Modelling small worlds

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Computational Modelling of Social Systems

So far

- **Block 1: Fundamentals of agent-based modelling**
  - Basics of agent-based modelling: the micro-macro gap
  - Modelling segregation: Schelling's model
  - Modelling cultures
- **Block 2: Opinion dynamics**
  - Basics of spreading: Granovetter's threshold model
  - Opinion dynamics
  - Modelling hyperpolarization and cognitive balance
- **Block 3: Fundamentals of agent-based modelling**
  - Basic network models
  - **Today: Modelling small worlds**
  - Scale-free networks
  - Growth processes

Overview

1. The small world phenomenon
2. The Watts-Strogatz model
3. Social small world model

The small world phenomenon

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2. The Watts-Strogatz model

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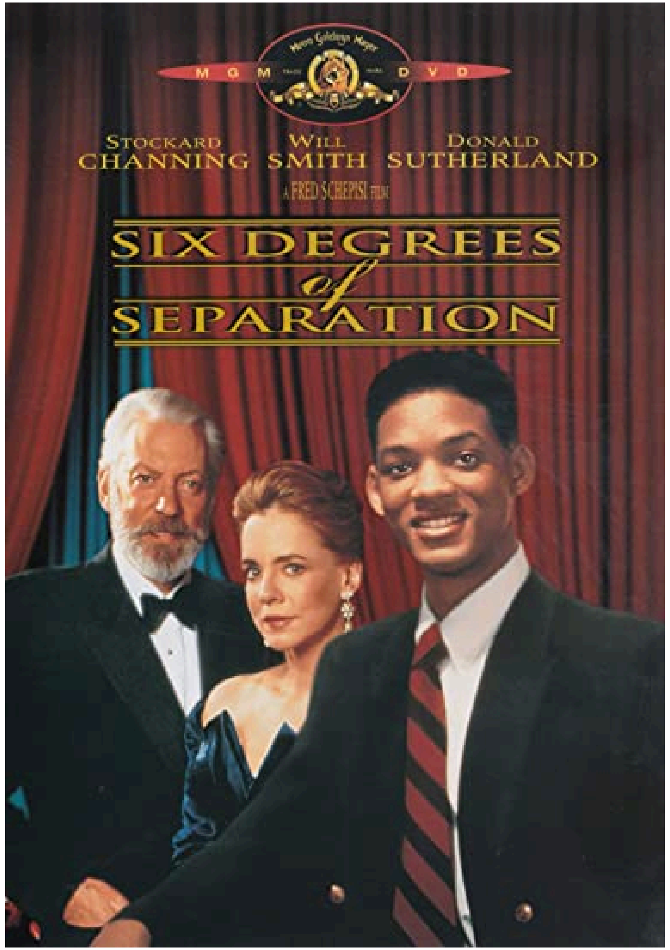
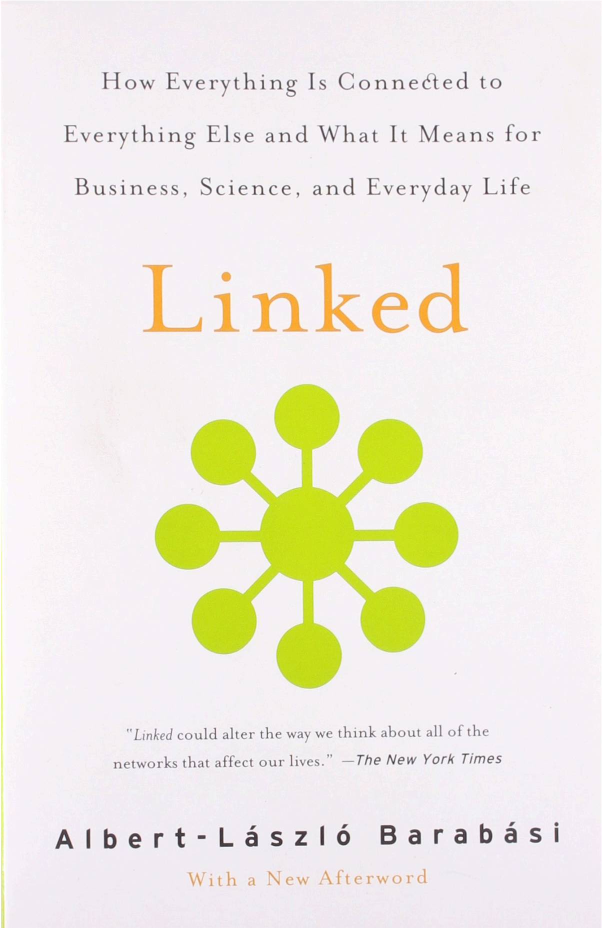
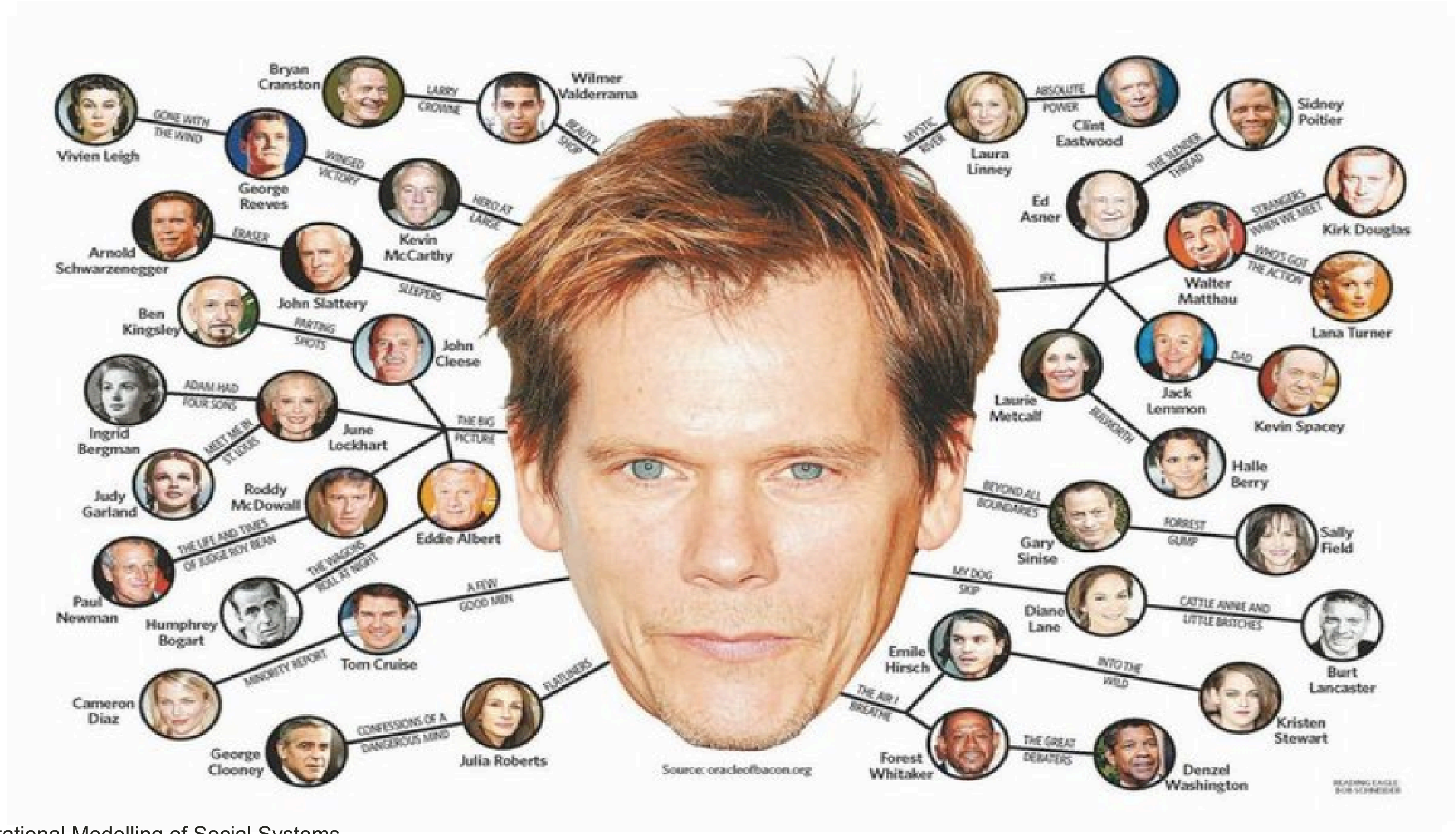
It's a small world!

