Periodicity of Geodesic Flows on the Necker Cube Surface

Pavel Javornik

November 12, 2017

Abstract

We study dynamical properties of geodesic flows on a flat, periodically constructed Euclidean cone surface by obtaining an infinite-type translation cover branched over conical singularities, of which admits a flat induced metric and a connection between any two points in the tangent vector bundle of the surface has trivial holonomy. A unit vector flow on the unit tangent bundle of such a surface is well-defined with a canonical vector representation in $\mathbb{R}^2$. We study this infinite-type surface as a $\mathbb{Z}^2$ cover of a compact Veech surface belonging to the stratum of $\mathcal{H}(2,2,2,2)$ translation surfaces, and use its $SL(2,\mathbb{R})-$commensurable Veech group to prove results that relate directions of flows to the periodicity and ergodicity of their lifts in the cover.

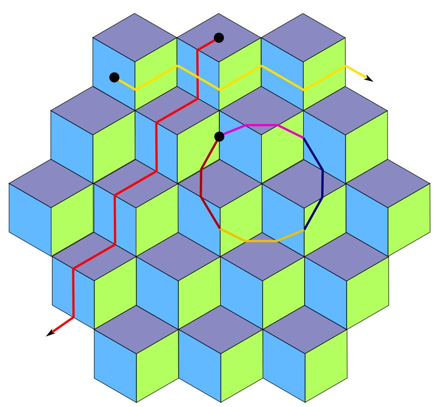


Figure 1: Periodic and drift-periodic flows on the Necker cube surface

1 Introduction

Put your Problem statement here! Example of a Citation[**?**, p.219]. Here's Another Citation [**?**]

Historically, the Necker cube[citation] has made numerous appearances in the work of mathematicians, crystallographers, and scientists interested in human visual systems prior to it being popularized in the works of illusionary artist M.C. Escher (pictured below). The crystallographer Louis Albert Necker was credited for having discovered the optical illusion, and studying its geometry[cite]. The solid presentation of the cube, when rendered as a flat surface with its back faces removed, achieves a similar effect when its three visible faces are shaded in a particular way. We are interested in the infinite tiling of this structure (figure **??**), which has the appearance of a rhombile tiling of the Euclidean plane, and refer to it as the Necker cube surface. Video game fans might even recognize it as the same surface that appears in the 1982 arcade game by Gottlieb, "Q*bert."

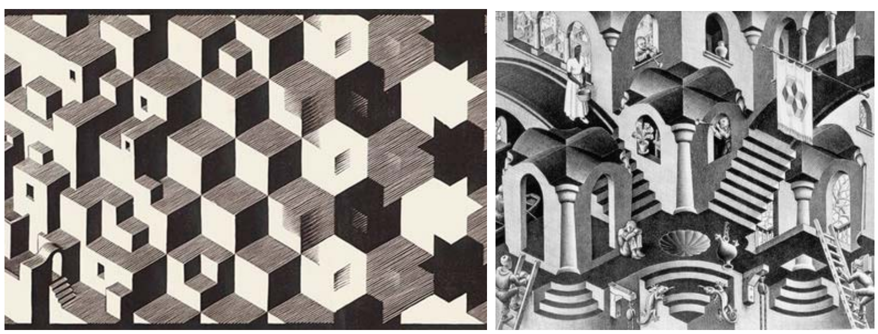


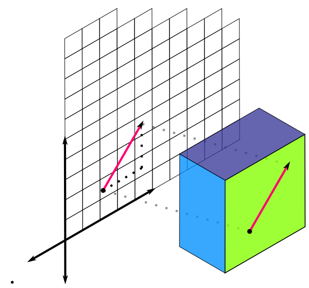
Figure 2: The Necker cube tiling as it appears briefly in "Metamorphosis I," and impressed into a banner in "Convex and Concave."[citation]

Flat, periodic surfaces such as the Necker cube surface are of much interest to differential geometers for two reasons in particular. The first is that, locally, neighborhoods are isometric to the plane, making it remarkably easy to describe parallel transports between two points in tangent vector bundles over the surface by linear group actions. The second is that Riemann surfaces constructed out of infinitely many polyhedra can be *flattened* and contained in the plane in such a way that the metric is preserved. The advantage in this case is that one obtains something akin to infinite billiard tables where a geodesic flows are represented by series of straight line segments in the plane. This paper will use such methods to prove dynamical results about Geodesic flows on the Necker cube surface.

Remark. *Familiarity is assumed on the part of the reader with covering space theory, translation (Veech) surfaces, and their associated Veech groups. For general surveys on these topics: [cite], [cite].*

1.1 Discussion of Results

Our initial experiments strongly supported the theory that there would be a correlation between a choice of trajectory angle and dynamical properties of a geodesic on such a symmetric object. Rightfully so, a surface composed of infinitely many cubes an induced flat metric where every neighborhood is locally isometric to the Euclidean plane. Via the parallel transport of a unit tangent vector over the sharp edges of the surface, it becomes obvious to think of the geodesic as a sequence of line segments contained in the faces of each cube on the surface's embedded form in $\mathbb{R}^3$. **??**.



Let x_0 be a point on the surface that is contained in an open neighborhood of a smooth section of the surface and consider a tangent unit vector $v \in \mathbb{R}^3$ protruding from x_0. Assuming the faces are parallel to every 2-dimensional subspace of $\mathbb{R}^3$ spanned by standard basis vectors, exactly one component of v would be 0. Projecting this vector to a parallel plane retains all necessary information about direction. Call this vector $v_0 \in \mathbb{R}^2$.

We say that v_0 is the *initial trajectory* of the geodesic and the angle it makes relative to our choice of basis is its *initial trajectory angle.* We consider a rational direction to fall into one of two categories.

Definition 1. *Let v_0 be a unit vector of the form $\frac{1}{k}(x, y) \in \mathbb{R}^2$ with $x, y \in \mathbb{Z}$ and $k = \sqrt{x^2 + y^2} \in \mathbb{R}$. We say v_0 is an* ***odd-odd*** *vector if its components are relatively prime and odd. We denote the* ***set of all odd-odd directions*** *by $\mathcal{O}$. We say that v_0 is an* ***even-odd*** *vector if its components are relatively prime and of opposite parity. We denote the* ***set of all even-odd directions*** *by $\mathcal{E}$.*

This paper will demonstrate how one reaches the following conclusion about the relationship between parity and periodicity of geodesic behavior on the Necker cube surface, denoted $\mathbf{S}$:

Theorem. *(Directional Dichotomy of Periodic Geodesics on $\mathbf{S}$) Let Φ_t be a unit-speed geodesic flowing on the unit tangent bundle of $\mathbf{S}$ for $t \in \mathbb{R}$ with initial point and vector $\Phi_0 = (x_0, v)$. Then the following is true:*

(i) Φ is periodic with period $T \in \mathbb{R}$ if and only if $v_0 \in \mathcal{O}$.

(ii) Φ is drift-periodic with period $T \in \mathbb{R}$ if and only if $v_0 \in \mathcal{E}$.

The proof for this theorem can be found in XXX and follows from Theorems YYYY and ZZZZZ.

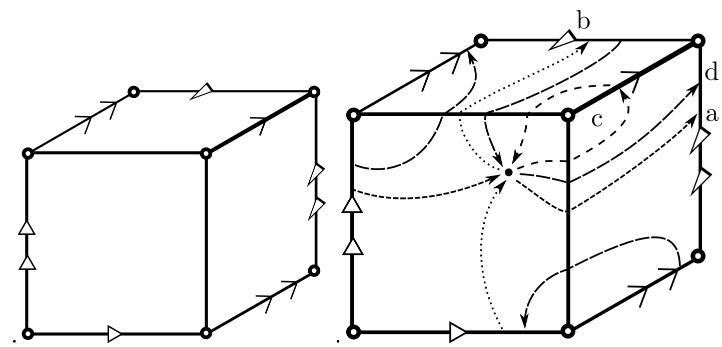
theorem about most directions being recurrent?

1.2 Acknowledgements

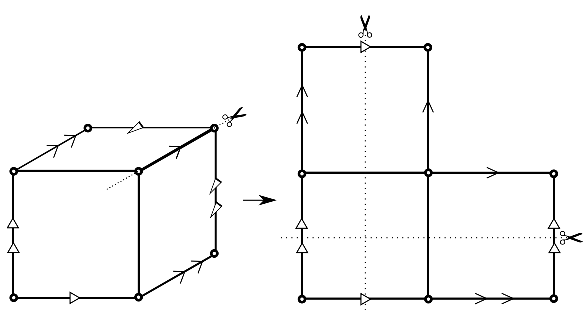
-Pat Hooper
- Vincent Delecroix, Ferrán Valdez, pascal hubert
-

2 Periodic Tiling of Necker Cubes

This section will detail how the Necker Cube surface is obtained as an infinite regular cover of a branched torus. Consider the Necker cube with the following identifications:



We denote this surface $\mathbf{G}$. $\mathbf{G}$ is obtained by taking three copies of the unit square and identifying the edges in order to resemble the visible portion of a unit cube. $\mathbf{G}$ is a genus one surface with three vertices homeomorphic to $\mathbb{T}^2$. Check that every vertex has a cone angle of either 3π or $\frac{3\pi}{2}$. These are the conical singularities of the surface, Σ. We denote the surface without its singularities $\mathbf{G}\backslash\Sigma = \mathbf{G}^\circ$. Thus, $\pi_1(\mathbf{G}^\circ) \cong \pi_1((\mathbb{T}^2 \text{ with 3 punctures})) \cong \pi_1(S^1 \vee S^1 \vee S^1 \vee S^1) \cong \mathbb{F}_4$, the free group of four generators. We may flatten $\mathbf{G}$ and $\mathbf{G}^\circ$ into an L-shaped torus:



This L-shaped torus is then cut along the dotted lines and pieced together to resemble a 2×2 torus with a unit square removed from the center:

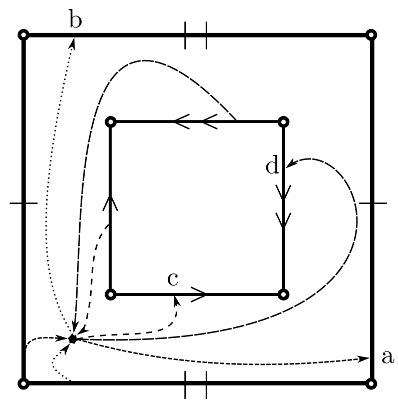


Figure 3: $\mathbf{G}^\circ$ with 4 independent, closed paths labeled identified

Denote its fundamental group by $\pi_1(\mathbf{G}^\circ) = \langle a, b, c, d\rangle$, where a, b, c, d are the labeled independent paths on the surface. Consider the following map:

Definition 2. *Let $\varphi_1 : \pi_1(\mathbf{G}^\circ) \to \mathbb{Z}^2$ be the group homomorphism defined on generators a, b, c, d:*
$c, d \mapsto (0, 0)$
$a \mapsto (1, 0)$
$b \mapsto (0, 1)$

Definition 3. *The punctured* ***Necker Cube Surface*** *is a regular cover of $\mathbf{G}^\circ$, and is denoted $\mathbf{U}^\circ$. It's fundamental group presentation is the subgroup of $\pi_1(\mathbf{G}^\circ)$, $\pi_1(\mathbf{U}^\circ) = \ker \varphi_1$.*

Thus $\Delta_{\varphi_1} = \pi_1(\mathbf{G}^\circ)/\pi_1(\mathbf{U}^\circ) \cong \varphi_1(\pi_1(\mathbf{G}^\circ)) = \mathbb{Z}^2$ is the associated *deck group* of the cover. $\mathbf{U}^\circ$ is the primary object of study.

2.1 Monodromy Group Representation

In this section we describe a translation cover of $\mathbf{G}^\circ$ which and $\mathbf{U}^\circ$ which is factored through as the kernel of a homomorphism from $\pi_1(\mathbf{G}^\circ) \to \mathrm{SO}(2, \mathbb{Z})$. This homomorphism represents the monodromy group as the fundamental group acting on the unit tangent bundles $T^1\mathbf{G}^\circ$ and $T^1\mathbf{U}^\circ$.

Definition 4. *Let $\varphi_2 : \pi_1(\mathbf{G}^\circ) \to SO(2, \mathbb{Z})$ be the group homomorphism defined on the generators a, b, c, d:*

$$a, b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$
$$c \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = R$$
$$d \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = R^3$$

Recall that a *translation surface* is a compact, locally flat surface that admits a discrete number of conical singularities and has trivial linear holonomy on any arbitrary path. If the total cone angle of the singularity is some integer multiple of 2π, $2\pi(d+1)$ for $d \geq 0$, we say that singularity has degree d. Two translation surfaces belong to the same stratum, or family, of surfaces characterized by the number and degree of conical singularities. Consider the effect that paths c, d have on holonomy:

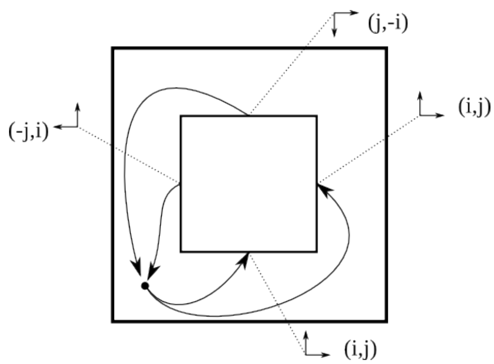


Figure 4: Effect a nontrivial loop has on arbitrary basis vectors (i,j).

We construct a cover of $\mathbf{G}^\circ$ by taking four copies of it and cyclically pairing edges, and show that this cover has trivial holonomy. Denote the unit tangent bundle of $\mathbf{G}^\circ$ by $T^1\mathbf{G}^\circ$. Because $\mathbf{G}^\circ$ can be contained in the plane, we consider vectors tangent to $\mathbf{G}^\circ$ as vectors in $\mathbb{R}^2$. Let $x_0 \in \mathbf{G}^\circ$. The fiber over x_0 under the projection $T^1\mathbf{G}^\circ \twoheadrightarrow \mathbf{G}^\circ$ is the set E_{x_0}. Let $\triangledown$ be a flat metric connection on $T^1\mathbf{G}^\circ$ and $p = (x_0, v_0) \in E_{x_0}$ be a single element in the fiber over x_0 with $v_0 \in \mathbb{R}^2$. We denote the *holonomy group* of the connection with base point p by $Hol_p(\triangledown) = \{A \in SO(E_{x_0}) \ : \ p = A \cdot p\}$ where p differs from $A \cdot p$ by some rotation of v_0. An element in $SO(E_{x_0})$ acts on p by rotation of the vector v_0. Denote the *holonomy bundle* based at p by $H(p) = \{p' \in E_{x_0} \ : \ p' = A \cdot p\}$, the set of elements in the orbit of p where a closed loop $\gamma : [0,1] \to \mathbf{G}^\circ$ defines a horizontal lift to $\gamma^* : [0,1] \to T^1\mathbf{G}^\circ$ by parallel transport of v_0 along the path such that $p = \gamma^*(0)$, and $p' = \gamma^*(1)$. Since $\mathbf{G}^\circ$ is flat and trivial paths have trivial holonomy, $Hol_p(\triangledown)$ acts on $H(p)$ monodromy group is discrete and well-defined on $\mathbf{G}^\circ$.

