Erez Zohar, 6-7.3.22

D_8

$$D_8 = \{g\} = \{a^p x^q \mid p = 0,\dots,3;\ q = 0,1\}$$

$$|D_8| = 8$$

$$a \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i\sigma_y, \qquad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z \rightarrow \underline{\text{Nontrivial rep.}}\ \underline{j=2.}$$

$\left(\frac{\pi}{2} \text{ rotation}\right)$ $\qquad$ (reflection)

Trivial representation, $j=0$: $D^0(g) = 1 \quad \forall g$

First Sign representation, $j=\bar{0}$: $D^{\bar{0}}(g) = \det(D^2(g)) =$

$$= \det(a)^p \det(x)^q = 1^p(-1)^q = (-1)^q.$$

Second Sign representation: $D^1(g) = (-1)^p$

Third Sign representation: $D^{\bar{1}}(g) = (-1)^{p+q}$

$$|D_8| = \sum_j \dim^2(j) = 2^2 + 4\cdot 1^2 = 8 \checkmark$$

$-\mathbb{1}$ is contained in any nontrivial representation.

In particular $D^2(a^2) = -\mathbb{1}_{2\times 2}$

From now on, we use $D(g) \equiv D^2(g)$.

Group Element States $\quad |g\rangle \equiv |p,q\rangle$

Group Multiplication Rules:

$$(p;q)\cdot(p';q') =$$
$$= a^p x^q a^{p'} x^{q'}$$

$$xa = i\sigma_y\sigma_x = -i\sigma_x\sigma_y = -ax = a^3 x \qquad (a^2=-1)$$

$\underline{q=0:}$ $\quad x^q a^{p'} = a^{p'} x^q$

$\underline{q=1:}$ $\quad x^q a^{p'} = a^3 x^q a^{p'-1} = a^6 x^q a^{p'-2} = \dots = a^{3p'} x^q$

$$\Rightarrow x^q a^{p'} = a^{2p'q} a^{p} x^q =$$

$$\Rightarrow (p,q)\cdot(p',q') = a^p x^q a^{p'} x^{q'} = a^{p+p'+2p'q} x^{q+q'}$$

$$\Rightarrow \boxed{(p,q)\cdot(p',q') = (p+p'+2p'q\,,\,q+q')}$$

Inversion: $\quad (p,q)^{-1} = (a^p x^q)^{-1} = (x^{-1})^q (a^{-1})^p = x^q a^{3p} = a^{3p+6pq} x^q \; \cancel{= a^{9p} x^q} \Rightarrow$

$x^{-1} = x$

$a^{-1} = a^3$

$\Rightarrow$ If $q=0$: $(p,0)^{-1} = a^{3p}$ ✓ $[a^4=1]$ $\cancel{= a^{3p} a^p x^q = a^p x^q}$

If $q=1$: $(p,1)^{-1} = a^{9p} x^q = a^p x^q = (p,1)$ $[a^{8p}=1]$

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Left multiplication:

$$\tilde{\Theta}(p',q')\,|p,q\rangle = |(p',q')^{-1}\cdot(p,q)\rangle$$

$$= |(3p'+6p'q',q')\cdot(p,q)\rangle =$$

$$= |(3p'+6p'q'+p+\cancel{2p(3p'+6p'q')}+2pq',\ q+q')\rangle$$

Right multiplication:

$$\Theta(p',q')\,|p,q\rangle = |(p,q)\cdot(p',q')^{-1}\rangle$$

$$= |(p,q)\cdot(3p'+6p'q',q')\rangle =$$

$$= |(p+3p'+6p'q'+2q(3p'+6p'q'),\ q+q')\rangle$$

To replace the fermions (2 comp. spinors) by hard-core bosons $\{\eta_m^{+}\}_{m=1,2}$, we need to find the operator P on each link, such that:

$$PUP = -U$$

P should correspond to either left or right multiplication by $a^2=-1$ (doesn't matter which).

