbf{U}}_0^\circ$ in the fiber $p^{-1}(x_0) = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$. Consider the closed paths around a hole in $\mathbf{U}^\circ$ in figure 8, and their subsequent lifts onto $\tilde{\mathbf{U}}_0^\circ$:

The kernel of the associated monodromy map $m : \pi_1(\mathbf{U}^\circ, x_0) \times p^{-1}(x_0) \to p^{-1}(x_0)$ of the cover $p$ is a cyclic permutation of 4 elements isomorphic to $SO(2,\mathbb{Z})$. The lifted kernel of this map is equal to $\pi_1(\tilde{\mathbf{U}}_0^\circ, \tilde{x}_0)$.

## 3.3 Translation Surface

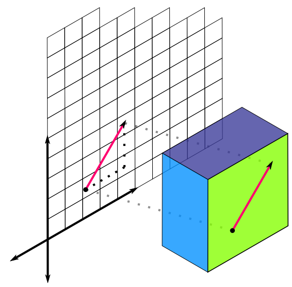
Now we consider rotating each plane about their origins so that the parallel transport of a unit vector over an edge acts trivially on that vector. We consider *North* on this new surface, call it $\tilde{\mathbf{U}}$, obtained in this way to respect $\mathbf{U}$'s notion of vertical direction (relative to the first plane). Let $P = 2m + i2n \in \mathbb{C}$ for some $m, n \in \mathbb{Z}$. Then the following diagrams illustrate the rotational relationships between directions in $\tilde{\mathbf{U}}$ and $\mathbf{U}$:

**Lemma 2.** *A geodesic on $\mathbf{U}$ is periodic if and only if it's lifted to a closed geodesic on $\tilde{\mathbf{U}}_0$.*

*Proof.* **Redo this maybe define the cover by monodromy map** Since a geodesic is undefined at conical singularities, it suffices to work with $\mathbf{U}^\circ$ and $\tilde{\mathbf{U}}_0^\circ$. Let $\gamma : [0,1] \to UT(\mathbf{U}^\circ)$ be a closed geodesic defined on the unit tangent bundle of $\mathbf{U}^\circ$ such that $\gamma(0) = \gamma(1) = (x_0, v_0)$ for constant unit vector $v_0 \in \mathbb{R}^3$. The covering map $p$ induces a map $p_* : UT(\tilde{\mathbf{U}}_0^\circ) \hookrightarrow UT(\mathbf{U}^\circ)$. The fiber $p_*^{-1}(x_0, v_0) = \{(\tilde{x}_0, V), (\tilde{x}_1, V), (\tilde{x}_2, V), (\tilde{x}_3, V)\}$ with $v_0 \in V$. Suppose that the lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{x_0}$ does not close, then $\tilde{\gamma}(1) \in \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$ Then $[\gamma] \notin \ker m$ and $m[\gamma] \neq \mathbf{I}_2$. Since $\tilde{\mathbf{U}}_0^\circ$ is simply connected, there is some $\tilde{\alpha} : [0,1] \to \tilde{\mathbf{U}}_0^\circ$ lifted from $\alpha \in \mathbf{U}^\circ$ such that $\tilde{x}_i \mapsto \tilde{x_0}$ for $i = 1,2,3$. Then $m[\alpha \circ \gamma] = m[\alpha] \cdot m[\gamma] \neq m[\alpha] \neq \mathbf{I}_2$ implies that $[\alpha] \in \ker m$, which is a contradiction since then $\tilde{\alpha}$ does not lift to a closed path.

Conversely let $\tilde{\gamma}$ be a closed geodesic on $\square$

**Definition 8.** *Take a trivial straight-line path in $S \subset \mathbb{R}^3$, $\ell : [0,1] \to \mathbf{S}$ such that for some $m, n, p \in \mathbb{Z}$, $\ell(t)$ is contained in exactly one of $\mathbf{A}_{m,n,p}, \mathbf{B}_{m,n,p},$ or $\mathbf{C}_{m,n,p}$ for all $t \in [0,1]$. The* ***initial trajectory angle*** *of a geodesic on* $\mathbf{S}$ *is given as $\theta = Arg(\Psi(\ell(1,s)) - \Psi(\ell(0,s)))$, where Arg is the principal argument of a vector in $\mathbb{C}$. The* ***initial trajectory*** *is the vector $\dot{v} = (\cos(\theta), \sin(\theta)) \in \mathbb{R}^2$.*
***Not sure if I should proceed with derivative or map***



We have two subsets of $\mathbb{Z}^2$ directions that partition the set of possible rational directions $\theta$.

If $v$ is rational, that is $v = k(x,y)$ for integers x,y, then we commonly refer to the pair $(x,y)$ as the *slope* of a given initial trajectory. For example, the following trajectory modeled on the surface is a closed geodesic on $\mathbf{U}$ with initial trajectory (51,31):

**Definition 9.** *A* ***geodesic flow*** *on a topological surface, $X$, is a function $\Phi_t^v : UT(X) \times \mathbb{R} \to UT(X)$ such that $\frac{d\Phi}{dt} = v \in UT(X)$.*

Experimental evidence indicates that geodesics in the odd-odd directions are periodic, while geodesics in the even-odd directions are drift-periodic. We will prove the theorem below.

**Theorem 3.** ***Dynamics of Geodesic Flow on the Necker cube surface.*** *Obtain $\theta$ and $v$ as described in Definition 3. Denote the non-singular unit-speed geodesic flow with initial point $s \in \mathbf{U}^0$ in direction $(\theta \mod \frac{\pi}{2}) \sim \phi \in UT(\mathbf{U}^0)$ by $\Phi_t^\phi : UT(\mathbf{U}^0) \times \mathbb{R} \to UT(\mathbf{U}^0)$ such that $\frac{d\Phi}{dt} = \phi \in UT(X)$ Then the following is true:*

*(i) (Periodic) There exists a $t_0 > 0$ such that $\Phi_{t+t_0}^\phi(s) = \Phi_t^\phi(s)$ if and only if $v \in \mathcal{O}$.*

*(ii) (Drift-Periodic) There exists a $t_0 > 0$ such that $\Phi_{t+t_0}^\phi(s) = \Phi_t^\phi \circ \Phi_{t_0}^\phi(s) \neq \Phi_t^\phi(s)$ if and only if $v \in \mathcal{E}$.*

## 3.4 Translation Surfaces

We recall some common definitions, theorems, and properties of translation surfaces and their $\mathbb{Z}^d$-covers.

**Definition 10.** *A translation surface is a pair $M = (S, \omega)$ formed of a connected Riemann surface $X$ and a holomorphic 1-form $\omega$ on $S$ which is not identically zero and satisfies:*

1. *$\Sigma \subset S$ is a discrete subset of cone singularities of $S$.*
2. *$\omega = z^k dz$ at points in $\Sigma$ of cone angle $2(k+1)\pi$.*
3. *$S \backslash \Sigma$ admits a translation surface $M^0 = (S\backslash\Sigma, dz)$, where $dz$ is a flat metric on $S\backslash\Sigma$.*

