Periodicity of Geodesics on the Necker Cube Surface

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Abstract

Although it is most recognizable as M.C. Escher's adaptation of an optical illusion popularized by Louis Albert Necker, the Necker cube surface has a complex structure and is, in analaytic language, described as an infinite, flat (locally isometric to $\mathbb{R}^2$ on smooth sections), $\mathbb{R}^3$-embeddable Euclidean cone surface with countably many singularities of cone angles 3π and $\frac{3\pi}{2}$. Its geometric construction is described as the gluing of infinitely many unit cubes along their edges so that every face shares an edge with exactly four others. A unit-speed geodesic on this surface has an initial trajectory angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ given up by the surface's local plane isometry. Given a rational vector direction in $\mathbb{R}^2$ (components are relatively prime integers) obtained from θ, one can classify every periodic/drift-periodic rational geodesic on the surface. In addition its total period/length and, in the case where it is drift-periodic, translational drift as Euclidean distance in its ambient space is given by solving a system of simple linear Diophantine equations.

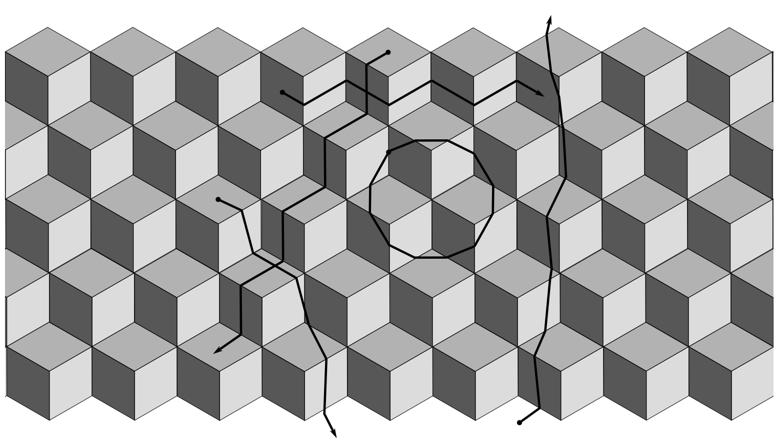


Figure 1: Periodic and drift-periodic flows on the Necker cube surface

1 Introduction

Put your Problem statement here! Example of a Citation[**?**, p.219]. Here's Another Citation [**?**]

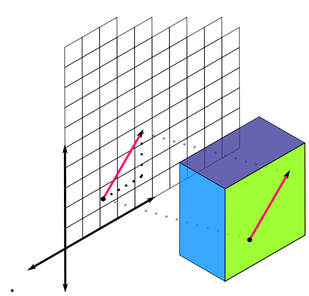
The Necker cube has made numerous appearances in the work of mathematicians, crystallographers, and scientists interested in human visual systems before and after appearing in the works of M.C. Escher [cite]. Louis Albert Necker was credited for having discovered the optical illusion, and studying its geometric properties [cite]. The solid presentation of the cube, when rendered as a flat surface with three rear faces removed, achieves a similar effect when its three visible faces are shaded in a particular way. The Necker cube surface is then obtained by gluing infinitely many copies along its outer edges as in Figure 1.

Dynamical classification of geodesics on flat, periodic surfaces such as the Necker cube surface ultimately boils down to studying the surface's various symmetries and how the rotational holonomy group acts on the unit tangent bundle of the surface as the unit vector (derivative of the geodesic) is parallel transported along its path. Loops on polyhedra punctured at its cone singularities tend to have discrete holonomy groups as its relative holonomy (holonomy of contractible loops) is trivial.

Remark. *Familiarity is assumed on the part of the reader with covering space theory, translation (Veech) surfaces, and their associated Veech groups. For general surveys on these topics: [cite], [cite].*

1.1 Discussion of Results

Our initial experiments strongly supported the theory that there would be a correlation between a choice of trajectory angle and dynamical properties of a geodesic on such a symmetric object. Rightfully so, a surface composed of infinitely many cubes an induced flat metric where every neighborhood is locally isometric to the Euclidean plane. Via the parallel transport of a unit tangent vector over the sharp edges of the surface denoted $\mathbf{U}^\circ$, it becomes necessary to think of the geodesic as a sequence of line segments contained in the faces of each cube on the surface's embedded form in $\mathbb{R}^3$. **??**.



Let u_0 be a point on the surface that is contained in the interior of a face and consider a tangent unit vector $v \in \mathbb{R}^3$ based at point u_0. Each face is parallel to a 2-dimensional subspace of $\mathbb{R}^3$ spanned by exactly two basis vectors, so one component of v is 0. Call this projected vector $v_0 \in \mathbb{R}^2$.

The angle v_0 makes relative to a choice of basis is the *initial trajectory angle* of the unit-speed geodesic on the surface, $\Phi_t : \mathbb{R} \to \mathbf{U}^\circ$. A rational initial trajectory angle in $\mathbb{R}^2$ falls into one of two categories:

Definition 1. *Let v_0 be a unit vector of the form $\frac{1}{k}(x, y) \in \mathbb{R}^2$ with $x, y \in \mathbb{Z}$ and $k = \sqrt{x^2 + y^2} \in \mathbb{R}$. We say v_0 is an* ***odd-odd*** *vector if its components are relatively prime and both odd. We denote the* ***set of all odd-odd directions*** *by $\mathcal{O}$. We say that v_0 is an* ***even-odd*** *vector if its components are relatively prime and of opposite parity. We denote the* ***set of all even-odd directions*** *by $\mathcal{E}$.*

For convenience we call the disjoint union of these sets $\mathcal{S}$. This paper aims to depict the relationship between initial trajectory angles and dynamical properties of a geodesic on the surface. Denote $\mathbf{U}^\circ$'s translational isometries by $\mathrm{Trans}(\mathbf{U}^\circ)$. Recall that a geodesic on the surface is *periodic* if there exists some $T > 0$ such that $\Phi(t+T) = \Phi(t)$ for all $t \in \mathbb{R}$ and a geodesic is *drift-periodic* if there exists some $T' > 0$ and non-trivial $p \in \mathrm{Trans}(\mathbf{U}^\circ)$ such that $\Phi(t+T') = p(\Phi(t))$ for all $t \in \mathbb{R}$. In either case we call this value the *period* of Φ if it is the *smallest* possible value that satisfies our definition. We additionally require that at no point does Φ encounter a cone singularity of the surface, in which case the geodesic behavior is non-deterministic.

$$\text{Let } \mathbb{X}' = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle \tag{1}$$

Section [blank] explains the significance of this group. Let $M = \begin{bmatrix} s & u \\ w & z \end{bmatrix} \in \mathbb{X}'$.

Theorem. *(Dynamical Properties of a Geodesic on $\mathbf{U}^\circ$)*
Let $\Phi_t : \mathbb{R} \to \mathbf{U}^\circ$ be a non-singular unit-speed geodesic on the Necker cube surface such that $\Phi(0) = u_0$ and $\Phi'(0) = v \in \mathbb{R}^3$, isometric to $v_0 \in \mathbb{R}^2$. Let $\overline{u}_t \in \mathbb{R}^3$ be Φ_t's coordinate in $\mathbf{U}^\circ$'s ambient space at time t. Then the following is true:

(i) *Φ is periodic if $v_0 \in \mathcal{O}$.*

(ii) *Suppose (i) is true. If $M(kv_0) = (1, 1)$ then Φ has period $T = 6\sqrt{2}||kv_0||$.*

(iii) *Φ is drift-periodic if $v_0 \in \mathcal{E}$.*

(iv) *Suppose (iii) is true. If $M(kv_0) = (1, 0)$ or $M(kv_0) = (0, 1)$ then Φ has period $T = 2||kv_0||$, and $d_E(\overline{u}_t, \overline{u}_{t+T}) = ||kv_0||\sqrt{2}$.*

Remember to prove that such an M exists.

Here $||\ ||$ is the standard Euclidean norm in $\mathbb{R}^2$, and $d_E : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is the Euclidean distance. The proof for this theorem can be found in XXX and follows from Theorems YYYY and ZZZZZ.

1.2 Acknowledgements

-Pat Hooper
- Vincent Delecroix, Ferrán Valdez, pascal hubert
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2 Periodic Tiling of Necker Cubes

This section will detail how the Necker Cube surface is constructed as a covering space of a thrice-punctured torus. Make the following identifications on 3 unit squares:

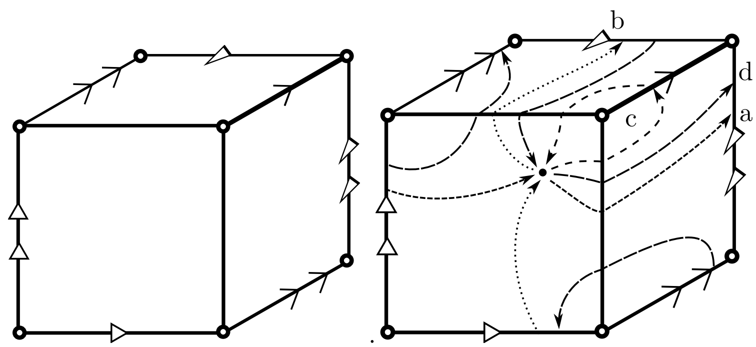
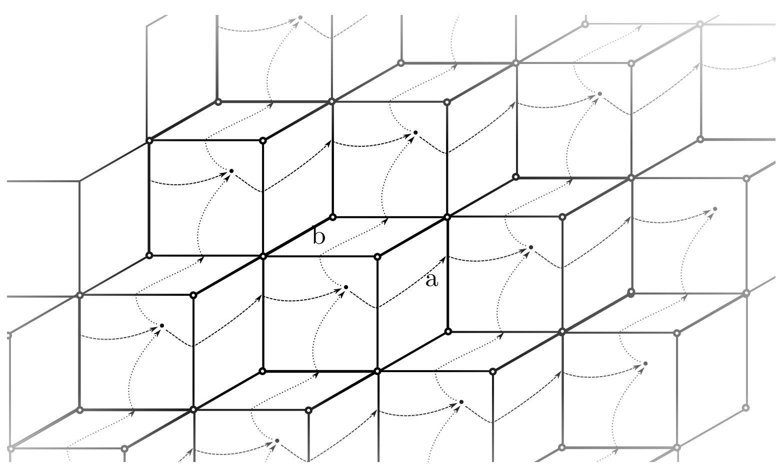


Figure 2: $\mathbf{G}^{\circ}$ with and without paths a, b, c, d.

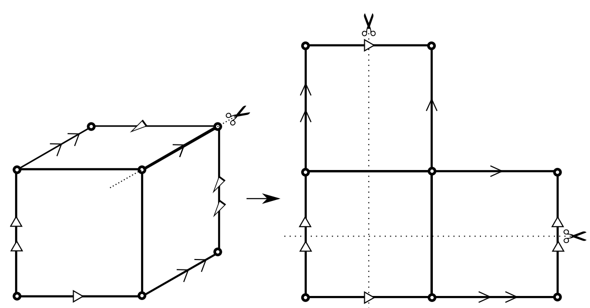
We will call this "half-cube" $\mathbf{G}$. It is a genus one piecewise smooth surface homeomorphic to $\mathbb{T}^2$. Every vertex has a cone angle of either 3π or $\frac{3\pi}{2}$. Let $\Sigma_{\mathbf{G}} \subset \mathbf{G}$ be the set of these three vertices. These are conical singularities of the surface. We denote the surface punctured at these points by $\mathbf{G}\backslash\Sigma_{\mathbf{G}} = \mathbf{G}^{\circ}$. The paths labeled a, b, c, d are independent of one another and span $\mathbf{G}^{\circ}$'s first fundamental group, $\pi_1(\mathbf{G}^{\circ}) \cong \langle a, b, c, d\rangle$, the free group of four generators. The unpunctured Necker cube surface $\mathbf{U}$ is embedded in $\mathbb{R}^3$ by tiling along $\mathbf{G}$'s outer edges:



From this diagram we see that $\mathbf{U}$ is a universal cover of $\mathbb{T}^2$. In this paper we are studying geodesic behavior on $\mathbf{U}$ and in order to do so, it is necessary to determine how the parallel transport of a unit vector around a cone singularity acts on the unit tangent bundle of the surface. But to study the path monodromy we require that these loops (see c, d in Fig 3) aren't contractible on the cube's sharp corners. Denote the countably infinite set of singularities of $\mathbf{U}$ by $\Sigma_{\mathbf{U}}$. The *Necker cube surface* is the punctured surface $\mathbf{U}^{\circ} = \mathbf{U} \backslash \Sigma_{\mathbf{U}}$. Observe from the previous diagram that $\pi_1(\mathbf{U}^{\circ})$ is the kernel of the following group homomorphism:

Definition 2. *Denote* $\varphi_1 : \pi_1(\mathbf{G}^{\circ}) \to \mathbb{Z}^2$ *as the group homomorphism such that* $c, d \mapsto (0,0)$*,* $a \mapsto (1,0)$*, and* $b \mapsto (0,1)$*.*

Take $\pi_1(\mathbf{U}^{\circ})$ to be $\ker \varphi_1$. Then $\Delta_{\varphi_1} = \pi_1(\mathbf{G}^{\circ})/\pi_1(\mathbf{U}^{\circ}) \cong \mathbb{Z}^2$ is the *deck group* of the covering map $\varphi_1^* : \mathbf{U}^{\circ} \to \mathbf{G}^{\circ}$. Now $\mathbf{G}^{\circ}$ is unfolded and flattened in order to study monodromy as it transports a vector in $\mathbb{R}^2$ along a path. Apply the following cutting and gluing operations to the surface:



This L-shaped torus is then taken apart and recovered as 2×2 torus missing a unit square with the following identifications:

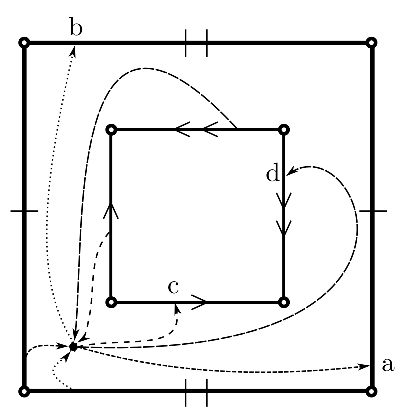


Figure 3: $\mathbf{G}^{\circ}$ and images of paths after reconstruction. The outermost vertex is *not* a puncture.

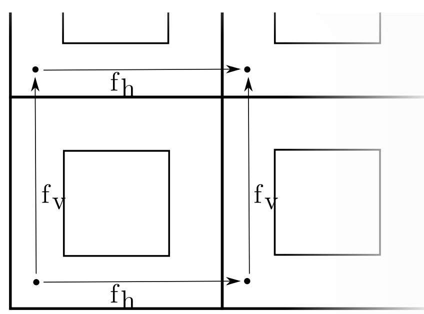
Consider a 2×2 sheet missing an open unit square and that square's vertices (punctures). Center this structure around $0 \in \mathbb{C}$ and recover $\mathbf{G}^\circ$ with the identifications made in Fig 3. Let $m, n \in \mathbb{Z}$. Similar identifications can be made on the following subset of $\mathbb{C}$ to recover $\mathbf{U}^\circ$:

$$\mathbf{P} = \mathbb{C} \setminus \bigcup_{m,n \in \mathbb{Z}} \{u + vi : \ |u - 2m| < \frac{1}{2}, \ |v - 2n| < \frac{1}{2}, \ u = 2m \pm \frac{1}{2} \ v = 2n \pm \frac{1}{2}\}. \tag{2}$$

Let $\aleph_{m,n} = 2m + i2n \in \mathbb{C}$ be the center of a removed square. Now let $\zeta_{m,n}(t) = (2m + t) + i(2n - \frac{1}{2})$ and $\xi_{m,n}(t) = (2m + t) + i(2n + \frac{1}{2})$ be paths on $\mathbf{P}$ that run along the edges of the squares, parameterized by $t \in (-\frac{1}{2}, \frac{1}{2})$ and $(m, n) \in \mathbb{Z}^2$. $\mathbf{U}^\circ$ is recovered as a topological quotient $\mathbf{P}/\sim$, where $\sim$ is the minimal relation on $\mathbf{P}$:

$$\begin{aligned} \zeta(t) &\sim e^{i\frac{\pi}{2}} \ \overline{\zeta(t) - \aleph} \\ \eta(t) &\sim e^{i\frac{\pi}{2}} \ \overline{\eta(t) - \aleph} \end{aligned} \tag{3}$$

The translations, $\pm 2, \pm 2i$, on $\mathbf{P}$ are isometries that preserve these identifications and induce isometries of $\mathbf{U}^\circ$. The group generated by these horizontal and vertical translations form the *group of Deck transformations*, $\mathrm{Deck}(\varphi_1^*)$, that is isomorphic to $\mathbb{Z}^2$:



Identify 2 with f_h, and $2i$ with f_v. Let $x_0 \in \mathbf{G}^\circ$ and fix a point in the fiber, $u_0 \in \varphi_1^* {-1}(x_0) \subset \mathbf{U}^\circ$. The action of $\mathbb{Z}^2$ on the fiber over x_0 is given by $(m, n) \cdot u_0 = (f_h^m \circ f_v^n)(u_0)$. The distance between u_0 and $(f_h^m \circ f_v^n)(u_0)$ is measured in its ambient space and taken to be the magnitude of $2m + i2n \in \mathbb{C}$.