Six degrees of separation





The Bacon number



Milgram's small world experiment

- 160 people in Omaha try to reach one person in Boston by mail to their acquaintances
- 44 letters reached the target with six steps on average
- Short path length as evidence of small-world
- However, some letters did not agree, are those paths infinite?

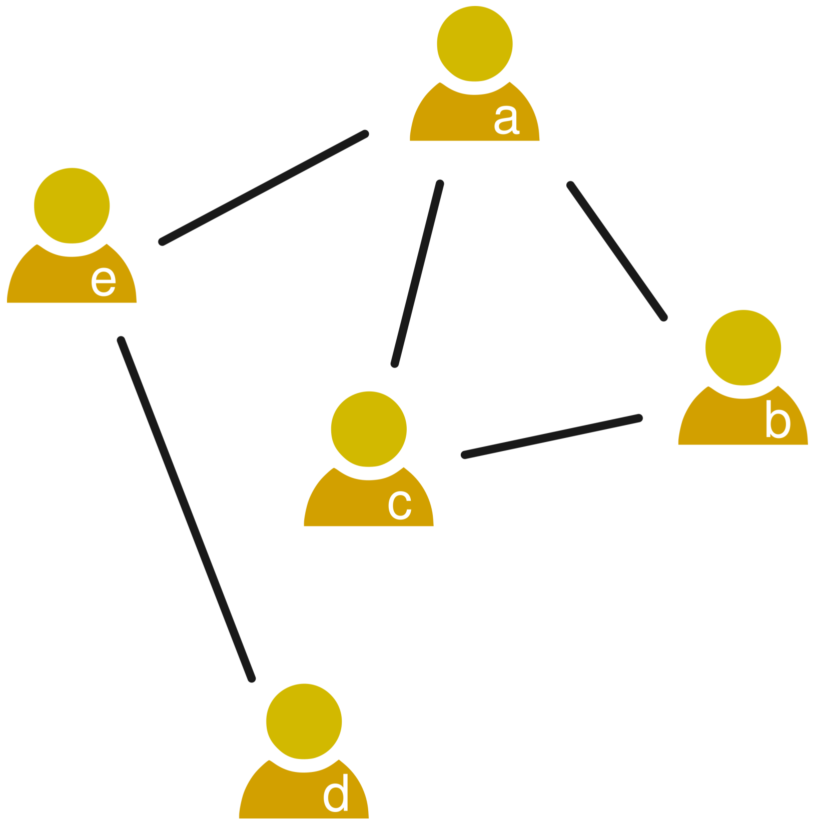
Refresher: Network distance

The **distance** between nodes v and w is denoted as $dist(v, w)$ and measures the minimum length among all the paths connecting v and w.

If there is no path between v and w, the distance between them is defined as $dist(v, w) := \infty$.