The set of all orientation-preserving affine automorphisms of $M$ forms the group $\mathrm{Aff}^+(M)$. The corresponding *Veech group* of M is the image of the group morphism $D : \mathrm{Aff}^+(M) \to SL(2,\mathbb{R})$, where $D$ is the derivative map of an element of $\mathrm{Aff}^+(M)$, and denote it by $V(M)$. A surface is said to by *Veech* if its Veech group is commensurable to $SL(2,\mathbb{R})$. The unit tangent bundle on $M\backslash\Sigma$, denoted $UT(M)$, is isomorphic to $M \times \mathbb{R}/2\pi\mathbb{Z}$. Let $\phi, \psi$ be two compatible charts between $M$ and $\mathbb{C}$ such that $\phi(x_0) \in \mathbb{C}$ for $x_0 \in M\backslash\Sigma$, and let $F^t_\theta : z \mapsto z + te^{i\theta}$ be a translation flow on $\mathbb{C}$. The non-singular geodesic flow on the unit tangent bundle of $M$, $G^t : UT(M) \times \mathbb{R} \to UT(M)$, is well defined on and given by $G^t(x_0, \theta) = (\psi^{-1} \circ F^t_\theta \circ \phi(x_0), \theta)$.
A *maximal geodesic* on $M$ is a re-parameterized closed geodesic $\gamma : [0,1] \to M$ obtained from $G_t$, and we consider $\gamma \in \pi_1(M, x_0)$. A normal subgroup $N \triangleleft \pi_1(M, x_0)$ is associated with the quotient group $\Delta = \pi_1(M, x_0)/N$. $\tilde{M}$ is called the $\Delta$-cover of $M$ if $p : \tilde{M} \to M$ is a cover of $M$ and for some fixed $\tilde{x}_0 \in p^{-1}(x_0)$, $\pi_1(M, x_0)/\Delta = \pi_1(\tilde{M}, \tilde{x}_0)$. $\Delta$ has a unique lift to the group of Deck transformations acting on $\tilde{x}_0 \in \tilde{M}$, or its orbit under $\mathrm{Aut}(p)$. Compact translation surfaces are oriented, hence why their fundamental groups are isomorphic to $\mathbb{Z}^{2g}$ (g is the genus of $M$). Therefore $\Delta \simeq \mathbb{Z}^d$, and $\tilde{M}$ is called a $\mathbb{Z}^d$ *translation cover* of $M$. Translation covers have Deck transformation groups isomorphic to $\mathbb{Z}^d$.
When $\gamma \in \pi_1(M, x_0)$ is a closed curve, we denote its abelianization as $[\gamma] \in H_1(M, R)$, with coefficients in unit ring $R$, and its unique lift to $\tilde{M}$ under the covering map $p$ as $\tilde{\gamma}$. Suppose you have some set of homology classes, $\{a_1, \ldots, a_n\} = \Gamma \subset H_1(M, R)$, such that $\mathrm{span}(\Gamma) = H_1(M, R)$. We call $\Gamma$ the *spanning set*, and express an element $[\beta] \in H_1(M, R)$ as $[\beta] = \Sigma^n_{j=1} x_j a_j$, where $x_1, \ldots x_n \in R$.

**Definition 11.** *Denote the non-degenerate, bi-linear* ***intersection number*** *between two homology classes $[\alpha], [\beta]$ as $i([\alpha], [\beta])$, where*

$$i : H_1(M, R) \times H_1(M, R) \to \mathbb{Z}.$$

*returns the total number of intersections and its sign corresponds to the sign of the angle that $[\beta]$ makes relative to $[\alpha]$.*

We mainly concern ourselves with $\mathbb{Z}^2$ covers for the purposes of this paper. Consider the set $Q \subset N \subset \pi_1(M, x_0)$ to be the spanning set of N.

**Definition 12.** *Consider two sets,* $q, r \subset \Gamma$*, such that* $span(q, r)$ *is the abelianization of* $N$*. There exists a group morphism* $s : N \to Aut(p)$ *that takes a generator of* $N$ *to a generator of* $Aut(p)$*, a translational symmetry of* $\tilde{M}$ *isometric to* $(m, n) \in \mathbb{Z}^2$*, call it* $T^{m,n}$*. Note that* $dim(N) = 2g - 2$*, and* $s$ *is an epimorphism when* $g > 2$*. Obtain* $q$ *as the abelianization of elements in* $Q$ *that are mapped to* $T^{0,\pm 1}$*, the vertical translations. Likewise, take* $r$ *to be the set of abelianized elements of* $Q$ *that are mapped to* $T^{\pm 1,0}$*, the horizontal translations. A path non-trivially intersecting with a curve in* $q$ *corresponds to a horizontal translation, while an intersection with* $r$ *corresponds to a translation in the vertical direction. This is because an intersection of a closed path* $\beta$ *with an element* $a \in s^{-1}(T^{0,\pm 1})$ *implies that* $\beta$ *at the time of intersection would have had to have been homotopic to* $b \in s^{-1}(T^{\pm 1,0})$ *instead (assuming that* $a, b$ *intersect).*
*For example, let* $t$ *be the time of that intersection, and* $t_0$ *be the time of the previous one. Then* $\tilde{\beta}(t) = T^{\pm 1,0}(\tilde{\beta}(t_0))$ *would have moved horizontally, and not vertically.*
*Since intersection number is bi-linear and non-degenerate, we obtain the* ***group homomorphism as a representation of a lifted path*** *on* $\tilde{M}$*:*

$$\Omega_{q,r} : \pi_1(M, x_0) \to \mathbb{Z}^2;\ \beta \mapsto (i(q, [\beta]), i(r, [\beta])).$$

**Lemma 3.** *Let* $\beta \in \pi_1(M, x_0)$*. Then* $\beta$ *lifts to* $\tilde{\beta} \in \pi_1(\tilde{M}, \tilde{x}_0)$ *if and only if* $\beta \in Ker\ \Omega_{q,r}$*.*

*Proof.* do this later □

This lemma becomes an integral part of the proof of the main theorem, and allows us to show how induced homomorphisms of a surface's homology by its affine automorphisms allows for us to generalize whenever a geodesic on the base surface lifts to a closed geodesic on the cover.

### 3.5 Acknowledgements

## 4 Four-fold Cover of U

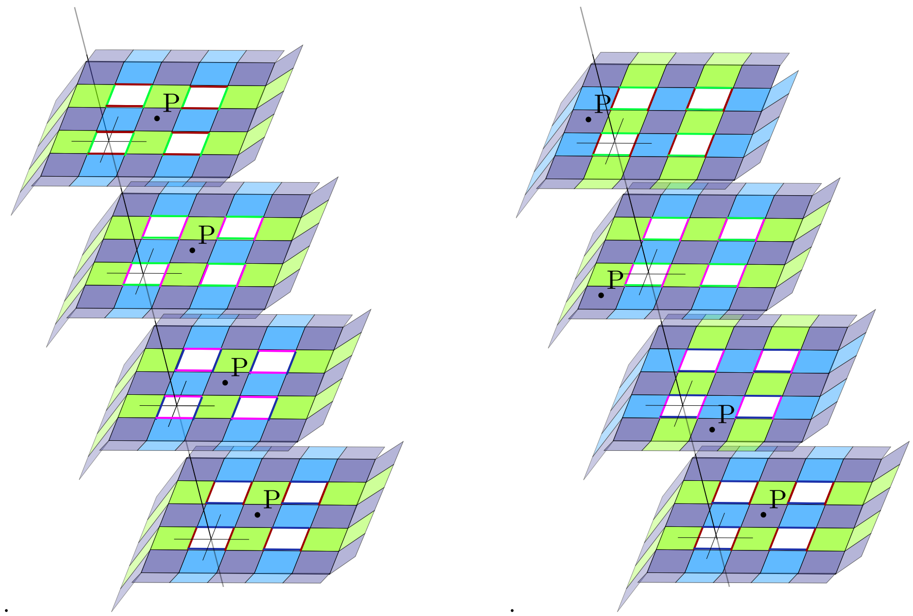



Figure 13: Four-fold cover isometry and the preimage of a point in $\mathbf{U} \backslash Sing(\mathbf{U})$.

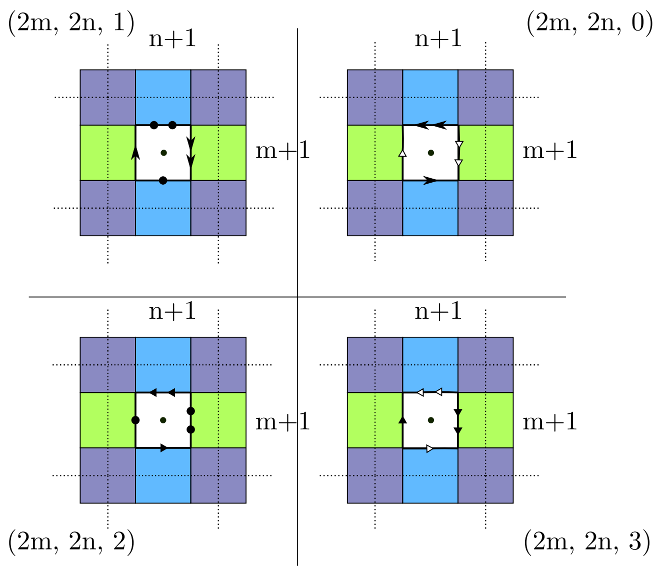



Figure 14: Branched cover associating every direction with one plane.