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For that, let's define the rep. basis states:

$$|0\rangle, |\bar{0}\rangle, |1\rangle, |\bar{1}\rangle, \left\{|2mn\rangle\right\}_{m,n=1,2}$$

$$\cancel{\langle j'm'n'|P|jmn\rangle = \sum_{g'g}}$$

$$\langle j'm'n'|P|jmn\rangle = \sum_{g'g} \langle j'm'n'|g'\rangle\overbrace{\langle g'|P|g\rangle}^{\delta_{g',a^2g}}\langle g|jmn\rangle$$

$$= \frac{di(j)}{|D_8|}\sum_g \langle j'm'n'|a^2g\rangle\langle g|jmn\rangle$$

$$= \frac{di(j)}{|D_8|}\sum_g \bar{D}^{j'}_{m'n'}(a^2g)\, D^j_{mn}(g) =$$

$$\cancel{= \frac{di(j)}{|D_8|}\, D}$$

$D^{j'}(a^2) = 1$ for all $j \neq 2$ ~~the 1 reps~~

-1 for $j=2$

$$= \frac{di(j)}{|D_8|} D^{j'}(a^2) \sum_g \bar{D}^{j'}_{m'n'}(g)\, D^j_{mn}(g)$$

$$\underset{\text{Gr. Orth. Thm.}}{=} \frac{di(j)}{|D_8|} D^{j'}(a^2)\frac{|D_8|}{di(j)}\delta_{j'j}\,\delta_{m'm}\,\delta_{n'n}$$

$$= D^{j}(a^2)\,\delta_{j'j}\,\delta_{m'm}\,\delta_{n'n}$$

$$\rightarrow P = \sum_j D^j(a^2)\, |jmn\rangle\langle jmn|$$

$$= \Pi_1 - \Pi_2$$

Π_1: proj. onto the $j\neq 2$ (1d) irreps

Π_2: proj onto the $j=2$ irrep

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$\Rightarrow$ The transformed gauge-matter interaction:

$$\boxed{\tilde{H}_{int} = \varepsilon \sum_x P(x-1)\, \eta_m^\dagger(x)\, U_{mn}(x)\, \eta_n(x+1) + h.c.}$$

Now we proceed and remove the fermions altogether. For that, we need two indep. Gauss laws that will allow us to solve for $n_1 = \eta_1^\dagger \eta_1$ and $n_2 = \eta_2^\dagger \eta_2$ separately.

~~We can wr~~

The group is finite, so we can write Gauss laws only in terms of finite transformations, and we have 8 such:

$\forall g \in D_8$, $\quad \tilde{\Theta}_g(x)\, \Theta_g^\dagger(x+1) |\psi\rangle = \hat{\Theta}_g(x) |\psi\rangle$

Where $\hat{\Theta}_g(x)$ is the matter's transformation:

$$\hat{\Theta}_g(x) = e^{i \eta_m^\dagger(x) (q_g)_{mn} \eta_n(x)}$$

(If we don't stagger which is pointless for a finite group).

$$q_g = -i \log\left(D^{j_2}(g)\right)$$

(Zohar, Burrello PRD 2015)

~~To take~~ We need two commuting, independent $g \in D_8$ $\rightarrow$ pick ax^2 and x.

$$q_{a^2x} = -i\log\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = -i\begin{pmatrix}i\pi & 0\\ 0 & 0\end{pmatrix} = \pi\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$q_x = -i\log\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = -i\begin{pmatrix}0 & 0\\ 0 & i\pi\end{pmatrix} = \pi\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

$$\Rightarrow \quad \hat{\theta}_{a^2x} = e^{i\pi(\eta_1^\dagger\eta_1+\eta_2^\dagger\eta_2)} = e^{i\pi n_1}\cancel{e^{i\pi n_2}}$$

$$= (1-2n_1)\cancel{(1-2n_2)}$$

$$\left(e^{i\pi n} = 1 + \left(\sum_{k=1} \frac{(i\pi)^k}{k!}\right) n = 1 + \left(e^{i\pi}-1\right)n = 1-2n \right)$$