Example: $dist(b, e) = 2$

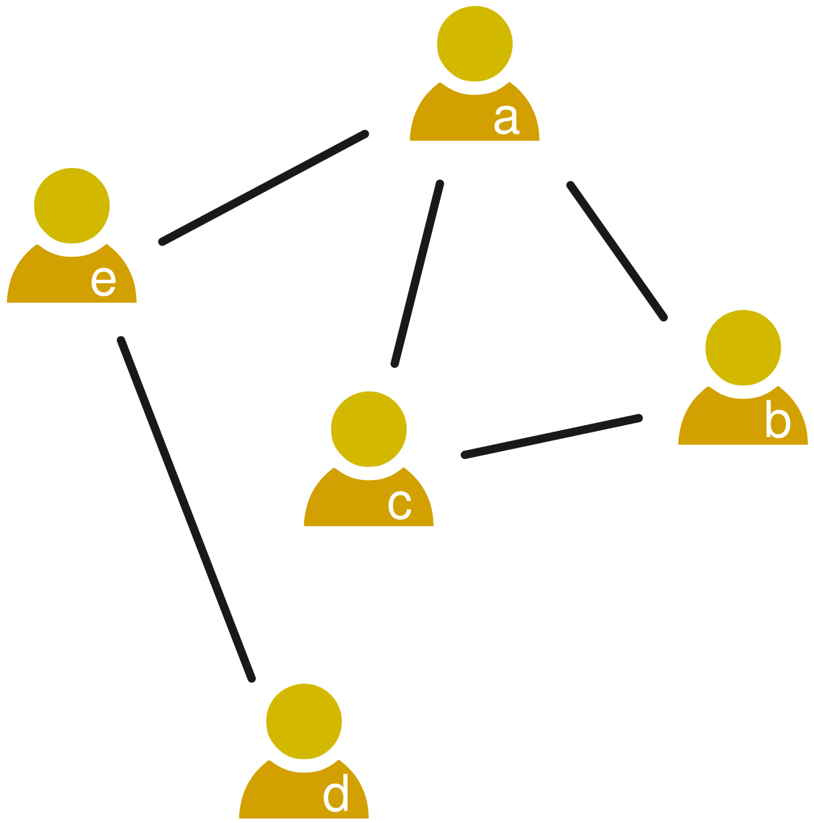
In directed networks, it might happen that $dist(v, w) \neq dist(w, v)$.



Average path length

$$\langle l \rangle = \frac{1}{N(N-1)} \sum_{u,v} dist(u,v)$$

- Global metric for the whole network
- It makes sense when network is connected, otherwise $\langle l \rangle = \infty$
- In the example, $\langle l \rangle = 1.7$



Refresher: Clustering coefficient

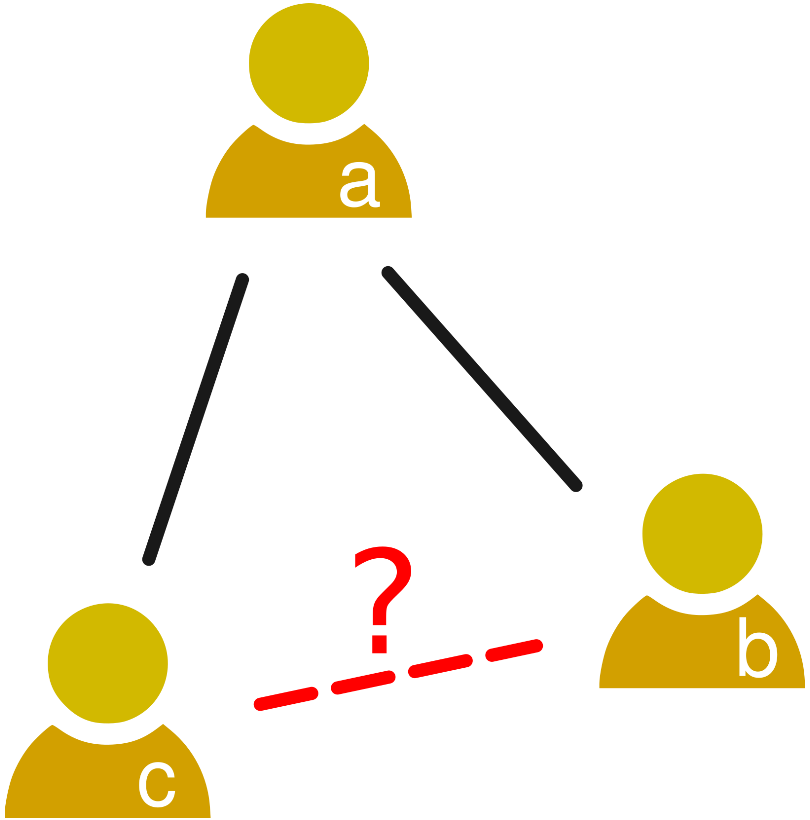
Local clustering coefficient:

$$C_i = \frac{2 * t(i)}{d_i * (d_i - 1)}$$

- $d_{out}(i)$ is the degree of i (>1)
- $t(i)$ is the number of pairs of neighbors of i that are connected

Average clustering coefficient:

$$C = \frac{1}{N} \sum_i C_i$$



The Watts-Strogatz model

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Collective dynamics of ‘small-world’ networks. Duncan J. Watts & Steven H. Strogatz. Nature (1998)

Clustering versus small distances

- Triangles reduce distances: all nodes in a triangle are at distance one
- Clustering and short path lengths appear to be opposing properties
- Social networks have high clustering (lots of triangles), can they also have short paths? Is the six degrees observation a robust one?

Research question: can a model produce networks with both high clustering and low average path distance?

Additional conditions:

- Network is large ($N \gg 1$)
- Network is sparse, like a social network ($\langle k \rangle \ll N$)

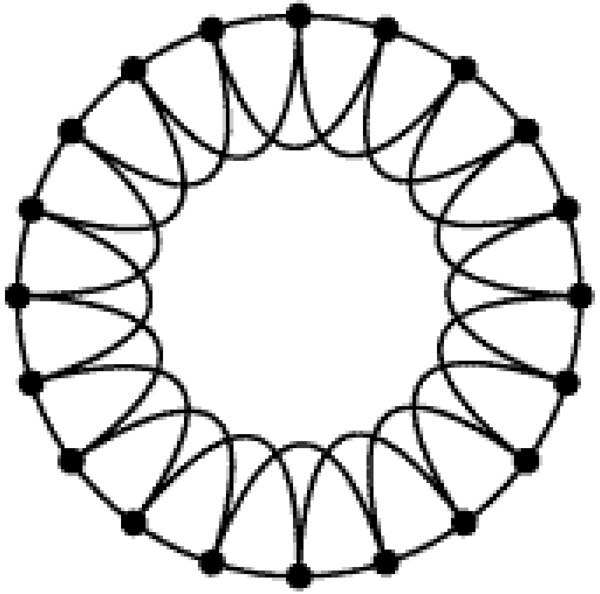
The Watts-Strogatz model

- Start with a fixed ring where N nodes are connected to k neighbors in the ring

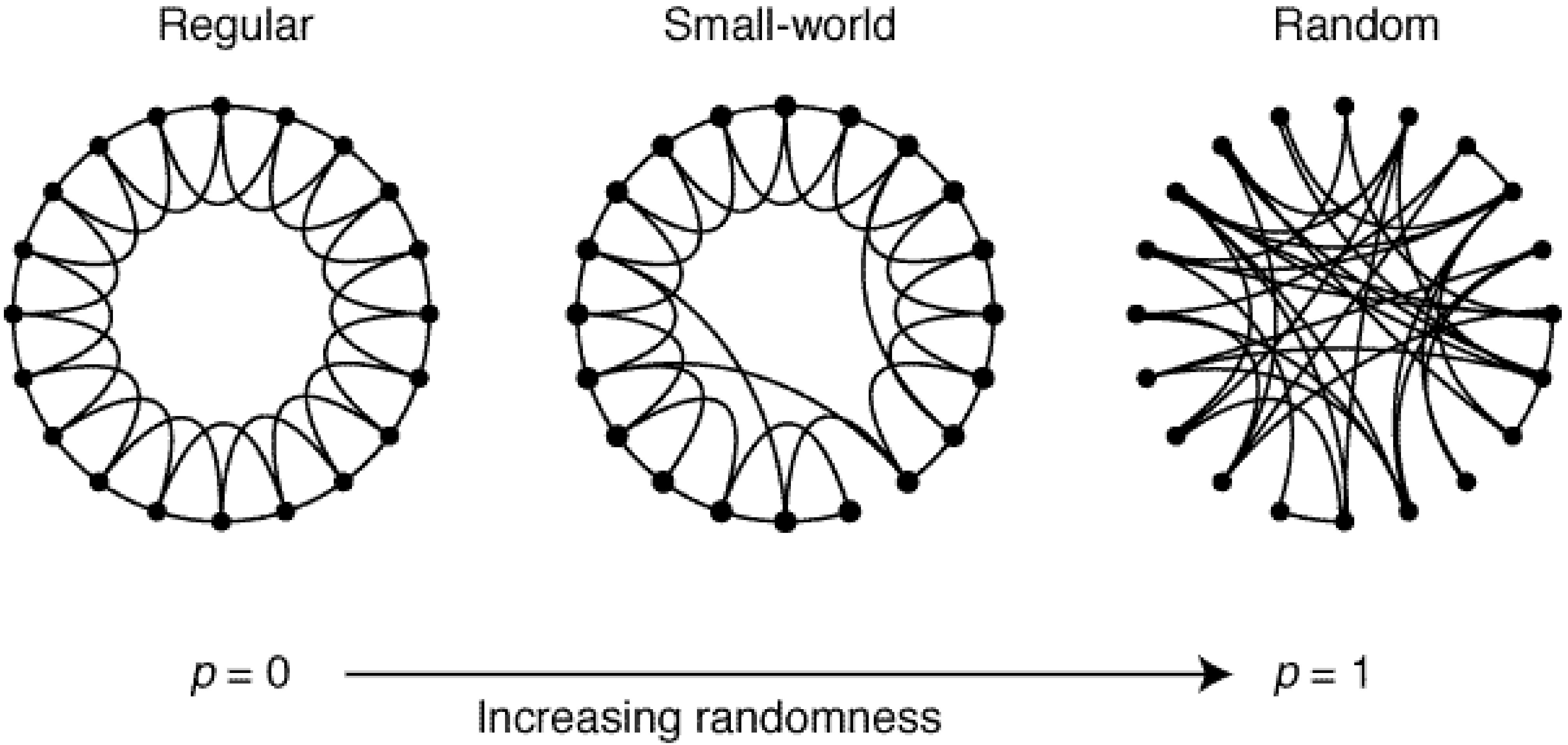
(example: $N = 20$, $k = 4$)

- For each edge:
  - With probability p: rewire the edge uniformly at random

(two versions: only one endpoint or both are rewired)



From order to randomness



Properties of the lattice ($p = 0$)

Average clustering coefficient:

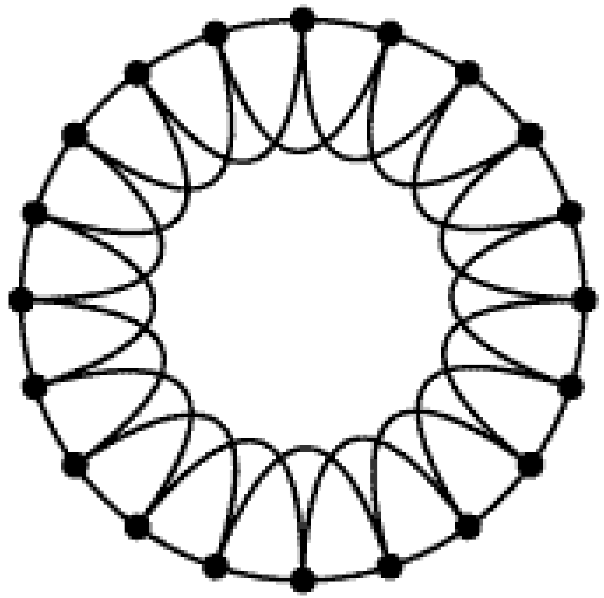
$$C(0) \sim 3/4$$

(tunable value of C based on k)

