

Periodicity of Geodesic Flows on the Necker Cube Surface

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Abstract

We study dynamical properties of geodesic flows on a flat, periodically constructed Euclidean cone surface by obtaining an infinite-type translation cover branched over conical singularities, of which admits a flat induced metric and a connection between any two points in the tangent vector bundle of the surface has trivial holonomy. A unit vector flow on the unit tangent bundle of such a surface is well-defined with a canonical vector representation in \mathbb{R}^2 . We study this infinite-type surface as a \mathbb{Z}^2 cover of a compact Veech surface belonging to the stratum of $\mathcal{H}(2, 2, 2, 2)$ translation surfaces, and use its $SL(2, \mathbb{R})$ -commensurable Veech group to prove results that relate directions of flows to the periodicity and ergodicity of their lifts in the cover.

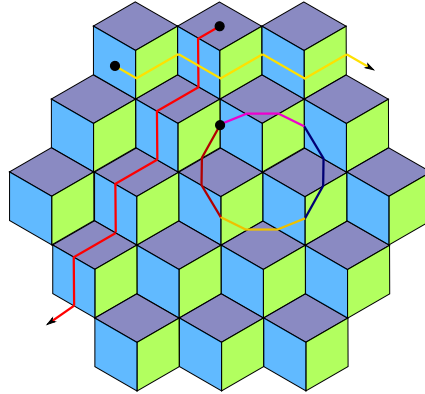


Figure 1: Periodic and drift-periodic flows on the Necker cube surface

1 Introduction

Put your Problem statement here! Example of a Citation[?, p.219]. Here's Another Citation [?]

Historically, the Necker cube[citation] has made numerous appearances in the work of mathematicians, crystallographers, and scientists interested in human visual systems prior to it being popularized in the works of illusionary artist M.C. Escher (pictured below). The crystallographer Louis Albert Necker was credited for having discovered the optical illusion, and studying its geometry[cite]. The solid presentation of the cube, when rendered as a flat surface with its back faces removed, achieves a similar effect when its three visible faces are shaded in a particular way. We are interested in the infinite tiling of this structure (figure ??), which has the appearance of a rhombile tiling of the Euclidean plane, and refer to it as the Necker cube surface. Video game fans might even recognize it as the same surface that appears in the 1982 arcade game by Gottlieb, “Q*bert.”

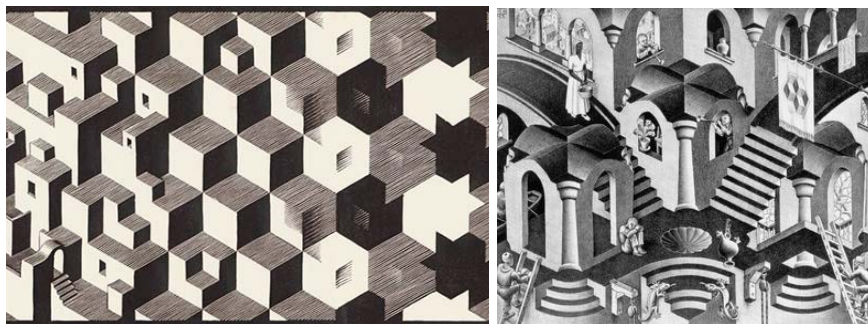


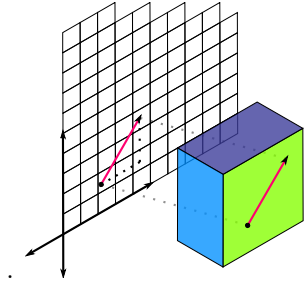
Figure 2: The Necker cube tiling as it appears briefly in “Metamorphosis I,” and impressed into a banner in “Convex and Concave.”[citation]

Flat, periodic surfaces such as the Necker cube surface are of much interest to differential geometers for two reasons in particular. The first is that, locally, neighborhoods are isometric to the plane, making it remarkably easy to describe parallel transports between two points in tangent vector bundles over the surface by linear group actions. The second is that Riemann surfaces constructed out of infinitely many polyhedra can be *flattened* and contained in the plane in such a way that the metric is preserved. The advantage in this case is that one obtains something akin to infinite billiard tables where a geodesic flows are represented by series of straight line segments in the plane. This paper will use such methods to prove dynamical results about Geodesic flows on the Necker cube surface.

Remark. *Familiarity is assumed on the part of the reader with covering space theory, translation (Veech) surfaces, and their associated Veech groups. For general surveys on these topics: [cite], [cite].*

1.1 Discussion of Results

Our initial experiments strongly supported the theory that there would be a correlation between a choice of trajectory angle and dynamical properties of a geodesic on such a symmetric object. Rightfully so, a surface composed of infinitely many cubes an induced flat metric where every neighborhood is locally isometric to the Euclidean plane. Via the parallel transport of a unit tangent vector over the sharp edges of the surface, it becomes obvious to think of the geodesic as a sequence of line segments contained in the faces of each cube on the surface's embedded form in \mathbb{R}^3 . ??.



Let x_0 be a point on the surface that is contained in an open neighborhood of a smooth section of the surface and consider a tangent unit vector $v \in \mathbb{R}^3$ protruding from x_0 . Assuming the faces are parallel to every 2-dimensional subspace of \mathbb{R}^3 spanned by standard basis vectors, exactly one component of v would be 0. Projecting this vector to a parallel plane retains all necessary information about direction. Call this vector $v_0 \in \mathbb{R}^2$.

We say that v_0 is the *initial trajectory* of the geodesic and the angle it makes relative to our choice of basis is its *initial trajectory angle*. We consider a rational direction to fall into one of two categories.

Definition 1. Let v_0 be a unit vector of the form $\frac{1}{k}(x, y) \in \mathbb{R}^2$ with $x, y \in \mathbb{Z}$ and $k = \sqrt{x^2 + y^2} \in \mathbb{R}$. We say v_0 is an **odd-odd** vector if its components are relatively prime and odd. We denote the **set of all odd-odd directions** by \mathcal{O} . We say that v_0 is an **even-odd** vector if its components are relatively prime and of opposite parity. We denote the **set of all even-odd directions** by \mathcal{E} .

This paper will demonstrate how one reaches the following conclusion about the relationship between parity and periodicity of geodesic behavior on the Necker cube surface, denoted \mathbf{S} :

Theorem. (*Directional Dichotomy of Periodic Geodesics on \mathbf{S}*) Let Φ_t be a unit-speed geodesic flowing on the unit tangent bundle of \mathbf{S} for $t \in \mathbb{R}$ with initial point and vector $\Phi_0 = (x_0, v)$. Then the following is true:

- (i) Φ is periodic with period $T \in \mathbb{R}$ if and only if $v_0 \in \mathcal{O}$.
- (ii) Φ is drift-periodic with period $T \in \mathbb{R}$ if and only if $v_0 \in \mathcal{E}$.

The proof for this theorem can be found in XXX and follows from Theorems YYYY and ZZZZ.

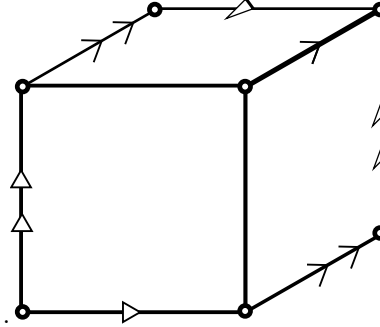
theorem about most directions being recurrent?

1.2 Acknowledgements

- Pat Hooper
- Vincent Delecroix, Ferrán Valdez, pascal hubert
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2 Periodic Tiling of Necker Cubes

This section will detail how the Necker Cube surface is obtained as an infinite regular cover of a branched torus. Consider the Necker cube with the following identifications:



We denote this surface \mathbf{G} . \mathbf{G} is obtained by taking three copies of the unit square and identifying the edges in order to resemble the visible portion of a unit cube. \mathbf{G} is a genus one surface with three vertices homeomorphic to \mathbb{T}^2 . Check that every vertex has a cone angle of either 3π or $\frac{3\pi}{2}$. These are the conical singularities of the surface, Σ . We denote the surface without its singularities $\mathbf{G} \setminus \Sigma = \mathbf{G}^\circ$. Thus, $\pi_1(\mathbf{G}^\circ) \cong \pi_1(\text{thrice-branched torus}) \cong \pi_1(S^1 \vee S^1 \vee S^1 \vee S^1) \cong \mathbb{F}_4$, the free group of four generators.

Definition 2. *The Necker cube surface \mathbf{S} is obtained as the regular cover of \mathbf{G} by periodically tiling along the edges with solid triangles.*

The branched Necker cube surface \mathbf{S}° is obtained similarly by tiling \mathbf{G}° .

Consider the following families of unit squares in \mathbb{R}^3 :

$$\begin{aligned} \mathbf{A}_{m,n,p} &= [m, m+1] \times [n, n+1] \times \{p\}, \\ \mathbf{B}_{m,n,p} &= \{m+1\} \times [n, n+1] \times [p-1, p], \\ \mathbf{C}_{m,n,p} &= [m, m+1] \times \{n+1\} \times [p-1, p]. \end{aligned}$$



With the proper identifications on $\mathbf{A}_{0,0,0} \cup \mathbf{B}_{0,0,0} \cup \mathbf{C}_{0,0,0}$, we have \mathbf{G} 's embedded form in \mathbb{R}^3 and a map $s : \mathbf{G} \hookrightarrow \mathbb{R}^3$. Similarly, removing the integer points from this set is an embedding of \mathbf{G}° . Check that tiling along the edges as previously described determines a set of points in \mathbb{R}^3 that coincides with points on \mathbf{S} with s^* as an extension of s to the cover. That is, $s(\mathbf{S}) = \bigcup \{\mathbf{A}_{m,n,p} \cup \mathbf{B}_{m,n,p} \cup \mathbf{C}_{m,n,p} : m+n+p=0\}$. Thus a point $x_0 \in \mathbf{S}$ is in one-to-one correspondence with this set. To avoid ambiguity we say U is a *neighborhood* of $s(x_0)$ if it is contained in exactly one of $\mathbf{A}_{m,n,p}, \mathbf{B}_{m,n,p}, \mathbf{C}_{m,n,p}$.

As U must contain an open ball contained on a single square there cannot be a neighborhood where $s(x_0) \in \mathbb{Z}^3$ and x_0 is a conical point. It follows then that a neighborhood in \mathbf{S} is a neighborhood in \mathbf{S}° as well.

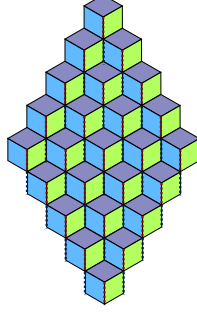


Figure 3: A section of \mathbf{S}

2.1 Flattening \mathbf{S}

At the moment it is difficult to describe how the parallel transport of a vector tangent to the surface along an arbitrary path on the surface acts on the vector in \mathbb{R}^3 . To simplify this problem we take \mathbf{S} to an isometric variant embedded in \mathbb{R}^2 by piecewise linear transformations on the sets

$$\begin{aligned}\mathbf{A} &= \bigcup \{ \mathbf{A}_{m,n,p} : m+n+p=0 \}, \\ \mathbf{B} &= \bigcup \{ \mathbf{B}_{m,n,p} : m+n+p=0 \}, \\ \mathbf{C} &= \bigcup \{ \mathbf{C}_{m,n,p} : m+n+p=0 \}.\end{aligned}$$

Definition 3. Let $\Psi : s^*(\mathbf{S}) \rightarrow \mathbb{R}^3$ be given as

$$\Psi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ z - \lfloor z \rfloor \end{bmatrix} & \text{if } (x, y, z) \in \mathbf{A} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ x - \lfloor x \rfloor \end{bmatrix} & \text{if } (x, y, z) \in \mathbf{B} \setminus (\mathbf{A} \cup \mathbf{C}) \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ y - \lfloor y \rfloor \end{bmatrix} & \text{if } (x, y, z) \in \mathbf{C} \setminus (\mathbf{A} \cup \mathbf{B}) \end{cases} \quad (1)$$

This map behaves as such:

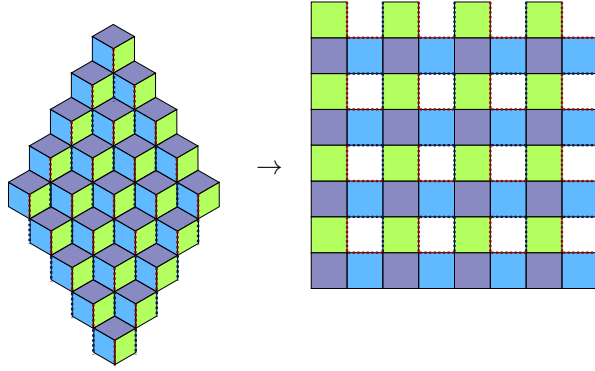


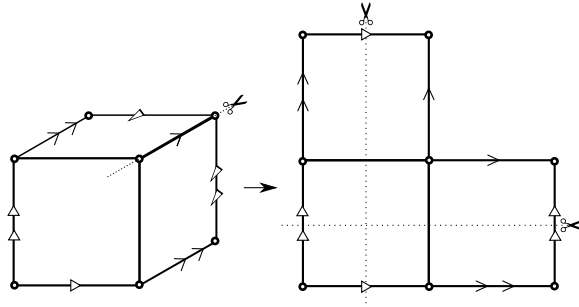
Figure 4: An isometry of the surface \mathbf{S} .