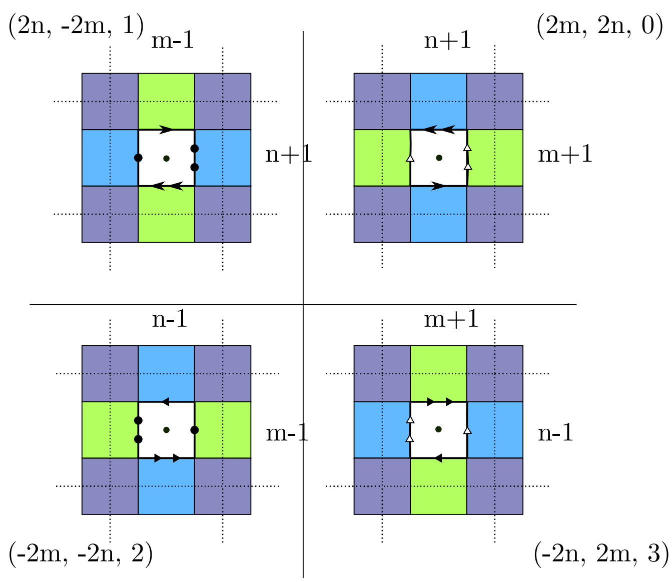



Figure 15: Infinite-type translation surface obtained by rotating each copy of the fundamental domain accordingly.

The quotient under the group action of translational symmetries is isomorphic to $\mathbb{Z}^2$ since the orbit of any point in the fundamental domain is a lattice in the space.

**Theorem 4.** *The translational symmetries of $\tilde{\mathbf{U}}$'s fundamental domain induce symmetries on the surface isomorphic to $\mathbb{Z}^2$.*

*Proof.* Let $(z; j) \in \mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$ and define the group action $T_0^{m,n} : \mathbb{C} \times \mathbb{Z}/4\mathbb{Z} \to \mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$ as $T^{m,n}(z; j) = (z + 2e^{-j\frac{i\pi}{2}}(m + in); j)$. This translation acts faithfully on the preimages of $\mathbf{U} \backslash Sing(\mathbf{U})$, and respects edge identifications of $\tilde{\mathbf{U}}$, thereby making it an isometry of the surface. Consider a group homomorphism, $T_0^{m,n} \mapsto m + in$ onto the plane of Gaussian integers, $\mathbb{Z}[i]$. The exponential function is never zero, so the identity of the translation group is $T_0^{0,0}$. This is an isomorphism since it is clearly surjective and any non-trivial element of $T^{m,n}$ could not possibly map to the identity element of $\mathbb{Z}[i]$, regardless of the value of j. Since $\mathbb{Z}^2$ is isomorphic to $\mathbb{Z}[i]$, it is isomorphic to $T_0^{m,n}$ as well. $\square$

**Definition 13.** *The automorphism $T^{m,n} : \tilde{\mathbf{U}} \to \tilde{\mathbf{U}}$ is an **induced translation** of $\tilde{\mathbf{U}}$ as a result of the previous theorem.*

We take the quotient of $\tilde{\mathbf{U}}$ by the action of the induced isometry of $T^{m,n}$, to get a translation base surface, $\mathbf{M}$.

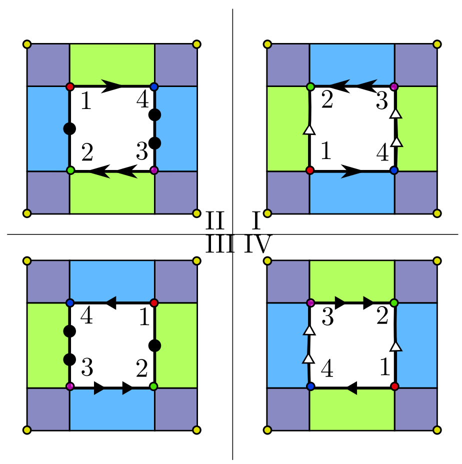



Figure 16: Compact translation surface, $\mathbf{M}$, covered by the infinite surface with edges and cone singularities (1,2,3,4) identified. The Roman numerals are meant to identify each quotient with a plane in the cover, $\tilde{\mathbf{U}}$.

It is clear that this is a square-tiled Veech surface when realized as a finite staircase:

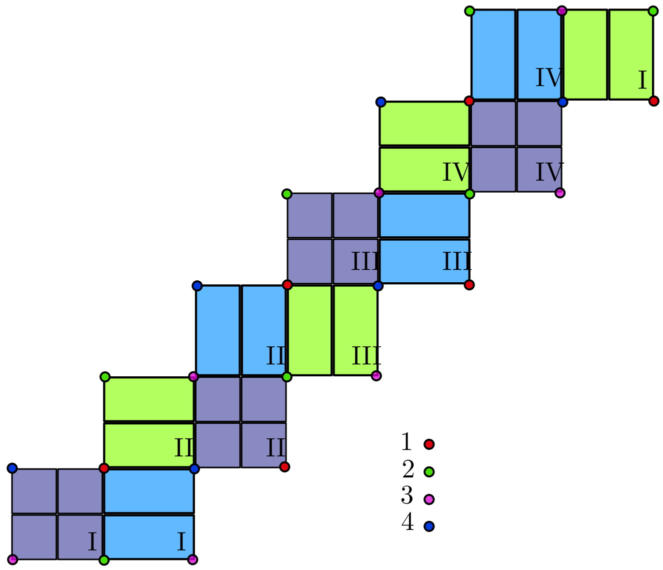



Figure 17: The staircase Veech surface with directional planes and vertices identified. All edges are paired by translations. Two adjacent squares have opposite edges identified. The top edge of the bottom-left square is glued to the bottom edge of the top-right square (both labeled I). Likewise, the bottom edge of the bottom-left square is identified with the top edge of the top-right square.

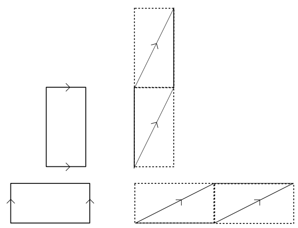
This surface is obtained as a ramified cover of the unit square torus. It is a translation surface and is therefore equipped with a **holomorphic one-form**, a collection of charts from neighborhoods of $\mathbf{M}$ to $\mathbb{C}$ such that any neighborhood away from $\mathrm{Sing}(\mathbf{M})$ has a *flat* induced Euclidean metric. A theorem of Gutkin and Judge tells us that its Veech group is commensurable to $\mathrm{SL}(2, \mathbb{Z})$ and is therefore a Veech surface. We look at some of its affine maps, and generate a subgroup $\mathbb{X} \subset \mathrm{Aff}^+(\mathbf{M})$ by the following transformations:

(i) Multi-twists of the surface as global diffeomorphisms given by Dehn-twists of its cylinder decomposition in horizontal and vertical directions with derivatives:

$$\left\{ \begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix} \right\}.$$

we call $\mathbf{A}^{\pm 1}$, $\mathbf{B}^{\pm 1}$, respectively.

A Dehn-twist on each cylinder in the cylinder decomposition of $\mathbf{M}$ in horizontal and vertical directions gives way to these global affine diffeomorphisms:



(ii) Rotation group generated by a $+\frac{\pi}{2}$ rotation of the surface fixed about the center of the second square on the bottom of the staircase, an order four isometry on $\mathbf{M}$ denoted $\mathbf{R}$.

(iii) Order 2 translation of the surface that moves the bottom left-most square to the square right next to it, denoted $\mathbf{H}$.

(iv) Order 2 translation of the surface that takes the bottom right-most square to the one right above it, denoted $\mathbf{V}$.

**Definition 14.** *The group $\mathbb{X}$ is the isometry group generated by affine maps $\mathbf{A}$,$\mathbf{B}$,$\mathbf{R}$, $\mathbf{H}$, and $\mathbf{V}$. The image of the derivative map on elements in $\mathbb{X}$ is denoted $\mathbb{X}'$ and generated by matrices*

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

*denoted $\mathbf{A}'$, $\mathbf{B}'$, and $\mathbf{R}'$ in that order.*

It is not immediately apparent if these affine maps generate $\mathrm{Aff}^+(\mathbf{M})$, or if their derivatives generate $V(\mathbf{M})$, its Veech group. We use these to induce homomorphisms on $H_1(X, \mathbb{Q})$. A spanning set of $H_1(\mathbf{M}, \mathbb{Q})$ is obtained as the set of homology classes of the core curves of X's cylinder decompositions in both vertical and horizontal directions:

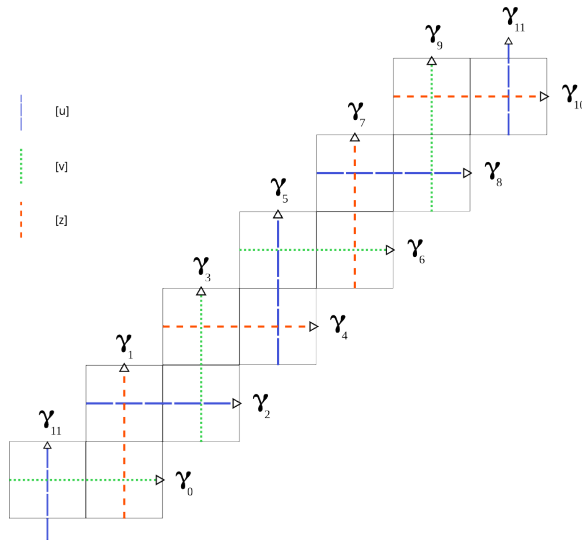



Figure 18: Cylinder core curves with u,v, and z homology classes that determines the $\mathbf{Z}^2$-cover.