Lemma 1. *$\varphi_2 : \pi_1(\mathbf{G}^\circ) \to \mathrm{SO}(2,\mathbb{Z})$ is a monodromy representation of the surface.*

Proof. It is clear from Figure 6 that this is the case as concatenating a closed loop with paths c or d would rotate a vector exactly by the cone angles of the singularities that they loops around: $\pm\frac{3\pi}{2}$. □

Definition 5. *Let $\mathbf{M}^\circ$ be the regular cover of $\mathbf{G}^\circ$ with fundamental group $\pi_1(\mathbf{M}^\circ) = \ker\varphi_2$.*

Showing that $\mathbf{M}^\circ$ is a degree four cover of $\mathbf{G}^\circ$ with trivial holonomy and deck group $\mathrm{SO}(2,\mathbb{Z})$ follows immediately from its construction.

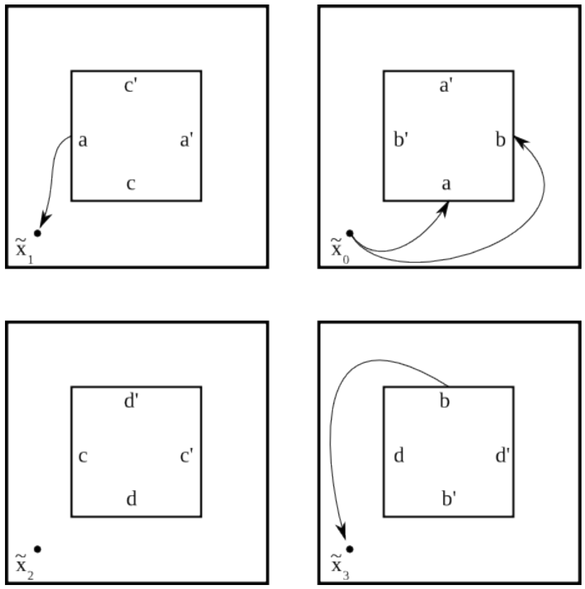
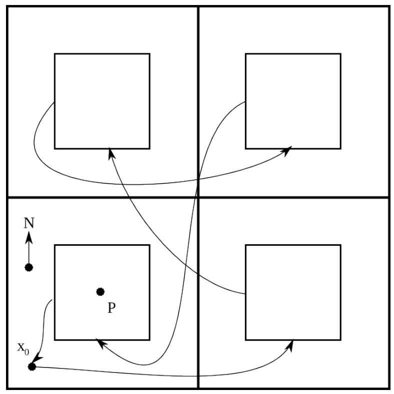


Figure 5: Lift of paths c, d to $\mathbf{M}^\circ$ (**RELABEL EDGES**)

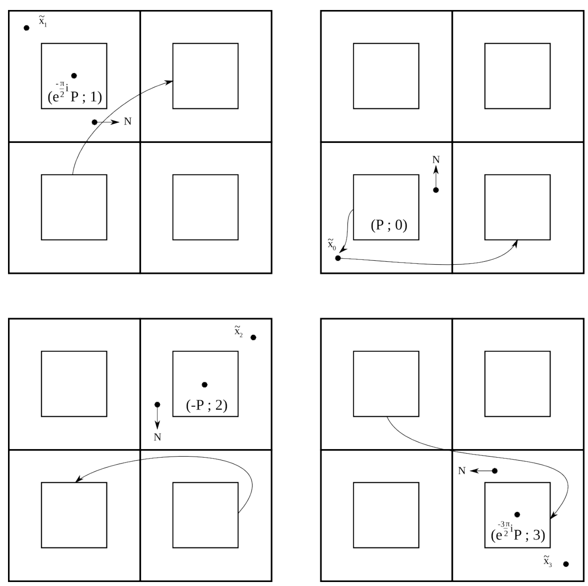
Definition 6. *Let $\tilde{\mathbf{U}}^\circ$ be the cover of $\mathbf{U}^\circ$ with fundamental group $\pi_1(\tilde{\mathbf{U}}^\circ) = \ker \varphi_2\big|_{\pi_1(\mathbf{U}^\circ)}$.*

Prove that $\ker \varphi_1\big|_{\pi_1(\mathbf{M}^\circ)} = \ker \varphi_2\big|_{\pi_1(\mathbf{U}^\circ)}$

$\tilde{\mathbf{U}}^\circ$ is also a cover of $\mathbf{M}^\circ$ with fundamental group $\ker \varphi_1\big|_{\pi_1(\mathbf{M}^\circ)}$. In the former case the deck group of the cover is $\mathrm{SO}(2,\mathbb{Z})$, while in the latter case it is $\mathbb{Z}^2$. The deck group of the cover $\tilde{\mathbf{U}}^\circ \to \mathbf{G}^\circ$ is $\mathbb{Z}^2 \rtimes \mathrm{SO}(2,\mathbb{Z})$. **worth talking about?**. $\tilde{\mathbf{U}}^\circ$ is a four-fold cover of $\mathbf{U}^\circ$ with trivial holonomy visualized as four copies of $\mathbf{U}^\circ$, which can be thought of as countably many copies of . We can identify a closed path on $\mathbf{G}^\circ$ as an element of the deck group $\mathbb{Z}^2 \times \mathrm{SO}(2,\mathbb{Z})$ or, equivalently, $2\mathbb{Z} + 2\mathbb{Z}i \times \mathbb{Z}/4\mathbb{Z}$ if we want to consider a metric on the ambient space $\mathbb{C}$ that is consistent with the tiling of $\mathbf{G}^\circ$. Further, we can rotate everything in $\mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$ along with $\tilde{\mathbf{U}}^\circ$ according to the element of $\mathrm{SO}(2,\mathbb{Z})$ that acts non-trivially on the holonomy of $\mathbf{U}^\circ$. For example, let $P \in 2\mathbb{Z} + 2\mathbb{Z}i$ and consider the following curve on $\mathbf{U}^\circ$:

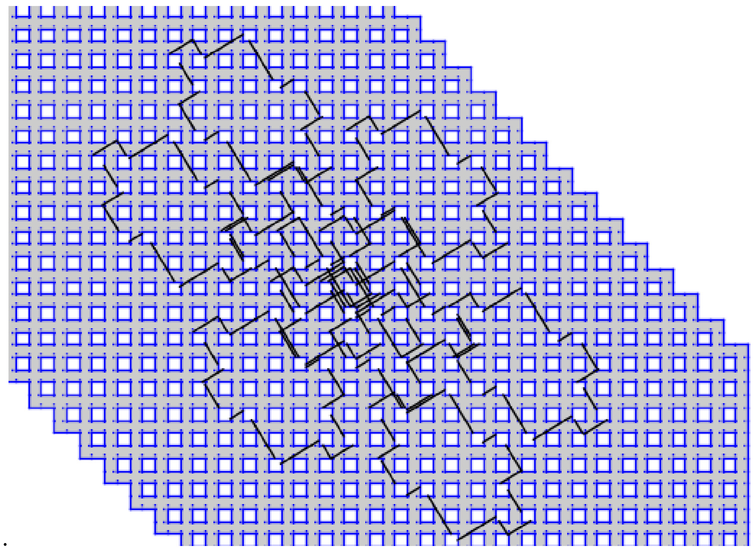


If we lift this curve to $\tilde{\mathbf{U}}^\circ$ we can identify it with an element of the deck group isomorphic to $2\mathbb{Z} + 2\mathbb{Z}i \times \mathbb{Z}/4\mathbb{Z}$. Observe that it will only close if the curve acts trivially on the holonomy of $\mathbf{U}^\circ$:



2.2 Geodesics on $\mathbf{U}^\circ$

The key connection made between surfaces $\mathbf{U}^\circ$ and $\tilde{\mathbf{U}}^\circ$ is that geodesics on $\mathbf{U}^\circ$ and $\tilde{\mathbf{U}}^\circ$ are identical in the sense that their images are in a one-to-one correspondence. Unless stated otherwise, a *geodesic* is a *unit-speed* locally distance minimizing curve on a smooth topological surface $\gamma : [0, \infty) \to S$ such that $\gamma'(0) = v_0 \in \mathbb{R}^2$ and $||v_0|| = 1$. A geodesic is *periodic* if for some $T \in \mathbb{R}^+$ $\gamma(t) = \gamma(t+T)$ for all t. A geodesic is *drift-periodic* if for some $T \in \mathbb{R}^+$, there is some non-trivial translation of the surface that takes $\gamma(t) \mapsto \gamma(t+T)$ for all t. In either case we say that T is the *period* or *length* of a segment; in the case where a geodesic does have a period, we restrict γ to the closed interval $[0, T]$ and call this a *geodesic segment*. In this paper we do not consider cases where a geodesic hits a cone singularity. For example, the following is a periodic geodesic on the surface with initial direction $kv_0 \in \mathcal{O}$:



..

Figure 6: Periodic geodesic path on $\mathbf{U}^\circ$ modeled with sage-flatsurf.

We refer later to the following commuting diagram of branched covering maps and inclusion maps. It is meant to illustrate the relationship between all of these spaces.

$$\begin{tikzcd}
\tilde{\mathbf{U}}^\circ \arrow[r, two heads, "r^\circ"] \arrow[d, hook, "pr_b"] \arrow[rrr, bend left, two heads, "f^\circ"] & \mathbf{U}^\circ \arrow[r, two heads, "q^\circ"] \arrow[d, hook, "pr_c"] & \mathbf{G}^\circ \arrow[d, hook, "pr_d"] & \mathbf{M}^\circ \arrow[l, two heads, "w^\circ"'] \arrow[d, hook, "pr_a"] \\
\tilde{\mathbf{U}} \arrow[r, two heads, "r"] \arrow[rrr, bend right, two heads, "f"'] & \mathbf{U} \arrow[r, two heads, "q"] & \mathbf{G} & \mathbf{M} \arrow[l, two heads, "w"']
\end{tikzcd} \tag{1}$$

Let $x_0 \in G^\circ$ and fix $u_0 \in q^{\circ -1}(x_0)$ Let $\alpha : [0, b] \to \mathbf{G}^\circ$ be a closed geodesic segment on $\mathbf{G}^\circ$ with initial and terminal point x_0, and $\alpha'(0) = v_0$. Denote α's unique lift to $\mathbf{U}^\circ$ under q° as $\gamma : [0, b]$, where $\gamma(0) = u_0$, $\gamma'(0) = \gamma'(b) = v_0$, and $\gamma(b) \in q^{\circ -1}(x_0)$. Fix an arbitrary point $\tilde{u}_0 \in r^{\circ -1}(u_0)$. We define the unique lift of γ under r as the curve $\tilde{\gamma} : [0, b] \to \tilde{\mathbf{U}}^\circ$ with $\tilde{\gamma}(0) = \tilde{u}_0$ and $\tilde{\gamma}'(0) = \tilde{\gamma}'(b) = v_0$. Denote their homotopy class representatives as $[\alpha], [\gamma]$, and $[\tilde{\gamma}]$.

A geodesic must have trivial holonomy upon returning to its initial point on $\mathbf{G}^\circ$. If a geodesic segment does not lift to a closed curve on $\mathbf{U}^\circ$, then $\gamma(b)$

would be a translation of u_0 by some element of the deck group of the cover, i.e. $\varphi_1([\alpha]) \in \mathbb{Z}^2$ is non-trivial, but $\varphi_2([\alpha]) = I_2$ as φ_2 is a monodromy map whose image acts on $Hol_p(\nabla)$ of $\mathbf{U}^\circ$. Therefore $\gamma'(b) = v_0$ as well. We can naturally extend these segments to the domain $[0, \infty)$ by continuously concatenating the segments.

Definition 7. *Let $m \in \mathbb{N}$. The concatenation of a geodesic segment lifted from α onto $\mathbf{U}^\circ$ is done in the following manner:*

$$\gamma^m = \begin{cases} \gamma(t) & 0 \leq t \leq b \\ \varphi_1([\alpha]) \cdot \gamma(t) & b \leq t \leq 2b \\ \vdots & \\ \varphi_1([\alpha]^{m-1}) \cdot \gamma(t) & (m-1)b \leq t \leq mb \end{cases} \tag{2}$$

*The new segment is a map $\gamma^m : [0, mb] \to \mathbf{U}^\circ$ that is well-defined on the interval. Let $\Phi = \lim\limits_{m\to\infty} \gamma^m$. We call $\Phi_t : [0, \infty) \to \mathbf{U}^\circ$ a **geodesic** on the Necker cube surface.*

Definition 8. *The geodesic $\tilde{\Phi}_t : [0, \infty) \to \tilde{\mathbf{U}}^\circ$ is a geodesic obtained in a similar manner to (2) by lifting γ under r° and continuously concatenating $\tilde{\gamma}$ with the exception that the element of the deck group acting after every iteration is $(\varphi_2([\alpha]), \varphi_1([\alpha])) \in \mathrm{SO}(2, \mathbb{Z}) \times \mathbb{Z}^2$.*

The geodesic Φ obtained in this manner has a period that depends entirely on the length of the closed segment on $\mathbf{G}^\circ$. In addition, $\Phi(t+mb) = m\varphi_1([\alpha]) \cdot \Phi(t)$ and $tilde\Phi(t+mb) = (m\varphi_1([\alpha]), I_2) \cdot \tilde{\Phi}(t)$ for all $t, m \geq 0$. The second statement holds since α and γ must have trivial holonomy at their terminal points. We make the following observations about these lifts:

Theorem 1. *Φ is periodic if and only if $\tilde{\Phi}$ is periodic. Further, both geodesics have period $||\alpha||$.*

Proof. Suppose Φ is periodic. Then $\gamma(0) = \gamma(b) = u_0$. Since γ has trivial holonomy, $\varphi_2\big|_{\ker \varphi_1}([\gamma]) = I_2$, and so $[\tilde{\gamma}] \in \pi_1(\tilde{\mathbf{U}}^\circ)$. Hence $\tilde{\gamma}(0) = \tilde{\gamma}(b) = \tilde{u}_0$. Since $\tilde{\Phi}$ is obtained by concatenating $\tilde{\gamma}$ under the action of the trivial deck group element of $[\alpha]$, $\tilde{\gamma}^m = \tilde{\gamma}(t)$ for all $t \in [0, mb]$. It follows that $\tilde{\Phi}$ is periodic as well. Conversely, the proof is trivial since $\tilde{\mathbf{U}}^\circ$ is a finite degree cover of $\mathbf{U}^\circ$. Since $||\alpha|| = b$, there can be no smaller period for its lifted paths as that would imply either α has non-trivial holonomy or does not lift to a closed segment on $\mathbf{U}^\circ$. $\square$

Theorem 2. *Φ is drift-periodic if and only if $\tilde{\Phi}$ is drift-periodic. Both geodesics have period $||\alpha||$.*

Proof. Suppose Φ is drift-periodic. Then $\varphi_1([\alpha]) = (x, y)$ is non-trivial, but $\varphi_2([\alpha]) = I_2$ is. The element of the deck group acting on $\tilde{\gamma}^m(t)$ for $m > 1$ is $(I_2, n(x, y))$ for $1 < n < m$. As $t \to mb$ and $m \to \infty$, $d(\tilde{\Phi}(t), \tilde{u}_0) =$

$d((I_2, m(x,y)) \cdot \tilde{u}_0, \tilde{u}_0) \to \infty$, and $\tilde{\Phi}(t+b) = (I_2, (x,y)) \cdot \tilde{\Phi}(t)$ is periodically translated by some element in $\mathbb{Z}^2$. Conversely, if $\tilde{\Phi}$ is drift-periodic there are three cases:
Case one: $\varphi_2([\alpha])$ is trivial, but $\varphi_1([\alpha])$ is not. Then Φ must be drift-periodic as well since γ would terminate on $\varphi_1([\alpha]) \cdot u_0 \neq u_0$.
Case two: $\varphi_1([\alpha])$ is trivial, but $\varphi_2([\alpha])$ is not. This is a contradiction since that would imply α has non-trivial holonomy. Case three:$\varphi_1([\alpha]), \varphi([\alpha])$ are both non-trivial. Again this is not possible since α, γ must both have non-trivial holonomy.
Now both geodesics must have the same period since they are obtained by concatenating unique lifts of α. $\square$