Lemma 1. *Let $[\alpha]$ be homotopy class not in the kernel of φ_1. It will lift to an unclosed path terminating on $\varphi_1([\alpha]) \cdot u_0$.*

2.1 Monodromy Group Representation

A *rotational monodromy map* is a morphism from the first fundamental group of a surface to the *monodromy group* acting on the unit tangent bundle via parallel transport of a vector along its path. Existence of a parallel transport map is a consequence of the metric connection on a surface with complex structure. The monodromy map is well defined when the *holonomy group* is *discrete*. When the *relative holonomy* of contractible loops is trivial, the holonomy group is the set of attainable rotations of a unit vector upon returning to the base point after transporting it along a non-trivial loop. A *flat surface* like $\mathbf{G}^\circ$ has trivial relative holonomy as its neighborhoods are isometric to the plane. Denote the *unit tangent bundles* of $\mathbf{U}^\circ$ and $\mathbf{G}^\circ$ by $T^1\mathbf{U}^\circ$ and $T^1\mathbf{G}^\circ$. If α is a loop on $\mathbf{G}^\circ$, then $[\alpha]$ acts on a fixed vector $v_0 \in \mathbb{R}^2$ by rotation. In the case where α is *not* a geodesic segment on the surface of length L, $v_0 = \alpha'(0) \in \mathbb{R}^2$ may differ from $\alpha'(L)$ by some rotation.

Definition 3. *Define $\varphi_2 : \pi_1(\mathbf{G}^\circ) \to SO(2,\mathbb{Z})$ to be the group homomorphism where $a, b \mapsto I_2$, $c \mapsto R$, and $d \mapsto R^3$, where I_2 is the identity matrix and R is a positive 90° rotation matrix.*

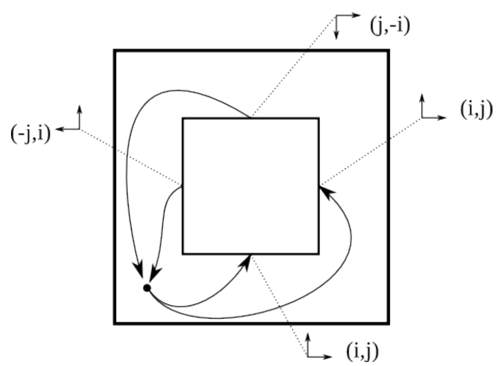


Figure 4: Effect a nontrivial loop has on basis vectors $(i, j) \in \mathbb{R}^2$.

Lemma 2. *$\varphi_2 : \pi_1(\mathbf{G}^\circ) \to \mathrm{SO}(2,\mathbb{Z})$ is the rotational monodromy representation of $\mathbf{G}^\circ$.*

Proof. From Figure 4 it's clear that c, d will rotate a unit vector by $\pm 90^\circ$ by parallel transport along their paths. These values are determined by the cone angles of the singularities they loop around. The parallel transport of a vector along a or b would be trivial since the outer edges they pass through are glued by translation. $\square$

Lemma 3. *Any element in Deck(φ_1^*) acting on $T^1\mathbf{U}^\circ$ has a trivial effect on holonomy.*

Proof. Since $\mathbf{G}^\circ = \mathbf{U}^\circ/\mathbb{Z}^2$, the deck group action lifts naturally to an action on their tangent vector spaces and unit tangent bundles via pushforward of deck transformations f_h, f_v on the bundle maps. i.e. $T^1\mathbf{G}^\circ = T^1\mathbf{U}^\circ/\mathbb{Z}^2$. This effect must be trivial since $\varphi_1(a) \sim f_h$ and $\varphi_1(b) \sim f_v$, and both homotopy classes

a, b have trivial holonomy.
How do f_h, f_v parallel transport a vector?
Well if $\varphi_1([a]) \cdot u_0 = f_h(u_0)$, then for ☐

Theorem 1. *Let $m = \varphi_2|_{\pi_1(\mathbf{U}^\circ)}$. Then m is the rotational monodromy map of $\mathbf{U}^\circ$.*

Proof. ☐

Corollary 1. *Let $\alpha : [0,1] \to \mathbf{G}^\circ$ be a loop on $\mathbf{G}^\circ$ with trivial holonomy and $\tilde{\alpha}$ its lift to $\mathbf{U}^\circ$. Then $\tilde{\alpha}$ has trivial holonomy as well.*

Proof. ☐

Definition 4. *Let $\mathbf{M}^\circ$ be the regular cover of $\mathbf{G}^\circ$ with fundamental group $\pi_1(\mathbf{M}^\circ) = \ker \varphi_2$.*

Showing that $\mathbf{M}^\circ$ is a degree four cover of $\mathbf{G}^\circ$ with trivial holonomy and deck group $\mathrm{SO}(2,\mathbb{Z})$ follows immediately from its construction.

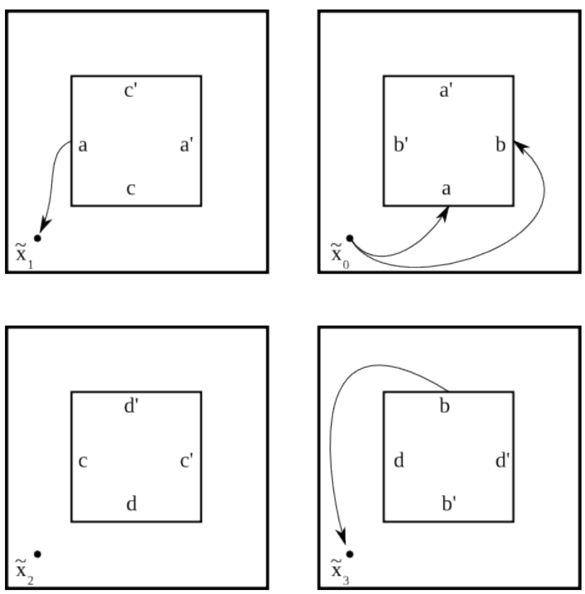


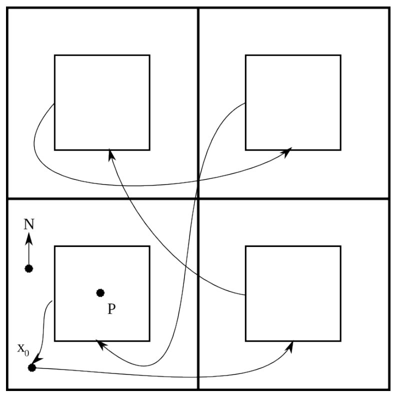
Figure 5: Lift of paths c, d to $\mathbf{M}^\circ$ (**RELABEL EDGES**)

2.2 A Four-Fold Cover

We now take these covers we built to construct a third one that covers them all. This is a four-fold cover of $\mathbf{U}^\circ$ and a $\mathbb{Z}^2$ translation cover of $\mathbf{M}^\circ$.

Definition 5. *Let $\tilde{\mathbf{U}}^\circ$ be the cover of $\mathbf{U}^\circ$ with fundamental group $\pi_1(\tilde{\mathbf{U}}^\circ) = \ker \varphi_2\big|_{\pi_1(\mathbf{U}^\circ)}$.*

It follows that $\ker \varphi_1\big|_{\pi_1(\mathbf{M}^\circ)} = \ker \varphi_2\big|_{\pi_1(\mathbf{U}^\circ)}$ and that $\tilde{\mathbf{U}}^\circ$ is a cover of these two surfaces. Edge identifications are made on four copies of $\mathbf{P}$, possibly indexed by the finite component of the direct product $\mathbf{P} \times \mathbb{Z}/4\mathbb{Z}$. The Deck group of the covering map from $\tilde{\mathbf{U}}^\circ$ to $\mathbf{G}^\circ$ is the set $\mathbb{Z}^2 \times \mathrm{SO}(2, \mathbb{Z})$. Consider a loop on $\mathbf{U}^\circ$:



If we lift this curve to $\tilde{\mathbf{U}}^\circ$ we can identify it with an element of the deck group, and check if its lift from $\mathbf{G}^\circ$ belongs to the kernel of both φ_1 and φ_2. Fix a direction on this surface pointing north that respects $\mathbf{U}^\circ$'s compass and rotate every plane by either 90°, 180°, or 270° as the curve travels from one plane (copy of $\mathbf{P}$) to the next. Observe that it will not only close if the curve acts trivially on the holonomy of $\mathbf{U}^\circ$, but with this construction we can see that $\tilde{\mathbf{U}}^\circ$ is actually an infinite-type translation surface:

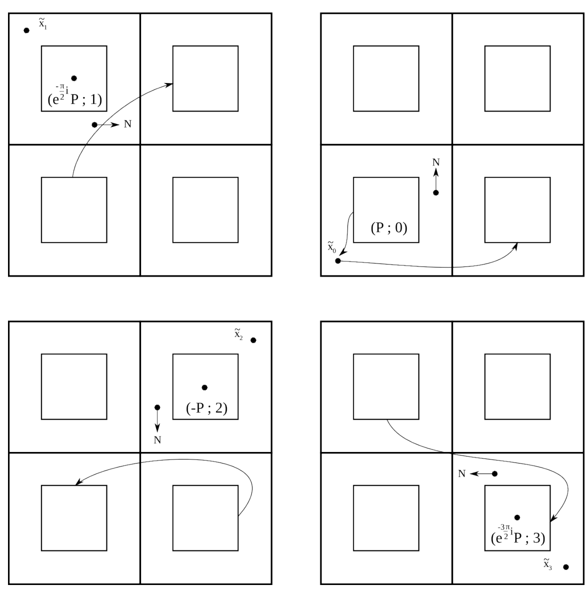


Figure 6: With $P \in 2\mathbb{Z} + 2\mathbb{Z}i$.

2.3 Geodesics on $\mathbf{U}^\circ$

The key connection made between surfaces $\mathbf{U}^\circ$ and $\tilde{\mathbf{U}}^\circ$ is that geodesics on $\mathbf{U}^\circ$ and $\tilde{\mathbf{U}}^\circ$ behave identically. Unless stated otherwise, a *geodesic* is a *unit-speed* locally distance minimizing curve on a smooth topological surface as a function parameterized by an element in $\mathbb{R}$. A *geodesic segment* is a restriction of the geodesic to some closed interval in $\mathbb{R}$. A geodesic has an initial trajectory defined at time 0, represented by a unit vector $v_0 \in \mathbb{R}^2$. If the vector can be scaled by $k \in \mathbb{R}$ such that $kv_0 \in \mathcal{S}$, the geodesic has a *rational* initial trajectory. For example, the following is a periodic geodesic on the surface with initial direction $kv_0 \in \mathcal{O}$:

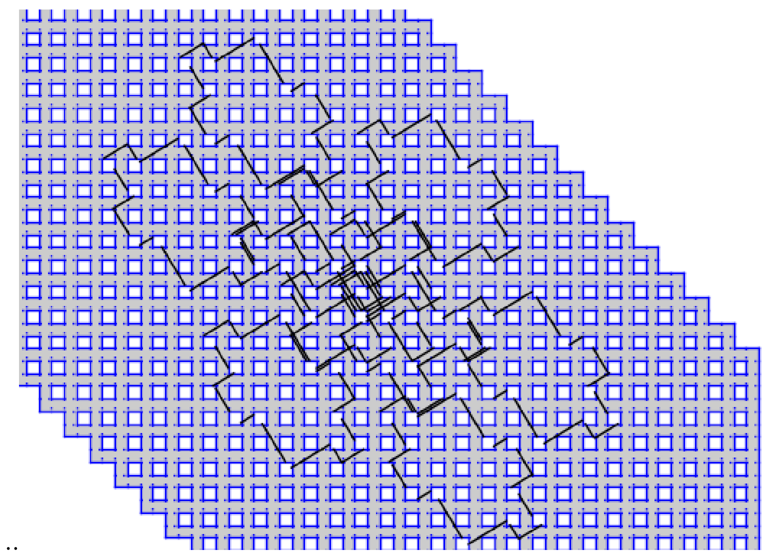


Figure 7: Periodic geodesic path on $\mathbf{U}^\circ$ modeled with sage-flatsurf.

We refer later to the following commuting diagram of branched covering maps and inclusion maps. It is meant to illustrate the relationship between all of these spaces.

$$\begin{tikzcd}
\tilde{\mathbf{U}}^\circ \arrow[r, two heads, "r^\circ"] \arrow[d, hook, "pr_b"] \arrow[rrr, bend left, two heads, "f^\circ"] & \mathbf{U}^\circ \arrow[r, two heads, "q^\circ"] \arrow[d, hook, "pr_c"] & \mathbf{G}^\circ \arrow[d, hook, "pr_d"] & \mathbf{M}^\circ \arrow[l, two heads, "w^\circ"] \arrow[d, hook, "pr_a"] \\
\tilde{\mathbf{U}} \arrow[r, two heads, "r"] \arrow[rrr, bend right, two heads, "f"'] & \mathbf{U} \arrow[r, two heads, "q"] & \mathbf{G} & \mathbf{M} \arrow[l, two heads, "w"]
\end{tikzcd} \tag{4}$$

Let $x_0 \in G^\circ$ and fix $u_0 \in \mathbf{U}^\circ$, an element in the fiber of the covering map.

A *geodesic* on G° is a closed unit-speed straight-line path given as $\eta : \mathbb{R} \to \mathbf{G}^\circ$. The *period/length* of the geodesic is some strictly positive $b \in \mathbb{R}$. Let $\eta|_{[0,b]} = \alpha : [0,b] \to \mathbf{G}^\circ$ be a closed geodesic segment on $\mathbf{G}^\circ$ with initial and terminal point x_0. Denote α's unique lift to $\mathbf{U}^\circ$ by $\gamma : [0,b] \to \mathbf{U}^\circ$, where $\gamma(0) = u_0$, and $\gamma(b)$ in the fiber over x_0. Fix a point $\tilde{u}_0 \in \tilde{\mathbf{U}}^\circ$ in the fiber over u_0. We define the unique lift of γ to be the curve $\tilde{\gamma} : [0,b] \to \tilde{\mathbf{U}}^\circ$ with $\tilde{\gamma}(0) = \tilde{u}_0$. Denote their homotopy class representatives by $[\alpha]$ and (if they exist) $[\gamma]$, $[\tilde{\gamma}]$.

The geodesic segment on $\mathbf{G}^\circ$ has the property that it returns to x_0 with a trivial effect on the holonomy of the curve. On the other hand, the lifted paths will close only if their homotopy classes belong in their fundamental groups, i.e. the kernel of homomorphisms φ_1, $\varphi_2|_{\ker \varphi_1}$. Otherwise they are mapped non-trivially to $\mathbb{Z}^2$ or $\mathbb{Z}^2 \times \mathrm{SO}(2,\mathbb{Z})$. Let $v_0 \in \mathbb{R}^2$ be a unit vector such that $\alpha's$ derivative is v_0 at times 0 and b. The derivative of γ is also v_0 at these times as per Corollary 1, as α is a geodesic with trivial holonomy. $\tilde{\gamma}'$ are also v_0 at their endpoints.

The Deck group of the map $\tilde{\mathbf{U}}^\circ \to \mathbf{U}^\circ$ is $\mathrm{SO}(2,\mathbb{Z})$. This deck group acts naturally on the tangent bundles via push-forward of rotational matrices. We obtain the identity $T^1\tilde{\mathbf{U}}^\circ/\mathrm{SO}(2,\mathbb{Z}) \cong T^1\mathbf{U}^\circ$. This action is *non-trivial.* Essentially, this means that there are 4 choices of a vector on the four-fold cover given a vector v_0 on $\mathbf{U}^\circ$. The choice of vector does not matter in the case where we flow it along a geodesic path, as a geodesic must return to a point in the fiber of u_0 in the same direction. That is to say that no matter where we start by lifting this path, we will always end up on the same plane (copy of $\mathbf{P}$) that we started in. In the case where a geodesic is periodic on $\mathbf{U}^\circ$, it would return to the same point in the fiber of u_0 as well. For simplicity's sake we will just lift to the vector mod the identity matrix, and call it v_0 as well.