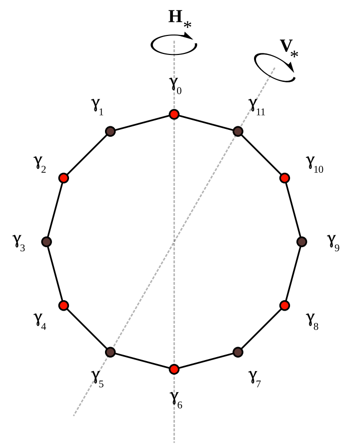
**Definition 15.** *The set of abelianized cylinder core curves is denoted as* $\Gamma = \{\gamma_i : i = 0, \ldots, 11\} \subset H_1(\mathbf{M}, \mathbb{Q})$.

**Remark.** *We use 12 elements to span homology, although a basis requires only 10. It's not impossible to determine the relations between these core curve classes, but it is not necessary. A* $12 \times 12$ *matrix of these core curve cylinder decompositions to their intersection numbers with adjacent curves is rank 10, as to be expected.*

The induced homomorphisms of $H_1(\mathbf{M}, \mathbb{Q})$ have come from affine maps that have various effects on these core-curves. We use a 12-gon to represent the set of curves, and show how these elements act on them. The multi-twists add curves to adjacent curves, and the translation maps permute them. The reader is encouraged to check these for themselves.

e.g. for $\mathbf{H}, \mathbf{V} \in \text{Aff}^+(\mathbf{M})$,

**H & V** The effect that these two translations have on the 12-gon is a reflection about these lines. Observed by keeping track of the squares and core curves after **H** and **V** have acted on **X**.




**Definition 16.** *The induced homomorphisms of $H_1(\mathbf{M}, \mathbb{Q})$ are obtained from the affine subgroup $\mathbb{X}$ and denoted $\mathbb{X}_*$. The associated homomorphisms on the spanning set $\Gamma$ are given as:*

$$\begin{aligned}
\mathbf{A}_*^k \circ [\gamma_i] =& [\gamma_i] + \frac{k}{2}(1-(-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}]) \\
\mathbf{B}_*^k \circ [\gamma_i] =& [\gamma_i] + \frac{k}{2}(1+(-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}]) \\
\mathbf{R}_* \circ [\gamma_i] =& (-1)^i [\gamma_{1-i \ mod \ 12}] \\
\mathbf{H}_* \circ [\gamma_i] =& [\gamma_{12-i \ mod \ 12}] \\
\mathbf{V}_* \circ [\gamma_i] =& [\gamma_{10-i \ mod \ 12}]
\end{aligned}$$

**Definition 17.** *The homology classes $u, v, z$ are given as the following sums of core curves:*

$$\begin{aligned}
[u] &= -[\gamma_2] + [\gamma_5] + [\gamma_8] - [\gamma_{11}], \\
[v] &= +[\gamma_0] - [\gamma_3] - [\gamma_6] + [\gamma_9], \\
[z] &= +[\gamma_1] + [\gamma_4] - [\gamma_7] - [\gamma_{10}].
\end{aligned}$$

**Theorem 5.** *The fundamental group of the $\mathbb{Z}^2$-cover is obtained by lifting the kernel of the closed paths of $\mathbf{M}$ of the homomorphism:*

$$\Omega_{u,v} : \pi_1(\mathbf{M}, x_0) \to \mathbb{Z}^2;\ \beta \mapsto (i(u,[\beta]), i(v,[\beta])),\ \textit{where}$$

$$i : H_1(\mathbf{M}, \mathbb{Q}) \times H_1(\mathbf{M}, \mathbb{Q}) \to \mathbb{Z}.$$

*is the intersection number of two homology classes.*

*Proof.* We know from Theorem 4 that the translational symmetries of $\tilde{\mathbf{U}}$ induced by $T^{m,n}$ is isometric to $\mathbb{Z}^2$. Since $\mathbf{M}$ is a genus 5 base surface, we know that $\pi_1(\mathbf{M}, x_0) \simeq \mathbb{Z}^{10}$, and the associated cover satisfies $\mathbf{M} = \tilde{\mathbf{U}}/(\pi_1(\mathbf{M}, x_0)/N)$, such that $N$ is a normal subgroup of $\pi_1(\mathbf{M}, x_0)$. This means that $N \simeq \mathbb{Z}^8$. The eight core curve classes are the abelianized forms of $\gamma_0, \gamma_2, \gamma_3, \gamma_5, \gamma_6, \gamma_8, \gamma_9$, and $\gamma_{11}$ that span N. The classes and their signs are obtained from Figure 16 as the outer regions identified by the translations of $T^{m,n}$. Thus any closed path on $\mathbf{M}$ is lifted to a closed path on the cover under the quotient map only when a path has a trivial intersection number with the classes. $\square$

Two paths are homologous if they return the same intersection number with the classes of closed core cylinder curves of $\mathbf{U}$ that span its homology. The classes u and v are obtained from the group group action of $T^{m,n}$ on the cover.

**Definition 18.** $\mathbf{hol} : \mathbf{M} \backslash \mathit{Sing}(\mathbf{M}) \to \mathbb{C}$ *is the holonomy vector pulled back from a non-singular path* $\gamma$ *in* $\mathbf{M}$ *onto the complex plane given by* $\mathbf{hol}(\gamma) = \int_\gamma dz$.

We denote the **closed path** $\alpha$, such that $\mathbf{hol}(\alpha) = 6 + 6i$, and show it is homologous to the closed geodesic with the same holonomy vector. The slope one direction also decomposes $\mathbf{M}$ into two cylinders by a series of saddle connections of length $\sqrt{2}$ between singularities:

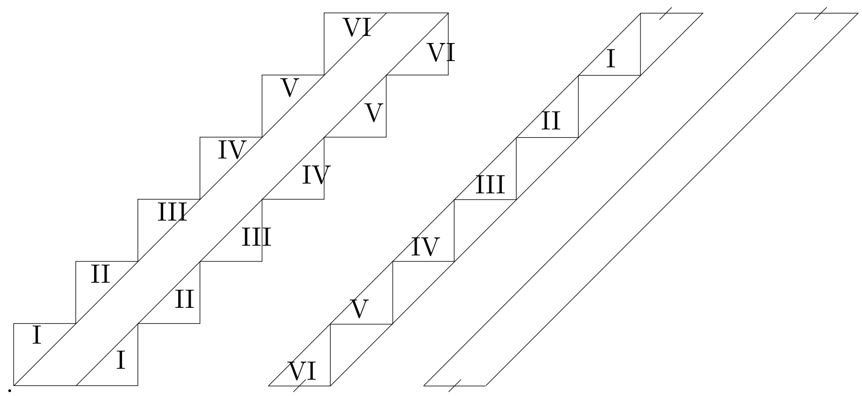



Figure 19: The two right-most cylinders $C_1$ (labeled) and $C_2$(unlabeled).

The circumferences of these two cylinders are $6\sqrt{2}$. Geodesic flows on this surface are well defined, and rational directions

**Definition 19.** *Let* $\omega_t^\theta : [0, 1] \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbf{M}$ *be the* ***maximal geodesic flow*** *on the surface in direction* $\theta$ *such that* $\omega_0^\theta = \omega_1^\theta = x_0 \in \mathbf{M} \backslash \mathit{Sing}(\mathbf{M})$.
$\omega_t^{\frac{\pi}{4}}$ *is the geodesic flow in the* ***slope one direction****, and* $\chi$ *is its image in* $\mathbf{M}$ *and element of* $\pi_1(\mathbf{M}, x_0)$.