These theorems essentially tell us that these kinds of geodesics behave identically on the Necker cube surface and its cover. Now we show that this is true for $\tilde{\mathbf{U}}^\circ$ and $\tilde{\mathbf{U}}$. For these proofs we will refer to the induced homomorphisms from (1):

$$\begin{tikzcd}
\pi_1(\tilde{\mathbf{U}}^\circ) \arrow[r, hook, "r_*^\circ"] \arrow[d, two heads, "PR_b"] \arrow[rrr, bend left, "f_*^\circ"] & \pi_1(\mathbf{U}^\circ) \arrow[r, hook, "q_*^\circ"] \arrow[d, two heads, "PR_c"] & \pi_1(\mathbf{G}^\circ) \arrow[d, two heads, "PR_d"] & \pi_1(\mathbf{M}^\circ) \arrow[l, hook', "w_*^\circ"'] \arrow[d, two heads, "PR_a"] \\
\pi_1(\tilde{\mathbf{U}}) \arrow[r, hook, "r_*"] \arrow[rrr, bend right, "f_*"'] & \pi_1(\mathbf{U}) \arrow[r, hook, "q_*"] & \pi_1(\mathbf{G}) & \pi_1(\mathbf{M}) \arrow[l, hook', "w_*"']
\end{tikzcd} \tag{3}$$

Definition 9. *Let $\tilde{\phi} : [0, \infty) \to \tilde{\mathbf{U}}^\circ$ be a geodesic with initial direction $v_\in$*

Theorem 3. *content...*

2.3 Translation Surface

Consider the projection $\tilde{\mathbf{U}} \to \mathbf{M}$ onto a quotient of the surface under its translational symmetries. What we obtain is the following compact translation surface belonging to the stratum $\mathcal{H}(2,2,2,2)$:

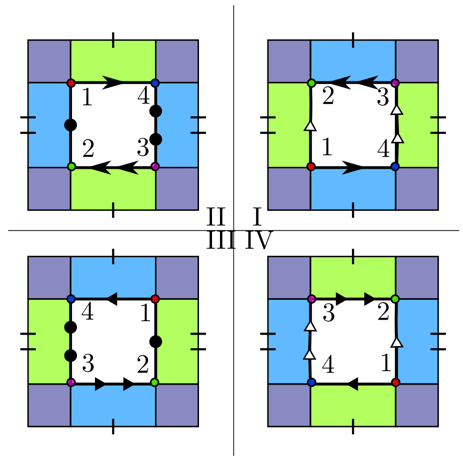


Figure 7: Compact translation surface, **M** with edges and cone singularities (1,2,3,4) identified. The Roman numerals are meant to identify every "copy" of **G** with a direction differing from v_0 by rotation of some element in $\mathrm{SO}(2,\mathbb{Z})$. The individual copies are then rotated so we have a canonical notion of direction

We rearrange **M** as such:

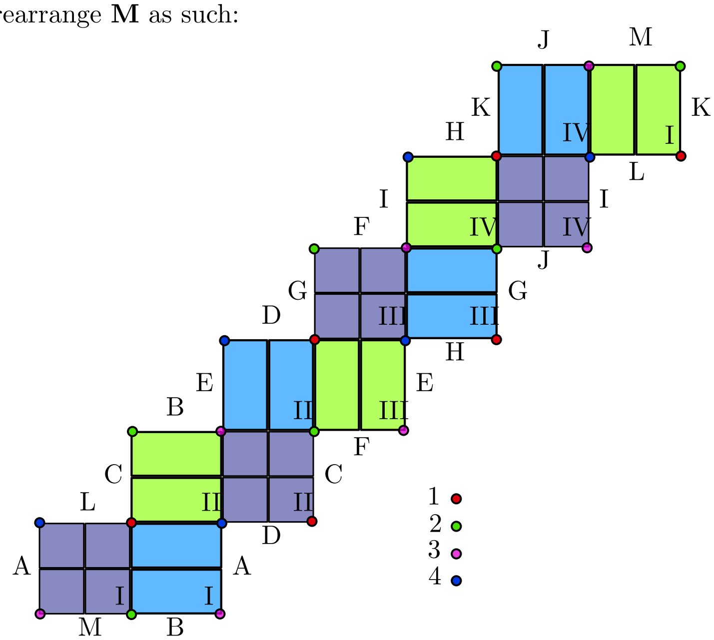


Figure 8: The staircase Veech surface with directional planes and vertices identified. All edges are paired by translation. Two adjacent squares have opposite edges identified. The top edge of the bottom-left square is glued to the bottom edge of the top-right square (both labeled I). Likewise, the bottom edge of the bottom-left square is identified with the top edge of the top-right square.

We recall some basic definitions and theorems about $\mathbb{Z}^2-$covers of translation surfaces as they apply to $\mathbf{M}$.

Definition 10. *Algebraic intersection number is a non-degenerate bilinear form:*

$$i : H_1(\mathbf{M}, \mathbb{Q}) \times H_1(\mathbf{M}, \mathbb{Q}) \to \mathbb{Q},$$

for $[\gamma], [\beta] \in H_1(\mathbf{M}, \mathbb{Q})$*,* $i([\beta], [\gamma])$ *returns the signed intersection number of two homology classes. We say a crossing at the instance of an intersection is positive if* γ *makes a positive angle relative to* β*.*

The set of all affine diffeomorphisms of $\mathbf{M}$ form the group $\mathrm{Aff}^+(\mathbf{M})$. The *Veech group* of $\mathbf{M}$ is the image in the co-domain of the homomorphism $D : \mathrm{Aff}^+(\mathbf{M}) \to SL(2, \mathbb{R})$ that takes every affine map to its derivative, which we denote $V(\mathbf{M})$. $\mathbf{M}$ is a square-tiled translation surface whose Veech group is a finite index subgroup of $SL(2, \mathbb{Z})$. We use automorphisms of the fundamental group in $\mathrm{Aut}(\pi_1(\mathbf{M}))$ induced by affine maps to prove our main results. But first, we define a group homomorphism from $\pi_1(\mathbf{M})$ to $\pi_1(\mathbf{M})/\pi_1(\tilde{\mathbf{U}})$ by algebraic intersection numbers pver a sum of linearly independent homology classes. Consider the following cylinder core curves on $\mathbf{M}$ labeled γ_i for $0 \leq i \leq 11$:

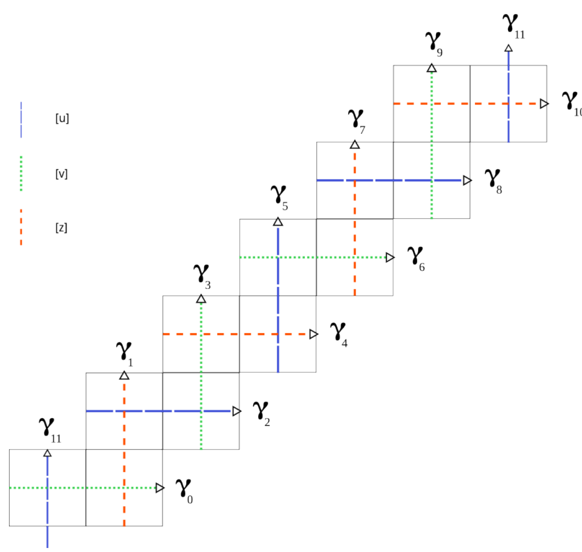


Figure 9: $\mathbf{M}$'s cylinder core curves with u,v, and z homology class labels.

Every curve intersects two others and if we take $i(\Sigma_{i=0}^{11}[\gamma_i])$ **show that $\gamma_0, \ldots, \gamma_{11}$ form a basis for $H(\mathbf{M}, \mathbb{Q})$**
show that u,v linearly independent

Definition 11. *The homology classes* u, v, z *are given as the following sums of* *core curves:*

$$u = -[\gamma_2] + [\gamma_5] + [\gamma_8] - [\gamma_{11}],$$
$$v = +[\gamma_0] - [\gamma_3] - [\gamma_6] + [\gamma_9],$$
$$z = +[\gamma_1] + [\gamma_4] - [\gamma_7] - [\gamma_{10}].$$

Definition 12. *Define the group homomorphism* $\Omega_{u,v} : \pi_1(\mathbf{M}) \to \mathbb{Q}^2$*, where* $\beta \mapsto (i(u, [\beta]), i(v, [\beta]))$*.*

Lemma 2. $\Omega_{u,v}(\pi_1(\mathbf{M})) = \mathbb{Z}^2$*.*

Lemma 3. ***show somehow that u,v determine the cover***

2.4 Induced Automorphisms of $H_1(\mathbf{M}, \mathbb{Q})$

We look at some important automorphisms induced by Affine maps on $\mathbf{M}$. Observe that $\mathbf{M}$ has a uniform cylinder decomposition in both horizontal and vertical directions as in figure 15. We define the *modulus*, μ, of a cylinder to be the ratio of the cylinder's width to its circumference, $\frac{w}{c}$. The *Dehn-twist* of a cylinder is an affine diffeomorphism that skews the cylinder and sends every vertex to itself.

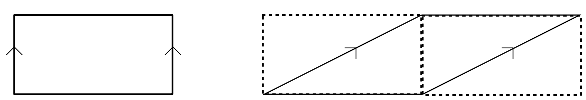


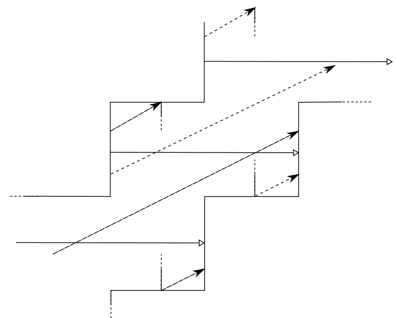
Figure 10: Dehn-twist of a cylinder in $\mathbf{M}$'s cylinder decomposition.

Such a map would have derivative $\begin{bmatrix} 1 & \pm\mu^{-1} \\ 0 & 1 \end{bmatrix}$. On $\mathbf{M}$ every cylinder in both vertical and horizontal decompositions has a modulus of $\frac{1}{2}$. These give way to global diffeomorphisms as *multi-twists* of $\mathbf{M}$.

Definition 13. *We call the global affine diffeomorphisms in* $\mathit{Aff}^+(\mathbf{M})$ *obtained as* ***multi-twists of the surface in horizontal and vertical directions*** $\mathbf{A}$ *and* $\mathbf{B}$*, respectively.*

We define the ***derivatives of*** $\mathbf{A}, \mathbf{B}$ *as the matrices* $D(\mathbf{A}) = \mathbf{A}' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ *, and* $D(\mathbf{B}) = \mathbf{B}' = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$*, where* $D : \mathit{Aff}^+(\mathbf{M}) \to V(\mathbf{M})$*.*

These are parabolic elements of $V(\mathbf{M})$ that are directly related to the dynamical dichotomy of rational trajectories on the Necker cube surface. To show this algebraically, we look at how skewing the surface affects our homology spanning core curves:



When skewing the surface under **A** in the horizontal direction, the horizontal curves are preserved, but the vertical curves (odd γ_i index) obtain two additional positive intersections with adjacent index (mod 12) horizontal curves. Similarly, the vertical **B** skews preserve the vertical curves, but the horizontal curves (even γ_i index) obtain two additional *negative* intersections with adjacent index (mod 12) vertical curves. The formulaic expressions of the $\mathbf{A}, \mathbf{B}$ induced automorphisms of $H_1(\mathbf{M}, \mathbb{Q})$ as powers of $k \in \mathbb{Z}$ are then:

$$\begin{aligned}\mathbf{A}_*^k([\gamma_i]) =& [\gamma_i] + \frac{k}{2}(1-(-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}])\\ \mathbf{B}_*^k([\gamma_i]) =& [\gamma_i] + \frac{k}{2}(1+(-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}])\end{aligned}$$

With all of these in mind we generate the following subgroups:

Definition 14. *We say* $\mathbb{X}$ *is the subgroup of* $Aff^+(\mathbf{M})$ *generated by* $\mathbf{A}$, $\mathbf{B}$.

Definition 15. *The images of* $\mathbb{X}$ *under their derivative maps to* $V(\mathbf{M})$ *is the subgroup of* $V(\mathbf{M})$ *generated by matrices* $\mathbf{A}'$, $\mathbf{B}'$. *We denote this group by* $\mathbb{X}'$.

Definition 16. *The induced automorphisms of* $\mathbb{X}$ *contained in* $Aut(H_1(\mathbf{M}, \mathbb{Q}))$ *is the subgroup generated by* $\mathbf{A}_*$, $\mathbf{B}_*$.

3 Proof of Main Theorem

$S = \mathcal{O} \cup \mathcal{E}$.

Lemma 4. *Let* $\mathbb{X}'$ *act on* $\mathcal{S}$ *by left matrix multiplication. Then* $\mathbb{X}'$ *acts faithfully and transitively on integer pairs in* $\mathcal{O}$, *partitions* $\mathcal{E}$ *into two sets and acts faithfully and transitively on them, and acts faithfully on* $\mathcal{S}$.

Proof. Let $v = kv_0 = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}$ with $||v_0|| = 1$ and $k \in \mathbb{R}$, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{X}'$. The kernel of the homomorphism $SL(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z})$ that maps a matrix to its $(mod\, 2)$ equivalent is precisely $\mathbb{X}'$ since it maps the generators trivially.

[quote this paper for the proof of this maybe? http://www.fysik.su.se/ ingemar/SL2.pdf]
x, y are relatively prime, so Bezout's identity states there exist $m, n \in \mathbb{Z}$ such that $mx + ny = 1$ $(*)$. It's obvious that this action is faithful on the sets since the stabilizer of any vector in $\mathbb{R}^2$ is trivial. Suppose $v \in \mathcal{O}$. Consider the orbit of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then there is some matrix $A \in \mathbb{X}'$ such that $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v$. Since $A \in \mathrm{SL}(2, \mathbb{Z})$, $\det A = ad - bc = 1$. Obtain from $1 = (x-b)(y-c) - bc = xy - xc - by$ that $xy - 1 = xc + by$. From $(*)$ we see that $xy - 1 = x(u(xy-1)) + y(v(xy-1))$. Let $c = u(xy-1)$ and $b = v(xy-1)$. Since $xy-1$ is even, c, b are even. It follows then that $a = x - b$ and $d = y - c$ are odd. Therefore $A \in \mathbb{X}'$ and the action is transitive on $\mathcal{O}$. If $v \in \mathcal{E}$, then either x or y is even. In either case this would mean that either a or d is even, which would mean $A \notin \mathbb{X}'$, a contradiction.
Now consider the orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Left multiplying by A would mean $a = x$ and $c = y$. Split $\mathcal{E}$ into sets $\mathcal{E}_1 = \{(x, y) \in \mathcal{E} : x \in 2\mathbb{Z},\ y \in (2\mathbb{Z}+1)\}$ and $\mathcal{E}_2 = \{(x, y) \in \mathcal{E} : x \in (2\mathbb{Z}+1),\ y \in 2\mathbb{Z}\}$. Since $\det A = 1 = da - bc = dx - by = mx + ny$, we have that $d = m$ and $b = -n$. We rule out when $v \in \mathcal{O}, \mathcal{E}_1$ since $c \neq 0(\mod 2)$ in those cases. Consider $v \in \mathcal{E}_2$. Then x is odd and y is even. d must be odd since $mx + ny = 1$ and y is even. And b must be even since $ad - bc = 1$
finish this and say the proof for $\mathcal{E}_1$ is analogous.