Definition 6. *Let $\Phi_t : \mathbb{R} \to \mathbf{U}^\circ$ be a unit-speed geodesic on the* Necker cube surface *with $\Phi(0) = u_0$ and $\Phi'(0) = v_0$.*
Similarly, let $\tilde{\Phi}_t : \mathbb{R} \to \tilde{\mathbf{U}}^\circ$ be a unit-speed geodesic on $\mathbf{U}^\circ$'s four-fold cover with $\tilde{\Phi}(0) = \tilde{u}_0$ and $\tilde{\Phi}'(0) = v_0$.

Let $\varphi = \varphi_1 \times \varphi_2 : \pi_1(\mathbf{G}^\circ) \to \mathbb{Z}^2 \times \mathrm{SO}(2,\mathbb{Z})$ be given by $[g] \mapsto (\varphi_1([g]), \varphi_2([g]))$. φ is a product of group homomorphisms that takes a loop to

Theorem 2. *Let $\delta = \varphi_1([\alpha]) \in \mathbb{Z}^2$ be an element of Δ_{φ_1} and $\tilde{\delta} = \varphi([\alpha]) \in \mathbb{Z}^2 \times \mathrm{SO}(2,\mathbb{Z})$ be an element of Δ_φ. Then:*
$\Phi(t+b) = \delta \cdot \Phi(t)$ for all $t \in \mathbb{R}$, and $\tilde{\Phi}(t+b) = \tilde{\delta} \cdot \tilde{\Phi}(t)$ for all $t \in \mathbb{R}$.

Corollary 2. *The following are equivalent:*

1. *Φ is periodic.*
2. *$\tilde{\Phi}$ is periodic.*
3. *$\delta = (0,0)$.*

Corollary 3. *The following are equivalent:*

1. Φ *is drift-periodic.*

2. $\tilde{\Phi}$ *is drift-periodic.*

3. $\delta \neq (0,0)$.

These theorems essentially tell us that these types of geodesics behave identically on the Necker cube surface and its cover. Now we show that this is true for $\tilde{\mathbf{U}}^{\circ}$ and its metric completion, $\tilde{\mathbf{U}}$.

$$\begin{tikzcd}
\pi_1(\tilde{\mathbf{U}}^{\circ}) \arrow[r, hook, "r_*^{\circ}"] \arrow[d, two heads, "PR_b"] \arrow[rrr, hook, bend left, "f_*^{\circ}"] & \pi_1(\mathbf{U}^{\circ}) \arrow[r, hook, "q_*^{\circ}"] \arrow[d, two heads, "PR_c"] & \pi_1(\mathbf{G}^{\circ}) \arrow[d, two heads, "PR_d"] & \pi_1(\mathbf{M}^{\circ}) \arrow[l, hook', "w_*^{\circ}"'] \arrow[d, two heads, "PR_a"] \\
\pi_1(\tilde{\mathbf{U}}) \arrow[r, hook, "r_*"] \arrow[rrr, hook, bend right, "f_*"'] & \pi_1(\mathbf{U}) \arrow[r, hook, "q_*"] & \pi_1(\mathbf{G}) & \pi_1(\mathbf{M}) \arrow[l, hook', "w_*"']
\end{tikzcd} \tag{5}$$

Definition 7. *Let $\tilde{\phi} : \mathbb{R} \to \tilde{\mathbf{U}}$ be a geodesic on the unpunctured four-fold surface such that $\tilde{\phi}(0) = \tilde{u}_0$ and $\tilde{\phi}'(0) = v_0$.*

Theorem 3. *$\tilde{\phi}$ is periodic if and only if $\tilde{\Phi}$ is periodic.*
$\tilde{\phi}$ is drift-periodic if and only if $\tilde{\Phi}$ is drift-periodic.

This follows readily from the fact that geodesics are undefined on cone singularities. A geodesic on $\tilde{\mathbf{U}}^{\circ}$ can be mapped bijectively to a geodesic on $\tilde{\mathbf{U}}$ and vice-a-versa.

2.4 Translation Surface

Consider the projection $\tilde{\mathbf{U}} \to \mathbf{M}$ onto a quotient of the surface under its translational symmetries. What we obtain is the following compact translation surface belonging to the stratum $\mathcal{H}(2,2,2,2)$:

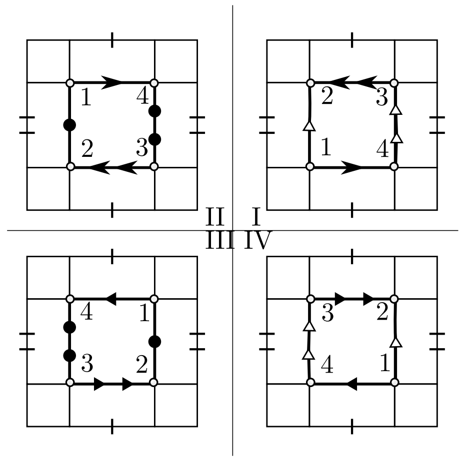


Figure 8: Compact translation surface, **M** with edges and cone singularities (1,2,3,4) identified. The Roman numerals are meant to identify every "copy" of **G** with a direction differing from v_0 by rotation of some element in $\mathrm{SO}(2,\mathbb{Z})$. The individual copies are then rotated so we have a canonical notion of direction

We rearrange **M** as such:

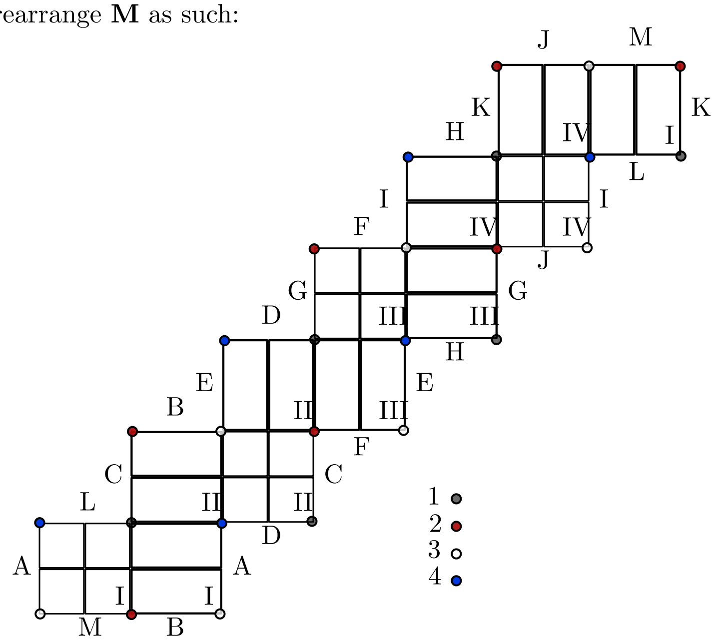


Figure 9: The staircase surface with directional planes and vertices identified. All edges are paired by translation. Two adjacent squares have opposite edges (A-M) identified. The top edge of the bottom-left square is glued to the bottom edge of the top-right square (both labeled I). Likewise, the bottom edge of the bottom-left square is identified with the top edge of the top-right square.

We recall some basic definitions and theorems concerning $\mathbb{Z}^2$covers of translation surfaces as they apply to $\mathbf{M}$.

Definition 8. *Algebraic intersection number is a non-degenerate bilinear form:*

$$i : H_1(\mathbf{M}, \mathbb{Q}) \times H_1(\mathbf{M}, \mathbb{Q}) \to \mathbb{Q},$$

for $[\gamma], [\beta] \in H_1(\mathbf{M}, \mathbb{Q})$*,* $i([\beta], [\gamma])$ *returns the signed intersection number of two homology classes. We say a crossing at the moment of intersection is positive if* γ *makes a positive angle relative to* β*.*

The set of all affine diffeomorphisms of $\mathbf{M}$ form the group $\text{Aff}^+(\mathbf{M})$. The *Veech group* of $\mathbf{M}$ is the image in the co-domain of the homomorphism D : $\text{Aff}^+(\mathbf{M}) \to SL(2, \mathbb{R})$ that takes every affine map to its derivative, which we denote $V(\mathbf{M})$. $\mathbf{M}$ is a square-tiled translation surface whose Veech group is a finite index subgroup of $SL(2, \mathbb{Z})$, thereby making it a *Veech surface*. We use automorphisms of the fundamental group in $\text{Aut}(\pi_1(\mathbf{M}))$ induced by affine maps to prove our main results. But first, we define a group homomorphism from $\pi_1(\mathbf{M})$ to $\pi_1(\mathbf{M})/\pi_1(\tilde{\mathbf{U}})$ by algebraic intersection numbers over a sum of linearly independent homology classes. Consider the following cylinder core curves on $\mathbf{M}$ labeled γ_i for $0 \leq i \leq 11$:

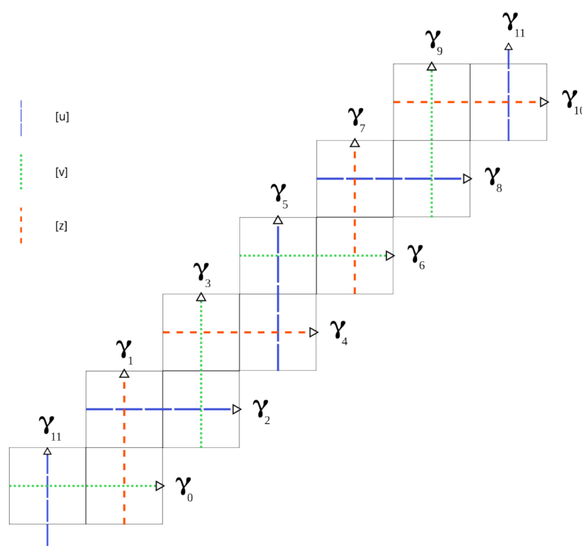


Figure 10: $\mathbf{M}$'s cylinder core curves with u,v, and z homology class labels.

The rank of a 12×12 intersection matrix of these curves is 10. Therefore these curves span $H_1(\mathbf{M}, \mathbb{Q})$.

Definition 9. *The homology classes* u, v, z *are given as the following sums of* *core curves:*

$$u = -[\gamma_2] + [\gamma_5] + [\gamma_8] - [\gamma_{11}],$$
$$v = +[\gamma_0] - [\gamma_3] - [\gamma_6] + [\gamma_9],$$
$$z = +[\gamma_1] + [\gamma_4] - [\gamma_7] - [\gamma_{10}].$$

Definition 10. *Define the group homomorphism* $\Omega_{u,v} : \pi_1(\mathbf{M}) \to \mathbb{Z}^2$, *where* $\beta \mapsto (i(u, [\beta]), i(v, [\beta]))$.

Lemma 4. ***show somehow that u,v determine the cover***

2.5 Induced Automorphisms of $H_1(\mathbf{M}, \mathbb{Q})$

We look at some important automorphisms induced by Affine maps on **M**. Observe that **M** has a uniform cylinder decomposition in both horizontal and vertical directions as in figure 15. We define the *modulus*, μ, of a cylinder to be the ratio of the cylinder's width to its circumference, $\frac{w}{c}$. The *Dehn-twist* of a cylinder is an affine diffeomorphism that skews the cylinder and sends every vertex to itself.

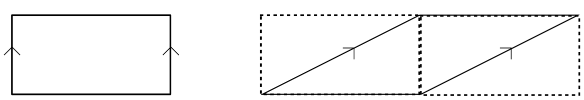


Figure 11: Dehn-twist of a cylinder in **M**'s cylinder decomposition.

Such a map has derivative $\begin{bmatrix} 1 & \pm\mu^{-1} \\ 0 & 1 \end{bmatrix}$. On **M** all cylinders in both vertical and horizontal decompositions have moduli $\mu = \frac{1}{2}$. These give way to global diffeomorphisms as *multi-twists* of **M**.

Definition 11. *We define the global affine diffeomorphisms of Aff*$^+(\mathbf{M})$ *obtained as* ***multi-twists of the surface in horizontal and vertical directions*** ψ_h *and* ψ_v, *respectively.*

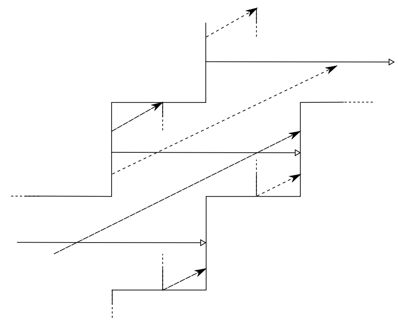
The *derivative map* of the affine group is the homomorphism $D : \text{Aff}^+(\mathbf{M}) \to \text{SL}(2, \mathbb{R})$. The *Veech group* of the surface is the image of the affine group under this map, $D(\text{Aff}^+(\mathbf{M}))$, denoted $V(\mathbf{M})$. The twists ψ_h and ψ_v under D are mapped to the same matrices as the Dehn-twists on its individual cylinders. i.e.

$$D(\psi_h) = \psi_h' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad D(\psi_v) = \psi_v' = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

These are parabolic elements of $V(\mathbf{M})$ that generate a free group of rank 2. [cite].

Definition 12. *We call* $\mathbb{X}$ *the generated parabolic subgroup of Aff*$^+(\mathbf{M})$, *where* $\mathbb{X} = \langle \psi_h, \psi_v \rangle$.

The image of this group under D is itself a subgroup of $V(\mathbf{M})$ and denoted $D(\mathbb{X}) = \mathbb{X}' = \langle \psi'_h, \psi'_v \rangle$ as in the introduction. Note that when skewing the surface and piecing it back together, the cylinder curves intersect each other:



By skewing in the horizontal direction, ψ_h preserves the even indexed cylinder curves, but the odd indexed vertical curves γ_i intersect the two horizontal curves (positive intersection), γ_{i-1} and γ_{i+1}. A similar observation can be made from applying ψ_v where the vertical curves are preserved, but the horizontal curves intersect with adjacent vertical curves (negative intersection). Let $i \in \mathbb{Z}/12\mathbb{Z}$. Then this convenient indexing scheme gives us the following formulas for ψ_h and ψ_v's counterparts in $\mathrm{Aut}(H_1(\mathbf{M}, \mathbb{Q}))$ raised to $k \in \mathbb{Z}\backslash\{0\}$:

$$\begin{aligned}
\psi_h^{*k}([\gamma_i]) =& [\gamma_i] + \frac{k}{2}(1 - (-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}]) \\
\psi_v^{*k}([\gamma_i]) =& [\gamma_i] - \frac{k}{2}(1 + (-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}])
\end{aligned}$$

We call this generated group $\mathbb{X}^* = \langle \psi_h^*, \psi_v^* \rangle$. There are no relations between its generators since i being even or odd has one act trivially and the other non-trivially. Therefore this group is also free and rank 2. Hence $\mathbb{X}^* \cong \mathbb{X}' \cong \mathbb{X}$. Let $g \in \mathbb{X}$, $g' \in \mathbb{X}'$, $g^* \in \mathbb{X}^*$ be isomorphic elements in their respective groups. Let $g = \prod_{j=0}^{n} \psi_h^{k_j} \circ \psi_v^{m_j}$ be a word in $\mathbb{X}$ as a finite left product of multi-twist pairs with $k_j, m_j \in \mathbb{Z}$. Let $\psi_j = \psi_h^{k_j} \circ \psi_v^{m_j}$ be the j^{th} word pair. Elements g', g^* have identical representations.

The j^{th} element pair that is mapped into $\mathrm{Aut}(H_1(\mathbf{M}, \mathbb{Q}))$ can be expressed as

$$\begin{aligned}
\psi_j^*([\gamma_i]) =& [\gamma_i] + \frac{k_j(1 - (-1)^i)}{2}([\gamma_{i-1}] + [\gamma_{i+1}]) \\
&- \frac{m_j(1 + (-1)^i)}{2}(k_j(2[\gamma_i] + [\gamma_{i-2}] + [\gamma_{i+2}]) + [\gamma_{i-1}] + [\gamma_{i+1}]).
\end{aligned}$$

Let $\sum \Gamma = \sum_{i=0}^{11}[\gamma_i]$, and denote the odd and even indexed sums as $\sum \Gamma_o$ and $\sum \Gamma_e$, respectively. We consider three kinds of closed geodesic segments on $\mathbf{M}$ and don't consider initial/terminal points for the time being. These are the slope one, horizontal, and vertical loops.

Call the geodesic in direction $\theta = \frac{\pi}{4}$ that runs up the staircase intersecting every γ_i once χ. Denote its homotopy class by $[\chi]$. Observe from Figure 10 that χ is homologous to the sum of cylinder curves divided by 2. In Figure 1, this is the closed geodesic that lifts to a closed loop that looks like a 12-gon on $\mathbf{U}^{\circ}$.

Lemma 5. $\Omega_{u,v}([\chi]) = (0,0)$.