$n^k = n \quad \forall k \in \aleph_0$

$$\hat{\theta}_x = e^{i\pi n_2} = (1-2n_2)$$

Define: $G_1^{(x)} = \tilde{\theta}_{a^2}(x)\,\theta^\dagger_{a^2}(x-1)$

$G_2^{(x)} = \tilde{\theta}_{x}(x)\,\theta^\dagger_{x}(x-1)$

We have:

$$G_2^{(x)}|\psi\rangle = \hat{\theta}_x(x)|\psi\rangle = (1-2n_2^{(x)})|\psi\rangle$$

$$\Rightarrow \quad n_2(x)|\psi\rangle = \underbrace{\left(\frac{1-G_2(x)}{2}\right)}_{\equiv\, O_2(x)}|\psi\rangle$$

Note that $G_2^2(x) = \left(\tilde{\theta}_x(x)\,\theta_x^\dagger(x-1)\right)^2$

$$= \tilde{\theta}_x^{2}(x)\,\theta_x^{\dagger 2}(x-1) = \mathbb{1}$$

Hence O_2 is a projector with spectrum 0,1 and we need no constraint for it $\longrightarrow$

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Similarly, $Q_1(x)|\Psi\rangle = (1-2n_1(x))|\Psi\rangle$

$$\Rightarrow n_1(x)|\Psi\rangle = \frac{1-Q_1(x)}{2}|\Psi\rangle$$

$$\equiv G_1(x)|\Psi\rangle$$

Note that $Q_i^2(x) = \mathbb{1}$
and thus $G_i(x)$ are projectors, ~~and~~
implying that

$$G_i(x)\left(G_i(x)-1\right)=0 \quad \forall i,x$$

holds as an operator equation,
and we don't need to impose these
local conditions!

$$\cancel{\left[G_i(x), U_{mn}(x)\right]} =$$

$$\Rightarrow H_{int} = \varepsilon \sum_{x,\xi,m,n} P(x-1)\,\hat{P}_1^m(x)\,U_{mn}\,\hat{P}_1^n(x) + h.c.$$

$\hat{P}_1^m(x)$ is the projector
onto $n_m(x)=1$.

AND THAT'S IT.

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$\cancel{\langle KNN' | U^{j}_{nn'} | JMM' \rangle}$

Clebsch Gordan coefficients.

$\cancel{|0\rangle \otimes |0\rangle = |0\rangle \rightarrow \langle 0 | 00 \rangle = 1}$

$\cancel{|0\rangle \otimes |\bar{0}\rangle = |\bar{0}\rangle \rightarrow \langle \bar{0} | 0\bar{0} \rangle = \langle \bar{0} | \bar{0} 0 \rangle = 1}$

$$\begin{cases} D^0(g)\,D^0(g) = D^0(g) \Rightarrow \langle 0|00\rangle = 1 \\ D^0(g)\,D^{\bar{0}}(g) = D^{\bar{0}}(g) \Rightarrow \langle \bar{0}|0\bar{0}\rangle = \langle \bar{0}|\bar{0}0\rangle = 1 \\ D^0(g)\,D^1(g) = D^1(g) \Rightarrow \langle 1|10\rangle = \langle 1|01\rangle = 1 \\ D^0(g)\,D^{\bar{1}}(g) = D^{\bar{1}}(g) \Rightarrow \langle \bar{1}|\bar{1}0\rangle = \langle \bar{1}|0\bar{1}\rangle = 1 \end{cases}$$