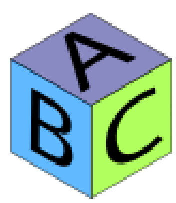
$\square$

Lemma 5. *Let $v_0\mathbb{R}^2$. Then for any geodesic α*

3.1 empty

Consider the following families of unit squares in $\mathbb{R}^3$:

$$\begin{aligned} \mathbf{A}_{m,n,p} &= [m, m+1] \times [n, n+1] \times \{p\}, \\ \mathbf{B}_{m,n,p} &= \{m+1\} \times [n, n+1] \times [p-1, p], \\ \mathbf{C}_{m,n,p} &= [m, m+1] \times \{n+1\} \times [p-1, p]. \end{aligned}$$



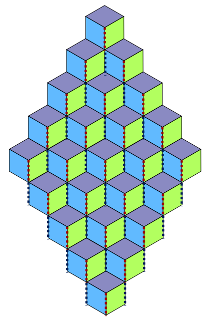


Figure 11: A section of **S**

3.2 Flattening S

At the moment it is difficult to describe how the parallel transport of a vector tangent to the surface along an arbitrary path on the surface acts on the vector in $\mathbb{R}^3$. To simplify this problem we take $\mathbf{S}$ to an isometric variant embedded in $\mathbb{R}^2$ by piecewise linear transformations on the sets

$$\mathbf{A} = \bigcup \{\mathbf{A}_{m,n,p} : m+n+p=0\},$$
$$\mathbf{B} = \bigcup \{\mathbf{B}_{m,n,p} : m+n+p=0\},$$
$$\mathbf{C} = \bigcup \{\mathbf{C}_{m,n,p} : m+n+p=0\}.$$

Definition 17. *Let $\Psi : s^*(\mathbf{S}) \to \mathbb{R}^3$ be given as*

$$\Psi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ z - \lfloor z \rfloor \end{bmatrix} & \text{if } (x,y,z) \in \mathbf{A} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ x - \lfloor x \rfloor \end{bmatrix} & \text{if } (x,y,z) \in \mathbf{B} \backslash (\mathbf{A} \cup \mathbf{C}) \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x - \lfloor x \rfloor \\ y - \lfloor y \rfloor \\ -(z - \lfloor z \rfloor) \end{bmatrix} + \begin{bmatrix} 2\lfloor x \rfloor - \frac{3}{2} \\ 2\lfloor y \rfloor - \frac{3}{2} \\ y - \lfloor y \rfloor \end{bmatrix} & \text{if } (x,y,z) \in \mathbf{C} \backslash (\mathbf{A} \cup \mathbf{B}) \end{cases} \tag{4}$$

This map behaves as such:

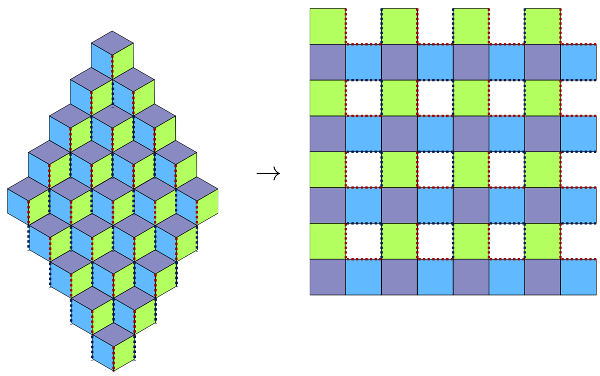


Figure 12: An isometry of the surface $\mathbf{S}$.

Lemma 6. $\Psi \circ s^*$ *is an isometry of the surface* $\mathbf{S}$

Proof. Ψ is well-defined on its domain $s^*(\mathbf{S})$, and s^* is an embedding of the surface that is a bijection when restricting its co-domain to its image. Since Ψ is a linear transformation composed of Euclidean matrices and translations it is also invertible and bijective. ☐

3.3 Translation Surface Cover of $\mathbf{G}^\circ$

3.4 A Four-Fold Cover of $\mathbf{U}^\circ$

Definition 18. $\mathbf{U}^\circ$ *is the cover of* $\mathbf{G}^\circ$ *such that* $\pi_1(\mathbf{U}^\circ) = \ker\varphi$.

Our cover of $\mathbf{G}^\circ$ has the fundamental group, $\pi_1(\mathbf{U}^\circ) = \langle\langle a^{-1}b^{-1}ab, c, d\rangle\rangle$, the conjugate subgroup of elements that map trivially under φ_1. The covering map $k_{\varphi_1} : \mathbf{U}^\circ \hookrightarrow \mathbf{G}^\circ$ takes every point on $\mathbf{U}^\circ$ to its modular equivalent under these translational symmetries. $\mathbf{U}^\circ$ is realized as an infinite $\mathbb{Z}^2$-tiling of $\mathbf{G}^\circ$ in $\mathbb{C}$:

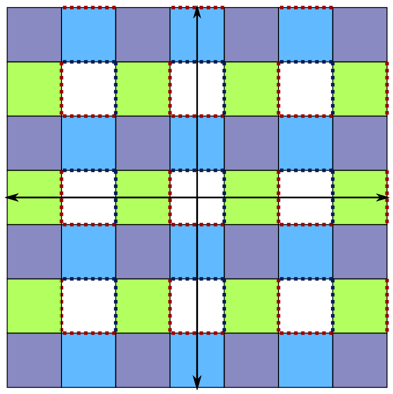


Figure 13: The infinite surface $\mathbf{U}^\circ$ embedded in $\mathbb{C}$ with edges identified.

Definition 19. $\mathbf{U}$ *is the completion of* $\mathbf{U}^\circ$ *that includes the branched vertices.*

Remark. *The surface* $\mathbf{G}$ *is homeomorphic to the torus, and its cover* $\mathbf{U}$ *is its universal cover as the paths labeled* c, d *are only non-trivial when* $\mathbf{G}$ *is branched at its three conical points.*

3.5

3.6 Isometry From S to U

The Necker cube surface can be *flattened* onto the plane by piecewise isometric maps onto a subspace of $\mathbb{R}^3$ with (open) unit squares removed at every even integer pair in the plane, what we claim to be $\mathbf{U}$.
The red/blue dotted lines represent the edges that are split on the plane. The map from one surface to the other is composed of piecewise isometries, $\Psi : \mathbf{S} \to \mathbb{R}^3$.

The flattened surface is contained entirely in $(x, y, 0) \in \mathbb{R}^3$, which is isometric to $\mathbb{C}$. $\mathbf{U}$ is recovered as a topological quotient on the domain

$$\mathbf{P} = \mathbb{C} \setminus \bigcup_{m,n\in\mathbb{Z}} \big\{u + vi : u \in (2m - \frac{1}{2}, 2m + \frac{1}{2}),\ v \in (2n - \frac{1}{2}, 2n + \frac{1}{2})\big\}. \quad (5)$$

Definition 20. $\mathbf{U}$ *is the surface obtained as the topological quotient* $\mathbf{P}/\sim_{\mathbf{P}}$, *where* $\mathbb{C}$ *is identified with* $\mathbb{R}^2$ *in the usual way and* $\sim_{\mathbf{P}}$ *is a minimal relation on* $\mathbf{P}$ *defined as follows:*

$$\text{Let } x_0 = (u_0, v_0), x_1 = (u_1, v_1) \in \mathbf{P}.\ \sim_{\mathbf{P}} \text{ is given as the relation}$$
$$x_0 \sim_{\mathbf{P}} x_1 \text{ iff } x_0 = x_1$$
$$\text{or, for some } m, n \in \mathbb{Z}\ x_0, x_1 \in \partial\left(\left[2m - \frac{1}{2}, 2m + \frac{1}{2}\right] \times \left[2n - \frac{1}{2}, 2n + \frac{1}{2}\right]\right)$$
$$\begin{bmatrix} u_1 - 2m \\ v_1 - 2n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_0 - 2m \\ v_0 - 2n \end{bmatrix}.$$

Remark. *Our map* Ψ *induces an isometry between* $\mathbf{S}$ *and* $\mathbf{U}$, *and its restriction to the branched surfaces induces an isometry between* $\mathbf{S}^\circ$ *and* $\mathbf{U}^\circ$. *This follows from these piecewise Euclidean transformations that preserves our induced flat metric.*

From here on we use $\mathbf{U}^\circ$ and $\mathbf{U}$ instead of $\mathbf{S}$ and $\mathbf{S}^\circ$.

3.7 Four-Fold Cover

We denote the *unit tangent bundle* on the surface $\mathbf{U}^\circ$ by $T^1\mathbf{U}^\circ$, and construct a cover with trivial holonomy. Initial experiments have shown us that any geodesic viewed as discontinuous line segments on $\mathbf{P}$ moves in at most four directions.

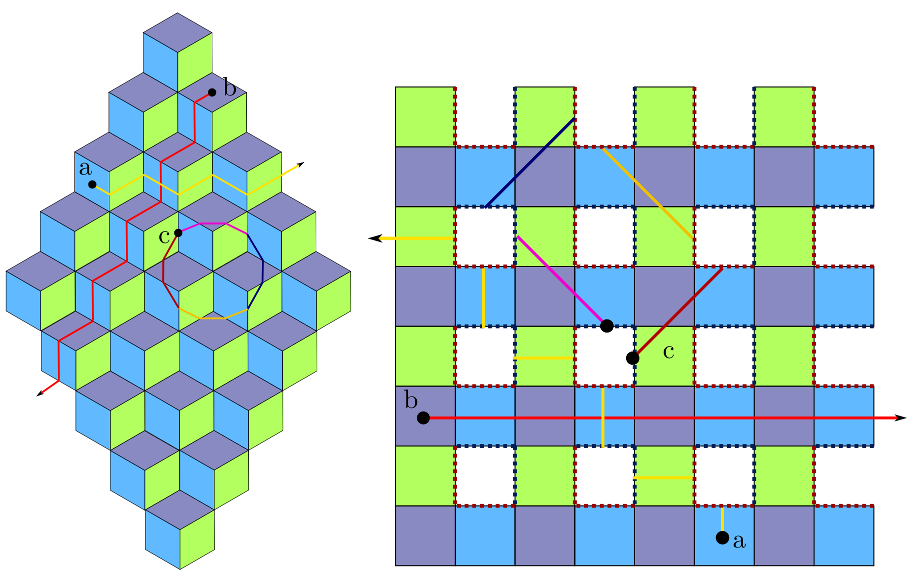


Figure 14: Simple periodic (c) and drift-periodic (a,b) trajectories represented as line segments on $\mathbf{S}$ and $\mathbf{U}$.**RELABEL THESE**

Observe that the parallel transport of a vector around a closed loop on $\mathbf{U}^\circ$ will act on vectors tangent to the surface by a rotation of an integer multiple of $\frac{\pi}{2}$ radians (since the surface is flat and embedded in $\mathbb{C}$, we work with vectors in $\mathbb{R}^2$).

Definition 21. *Define $\varphi_2 : \pi_1(\mathbf{G}^\circ) \to SO(2,\mathbb{Z})$ as the group homomorphism on generators a, b, c, d such that:*

$$a, b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \quad c \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = R, \quad d \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = R^3$$

The homomorphism restricted to $\pi_1(\mathbf{U}^\circ)$ factors through $\pi_1(\mathbf{G}^\circ)$ and we show that:

Lemma 7. *The group $\varphi_2(\pi_1(\mathbf{U}^\circ, x_0)) = Hol_p(\omega)$.*

Proof. Since $c \mapsto R$ and $d \mapsto R^3$, the image of φ_2 is generated by R, R^3 and isomorphic to $SO(2,\mathbb{Z})$. We see from figure 14 that parallel transports of vectors by non-trivial paths produce clockwise/counter-clockwise rotations equal to that of of the cone angles of the singularities they loop around, all integer multiples of $\frac{\pi}{2}$ that coincide with paths generated by elements c, d. Further, if there is an element in $\pi_1(\mathbf{U}^\circ, x_0)$ conjugated by $[a, b]$, the effect this would have on holonomy would be trivial as the total cone angle sum around each cut out square is an integer multiple of 2π, which agrees with a and b's trivial images under ϕ_2. $\square$

Hence, we obtain a natural monodromy representation with the map $m : \pi_1(\mathbf{U}^\circ, x_0) \to SO(2,\mathbb{Z}) \cong Hol_p(\omega)$, where for $[\gamma] \in \pi_1(\mathbf{U}^\circ, x_0)$ we have that $\gamma^*(1) = (x_0, m([\gamma])v_0)$. It follows that since ω is a flat connection, trivial loops have trivial holonomy and $Hol_p(\omega)$ acts on $H(p)$.

Definition 22. *Let $\tilde{\mathbf{U}}_0^\circ$ be the cover of $\mathbf{U}^\circ$ with fundamental group $\pi_1(\tilde{\mathbf{U}}_0^\circ) = \ker \varphi_2|_{\pi_1(\mathbf{U}^\circ)} \trianglelefteq \pi_1(\mathbf{U}^\circ)$.*

Lemma 8. *Let $\Phi_t : \mathbf{U}^\circ \times \mathbb{R} \to \mathbf{U}^\circ$ be a unit-speed geodesic flow on $\mathbf{U}^\circ$, with a parallel transport map induced by ω. Then the following is true:*

(i) Φ is periodic if and only if its lift to $\tilde{\mathbf{U}}_0^\circ$ is.

(ii) Φ is drift-periodic if and only if its lift to $\tilde{\mathbf{U}}_0^\circ$ is.

Proof. Let $x_0 = \Phi(0)$ be an initial point in $\mathbf{U}^\circ$, and let $v_0 = \frac{d}{dt}\Phi|_{t=0} \in \mathbb{R}^2$ be its initial direction. Then $\Phi^* : T^1\mathbf{U}^\circ \times \mathbb{R} \to T^1\mathbf{U}^\circ$ is a well-defined lift to the unit tangent bundle with initial point $p = (x_0, v_0)$.
(i). Suppose Φ is periodic with period $T \in \mathbb{R}$. Then Φ is reparameterized as $\alpha : [0,1] \to \mathbf{U}^\circ$, where $[\alpha] \in \pi_1(\mathbf{U}^\circ, x_0)$. It follows then that if α does not lift to a closed path, then α must have non-trivial holonomy since $[\alpha] \notin \ker \varphi_2$. That is when lifted to the unit tangent bundle with base point p, $\alpha^*(1) \neq p$ since $m([\alpha]) \neq I_2$. But this implies that $\Phi^*(T) \neq p$, which is impossible. Hence

$[\alpha] \in \ker \varphi_2 = \pi_1(\tilde{\mathbf{U}}_0^\circ, \tilde{x}_0)$, where $\tilde{x}_0$ belongs to the fiber over x_0 under the covering map $\tilde{\mathbf{U}}_0^\circ \hookrightarrow \mathbf{U}^\circ$. The converse holds trivially.
(ii). Suppose Φ is drift-periodic with period $T \in \mathbb{R}$ and non-trivial $f : \mathbf{U}^\circ \to \mathbf{U}^\circ \in Trans(\mathbf{U}^\circ) \cong \pi_1(\mathbf{G}^\circ)/\pi_1(\mathbf{U}^\circ) \cong \mathbb{Z}^2$ such that $\Phi(T) = f(x_0)$. Reparameterize this as $\alpha : [0,1] \to \mathbf{U}^\circ$ with $[\alpha] \in \pi_1(\mathbf{G}^\circ)$. When lifted to $T^1\mathbf{U}^\circ$, $\Phi^*(0) = p$ and $\Phi^*(T) = (f(x_0), v_0)$. Let $\alpha : [0,1] \to \mathbf{U}^\circ$ be its reparameterization up to time T. Thus $[\alpha] \notin \pi_1(\mathbf{U}^\circ, x_0), \pi_1(\tilde{\mathbf{U}}_0^\circ, \tilde{x}_0)$. Further, f has a unique, lift to $\tilde{f} \in Trans(\tilde{\mathbf{U}}_0^\circ)$ because the space is connected. (a bit stuck here).
Conversely, ... $\square$

A visual representation of $\tilde{\mathbf{U}}_0^\circ$ is as a four-fold cover of $\mathbf{U}^\circ$ with trivial holonomy on all closed paths. Arbitrary paths do *not* have trivial holonomy:

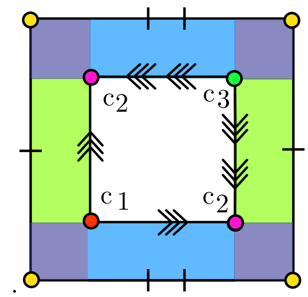


Figure 15: A 2×2 cut out section centered at each missing square. Edges and vertices identified.