Proof. Observe that $\Omega_{u,v}([\chi]) = \Omega_{u,v}(\frac{1}{2}\sum\Gamma) = \frac{1}{2}(i(u,\sum\Gamma), i(v,\sum\Gamma))$. Since only adjacent curves intersect, $i(u,\sum\Gamma) = i(-[\gamma_2],[\gamma_1]+[\gamma_3]) + i([\gamma_5],[\gamma_4]+[\gamma_6]) + i([\gamma_8],[\gamma_7]+[\gamma_9]) - i([\gamma_{11}],[\gamma_{10}]+[\gamma_0]) = -2+(-2)+2-(-2) = 0$.
Similarly, $i(u,\sum\Gamma) = i([\gamma_0],[\gamma_{11}]+[\gamma_1]) - i([\gamma_3],[\gamma_2]+[\gamma_4]) - i([\gamma_6],[\gamma_5]+[\gamma_7]) + i([\gamma_9],[\gamma_8]+[\gamma_{10}]) = 2-(-2)-2+(-2) = 0$. Therefore, $\Omega_{u,v}([\chi]) = (0,0)$. $\square$

Lemma 6. *Let g be as above. If $\alpha = g(\chi)$, then $[\alpha] = \frac{1}{2}(c_1\sum_e + c_2\sum_o)$ for $c_1, c_2 \in \mathbb{Z}$.*

Proof. (By Induction on $\mathbb{X}^*$).
Consider $\psi_0(\chi)$.
$[\psi_0(\chi)] = \psi_0^*[\chi] = \frac{1}{2}\psi_0^*(\sum\Gamma) = \frac{1}{2}\psi_0^*(\sum\Gamma_e + \sum\Gamma_o) = \frac{1}{2}(\psi_0^*(\sum\Gamma_e) + \psi_0^*(\sum\Gamma_o))$.
Now
$\psi_0^*(\sum\Gamma_e) = \sum_{i=0}^{11}\psi_0^*([\gamma_{2i}]) = \sum_{i=0}^{11}([\gamma_{2i}] - 2m_0k_0[\gamma_{2i}] - m_0k_0[\gamma_{2i-2}]$
$-m_0k_0[\gamma_{2i+2}] - m_0[\gamma_{2i+1}] - m_0[\gamma_{2i-1}]) = (1-4m_0k_0)\sum\Gamma_e + (-2m_0)\sum\Gamma_o$.
Now
$\psi_0^*(\sum\Gamma_o) = \sum_{i=0}^{11}\psi_0([\gamma_{2i+1}]) = \sum_{i=0}^{11}([\gamma_{2i+1}] + k_0([\gamma_{2i}] + [\gamma_{2i+2}])) = \sum\Gamma_o + 2k_0\sum\Gamma_e$.
And so $\frac{1}{2}\psi_0^*(\sum\Gamma) = \frac{1}{2}((1+2k_0-4k_0j_0)\sum\Gamma_e + (1-2m_0)\sum\Gamma_o)$.

Induction: Suppose $\frac{1}{2}\psi_j^*(\sum\Gamma) = c_1\sum\Gamma_e + c_2\sum\Gamma_o$.
Then $\psi_{j+1}^*(\psi_j^*(\sum\Gamma)) = c_1\psi_{j+1}^*(\sum\Gamma_e) + c_2\psi_{j+1}^*(\sum\Gamma_o) =$
$c_1((1-4m_{j+1}k_{j+1})\sum\Gamma_e + (-2m_{j+1})\sum\Gamma_o) + c_2(\sum\Gamma_o + 2k_{j+1}\sum\Gamma_e) =$
$(c_1(1-4m_{j+1}k_{j+1}) + 2k_{j+1}c_2)\sum\Gamma_e + (-2m_{j+1}c_1 + c_2)\sum\Gamma_o$.
$\square$

Theorem 4. *Let g be as above. Suppose $\alpha = g(\chi)$. Then $\Omega_{u,v}([\alpha]) = (0,0)$.*

Proof. $\Omega_{u,v}([\alpha]) = \Omega_{u,v}([g(\chi)]) = \Omega_{u,v}(g^*([\chi])) = \frac{1}{2}(i(u,g^*(\sum\Gamma))i(v,g^*(\sum\Gamma))$.
$i(u,g^*(\sum\Gamma) = c_1 i(u,\sum\Gamma_e) + c_2 i(u,\sum\Gamma_o) =$
$c_1(i([\gamma_5],[\gamma_4] - i([\gamma_{11}],[\gamma_{10}]+[\gamma_0]))$
$+ c_2(i(-[\gamma_2],[\gamma_1]+[\gamma_3]) + i([\gamma_8],[\gamma_7]+[\gamma_9])) =$
$c_1((-2)-(-2)) + c_2(-2+2) = 0$
Holds similarly for $i(v,g^*(\sum\Gamma))$.
$\therefore \Omega_{u,v}([\alpha]) = (0,0)$. $\square$

Lemma 7. $\Omega_{u,v}([\gamma_i]) \neq (0,0)$.

Proof. every curve intersects nontrivially with exactly one of either u or v. $\square$

Theorem 5. *Suppose $\beta = g(\gamma_i)$. Then $\Omega_{u,v}([\beta]) \neq (0,0)$.*

Proof. $g^*([\gamma_i]) = [\gamma_i] + C$. If $i(u, g^*([\gamma_i])) = i(u, [\gamma_i] + C) = i(u, [\gamma_i]) + i(u, C) = 0$, then $\square$

3 Proof of Main Theorem

$S = \mathcal{O} \cup \mathcal{E}$. Let $\mathbb{X}'$ act on $\mathcal{S}$ by linear transformations.

Lemma 8.

Lemma 9. $\mathbb{X}'$ *acts faithfully and transitively on integer pairs in $\mathcal{O}$, partitions $\mathcal{E}$ into two sets and acts faithfully and transitively on them, and acts faithfully on $\mathcal{S}$.*

Proof. Let $v = kv_0 = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}$ with $||v_0|| = 1$ and $k \in \mathbb{R}$, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{X}'$. The kernel of the homomorphism $SL(2,\mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z})$ that maps a matrix to its $(mod\ 2)$ equivalent is precisely $\mathbb{X}'$ since it maps the generators trivially. [quote this paper for the proof of this maybe? http://www.fysik.su.se/ ingemar/SL2.pdf]
x, y are relatively prime, so Bezout's identity states there exist $m, n \in \mathbb{Z}$ such that $mx + ny = 1$ $(*)$. It's obvious that this action is faithful on the sets since the stabilizer of any vector in $\mathbb{R}^2$ is trivial. Suppose $v \in \mathcal{O}$. Consider the orbit of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then there is some matrix $A \in \mathbb{X}'$ such that $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v$. Since $A \in \mathrm{SL}(2,\mathbb{Z})$, $\det A = ad - bc = 1$. Obtain from $1 = (x-b)(y-c) - bc = xy - xc - by$ that $xy - 1 = xc + by$. From $(*)$ we see that $xy - 1 = x(u(xy-1)) + y(v(xy-1))$. Let $c = u(xy-1)$ and $b = v(xy-1)$. Since $xy - 1$ is even, c, b are even. It follows then that $a = x - b$ and $d = y - c$ are odd. Therefore $A \in \mathbb{X}'$ and the action is transitive on $\mathcal{O}$. If $v \in \mathcal{E}$, then either x or y is even. In either case this would mean that either a or d is even, which would mean $A \notin \mathbb{X}'$, a contradiction.
Now consider the orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Left multiplying by A would mean $a = x$ and $c = y$. Split $\mathcal{E}$ into sets $\mathcal{E}_1 = \{(x,y) \in \mathcal{E} : x \in 2\mathbb{Z},\ y \in (2\mathbb{Z}+1)\}$ and $\mathcal{E}_2 = \{(x,y) \in \mathcal{E} : x \in (2\mathbb{Z}+1),\ y \in 2\mathbb{Z}\}$. Since $\det A = 1 = da - bc = dx - by = mx + ny$, we have that $d = m$ and $b = -n$. We rule out when $v \in \mathcal{O}, \mathcal{E}_1$ since $c \neq 0(\mod 2)$ in those cases. Consider $v \in \mathcal{E}_2$. Then x is odd and y is even. d must be odd since $mx + ny = 1$ and y is even. And b must be even since $ad - bc = 1$
finish this and say the proof for $\mathcal{E}_1$ is analogous.

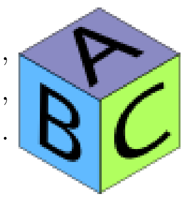
$\square$

Definition 13. *Let $\phi_t^\theta : \mathbb{R} \to \mathbf{M}$ be a unit-speed geodesic on $\mathbf{M}$ in direction θ.*

3.1 empty

Consider the following families of unit squares in $\mathbb{R}^3$:

$$\begin{aligned}
\mathbf{A}_{m,n,p} &= [m, m+1] \times [n, n+1] \times \{p\}, \\
\mathbf{B}_{m,n,p} &= \{m+1\} \times [n, n+1] \times [p-1, p], \\
\mathbf{C}_{m,n,p} &= [m, m+1] \times \{n+1\} \times [p-1, p].
\end{aligned}$$



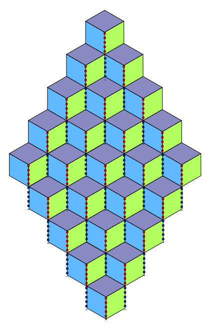


Figure 12: A section of **S**

3.2 Flattening S

At the moment it is difficult to describe how the parallel transport of a vector tangent to the surface along an arbitrary path on the surface acts on the vector in $\mathbb{R}^3$. To simplify this problem we take $\mathbf{S}$ to an isometric variant embedded in $\mathbb{R}^2$ by piecewise linear transformations on the sets

$$\begin{aligned}
\mathbf{A} &= \bigcup \big\{\mathbf{A}_{m,n,p} : m+n+p = 0\big\}, \\
\mathbf{B} &= \bigcup \big\{\mathbf{B}_{m,n,p} : m+n+p = 0\big\}, \\
\mathbf{C} &= \bigcup \big\{\mathbf{C}_{m,n,p} : m+n+p = 0\big\}.
\end{aligned}$$

Definition 14. *Let $\Psi : s^*(\mathbf{S}) \to \mathbb{R}^3$ be given as*

$$\Psi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ z - \lfloor z \rfloor \end{bmatrix} & \textit{if } (x,y,z) \in \mathbf{A} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ x - \lfloor x \rfloor \end{bmatrix} & \textit{if } (x,y,z) \in \mathbf{B} \backslash (\mathbf{A} \cup \mathbf{C}) \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ y - \lfloor y \rfloor \end{bmatrix} & \textit{if } (x,y,z) \in \mathbf{C} \backslash (\mathbf{A} \cup \mathbf{B}) \end{cases} \tag{6}$$

This map behaves as such:

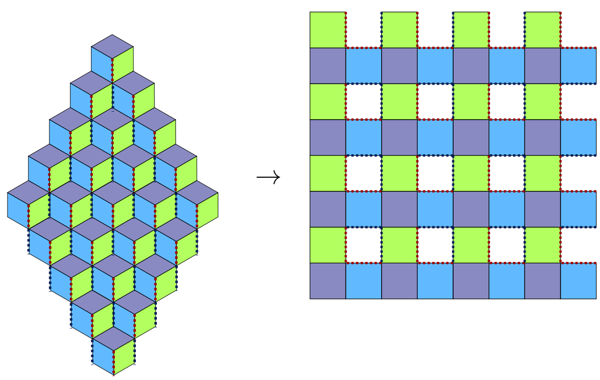


Figure 13: An isometry of the surface **S**.

Lemma 10. $\Psi \circ s^*$ *is an isometry of the surface* **S**

Proof. Ψ is well-defined on its domain $s^*(\mathbf{S})$, and s^* is an embedding of the surface that is a bijection when restricting its co-domain to its image. Since Ψ is a linear transformation composed of Euclidean matrices and translations it is also invertible and bijective. □

3.3 Translation Surface Cover of $\mathbf{G}^\circ$

3.4 A Four-Fold Cover of $\mathbf{U}^\circ$

Definition 15. $\mathbf{U}^\circ$ *is the cover of* $\mathbf{G}^\circ$ *such that* $\pi_1(\mathbf{U}^\circ) = \ker \varphi$.

Our cover of $\mathbf{G}^\circ$ has the fundamental group, $\pi_1(\mathbf{U}^\circ) = \langle\langle a^{-1}b^{-1}ab, c, d\rangle\rangle$, the conjugate subgroup of elements that map trivially under φ_1. The covering map $k_{\varphi_1} : \mathbf{U}^\circ \hookrightarrow \mathbf{G}^\circ$ takes every point on $\mathbf{U}^\circ$ to its modular equivalent under these translational symmetries. $\mathbf{U}^\circ$ is realized as an infinite $\mathbb{Z}^2$-tiling of $\mathbf{G}^\circ$ in $\mathbb{C}$:

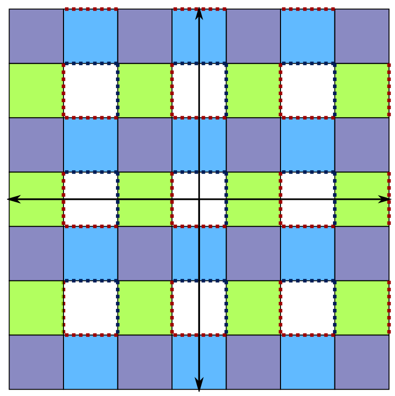


Figure 14: The infinite surface $\mathbf{U}^\circ$ embedded in $\mathbb{C}$ with edges identified.

Definition 16. $\mathbf{U}$ *is the completion of* $\mathbf{U}^\circ$ *that includes the branched vertices.*

Remark. *The surface* $\mathbf{G}$ *is homeomorphic to the torus, and its cover* $\mathbf{U}$ *is its universal cover as the paths labeled* c, d *are only non-trivial when* $\mathbf{G}$ *is branched at its three conical points.*

3.5

3.6 Isometry From S to U

The Necker cube surface can be *flattened* onto the plane by piecewise isometric maps onto a subspace of $\mathbb{R}^3$ with (open) unit squares removed at every even integer pair in the plane, what we claim to be $\mathbf{U}$.
The red/blue dotted lines represent the edges that are split on the plane. The map from one surface to the other is composed of piecewise isometries, $\Psi : \mathbf{S} \to \mathbb{R}^3$.

The flattened surface is contained entirely in $(x, y, 0) \in \mathbb{R}^3$, which is isometric to $\mathbb{C}$. $\mathbf{U}$ is recovered as a topological quotient on the domain

Definition 17. $\mathbf{U}$ *is the surface obtained as the topological quotient* $\mathbf{P}/\sim_{\mathbf{P}}$, *where* $\mathbb{C}$ *is identified with* $\mathbb{R}^2$ *in the usual way and* $\sim_{\mathbf{P}}$ *is a minimal relation on*

P *defined as follows:*

$$\textit{Let } x_0 = (u_0, v_0), x_1 = (u_1, v_1) \in \mathbf{P}. \ \sim_{\mathbf{P}} \ \textit{is given as the relation}$$
$$x_0 \sim_{\mathbf{P}} x_1 \textit{ iff } x_0 = x_1$$
$$\textit{or, for some } m, n \in \mathbb{Z} \ x_0, x_1 \in \partial\left(\left[2m - \frac{1}{2}, 2m + \frac{1}{2}\right] \times \left[2n - \frac{1}{2}, 2n + \frac{1}{2}\right]\right)$$
$$\begin{bmatrix} u_1 - 2m \\ v_1 - 2n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_0 - 2m \\ v_0 - 2n \end{bmatrix}.$$

Remark. *Our map* Ψ *induces an isometry between* **S** *and* **U**, *and its restriction to the branched surfaces induces an isometry between* $\mathbf{S}^\circ$ *and* $\mathbf{U}^\circ$. *This follows from these piecewise Euclidean transformations that preserves our induced flat metric.*

From here on we use $\mathbf{U}^\circ$ and **U** instead of **S** and $\mathbf{S}^\circ$.

3.7 Four-Fold Cover

We denote the *unit tangent bundle* on the surface $\mathbf{U}^\circ$ by $T^1\mathbf{U}^\circ$, and construct a cover with trivial holonomy. Initial experiments have shown us that any geodesic viewed as discontinuous line segments on **P** moves in at most four directions.