Lemma 1. $\Psi \circ s^*$ is an isometry of the surface \mathbf{S}

Proof. Ψ is well-defined on its domain $s^*(\mathbf{S})$, and s^* is an embedding of the surface that is a bijection when restricting its co-domain to its image. Since Ψ is a linear transformation composed of Euclidean matrices and translations it is also invertible and bijective. \square

2.2 Translation Surface Cover of \mathbf{G}°

Reconstruct \mathbf{G} and \mathbf{G}° to resemble a quotient of the flattened Necker cube surface by a series of cutting and gluing operations:



This L-shaped torus is then cut along the dotted lines and pieced together to look like a 2×2 torus with a unit square removed from the center:

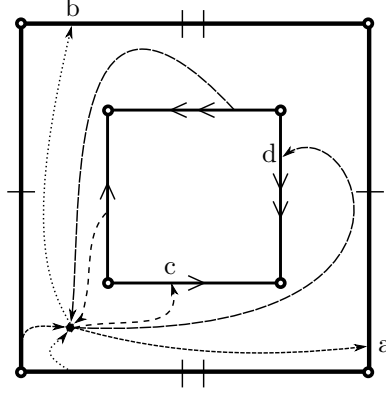


Figure 5: \mathbf{G}° with 4 independent, closed paths labeled identified

The paths labeled a, b, c, d are independent. Let $\pi_1(\mathbf{G}^\circ) = \langle a, b, c, d \rangle$. \mathbf{G}° has a \mathbb{Z}^2 tiling by traversing along paths a, b and attaching infinitely many copies of the surface. It's apparent that this \mathbb{Z}^2 tiling is the flattened Necker cube surface. We use group homomorphisms on this free group to construct translation surfaces and \mathbb{Z}^2 -covers to better understand geodesic behavior on the Necker cube surface.

Definition 4. Let $\varphi_1 : \pi_1(\mathbf{G}^\circ) \rightarrow \mathbb{Z}^2$ be the group homomorphism defined on generators a, b, c, d :

$$c, d \mapsto (0, 0)$$

$$a \mapsto (1, 0)$$

$$b \mapsto (0, 1)$$

$$ab, ba \mapsto (1, 1)$$

Definition 5. Let $\varphi_2 : \pi_1(\mathbf{G}^\circ) \rightarrow SO(2, \mathbb{Z})$ be the group homomorphism defined on the generators a, b, c, d :

$$\begin{aligned} a, b &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \\ c &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = R \\ d &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = R^3 \end{aligned}$$

Recall that a *translation surface* is a compact, locally flat surface that admits a discrete number of conical singularities and has trivial linear holonomy on any arbitrary path. If the total cone angle of the singularity is some integer multiple of 2π , $2\pi(d+1)$ for $d \geq 0$, we say that singularity has degree d . Two translation surfaces belong to the same stratum, or family, of surfaces characterized by the number and degree of conical singularities. Consider the effect that paths c, d have on holonomy:

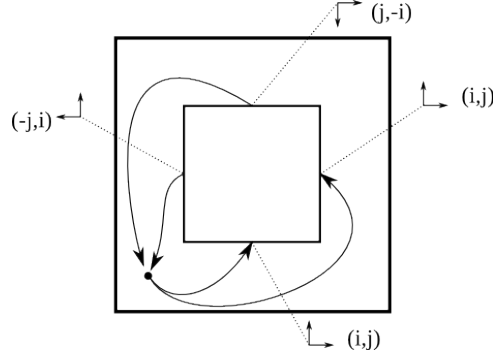


Figure 6: Effect a nontrivial loop has on arbitrary basis vectors (i,j) .

We construct a cover of \mathbf{G}° by taking four copies of it and cyclically pairing edges, and show that this cover has trivial holonomy. Denote the unit tangent bundle of \mathbf{G}° by $T^1\mathbf{G}^\circ$. Because \mathbf{G}° can be contained in the plane, we consider vectors tangent to \mathbf{G}° to be in \mathbb{R}^2 . Let $x_0 \in \mathbf{G}^\circ$. The fiber over x_0 under the projection $T^1\mathbf{G}^\circ \rightarrow \mathbf{G}^\circ$ is the set E_{x_0} . Let ∇ be a flat metric connection on $T^1\mathbf{G}^\circ$ and $p = (x_0, v_0) \in E_{x_0}$ be a single element in the fiber over x_0 with $v_0 \in \mathbb{R}^2$. We denote the *holonomy group* of the connection with base point p by $\text{Hol}_p(\nabla) = \{A \in \text{SO}(E_{x_0}) : p = A \cdot p\}$ where p differs from $A \cdot p$ by some rotation of v_0 . An element in $\text{SO}(E_{x_0})$ acts on p by rotation of the vector v_0 . Denote the *holonomy bundle* based at p by $H(p) = \{p' \in E_{x_0} : p' = A \cdot p\}$, the set of elements in the orbit of p where a closed loop $\gamma : [0, 1] \rightarrow \mathbf{G}^\circ$ defines a horizontal lift to $\gamma^* : [0, 1] \rightarrow T^1\mathbf{G}^\circ$ by parallel transport of v_0 along the path such that $p = \gamma^*(0)$, and $p' = \gamma^*(1)$. Since \mathbf{G}° is flat and trivial paths have trivial holonomy, $\text{Hol}_p(\nabla)$ acts on $H(p)$ the monodromy group is discrete and well-defined on \mathbf{G}° .

Lemma 2. $\varphi_2 : \pi_1(\mathbf{G}^\circ) \rightarrow \text{SO}(2, \mathbb{Z})$ is a monodromy representation of the surface.

Proof. It is clear from Figure 6 that this is the case as concatenating a closed loop with paths c or d would rotate a vector exactly by the cone angles of the singularities it loops around: $\pm \frac{3\pi}{2}$. \square

Definition 6. Let \mathbf{M}° be the regular cover of \mathbf{G}° with fundamental group $\pi_1(\mathbf{M}^\circ) = \ker \varphi_2$.

Showing that \mathbf{M}° is a degree four cover of \mathbf{G}° with trivial holonomy and deck group $\text{SO}(2, \mathbb{Z})$ follows immediately from its construction.

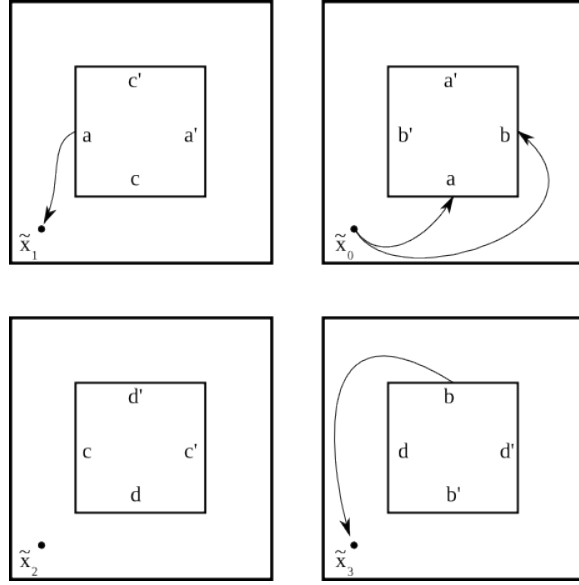
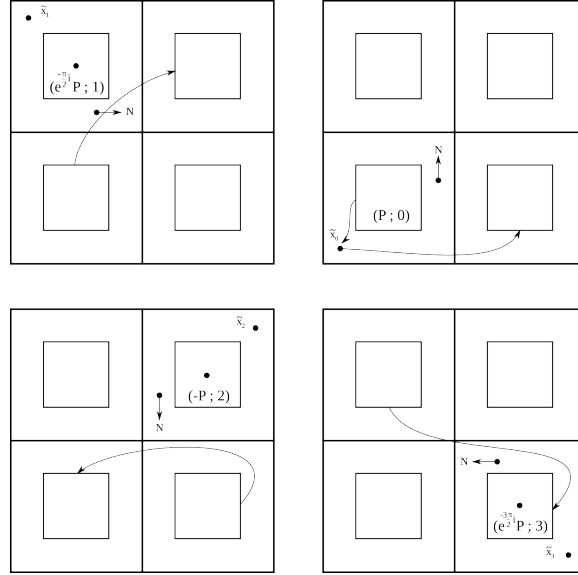


Figure 7: Lift of paths c, d to \mathbf{M}° (**RELABEL EDGES**)

Definition 7. *The surface \mathbf{U}° is the regular cover of \mathbf{G}° such that $\pi_1(\mathbf{U}^\circ) = \ker \varphi_1$.*

The deck group of this cover is isomorphic to \mathbb{Z}^2 . By reversing the cutting and gluing done previously to \mathbf{G}° and \mathbf{G} , one can easily see that a cover defined as the kernel of this homomorphism is identical to the tiling of Necker cubes along two of its edges. For this reason, $\mathbf{U} \cong \mathbf{S}$.

2.3 A Four-Fold Cover of U°



Definition 8. \mathbf{U}° is the cover of \mathbf{G}° such that $\pi_1(\mathbf{U}^\circ) = \ker \varphi$.

Our cover of \mathbf{G}° has the fundamental group, $\pi_1(\mathbf{U}^\circ) = \langle\langle a^{-1}b^{-1}ab, c, d \rangle\rangle$, the conjugate subgroup of elements that map trivially under φ_1 . The covering map $k_{\varphi_1} : \mathbf{U}^\circ \hookrightarrow \mathbf{G}^\circ$ takes every point on \mathbf{U}° to its modular equivalent under these translational symmetries. \mathbf{U}° is realized as an infinite \mathbb{Z}^2 -tiling of \mathbf{G}° in \mathbb{C} :

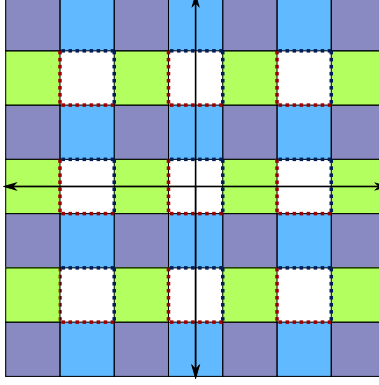


Figure 8: The infinite surface \mathbf{U}° embedded in \mathbb{C} with edges identified.

Definition 9. \mathbf{U} is the completion of \mathbf{U}° that includes the branched vertices.

Remark. The surface \mathbf{G} is homeomorphic to the torus, and its cover \mathbf{U} is its universal cover as the paths labeled c, d are only non-trivial when \mathbf{G} is branched at its three conical points.

2.4

2.5 Isometry From \mathbf{S} to \mathbf{U}

The Necker cube surface can be *flattened* onto the plane by piecewise isometric maps onto a subspace of \mathbb{R}^3 with (open) unit squares removed at every even integer pair in the plane, what we claim to be \mathbf{U} .

The red/blue dotted lines represent the edges that are split on the plane. The map from one surface to the other is composed of piecewise isometries, $\Psi : \mathbf{S} \rightarrow \mathbb{R}^3$.

The flattened surface is contained entirely in $(x, y, 0) \in \mathbb{R}^3$, which is isometric to \mathbb{C} . \mathbf{U} is recovered as a topological quotient on the domain

$$\mathbf{P} = \mathbb{C} \setminus \bigcup_{m,n \in \mathbb{Z}} \left\{ u + vi : u \in (2m - \frac{1}{2}, 2m + \frac{1}{2}), v \in (2n - \frac{1}{2}, 2n + \frac{1}{2}) \right\}. \quad (2)$$

Definition 10. \mathbf{U} is the surface obtained as the topological quotient $\mathbf{P}/\sim_{\mathbf{P}}$, where \mathbb{C} is identified with \mathbb{R}^2 in the usual way and $\sim_{\mathbf{P}}$ is a minimal relation on \mathbf{P} defined as follows:

$$\begin{aligned} \text{Let } x_0 = (u_0, v_0), x_1 = (u_1, v_1) \in \mathbf{P}. \quad \sim_{\mathbf{P}} \text{ is given as the relation} \\ x_0 \sim_{\mathbf{P}} x_1 \text{ iff } x_0 = x_1 \\ \text{or, for some } m, n \in \mathbb{Z} \quad x_0, x_1 \in \partial \left(\left[2m - \frac{1}{2}, 2m + \frac{1}{2} \right] \times \left[2n - \frac{1}{2}, 2n + \frac{1}{2} \right] \right) \\ \begin{bmatrix} u_1 - 2m \\ v_1 - 2n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_0 - 2m \\ v_0 - 2n \end{bmatrix}. \end{aligned}$$

Remark. Our map Ψ induces an isometry between \mathbf{S} and \mathbf{U} , and its restriction to the branched surfaces induces an isometry between \mathbf{S}° and \mathbf{U}° . This follows from these piecewise Euclidean transformations that preserves our induced flat metric.

From here on we use \mathbf{U}° and \mathbf{U} instead of \mathbf{S} and \mathbf{S}° .

2.6 Four-Fold Cover

We denote the *unit tangent bundle* on the surface \mathbf{U}° by $T^1\mathbf{U}^\circ$, and construct a cover with trivial holonomy. Initial experiments have shown us that any geodesic viewed as discontinuous line segments on \mathbf{P} moves in at most four directions.