3.8 Translation Surface

A better picture of $\tilde{\mathbf{U}}_0^\circ$ is obtained by making cyclic edge identifications on $\mathbf{P}' = \mathbf{P} \times \mathbb{Z}/4\mathbb{Z}$. This comes from $\tilde{\mathbf{U}}_0^\circ$ inheriting the topological properties of $\mathbf{U}^\circ \times \pi_1(\mathbf{U}^\circ)/\pi_1(\tilde{\mathbf{U}}_0^\circ) \cong \mathbf{U}^\circ \times SO(2, \mathbb{Z})$.

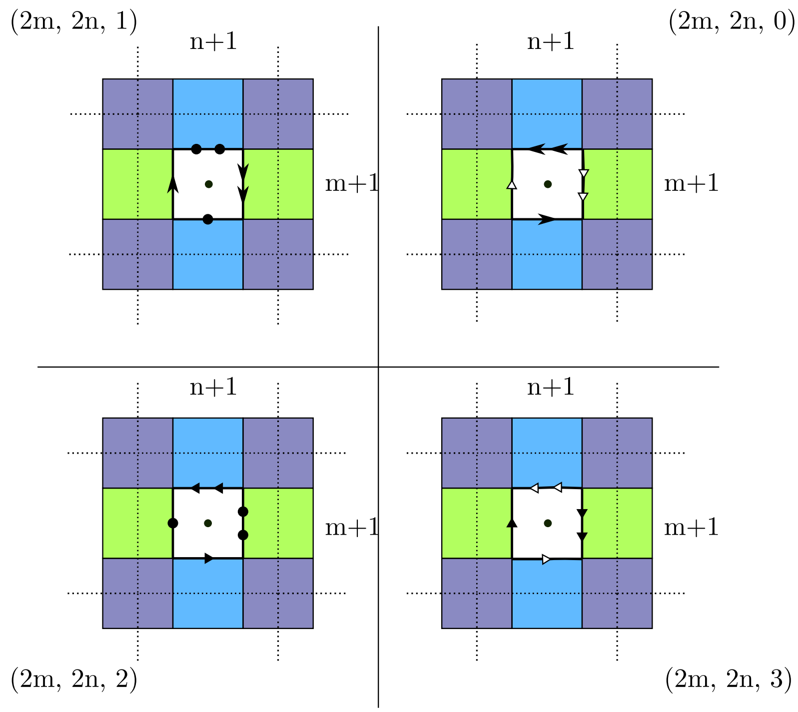


Figure 16: Branched cover of $\mathbf{U}$ of degree four.

Definition 23. *$\tilde{\mathbf{U}}_0$ is the surface obtained as the quotient $\mathbf{P}'/\sim_{\mathbf{P}'}$. Denote paths ζ and η on $\mathbf{P}$, parameterized by integers $m, n \in \mathbb{Z}$ and $t \in [-\frac{1}{4}, \frac{1}{4}]$, and defined:*
$\zeta(t) = (2m + t) + i(2n - \frac{1}{2})$,
$\eta(t) = (2m + t) + i(2n + \frac{1}{2})$.
Let $\aleph = 2m + i2n \in \mathbb{C}$. The minimal relation $\sim_{\mathbf{P}'}$ is given as:

$$\begin{aligned} (\zeta(t); j) &\sim_{\mathbf{P}'} (e^{i\frac{\pi}{2}}\ \overline{\zeta(t) - \aleph}; j+1) \\ (\eta(t); j) &\sim_{\mathbf{P}'} (e^{i\frac{\pi}{2}}\ \overline{\eta(t) - \aleph}; j+1) \end{aligned} \tag{6}$$

This relation is similar to $\sim_{\mathbf{P}}$ in that it translates line segments of squares surrounding even integer pairs to the origin and relates points on one edge of a square to points on an adjacent edge. Where it differs is that these adjacent edges now belong to a different "copy" of $\mathbf{P}$. It is in this cyclic manner that edges are glued that allows for trivial linear holonomy on arbitrary paths by rotating each copy of $\mathbf{P}$ accordingly. For example, here is a path on a section of the surface in the neighborhood of $P = 2m + i2n \in \mathbb{C}$ $(m, n \in \mathbb{Z})$ after rotation: and its subsequent projection onto $\mathbf{U}^{\circ}$:

Definition 24. *Let $r : \mathbf{P}' \to \mathbf{P}'$ be the isometric map of $\tilde{\mathbf{U}}_0$'s domain given $r(z; \bar{j}) = (e^{(-j)i\frac{\pi}{2}} z; \bar{j})$.*

Observe that when r acts on the relations (6) we get the following:

$$\begin{aligned}(e^{(-j)i\frac{\pi}{2}}\zeta(t);j) \sim_{r(\mathbf{P}')} (e^{(-j)i\frac{\pi}{2}}\ \zeta(t)-\aleph;j+1)\\ (e^{(-j)i\frac{\pi}{2}}\eta(t);j) \sim_{r(\mathbf{P}')} (e^{(-j)i\frac{\pi}{2}}\ \eta(t)-\aleph;j+1)\end{aligned} \tag{7}$$

$\tilde{\mathbf{U}}$ is then recovered as $r \cdot (\tilde{\mathbf{U}}_0) = \mathbf{P}'/\sim_{r(\mathbf{P}')}$, where all identifications are translations made on copies of $\mathbf{P}$.

3.9 Translation Surfaces and $\mathbb{Z}^2$-Covers

Definition 25. *Algebraic intersection number is a non-degenerate bilinear form:*

$$i : H_1(S, \mathbf{R}) \times H_1(S, \mathbf{R}) \to \mathbf{R},$$

where $\mathbf{R}$ *is a ring and for* $[\gamma], [\beta] \in H_1(S, \mathbf{R})$, $i([\beta], [\gamma])$ *returns the intersection number of two homology classes.*

Algebraic intersections are signed and follow some convention such as the right-hand rule.

Definition 26. *Let* $u, v \in H_1(S, \mathbb{Q})$ *be linearly independent homology classes of curves on* S. *Then the group homomorphism from* $\pi_1(S)$ *to* $\mathbb{Q}^2$ *is given as:*

$$\Omega_{u,v} : \pi_1(S) \to \mathbf{R}^2;\ \beta \mapsto (i(u, [\beta]), i(v, [\beta])).$$

The set of all orientation-preserving affine diffeomorphisms of S forms the group $\mathrm{Aff}^+(S)$. The corresponding *Veech group*, $V(S)$ of S is the image of the group morphism $D : \mathrm{Aff}^+(S) \to SL(2, \mathbb{R})$ that takes an affine map to its derivative. A surface is said to be *Veech* if its Veech group is commensurable to $SL(2, \mathbb{R})$. It is well known that origami, or square-tiled, surfaces have Veech groups commensurable to $SL(2, \mathbb{Z})$. [cite] When $\mathbf{R} = \mathbb{Z}$, $\Omega_{u,v}$ takes an element of $\pi_1(S)$ to Δ. Thus, $\gamma \in \pi_1(S)$ lifts to $\tilde{\gamma} \in \pi_1(\tilde{S})$ if and only if $\gamma \in \ker \Omega_{u,v}$.

Definition 27. *Let* $\alpha : [0, 1] \to S$ *be a closed, non-singular geodesic path on* S. *The holonomy map* $\mathbf{hol} : H_1(S, \mathbf{R}) \to \mathbb{C}$ *returns the holonomy vector of a closed path as a difference of the starting and endpoints of a flow by*

$$\mathbf{hol}([\alpha]) = \int_\alpha dz.$$

Since α is non-singular, it can be mapped to S° which admits a flat holomorphic one-form dz. Let $\theta = Arg(\mathbf{hol}([\alpha]))$. We say that $\phi_t^\theta : \mathbb{R} \times S^\circ \to S^\circ$ is the unit-speed geodesic flow on S° in direction θ given by the $[\alpha]$ such that $\phi_0^\theta = \alpha(0)$. In local coordinates this corresponds to $z + te^{i\theta} \in \mathbb{C}$.

Lemma 9. ϕ_t^θ *has a period of* $T = |\mathbf{hol}([\alpha])|$

Proof. This just follows from the fact that ϕ_t^θ flows at unit-speed in the direction of $\frac{\mathbf{hol}([\alpha])}{|\mathbf{hol}([\alpha])|}$. $\square$

And so the length of a vector determines the period of a flow on S. More importantly, we have the following:

Lemma 10. *Denote the lifted flow of ϕ_t^θ on $\tilde{S}$ by $\tilde{\phi}_t^\theta$. Then $\tilde{\phi}_t^\theta$ is periodic on $\tilde{S}$ and $\tilde{S}^\circ$ if and only if $[\alpha] \in \ker \Omega_{u,v}$. Further, $\tilde{\phi}_t^\theta$ has period $T = |\mathbf{hol}([\alpha])|$.*

Proof. Suppose that $[\alpha] \notin \ker \Omega_{u,v}$. Since $\alpha([0,1]) = \phi^\theta_{[0,\mathbf{hol}([\alpha])]}$, their homology classes are equivalent. Hence $\tilde{\phi}_t^\theta$ could not close on $\tilde{S}$ or $\tilde{S}^\circ$, or else $\Omega_{u,v}(k[\alpha]) = k(i(u,[\alpha]), i(v,[\alpha])) = (0,0)$ for some non-zero $k \in \mathbb{Z}$. Conversely, suppose $[\alpha] \in \ker \Omega_{u,v}$. Then $\square$

Corollary 1. *If $[\alpha] \notin \ker \Omega_{u,v}$, then $\tilde{\phi}_t^\theta$ is drift-periodic with period $T = \mathbf{hol}([\alpha])$.*

Proof. This follows immediately from the previous lemma since $\tilde{S}, \tilde{S}^\circ$ have translational $\mathbb{Z}^2$ symmetries and a well-defined $\mathbb{Z}^2$ action on an element in the fiber of a basepoint in S under f. The period is T since a geodesic closes on S with period T. $\square$

Lemma 11. *Let $h \in \mathit{Aff}^+(S)$. If $h(\beta) = \alpha$ for closed geodesics α, β on S and β lifts to a closed path on $\tilde{S}$, then so does α. (not sure about this)*

Proof. If the premise is true, then $[\beta] \in \ker \Omega_{u,v}$. Denote the group automorphism induced by h as h_*. Then $\Omega_{u,v}([\alpha]) = \Omega_{u,v}([h(\beta)]) = \Omega_{u,v}(h_* \cdot [\beta]) = (i([\beta], h_*^{-1} \cdot u), i([\beta], h_*^{-1} \cdot v)) = (0,0)$ as automorphisms . $\square$

Lemma 12. *Let $\beta = h(\alpha)$ as before. If $D(h) = h' \in V(S)$ and S is Veech, then $\tilde{\phi}_t^\theta$ has period $T = h' \cdot \mathbf{hol}([\beta])$. (also not sure)*

Proof. $\square$

3.10 Symmetries of M

4 Four-fold Cover of U

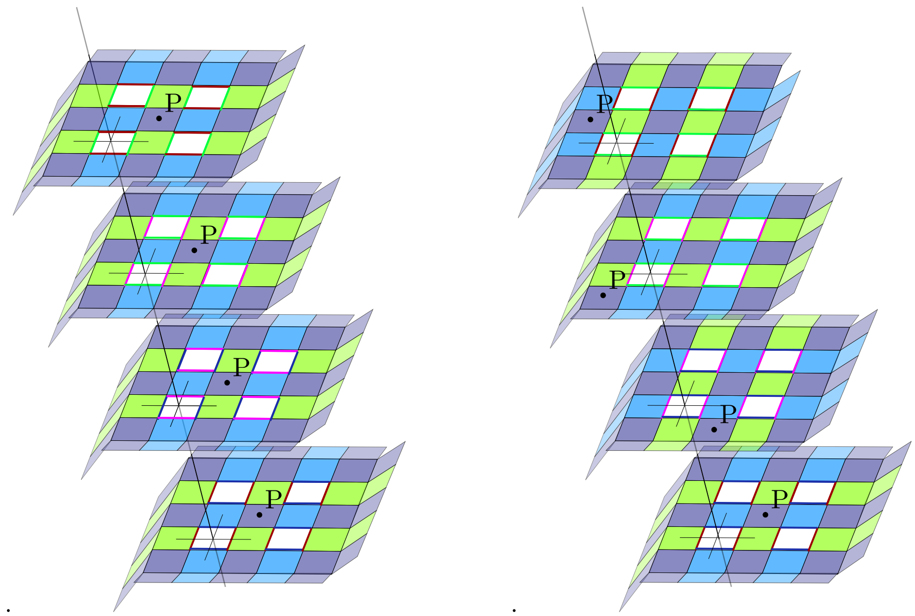


Figure 17: Four-fold cover isometry and the preimage of a point in $\mathbf{U}\backslash Sing(\mathbf{U})$.

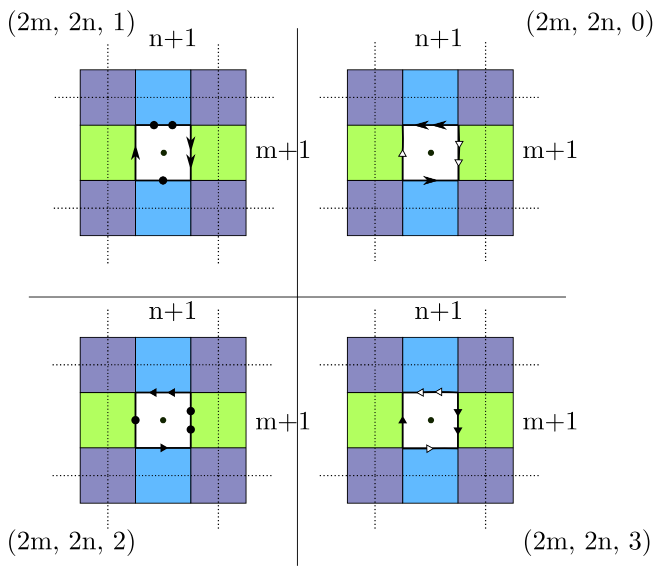


Figure 18: Branched cover associating every direction with one plane.