**Lemma 4.** $\alpha$ *is homologous to* $\chi$ *in* $\mathbf{M} \backslash \mathit{Sing}(\mathbf{M})$.

*Proof.* Let $\chi$ be a geodesic contained in either $C_1$ or $C_2$. Since a geodesic does not admit singularities, it is the image of a closed path on $X\backslash\mathrm{Sing}(X)$ with initial point $x_0$ on the strips of $C_1$ and $C_2$ with boundaries removed, denoted $C_1'$, $C_2'$. Express $[\alpha]$ as $\Sigma_{j=0}^{11} \frac{1}{2}\gamma_j$ (a closed path climbing up the staircase). We show that the intersection numbers of $[\alpha]$ and $[\chi]$ are the same for every core cylinder curve $\gamma$, i.e. $i([\gamma_k], \Sigma_{j=0}^{11} \frac{1}{2}[\gamma_j]) = i([\gamma_k], [\chi])\ \forall k = 0, \ldots, 11$.
**Case one:** k is even. If k is even, then every curve $\gamma_k$ is oriented to the right. Since $\chi$ intersects every curve once, $i([\gamma_k], [\chi]) = 1$. No even indexed curves intersect eachother, so we need only consider when $j$ is odd. Now if $j$ is odd, it is incident (positively crossing) with only two horizontal curves, namely $\gamma_{j+1}, \gamma_{j-1}$. Therefore $i([\gamma_k], [\alpha]) = i([\gamma_{j-1}], \frac{1}{2}[\gamma_j]) + i([\gamma_{j+1}], \frac{1}{2}[\gamma_j]) = \frac{1}{2}(i([\gamma_{j-1}], [\gamma_j]) + i([\gamma_{j+1}], [\gamma_j])) = \frac{1}{2}(1+1) = 1$.
**Case two:** k is odd. If k is odd, then $[\chi]$ will have an intersection number of $-1$ with $[\gamma_k]$ since odd-indexed core curves are oriented upwards. Now since k is odd, we only consider when $j$ is even. Similarly, this means that $\gamma_j$ negatively intersects the two vertical core curves with adjacent indices. Hence, $i([\gamma_k], [\alpha]) = i([\gamma_{j-1}], \frac{1}{2}[\gamma_j]) + i([\gamma_{j+1}], \frac{1}{2}[\gamma_j]) = \frac{1}{2}(i([\gamma_{j-1}], [\gamma_j]) + i([\gamma_{j+1}], [\gamma_j])) = \frac{1}{2}(-1-1) = -1$.
We know intersection number to be bilinear and non-degenerate on homology. So if $\alpha$ and $\chi$'s abelianizations admit the same intersection numbers for every curve in the spanning set of $H_1(\mathbf{M}, \mathbb{Q})$, then $[\alpha] = [\chi]$. $\square$

**Theorem 6.** *$\chi \in \pi_1(\mathbf{M}, x_0)$ lifts to $\tilde{\chi} \in \pi_1(\tilde{\mathbf{U}}, \tilde{x}_0)$*

*Proof.* From Lemma 2, $[\chi] = [\alpha]$, so $\Omega_{u,v}(\chi) = \Omega_{u,v}(\alpha)$. Since $i([u], [\alpha]) = -i([\gamma_2], [\alpha] + i([\gamma_5], [\alpha]) + i([\gamma_8], [\alpha]) - i([\gamma_{11}], [\alpha]) = -1 + (-1) + 1 - (-1) = 0$ and $i([v], [\alpha]) = 1 - (-1) - 1 + (-1) = 0$, it follows that $\alpha, \chi \in \mathrm{Ker}\ \Omega_{u,v}$, and $\chi$ lifts to a closed geodesic on $\tilde{\mathbf{U}}$. $\square$

**Corollary 1.** *content...*

From here, we use $\alpha$ to show that the *only* trajectories that close on the Necker cube surface are those that are in vector direction $(a, b)$ such that $\gcd(a, b) = 1$ and $a, b$ are both odd. We call these **odd-odd** directions. We can make this claim because the group generated by the matrices

$$\begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix}$$

is the *Sanov subgroup* of $SL(2, \mathbb{Z})$ and only sends elements in the odd-odd set to itself. There are dualizations made between how these matrices skew a geodesic direction, and how their original affine transformations induce an effect homology. In a sense the kernel is obtained by the orbit of $\chi$ under $\mathbb{X}$ and its holonomy vector under $\mathbb{X}'$.

**Lemma 5.** *The actions of $\mathbb{X}'$ on $\mathcal{O}$ and $\mathcal{E}$ are closed in their respective sets.*

*Proof.* Since $\mathbb{X}'$ is generated by the elements $\mathbf{A}'$, $\mathbf{B}'$, and $\mathbf{R}'$, any matrix $G' \in \mathbb{X}'$ is of the form $G' = (\mathbf{A}')^{1_1} \circ (\mathbf{B}')^{1_2} \circ (\mathbf{R}')^{1_3} \circ (\mathbf{A}')^{2_1} \circ \ldots (\mathbf{A}')^{n_1} \circ (\mathbf{B}')^{n_2} \circ (\mathbf{R}')^{n_3}$, where $i_k \in \mathbb{Z}$ for $i = 1, \ldots, n$ and $k = 1, 2, 3$. Let $x = \begin{pmatrix} p \\ q \end{pmatrix}, y \in \mathcal{O}$, and consider the equation $G'x = y$. Observe that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^l x = \begin{pmatrix} p + 2jq \\ q \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^m x = \begin{pmatrix} p \\ q + 2mp \end{pmatrix}$ for any $l, m \in \mathbb{Z}$. Also note that for any $j \in \mathbb{Z}$, $(\mathbf{R}')^m x = \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} -q \\ p \end{pmatrix}, \begin{pmatrix} -p \\ -q \end{pmatrix}, \begin{pmatrix} q \\ -p \end{pmatrix}$ when $j \mod 4 \equiv 0, 1, 2, 3$, respectively. In any case, the product of any power of a generator of $\mathbb{X}'$ and any $x \in \mathcal{O}$ is an element of $\mathcal{O}$ . By letting $l = i_1, m = i_2$, and $j = i_3$, we first consider the base case when $i = n$. Let $G' = G'_1 \circ \cdots \circ G'_n$, such that $G'_i = (\mathbf{A}')^{i_1} \circ (\mathbf{B}')^{i_2} \circ (\mathbf{R}')^{i_3}$. Since $n_1, n_2, n_3$ are arbitrary integers, $G'_n x \in \mathcal{O}$. Suppose for some $b < n - 1$, $G'_{n-b} \circ \cdots \circ G'_n x = y' \in \mathcal{O}$. Therefore $y' = (G'_1 \circ \cdots \circ G'_b)^{-1} y$, which implies that $(G'_1 \circ \cdots \circ G'_b)^{-1}$ preserves the set $\mathcal{O}$. Otherwise, if $y \in \mathcal{E}$, there exists at least one $G'_i$ for $1 < i < b$ and $\tau \in \mathcal{E}$ such that $G_i'^{-1} \tau = (\mathbf{R}')^{-i_3} \circ (\mathbf{B}')^{-i_2} \circ (\mathbf{A}')^{-i_1} \tau \in \mathcal{O}$, a contradiction. Since elements in $\mathbb{X}'$ are invertible, $G'_1 \circ \cdots \circ G'_b$ must also map $\mathcal{O}$ to itself. Left multiply both sides of the equation to show that $G'_1 \circ \cdots \circ G'_n x = G'x = y$. By the principle of strong induction, this holds for all $0 < b \leq n$. Since $G'$ is invertible and an arbitrarily chosen element of $\mathbb{X}'$, it follows that $x \in \mathcal{O}$ if and only if $y \in \mathcal{O}$ and $\mathcal{O}$ is closed under $\mathbb{X}'$.
The proof for when $x \in \mathcal{E}$ is made in the same way. □

Now a trajectory in the horizontal direction has a directional vector of (1,0). The orbit of this vector by the Veech group is the set of all **even-odd** vectors. We also know that in this direction a geodesic is drift-periodic (See figure 1). The Veech group of $\mathbf{M}$ preserves these properties. Suppose you had some closed geodesic on $\mathbf{M} \backslash Sing(\mathbf{M})$ called $\beta$ such that $\beta = h(\alpha)$, where $h \in \mathrm{Aff}^+(\mathbf{M})$, and $h_*$ is its induced homomorphism. Then we want to show that

$$(i([\beta], [u]), i([\beta], [v])) = (i([\alpha], h_*^{-1}[u]), i([\alpha], h_*^{-1}[v])) = (0, 0).$$

But first, we look at some of the properties of the group $\mathbb{X}_*$.