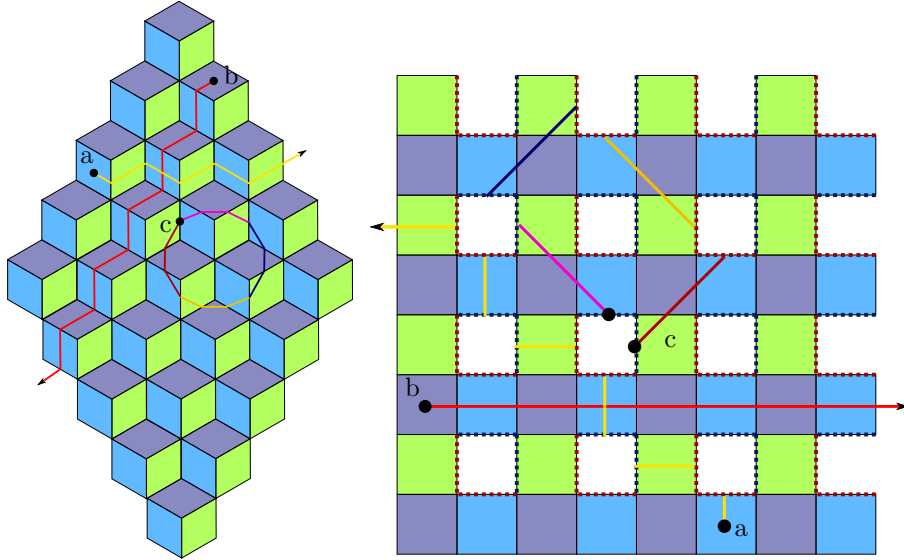


Figure 9: Simple periodic (c) and drift-periodic (a,b) trajectories represented as line segments on \mathbf{S} and \mathbf{U} . **RELABEL THESE**

Observe that the parallel transport of a vector around a closed loop on \mathbf{U}° will act on vectors tangent to the surface by a rotation of an integer multiple of $\frac{\pi}{2}$ radians (since the surface is flat and embedded in \mathbb{C} , we work with vectors in \mathbb{R}^2).

Definition 11. Define $\varphi_2 : \pi_1(\mathbf{G}^\circ) \rightarrow SO(2, \mathbb{Z})$ as the group homomorphism on generators a, b, c, d such that:

$$a, b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \quad c \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = R, \quad d \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = R^3$$

The homomorphism restricted to $\pi_1(\mathbf{U}^\circ)$ factors through $\pi_1(\mathbf{G}^\circ)$ and we show that:

Lemma 3. The group $\varphi_2(\pi_1(\mathbf{U}^\circ, x_0)) = Hol_p(\omega)$.

Proof. Since $c \mapsto R$ and $d \mapsto R^3$, the image of φ_2 is generated by R, R^3 and isomorphic to $SO(2, \mathbb{Z})$. We see from figure 9 that parallel transports of vectors by non-trivial paths produce clockwise/counter-clockwise rotations equal to that of the cone angles of the singularities they loop around, all integer multiples of $\frac{\pi}{2}$ that coincide with paths generated by elements c, d . Further, if there is an element in $\pi_1(\mathbf{U}^\circ, x_0)$ conjugated by $[a, b]$, the effect this would have on holonomy would be trivial as the total cone angle sum around each cut out square is an integer multiple of 2π , which agrees with a and b 's trivial images under ϕ_2 . \square

Hence, we obtain a natural monodromy representation with the map $m : \pi_1(\mathbf{U}^\circ, x_0) \rightarrow SO(2, \mathbb{Z}) \cong Hol_p(\omega)$, where for $[\gamma] \in \pi_1(\mathbf{U}^\circ, x_0)$ we have that $\gamma^*(1) = (x_0, m([\gamma])v_0)$. It follows that since ω is a flat connection, trivial loops have trivial holonomy and $Hol_p(\omega)$ acts on $H(p)$.

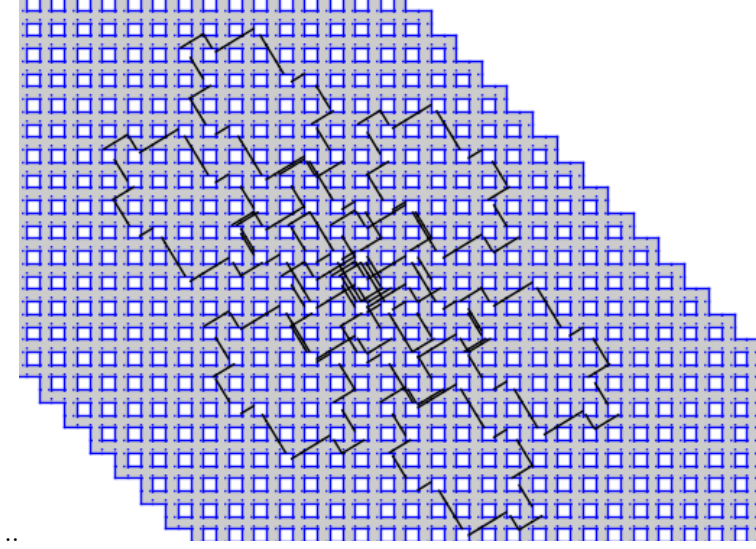


Figure 10: Geodesic flow on \mathbf{U} modeled using sage-flatsurf.

Definition 12. Let $\tilde{\mathbf{U}}_0^\circ$ be the cover of \mathbf{U}° with fundamental group $\pi_1(\tilde{\mathbf{U}}_0^\circ) = \ker \varphi_2|_{\pi_1(\mathbf{U}^\circ)} \trianglelefteq \pi_1(\mathbf{U}^\circ)$.

Lemma 4. Let $\Phi_t : \mathbf{U}^\circ \times \mathbb{R} \rightarrow \mathbf{U}^\circ$ be a unit-speed geodesic flow on \mathbf{U}° , with a parallel transport map induced by ω . Then the following is true:

- (i) Φ is periodic if and only if its lift to $\tilde{\mathbf{U}}_0^\circ$ is.
- (ii) Φ is drift-periodic if and only if its lift to $\tilde{\mathbf{U}}_0^\circ$ is.

Proof. Let $x_0 = \Phi(0)$ be an initial point in \mathbf{U}° , and let $v_0 = \frac{d}{dt}\Phi|_{t=0} \in \mathbb{R}^2$ be its initial direction. Then $\Phi^* : T^1\mathbf{U}^\circ \times \mathbb{R} \rightarrow T^1\mathbf{U}^\circ$ is a well-defined lift to the unit tangent bundle with initial point $p = (x_0, v_0)$.

(i). Suppose Φ is periodic with period $T \in \mathbb{R}$. Then Φ is reparameterized as $\alpha : [0, 1] \rightarrow \mathbf{U}^\circ$, where $[\alpha] \in \pi_1(\mathbf{U}^\circ, x_0)$. It follows then that if α does not lift to a closed path, then α must have non-trivial holonomy since $[\alpha] \notin \ker \varphi_2$. That is when lifted to the unit tangent bundle with base point p , $\alpha^*(1) \neq p$ since $m([\alpha]) \neq I_2$. But this implies that $\Phi^*(T) \neq p$, which is impossible. Hence $[\alpha] \in \ker \varphi_2 = \pi_1(\tilde{\mathbf{U}}_0^\circ, \tilde{x}_0)$, where \tilde{x}_0 belongs to the fiber over x_0 under the covering map $\tilde{\mathbf{U}}_0^\circ \hookrightarrow \mathbf{U}^\circ$. The converse holds trivially.

(ii). Suppose Φ is drift-periodic with period $T \in \mathbb{R}$ and non-trivial $f : \mathbf{U}^\circ \rightarrow \mathbf{U}^\circ \in \text{Trans}(\mathbf{U}^\circ) \cong \pi_1(\mathbf{G}^\circ)/\pi_1(\mathbf{U}^\circ) \cong \mathbb{Z}^2$ such that $\Phi(T) = f(x_0)$. Reparameterize this as $\alpha : [0, 1] \rightarrow \mathbf{U}^\circ$ with $[\alpha] \in \pi_1(\mathbf{G}^\circ)$. When lifted to $T^1\mathbf{U}^\circ$, $\Phi^*(0) = p$ and $\Phi^*(T) = (f(x_0), v_0)$. Let $\alpha : [0, 1] \rightarrow \mathbf{U}^\circ$ be its reparameterization up to time T . Thus $[\alpha] \notin \pi_1(\mathbf{U}^\circ, x_0), \pi_1(\tilde{\mathbf{U}}_0^\circ, \tilde{x}_0)$. Further, f has a unique lift to $\tilde{f} \in \text{Trans}(\tilde{\mathbf{U}}_0^\circ)$ because the space is connected. (a bit stuck here).

Conversely, ... □

A visual representation of $\tilde{\mathbf{U}}_0^\circ$ is as a four-fold cover of \mathbf{U}° with trivial holonomy on all closed paths. Arbitrary paths do *not* have trivial holonomy:

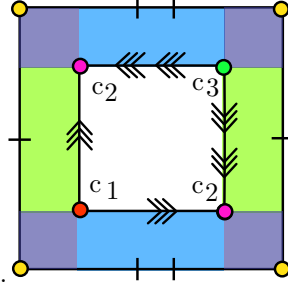


Figure 11: A 2×2 cut out section centered at each missing square. Edges and vertices identified.

2.7 Translation Surface

A better picture of $\tilde{\mathbf{U}}_0^\circ$ is obtained by making cyclic edge identifications on $\mathbf{P}' = \mathbf{P} \times \mathbb{Z}/4\mathbb{Z}$. This comes from $\tilde{\mathbf{U}}_0^\circ$ inheriting the topological properties of $\mathbf{U}^\circ \times \pi_1(\mathbf{U}^\circ)/\pi_1(\tilde{\mathbf{U}}_0^\circ) \cong \mathbf{U}^\circ \times SO(2, \mathbb{Z})$.

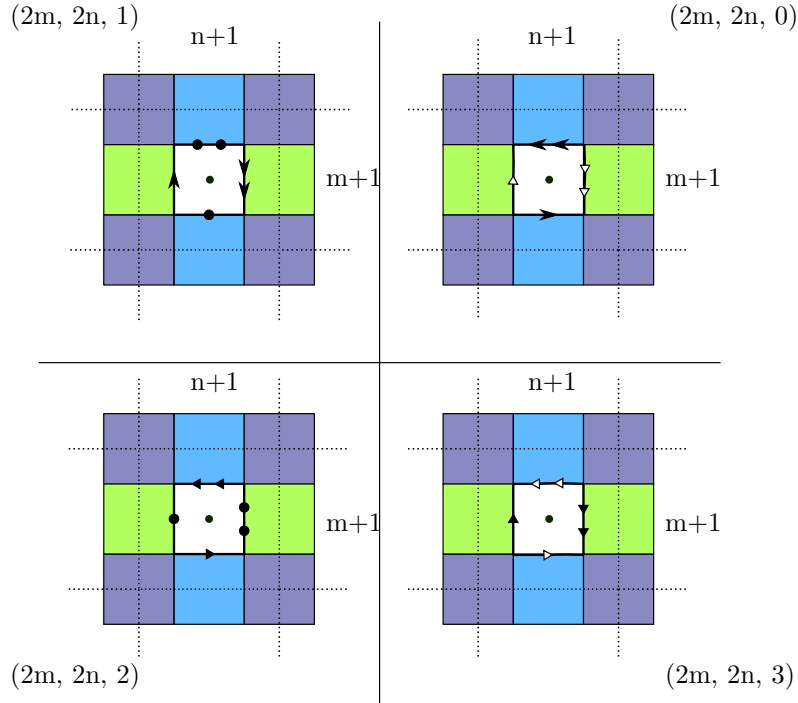


Figure 12: Branched cover of \mathbf{U} of degree four.

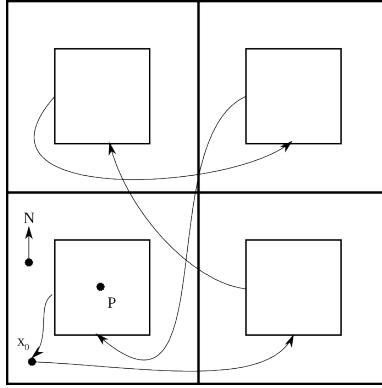
Definition 13. $\tilde{\mathbf{U}}_0$ is the surface obtained as the quotient $\mathbf{P}' / \sim_{\mathbf{P}'}$. Denote paths ζ and η on \mathbf{P} , parameterized by integers $m, n \in \mathbb{Z}$ and $t \in [-\frac{1}{4}, \frac{1}{4}]$, and defined:

$$\begin{aligned}\zeta(t) &= (2m + t) + i(2n - \tfrac{1}{2}), \\ \eta(t) &= (2m + t) + i(2n + \tfrac{1}{2}).\end{aligned}$$

Let $\aleph = 2m + i2n \in \mathbb{C}$. The minimal relation $\sim_{\mathbf{P}'}$ is given as:

$$\begin{aligned}(\zeta(t); j) &\sim_{\mathbf{P}'} (e^{i\frac{\pi}{2}} \overline{\zeta(t) - \aleph}; j + 1) \\ (\eta(t); j) &\sim_{\mathbf{P}'} (e^{i\frac{\pi}{2}} \overline{\eta(t) - \aleph}; j + 1)\end{aligned}\tag{3}$$

This relation is similar to $\sim_{\mathbf{P}}$ in that it translates line segments of squares surrounding even integer pairs to the origin and relates points on one edge of a square to points on an adjacent edge. Where it differs is that these adjacent edges now belong to a different “copy” of \mathbf{P} . It is in this cyclic manner that edges are glued that allows for trivial linear holonomy on arbitrary paths by rotating each copy of \mathbf{P} accordingly. For example, here is a path on a section of the surface in the neighborhood of $P = 2m + i2n \in \mathbb{C}$ ($m, n \in \mathbb{Z}$) after rotation: and its subsequent projection onto \mathbf{U}° :



Definition 14. Let $r : \mathbf{P}' \rightarrow \mathbf{P}'$ be the isometric map of $\tilde{\mathbf{U}}_0$'s domain given $r(z; \bar{j}) = (e^{(-j)i\frac{\pi}{2}} z; \bar{j})$.