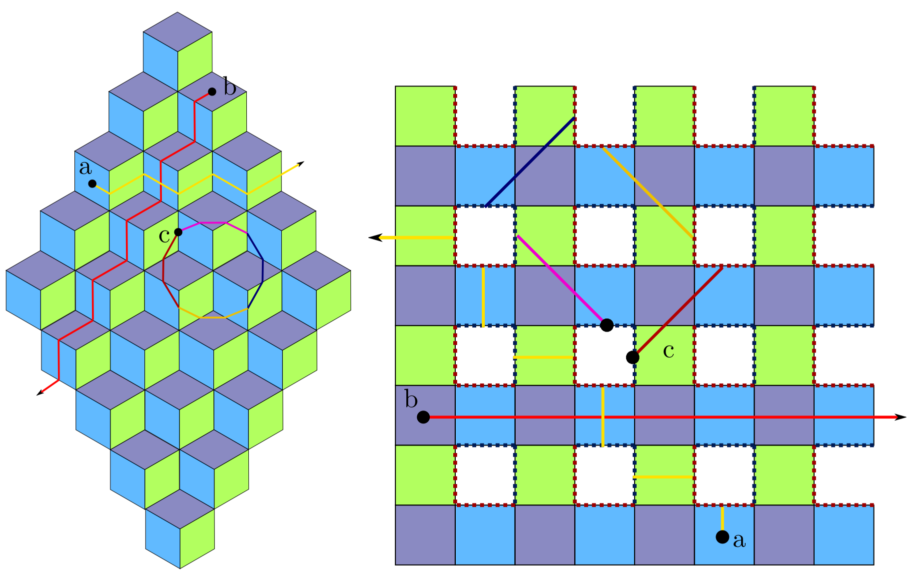


Figure 15: Simple periodic (c) and drift-periodic (a,b) trajectories represented as line segments on **S** and **U**.**RELABEL THESE**

Observe that the parallel transport of a vector around a closed loop on $\mathbf{U}^\circ$ will act on vectors tangent to the surface by a rotation of an integer multiple of

$\frac{\pi}{2}$ radians (since the surface is flat and embedded in $\mathbb{C}$, we work with vectors in $\mathbb{R}^2$).

Definition 18. *Define $\varphi_2 : \pi_1(\mathbf{G}^\circ) \to SO(2,\mathbb{Z})$ as the group homomorphism on generators a, b, c, d such that:*

$$a, b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \quad c \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = R, \quad d \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = R^3$$

The homomorphism restricted to $\pi_1(\mathbf{U}^\circ)$ factors through $\pi_1(\mathbf{G}^\circ)$ and we show that:

Lemma 11. *The group $\varphi_2(\pi_1(\mathbf{U}^\circ, x_0)) = Hol_p(\omega)$.*

Proof. Since $c \mapsto R$ and $d \mapsto R^3$, the image of φ_2 is generated by R, R^3 and isomorphic to $SO(2,\mathbb{Z})$. We see from figure 15 that parallel transports of vectors by non-trivial paths produce clockwise/counter-clockwise rotations equal to that of of the cone angles of the singularities they loop around, all integer multiples of $\frac{\pi}{2}$ that coincide with paths generated by elements c, d. Further, if there is an element in $\pi_1(\mathbf{U}^\circ, x_0)$ conjugated by $[a, b]$, the effect this would have on holonomy would be trivial as the total cone angle sum around each cut out square is an integer multiple of 2π, which agrees with a and b's trivial images under ϕ_2. $\square$

Hence, we obtain a natural monodromy representation with the map $m : \pi_1(\mathbf{U}^\circ, x_0) \to SO(2,\mathbb{Z}) \cong Hol_p(\omega)$, where for $[\gamma] \in \pi_1(\mathbf{U}^\circ, x_0)$ we have that $\gamma^*(1) = (x_0, m([\gamma])v_0)$. It follows that since ω is a flat connection, trivial loops have trivial holonomy and $Hol_p(\omega)$ acts on $H(p)$.

Definition 19. *Let $\tilde{\mathbf{U}}_0^\circ$ be the cover of $\mathbf{U}^\circ$ with fundamental group $\pi_1(\tilde{\mathbf{U}}_0^\circ) = \ker \varphi_2|_{\pi_1(\mathbf{U}^\circ)} \trianglelefteq \pi_1(\mathbf{U}^\circ)$.*

Lemma 12. *Let $\Phi_t : \mathbf{U}^\circ \times \mathbb{R} \to \mathbf{U}^\circ$ be a unit-speed geodesic flow on $\mathbf{U}^\circ$, with a parallel transport map induced by ω. Then the following is true:*

(i) Φ is periodic if and only if its lift to $\tilde{\mathbf{U}}_0^\circ$ is.

(ii) Φ is drift-periodic if and only if its lift to $\tilde{\mathbf{U}}_0^\circ$ is.

Proof. Let $x_0 = \Phi(0)$ be an initial point in $\mathbf{U}^\circ$, and let $v_0 = \frac{d}{dt}\Phi|_{t=0} \in \mathbb{R}^2$ be its initial direction. Then $\Phi^* : T^1\mathbf{U}^\circ \times \mathbb{R} \to T^1\mathbf{U}^\circ$ is a well-defined lift to the unit tangent bundle with initial point $p = (x_0, v_0)$.
(i). Suppose Φ is periodic with period $T \in \mathbb{R}$. Then Φ is reparameterized as $\alpha : [0,1] \to \mathbf{U}^\circ$, where $[\alpha] \in \pi_1(\mathbf{U}^\circ, x_0)$. It follows then that if α does not lift to a closed path, then α must have non-trivial holonomy since $[\alpha] \notin \ker \varphi_2$. That is when lifted to the unit tangent bundle with base point p, $\alpha^*(1) \neq p$ since $m([\alpha]) \neq I_2$. But this implies that $\Phi^*(T) \neq p$, which is impossible. Hence $[\alpha] \in \ker \varphi_2 = \pi_1(\tilde{\mathbf{U}}_0^\circ, \tilde{x}_0)$, where $\tilde{x}_0$ belongs to the fiber over x_0 under the covering map $\tilde{\mathbf{U}}_0^\circ \hookrightarrow \mathbf{U}^\circ$. The converse holds trivially.

(ii). Suppose Φ is drift-periodic with period $T \in \mathbb{R}$ and non-trivial $f : \mathbf{U}^\circ \to \mathbf{U}^\circ \in Trans(\mathbf{U}^\circ) \cong \pi_1(\mathbf{G}^\circ)/\pi_1(\mathbf{U}^\circ) \cong \mathbb{Z}^2$ such that $\Phi(T) = f(x_0)$. Reparameterize this as $\alpha : [0,1] \to \mathbf{U}^\circ$ with $[\alpha] \in \pi_1(\mathbf{G}^\circ)$. When lifted to $T^1\mathbf{U}^\circ$, $\Phi^*(0) = p$ and $\Phi^*(T) = (f(x_0), v_0)$. Let $\alpha : [0,1] \to \mathbf{U}^\circ$ be its reparameterization up to time T. Thus $[\alpha] \notin \pi_1(\mathbf{U}^\circ, x_0), \pi_1(\tilde{\mathbf{U}}_0^\circ, \tilde{x}_0)$. Further, f has a unique, lift to $\tilde{f} \in Trans(\tilde{\mathbf{U}}_0^\circ)$ because the space is connected. (a bit stuck here).
Conversely, ... □

A visual representation of $\tilde{\mathbf{U}}_0^\circ$ is as a four-fold cover of $\mathbf{U}^\circ$ with trivial holonomy on all closed paths. Arbitrary paths do *not* have trivial holonomy:

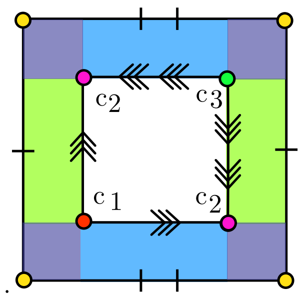


Figure 16: A 2×2 cut out section centered at each missing square. Edges and vertices identified.

3.8 Translation Surface

A better picture of $\tilde{\mathbf{U}}_0^\circ$ is obtained by making cyclic edge identifications on $\mathbf{P}' = \mathbf{P} \times \mathbb{Z}/4\mathbb{Z}$. This comes from $\tilde{\mathbf{U}}_0^\circ$ inheriting the topological properties of $\mathbf{U}^\circ \times \pi_1(\mathbf{U}^\circ)/\pi_1(\tilde{\mathbf{U}}_0^\circ) \cong \mathbf{U}^\circ \times SO(2, \mathbb{Z})$.

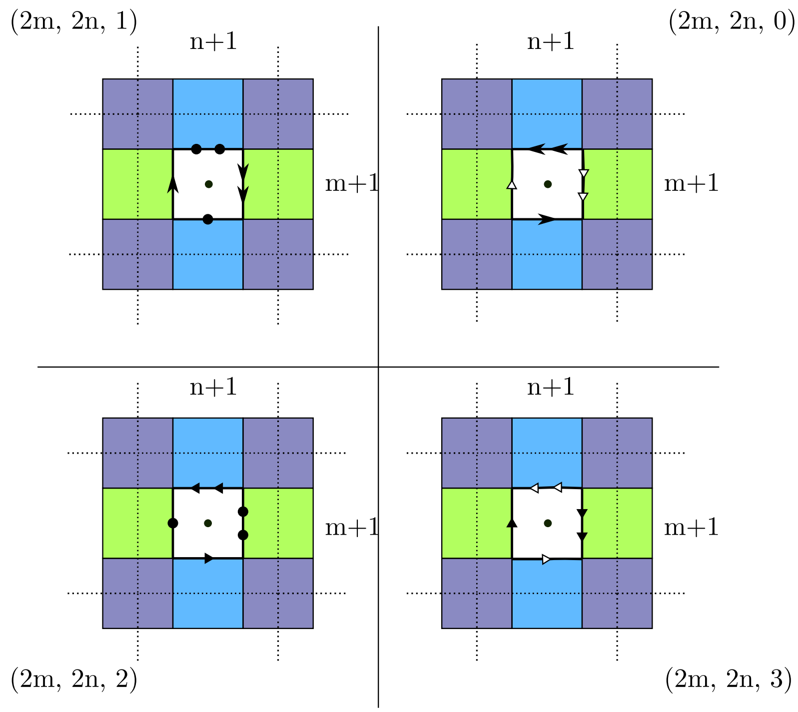


Figure 17: Branched cover of $\mathbf{U}$ of degree four.

Definition 20. *$\tilde{\mathbf{U}}_0$ is the surface obtained as the quotient $\mathbf{P}'/\sim_{\mathbf{P}'}$. Denote paths ζ and η on $\mathbf{P}$, parameterized by integers $m, n \in \mathbb{Z}$ and $t \in [-\frac{1}{4}, \frac{1}{4}]$, and defined:*
$\zeta(t) = (2m + t) + i(2n - \frac{1}{2})$,
$\eta(t) = (2m + t) + i(2n + \frac{1}{2})$.

Let $\aleph = 2m + i2n \in \mathbb{C}$. The minimal relation $\sim_{\mathbf{P}'}$ is given as:

$$\begin{aligned}(\zeta(t); j) &\sim_{\mathbf{P}'} (e^{i\frac{\pi}{2}}\ \overline{\zeta(t) - \aleph}; j+1) \\ (\eta(t); j) &\sim_{\mathbf{P}'} (e^{i\frac{\pi}{2}}\ \overline{\eta(t) - \aleph}; j+1)\end{aligned} \tag{7}$$

This relation is similar to $\sim_{\mathbf{P}}$ in that it translates line segments of squares surrounding even integer pairs to the origin and relates points on one edge of a square to points on an adjacent edge. Where it differs is that these adjacent edges now belong to a different "copy" of $\mathbf{P}$. It is in this cyclic manner that edges are glued that allows for trivial linear holonomy on arbitrary paths by rotating each copy of $\mathbf{P}$ accordingly. For example, here is a path on a section of the surface in the neighborhood of $P = 2m + i2n \in \mathbb{C}$ ($m, n \in \mathbb{Z}$) after rotation: and its subsequent projection onto $\mathbf{U}^{\circ}$:

Definition 21. *Let $r : \mathbf{P}' \to \mathbf{P}'$ be the isometric map of $\tilde{\mathbf{U}}_0$'s domain given $r(z; \bar{j}) = (e^{(-j)i\frac{\pi}{2}} z; \bar{j})$.*

Observe that when r acts on the relations (3) we get the following:

$$
\begin{aligned}
(e^{(-j)i\frac{\pi}{2}}\zeta(t);j) &\sim_{r(\mathbf{P}')} (e^{(-j)i\frac{\pi}{2}}\ \zeta(t)-\aleph;j+1)\\
(e^{(-j)i\frac{\pi}{2}}\eta(t);j) &\sim_{r(\mathbf{P}')} (e^{(-j)i\frac{\pi}{2}}\ \eta(t)-\aleph;j+1)
\end{aligned}
\tag{8}
$$

$\tilde{\mathbf{U}}$ is then recovered as $r\cdot(\tilde{\mathbf{U}}_0)=\mathbf{P}'/\sim_{r(\mathbf{P}')}$, where all identifications are translations made on copies of $\mathbf{P}$.

3.9 Translation Surfaces and $\mathbb{Z}^2$-Covers

Definition 22. *Algebraic intersection number is a non-degenerate bilinear form:*

$$i: H_1(S,\mathbf{R})\times H_1(S,\mathbf{R})\to\mathbf{R},$$

where $\mathbf{R}$ *is a ring and for* $[\gamma],[\beta]\in H_1(S,\mathbf{R})$, $i([\beta],[\gamma])$ *returns the intersection number of two homology classes.*

Algebraic intersections are signed and follow some convention such as the right-hand rule.

Definition 23. *Let* $u,v\in H_1(S,\mathbb{Q})$ *be linearly independent homology classes of curves on* S*. Then the group homomorphism from* $\pi_1(S)$ *to* $\mathbb{Q}^2$ *is given as:*

$$\Omega_{u,v}:\pi_1(S)\to\mathbf{R}^2;\ \beta\mapsto(i(u,[\beta]),i(v,[\beta])).$$

The set of all orientation-preserving affine diffeomorphisms of S forms the group $\mathrm{Aff}^+(S)$. The corresponding *Veech group*, $V(S)$ of S is the image of the group morphism $D:\mathrm{Aff}^+(S)\to SL(2,\mathbb{R})$ that takes an affine map to its derivative. A surface is said to be *Veech* if its Veech group is commensurable to $SL(2,\mathbb{R})$. It is well known that origami, or square-tiled, surfaces have Veech groups commensurable to $SL(2,\mathbb{Z})$. [cite] When $\mathbf{R}=\mathbb{Z}$, $\Omega_{u,v}$ takes an element of $\pi_1(S)$ to Δ. Thus, $\gamma\in\pi_1(S)$ lifts to $\tilde{\gamma}\in\pi_1(\tilde{S})$ if and only if $\gamma\in\ker\Omega_{u,v}$.

Definition 24. *Let* $\alpha:[0,1]\to S$ *be a closed, non-singular geodesic path on* S*. The holonomy map* $\mathbf{hol}:H_1(S,\mathbf{R})\to\mathbb{C}$ *returns the holonomy vector of a closed path as a difference of the starting and endpoints of a flow by*

$$\mathbf{hol}([\alpha])=\int_\alpha dz.$$

Since α is non-singular, it can be mapped to S° which admits a flat holomorphic one-form dz. Let $\theta=Arg(\mathbf{hol}([\alpha]))$. We say that $\phi_t^\theta:\mathbb{R}\times S^\circ\to S^\circ$ is the unit-speed geodesic flow on S° in direction θ given by the $[\alpha]$ such that $\phi_0^\theta=\alpha(0)$. In local coordinates this corresponds to $z+te^{i\theta}\in\mathbb{C}$.