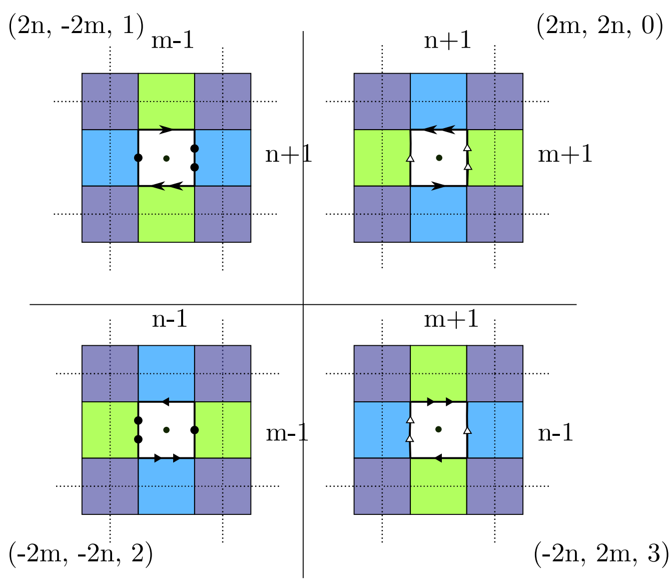


Figure 19: Infinite-type translation surface obtained by rotating each copy of the fundamental domain accordingly.

The quotient under the group action of translational symmetries is isomorphic to $\mathbb{Z}^2$ since the orbit of any point in the fundamental domain is a lattice in the space.

Theorem 4. *The translational symmetries of* $\tilde{\mathbf{U}}$*'s fundamental domain induce symmetries on the surface isomorphic to* $\mathbb{Z}^2$*.*

Proof. Let $(z;j) \in \mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$ and define the group action $T_0^{m,n} : \mathbb{C} \times \mathbb{Z}/4\mathbb{Z} \to \mathbb{C} \times \mathbb{Z}/4\mathbb{Z}$ as $T^{m,n}(z;j) = (z + 2e^{-j\frac{i\pi}{2}}(m+in);j)$. This translation acts faithfully on the preimages of $\mathbf{U}\backslash Sing(\mathbf{U})$, and respects edge identifications of $\tilde{\mathbf{U}}$, thereby making it an isometry of the surface. Consider a group homomorphism, $T_0^{m,n} \mapsto m+in$ onto the plane of Gaussian integers, $\mathbb{Z}[i]$. The exponential function is never zero, so the identity of the translation group is $T_0^{0,0}$. This is an isomorphism since it is clearly surjective and any non-trivial element of $T^{m,n}$ could not possibly map to the identity element of $\mathbb{Z}[i]$, regardless of the value of j. Since $\mathbb{Z}^2$ is isomorphic to $\mathbb{Z}[i]$, it is isomorphic to $T_0^{m,n}$ as well. $\square$

Definition 28. *The automorphism* $T^{m,n} : \tilde{\mathbf{U}} \to \tilde{\mathbf{U}}$ *is an* ***induced translation*** *of* $\tilde{\mathbf{U}}$ *as a result of the previous theorem.*

This surface is obtained as a ramified cover of the unit square torus. It is a translation surface and is therefore equipped with a **holomorphic one-form**, a collection of charts from neighborhoods of $\mathbf{M}$ to $\mathbb{C}$ such that any neighborhood away from Sing($\mathbf{M}$) has a *flat* induced Euclidean metric. A theorem of Gutkin and Judge tells us that its Veech group is commensurable to $\mathrm{SL}(2,\mathbb{Z})$ and is therefore a Veech surface. We look at some of its affine maps, and generate a subgroup $\mathbb{X} \subset \mathrm{Aff}^+(\mathbf{M})$ by the following transformations:

(i) Multi-twists of the surface as global diffeomorphisms given by Dehn-twists of its cylinder decomposition in horizontal and vertical directions with derivatives:
$$\left\{ \begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix} \right\}.$$
we call $\mathbf{A}^{\pm 1}$, $\mathbf{B}^{\pm 1}$, respectively.
A Dehn-twist on each cylinder in the cylinder decomposition of $\mathbf{M}$ in horizontal and vertical directions gives way to these global affine diffeomorphisms:

(ii) Rotation group generated by a $+\frac{\pi}{2}$ rotation of the surface fixed about the center of the second square on the bottom of the staircase, an order four isometry on $\mathbf{M}$ denoted $\mathbf{R}$.

(iii) Order 2 translation of the surface that moves the bottom left-most square to the square right next to it, denoted $\mathbf{H}$.

(iv) Order 2 translation of the surface that takes the bottom right-most square to the one right above it, denoted $\mathbf{V}$.

Definition 29. *The group* $\mathbb{X}$ *is the isometry group generated by affine maps* $\mathbf{A}$,$\mathbf{B}$,$\mathbf{R}$*,* $\mathbf{H}$*, and* $\mathbf{V}$*. The image of the derivative map on elements in* $\mathbb{X}$ *is* *denoted $\mathbb{X}'$ and generated by matrices*

$$\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix},$$

denoted $\mathbf{A}'$, $\mathbf{B}'$, *and* $\mathbf{R}'$ *in that order.*

It is not immediately apparent if these affine maps generate $\mathrm{Aff}^+(\mathbf{M})$, or if their derivatives generate $V(\mathbf{M})$, its Veech group. We use these to induce homomorphisms on $H_1(X, \mathbb{Q})$. A spanning set of $H_1(\mathbf{M}, \mathbb{Q})$ is obtained as the set of homology classes of the core curves of X's cylinder decompositions in both vertical and horizontal directions:

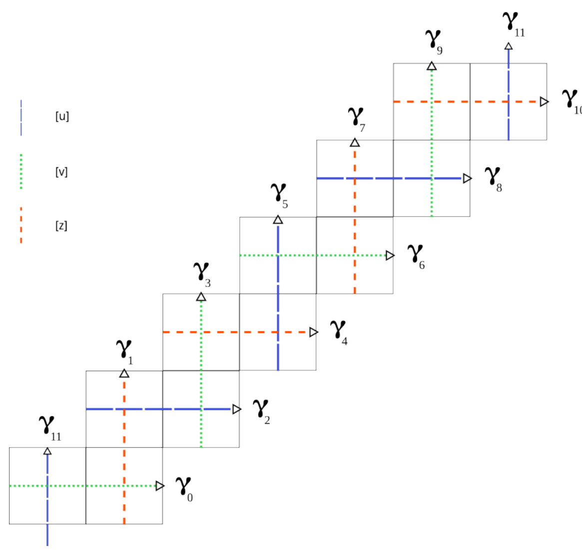


Figure 20: Cylinder core curves with u,v, and z homology classes that determines the $\mathbf{Z}^2$-cover.

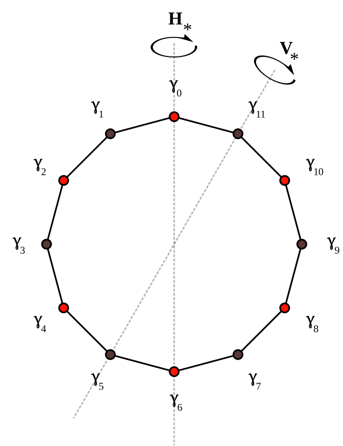
Definition 30. *The set of abelianized cylinder core curves is denoted as* $\Gamma = \{\gamma_i : i = 0, \ldots, 11\} \subset H_1(\mathbf{M}, \mathbb{Q})$.

Remark. *We use 12 elements to span homology, although a basis requires only 10. It's not impossible to determine the relations between these core curve classes, but it is not necessary. A* 12×12 *matrix of these core curve cylinder decompositions to their intersection numbers with adjacent curves is rank 10, as to be expected.*

The induced homomorphisms of $H_1(\mathbf{M}, \mathbb{Q})$ have come from affine maps that have various effects on these core-curves. We use a 12-gon to represent the set of curves, and show how these elements act on them. The multi-twists add curves to adjacent curves, and the translation maps permute them. The reader is encouraged to check these for themselves.

e.g. for $\mathbf{H}, \mathbf{V} \in \mathrm{Aff}^{+}(\mathbf{M})$,

H & V The effect that these two translations have on the 12-gon is a reflection about these lines. Observed by keeping track of the squares and core curves after $\mathbf{H}$ and $\mathbf{V}$ have acted on $\mathbb{X}$.



Definition 31. *The induced homomorphisms of $H_1(\mathbf{M}, \mathbb{Q})$ are obtained from the affine subgroup $\mathbb{X}$ and denoted $\mathbb{X}_*$. The associated homomorphisms on the spanning set Γ are given as:*

$$
\begin{aligned}
\mathbf{A}_*^k \circ [\gamma_i] =& [\gamma_i] + \frac{k}{2}(1 - (-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}]) \\
\mathbf{B}_*^k \circ [\gamma_i] =& [\gamma_i] + \frac{k}{2}(1 + (-1)^i)([\gamma_{i-1}] + [\gamma_{i+1}]) \\
\mathbf{R}_* \circ [\gamma_i] =& (-1)^i [\gamma_{1-i \ mod \ 12}] \\
\mathbf{H}_* \circ [\gamma_i] =& [\gamma_{12-i \ mod \ 12}] \\
\mathbf{V}_* \circ [\gamma_i] =& [\gamma_{10-i \ mod \ 12}]
\end{aligned}
$$

Definition 32. *The homology classes u, v, z are given as the following sums of core curves:*

$$
\begin{aligned}
[u] &= -[\gamma_2] + [\gamma_5] + [\gamma_8] - [\gamma_{11}], \\
[v] &= +[\gamma_0] - [\gamma_3] - [\gamma_6] + [\gamma_9], \\
[z] &= +[\gamma_1] + [\gamma_4] - [\gamma_7] - [\gamma_{10}].
\end{aligned}
$$

Theorem 5. *The fundamental group of the $\mathbb{Z}^2$-cover is obtained by lifting the kernel of the closed paths of $\mathbf{M}$ of the homomorphism:*

$$\Omega_{u,v} : \pi_1(\mathbf{M}, x_0) \to \mathbb{Z}^2; \ \beta \mapsto (i(u, [\beta]), i(v, [\beta])), \ \textit{where}$$

$$i : H_1(\mathbf{M}, \mathbb{Q}) \times H_1(\mathbf{M}, \mathbb{Q}) \to \mathbb{Z}.$$

is the intersection number of two homology classes.

Proof. We know from Theorem 4 that the translational symmetries of $\tilde{\mathbf{U}}$ induced by $T^{m,n}$ is isometric to $\mathbb{Z}^2$. Since $\mathbf{M}$ is a genus 5 base surface, we know that $\pi_1(\mathbf{M}, x_0) \simeq \mathbb{Z}^{10}$, and the associated cover satisfies $\mathbf{M} = \tilde{\mathbf{U}}/(\pi_1(\mathbf{M}, x_0)/N)$, such that N is a normal subgroup of $\pi_1(\mathbf{M}, x_0)$. This means that $N \simeq \mathbb{Z}^8$. The eight core curve classes are the abelianized forms of $\gamma_0, \gamma_2, \gamma_3, \gamma_5, \gamma_6, \gamma_8, \gamma_9$, and γ_{11} that span N. The classes and their signs are obtained from Figure 7 as the outer regions identified by the translations of $T^{m,n}$. Thus any closed path on $\mathbf{M}$ is lifted to a closed path on the cover under the quotient map only when a path has a trivial intersection number with the classes. □

Two paths are homologous if they return the same intersection number with the classes of closed core cylinder curves of $\mathbf{U}$ that span its homology. The classes u and v are obtained from the group group action of $T^{m,n}$ on the cover.

Definition 33. $\mathbf{hol} : \mathbf{M}\backslash Sing(\mathbf{M}) \to \mathbb{C}$ *is the holonomy vector pulled back from a non-singular path* γ *in* $\mathbf{M}$ *onto the complex plane given by* $\mathbf{hol}(\gamma) = \int_\gamma dz$.

We denote the **closed path** α, such that $\mathbf{hol}(\alpha) = 6 + 6i$, and show it is homologous to the closed geodesic with the same holonomy vector. The slope one direction also decomposes $\mathbf{M}$ into two cylinders by a series of saddle connections of length $\sqrt{2}$ between singularities:

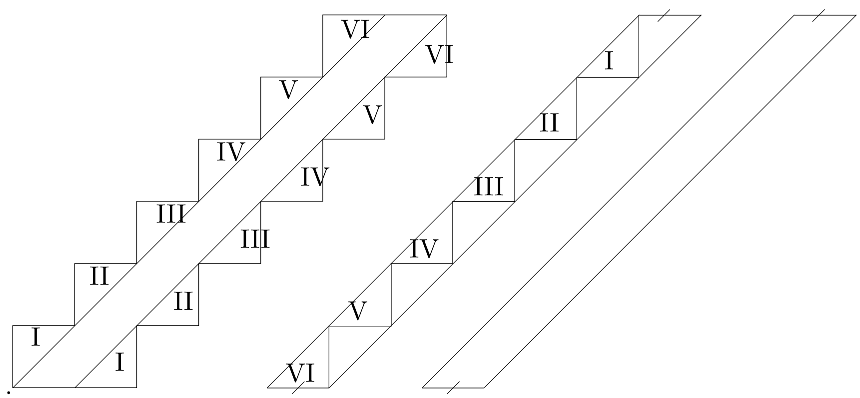


Figure 21: The two right-most cylinders C_1 (labeled) and C_2(unlabeled).

The circumferences of these two cylinders are $6\sqrt{2}$. Geodesic flows on this surface are well defined, and rational directions

Definition 34. *Let* $\omega_t^\theta : [0,1] \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbf{M}$ *be the* ***maximal geodesic flow*** *on the surface in direction* θ *such that* $\omega_0^\theta = \omega_1^\theta = x_0 \in \mathbf{M}\backslash Sing(\mathbf{M})$.
$\omega_t^{\frac{\pi}{4}}$ *is the geodesic flow in the* ***slope one direction****, and* χ *is its image in* $\mathbf{M}$ *and element of* $\pi_1(\mathbf{M}, x_0)$.

Lemma 13. α *is homologous to* χ *in* $\mathbf{M}\backslash Sing(\mathbf{M})$.