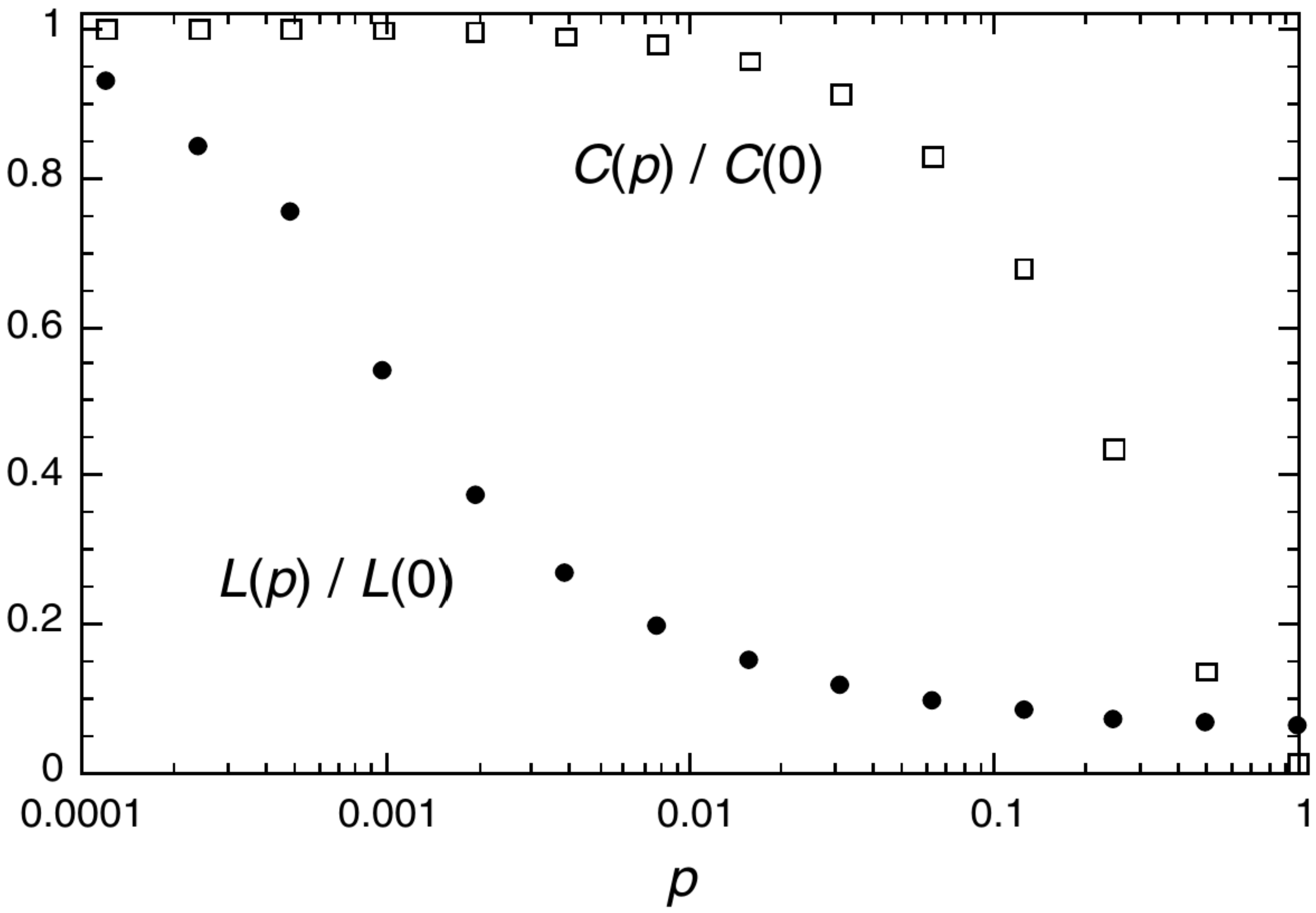
Average path length:

$$L(0) = \frac{N}{2k}$$

(very high path length, grows linearly with network size)