Lemma 13. ϕ_t^θ *has a period of* $T=|\mathbf{hol}([\alpha])|$

Proof. This just follows from the fact that ϕ_t^θ flows at unit-speed in the direction of $\frac{\mathbf{hol}([\alpha])}{|\mathbf{hol}([\alpha])|}$. $\square$

And so the length of a vector determines the period of a flow on S. More importantly, we have the following:

Lemma 14. *Denote the lifted flow of ϕ_t^θ on $\tilde{S}$ by $\tilde{\phi}_t^\theta$. Then $\tilde{\phi}_t^\theta$ is periodic on $\tilde{S}$ and $\tilde{S}^\circ$ if and only if $[\alpha] \in \ker \Omega_{u,v}$. Further, $\tilde{\phi}_t^\theta$ has period $T = |\mathbf{hol}([\alpha])|$.*

Proof. Suppose that $[\alpha] \notin \ker \Omega_{u,v}$. Since $\alpha([0,1]) = \phi^\theta_{[0,\mathbf{hol}([\alpha])]}$, their homology classes are equivalent. Hence $\tilde{\phi}_t^\theta$ could not close on $\tilde{S}$ or $\tilde{S}^\circ$, or else $\Omega_{u,v}(k[\alpha]) = k(i(u,[\alpha]), i(v,[\alpha])) = (0,0)$ for some non-zero $k \in \mathbb{Z}$. Conversely, suppose $[\alpha] \in \ker \Omega_{u,v}$. Then $\square$

Corollary 4. *If $[\alpha] \notin \ker \Omega_{u,v}$, then $\tilde{\phi}_t^\theta$ is drift-periodic with period $T = \mathbf{hol}([\alpha])$.*

Proof. This follows immediately from the previous lemma since $\tilde{S}, \tilde{S}^\circ$ have translational $\mathbb{Z}^2$ symmetries and a well-defined $\mathbb{Z}^2$ action on an element in the fiber of a basepoint in S under f. The period is T since a geodesic closes on S with period T. $\square$

Lemma 15. *Let $h \in Aff^+(S)$. If $h(\beta) = \alpha$ for closed geodesics α, β on S and β lifts to a closed path on $\tilde{S}$, then so does α. (not sure about this)*

Proof. If the premise is true, then $[\beta] \in \ker \Omega_{u,v}$. Denote the group automorphism induced by h as h_*. Then $\Omega_{u,v}([\alpha]) = \Omega_{u,v}([h(\beta)]) = \Omega_{u,v}(h_* \cdot [\beta]) = (i([\beta], h_*^{-1} \cdot u), i([\beta], h_*^{-1} \cdot v)) = (0,0)$ as automorphisms . $\square$

Lemma 16. *Let $\beta = h(\alpha)$ as before. If $D(h) = h' \in V(S)$ and S is Veech, then $\tilde{\phi}_t^\theta$ has period $T = h' \cdot \mathbf{hol}([\beta])$. (also not sure)*

Proof. $\square$

3.10 Symmetries of M

4 Four-fold Cover of U

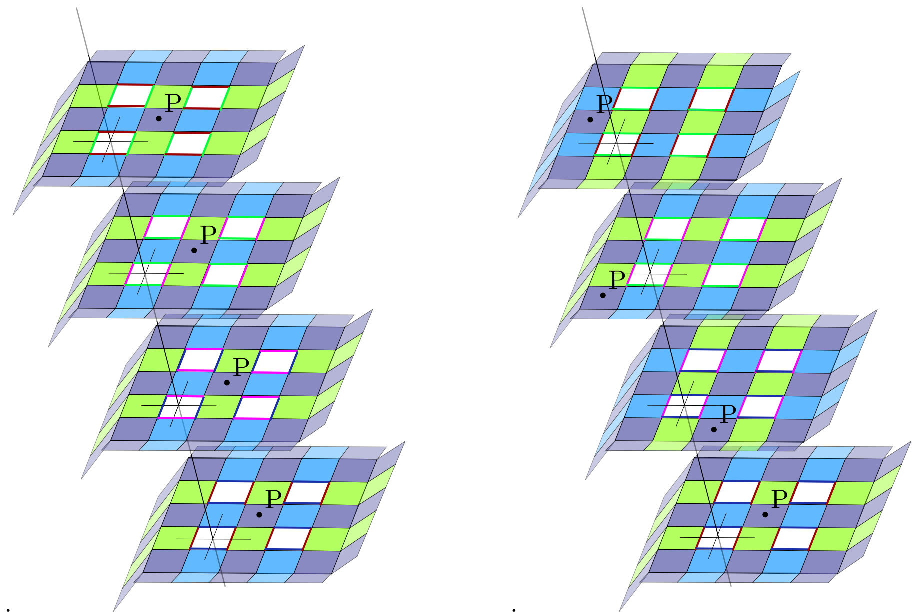


Figure 18: Four-fold cover isometry and the preimage of a point in $\mathbf{U}\backslash Sing(\mathbf{U})$.

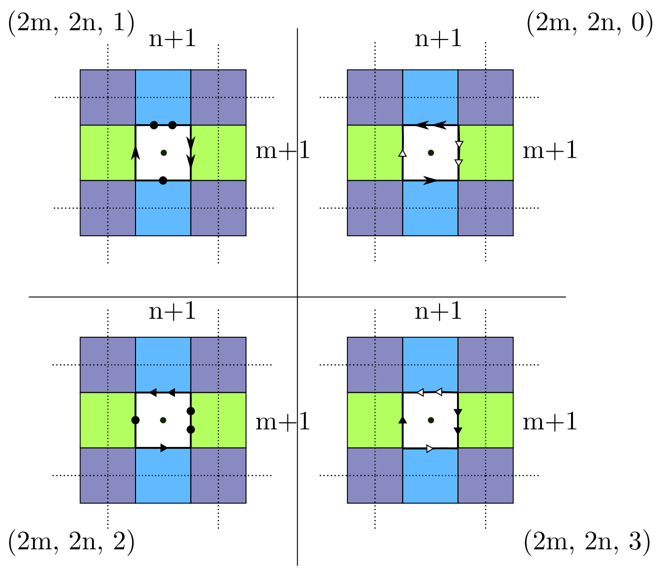


Figure 19: Branched cover associating every direction with one plane.

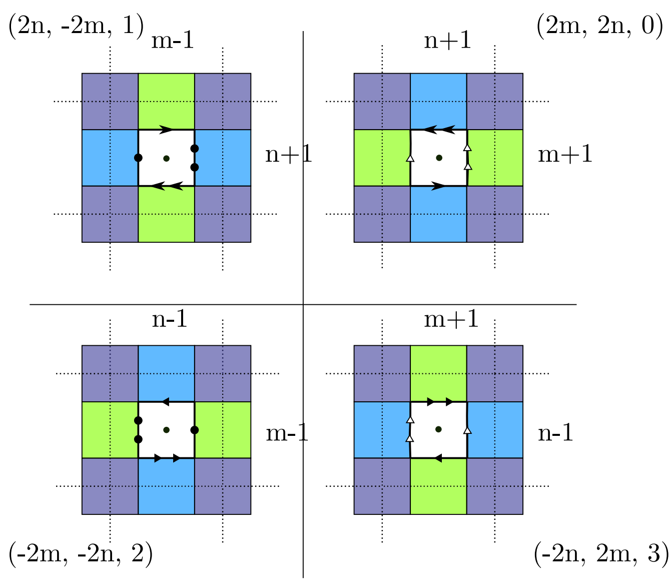


Figure 20: Infinite-type translation surface obtained by rotating each copy of the fundamental domain accordingly.

The quotient under the group action of translational symmetries is isomorphic to $\mathbb{Z}^2$ since the orbit of any point in the fundamental domain is a lattice in the space.

Theorem 6. *The translational symmetries of $\tilde{\mathbf{U}}$'s fundamental domain induce symmetries on the surface isomorphic to $\mathbb{Z}^2$.*

Proof. Let $(z;j) \in \mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$ and define the group action $T_0^{m,n} : \mathbb{C} \times \mathbb{Z}/4\mathbb{Z} \to \mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$ as $T^{m,n}(z;j) = (z + 2e^{-j\frac{i\pi}{2}}(m+in);j)$. This translation acts faithfully on the preimages of $\mathbf{U} \backslash Sing(\mathbf{U})$, and respects edge identifications of $\tilde{\mathbf{U}}$, thereby making it an isometry of the surface. Consider a group homomorphism, $T_0^{m,n} \mapsto m+in$ onto the plane of Gaussian integers, $\mathbb{Z}[i]$. The exponential function is never zero, so the identity of the translation group is $T_0^{0,0}$. This is an isomorphism since it is clearly surjective and any non-trivial element of $T^{m,n}$ could not possibly map to the identity element of $\mathbb{Z}[i]$, regardless of the value of j. Since $\mathbb{Z}^2$ is isomorphic to $\mathbb{Z}[i]$, it is isomorphic to $T_0^{m,n}$ as well. □

Definition 25. *The automorphism $T^{m,n} : \tilde{\mathbf{U}} \to \tilde{\mathbf{U}}$ is an **induced translation** of $\tilde{\mathbf{U}}$ as a result of the previous theorem.*

This surface is obtained as a ramified cover of the unit square torus. It is a translation surface and is therefore equipped with a **holomorphic one-form**, a collection of charts from neighborhoods of $\mathbf{M}$ to $\mathbb{C}$ such that any neighborhood away from Sing($\mathbf{M}$) has a *flat* induced Euclidean metric. A theorem of Gutkin and Judge tells us that its Veech group is commensurable to $\mathrm{SL}(2,\mathbb{Z})$ and is therefore a Veech surface. We look at some of its affine maps, and generate a subgroup $\mathbb{X} \subset \mathrm{Aff}^+(\mathbf{M})$ by the following transformations:

(i) Multi-twists of the surface as global diffeomorphisms given by Dehn-twists of its cylinder decomposition in horizontal and vertical directions with derivatives:
$$\left\{ \begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix} \right\}.$$
we call $\mathbf{A}^{\pm 1}$, $\mathbf{B}^{\pm 1}$, respectively.
A Dehn-twist on each cylinder in the cylinder decomposition of $\mathbf{M}$ in horizontal and vertical directions gives way to these global affine diffeomorphisms:

(ii) Rotation group generated by a $+\frac{\pi}{2}$ rotation of the surface fixed about the center of the second square on the bottom of the staircase, an order four isometry on $\mathbf{M}$ denoted $\mathbf{R}$.

(iii) Order 2 translation of the surface that moves the bottom left-most square to the square right next to it, denoted $\mathbf{H}$.

(iv) Order 2 translation of the surface that takes the bottom right-most square to the one right above it, denoted $\mathbf{V}$.

Definition 26. *The group $\mathbb{X}$ is the isometry group generated by affine maps* $\mathbf{A}$,$\mathbf{B}$,$\mathbf{R}$, *and* $\mathbf{H}$, *and* $\mathbf{V}$. *The image of the derivative map on elements in $\mathbb{X}$ is* *denoted $\mathbb{X}'$ and generated by matrices*

$$\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix},$$

denoted $\mathbf{A}'$, $\mathbf{B}'$, and $\mathbf{R}'$ in that order.

It is not immediately apparent if these affine maps generate $\mathrm{Aff}^+(\mathbf{M})$, or if their derivatives generate $V(\mathbf{M})$, its Veech group. We use these to induce homomorphisms on $H_1(X, \mathbb{Q})$. A spanning set of $H_1(\mathbf{M}, \mathbb{Q})$ is obtained as the set of homology classes of the core curves of X's cylinder decompositions in both vertical and horizontal directions:

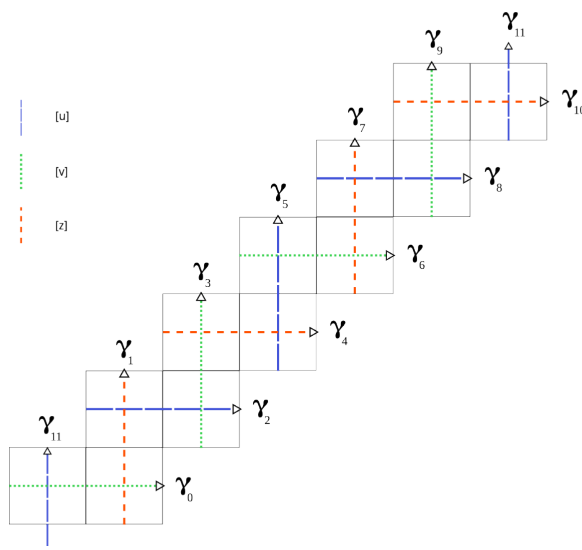


Figure 21: Cylinder core curves with u,v, and z homology classes that determines the $\mathbf{Z}^2$-cover.

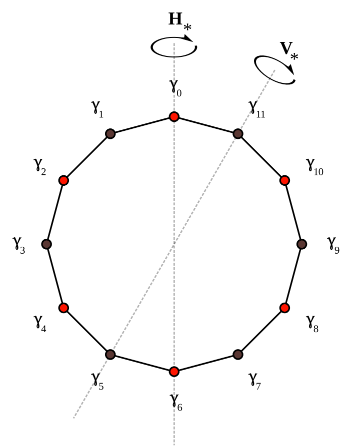
Definition 27. *The set of abelianized cylinder core curves is denoted as $\Gamma = \{\gamma_i : i = 0, \ldots, 11\} \subset H_1(\mathbf{M}, \mathbb{Q})$.*

Remark. *We use 12 elements to span homology, although a basis requires only 10. It's not impossible to determine the relations between these core curve classes, but it is not necessary. A 12×12 matrix of these core curve cylinder decompositions to their intersection numbers with adjacent curves is rank 10, as to be expected.*

The induced homomorphisms of $H_1(\mathbf{M}, \mathbb{Q})$ have come from affine maps that have various effects on these core-curves. We use a 12-gon to represent the set of curves, and show how these elements act on them. The multi-twists add curves to adjacent curves, and the translation maps permute them. The reader is encouraged to check these for themselves.

e.g. for $\mathbf{H}, \mathbf{V} \in \mathrm{Aff}^+(\mathbf{M})$,

H & V The effect that these two translations have on the 12-gon is a reflection about these lines. Observed by keeping track of the squares and core curves after **H** and **V** have acted on **X**.



Definition 28. *The induced homomorphisms of $H_1(\mathbf{M}, \mathbb{Q})$ are obtained from the affine subgroup $\mathbb{X}$ and denoted $\mathbb{X}_*$. The associated homomorphisms on the spanning set Γ are given as:*

$$\begin{aligned}
\mathbf{A}_*^k \circ [\gamma_i] =& [\gamma_i] + \frac{k}{2}(1 - (-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}]) \\
\mathbf{B}_*^k \circ [\gamma_i] =& [\gamma_i] + \frac{k}{2}(1 + (-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}]) \\
\mathbf{R}_* \circ [\gamma_i] =& (-1)^i [\gamma_{1-i \ mod \ 12}] \\
\mathbf{H}_* \circ [\gamma_i] =& [\gamma_{12-i \ mod \ 12}] \\
\mathbf{V}_* \circ [\gamma_i] =& [\gamma_{10-i \ mod \ 12}]
\end{aligned}$$

Definition 29. *The homology classes u, v, z are given as the following sums of core curves:*

$$\begin{aligned}
[u] &= -[\gamma_2] + [\gamma_5] + [\gamma_8] - [\gamma_{11}], \\
[v] &= +[\gamma_0] - [\gamma_3] - [\gamma_6] + [\gamma_9], \\
[z] &= +[\gamma_1] + [\gamma_4] - [\gamma_7] - [\gamma_{10}].
\end{aligned}$$

Theorem 7. *The fundamental group of the $\mathbb{Z}^2$-cover is obtained by lifting the kernel of the closed paths of $\mathbf{M}$ of the homomorphism:*

$$\Omega_{u,v} : \pi_1(\mathbf{M}, x_0) \to \mathbb{Z}^2;\ \beta \mapsto (i(u, [\beta]), i(v, [\beta])),\ \textit{where}$$

$$i : H_1(\mathbf{M}, \mathbb{Q}) \times H_1(\mathbf{M}, \mathbb{Q}) \to \mathbb{Z}.$$

is the intersection number of two homology classes.

Proof. We know from Theorem 6 that the translational symmetries of $\tilde{\mathbf{U}}$ induced by $T^{m,n}$ is isometric to $\mathbb{Z}^2$. Since $\mathbf{M}$ is a genus 5 base surface, we know that $\pi_1(\mathbf{M}, x_0) \simeq \mathbb{Z}^{10}$, and the associated cover satisfies $\mathbf{M} = \tilde{\mathbf{U}}/(\pi_1(\mathbf{M}, x_0)/N)$, such that N is a normal subgroup of $\pi_1(\mathbf{M}, x_0)$. This means that $N \simeq \mathbb{Z}^8$. The eight core curve classes are the abelianized forms of $\gamma_0, \gamma_2, \gamma_3, \gamma_5, \gamma_6, \gamma_8, \gamma_9$, and γ_{11} that span N. The classes and their signs are obtained from Figure 8 as the outer regions identified by the translations of $T^{m,n}$. Thus any closed path on $\mathbf{M}$ is lifted to a closed path on the cover under the quotient map only when a path has a trivial intersection number with the classes. □

Two paths are homologous if they return the same intersection number with the classes of closed core cylinder curves of $\mathbf{U}$ that span its homology. The classes u and v are obtained from the group group action of $T^{m,n}$ on the cover.

Definition 30. $\mathbf{hol} : \mathbf{M}\backslash Sing(\mathbf{M}) \to \mathbb{C}$ *is the holonomy vector pulled back from a non-singular path* γ *in* $\mathbf{M}$ *onto the complex plane given by* $\mathbf{hol}(\gamma) = \int_\gamma dz$.