Proof. Let χ be a geodesic contained in either C_1 or C_2. Since a geodesic does not admit singularities, it is the image of a closed path on $X\backslash\mathrm{Sing}(X)$ with initial point x_0 on the strips of C_1 and C_2 with boundaries removed, denoted C_1', C_2'. Express $[\alpha]$ as $\Sigma_{j=0}^{11} \frac{1}{2}\gamma_j$ (a closed path climbing up the staircase). We show that the intersection numbers of $[\alpha]$ and $[\chi]$ are the same for every core cylinder curve γ, i.e. $i([\gamma_k], \Sigma_{j=0}^{11} \frac{1}{2}[\gamma_j]) = i([\gamma_k], [\chi])\ \forall k = 0, \ldots, 11$.
Case one: k is even. If k is even, then every curve γ_k is oriented to the right. Since χ intersects every curve once, $i([\gamma_k], [\chi]) = 1$. No even indexed curves intersect eachother, so we need only consider when j is odd. Now if j is odd, it is incident (positively crossing) with only two horizontal curves, namely $\gamma_{j+1}, \gamma_{j-1}$. Therefore $i([\gamma_k], [\alpha]) = i([\gamma_{j-1}], \frac{1}{2}[\gamma_j]) + i([\gamma_{j+1}], \frac{1}{2}[\gamma_j]) = \frac{1}{2}(i([\gamma_{j-1}], [\gamma_j]) + i([\gamma_{j+1}], [\gamma_j])) = \frac{1}{2}(1+1) = 1$.
Case two: k is odd. If k is odd, then $[\chi]$ will have an intersection number of -1 with $[\gamma_k]$ since odd-indexed core curves are oriented upwards. Now since k is odd, we only consider when j is even. Similarly, this means that γ_j negatively intersects the two vertical core curves with adjacent indices. Hence, $i([\gamma_k], [\alpha]) = i([\gamma_{j-1}], \frac{1}{2}[\gamma_j]) + i([\gamma_{j+1}], \frac{1}{2}[\gamma_j]) = \frac{1}{2}(i([\gamma_{j-1}], [\gamma_j]) + i([\gamma_{j+1}], [\gamma_j])) = \frac{1}{2}(-1-1) = -1$.
We know intersection number to be bilinear and non-degenerate on homology. So if α and χ's abelianizations admit the same intersection numbers for every curve in the spanning set of $H_1(\mathbf{M}, \mathbb{Q})$, then $[\alpha] = [\chi]$. $\square$

Theorem 6. *$\chi \in \pi_1(\mathbf{M}, x_0)$ lifts to $\tilde{\chi} \in \pi_1(\tilde{\mathbf{U}}, \tilde{x}_0)$*

Proof. From Lemma 2, $[\chi] = [\alpha]$, so $\Omega_{u,v}(\chi) = \Omega_{u,v}(\alpha)$. Since $i([u], [\alpha]) = -i([\gamma_2], [\alpha] + i([\gamma_5], [\alpha]) + i([\gamma_8], [\alpha]) - i([\gamma_{11}], [\alpha]) = -1 + (-1) + 1 - (-1) = 0$ and $i([v], [\alpha]) = 1 - (-1) - 1 + (-1) = 0$, it follows that $\alpha, \chi \in \mathrm{Ker}\ \Omega_{u,v}$, and χ lifts to a closed geodesic on $\tilde{\mathbf{U}}$. $\square$

Corollary 2. *content...*

From here, we use α to show that the *only* trajectories that close on the Necker cube surface are those that are in vector direction (a, b) such that $\gcd(a, b) = 1$ and a, b are both odd. We call these **odd-odd** directions. We can make this claim because the group generated by the matrices

$$\begin{bmatrix} 1 & \pm 2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ \pm 2 & 1 \end{bmatrix}$$

is the *Sanov subgroup* of $SL(2, \mathbb{Z})$ and only sends elements in the odd-odd set to itself. There are dualizations made between how these matrices skew a geodesic direction, and how their original affine transformations induce an effect homology. In a sense the kernel is obtained by the orbit of χ under $\mathbb{X}$ and its holonomy vector under $\mathbb{X}'$.

Lemma 14. *The actions of $\mathbb{X}'$ on $\mathcal{O}$ and $\mathcal{E}$ are closed in their respective sets.*

Proof. Since $\mathbb{X}'$ is generated by the elements $\mathbf{A}'$, $\mathbf{B}'$, and $\mathbf{R}'$, any matrix $G' \in \mathbb{X}'$ is of the form $G' = (\mathbf{A}')^{1_1} \circ (\mathbf{B}')^{1_2} \circ (\mathbf{R}')^{1_3} \circ (\mathbf{A}')^{2_1} \circ \ldots (\mathbf{A}')^{n_1} \circ (\mathbf{B}')^{n_2} \circ (\mathbf{R}')^{n_3}$, where $i_k \in \mathbb{Z}$ for $i = 1, \ldots, n$ and $k = 1, 2, 3$. Let $x = \begin{pmatrix} p \\ q \end{pmatrix}, y \in \mathcal{O}$, and consider the equation $G'x = y$. Observe that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^l x = \begin{pmatrix} p + 2jq \\ q \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^m x = \begin{pmatrix} p \\ q + 2mp \end{pmatrix}$ for any $l, m \in \mathbb{Z}$. Also note that for any $j \in \mathbb{Z}$, $(\mathbf{R}')^m x = \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} -q \\ p \end{pmatrix}, \begin{pmatrix} -p \\ -q \end{pmatrix}, \begin{pmatrix} q \\ -p \end{pmatrix}$ when $j \mod 4 \equiv 0, 1, 2, 3$, respectively. In any case, the product of any power of a generator of $\mathbb{X}'$ and any $x \in \mathcal{O}$ is an element of $\mathcal{O}$. By letting $l = i_1, m = i_2$, and $j = i_3$, we first consider the base case when $i = n$. Let $G' = G'_1 \circ \cdots \circ G'_n$, such that $G'_i = (\mathbf{A}')^{i_1} \circ (\mathbf{B}')^{i_2} \circ (\mathbf{R}')^{i_3}$. Since n_1, n_2, n_3 are arbitrary integers, $G'_n x \in \mathcal{O}$. Suppose for some $b < n - 1$, $G'_{n-b} \circ \cdots \circ G'_n x = y' \in \mathcal{O}$. Therefore $y' = (G'_1 \circ \cdots \circ G'_b)^{-1} y$, which implies that $(G'_1 \circ \cdots \circ G'_b)^{-1}$ preserves the set $\mathcal{O}$. Otherwise, if $y \in \mathcal{E}$, there exists at least one G'_i for $1 < i < b$ and $\tau \in \mathcal{E}$ such that $G_i'^{-1} \tau = (\mathbf{R}')^{-i_3} \circ (\mathbf{B}')^{-i_2} \circ (\mathbf{A}')^{-i_1} \tau \in \mathcal{O}$, a contradiction. Since elements in $\mathbb{X}'$ are invertible, $G'_1 \circ \cdots \circ G'_b$ must also map $\mathcal{O}$ to itself. Left multiply both sides of the equation to show that $G'_1 \circ \cdots \circ G'_n x = G'x = y$. By the principle of strong induction, this holds for all $0 < b \leq n$. Since G' is invertible and an arbitrarily chosen element of $\mathbb{X}'$, it follows that $x \in \mathcal{O}$ if and only if $y \in \mathcal{O}$ and $\mathcal{O}$ is closed under $\mathbb{X}'$.
The proof for when $x \in \mathcal{E}$ is made in the same way. $\square$

Now a trajectory in the horizontal direction has a directional vector of (1,0). The orbit of this vector by the Veech group is the set of all **even-odd** vectors. We also know that in this direction a geodesic is drift-periodic (See figure 1). The Veech group of $\mathbf{M}$ preserves these properties. Suppose you had some closed geodesic on $\mathbf{M} \backslash Sing(\mathbf{M})$ called β such that $\beta = h(\alpha)$, where $h \in \mathrm{Aff}^+(\mathbf{M})$, and h_* is its induced homomorphism. Then we want to show that

$$(i([\beta], [u]), i([\beta], [v])) = (i([\alpha], h_*^{-1}[u]), i([\alpha], h_*^{-1}[v])) = (0, 0).$$

But first, we look at some of the properties of the group $\mathbb{X}_*$.

Theorem 7. *Let $\mathbb{X}_*$ be the group generated by $\mathbf{A}_*, \mathbf{B}_*, \mathbf{R}_*, \mathbf{H}_*,$ and $\mathbf{V}_*$. Let $G = \langle \mathbf{A}_*, \mathbf{B}_* \rangle$, $T = \langle \mathbf{H}_*, \mathbf{V}_* \rangle$, and $R = \langle \mathbf{R}_* \rangle$. Then the following is true:*

(i) G is a free subgroup of $\mathbb{X}_*$ of rank two.

(ii) T is a finite cyclic subgroup of $\mathbb{X}_*$ and a centralizer of G.

(iii) R is a finite cyclic subgroup of $\mathbb{X}_*$, and a normalizer of G.

Proof. Let $h_*^j = \mathbf{A}_*^{k_j} \circ \mathbf{B}_*^{g_j} \in G$ for $k_j, g_j \in \mathbb{Z}$, $j = 1, \ldots, n$.
(i). When $\mathbf{A}_*$ and $\mathbf{B}_*$ act on γ_i, it is only ever trivial if i is even for $\mathbf{A}_*$ or i is odd on $\mathbf{B}_*$. Since i cannot be both odd and even at the same time, there is no

relation between the two generators and therefore G is free.
(ii) It is up to the reader to show that T has the relations $\mathbf{H}_*^2 = \mathbf{V}_*^2 = (\mathbf{H}_*\mathbf{V}_*)^3 = id_*$, and is isomorphic to the rotational group of the hexagon generated by reflections about adjacent vertices of a 12-gon. Observe that $\mathbf{H}_* \circ \mathbf{A}_*^{k_j} \circ [\gamma_i] = \mathbf{H}_* \circ [\gamma_i] + \frac{k_j}{2}(1-(-1)^i)(\mathbf{H}_* \circ [\gamma_{i-1}] + \mathbf{H}_* \circ [\gamma_{i+1}]) = [\gamma_{-i}] + \frac{k_j}{2}(1-(-1)^i)([\gamma_{1-i}] + [\gamma_{-i-1}]) = \mathbf{A}^{k_j} \circ [\gamma_- i] = \mathbf{A}^{k_j} \circ \mathbf{H}_* \circ [\gamma_i]$, and $\mathbf{V}_* \circ \mathbf{A}_*^{k_j} \circ [\gamma_i] = \mathbf{V}_* \circ [\gamma_i] + \frac{k_j}{2}(1-(-1)^i)(\mathbf{V}_* \circ [\gamma_{i-1}] + \mathbf{V}_* \circ [\gamma_{i+1}]) = [\gamma_{10-i}] + \frac{k_j}{2}(1-(-1)^i)([\gamma_{11-i}] + [\gamma_{9-i}]) = \mathbf{A}^{k_j} \circ [\gamma_1 0 - i] = \mathbf{A}^{k_j} \circ \mathbf{V}_* \circ [\gamma_i]$. In the same way one can show this to be true for $\mathbf{B}_*^{g_j}$, and we can see that T is a centralizer of G.
(iii) R is obviously cyclic and finite since an isomorphism is obtained as $\mathbf{R}_* \mapsto \mathbf{R}' \in SO(2, \mathbb{Z})$.
Note that $\mathbf{R}_* \circ \mathbf{A}_*^{k_j} \circ [\gamma_i] = \mathbf{R}_* \circ [\gamma_i] + \frac{k_j}{2}(1-(-1)^i)(\mathbf{R}_* \circ [\gamma_{i-1}] + \mathbf{R}_* \circ [\gamma_{i+1}])$
$= (-1)^i[\gamma_{1-i}] + \frac{k_j}{2}(1-(-1)^i)((-1)^{i-1}[\gamma_{2-i}] + (-1)^{i+1}[\gamma_{-i}])$
$= (-1)^{1-i}([\gamma_{1-i}] - \frac{k_j}{2}(1+(-1)^{1-i})([\gamma_{2-i}] + [\gamma_{-i}]))$
$= (-1)^{1-i}\mathbf{B}_*^{-k_j} \circ [\gamma_{1-i}] = \mathbf{B}_*^{-k_j} \circ (-1)^{1-i}[\gamma_{1-i}] = \mathbf{B}_*^{-k_j} \circ \mathbf{R}_* \circ [\gamma_i]$.
Likewise, $\mathbf{R}_* \circ \mathbf{B}_*^{g_j} \circ [\gamma_i] = \mathbf{A}_*^{-g_j} \circ \mathbf{R}_* \circ [\gamma_i]$. □

Remark. *It can be easily shown that $\mathbb{X}'$ has similar properties.*

Lemma 15. *Let $h_* \in \langle \mathbf{A}_*, \mathbf{B}_* \rangle$. Then $h_* \circ [\alpha] \in H_1(\mathbf{M}, \mathbb{Q})$ can be expressed as $h_* \circ [\alpha] = \frac{1}{2}(c_1 \Sigma_{j=0}^5 [\gamma_{2j}] + c_2 \Sigma_{j=0}^5 [\gamma_{2j+1}])$ for $c_1, c_2 \in \mathbb{Z}$.*

Proof. Let $\Sigma_{j=0}^5[\gamma_{2j}] = \Sigma\Gamma_{even}$, $\Sigma_{j=0}^5[\gamma_{2j+1}] = \Sigma\Gamma_{odd}$, and $\Sigma_{j=0}^{11}[\gamma_j] = \Sigma\Gamma$. Let $h_* = h_*^n \circ \cdots \circ h_*^1$, and $h_*^i = \mathbf{A}_*^{k_i} \circ \mathbf{B}_*^{g_i}$ for $k_i, g_i \in \mathbb{Z}$, $i = 1, \ldots, n$. Compose these two homomorphisms and obtain $\mathbf{A}_*^{k_i} \circ \mathbf{B}_*^{g_i}(\Sigma\Gamma) = (4g_ik_i + 2k_i)\Sigma\Gamma_{even} + 2g_i\Sigma\Gamma_{odd} + \Sigma\Gamma$. Let $c_i^1 = (4g_ik_i + 2k_i), c_i^2 = 2g_i$, and solve for $h_*^{i+1} \circ h_*^i \circ \Sigma\Gamma$:

$$\begin{aligned}
h_*^{i+1} \circ h_*^i \circ (\Sigma\Gamma) &= h_*^{i+1} \circ (c_i^1\Sigma\Gamma_{even} + c_i^2\Sigma\Gamma_{odd} + \Sigma\Gamma) \\
&= c_i^1 h_*^{i+1} \circ (\Sigma\Gamma_{even}) + c_i^2 h_*^{i+1} \circ (\Sigma\Gamma_{odd}) + h_*^{i+1} \circ (\Sigma\Gamma) \\
&= 2g_{i+1}\Sigma\Gamma_{odd} + (4g_{i+1}k_{i+1} + 2k_{i+1})\Sigma\Gamma_{even} + \Sigma\Gamma \\
&\quad + c_i^1(4g_{i+1}k_{i+1}\Sigma\Gamma_{even} + 2g_{i+1}\Sigma\Gamma_{odd} + \Sigma\Gamma_{even}) \\
&\quad + c_i^2(2k_{i+1}\Sigma\Gamma_{even} + \Sigma\Gamma_{odd}) \\
&= \Sigma\Gamma + (c_i^1 + (c_i^1+1)(4g_{i+1}k_{i+1}) + (c_i^2+1)2k_{i+1})\Sigma\Gamma_{even} \\
&\quad + (c_i^2 + (c_i^1+1)2g_{i+1})\Sigma\Gamma_{odd} \\
\text{Let } c_{i+1}^1 &:= (c_i^1 + (c_i^1+1)(4g_{i+1}k_{i+1}) + (c_i^2+1)2k_{i+1}), \\
c_{i+1}^2 &:= (c_i^2 + (c_i^1+1)2g_{i+1}).
\end{aligned}$$

From these recursive definitions and a finite sequence of integers, $\{k\}_i, \{g\}_i$, observe then that
$h_* \circ [\alpha] = h_* \circ [\frac{1}{2}\Sigma\Gamma] = \frac{1}{2}h_* \circ [\Sigma\Gamma] = \frac{1}{2}[c_n^1\Sigma\Gamma_{even} + c_n^2\Sigma\Gamma_{odd} + \Sigma\Gamma]$
$= \frac{1}{2}[(c_n^1+1)\Sigma\Gamma_{even} + (c_n^2+1)\Sigma\Gamma_{odd}]$. Further simplify by letting $c_1 = c_n^1 + 1, c_2 = c_n^2 + 1$. □