$$\begin{cases} D^{\bar{0}}(g)\,D^{\bar{0}}(g) = (-1)^{2q} = D^0(g) \Rightarrow \langle 0|\bar{0}\bar{0}\rangle = 1 \\ D^{\bar{0}}(g)\,D^1(g) = (-1)^{p+q} = D^{\bar{1}}(g) \Rightarrow \langle \bar{1}|0\bar{1}\rangle = \langle \bar{1}|\bar{1}0\rangle = 1 \\ D^{\bar{0}}(g)\,D^{\bar{1}}(g) = (-1)^{p+2q} = D^1(g) \Rightarrow \langle 1|\bar{0}\bar{1}\rangle = \langle 1|\bar{1}\bar{0}\rangle = 1 \end{cases}$$

$$\begin{cases} D^1(g)\,D^{\bar{1}}(g) = (-1)^{2p+q} = D^{\bar{0}}(g) \Rightarrow \langle \bar{0}|1\bar{1}\rangle = \langle \bar{0}|\bar{1}1\rangle = 1 \end{cases}$$

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$$D^{0}(g)\, D^{2}_{mn}(g) = D^{2}_{mn}(g) \Rightarrow \cancel{\langle 2\,mn|}$$

$$\Rightarrow \langle 0, 2m | 2n \rangle = \langle 2m, 0 | 2n \rangle = \delta_{mn} \quad \cancel{\langle 2mn | 0, 2mn \rangle} = \langle 2mn | 2mn, 0 \rangle = 1$$

$$\cancel{D^{\bar{0}}(g)\, D^{2}_{mn}(g) = (-1)^{q}\, D^{2}_{mn}(g)} =$$

~~C.G. Series $\Rightarrow$ #~~

C-G Series:

$$\cancel{D^{J}_{MN}(g)} \qquad D^{j_1(g)}_{m_1 m_1'}(g)\, D^{j_2(g)}_{m_2 m_2'} = \sum_{J} \langle j_1 m_1, j_2 m_2 | JM \rangle D^{J}_{MM'}(g) \cancel{\langle j_1 m_1'} \langle JM' | j_1 m_1' j_2 m_2' \rangle$$

$$D^{\bar{0}}(g)\, D^{2}_{mn}(g) = \quad \langle \bar{0}, 2mn | 2m'n' \rangle\, D^{2}_{m'n'}(g)$$

$$\cancel{(-1)^{q} A^{p} X^{q} = A^{2q+p} X^{q}}$$

$$(-1)^{q}\, D^{2}(g) = D^{2}(a^{2q})\, D^{2}(g) =$$

$$= \cancel{D^{2}(a^{2q}\, a^{p}\, X^{q})} =$$

$$= \cancel{D^{2}(a^{p}}$$

$$(-D)$$

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$A \equiv D^2(a)$
$X = D^2(x)$

$$(-1)^q A^p X^q = \cancel{A^p (\varepsilon X)^q}$$

$\det X$

$$= A^p (\det X \cdot X)^q =$$

$$\underset{X=X^{-1}}{=} A^p (\det X \cdot X^{-1})^q =$$

$$= A^p (\varepsilon X^T \varepsilon^T)^q$$

$$\cancel{= A^p} \underset{X=X^T}{=} A^p (\varepsilon X \varepsilon^T)^q$$

$$= A^p \varepsilon X^q \varepsilon^T$$

$$\varepsilon = -i\sigma_y = -A \Rightarrow [A^p, \varepsilon] = 0$$

$$\Rightarrow (-1)^q A^p X^q = \varepsilon A^p X^q \varepsilon^T$$

$$\Rightarrow D^{\bar{0}}(g) D^2_{mn}(g) = \varepsilon_{mm'} D^2_{m'n'}(g) \varepsilon^T_{n'n} =$$

$$= \varepsilon_{mm'} \varepsilon_{nn'} D^2_{m'n'}(g)$$

$$\underset{\text{C.G. series}}{=} \langle \bar{0}, 2m | 2m' \rangle D^2_{m'n'}(g)$$

$$\langle \bar{0}, 2n | 2n' \rangle$$

$$\Rightarrow \boxed{\langle \bar{0}, 2mn | 2m'n' \rangle = \varepsilon_{m'm'} \varepsilon_{nn'}}$$

$$\boxed{\cancel{\langle 2mn, \bar{0} | 2m'n' \rangle = \varepsilon_{mm'} \varepsilon_{nn'}}}$$

$$= -(\sigma_y)_{mm'} (\sigma_y)_{nn'}$$

$$\boxed{\begin{aligned} \langle \bar{0}, 2m | 2m' \rangle &= \langle 2m, \bar{0} | 2m' \rangle \\ &= \varepsilon_{mm'} \end{aligned}}$$