Observe that when r acts on the relations (3) we get the following:

$$\begin{aligned}(e^{(-j)i\frac{\pi}{2}} \zeta(t); j) &\sim_{r(\mathbf{P}')} (e^{(-j)i\frac{\pi}{2}} \zeta(t) - \aleph; j + 1) \\ (e^{(-j)i\frac{\pi}{2}} \eta(t); j) &\sim_{r(\mathbf{P}')} (e^{(-j)i\frac{\pi}{2}} \eta(t) - \aleph; j + 1)\end{aligned}\tag{4}$$

$\tilde{\mathbf{U}}$ is then recovered as $r \cdot (\tilde{\mathbf{U}}_0) = \mathbf{P}' / \sim_{r(\mathbf{P}')}$, where all identifications are translations made on copies of \mathbf{P} .

2.8 Veech Staircase Model

Let $f : \tilde{\mathbf{U}} \rightarrow \mathbf{M}$ be a projection onto the quotient surface that takes every point to its modular equivalent by its \mathbb{Z}^2 symmetries, and what we obtain is the following compact translation surface belonging to the stratum $\mathcal{H}(2, 2, 2, 2)$:

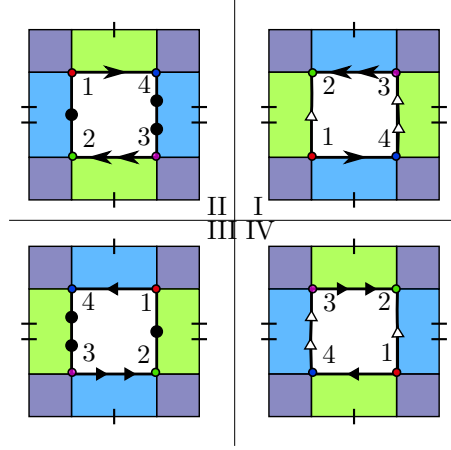


Figure 13: Compact translation surface, \mathbf{M} , covered by the infinite surface with edges and cone singularities (1,2,3,4) identified. The Roman numerals are meant to identify each quotient with a plane in the cover, $\tilde{\mathbf{U}}$.

We rearrange \mathbf{M} as such:

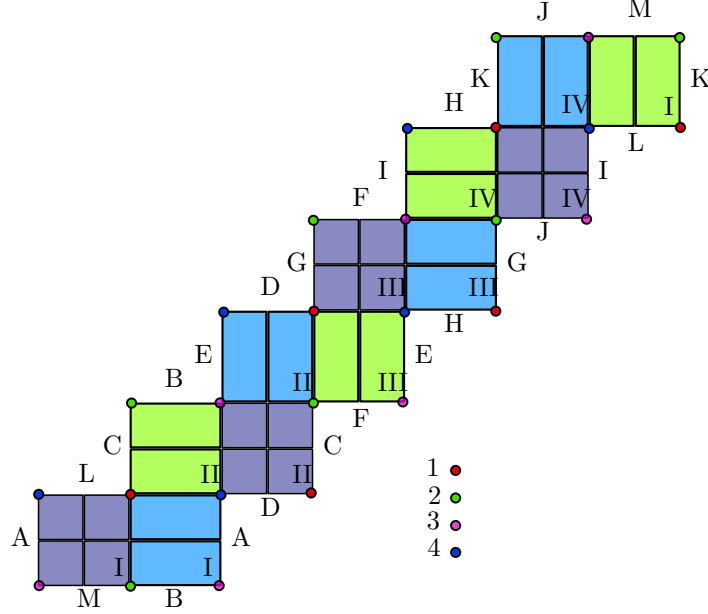


Figure 14: The staircase Veech surface with directional planes and vertices identified. All edges are paired by translation. Two adjacent squares have opposite edges identified. The top edge of the bottom-left square is glued to the bottom edge of the top-right square (both labeled I). Likewise, the bottom edge of the bottom-left square is identified with the top edge of the top-right square.

We recall some basic definitions and theorems about \mathbb{Z}^2 -covers of translation surfaces as they apply to \mathbf{M} .

Definition 15. *Algebraic intersection number is a non-degenerate bilinear form:*

$$i : H_1(\mathbf{M}, \mathbb{Q}) \times H_1(\mathbf{M}, \mathbb{Q}) \rightarrow \mathbb{Q},$$

for $[\gamma], [\beta] \in H_1(\mathbf{M}, \mathbb{Q})$, $i([\beta], [\gamma])$ returns the signed intersection number of two homology classes. We say a crossing at the instance of an intersection is positive if γ makes a positive angle relative to β .

The set of all affine diffeomorphisms of \mathbf{M} form the group $\text{Aff}^+(\mathbf{M})$. The Veech group of \mathbf{M} is the image in the co-domain of the homomorphism $D : \text{Aff}^+(\mathbf{M}) \rightarrow SL(2, \mathbb{R})$ that takes every affine map to its derivative, which we denote $V(\mathbf{M})$. \mathbf{M} is a square-tiled translation surface whose Veech group is a finite index subgroup of $SL(2, \mathbb{Z})$. We use automorphisms of the fundamental group in $\text{Aut}(\pi_1(\mathbf{M}))$ induced by affine maps to prove our main results. But first, we define a group homomorphism from $\pi_1(\mathbf{M})$ to $\pi_1(\mathbf{M})/\pi_1(\tilde{\mathbf{U}})$ by algebraic intersection numbers over a sum of linearly independent homology classes. Consider the following cylinder core curves on \mathbf{M} labeled γ_i for $0 \leq i \leq 11$:

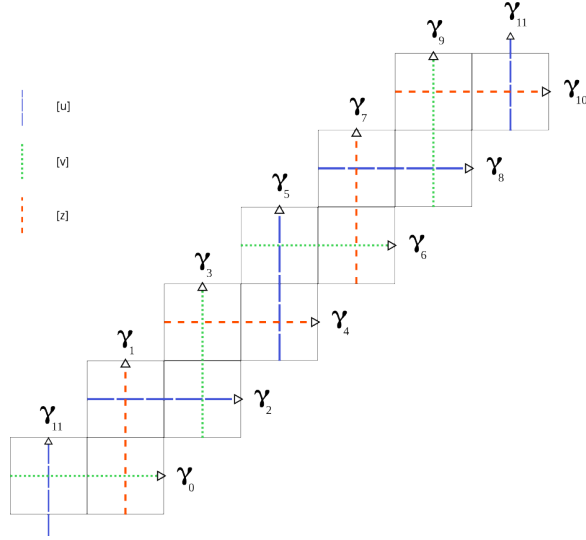


Figure 15: \mathbf{M} 's cylinder core curves with u, v , and z homology class labels.

Every curve intersects two others and if we take $i(\Sigma_{i=0}^{11}[\gamma_i])$ **show that**
 $\gamma_0, \dots, \gamma_{11}$ **form a basis for** $H(\mathbf{M}, \mathbb{Q})$
show that u, v **linearly independent**

Definition 16. *The homology classes u, v, z are given as the following sums of core curves:*

$$\begin{aligned} u &= -[\gamma_2] + [\gamma_5] + [\gamma_8] - [\gamma_{11}], \\ v &= +[\gamma_0] - [\gamma_3] - [\gamma_6] + [\gamma_9], \\ z &= +[\gamma_1] + [\gamma_4] - [\gamma_7] - [\gamma_{10}]. \end{aligned}$$

Definition 17. *Define the group homomorphism $\Omega_{u,v} : \pi_1(\mathbf{M}) \rightarrow \mathbb{Q}^2$, where $\beta \mapsto (i(u, [\beta]), i(v, [\beta]))$.*

Lemma 5. $\Omega_{u,v}(\pi_1(\mathbf{M})) = \mathbb{Z}^2$.

Lemma 6. **show somehow that u, v determine the cover**

2.9 Induced Automorphisms of $H_1(\mathbf{M}, \mathbb{Q})$

We look at some important automorphisms induced by Affine maps on \mathbf{M} . Observe that \mathbf{M} has a uniform cylinder decomposition in both horizontal and vertical directions as in figure 15. We define the *modulus*, μ , of a cylinder to be the ratio of the cylinder's width to its circumference, $\frac{w}{c}$. The *Dehn-twist* of a cylinder is an affine diffeomorphism that skews the cylinder and sends every vertex to itself.

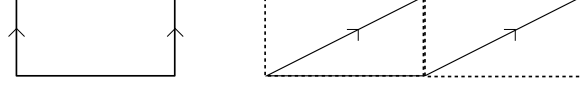


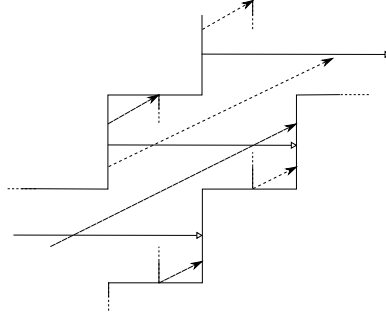
Figure 16: Dehn-twist of a cylinder in \mathbf{M} 's cylinder decomposition.

Such a map would have derivative $\begin{bmatrix} 1 & \pm\mu^{-1} \\ 0 & 1 \end{bmatrix}$. On \mathbf{M} every cylinder in both vertical and horizontal decompositions has a modulus of $\frac{1}{2}$. These give way to global diffeomorphisms as *multi-twists* of \mathbf{M} .

Definition 18. We call the global affine diffeomorphisms in $\text{Aff}^+(\mathbf{M})$ obtained as *multi-twists of the surface in horizontal and vertical directions* \mathbf{A} and \mathbf{B} , respectively.

We define the *derivatives of \mathbf{A}, \mathbf{B}* as the matrices $D(\mathbf{A}) = \mathbf{A}' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and $D(\mathbf{B}) = \mathbf{B}' = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, where $D : \text{Aff}^+(\mathbf{M}) \rightarrow V(\mathbf{M})$.

These are parabolic elements of $V(\mathbf{M})$ that are directly related to the dynamical dichotomy of rational trajectories on the Necker cube surface. To show this algebraically, we look at how skewing the surface affects our homology spanning core curves:



When skewing the surface under \mathbf{A} in the horizontal direction, the horizontal curves are preserved, but the vertical curves (odd γ_i index) obtain two additional positive intersections with adjacent index (mod 12) horizontal curves. Similarly, the vertical \mathbf{B} skews preserve the vertical curves, but the horizontal curves (even γ_i index) obtain two additional *negative* intersections with adjacent index (mod 12) vertical curves. The formulaic expressions of the \mathbf{A}, \mathbf{B} induced automorphisms of $H_1(\mathbf{M}, \mathbb{Q})$ as powers of $k \in \mathbb{Z}$ are then:

$$\begin{aligned}\mathbf{A}_*^k([\gamma_i]) &= [\gamma_i] + \frac{k}{2}(1 - (-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}]) \\ \mathbf{B}_*^k([\gamma_i]) &= [\gamma_i] + \frac{k}{2}(1 + (-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}])\end{aligned}$$

With all of these in mind we generate the following subgroups:

Definition 19. We say \mathbb{X} is the subgroup of $\text{Aff}^+(\mathbf{M})$ generated by \mathbf{A}, \mathbf{B} .

Definition 20. The images of \mathbb{X} under their derivative maps to $V(\mathbf{M})$ is the subgroup of $V(\mathbf{M})$ generated by matrices \mathbf{A}', \mathbf{B}' . We denote this group by \mathbb{X}' .

Definition 21. The induced automorphisms of \mathbb{X} contained in $\text{Aut}(H_1(\mathbf{M}, \mathbb{Q}))$ is the subgroup generated by $\mathbf{A}_*, \mathbf{B}_*$.

3 Proof of Main Theorem

3.1 Translation Surfaces and \mathbb{Z}^2 -Covers

Definition 22. Algebraic intersection number is a non-degenerate bilinear form:

$$i : H_1(S, \mathbf{R}) \times H_1(S, \mathbf{R}) \rightarrow \mathbf{R},$$

where \mathbf{R} is a ring and for $[\gamma], [\beta] \in H_1(S, \mathbf{R})$, $i([\beta], [\gamma])$ returns the intersection number of two homology classes.

Algebraic intersections are signed and follow some convention such as the right-hand rule.

Definition 23. Let $u, v \in H_1(S, \mathbb{Q})$ be linearly independent homology classes of curves on S . Then the group homomorphism from $\pi_1(S)$ to \mathbb{Q}^2 is given as:

$$\Omega_{u,v} : \pi_1(S) \rightarrow \mathbf{R}^2; \beta \mapsto (i(u, [\beta]), i(v, [\beta])).$$

The set of all orientation-preserving affine diffeomorphisms of S forms the group $\text{Aff}^+(S)$. The corresponding *Veech group*, $V(S)$ of S is the image of the group morphism $D : \text{Aff}^+(S) \rightarrow SL(2, \mathbb{R})$ that takes an affine map to its derivative. A surface is said to be *Veech* if its Veech group is commensurable to $SL(2, \mathbb{R})$. It is well known that origami, or square-tiled, surfaces have Veech groups commensurable to $SL(2, \mathbb{Z})$. [cite] When $\mathbf{R} = \mathbb{Z}$, $\Omega_{u,v}$ takes an element of $\pi_1(S)$ to Δ . Thus, $\gamma \in \pi_1(S)$ lifts to $\tilde{\gamma} \in \pi_1(\tilde{S})$ if and only if $\gamma \in \ker \Omega_{u,v}$.