We denote the **closed path** α, such that $\mathbf{hol}(\alpha) = 6 + 6i$, and show it is homologous to the closed geodesic with the same holonomy vector. The slope one direction also decomposes $\mathbf{M}$ into two cylinders by a series of saddle connections of length $\sqrt{2}$ between singularities:

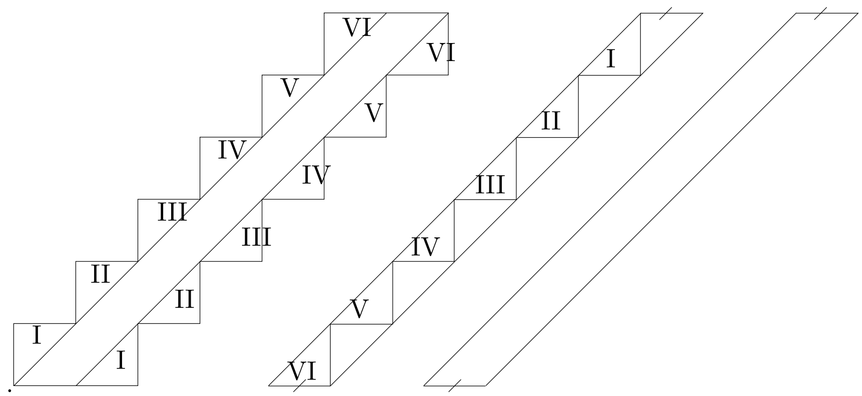


Figure 22: The two right-most cylinders C_1 (labeled) and C_2(unlabeled).

The circumferences of these two cylinders are $6\sqrt{2}$. Geodesic flows on this surface are well defined, and rational directions

Definition 31. *Let* $\omega_t^\theta : [0,1] \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbf{M}$ *be the* ***maximal geodesic flow*** *on the surface in direction* θ *such that* $\omega_0^\theta = \omega_1^\theta = x_0 \in \mathbf{M}\backslash Sing(\mathbf{M})$.
$\omega_t^{\frac{\pi}{4}}$ *is the geodesic flow in the* ***slope one direction****, and* χ *is its image in* $\mathbf{M}$ *and element of* $\pi_1(\mathbf{M}, x_0)$.

Lemma 17. α *is homologous to* χ *in* $\mathbf{M}\backslash Sing(\mathbf{M})$.

Proof. Let χ be a geodesic contained in either C_1 or C_2. Since a geodesic does not admit singularities, it is the image of a closed path on $X\backslash\mathrm{Sing}(X)$ with initial point x_0 on the strips of C_1 and C_2 with boundaries removed, denoted C_1', C_2'. Express $[\alpha]$ as $\Sigma_{j=0}^{11} \frac{1}{2}\gamma_j$ (a closed path climbing up the staircase). We show that the intersection numbers of $[\alpha]$ and $[\chi]$ are the same for every core cylinder curve γ, i.e. $i([\gamma_k], \Sigma_{j=0}^{11} \frac{1}{2}[\gamma_j]) = i([\gamma_k], [\chi])\ \forall k = 0, \ldots, 11$.
Case one: k is even. If k is even, then every curve γ_k is oriented to the right. Since χ intersects every curve once, $i([\gamma_k], [\chi]) = 1$. No even indexed curves intersect eachother, so we need only consider when j is odd. Now if j is odd, it is incident (positively crossing) with only two horizontal curves, namely $\gamma_{j+1}, \gamma_{j-1}$. Therefore $i([\gamma_k], [\alpha]) = i([\gamma_{j-1}], \frac{1}{2}[\gamma_j]) + i([\gamma_{j+1}], \frac{1}{2}[\gamma_j]) = \frac{1}{2}(i([\gamma_{j-1}], [\gamma_j]) + i([\gamma_{j+1}], [\gamma_j])) = \frac{1}{2}(1+1) = 1$.
Case two: k is odd. If k is odd, then $[\chi]$ will have an intersection number of -1 with $[\gamma_k]$ since odd-indexed core curves are oriented upwards. Now since k is odd, we only consider when j is even. Similarly, this means that γ_j negatively intersects the two vertical core curves with adjacent indices. Hence, $i([\gamma_k], [\alpha]) = i([\gamma_{j-1}], \frac{1}{2}[\gamma_j]) + i([\gamma_{j+1}], \frac{1}{2}[\gamma_j]) = \frac{1}{2}(i([\gamma_{j-1}], [\gamma_j]) + i([\gamma_{j+1}], [\gamma_j])) = \frac{1}{2}(-1-1) = -1$.
We know intersection number to be bilinear and non-degenerate on homology. So if α and χ's abelianizations admit the same intersection numbers for every curve in the spanning set of $H_1(\mathbf{M}, \mathbb{Q})$, then $[\alpha] = [\chi]$. □

Theorem 8. *$\chi \in \pi_1(\mathbf{M}, x_0)$ lifts to $\tilde{\chi} \in \pi_1(\tilde{\mathbf{U}}, \tilde{x}_0)$*

Proof. From Lemma 2, $[\chi] = [\alpha]$, so $\Omega_{u,v}(\chi) = \Omega_{u,v}(\alpha)$. Since $i([u], [\alpha]) = -i([\gamma_2], [\alpha] + i([\gamma_5], [\alpha]) + i([\gamma_8], [\alpha]) - i([\gamma_{11}], [\alpha]) = -1 + (-1) + 1 - (-1) = 0$ and $i([v], [\alpha]) = 1 - (-1) - 1 + (-1) = 0$, it follows that $\alpha, \chi \in \mathrm{Ker}\ \Omega_{u,v}$, and χ lifts to a closed geodesic on $\tilde{\mathbf{U}}$. □

Corollary 5. *content...*

From here, we use α to show that the *only* trajectories that close on the Necker cube surface are those that are in vector direction (a, b) such that $\gcd(a, b) = 1$ and a, b are both odd. We call these **odd-odd** directions. We can make this claim because the group generated by the matrices

$$\begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix}$$

is the *Sanov subgroup* of $SL(2, \mathbb{Z})$ and only sends elements in the odd-odd set to itself. There are dualizations made between how these matrices skew a geodesic direction, and how their original affine transformations induce an effect homology. In a sense the kernel is obtained by the orbit of χ under $\mathbb{X}$ and its holonomy vector under $\mathbb{X}'$.

Lemma 18. *The actions of $\mathbb{X}'$ on $\mathcal{O}$ and $\mathcal{E}$ are closed in their respective sets.*

Proof. Since $\mathbb{X}'$ is generated by the elements $\mathbf{A}'$, $\mathbf{B}'$, and $\mathbf{R}'$, any matrix $G' \in \mathbb{X}'$ is of the form $G' = (\mathbf{A}')^{1_1} \circ (\mathbf{B}')^{1_2} \circ (\mathbf{R}')^{1_3} \circ (\mathbf{A}')^{2_1} \circ \dots (\mathbf{A}')^{n_1} \circ (\mathbf{B}')^{n_2} \circ (\mathbf{R}')^{n_3}$, where $i_k \in \mathbb{Z}$ for $i = 1, \dots, n$ and $k = 1, 2, 3$. Let $x = \begin{pmatrix} p \\ q \end{pmatrix}, y \in \mathcal{O}$, and consider the equation $G'x = y$. Observe that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^l x = \begin{pmatrix} p + 2jq \\ q \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^m x = \begin{pmatrix} p \\ q + 2mp \end{pmatrix}$ for any $l, m \in \mathbb{Z}$. Also note that for any $j \in \mathbb{Z}$, $(\mathbf{R}')^m x = \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} -q \\ p \end{pmatrix}, \begin{pmatrix} -p \\ -q \end{pmatrix}, \begin{pmatrix} q \\ -p \end{pmatrix}$ when $j \mod 4 \equiv 0, 1, 2, 3$, respectively. In any case, the product of any power of a generator of $\mathbb{X}'$ and any $x \in \mathcal{O}$ is an element of $\mathcal{O}$. By letting $l = i_1, m = i_2$, and $j = i_3$, we first consider the base case when $i = n$. Let $G' = G'_1 \circ \dots \circ G'_n$, such that $G'_i = (\mathbf{A}')^{i_1} \circ (\mathbf{B}')^{i_2} \circ (\mathbf{R}')^{i_3}$. Since n_1, n_2, n_3 are arbitrary integers, $G'_n x \in \mathcal{O}$. Suppose for some $b < n - 1$, $G'_{n-b} \circ \dots \circ G'_n x = y' \in \mathcal{O}$. Therefore $y' = (G'_1 \circ \dots \circ G'_b)^{-1} y$, which implies that $(G'_1 \circ \dots \circ G'_b)^{-1}$ preserves the set $\mathcal{O}$. Otherwise, if $y \in \mathcal{E}$, there exists at least one G'_i for $1 < i < b$ and $\tau \in \mathcal{E}$ such that $G'^{-1}_i \tau = (\mathbf{R}')^{-i_3} \circ (\mathbf{B}')^{-i_2} \circ (\mathbf{A}')^{-i_1} \tau \in \mathcal{O}$, a contradiction. Since elements in $\mathbb{X}'$ are invertible, $G'_1 \circ \dots \circ G'_b$ must also map $\mathcal{O}$ to itself. Left multiply both sides of the equation to show that $G'_1 \circ \dots \circ G'_n x = G'x = y$. By the principle of strong induction, this holds for all $0 < b \leq n$. Since G' is invertible and an arbitrarily chosen element of $\mathbb{X}'$, it follows that $x \in \mathcal{O}$ if and only if $y \in \mathcal{O}$ and $\mathcal{O}$ is closed under $\mathbb{X}'$.
The proof for when $x \in \mathcal{E}$ is made in the same way. □

Now a trajectory in the horizontal direction has a directional vector of (1,0). The orbit of this vector by the Veech group is the set of all **even-odd** vectors. We also know that in this direction a geodesic is drift-periodic (See figure 1). The Veech group of $\mathbf{M}$ preserves these properties. Suppose you had some closed geodesic on $\mathbf{M} \backslash Sing(\mathbf{M})$ called β such that $\beta = h(\alpha)$, where $h \in \text{Aff}^+(\mathbf{M})$, and h_* is its induced homomorphism. Then we want to show that

$$(i([\beta], [u]), i([\beta], [v])) = (i([\alpha], h_*^{-1}[u]), i([\alpha], h_*^{-1}[v])) = (0, 0).$$

But first, we look at some of the properties of the group $\mathbb{X}_*$.

Theorem 9. *Let $\mathbb{X}_*$ be the group generated by $\mathbf{A}_*, \mathbf{B}_*, \mathbf{R}_*, \mathbf{H}_*$, and $\mathbf{V}_*$. Let $G = \langle \mathbf{A}_*, \mathbf{B}_* \rangle$, $T = \langle \mathbf{H}_*, \mathbf{V}_* \rangle$, and $R = \langle \mathbf{R}_* \rangle$. Then the following is true:*

(i) G is a free subgroup of $\mathbb{X}_*$ of rank two.

(ii) T is a finite cyclic subgroup of $\mathbb{X}_*$ and a centralizer of G.

(iii) R is a finite cyclic subgroup of $\mathbb{X}_*$, and a normalizer of G.

Proof. Let $h_*^j = \mathbf{A}_*^{k_j} \circ \mathbf{B}_*^{g_j} \in G$ for $k_j, g_j \in \mathbb{Z}$, $j = 1, \dots, n$.
(i). When $\mathbf{A}_*$ and $\mathbf{B}_*$ act on γ_i, it is only ever trivial if i is even for $\mathbf{A}_*$ or i is odd on $\mathbf{B}_*$. Since i cannot be both odd and even at the same time, there is no

relation between the two generators and therefore G is free.
(ii) It is up to the reader to show that T has the relations $\mathbf{H}_*^2 = \mathbf{V}_*^2 = (\mathbf{H}_*\mathbf{V}_*)^3 = id_*$, and is isomorphic to the rotational group of the hexagon generated by reflections about adjacent vertices of a 12-gon. Observe that $\mathbf{H}_*\circ\mathbf{A}_*^{k_j}\circ[\gamma_i] = \mathbf{H}_*\circ[\gamma_i]+\frac{k_j}{2}(1-(-1)^i)(\mathbf{H}_*\circ[\gamma_{i-1}]+\mathbf{H}_*\circ[\gamma_{i+1}]) = [\gamma_{-i}]+\frac{k_j}{2}(1-(-1)^i)([\gamma_{1-i}]+[\gamma_{-i-1}]) = \mathbf{A}^{k_j}\circ[\gamma_-i] = \mathbf{A}^{k_j}\circ\mathbf{H}_*\circ[\gamma_i]$, and $\mathbf{V}_*\circ\mathbf{A}_*^{k_j}\circ[\gamma_i] = \mathbf{V}_*\circ[\gamma_i]+\frac{k_j}{2}(1-(-1)^i)(\mathbf{V}_*\circ[\gamma_{i-1}]+\mathbf{V}_*\circ[\gamma_{i+1}]) = [\gamma_{10-i}]+\frac{k_j}{2}(1-(-1)^i)([\gamma_{11-i}]+[\gamma_{9-i}]) = \mathbf{A}^{k_j}\circ[\gamma_10-i] = \mathbf{A}^{k_j}\circ\mathbf{V}_*\circ[\gamma_i]$. In the same way one can show this to be true for $\mathbf{B}_*^{g_j}$, and we can see that T is a centralizer of G.
(iii) R is obviously cyclic and finite since an isomorphism is obtained as $\mathbf{R}_*\mapsto\mathbf{R}'\in SO(2,\mathbb{Z})$.
Note that $\mathbf{R}_*\circ\mathbf{A}_*^{k_j}\circ[\gamma_i] = \mathbf{R}_*\circ[\gamma_i]+\frac{k_j}{2}(1-(-1)^i)(\mathbf{R}_*\circ[\gamma_{i-1}]+\mathbf{R}_*\circ[\gamma_{i+1}])$
$= (-1)^i[\gamma_{1-i}]+\frac{k_j}{2}(1-(-1)^i)((-1)^{i-1}[\gamma_{2-i}]+(-1)^{i+1}[\gamma_{-i}])$
$= (-1)^{1-i}([\gamma_{1-i}]-\frac{k_j}{2}(1+(-1)^{1-i})([\gamma_{2-i}]+[\gamma_{-i}]))$
$= (-1)^{1-i}\mathbf{B}_*^{-k_j}\circ[\gamma_{1-i}] = \mathbf{B}_*^{-k_j}\circ(-1)^{1-i}[\gamma_{1-i}] = \mathbf{B}_*^{-k_j}\circ\mathbf{R}_*\circ[\gamma_i]$.
Likewise, $\mathbf{R}_*\circ\mathbf{B}_*^{g_j}\circ[\gamma_i] = \mathbf{A}_*^{-g_j}\circ\mathbf{R}_*\circ[\gamma_i]$. $\square$

Remark. *It can be easily shown that $\mathbb{X}'$ has similar properties.*

Lemma 19. *Let $h_*\in\langle\mathbf{A}_*,\mathbf{B}_*\rangle$. Then $h_*\circ[\alpha]\in H_1(\mathbf{M},\mathbb{Q})$ can be expressed as $h_*\circ[\alpha] = \frac{1}{2}(c_1\Sigma_{j=0}^5[\gamma_{2j}]+c_2\Sigma_{j=0}^5[\gamma_{2j+1}])$ for $c_1, c_2\in\mathbb{Z}$.*

Proof. Let $\Sigma_{j=0}^5[\gamma_{2j}] = \Sigma\Gamma_{even}$, $\Sigma_{j=0}^5[\gamma_{2j+1}] = \Sigma\Gamma_{odd}$, and $\Sigma_{j=0}^{11}[\gamma_j] = \Sigma\Gamma$. Let $h_* = h_*^n\circ\cdots\circ h_*^1$, and $h_*^i = \mathbf{A}_*^{k_i}\circ\mathbf{B}_*^{g_i}$ for $k_i, g_i\in\mathbb{Z}$, $i = 1,\ldots,n$. Compose these two homomorphisms and obtain $\mathbf{A}_*^{k_i}\circ\mathbf{B}_*^{g_i}(\Sigma\Gamma) = (4g_ik_i+2k_i)\Sigma\Gamma_{even}+2g_i\Sigma\Gamma_{odd}+\Sigma\Gamma$. Let $c_i^1 = (4g_ik_i+2k_i), c_i^2 = 2g_i$, and solve for $h_*^{i+1}\circ h_*^i\circ\Sigma\Gamma$:

$$\begin{aligned}
h_*^{i+1}\circ h_*^i\circ(\Sigma\Gamma) &= h_*^{i+1}\circ(c_i^1\Sigma\Gamma_{even}+c_i^2\Sigma\Gamma_{odd}+\Sigma\Gamma)\\
&= c_i^1h_*^{i+1}\circ(\Sigma\Gamma_{even})+c_i^2h_*^{i+1}\circ(\Sigma\Gamma_{odd})+h_*^{i+1}\circ(\Sigma\Gamma)\\
&= 2g_{i+1}\Sigma\Gamma_{odd}+(4g_{i+1}k_{i+1}+2k_{i+1})\Sigma\Gamma_{even}+\Sigma\Gamma\\
&\quad +c_i^1(4g_{i+1}k_{i+1}\Sigma\Gamma_{even}+2g_{i+1}\Sigma\Gamma_{odd}+\Sigma\Gamma_{even})\\
&\quad +c_i^2(2k_{i+1}\Sigma\Gamma_{even}+\Sigma\Gamma_{odd})\\
= \Sigma\Gamma+(c_i^1+(c_i^1+1)&(4g_{i+1}k_{i+1})+(c_i^2+1)2k_{i+1})\Sigma\Gamma_{even}\\
&\quad +(c_i^2+(c_i^1+1)2g_{i+1})\Sigma\Gamma_{odd}\\
\text{Let } c_{i+1}^1 &:= (c_i^1+(c_i^1+1)(4g_{i+1}k_{i+1})+(c_i^2+1)2k_{i+1}),\\
c_{i+1}^2 &:= (c_i^2+(c_i^1+1)2g_{i+1}).
\end{aligned}$$