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$$D^1(g)\, D^2_{mn}(g) = (-1)^p A^p X^q = (-A)^p X^q$$

$$\cancel{= (-A)^p X^q = (A^T)^p X^q}$$

$$\cancel{= (\det A\, \varepsilon^T A^{-1} \varepsilon)^p X^q =}$$

$$-A = A^{-1} = XAX$$

$$D^1(g)\, D^2_{mn}(g) = (XAX)^p X^q =$$

$$= X A^p X^{q+1} =$$

$$= X A^p X^q X = \left(X D^2(g) X\right)_{mn}$$

$$D^1(g)\, D^2_{mn}(g) = X_{mm'} D^2_{m'n'}(g) X^T_{n'n} =$$

$$= X_{mm'} X_{nn'} D^2_{m'n'}(g)$$

$$\Rightarrow \langle 1, 2m | 2m' \rangle = \langle 2m, 1 | 2m' \rangle = (\sigma_z)_{mm'}$$

$$\Rightarrow \boxed{\begin{array}{l} \cancel{\langle 1, 2mn | 2m'n' \rangle = X_{mm'} X_{nn'} = (\sigma_z)_{mm'} (\sigma_z)_{nn'}} \\ \cancel{\langle 2mn, 1 | 2m'n' \rangle = X_{mm'} X_{nn'} =} \end{array}}$$

$$D^{\bar{1}}(g)\, D^2_{mn}(g) = (-1)^{p+q} D_{mn}(g) =$$

$$= (-1)^p \varepsilon_{mm'} \varepsilon_{nn'} D^2_{m'n'}(g)$$

$$= \varepsilon_{mm'} \varepsilon_{nn'} X_{m'm''} X_{n'n''} D^2_{m''n''}(g)$$

$$= (\varepsilon X)_{mm'} (\varepsilon X)_{nn'} D^2_{m'n'}(g)$$

$$\varepsilon X = i\sigma_y \sigma_z = \sigma_x \Rightarrow \langle \bar{1}, 2m | 2m' \rangle = \langle 2m, \bar{1} | 2m' \rangle = (\sigma_x)_{mm'}$$

$$\Rightarrow \boxed{\cancel{\langle \bar{1}, 2mn | 2m'n' \rangle = \langle 2mn, \bar{1} | 2mn \rangle = (\sigma_x)_{mm'} (\sigma_x)_{nn'}}}$$

$$D^2_{mn}(g)\, D^2_{m'n'}(g) = \underline{(A\otimes A)^p\, (X\otimes X)^q}$$

$$[A\otimes A,\, X\otimes X] = 0$$

$$(A\otimes A)^2 = (X\otimes X)^2 = \mathbb{1}_4$$

$\Rightarrow$ Forming a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group $\downarrow$ Only Abelian irreps are included.

$$A\otimes A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\sigma_y \otimes \sigma_y$$

$$\sigma_y = u\sigma_z u^\dagger \qquad u = e^{i\frac{\pi}{4}\sigma_x} \qquad U = u\otimes u$$

$$A\otimes A = U(-\sigma_z\otimes\sigma_z)U^\dagger$$

$$(A\otimes A)^p = (-1)^p\, U(\sigma_z\otimes\sigma_z)^p U^\dagger$$

$$X\otimes X = \sigma_z\otimes\sigma_z$$

$$D^2 \otimes D^2 = (-1)^p\, U(\sigma_z\otimes\sigma_z)^p U^\dagger\, (\sigma_z\otimes\sigma_z)^q$$

$$\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad \varphi^T\varphi = \mathbb{1}$$