Definition 24. Let $\alpha : [0, 1] \rightarrow S$ be a closed, non-singular geodesic path on S . The holonomy map $\mathbf{hol} : H_1(S, \mathbf{R}) \rightarrow \mathbb{C}$ returns the holonomy vector of a closed path as a difference of the starting and endpoints of a flow by

$$\mathbf{hol}([\alpha]) = \int_{\alpha} dz.$$

Since α is non-singular, it can be mapped to S° which admits a flat holomorphic one-form dz . Let $\theta = \text{Arg}(\mathbf{hol}([\alpha]))$. We say that $\phi_t^\theta : \mathbb{R} \times S^\circ \rightarrow S^\circ$ is the unit-speed geodesic flow on S° in direction θ given by the $[\alpha]$ such that $\phi_0^\theta = \alpha(0)$. In local coordinates this corresponds to $z + te^{i\theta} \in \mathbb{C}$.

Lemma 7. ϕ_t^θ has a period of $T = |\mathbf{hol}([\alpha])|$

Proof. This just follows from the fact that ϕ_t^θ flows at unit-speed in the direction of $\frac{\mathbf{hol}([\alpha])}{|\mathbf{hol}([\alpha])|}$. \square

And so the length of a vector determines the period of a flow on S . More importantly, we have the following:

Lemma 8. Denote the lifted flow of ϕ_t^θ on \tilde{S} by $\tilde{\phi}_t^\theta$. Then $\tilde{\phi}_t^\theta$ is periodic on \tilde{S} and \tilde{S}° if and only if $[\alpha] \in \ker \Omega_{u,v}$. Further, $\tilde{\phi}_t^\theta$ has period $T = |\mathbf{hol}([\alpha])|$.

Proof. Suppose that $[\alpha] \notin \ker \Omega_{u,v}$. Since $\alpha([0, 1]) = \phi_{[0, \mathbf{hol}([\alpha])]}^\theta$, their homology classes are equivalent. Hence $\tilde{\phi}_t^\theta$ could not close on \tilde{S} or \tilde{S}° , or else $\Omega_{u,v}(k[\alpha]) = k(i(u, [\alpha]), i(v, [\alpha])) = (0, 0)$ for some non-zero $k \in \mathbb{Z}$. Conversely, suppose $[\alpha] \in \ker \Omega_{u,v}$. Then \square

Corollary 1. If $[\alpha] \notin \ker \Omega_{u,v}$, then $\tilde{\phi}_t^\theta$ is drift-periodic with period $T = |\mathbf{hol}([\alpha])|$.

Proof. This follows immediately from the previous lemma since $\tilde{S}, \tilde{S}^\circ$ have translational \mathbb{Z}^2 symmetries and a well-defined \mathbb{Z}^2 action on an element in the fiber of a basepoint in S under f . The period is T since a geodesic closes on S with period T . \square

Lemma 9. Let $h \in \text{Aff}^+(S)$. If $h(\beta) = \alpha$ for closed geodesics α, β on S and β lifts to a closed path on \tilde{S} , then so does α . (not sure about this)

Proof. If the premise is true, then $[\beta] \in \ker \Omega_{u,v}$. Denote the group automorphism induced by h as h_* . Then $\Omega_{u,v}([\alpha]) = \Omega_{u,v}([h(\beta)]) = \Omega_{u,v}(h_* \cdot [\beta]) = (i([\beta], h_*^{-1} \cdot u), i([\beta], h_*^{-1} \cdot v)) = (0, 0)$ as automorphisms. \square

Lemma 10. Let $\beta = h(\alpha)$ as before. If $D(h) = h' \in V(S)$ and S is Veech, then $\tilde{\phi}_t^\theta$ has period $T = h' \cdot |\mathbf{hol}([\beta])|$. (also not sure)

Proof. \square

3.2 Symmetries of M

4 Four-fold Cover of U

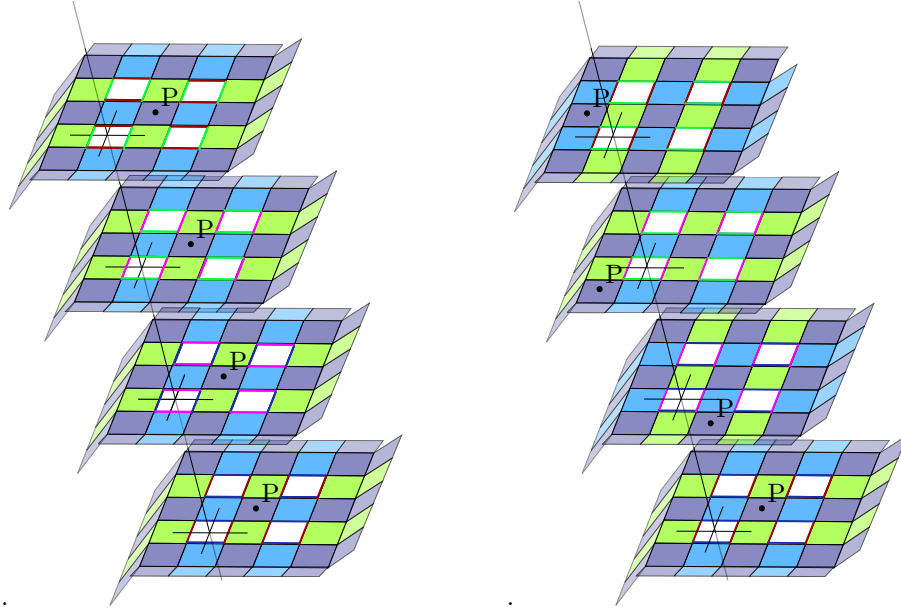


Figure 17: Four-fold cover isometry and the preimage of a point in $U \setminus \text{Sing}(U)$.

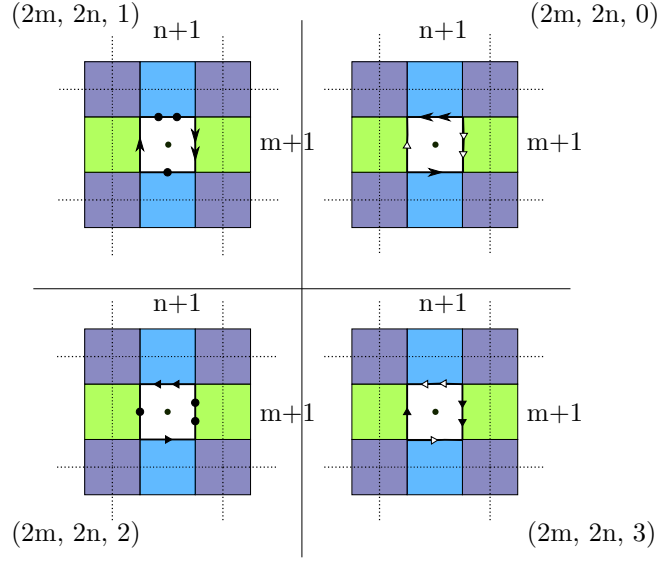


Figure 18: Branched cover associating every direction with one plane.

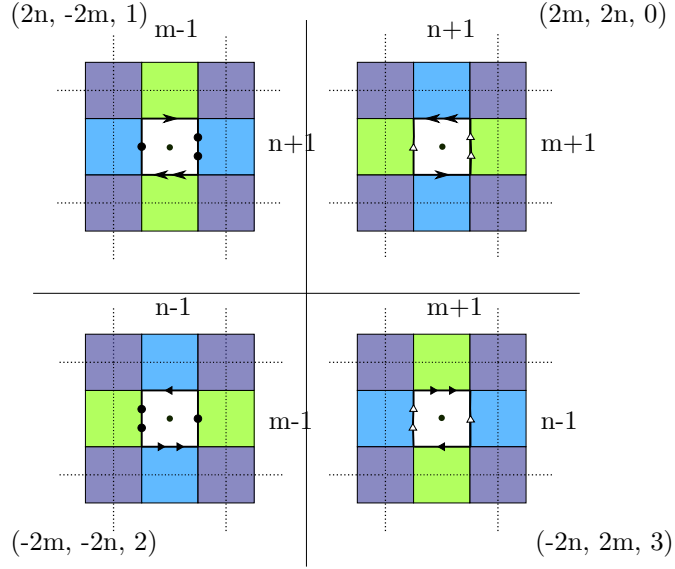


Figure 19: Infinite-type translation surface obtained by rotating each copy of the fundamental domain accordingly.

The quotient under the group action of translational symmetries is isomorphic to \mathbb{Z}^2 since the orbit of any point in the fundamental domain is a lattice in the space.

Theorem 1. *The translational symmetries of $\tilde{\mathbf{U}}$'s fundamental domain induce symmetries on the surface isomorphic to \mathbb{Z}^2 .*

Proof. Let $(z; j) \in \mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$ and define the group action $T_0^{m,n} : \mathbb{C} \times \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$ as $T_0^{m,n}(z; j) = (z + 2e^{-j\frac{i\pi}{2}}(m + in); j)$. This translation acts faithfully on the preimages of $\mathbf{U} \setminus \text{Sing}(\mathbf{U})$, and respects edge identifications of $\tilde{\mathbf{U}}$, thereby making it an isometry of the surface. Consider a group homomorphism, $T_0^{m,n} \mapsto m + in$ onto the plane of Gaussian integers, $\mathbb{Z}[i]$. The exponential function is never zero, so the identity of the translation group is $T_0^{0,0}$. This is an isomorphism since it is clearly surjective and any non-trivial element of $T_0^{m,n}$ could not possibly map to the identity element of $\mathbb{Z}[i]$, regardless of the value of j . Since \mathbb{Z}^2 is isomorphic to $\mathbb{Z}[i]$, it is isomorphic to $T_0^{m,n}$ as well. \square

Definition 25. *The automorphism $T_0^{m,n} : \tilde{\mathbf{U}} \rightarrow \tilde{\mathbf{U}}$ is an **induced translation** of $\tilde{\mathbf{U}}$ as a result of the previous theorem.*

This surface is obtained as a ramified cover of the unit square torus. It is a translation surface and is therefore equipped with a **holomorphic one-form**, a collection of charts from neighborhoods of \mathbf{M} to \mathbb{C} such that any neighborhood away from $\text{Sing}(\mathbf{M})$ has a *flat* induced Euclidean metric. A theorem of Gutkin and Judge tells us that its Veech group is commensurable to $\text{SL}(2, \mathbb{Z})$ and is therefore a Veech surface. We look at some of its affine maps, and generate a subgroup $\mathbb{X} \subset \text{Aff}^+(\mathbf{M})$ by the following transformations:

- (i) Multi-twists of the surface as global diffeomorphisms given by Dehn-twists of its cylinder decomposition in horizontal and vertical directions with derivatives:

$$\left\{ \begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix} \right\}.$$

we call $\mathbf{A}^{\pm 1}$, $\mathbf{B}^{\pm 1}$, respectively.

A Dehn-twist on each cylinder in the cylinder decomposition of \mathbf{M} in horizontal and vertical directions gives way to these global affine diffeomorphisms:

- (ii) Rotation group generated by a $+\frac{\pi}{2}$ rotation of the surface fixed about the center of the second square on the bottom of the staircase, an order four isometry on \mathbf{M} denoted \mathbf{R} .
- (iii) Order 2 translation of the surface that moves the bottom left-most square to the square right next to it, denoted \mathbf{H} .
- (iv) Order 2 translation of the surface that takes the bottom right-most square to the one right above it, denoted \mathbf{V} .

Definition 26. *The group \mathbb{X} is the isometry group generated by affine maps $\mathbf{A}, \mathbf{B}, \mathbf{R}, \mathbf{H}$, and \mathbf{V} . The image of the derivative map on elements in \mathbb{X} is*

denoted \mathbb{X}' and generated by matrices

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

denoted \mathbf{A}' , \mathbf{B}' , and \mathbf{R}' in that order.

It is not immediately apparent if these affine maps generate $\text{Aff}^+(\mathbf{M})$, or if their derivatives generate $V(\mathbf{M})$, its Veech group. We use these to induce homomorphisms on $H_1(X, \mathbb{Q})$. A spanning set of $H_1(\mathbf{M}, \mathbb{Q})$ is obtained as the set of homology classes of the core curves of X 's cylinder decompositions in both vertical and horizontal directions:

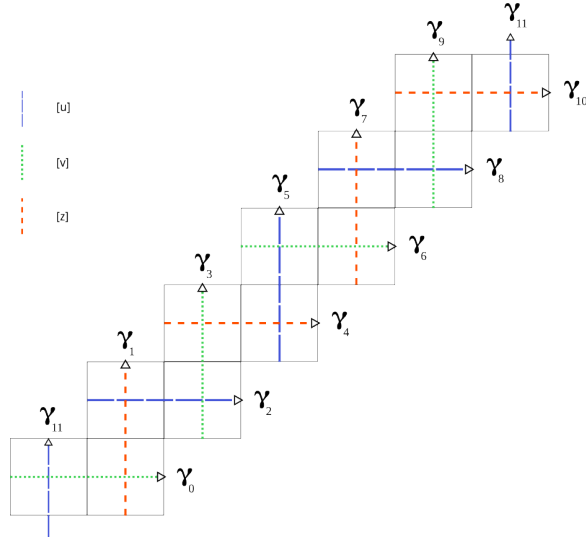


Figure 20: Cylinder core curves with u, v , and z homology classes that determines the \mathbb{Z}^2 -cover.