**Theorem 7.** *Let $\mathbb{X}_*$ be the group generated by $\mathbf{A}_*, \mathbf{B}_*, \mathbf{R}_*, \mathbf{H}_*$, and $\mathbf{V}_*$. Let $G = \langle \mathbf{A}_*, \mathbf{B}_* \rangle$, $T = \langle \mathbf{H}_*, \mathbf{V}_* \rangle$, and $R = \langle \mathbf{R}_* \rangle$. Then the following is true:*

*(i) $G$ is a free subgroup of $\mathbb{X}_*$ of rank two.*

*(ii) $T$ is a finite cyclic subgroup of $\mathbb{X}_*$ and a centralizer of $G$.*

*(iii) $R$ is a finite cyclic subgroup of $\mathbb{X}_*$, and a normalizer of $G$.*

*Proof.* Let $h_*^j = \mathbf{A}_*^{k_j} \circ \mathbf{B}_*^{g_j} \in G$ for $k_j, g_j \in \mathbb{Z}$, $j = 1, \ldots, n$.
*(i).* When $\mathbf{A}_*$ and $\mathbf{B}_*$ act on $\gamma_i$, it is only ever trivial if i is even for $\mathbf{A}_*$ or i is odd on $\mathbf{B}_*$. Since i cannot be both odd and even at the same time, there is no

relation between the two generators and therefore $G$ is free.
*(ii)* It is up to the reader to show that $T$ has the relations $\mathbf{H}_*^2 = \mathbf{V}_*^2 = (\mathbf{H}_*\mathbf{V}_*)^3 = id_*$, and is isomorphic to the rotational group of the hexagon generated by reflections about adjacent vertices of a 12-gon. Observe that $\mathbf{H}_* \circ \mathbf{A}_*^{k_j} \circ [\gamma_i] = \mathbf{H}_* \circ [\gamma_i] + \frac{k_j}{2}(1-(-1)^i)(\mathbf{H}_* \circ [\gamma_{i-1}] + \mathbf{H}_* \circ [\gamma_{i+1}]) = [\gamma_{-i}] + \frac{k_j}{2}(1-(-1)^i)([\gamma_{1-i}] + [\gamma_{-i-1}]) = \mathbf{A}^{k_j} \circ [\gamma_- i] = \mathbf{A}^{k_j} \circ \mathbf{H}_* \circ [\gamma_i]$, and $\mathbf{V}_* \circ \mathbf{A}_*^{k_j} \circ [\gamma_i] = \mathbf{V}_* \circ [\gamma_i] + \frac{k_j}{2}(1-(-1)^i)(\mathbf{V}_* \circ [\gamma_{i-1}] + \mathbf{V}_* \circ [\gamma_{i+1}]) = [\gamma_{10-i}] + \frac{k_j}{2}(1-(-1)^i)([\gamma_{11-i}] + [\gamma_{9-i}]) = \mathbf{A}^{k_j} \circ [\gamma_1 0 - i] = \mathbf{A}^{k_j} \circ \mathbf{V}_* \circ [\gamma_i]$. In the same way one can show this to be true for $\mathbf{B}_*^{g_j}$, and we can see that $T$ is a centralizer of $G$.
*(iii)* $R$ is obviously cyclic and finite since an isomorphism is obtained as $\mathbf{R}_* \mapsto \mathbf{R}' \in SO(2, \mathbb{Z})$.
Note that $\mathbf{R}_* \circ \mathbf{A}_*^{k_j} \circ [\gamma_i] = \mathbf{R}_* \circ [\gamma_i] + \frac{k_j}{2}(1-(-1)^i)(\mathbf{R}_* \circ [\gamma_{i-1}] + \mathbf{R}_* \circ [\gamma_{i+1}])$
$= (-1)^i[\gamma_{1-i}] + \frac{k_j}{2}(1-(-1)^i)((-1)^{i-1}[\gamma_{2-i}] + (-1)^{i+1}[\gamma_{-i}])$
$= (-1)^{1-i}([\gamma_{1-i}] - \frac{k_j}{2}(1+(-1)^{1-i})([\gamma_{2-i}] + [\gamma_{-i}]))$
$= (-1)^{1-i}\mathbf{B}_*^{-k_j} \circ [\gamma_{1-i}] = \mathbf{B}_*^{-k_j} \circ (-1)^{1-i}[\gamma_{1-i}] = \mathbf{B}_*^{-k_j} \circ \mathbf{R}_* \circ [\gamma_i]$.
Likewise, $\mathbf{R}_* \circ \mathbf{B}_*^{g_j} \circ [\gamma_i] = \mathbf{A}_*^{-g_j} \circ \mathbf{R}_* \circ [\gamma_i]$. □

**Remark.** *It can be easily shown that $\mathbb{X}'$ has similar properties.*

**Lemma 6.** *Let $h_* \in \langle \mathbf{A}_*, \mathbf{B}_* \rangle$. Then $h_* \circ [\alpha] \in H_1(\mathbf{M}, \mathbb{Q})$ can be expressed as $h_* \circ [\alpha] = \frac{1}{2}(c_1\Sigma_{j=0}^5[\gamma_{2j}] + c_2\Sigma_{j=0}^5[\gamma_{2j+1}])$ for $c_1, c_2 \in \mathbb{Z}$.*

*Proof.* Let $\Sigma_{j=0}^5[\gamma_{2j}] = \Sigma\Gamma_{even}$, $\Sigma_{j=0}^5[\gamma_{2j+1}] = \Sigma\Gamma_{odd}$, and $\Sigma_{j=0}^{11}[\gamma_j] = \Sigma\Gamma$. Let $h_* = h_*^n \circ \cdots \circ h_*^1$, and $h_*^i = \mathbf{A}_*^{k_i} \circ \mathbf{B}_*^{g_i}$ for $k_i, g_i \in \mathbb{Z}$, $i = 1, \ldots, n$. Compose these two homomorphisms and obtain $\mathbf{A}_*^{k_i} \circ \mathbf{B}_*^{g_i}(\Sigma\Gamma) = (4g_ik_i + 2k_i)\Sigma\Gamma_{even} + 2g_i\Sigma\Gamma_{odd} + \Sigma\Gamma$. Let $c_i^1 = (4g_ik_i + 2k_i), c_i^2 = 2g_i$, and solve for $h_*^{i+1} \circ h_*^i \circ \Sigma\Gamma$:

$$
\begin{aligned}
h_*^{i+1} \circ h_*^i \circ (\Sigma\Gamma) &= h_*^{i+1} \circ (c_i^1\Sigma\Gamma_{even} + c_i^2\Sigma\Gamma_{odd} + \Sigma\Gamma) \\
&= c_i^1 h_*^{i+1} \circ (\Sigma\Gamma_{even}) + c_i^2 h_*^{i+1} \circ (\Sigma\Gamma_{odd}) + h_*^{i+1} \circ (\Sigma\Gamma) \\
&= 2g_{i+1}\Sigma\Gamma_{odd} + (4g_{i+1}k_{i+1} + 2k_{i+1})\Sigma\Gamma_{even} + \Sigma\Gamma \\
&\quad + c_i^1(4g_{i+1}k_{i+1}\Sigma\Gamma_{even} + 2g_{i+1}\Sigma\Gamma_{odd} + \Sigma\Gamma_{even}) \\
&\quad + c_i^2(2k_{i+1}\Sigma\Gamma_{even} + \Sigma\Gamma_{odd}) \\
&= \Sigma\Gamma + (c_i^1 + (c_i^1 + 1)(4g_{i+1}k_{i+1}) + (c_i^2 + 1)2k_{i+1})\Sigma\Gamma_{even} \\
&\quad + (c_i^2 + (c_i^1 + 1)2g_{i+1})\Sigma\Gamma_{odd} \\
\text{Let } c_{i+1}^1 &:= (c_i^1 + (c_i^1 + 1)(4g_{i+1}k_{i+1}) + (c_i^2 + 1)2k_{i+1}), \\
c_{i+1}^2 &:= (c_i^2 + (c_i^1 + 1)2g_{i+1}).
\end{aligned}
$$