From these recursive definitions and a finite sequence of integers, $\{k\}_i, \{g\}_i$, observe then that
$h_*\circ[\alpha] = h_*\circ[\frac{1}{2}\Sigma\Gamma] = \frac{1}{2}h_*\circ[\Sigma\Gamma] = \frac{1}{2}[c_n^1\Sigma\Gamma_{even}+c_n^2\Sigma\Gamma_{odd}+\Sigma\Gamma]$
$= \frac{1}{2}[(c_n^1+1)\Sigma\Gamma_{even}+(c_n^2+1)\Sigma\Gamma_{odd}]$. Further simplify by letting $c_1 = c_n^1+1, c_2 = c_n^2+1$. $\square$

Lemma 20. *Let $h_* \circ [\alpha] \in H_1(\mathbf{M}, \mathbb{Q})$. Then for $a \in \langle \mathbf{H}_*, \mathbf{V}_* \rangle$ and $b \in \langle \mathbf{R}_* \rangle$, the following is true:*

$$a \circ h_* \circ [\alpha] = h_* \circ [\alpha]$$
$$b \circ h_* \circ [\alpha] = \frac{1}{2}[c_1' \Sigma\Gamma_{even} + c_2' \Sigma\Gamma_{odd}]$$
$$h_* \circ b \circ [\alpha] = \frac{1}{2}[c_1'' \Sigma\Gamma_{even} + c_2'' \Sigma\Gamma_{odd}]$$

Proof. By Theorem 4, a is a centralizer of the group so $a \circ h_* \circ [\alpha] = h_* \circ a \circ [\alpha] = h_* \circ \frac{1}{2} a \circ [\Sigma\Gamma]$. Since a is a cyclic permutation of the set Γ, it acts trivially on $\Sigma\Gamma$. Therefore, $a \circ h_* \circ [\alpha] = h_* \circ \frac{1}{2}[\Sigma\Gamma] = h_* \circ [\alpha]$.
By theorem 4, $a \circ \mathbf{A}_*^{k_i} \circ \mathbf{B}_*^{g_i} = \mathbf{B}_*^{-k_i} \circ \mathbf{A}_*^{-g_i} \circ a$. Extend this property to h_*, and denote the normalized element as h_{**}, such that $b \circ h_* = h_{**} \circ b$. Note that $b(\Sigma\Gamma) = b(\Sigma\Gamma_{even} + \Sigma\Gamma_{odd}) = \Sigma\Gamma_{odd} - \Sigma\Gamma_{even}$. $b \circ h_* \circ [\Sigma\Gamma] = c_1 b \circ \Sigma\Gamma_{even} + c_2 b \circ \Sigma\Gamma_{odd} = c_1 \Sigma\Gamma_{odd} - c_2 \Sigma\Gamma_{even}$. So, $c_1' = -c_2$ and $c_2' = c_1$.
Since h_* is arbitrary, let $h_{**} = g_*$ be generated by an integer sequence that defines the word and consider $h_* \circ b \circ [\Sigma\Gamma] = b \circ g_* \circ [\Sigma\Gamma] = c_1^* b \circ \Sigma\Gamma_{even} + c_2^* b \circ \Sigma\Gamma_{odd} = c_1^* \Sigma\Gamma_{odd} - c_2^* \Sigma\Gamma_{even}$. So, $c_1'' = -c_2^*$ and $c_2'' = c_1^*$. □

Now that every element in the orbit of $[\alpha]$ can be expressed as a linear combination of integers, it is simple to show they lift to a closed trajectory in the cover.

Definition 32. *Let* $\mathbf{dir} : UT(\mathbf{M} \backslash Sing(\mathbf{M})) \to \mathcal{O} \cup \mathcal{E}$ *be the injective map from* $\mathbb{R}/2\pi\mathbb{Z}$ *to* $\mathbb{Z}^2$ *given as* $\mathbf{dir}(\theta) = (k_1 \cos(\theta), k_2 \sin(\theta)), k_1, k_2 \in \mathbb{R}$ *such that* $\gcd(k_1 \cos(\theta), k_2 \sin(\theta)) = 1$.

Theorem 10. *(Sketch)*
Any geodesic, β, in $\mathbf{M}$ lifts to a closed geodesic $\tilde{\beta}$ on $\tilde{\mathbf{U}}$ if and only if $\mathbf{dir}(Arg(\mathbf{hol}(\beta))) \in \mathcal{O}$.

Proof. Call the quotient cover $p : \tilde{\mathbf{U}} \to \mathbf{M}$, and fix a point $\tilde{x_0} \in p^{-1}(x_0)$. Let $\beta = h(\chi)$, where $h \in \mathbb{X}$. We also obtain $[\beta] = h_* \circ [\alpha]$ from Lemma 2. Since h sends geodesics to geodesics, h induces the following: $\mathbf{hol}(h(\chi)) = h'(\mathbf{hol}(\chi)) = h'(6 + 6i)$ for $h' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{X}'$. So, $\mathrm{Arg}(h'(\mathbf{hol}(\chi))) = \mathrm{Arg}(6[(a+b) + i(c+d)]) = \mathrm{Arg}(6h'(1+i))$. Lemma 3 states that for any $h' \in \mathbb{X}'$, $h'(\mathcal{O}) = \mathcal{O}$. Therefore there is no such geodesic of **even-odd** slope in the orbit of χ. Otherwise $h', h \notin \mathbb{X}', \mathbb{X}$. Consequently, $\mathbf{dir}(Arg(\mathbf{hol}(\beta))) \in \mathcal{O}$.
From Lemma 5 we see that $[h(\chi)] = h_* \circ [\chi] = h_* \circ [\alpha] = \frac{1}{2}(c_1 \Sigma_{j=0}^5 [\gamma_{2j}] + c_2 \Sigma_{j=0}^5 [\gamma_{2j+1}])$ for $c_1, c_2 \in \mathbb{Z}$. Denote the sums as $\Sigma\Gamma_{even}$ and $\Sigma\Gamma_{odd}$.
Therefore, $2i([u], h_* \circ [\alpha]) = c_1 i([u], \Sigma\Gamma_{even}) + c_2 i([u], \Sigma\Gamma_{odd})$
$= c_1(-i([\gamma_2], 0) + i([\gamma_5], [\gamma_6] + [\gamma_4]) + i([\gamma_8], 0) - i([\gamma_{11}], [\gamma_{10}] + [\gamma_0]))$
$+ c_2(-i([\gamma_2], [\gamma_1] + [\gamma_3]) + i([\gamma_5], 0) + i([\gamma_8], [\gamma_7] + [\gamma_9]) - i([\gamma_{11}], 0))$
$= c_1(-(0) + (-1-1) + (0) - (-1-1)) + c_2(-(1+1) + (0) + (1+1) - (0)) = 0.$

Similarly, $2i([v], h_* \circ [\alpha]) = c_1 i([v], \Sigma\Gamma_{even}) + c_2 i([v], \Sigma\Gamma_{odd})$
$= c_1(i([\gamma_0], 0) - i([\gamma_3], [\gamma_2] + [\gamma_4]) - i([\gamma_6], 0) + i([\gamma_9], [\gamma_8] + [\gamma_{10}])$
$+ c_2(i([\gamma_0], [\gamma_{11}] + [\gamma_1]) - i([\gamma_3], 0) - i([\gamma_6], [\gamma_5] + [\gamma_7]) + i([\gamma_9], 0)$
$= c_1((0) - (-2) - (0) + (-2)) + c_2((2) - (0) - (2) + (0)) = 0.$
Therefore, $\Omega_{u,v}(h(\chi)) = (0,0)$, and $h(\chi) = \beta \in \text{Ker } \Omega_{u,v}$ for all $h \in \mathbb{X}$. By Theorem 2, β lifts to $\tilde{\beta} \in \pi_1(\tilde{\mathbf{U}}, \tilde{x}_0)$. Let $\theta =$Arg($\mathbf{hol}(\beta)$). Then ω_t^θ at x_0 lifts to $\tilde{\omega}_t^{p^{-1}(\theta)} \in \tilde{\mathbf{U}} \backslash \text{Sing}(\tilde{\mathbf{U}})$.
Now suppose instead that $\beta = h(\gamma_i)$. Then $\mathbf{dir}(\beta) = \frac{1}{2}(1 + (-1)^i, 1 - (-1)^i)$. According to Lemma 3, $h'(\mathcal{E}) = \mathcal{E}$. Thus we have no geodesic in the **odd-odd** directions obtained from the orbits of $(1,0)$ and $(0,1)$. For contradiction, suppose that $h(\gamma_i) \in \mathbf{Ker}\ \Omega u, v$. Then $(i(h_* \circ [\gamma_i], [u]), i(h_* \circ [\gamma_i], [v])) = (i([\gamma_i], h_*^{-1} \circ [u]), i([\gamma_i], h_*^{-1} \circ [v])) = (0,0)$. Let $h_*^{-1} \circ [u] = \Sigma_{j=0}^{11} x_j [\gamma_j]$, and $h_*^{-1} \circ [v] = \Sigma_{j=0}^{11} y_j [\gamma_j]$. Note that since γ_i intersects $\gamma_{i\pm1}$, $i([\gamma_i], h_*^{-1} \circ [u]) = (-1)^{i+1}(x_{i-1} + x_{i+1})$ and $i([\gamma_i], h_*^{-1} \circ [v]) = (-1)^{i+1}(y_{i-1} + y_{i+1})$.
Unfinished.. □

Conjecture. ***Dynamics of Geodesic Flow on the Necker cube surface.*** *Obtain θ and $\vec{\theta}$ as described in Definition 3. Denote the non-singular unit-speed geodesic flow with initial point $s \in (\mathbf{U} \backslash Sing(U))$ in direction $[\theta] \sim \phi \in UT(\mathbf{U} \backslash Sing(\mathbf{U}))$ by $F_t^\phi : \mathbf{U} \times \mathbb{R}_0^+ \to \mathbf{U}$ on $(\mathbf{U}, \mu)$, where μ is a flow-invariant measure. Then the following is true:*

(i) (Periodic) There exists a $t_0 > 0$ such that $F_{t+t_0}^\phi(s) = F_t^\phi(s)$ if and only if $\vec{\theta} \in \mathcal{O}$.

(ii) (Drift-Periodic) There exists a $t_0 > 0$ such that $F_{t+t_0}^\phi(s) = F_t^\phi(s) + c$, where $c \in \mathbf{U}$ is a non-trivial translation of a point in $\mathbf{U}$, if and only if $\vec{\theta} \in \mathcal{E}$.

Proof. (Sketch)
Denote the covering maps $f : \tilde{\mathbf{U}} \to \mathbf{U}$, $p : \tilde{\mathbf{U}} \to \mathbf{M}$, and fix a point $\tilde{x}_0 \in f^{-1}(s), p^{-1}(x_0)$, for $x_0 \in \mathbf{M}$. $f^{-1}([\theta]) = \{x : x = \theta + n\frac{\pi}{2}, n \in \mathbb{Z}\} = [\theta] \subset UT(\tilde{\mathbf{U}} \backslash \text{Sing}(\tilde{\mathbf{U}}))$ given by the four-fold cover and rotations of each individual plane. This gives us a relation between the two tangent bundles, where the translation four-fold cover has the standard $\mathbb{R}/2\pi\mathbb{Z}$ unit tangent fiber. θ' is the direction associated to the flow $\omega_t^{\theta'} : [0,1] \to \mathbf{M}$ on the translation surface. Since the cover is translation, $p^{-1}(\theta') = \theta' = \theta + n\frac{\pi}{2}$. First suppose that $\vec{\theta} \in \mathcal{O}$. Then θ is identified with the set of directions that close on $\tilde{\mathbf{U}}$. From Theorem 5, $\omega_t^{\theta'}$ lifts to a closed geodesic $\tilde{\omega}_t^{\theta'}$. Given $\mathbf{hol}(\omega) = \int_\omega dz$, we obtain a period for the unit-speed flow, $t_0 = |\mathbf{hol}(\omega)|$. That is, $\tilde{F}_t^{\theta'} : \mathbb{R}_0^+ \to \tilde{\mathbf{U}}$ such that $\frac{d}{dt}\tilde{F}_t = \frac{1}{|\mathbf{hol}(\omega)|}$. Then $F_t^\phi = F_t^{[\theta']} = f \circ \tilde{F}_t^{\theta'}$. The period carries over since there is no concern over a trajectory returning to $\tilde{x}_0$ in a different direction. Otherwise, the geodesic $\omega_t^{\theta'}$ on $\mathbf{M}$ would have closed in $0 < t < 1$. Now suppose that $\vec{\theta} \in \mathcal{E}$. Identifying it with θ', we see that ω in direction θ' is not an element of $\mathbf{Ker}\ \Omega_{u,v}$ from Theorem 5. Therefore, $\Omega_{u,v}(\omega) = (m,n) \simeq T^{m,n}$ and lifting the terminal point

$\omega(1)$, $\tilde{\omega}(1) = T^{m,n}(\tilde{\omega}(0)) = T^{m,n}(\tilde{x}_0)$. The period remains unchanged, in that $\tilde{F}^{\theta'}_{t+\mathbf{hol}(\omega)} = \tilde{F}^{\theta'}_t + T^{m,n}(\tilde{x}_0)$. Therefore, $F^{\phi}_{t+t_0}(s) = f \circ \tilde{F}^{\theta'}_t(\tilde{x}_0) + f \circ T^{m,n}(\tilde{x}_0)$. Conversely, suppose F_t is periodic. Then $[\theta] = \phi = [\theta']$, which defines directional flows $\tilde{F}^{\phi}_t$. According to Theorem 5, $\tilde{F}^{\phi}_t$ will close if and only if $\phi \subset \mathcal{O}$. ϕ is the orbit of $\vec{\theta'}$ under the 90 degree rotational matrix. This matrix does not alter the length or period of a geodesic. Thus, F^{ϕ}_t is exactly one of the flows $\tilde{F}^{\phi}_t$. Likewise, if F^{ϕ}_t is drift-periodic then $F^{\phi}_{t+t_0} = f \circ \tilde{F}^{\phi}_t + f \circ T^{m,n}$. $T^{m,n}$ is trivial if and only if $\theta' \in \mathcal{O}$. Therefore, $\theta' \in \mathcal{E}$, and $[\theta'] = \phi$. $\square$

There is still much work to do in terms of cleaning up the proofs and organizing the final paper.

Conclusion

What I ultimately aim to do is port these results on X's homology back to the Necker cube surface. I want do it in such a way that the final theorem is bi-conditional. To do so, I imagine I can take a vector image of a small segment of a geodesic in $\mathbb{R}^3$ and project it onto the isometric flattening of the Necker Cube surface to obtain a direction (or classes of equivalent directions), and relate it to the unit tangent bundle of $\mathbf{M}$.

In addition, I would also like to find a formula for the arc-length of a geodesic based on direction alone. Knowing that $\text{hol}(\alpha) = 6 + 6i$ means that the induced Euclidean metric on $\mathbf{M}\backslash\text{Sing}(\mathbf{M})$ gives the geodesic an arc-length of $6\sqrt{2}$. I would like to show that:

$$\int_\beta |dz| = |hol(h(\alpha))| = |h'(hol(\alpha))|,$$

where $h' \in V(\mathbf{M})$ is the derivative of h, and $\beta = h(\alpha)$. We know that is true on the translation surface, but it's a matter of then showing the translation quotient, branch-cover, and the Necker cube surface have the same induced Euclidean metric of these non-singular geodesics. (It would not be surprising considering that the surface is built out of subsets of planes.) Even more of a problem is finding a way to solve for a matrix in the Sanov subgroup that brings (1,1) to the desired odd-odd slope.

References

[1]

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