(Orthogonal)

$$\varphi(A\otimes A)\varphi^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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$$\varphi (X\otimes X)\varphi^T = \begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&1\end{pmatrix}$$

$$D^2(g)\,D^2(g) = (A\otimes A)^p\,(\cancel{A}X\otimes \cancel{A}X)^q =$$

$$= \varphi \begin{pmatrix}1&&&\\&(-1)^{p+q}&&\\&&(-1)^q&\\&&&(-1)^p\end{pmatrix}\varphi^T$$

$\downarrow$

The four 1d irreps.

Ordered as: 0, $\bar{1}$, $\bar{0}$, 1

$\Rightarrow$ $\cancel{D^2(g)}$ $\cancel{\varphi}$

$$D^2_{mn}(g)\,D^2_{m'n'}(g) = \underset{\substack{\downarrow\\ \text{Direct sum}}}{\sum_\alpha} \varphi_{\underset{\text{row}}{(mm')},\underset{\text{column}}{\alpha}}\; \varphi_{(nn'\cancel{\alpha})}\; D^\alpha(g)$$

φ - Clebsch-Gordan coefficients!

$$\boxed{\varphi^\alpha_{mm'} = \langle \alpha \,|\, \cancel{2\,mm'}\; 2m, 2m' \rangle}$$

$\cancel{\cdot\ \langle \alpha | 21, 21\rangle = \frac{1}{\sqrt{2}}}$

$\cancel{\langle \alpha | 2}$

$\alpha = 0, \bar{1}, \bar{0}, 1$

Order $(m, m') = (1,1), (1,2), (2,1), (2,2)$

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$$\begin{array}{c|cccc} mm' & 1,1 & 1,2 & 2,1 & 2,2 \end{array}$$

First row:

$$\langle 0 | 2m, 2m' \rangle = \varphi^{0}_{mm'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

Second row:

$$\langle \bar{1} | 2m, 2m' \rangle = \varphi^{\bar{1}}_{mm'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}$$

Third row:

$$\langle \bar{0} | 2m\ 2m' \rangle = \varphi^{\bar{0}}_{mm'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}$$

Fourth row:

$$\langle 1 | 2m\ 2m' \rangle = \varphi^{1}_{mm'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}$$

All C-G coefficients of D_8:

$$\langle 0 | 00 \rangle = 1$$
$$\langle \bar{0} | \bar{0} 0 \rangle = \langle \bar{0} | 0 \bar{0} \rangle = 1$$
$$\langle 1 | 10 \rangle = \langle 1 | 01 \rangle = 1$$
$$\langle \bar{1} | \bar{1} 0 \rangle = \langle \bar{1} | 0 \bar{1} \rangle = 1$$
$$\langle 0 | \bar{0} \bar{0} \rangle = 1$$

$$\langle \bar{1} | 0 \bar{1} \rangle = \langle \bar{1} | \bar{1} 0 \rangle = 1$$

$$\langle 1 | \bar{0} \bar{1} \rangle = \langle 1 | \bar{1} \bar{0} \rangle = 1$$

$$\langle \bar{0} | 1 \bar{1} \rangle = \langle \bar{0} | \bar{1} 1 \rangle = 1$$

$$\langle \bar{0}\ 2m | 2m' \rangle = \langle 2m, \bar{0} | 2m' \rangle = \varepsilon_{mm'}$$
$$\langle 1\ 2m | 2m' \rangle = \langle 2m, 1 | 2m' \rangle = (\sigma_z)_{mm'}$$
$$\langle 0\ 2m | 2m' \rangle = \langle 2m\ 0 | 2m' \rangle = \delta_{mm'}$$

$$\langle \bar{1}\ 2m | 2m' \rangle = \langle 2m\ \bar{1} | 2m' \rangle = (\sigma_x)_{mm'}$$

The rest - given on top of this page, or zero.