From these recursive definitions and a finite sequence of integers, $\{k\}_i, \{g\}_i$, observe then that
$h_* \circ [\alpha] = h_* \circ [\frac{1}{2}\Sigma\Gamma] = \frac{1}{2}h_* \circ [\Sigma\Gamma] = \frac{1}{2}[c_n^1\Sigma\Gamma_{even} + c_n^2\Sigma\Gamma_{odd} + \Sigma\Gamma]$
$= \frac{1}{2}[(c_n^1+1)\Sigma\Gamma_{even} + (c_n^2+1)\Sigma\Gamma_{odd}]$. Further simplify by letting $c_1 = c_n^1+1, c_2 = c_n^2 + 1$. □

**Lemma 7.** *Let $h_* \circ [\alpha] \in H_1(\mathbf{M}, \mathbb{Q})$. Then for $a \in \langle \mathbf{H}_*, \mathbf{V}_* \rangle$ and $b \in \langle \mathbf{R}_* \rangle$, the following is true:*

$$\begin{aligned} a \circ h_* \circ [\alpha] &= h_* \circ [\alpha] \\ b \circ h_* \circ [\alpha] &= \frac{1}{2}[c_1' \Sigma\Gamma_{even} + c_2' \Sigma\Gamma_{odd}] \\ h_* \circ b \circ [\alpha] &= \frac{1}{2}[c_1'' \Sigma\Gamma_{even} + c_2'' \Sigma\Gamma_{odd}] \end{aligned}$$

*Proof.* By Theorem 4, a is a centralizer of the group so $a \circ h_* \circ [\alpha] = h_* \circ a \circ [\alpha] = h_* \circ \frac{1}{2} a \circ [\Sigma\Gamma]$. Since a is a cyclic permutation of the set $\Gamma$, it acts trivially on $\Sigma\Gamma$. Therefore, $a \circ h_* \circ [\alpha] = h_* \circ \frac{1}{2}[\Sigma\Gamma] = h_* \circ [\alpha]$.
By theorem 4, $a \circ \mathbf{A}_*^{k_i} \circ \mathbf{B}_*^{g_i} = \mathbf{B}_*^{-k_i} \circ \mathbf{A}_*^{-g_i} \circ a$. Extend this property to $h_*$, and denote the normalized element as $h_{**}$, such that $b \circ h_* = h_{**} \circ b$. Note that $b(\Sigma\Gamma) = b(\Sigma\Gamma_{even} + \Sigma\Gamma_{odd}) = \Sigma\Gamma_{odd} - \Sigma\Gamma_{even}$. $b \circ h_* \circ [\Sigma\Gamma] = c_1 b \circ \Sigma\Gamma_{even} + c_2 b \circ \Sigma\Gamma_{odd} = c_1 \Sigma\Gamma_{odd} - c_2 \Sigma\Gamma_{even}$. So, $c_1' = -c_2$ and $c_2' = c_1$.
Since $h_*$ is arbitrary, let $h_{**} = g_*$ be generated by an integer sequence that defines the word and consider $h_* \circ b \circ [\Sigma\Gamma] = b \circ g_* \circ [\Sigma\Gamma] = c_1^* b \circ \Sigma\Gamma_{even} + c_2^* b \circ \Sigma\Gamma_{odd} = c_1^* \Sigma\Gamma_{odd} - c_2^* \Sigma\Gamma_{even}$. So, $c_1'' = -c_2^*$ and $c_2'' = c_1^*$. □

Now that every element in the orbit of $[\alpha]$ can be expressed as a linear combination of integers, it is simple to show they lift to a closed trajectory in the cover.

**Definition 20.** *Let* $\mathbf{dir} : UT(\mathbf{M} \backslash Sing(\mathbf{M})) \to \mathcal{O} \cup \mathcal{E}$ *be the injective map from $\mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{Z}^2$ given as $\mathbf{dir}(\theta) = (k_1 \cos(\theta), k_2 \sin(\theta)), k_1, k_2 \in \mathbb{R}$ such that* $\gcd(k_1 \cos(\theta), k_2 \sin(\theta)) = 1$.

**Theorem 8.** *(Sketch)*
*Any geodesic, $\beta$, in $\mathbf{M}$ lifts to a closed geodesic $\tilde{\beta}$ on $\tilde{\mathbf{U}}$ if and only if $\mathbf{dir}(Arg(\mathbf{hol}(\beta))) \in \mathcal{O}$.*

*Proof.* Call the quotient cover $p : \tilde{\mathbf{U}} \to \mathbf{M}$, and fix a point $\tilde{x_0} \in p^{-1}(x_0)$. Let $\beta = h(\chi)$, where $h \in \mathbb{X}$. We also obtain $[\beta] = h_* \circ [\alpha]$ from Lemma 2. Since $h$ sends geodesics to geodesics, h induces the following: $\mathbf{hol}(h(\chi)) = h'(\mathbf{hol}(\chi)) = h'(6+6i)$ for $h' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{X}'$. So, $\text{Arg}(h'(\mathbf{hol}(\chi)))$ $=\text{Arg}(6[(a+b)+i(c+d)])$ $=\text{Arg}(6h'(1+i))$. Lemma 3 states that for any $h' \in \mathbb{X}'$, $h'(\mathcal{O}) = \mathcal{O}$. Therefore there is no such geodesic of **even-odd** slope in the orbit of $\chi$. Otherwise $h', h \notin \mathbb{X}', \mathbb{X}$. Consequently, $\mathbf{dir}(Arg(\mathbf{hol}(\beta))) \in \mathcal{O}$.
From Lemma 5 we see that $[h(\chi)] = h_* \circ [\chi] = h_* \circ [\alpha] = \frac{1}{2}(c_1 \Sigma_{j=0}^5 [\gamma_{2j}] + c_2 \Sigma_{j=0}^5 [\gamma_{2j+1}])$ for $c_1, c_2 \in \mathbb{Z}$. Denote the sums as $\Sigma\Gamma_{even}$ and $\Sigma\Gamma_{odd}$.
Therefore, $2i([u], h_* \circ [\alpha]) = c_1 i([u], \Sigma\Gamma_{even}) + c_2 i([u], \Sigma\Gamma_{odd})$
$= c_1(-i([\gamma_2], 0) + i([\gamma_5], [\gamma_6] + [\gamma_4]) + i([\gamma_8], 0) - i([\gamma_{11}], [\gamma_{10}] + [\gamma_0]))$
$+ c_2(-i([\gamma_2], [\gamma_1] + [\gamma_3]) + i([\gamma_5], 0) + i([\gamma_8], [\gamma_7] + [\gamma_9]) - i([\gamma_{11}], 0))$
$= c_1(-(0) + (-1-1) + (0) - (-1-1)) + c_2(-(1+1) + (0) + (1+1) - (0)) = 0.$

Similarly, $2i([v], h_* \circ [\alpha]) = c_1 i([v], \Sigma\Gamma_{even}) + c_2 i([v], \Sigma\Gamma_{odd})$
$= c_1(i([\gamma_0], 0) - i([\gamma_3], [\gamma_2] + [\gamma_4]) - i([\gamma_6], 0) + i([\gamma_9], [\gamma_8] + [\gamma_{10}])$
$+ c_2(i([\gamma_0], [\gamma_{11}] + [\gamma_1]) - i([\gamma_3], 0) - i([\gamma_6], [\gamma_5] + [\gamma_7]) + i([\gamma_9], 0)$
$= c_1((0) - (-2) - (0) + (-2)) + c_2((2) - (0) - (2) + (0)) = 0.$
Therefore, $\Omega_{u,v}(h(\chi)) = (0,0)$, and $h(\chi) = \beta \in$ Ker $\Omega_{u,v}$ for all $h \in \mathbb{X}$. By Theorem 2, $\beta$ lifts to $\tilde{\beta} \in \pi_1(\tilde{\mathbf{U}}, \tilde{x}_0)$. Let $\theta =$Arg($\mathbf{hol}(\beta)$). Then $\omega_t^\theta$ at $x_0$ lifts to $\tilde{\omega}_t^{p^{-1}(\theta)} \in \tilde{\mathbf{U}}\backslash\text{Sing}(\tilde{\mathbf{U}})$.
Now suppose instead that $\beta = h(\gamma_i)$. Then $\mathbf{dir}(\beta) = \frac{1}{2}(1 + (-1)^i, 1 - (-1)^i)$. According to Lemma 3, $h'(\mathcal{E}) = \mathcal{E}$. Thus we have no geodesic in the **odd-odd** directions obtained from the orbits of $(1,0)$ and $(0,1)$. For contradiction, suppose that $h(\gamma_i) \in$ **Ker** $\Omega u, v$. Then $(i(h_* \circ [\gamma_i], [u]), i(h_* \circ [\gamma_i], [v])) = (i([\gamma_i], h_*^{-1} \circ [u]), i([\gamma_i], h_*^{-1} \circ [v])) = (0,0)$. Let $h_*^{-1} \circ [u] = \Sigma_{j=0}^{11} x_j [\gamma_j]$, and $h_*^{-1} \circ [v] = \Sigma_{j=0}^{11} y_j [\gamma_j]$. Note that since $\gamma_i$ intersects $\gamma_{i\pm 1}$, $i([\gamma_i], h_*^{-1} \circ [u]) = (-1)^{i+1}(x_{i-1} + x_{i+1})$ and $i([\gamma_i], h_*^{-1} \circ [v]) = (-1)^{i+1}(y_{i-1} + y_{i+1})$.
**Unfinished**.. □