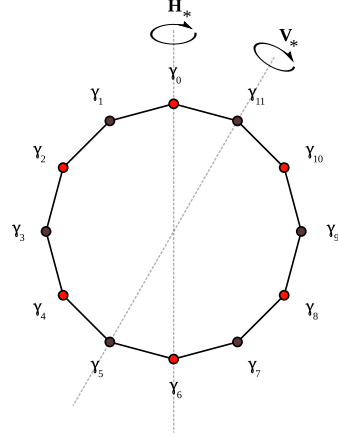
Definition 27. The set of abelianized cylinder core curves is denoted as $\Gamma = \{\gamma_i : i = 0, \dots, 11\} \subset H_1(\mathbf{M}, \mathbb{Q})$.

Remark. We use 12 elements to span homology, although a basis requires only 10. It's not impossible to determine the relations between these core curve classes, but it is not necessary. A 12×12 matrix of these core curve cylinder decompositions to their intersection numbers with adjacent curves is rank 10, as to be expected.

The induced homomorphisms of $H_1(\mathbf{M}, \mathbb{Q})$ have come from affine maps that have various effects on these core-curves. We use a 12-gon to represent the set of curves, and show how these elements act on them. The multi-twists add curves to adjacent curves, and the translation maps permute them. The reader is encouraged to check these for themselves.

e.g. for $\mathbf{H}, \mathbf{V} \in \text{Aff}^+(\mathbf{M})$,

H & V The effect that these two translations have on the 12-gon is a reflection about these lines. Observed by keeping track of the squares and core curves after \mathbf{H} and \mathbf{V} have acted on \mathbf{X} .



Definition 28. The induced homomorphisms of $H_1(\mathbf{M}, \mathbb{Q})$ are obtained from the affine subgroup \mathbb{X} and denoted \mathbb{X}_* . The associated homomorphisms on the spanning set Γ are given as:

$$\begin{aligned} \mathbf{A}_*^k \circ [\gamma_i] &= [\gamma_i] + \frac{k}{2}(1 - (-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}]) \\ \mathbf{B}_*^k \circ [\gamma_i] &= [\gamma_i] + \frac{k}{2}(1 + (-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}]) \\ \mathbf{R}_* \circ [\gamma_i] &= (-1)^i [\gamma_{1-i \bmod 12}] \\ \mathbf{H}_* \circ [\gamma_i] &= [\gamma_{12-i \bmod 12}] \\ \mathbf{V}_* \circ [\gamma_i] &= [\gamma_{10-i \bmod 12}] \end{aligned}$$

Definition 29. The homology classes u, v, z are given as the following sums of core curves:

$$\begin{aligned} [u] &= -[\gamma_2] + [\gamma_5] + [\gamma_8] - [\gamma_{11}], \\ [v] &= +[\gamma_0] - [\gamma_3] - [\gamma_6] + [\gamma_9], \\ [z] &= +[\gamma_1] + [\gamma_4] - [\gamma_7] - [\gamma_{10}]. \end{aligned}$$

Theorem 2. The fundamental group of the \mathbb{Z}^2 -cover is obtained by lifting the kernel of the closed paths of \mathbf{M} of the homomorphism:

$$\Omega_{u,v} : \pi_1(\mathbf{M}, x_0) \rightarrow \mathbb{Z}^2; \beta \mapsto (i(u, [\beta]), i(v, [\beta])), \text{ where}$$

$$i : H_1(\mathbf{M}, \mathbb{Q}) \times H_1(\mathbf{M}, \mathbb{Q}) \rightarrow \mathbb{Z}.$$

is the intersection number of two homology classes.

Proof. We know from Theorem 1 that the translational symmetries of $\tilde{\mathbf{U}}$ induced by $T^{m,n}$ is isometric to \mathbb{Z}^2 . Since \mathbf{M} is a genus 5 base surface, we know that $\pi_1(\mathbf{M}, x_0) \simeq \mathbb{Z}^{10}$, and the associated cover satisfies $\mathbf{M} = \tilde{\mathbf{U}}/(\pi_1(\mathbf{M}, x_0)/N)$, such that N is a normal subgroup of $\pi_1(\mathbf{M}, x_0)$. This means that $N \simeq \mathbb{Z}^8$. The eight core curve classes are the abelianized forms of $\gamma_0, \gamma_2, \gamma_3, \gamma_5, \gamma_6, \gamma_8, \gamma_9$, and γ_{11} that span N . The classes and their signs are obtained from Figure 13 as the outer regions identified by the translations of $T^{m,n}$. Thus any closed path on \mathbf{M} is lifted to a closed path on the cover under the quotient map only when a path has a trivial intersection number with the classes. \square

Two paths are homologous if they return the same intersection number with the classes of closed core cylinder curves of \mathbf{U} that span its homology. The classes u and v are obtained from the group group action of $T^{m,n}$ on the cover.

Definition 30. $\mathbf{hol} : \mathbf{M} \setminus \text{Sing}(\mathbf{M}) \rightarrow \mathbb{C}$ is the holonomy vector pulled back from a non-singular path γ in \mathbf{M} onto the complex plane given by $\mathbf{hol}(\gamma) = \int_{\gamma} dz$.

We denote the **closed path** α , such that $\mathbf{hol}(\alpha) = 6 + 6i$, and show it is homologous to the closed geodesic with the same holonomy vector. The slope one direction also decomposes \mathbf{M} into two cylinders by a series of saddle connections of length $\sqrt{2}$ between singularities:

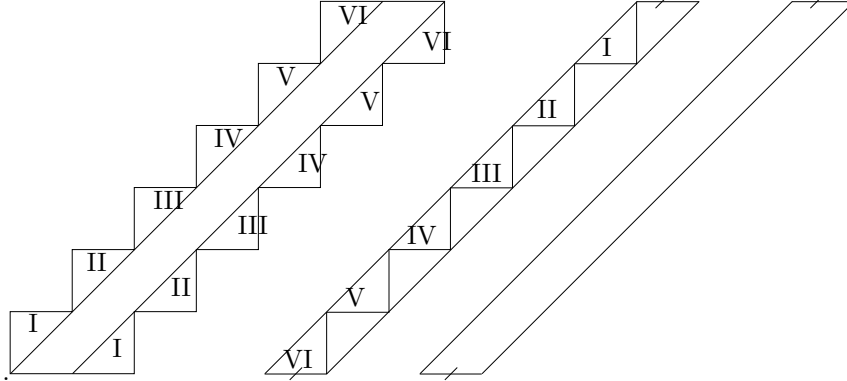


Figure 21: The two right-most cylinders C_1 (labeled) and C_2 (unlabeled).

The circumferences of these two cylinders are $6\sqrt{2}$. Geodesic flows on this surface are well defined, and rational directions

Definition 31. Let $\omega_t^\theta : [0, 1] \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbf{M}$ be the **maximal geodesic flow** on the surface in direction θ such that $\omega_0^\theta = \omega_1^\theta = x_0 \in \mathbf{M} \setminus \text{Sing}(\mathbf{M})$.

$\omega_t^{\frac{\pi}{4}}$ is the geodesic flow in the **slope one direction**, and χ is its image in \mathbf{M} and element of $\pi_1(\mathbf{M}, x_0)$.

Lemma 11. α is homologous to χ in $\mathbf{M} \setminus \text{Sing}(\mathbf{M})$.

Proof. Let χ be a geodesic contained in either C_1 or C_2 . Since a geodesic does not admit singularities, it is the image of a closed path on $X \setminus \text{Sing}(X)$ with initial point x_0 on the strips of C_1 and C_2 with boundaries removed, denoted C'_1, C'_2 . Express $[\alpha]$ as $\sum_{j=0}^{11} \frac{1}{2} \gamma_j$ (a closed path climbing up the staircase). We show that the intersection numbers of $[\alpha]$ and $[\chi]$ are the same for every core cylinder curve γ , i.e. $i([\gamma_k], \sum_{j=0}^{11} \frac{1}{2} [\gamma_j]) = i([\gamma_k], [\chi]) \forall k = 0, \dots, 11$.

Case one: k is even. If k is even, then every curve γ_k is oriented to the right. Since χ intersects every curve once, $i([\gamma_k], [\chi]) = 1$. No even indexed curves intersect each other, so we need only consider when j is odd. Now if j is odd, it is incident (positively crossing) with only two horizontal curves, namely $\gamma_{j+1}, \gamma_{j-1}$. Therefore $i([\gamma_k], [\alpha]) = i([\gamma_{j-1}], \frac{1}{2} [\gamma_j]) + i([\gamma_{j+1}], \frac{1}{2} [\gamma_j]) = \frac{1}{2}(i([\gamma_{j-1}], [\gamma_j]) + i([\gamma_{j+1}], [\gamma_j])) = \frac{1}{2}(1 + 1) = 1$.

Case two: k is odd. If k is odd, then $[\chi]$ will have an intersection number of -1 with $[\gamma_k]$ since odd-indexed core curves are oriented upwards. Now since k is odd, we only consider when j is even. Similarly, this means that γ_j negatively intersects the two vertical core curves with adjacent indices. Hence, $i([\gamma_k], [\alpha]) = i([\gamma_{j-1}], \frac{1}{2} [\gamma_j]) + i([\gamma_{j+1}], \frac{1}{2} [\gamma_j]) = \frac{1}{2}(i([\gamma_{j-1}], [\gamma_j]) + i([\gamma_{j+1}], [\gamma_j])) = \frac{1}{2}(-1 - 1) = -1$.

We know intersection number to be bilinear and non-degenerate on homology. So if α and χ 's abelianizations admit the same intersection numbers for every curve in the spanning set of $H_1(\mathbf{M}, \mathbb{Q})$, then $[\alpha] = [\chi]$. \square

Theorem 3. $\chi \in \pi_1(\mathbf{M}, x_0)$ lifts to $\tilde{\chi} \in \pi_1(\tilde{\mathbf{U}}, \tilde{x}_0)$

Proof. From Lemma 2, $[\chi] = [\alpha]$, so $\Omega_{u,v}(\chi) = \Omega_{u,v}(\alpha)$. Since $i([u], [\alpha]) = -i([\gamma_2], [\alpha]) + i([\gamma_5], [\alpha]) + i([\gamma_8], [\alpha]) - i([\gamma_{11}], [\alpha]) = -1 + (-1) + 1 - (-1) = 0$ and $i([v], [\alpha]) = 1 - (-1) - 1 + (-1) = 0$, it follows that $\alpha, \chi \in \text{Ker } \Omega_{u,v}$, and χ lifts to a closed geodesic on $\tilde{\mathbf{U}}$. \square

Corollary 2. *content...*

From here, we use α to show that the *only* trajectories that close on the Necker cube surface are those that are in vector direction (a, b) such that $\gcd(a, b) = 1$ and a, b are both odd. We call these **odd-odd** directions. We can make this claim because the group generated by the matrices

$$\begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix}$$

is the *Sanov subgroup* of $SL(2, \mathbb{Z})$ and only sends elements in the odd-odd set to itself. There are dualizations made between how these matrices skew a geodesic direction, and how their original affine transformations induce an effect homology. In a sense the kernel is obtained by the orbit of χ under \mathbb{X} and its holonomy vector under \mathbb{X}' .