Lemma 16. *Let $h_* \circ [\alpha] \in H_1(\mathbf{M}, \mathbb{Q})$. Then for $a \in \langle \mathbf{H}_*, \mathbf{V}_* \rangle$ and $b \in \langle \mathbf{R}_* \rangle$, the following is true:*

$$\begin{aligned}
a \circ h_* \circ [\alpha] &= h_* \circ [\alpha] \\
b \circ h_* \circ [\alpha] &= \frac{1}{2}[c_1' \Sigma\Gamma_{even} + c_2' \Sigma\Gamma_{odd}] \\
h_* \circ b \circ [\alpha] &= \frac{1}{2}[c_1'' \Sigma\Gamma_{even} + c_2'' \Sigma\Gamma_{odd}]
\end{aligned}$$

Proof. By Theorem 4, a is a centralizer of the group so $a \circ h_* \circ [\alpha] = h_* \circ a \circ [\alpha] = h_* \circ \frac{1}{2} a \circ [\Sigma\Gamma]$. Since a is a cyclic permutation of the set Γ, it acts trivially on $\Sigma\Gamma$. Therefore, $a \circ h_* \circ [\alpha] = h_* \circ \frac{1}{2}[\Sigma\Gamma] = h_* \circ [\alpha]$.
By theorem 4, $a \circ \mathbf{A}_*^{k_i} \circ \mathbf{B}_*^{g_i} = \mathbf{B}_*^{-k_i} \circ \mathbf{A}_*^{-g_i} \circ a$. Extend this property to h_*, and denote the normalized element as h_{**}, such that $b \circ h_* = h_{**} \circ b$. Note that $b(\Sigma\Gamma) = b(\Sigma\Gamma_{even} + \Sigma\Gamma_{odd}) = \Sigma\Gamma_{odd} - \Sigma\Gamma_{even}$. $b \circ h_* \circ [\Sigma\Gamma] = c_1 b \circ \Sigma\Gamma_{even} + c_2 b \circ \Sigma\Gamma_{odd} = c_1 \Sigma\Gamma_{odd} - c_2 \Sigma\Gamma_{even}$. So, $c_1' = -c_2$ and $c_2' = c_1$.
Since h_* is arbitrary, let $h_{**} = g_*$ be generated by an integer sequence that defines the word and consider $h_* \circ b \circ [\Sigma\Gamma] = b \circ g_* \circ [\Sigma\Gamma] = c_1^* b \circ \Sigma\Gamma_{even} + c_2^* b \circ \Sigma\Gamma_{odd} = c_1^* \Sigma\Gamma_{odd} - c_2^* \Sigma\Gamma_{even}$. So, $c_1'' = -c_2^*$ and $c_2'' = c_1^*$. $\square$

Now that every element in the orbit of $[\alpha]$ can be expressed as a linear combination of integers, it is simple to show they lift to a closed trajectory in the cover.

Definition 35. *Let* $\mathbf{dir}$: $UT(\mathbf{M} \backslash Sing(\mathbf{M})) \to \mathcal{O} \cup \mathcal{E}$ *be the injective map from $\mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{Z}^2$ given as $\mathbf{dir}(\theta) = (k_1 \cos(\theta), k_2 \sin(\theta)), k_1, k_2 \in \mathbb{R}$ such that* $\gcd(k_1 \cos(\theta), k_2 \sin(\theta)) = 1$.

Theorem 8. *(Sketch)*
Any geodesic, β, in $\mathbf{M}$ lifts to a closed geodesic $\tilde{\beta}$ on $\tilde{\mathbf{U}}$ if and only if $\mathbf{dir}(Arg(\mathbf{hol}(\beta))) \in \mathcal{O}$.

Proof. Call the quotient cover $p : \tilde{\mathbf{U}} \to \mathbf{M}$, and fix a point $\tilde{x_0} \in p^{-1}(x_0)$. Let $\beta = h(\chi)$, where $h \in \mathbb{X}$. We also obtain $[\beta] = h_* \circ [\alpha]$ from Lemma 2. Since h sends geodesics to geodesics, h induces the following: $\mathbf{hol}(h(\chi)) = h'(\mathbf{hol}(\chi)) = h'(6 + 6i)$ for $h' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{X}'$. So, $\text{Arg}(h'(\mathbf{hol}(\chi))) = \text{Arg}(6[(a + b) + i(c + d)]) = \text{Arg}(6h'(1 + i))$. Lemma 3 states that for any $h' \in \mathbb{X}'$, $h'(\mathcal{O}) = \mathcal{O}$. Therefore there is no such geodesic of **even-odd** slope in the orbit of χ. Otherwise $h', h \notin \mathbb{X}', \mathbb{X}$. Consequently, $\mathbf{dir}(Arg(\mathbf{hol}(\beta))) \in \mathcal{O}$.
From Lemma 5 we see that $[h(\chi)] = h_* \circ [\chi] = h_* \circ [\alpha] = \frac{1}{2}(c_1 \Sigma_{j=0}^5 [\gamma_{2j}] + c_2 \Sigma_{j=0}^5 [\gamma_{2j+1}])$ for $c_1, c_2 \in \mathbb{Z}$. Denote the sums as $\Sigma\Gamma_{even}$ and $\Sigma\Gamma_{odd}$.
Therefore, $2i([u], h_* \circ [\alpha]) = c_1 i([u], \Sigma\Gamma_{even}) + c_2 i([u], \Sigma\Gamma_{odd})$
$= c_1(-i([\gamma_2], 0) + i([\gamma_5], [\gamma_6] + [\gamma_4]) + i([\gamma_8], 0) - i([\gamma_{11}], [\gamma_{10}] + [\gamma_0]))$
$+ c_2(-i([\gamma_2], [\gamma_1] + [\gamma_3]) + i([\gamma_5], 0) + i([\gamma_8], [\gamma_7] + [\gamma_9]) - i([\gamma_{11}], 0))$
$= c_1(-(0) + (-1 - 1) + (0) - (-1 - 1)) + c_2(-(1 + 1) + (0) + (1 + 1) - (0)) = 0.$

Similarly, $2i([v], h_* \circ [\alpha]) = c_1 i([v], \Sigma\Gamma_{even}) + c_2 i([v], \Sigma\Gamma_{odd})$
$= c_1(i([\gamma_0], 0) - i([\gamma_3], [\gamma_2] + [\gamma_4]) - i([\gamma_6], 0) + i([\gamma_9], [\gamma_8] + [\gamma_{10}])$
$+ c_2(i([\gamma_0], [\gamma_{11}] + [\gamma_1]) - i([\gamma_3], 0) - i([\gamma_6], [\gamma_5] + [\gamma_7]) + i([\gamma_9], 0)$
$= c_1((0) - (-2) - (0) + (-2)) + c_2((2) - (0) - (2) + (0)) = 0.$
Therefore, $\Omega_{u,v}(h(\chi)) = (0,0)$, and $h(\chi) = \beta \in \text{Ker } \Omega_{u,v}$ for all $h \in \mathbb{X}$. By Theorem 2, β lifts to $\tilde{\beta} \in \pi_1(\tilde{\mathbf{U}}, \tilde{x}_0)$. Let $\theta =$Arg($\mathbf{hol}(\beta)$). Then ω_t^{θ} at x_0 lifts to $\tilde{\omega}_t^{p^{-1}(\theta)} \in \tilde{\mathbf{U}} \backslash \text{Sing}(\tilde{\mathbf{U}})$.
Now suppose instead that $\beta = h(\gamma_i)$. Then $\mathbf{dir}(\beta) = \frac{1}{2}(1 + (-1)^i, 1 - (-1)^i)$. According to Lemma 3, $h'(\mathcal{E}) = \mathcal{E}$. Thus we have no geodesic in the **odd-odd** directions obtained from the orbits of $(1,0)$ and $(0,1)$. For contradiction, suppose that $h(\gamma_i) \in \mathbf{Ker}\ \Omega u, v$. Then $(i(h_* \circ [\gamma_i], [u]), i(h_* \circ [\gamma_i], [v])) = (i([\gamma_i], h_*^{-1} \circ [u]), i([\gamma_i], h_*^{-1} \circ [v])) = (0,0)$. Let $h_*^{-1} \circ [u] = \Sigma_{j=0}^{11} x_j [\gamma_j]$, and $h_*^{-1} \circ [v] = \Sigma_{j=0}^{11} y_j [\gamma_j]$. Note that since γ_i intersects $\gamma_{i\pm1}$, $i([\gamma_i], h_*^{-1} \circ [u]) = (-1)^{i+1}(x_{i-1} + x_{i+1})$ and $i([\gamma_i], h_*^{-1} \circ [v]) = (-1)^{i+1}(y_{i-1} + y_{i+1})$.
Unfinished.. □

Conjecture. ***Dynamics of Geodesic Flow on the Necker cube surface.*** *Obtain θ and $\vec{\theta}$ as described in Definition 3. Denote the non-singular unit-speed geodesic flow with initial point $s \in (\mathbf{U} \backslash Sing(U))$ in direction $[\theta] \sim \phi \in UT(\mathbf{U} \backslash Sing(\mathbf{U}))$ by $F_t^{\phi} : \mathbf{U} \times \mathbb{R}_0^+ \to \mathbf{U}$ on $(\mathbf{U}, \mu)$, where μ is a flow-invariant measure. Then the following is true:*

(i) (Periodic) There exists a $t_0 > 0$ such that $F_{t+t_0}^{\phi}(s) = F_t^{\phi}(s)$ if and only if $\vec{\theta} \in \mathcal{O}$.

(ii) (Drift-Periodic) There exists a $t_0 > 0$ such that $F_{t+t_0}^{\phi}(s) = F_t^{\phi}(s) + c$, where $c \in \mathbf{U}$ is a non-trivial translation of a point in $\mathbf{U}$, if and only if $\vec{\theta} \in \mathcal{E}$.

Proof. (Sketch)
Denote the covering maps $f : \tilde{\mathbf{U}} \to \mathbf{U}$, $p : \tilde{\mathbf{U}} \to \mathbf{M}$, and fix a point $\tilde{x}_0 \in f^{-1}(s), p^{-1}(x_0)$, for $x_0 \in \mathbf{M}$. $f^{-1}([\theta]) = \{x : x = \theta + n\frac{\pi}{2}, n \in \mathbb{Z}\} = [\theta] \subset UT(\tilde{\mathbf{U}} \backslash \text{Sing}(\tilde{\mathbf{U}}))$ given by the four-fold cover and rotations of each individual plane. This gives us a relation between the two tangent bundles, where the translation four-fold cover has the standard $\mathbb{R}/2\pi\mathbb{Z}$ unit tangent fiber. θ' is the direction associated to the flow $\omega_t^{\theta'} : [0,1] \to \mathbf{M}$ on the translation surface. Since the cover is translation, $p^{-1}(\theta') = \theta' = \theta + n\frac{\pi}{2}$. First suppose that $\vec{\theta} \in \mathcal{O}$. Then θ is identified with the set of directions that close on $\tilde{\mathbf{U}}$. From Theorem 5, $\omega_t^{\theta'}$ lifts to a closed geodesic $\tilde{\omega}_t^{\theta'}$. Given $\mathbf{hol}(\omega) = \int_{\omega} dz$, we obtain a period for the unit-speed flow, $t_0 = |\mathbf{hol}(\omega)|$. That is, $\tilde{F}_t^{\theta'} : \mathbb{R}_0^+ \to \tilde{\mathbf{U}}$ such that $\frac{d}{dt}\tilde{F}_t = \frac{1}{|\mathbf{hol}(\omega)|}$. Then $F_t^{\phi} = F_t^{[\theta']} = f \circ \tilde{F}_t^{\theta'}$. The period carries over since there is no concern over a trajectory returning to $\tilde{x}_0$ in a different direction. Otherwise, the geodesic $\omega_t^{\theta'}$ on $\mathbf{M}$ would have closed in $0 < t < 1$. Now suppose that $\vec{\theta} \in \mathcal{E}$. Identifying it with θ', we see that ω in direction θ' is not an element of $\mathbf{Ker}\ \Omega_{u,v}$ from Theorem 5. Therefore, $\Omega_{u,v}(\omega) = (m,n) \simeq T^{m,n}$ and lifting the terminal point

$\omega(1)$, $\tilde{\omega}(1) = T^{m,n}(\tilde{\omega}(0)) = T^{m,n}(\tilde{x}_0)$. The period remains unchanged, in that $\tilde{F}^{\theta'}_{t+\mathbf{hol}(\omega)} = \tilde{F}^{\theta'}_t + T^{m,n}(\tilde{x}_0)$. Therefore, $F^{\phi}_{t+t_0}(s) = f \circ \tilde{F}^{\theta'}_t(\tilde{x}_0) + f \circ T^{m,n}(\tilde{x}_0)$. Conversely, suppose F_t is periodic. Then $[\theta] = \phi = [\theta']$, which defines directional flows $\tilde{F}^{\phi}_t$. According to Theorem 5, $\tilde{F}^{\phi}_t$ will close if and only if $\phi \subset \mathcal{O}$. ϕ is the orbit of $\vec{\theta}'$ under the 90 degree rotational matrix. This matrix does not alter the length or period of a geodesic. Thus, F^{ϕ}_t is exactly one of the flows $\tilde{F}^{\phi}_t$. Likewise, if F^{ϕ}_t is drift-periodic then $F^{\phi}_{t+t_0} = f \circ \tilde{F}^{\phi}_t + f \circ T^{m,n}$. $T^{m,n}$ is trivial if and only if $\theta' \in \mathcal{O}$. Therefore, $\theta' \in \mathcal{E}$, and $[\theta'] = \phi$. $\square$

There is still much work to do in terms of cleaning up the proofs and organizing the final paper.

Conclusion

What I ultimately aim to do is port these results on X's homology back to the Necker cube surface. I want do it in such a way that the final theorem is bi-conditional. To do so, I imagine I can take a vector image of a small segment of a geodesic in $\mathbb{R}^3$ and project it onto the isometric flattening of the Necker Cube surface to obtain a direction (or classes of equivalent directions), and relate it to the unit tangent bundle of $\mathbf{M}$.

In addition, I would also like to find a formula for the arc-length of a geodesic based on direction alone. Knowing that $\mathrm{hol}(\alpha) = 6 + 6i$ means that the induced Euclidean metric on $\mathbf{M} \backslash \mathrm{Sing}(\mathbf{M})$ gives the geodesic an arc-length of $6\sqrt{2}$. I would like to show that:

$$\int_{\beta} |dz| = |hol(h(\alpha))| = |h'(hol(\alpha))|,$$

where $h' \in V(\mathbf{M})$ is the derivative of h, and $\beta = h(\alpha)$. We know that is true on the translation surface, but it's a matter of then showing the translation quotient, branch-cover, and the Necker cube surface have the same induced Euclidean metric of these non-singular geodesics. (It would not be surprising considering that the surface is built out of subsets of planes.) Even more of a problem is finding a way to solve for a matrix in the Sanov subgroup that brings (1,1) to the desired odd-odd slope.

References

[1]

[2] Necker, L.A., *Observations on some remarkable optical phaenomena seen in Switzerland; and on an optical phaenomenon which occurs on viewing a figure of a crystal or geometrical solid.* London and Edinburgh Philosophical Magazine and Journal of Science. 1832. doi:10.1080/14786443208647909

[3] Gottlieb., *Q*bert* [Arcade], Gottlieb, 1982.

[4] Escher, M.C., *Metamorphosis I*, 1937.

[5] Escher, M.C., *Convex and Concave*, 1955.