**Conjecture.** ***Dynamics of Geodesic Flow on the Necker cube surface.*** *Obtain $\theta$ and $\vec{\theta}$ as described in Definition 3. Denote the non-singular unit-speed geodesic flow with initial point $s \in (\mathbf{U}\backslash Sing(U))$ in direction $[\theta] \sim \phi \in UT(\mathbf{U}\backslash Sing(\mathbf{U}))$ by $F_t^\phi : \mathbf{U} \times \mathbb{R}_0^+ \to \mathbf{U}$ on $(\mathbf{U}, \mu)$, where $\mu$ is a flow-invariant measure. Then the following is true:*

- *(i) (Periodic) There exists a $t_0 > 0$ such that $F_{t+t_0}^\phi(s) = F_t^\phi(s)$ if and only if $\vec{\theta} \in \mathcal{O}$.*
- *(ii) (Drift-Periodic) There exists a $t_0 > 0$ such that $F_{t+t_0}^\phi(s) = F_t^\phi(s) + c$, where $c \in \mathbf{U}$ is a non-trivial translation of a point in $\mathbf{U}$, if and only if $\vec{\theta} \in \mathcal{E}$.*

*Proof.* (Sketch)
Denote the covering maps $f : \tilde{\mathbf{U}} \to \mathbf{U}$, $p : \tilde{\mathbf{U}} \to \mathbf{M}$, and fix a point $\tilde{x}_0 \in f^{-1}(s), p^{-1}(x_0)$, for $x_0 \in \mathbf{M}$. $f^{-1}([\theta]) = \{x : x = \theta + n\frac{\pi}{2}, n \in \mathbb{Z}\} = [\theta] \subset UT(\tilde{\mathbf{U}}\backslash\text{Sing}(\tilde{\mathbf{U}}))$ given by the four-fold cover and rotations of each individual plane. This gives us a relation between the two tangent bundles, where the translation four-fold cover has the standard $\mathbb{R}/2\pi\mathbb{Z}$ unit tangent fiber. $\theta'$ is the direction associated to the flow $\omega_t^{\theta'} : [0,1] \to \mathbf{M}$ on the translation surface. Since the cover is translation, $p^{-1}(\theta') = \theta' = \theta + n\frac{\pi}{2}$. First suppose that $\vec{\theta} \in \mathcal{O}$. Then $\theta$ is identified with the set of directions that close on $\tilde{\mathbf{U}}$. From Theorem 5, $\omega_t^{\theta'}$ lifts to a closed geodesic $\tilde{\omega}_t^{\theta'}$. Given $\mathbf{hol}(\omega) = \int_\omega dz$, we obtain a period for the unit-speed flow, $t_0 = |\mathbf{hol}(\omega)|$. That is, $\tilde{F}_t^{\theta'} : \mathbb{R}_0^+ \to \tilde{\mathbf{U}}$ such that $\frac{d}{dt}\tilde{F}_t = \frac{1}{|\mathbf{hol}(\omega)|}$. Then $F_t^\phi = F_t^{[\theta']} = f \circ \tilde{F}_t^{\theta'}$. The period carries over since there is no concern over a trajectory returning to $\tilde{x}_0$ in a different direction. Otherwise, the geodesic $\omega_t^{\theta'}$ on $\mathbf{M}$ would have closed in $0 < t < 1$. Now suppose that $\vec{\theta} \in \mathcal{E}$. Identifying it with $\theta'$, we see that $\omega$ in direction $\theta'$ is not an element of **Ker** $\Omega_{u,v}$ from Theorem 5. Therefore, $\Omega_{u,v}(\omega) = (m,n) \simeq T^{m,n}$ and lifting the terminal point

$\omega(1)$, $\tilde{\omega}(1) = T^{m,n}(\tilde{\omega}(0)) = T^{m,n}(\tilde{x}_0)$. The period remains unchanged, in that $\tilde{F}^{\theta'}_{t+\mathbf{hol}(\omega)} = \tilde{F}^{\theta'}_t + T^{m,n}(\tilde{x}_0)$. Therefore, $F^{\phi}_{t+t_0}(s) = f \circ \tilde{F}^{\theta'}_t(\tilde{x}_0) + f \circ T^{m,n}(\tilde{x}_0)$. Conversely, suppose $F_t$ is periodic. Then $[\theta] = \phi = [\theta']$, which defines directional flows $\tilde{F}^{\phi}_t$. According to Theorem 5, $\tilde{F}^{\phi}_t$ will close if and only if $\phi \subset \mathcal{O}$. $\phi$ is the orbit of $\vec{\theta'}$ under the 90 degree rotational matrix. This matrix does not alter the length or period of a geodesic. Thus, $F^{\phi}_t$ is exactly one of the flows $\tilde{F}^{\phi}_t$. Likewise, if $F^{\phi}_t$ is drift-periodic then $F^{\phi}_{t+t_0} = f \circ \tilde{F}^{\phi}_t + f \circ T^{m,n}$. $T^{m,n}$ is trivial if and only if $\theta' \in \mathcal{O}$. Therefore, $\theta' \in \mathcal{E}$, and $[\theta'] = \phi$. $\square$

There is still much work to do in terms of cleaning up the proofs and organizing the final paper.

# Conclusion

What I ultimately aim to do is port these results on X's homology back to the Necker cube surface. I want do it in such a way that the final theorem is bi-conditional. To do so, I imagine I can take a vector image of a small segment of a geodesic in $\mathbb{R}^3$ and project it onto the isometric flattening of the Necker Cube surface to obtain a direction (or classes of equivalent directions), and relate it to the unit tangent bundle of $\mathbf{M}$.

In addition, I would also like to find a formula for the arc-length of a geodesic based on direction alone. Knowing that $\text{hol}(\alpha) = 6 + 6i$ means that the induced Euclidean metric on $\mathbf{M}\backslash\text{Sing}(\mathbf{M})$ gives the geodesic an arc-length of $6\sqrt{2}$. I would like to show that:

$$\int_\beta |dz| = |hol(h(\alpha))| = |h'(hol(\alpha))|,$$

where $h' \in V(\mathbf{M})$ is the derivative of h, and $\beta = h(\alpha)$. We know that is true on the translation surface, but it's a matter of then showing the translation quotient, branch-cover, and the Necker cube surface have the same induced Euclidean metric of these non-singular geodesics. (It would not be surprising considering that the surface is built out of subsets of planes.) Even more of a problem is finding a way to solve for a matrix in the Sanov subgroup that brings (1,1) to the desired odd-odd slope.

Even though the main theorem remains unproven for now, it is an enjoyable property of the surface and venturing in its understanding involves exploring some fascinating niches in algebraic topology and dynamical systems.

Needless to say, it has been a very busy summer for me. I am grateful for the opportunity to really take some time to delve deep into numerous topics that I have been very interested in for a while. I sincerely doubt it would have been possible without the summer internship, and I have the Rich scholarship committee to thank for that. I would also like to extend my gratitude to Dr. Hooper. He has been incredibly patient and kind throughout these last few months, and I would not have come this far in so short a time without his help.

# References


[1] Fred G. Martin *Robotics Explorations: A Hands-On Introduction to Engineering*. New Jersey: Prentice Hall.

[2] Flueck, Alexander J. 2005. *ECE 100*[online]. Chicago: Illinois Institute of Technology, Electrical and Computer Engineering Department, 2005 [cited 30 August 2005]. Available from World Wide Web: (http://www.ece.iit.edu/ flueck/ece100).