Lemma 12. *The actions of \mathbb{X}' on \mathcal{O} and \mathcal{E} are closed in their respective sets.*

Proof. Since \mathbb{X}' is generated by the elements \mathbf{A}' , \mathbf{B}' , and \mathbf{R}' , any matrix $G' \in \mathbb{X}'$ is of the form $G' = (\mathbf{A}')^{i_1} \circ (\mathbf{B}')^{i_2} \circ (\mathbf{R}')^{i_3} \circ (\mathbf{A}')^{i_4} \circ \dots \circ (\mathbf{A}')^{i_n} \circ (\mathbf{B}')^{i_{n+1}} \circ (\mathbf{R}')^{i_{n+2}}$, where $i_k \in \mathbb{Z}$ for $i = 1, \dots, n$ and $k = 1, 2, 3$. Let $x = \begin{pmatrix} p \\ q \end{pmatrix}$, $y \in \mathcal{O}$, and consider the equation $G'x = y$. Observe that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^l x = \begin{pmatrix} p+2jq \\ q \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^m x = \begin{pmatrix} p \\ q+2mp \end{pmatrix}$ for any $l, m \in \mathbb{Z}$. Also note that for any $j \in \mathbb{Z}$, $(\mathbf{R}')^m x = \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} -q \\ p \end{pmatrix}, \begin{pmatrix} -p \\ -q \end{pmatrix}, \begin{pmatrix} q \\ -p \end{pmatrix}$ when $j \bmod 4 \equiv 0, 1, 2, 3$, respectively. In any case, the product of any power of a generator of \mathbb{X}' and any $x \in \mathcal{O}$ is an element of \mathcal{O} . By letting $l = i_1, m = i_2$, and $j = i_3$, we first consider the base case when $i = n$. Let $G' = G'_1 \circ \dots \circ G'_n$, such that $G'_i = (\mathbf{A}')^{i_1} \circ (\mathbf{B}')^{i_2} \circ (\mathbf{R}')^{i_3}$. Since n_1, n_2, n_3 are arbitrary integers, $G'_n x \in \mathcal{O}$. Suppose for some $b < n - 1$, $G'_{n-b} \circ \dots \circ G'_n x = y' \in \mathcal{O}$. Therefore $y' = (G'_1 \circ \dots \circ G'_b)^{-1} y$, which implies that $(G'_1 \circ \dots \circ G'_b)^{-1}$ preserves the set \mathcal{O} . Otherwise, if $y \in \mathcal{E}$, there exists at least one G'_i for $1 < i < b$ and $\tau \in \mathcal{E}$ such that $G'_i^{-1} \tau = (\mathbf{R}')^{-i_3} \circ (\mathbf{B}')^{-i_2} \circ (\mathbf{A}')^{-i_1} \tau \in \mathcal{O}$, a contradiction. Since elements in \mathbb{X}' are invertible, $G'_1 \circ \dots \circ G'_b$ must also map \mathcal{O} to itself. Left multiply both sides of the equation to show that $G'_1 \circ \dots \circ G'_n x = G'x = y$. By the principle of strong induction, this holds for all $0 < b \leq n$. Since G' is invertible and an arbitrarily chosen element of \mathbb{X}' , it follows that $x \in \mathcal{O}$ if and only if $y \in \mathcal{O}$ and \mathcal{O} is closed under \mathbb{X}' . The proof for when $x \in \mathcal{E}$ is made in the same way. \square

Now a trajectory in the horizontal direction has a directional vector of $(1, 0)$. The orbit of this vector by the Veech group is the set of all **even-odd** vectors. We also know that in this direction a geodesic is drift-periodic (See figure 1). The Veech group of \mathbf{M} preserves these properties. Suppose you had some closed geodesic on $\mathbf{M} \setminus \text{Sing}(\mathbf{M})$ called β such that $\beta = h(\alpha)$, where $h \in \text{Aff}^+(\mathbf{M})$, and h_* is its induced homomorphism. Then we want to show that

$$(i([\beta], [u]), i([\beta], [v])) = (i([\alpha], h_*^{-1}[u]), i([\alpha], h_*^{-1}[v])) = (0, 0).$$

But first, we look at some of the properties of the group \mathbb{X}_* .

Theorem 4. *Let \mathbb{X}_* be the group generated by $\mathbf{A}_*, \mathbf{B}_*, \mathbf{R}_*, \mathbf{H}_*$, and \mathbf{V}_* . Let $G = \langle \mathbf{A}_*, \mathbf{B}_* \rangle$, $T = \langle \mathbf{H}_*, \mathbf{V}_* \rangle$, and $R = \langle \mathbf{R}_* \rangle$. Then the following is true:*

- (i) G is a free subgroup of \mathbb{X}_* of rank two.
- (ii) T is a finite cyclic subgroup of \mathbb{X}_* and a centralizer of G .
- (iii) R is a finite cyclic subgroup of \mathbb{X}_* , and a normalizer of G .

Proof. Let $h_*^j = \mathbf{A}_*^{k_j} \circ \mathbf{B}_*^{g_j} \in G$ for $k_j, g_j \in \mathbb{Z}$, $j = 1, \dots, n$.

(i). When \mathbf{A}_* and \mathbf{B}_* act on γ_i , it is only ever trivial if i is even for \mathbf{A}_* or i is odd on \mathbf{B}_* . Since i cannot be both odd and even at the same time, there is no

relation between the two generators and therefore G is free.

(ii) It is up to the reader to show that T has the relations $\mathbf{H}_*^2 = \mathbf{V}_*^2 = (\mathbf{H}_* \mathbf{V}_*)^3 = id_*$, and is isomorphic to the rotational group of the hexagon generated by reflections about adjacent vertices of a 12-gon. Observe that $\mathbf{H}_* \circ \mathbf{A}_*^{k_j} \circ [\gamma_i] = \mathbf{H}_* \circ [\gamma_i] + \frac{k_j}{2}(1 - (-1)^i)(\mathbf{H}_* \circ [\gamma_{i-1}] + \mathbf{H}_* \circ [\gamma_{i+1}]) = [\gamma_{-i}] + \frac{k_j}{2}(1 - (-1)^i)([\gamma_{1-i}] + [\gamma_{-i-1}]) = \mathbf{A}_*^{k_j} \circ [\gamma_{-i}] = \mathbf{A}_*^{k_j} \circ \mathbf{H}_* \circ [\gamma_i]$, and $\mathbf{V}_* \circ \mathbf{A}_*^{k_j} \circ [\gamma_i] = \mathbf{V}_* \circ [\gamma_i] + \frac{k_j}{2}(1 - (-1)^i)(\mathbf{V}_* \circ [\gamma_{i-1}] + \mathbf{V}_* \circ [\gamma_{i+1}]) = [\gamma_{10-i}] + \frac{k_j}{2}(1 - (-1)^i)([\gamma_{11-i}] + [\gamma_{9-i}]) = \mathbf{A}_*^{k_j} \circ [\gamma_{10-i}] = \mathbf{A}_*^{k_j} \circ \mathbf{V}_* \circ [\gamma_i]$. In the same way one can show this to be true for $\mathbf{B}_*^{g_j}$, and we can see that T is a centralizer of G .

(iii) R is obviously cyclic and finite since an isomorphism is obtained as $\mathbf{R}_* \mapsto \mathbf{R}' \in SO(2, \mathbb{Z})$.

Note that $\mathbf{R}_* \circ \mathbf{A}_*^{k_j} \circ [\gamma_i] = \mathbf{R}_* \circ [\gamma_i] + \frac{k_j}{2}(1 - (-1)^i)(\mathbf{R}_* \circ [\gamma_{i-1}] + \mathbf{R}_* \circ [\gamma_{i+1}]) = (-1)^i[\gamma_{1-i}] + \frac{k_j}{2}(1 - (-1)^i)((-1)^{i-1}[\gamma_{2-i}] + (-1)^{i+1}[\gamma_{-i}]) = (-1)^{1-i}([\gamma_{1-i}] - \frac{k_j}{2}(1 + (-1)^{1-i})([\gamma_{2-i}] + [\gamma_{-i}])) = (-1)^{1-i}\mathbf{B}_*^{-k_j} \circ [\gamma_{1-i}] = \mathbf{B}_*^{-k_j} \circ (-1)^{1-i}[\gamma_{1-i}] = \mathbf{B}_*^{-k_j} \circ \mathbf{R}_* \circ [\gamma_i]$. Likewise, $\mathbf{R}_* \circ \mathbf{B}_*^{g_j} \circ [\gamma_i] = \mathbf{A}_*^{-g_j} \circ \mathbf{R}_* \circ [\gamma_i]$. \square

Remark. It can be easily shown that \mathbb{X}' has similar properties.

Lemma 13. Let $h_* \in \langle \mathbf{A}_*, \mathbf{B}_* \rangle$. Then $h_* \circ [\alpha] \in H_1(\mathbf{M}, \mathbb{Q})$ can be expressed as $h_* \circ [\alpha] = \frac{1}{2}(c_1 \Sigma_{j=0}^5 [\gamma_{2j}] + c_2 \Sigma_{j=0}^5 [\gamma_{2j+1}])$ for $c_1, c_2 \in \mathbb{Z}$.

Proof. Let $\Sigma_{j=0}^5 [\gamma_{2j}] = \Sigma \Gamma_{\text{even}}$, $\Sigma_{j=0}^5 [\gamma_{2j+1}] = \Sigma \Gamma_{\text{odd}}$, and $\Sigma_{j=0}^{11} [\gamma_j] = \Sigma \Gamma$. Let $h_* = h_*^n \circ \dots \circ h_*^1$, and $h_*^i = \mathbf{A}_*^{k_i} \circ \mathbf{B}_*^{g_i}$ for $k_i, g_i \in \mathbb{Z}$, $i = 1, \dots, n$. Compose these two homomorphisms and obtain $\mathbf{A}_*^{k_i} \circ \mathbf{B}_*^{g_i}(\Sigma \Gamma) = (4g_i k_i + 2k_i)\Sigma \Gamma_{\text{even}} + 2g_i \Sigma \Gamma_{\text{odd}} + \Sigma \Gamma$. Let $c_i^1 = (4g_i k_i + 2k_i)$, $c_i^2 = 2g_i$, and solve for $h_*^{i+1} \circ h_*^i \circ \Sigma \Gamma$:

$$\begin{aligned} h_*^{i+1} \circ h_*^i \circ (\Sigma \Gamma) &= h_*^{i+1} \circ (c_i^1 \Sigma \Gamma_{\text{even}} + c_i^2 \Sigma \Gamma_{\text{odd}} + \Sigma \Gamma) \\ &= c_i^1 h_*^{i+1} \circ (\Sigma \Gamma_{\text{even}}) + c_i^2 h_*^{i+1} \circ (\Sigma \Gamma_{\text{odd}}) + h_*^{i+1} \circ (\Sigma \Gamma) \\ &= 2g_{i+1} \Sigma \Gamma_{\text{odd}} + (4g_{i+1} k_{i+1} + 2k_{i+1}) \Sigma \Gamma_{\text{even}} + \Sigma \Gamma \\ &\quad + c_i^1 (4g_{i+1} k_{i+1} \Sigma \Gamma_{\text{even}} + 2g_{i+1} \Sigma \Gamma_{\text{odd}} + \Sigma \Gamma_{\text{even}}) \\ &\quad + c_i^2 (2k_{i+1} \Sigma \Gamma_{\text{even}} + \Sigma \Gamma_{\text{odd}}) \\ &= \Sigma \Gamma + (c_i^1 + (c_i^1 + 1)(4g_{i+1} k_{i+1}) + (c_i^2 + 1)2k_{i+1}) \Sigma \Gamma_{\text{even}} \\ &\quad + (c_i^2 + (c_i^1 + 1)2g_{i+1}) \Sigma \Gamma_{\text{odd}} \\ \text{Let } c_{i+1}^1 &:= (c_i^1 + (c_i^1 + 1)(4g_{i+1} k_{i+1}) + (c_i^2 + 1)2k_{i+1}), \\ c_{i+1}^2 &:= (c_i^2 + (c_i^1 + 1)2g_{i+1}). \end{aligned}$$

From these recursive definitions and a finite sequence of integers, $\{k\}_i, \{g\}_i$, observe then that

$$\begin{aligned} h_* \circ [\alpha] &= h_* \circ [\tfrac{1}{2} \Sigma \Gamma] = \tfrac{1}{2} h_* \circ [\Sigma \Gamma] = \tfrac{1}{2} [c_n^1 \Sigma \Gamma_{\text{even}} + c_n^2 \Sigma \Gamma_{\text{odd}} + \Sigma \Gamma] \\ &= \tfrac{1}{2} [(c_n^1 + 1) \Sigma \Gamma_{\text{even}} + (c_n^2 + 1) \Sigma \Gamma_{\text{odd}}]. \end{aligned}$$

Further simplify by letting $c_1 = c_n^1 + 1$, $c_2 = c_n^2 + 1$. \square

Lemma 14. *Let $h_* \circ [\alpha] \in H_1(\mathbf{M}, \mathbb{Q})$. Then for $a \in \langle \mathbf{H}_*, \mathbf{V}_* \rangle$ and $b \in \langle \mathbf{R}_* \rangle$, the following is true:*

$$\begin{aligned} a \circ h_* \circ [\alpha] &= h_* \circ [\alpha] \\ b \circ h_* \circ [\alpha] &= \frac{1}{2}[c'_1 \Sigma \Gamma_{\text{even}} + c'_2 \Sigma \Gamma_{\text{odd}}] \\ h_* \circ b \circ [\alpha] &= \frac{1}{2}[c''_1 \Sigma \Gamma_{\text{even}} + c''_2 \Sigma \Gamma_{\text{odd}}] \end{aligned}$$

Proof. By Theorem 4, a is a centralizer of the group so $a \circ h_* \circ [\alpha] = h_* \circ a \circ [\alpha] = h_* \circ \frac{1}{2}a \circ [\Sigma \Gamma]$. Since a is a cyclic permutation of the set Γ , it acts trivially on $\Sigma \Gamma$. Therefore, $a \circ h_* \circ [\alpha] = h_* \circ \frac{1}{2}[\Sigma \Gamma] = h_* \circ [\alpha]$.

By theorem 4, $a \circ \mathbf{A}_*^{k_i} \circ \mathbf{B}_*^{g_i} = \mathbf{B}_*^{-k_i} \circ \mathbf{A}_*^{-g_i} \circ a$. Extend this property to h_* , and denote the normalized element as h_{**} , such that $b \circ h_* = h_{**} \circ b$. Note that $b(\Sigma \Gamma) = b(\Sigma \Gamma_{\text{even}} + \Sigma \Gamma_{\text{odd}}) = \Sigma \Gamma_{\text{odd}} - \Sigma \Gamma_{\text{even}}$. $b \circ h_* \circ [\Sigma \Gamma] = c_1 b \circ \Sigma \Gamma_{\text{even}} + c_2 b \circ \Sigma \Gamma_{\text{odd}} = c_1 \Sigma \Gamma_{\text{odd}} - c_2 \Sigma \Gamma_{\text{even}}$. So, $c'_1 = -c_2$ and $c'_2 = c_1$. Since h_* is arbitrary, let $h_{**} = g_*$ be generated by an integer sequence that defines the word and consider $h_* \circ b \circ [\Sigma \Gamma] = b \circ g_* \circ [\Sigma \Gamma] = c_1^* b \circ \Sigma \Gamma_{\text{even}} + c_2^* b \circ \Sigma \Gamma_{\text{odd}} = c_1^* \Sigma \Gamma_{\text{odd}} - c_2^* \Sigma \Gamma_{\text{even}}$. So, $c''_1 = -c_2^*$ and $c''_2 = c_1^*$. \square

Now that every element in the orbit of $[\alpha]$ can be expressed as a linear combination of integers, it is simple to show they lift to a closed trajectory in the cover.

Definition 32. *Let $\mathbf{dir} : UT(\mathbf{M} \setminus \text{Sing}(\mathbf{M})) \rightarrow \mathcal{O} \cup \mathcal{E}$ be the injective map from $\mathbb{R}/2\pi\mathbb{Z}$ to \mathbb{Z}^2 given as $\mathbf{dir}(\theta) = (k_1 \cos(\theta), k_2 \sin(\theta))$, $k_1, k_2 \in \mathbb{R}$ such that $\gcd(k_1 \cos(\theta), k_2 \sin(\theta)) = 1$.*

Theorem 5. *(Sketch)*

Any geodesic, β , in \mathbf{M} lifts to a closed geodesic $\tilde{\beta}$ on $\tilde{\mathbf{U}}$ if and only if $\mathbf{dir}(\text{Arg}(\mathbf{hol}(\beta))) \in \mathcal{O}$.

Proof. Call the quotient cover $p : \tilde{\mathbf{U}} \rightarrow \mathbf{M}$, and fix a point $\tilde{x}_0 \in p^{-1}(x_0)$. Let $\beta = h(\chi)$, where $h \in \mathbb{X}$. We also obtain $[\beta] = h_* \circ [\alpha]$ from Lemma 2. Since h sends geodesics to geodesics, h induces the following: $\mathbf{hol}(h(\chi)) = h'(\mathbf{hol}(\chi)) = h'(6 + 6i)$ for $h' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{X}'$. So, $\text{Arg}(h'(\mathbf{hol}(\chi))) = \text{Arg}(6[(a + b) + i(c + d)]) = \text{Arg}(6h'(1 + i))$. Lemma 3 states that for any $h' \in \mathbb{X}'$, $h'(\mathcal{O}) = \mathcal{O}$. Therefore there is no such geodesic of **even-odd** slope in the orbit of χ . Otherwise $h', h \notin \mathbb{X}', \mathbb{X}$. Consequently, $\mathbf{dir}(\text{Arg}(\mathbf{hol}(\beta))) \in \mathcal{O}$.

From Lemma 5 we see that $[h(\chi)] = h_* \circ [\chi] = h_* \circ [\alpha] = \frac{1}{2}(c_1 \Sigma_{j=0}^5 [\gamma_{2j}] + c_2 \Sigma_{j=0}^5 [\gamma_{2j+1}])$ for $c_1, c_2 \in \mathbb{Z}$. Denote the sums as $\Sigma \Gamma_{\text{even}}$ and $\Sigma \Gamma_{\text{odd}}$. Therefore, $2i([u], h_* \circ [\alpha]) = c_1 i([u], \Sigma \Gamma_{\text{even}}) + c_2 i([u], \Sigma \Gamma_{\text{odd}})$
 $= c_1 (-i([\gamma_2], 0) + i([\gamma_5], [\gamma_6] + [\gamma_4]) + i([\gamma_8], 0) - i([\gamma_{11}], [\gamma_{10}] + [\gamma_0]))$
 $+ c_2 (-i([\gamma_2], [\gamma_1] + [\gamma_3]) + i([\gamma_5], 0) + i([\gamma_8], [\gamma_7] + [\gamma_9]) - i([\gamma_{11}], 0))$
 $= c_1 (- (0) + (-1 - 1) + (0) - (-1 - 1)) + c_2 (- (1 + 1) + (0) + (1 + 1) - (0)) = 0.$

$$\begin{aligned}
& \text{Similarly, } 2i([v], h_* \circ [\alpha]) = c_1 i([v], \Sigma \Gamma_{\text{even}}) + c_2 i([v], \Sigma \Gamma_{\text{odd}}) \\
& = c_1 (i([\gamma_0], 0) - i([\gamma_3], [\gamma_2] + [\gamma_4]) - i([\gamma_6], 0) + i([\gamma_9], [\gamma_8] + [\gamma_{10}]) \\
& + c_2 (i([\gamma_0], [\gamma_{11}] + [\gamma_1]) - i([\gamma_3], 0) - i([\gamma_6], [\gamma_5] + [\gamma_7]) + i([\gamma_9], 0) \\
& = c_1 ((0) - (-2) - (0) + (-2)) + c_2 ((2) - (0) - (2) + (0)) = 0.
\end{aligned}$$

Therefore, $\Omega_{u,v}(h(\chi)) = (0, 0)$, and $h(\chi) = \beta \in \text{Ker } \Omega_{u,v}$ for all $h \in \mathbb{X}$. By Theorem 2, β lifts to $\tilde{\beta} \in \pi_1(\tilde{\mathbf{U}}, \tilde{x}_0)$. Let $\theta = \text{Arg}(\mathbf{hol}(\beta))$. Then ω_t^θ at x_0 lifts to $\tilde{\omega}_t^{p^{-1}(\theta)} \in \tilde{\mathbf{U}} \setminus \text{Sing}(\tilde{\mathbf{U}})$.

Now suppose instead that $\beta = h(\gamma_i)$. Then $\mathbf{dir}(\beta) = \frac{1}{2}(1 + (-1)^i, 1 - (-1)^i)$. According to Lemma 3, $h'(\mathcal{E}) = \mathcal{E}$. Thus we have no geodesic in the **odd-odd** directions obtained from the orbits of $(1, 0)$ and $(0, 1)$. For contradiction, suppose that $h(\gamma_i) \in \mathbf{Ker } \Omega_{u,v}$. Then $(i(h_* \circ [\gamma_i], [u]), i(h_* \circ [\gamma_i], [v])) = (i([\gamma_i], h_*^{-1} \circ [u]), i([\gamma_i], h_*^{-1} \circ [v])) = (0, 0)$. Let $h_*^{-1} \circ [u] = \Sigma_{j=0}^{11} x_j [\gamma_j]$, and $h_*^{-1} \circ [v] = \Sigma_{j=0}^{11} y_j [\gamma_j]$. Note that since γ_i intersects $\gamma_{i \pm 1}$, $i([\gamma_i], h_*^{-1} \circ [u]) = (-1)^{i+1}(x_{i-1} + x_{i+1})$ and $i([\gamma_i], h_*^{-1} \circ [v]) = (-1)^{i+1}(y_{i-1} + y_{i+1})$.

Unfinished.. □

Conjecture. Dynamics of Geodesic Flow on the Necker cube surface. Obtain θ and $\tilde{\theta}$ as described in Definition 3. Denote the non-singular unit-speed geodesic flow with initial point $s \in (\mathbf{U} \setminus \text{Sing}(\mathbf{U}))$ in direction $[\theta] \sim \phi \in UT(\mathbf{U} \setminus \text{Sing}(\mathbf{U}))$ by $F_t^\phi : \mathbf{U} \times \mathbb{R}_0^+ \rightarrow \mathbf{U}$ on (\mathbf{U}, μ) , where μ is a flow-invariant measure. Then the following is true:

- (i) (Periodic) There exists a $t_0 > 0$ such that $F_{t+t_0}^\phi(s) = F_t^\phi(s)$ if and only if $\tilde{\theta} \in \mathcal{O}$.
- (ii) (Drift-Periodic) There exists a $t_0 > 0$ such that $F_{t+t_0}^\phi(s) = F_t^\phi(s) + c$, where $c \in \mathbf{U}$ is a non-trivial translation of a point in \mathbf{U} , if and only if $\tilde{\theta} \in \mathcal{E}$.

Proof. (Sketch)

Denote the covering maps $f : \tilde{\mathbf{U}} \rightarrow \mathbf{U}$, $p : \tilde{\mathbf{U}} \rightarrow \mathbf{M}$, and fix a point $\tilde{x}_0 \in f^{-1}(s), p^{-1}(x_0)$, for $x_0 \in \mathbf{M}$. $f^{-1}([\theta]) = \{x : x = \theta + n\frac{\pi}{2}, n \in \mathbb{Z}\} = [\theta] \subset UT(\tilde{\mathbf{U}} \setminus \text{Sing}(\tilde{\mathbf{U}}))$ given by the four-fold cover and rotations of each individual plane. This gives us a relation between the two tangent bundles, where the translation four-fold cover has the standard $\mathbb{R}/2\pi\mathbb{Z}$ unit tangent fiber. θ' is the direction associated to the flow $\omega_t^{\theta'} : [0, 1] \rightarrow \mathbf{M}$ on the translation surface. Since the cover is translation, $p^{-1}(\theta') = \theta' = \theta + n\frac{\pi}{2}$. First suppose that $\tilde{\theta} \in \mathcal{O}$. Then θ is identified with the set of directions that close on $\tilde{\mathbf{U}}$. From Theorem 5, $\omega_t^{\theta'}$ lifts to a closed geodesic $\tilde{\omega}_t^{\theta'}$. Given $\mathbf{hol}(\omega) = \int_\omega dz$, we obtain a period for the unit-speed flow, $t_0 = |\mathbf{hol}(\omega)|$. That is, $\tilde{F}_t^{\theta'} : \mathbb{R}_0^+ \rightarrow \tilde{\mathbf{U}}$ such that $\frac{d}{dt} \tilde{F}_t = \frac{1}{|\mathbf{hol}(\omega)|}$. Then $F_t^\phi = F_t^{[\theta']} = f \circ \tilde{F}_t^{\theta'}$. The period carries over since there is no concern over a trajectory returning to \tilde{x}_0 in a different direction. Otherwise, the geodesic $\omega_t^{\theta'}$ on \mathbf{M} would have closed in $0 < t < 1$. Now suppose that $\tilde{\theta} \in \mathcal{E}$. Identifying it with θ' , we see that ω in direction θ' is not an element of $\mathbf{Ker } \Omega_{u,v}$ from Theorem 5. Therefore, $\Omega_{u,v}(\omega) = (m, n) \simeq T^{m,n}$ and lifting the terminal point

$\omega(1)$, $\tilde{\omega}(1) = T^{m,n}(\tilde{\omega}(0)) = T^{m,n}(\tilde{x}_0)$. The period remains unchanged, in that $\tilde{F}_{t+\text{hol}(\omega)}^{\theta'} = \tilde{F}_t^{\theta'} + T^{m,n}(\tilde{x}_0)$. Therefore, $F_{t+t_0}^\phi(s) = f \circ \tilde{F}_t^{\theta'}(\tilde{x}_0) + f \circ T^{m,n}(\tilde{x}_0)$. Conversely, suppose F_t is periodic. Then $[\theta] = \phi = [\theta']$, which defines directional flows \tilde{F}_t^ϕ . According to Theorem 5, \tilde{F}_t^ϕ will close if and only if $\phi \subset \mathcal{O}$. ϕ is the orbit of $\vec{\theta}'$ under the 90 degree rotational matrix. This matrix does not alter the length or period of a geodesic. Thus, F_t^ϕ is exactly one of the flows \tilde{F}_t^ϕ . Likewise, if F_t^ϕ is drift-periodic then $F_{t+t_0}^\phi = f \circ \tilde{F}_t^\phi + f \circ T^{m,n}$. $T^{m,n}$ is trivial if and only if $\theta' \in \mathcal{O}$. Therefore, $\theta' \in \mathcal{E}$, and $[\theta'] = \phi$. \square

There is still much work to do in terms of cleaning up the proofs and organizing the final paper.

Conclusion

What I ultimately aim to do is port these results on X 's homology back to the Necker cube surface. I want to do it in such a way that the final theorem is bi-conditional. To do so, I imagine I can take a vector image of a small segment of a geodesic in \mathbb{R}^3 and project it onto the isometric flattening of the Necker Cube surface to obtain a direction (or classes of equivalent directions), and relate it to the unit tangent bundle of \mathbf{M} .

In addition, I would also like to find a formula for the arc-length of a geodesic based on direction alone. Knowing that $\text{hol}(\alpha) = 6 + 6i$ means that the induced Euclidean metric on $\mathbf{M} \setminus \text{Sing}(\mathbf{M})$ gives the geodesic an arc-length of $6\sqrt{2}$. I would like to show that:

$$\int_{\beta} |dz| = |\text{hol}(h(\alpha))| = |h'(\text{hol}(\alpha))|,$$

where $h' \in V(\mathbf{M})$ is the derivative of h , and $\beta = h(\alpha)$. We know that is true on the translation surface, but it's a matter of then showing the translation quotient, branch-cover, and the Necker cube surface have the same induced Euclidean metric of these non-singular geodesics. (It would not be surprising considering that the surface is built out of subsets of planes.) Even more of a problem is finding a way to solve for a matrix in the Sanov subgroup that brings (1,1) to the desired odd-odd slope.

References

- [1]
- [2] Necker, L.A., *Observations on some remarkable optical phaenomena seen in Switzerland; and on an optical phaenomenon which occurs on viewing a figure of a crystal or geometrical solid*. London and Edinburgh Philosophical Magazine and Journal of Science. 1832. doi:10.1080/14786443208647909
- [3] Gottlieb., *Q*bert* [Arcade], Gottlieb, 1982.
- [4] Escher, M.C., *Metamorphosis I*, 1937.
- [5] Escher, M.C., *Convex and Concave*, 1955.