C(p) and L(p) versus C(0) and L(0)



Empirical versus random networks

Empirical network analysis of L and C compared to random networks with the same number of nodes and edges (null model, $G(n, m)$)

Network	L_{actual}	L_{random}	C_{actual}	C_{random}
Film actors	3.65	2.99	0.79	0.00027
Power grid	18.7	12.4	0.080	0.005
C. elegans	2.65	2.25	0.28	0.05

Evidence of small world networks: empirical networks have average path lenghts close to random networks but much higher average clustering

Social small world model

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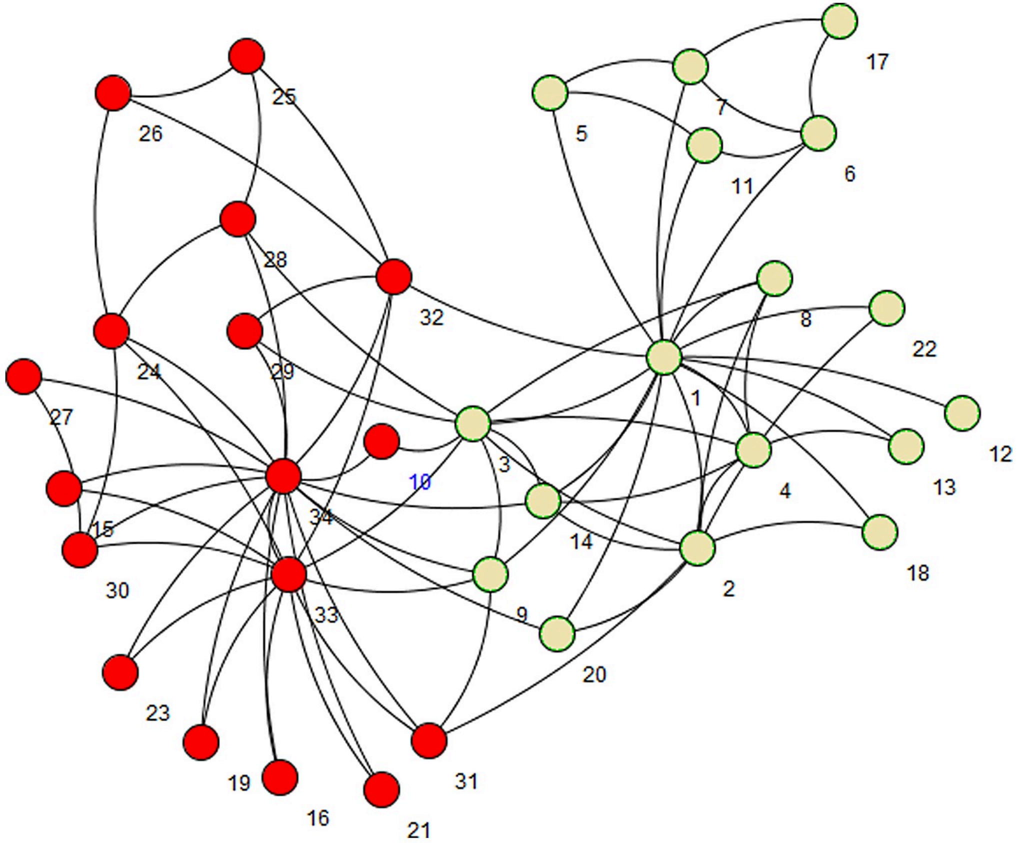
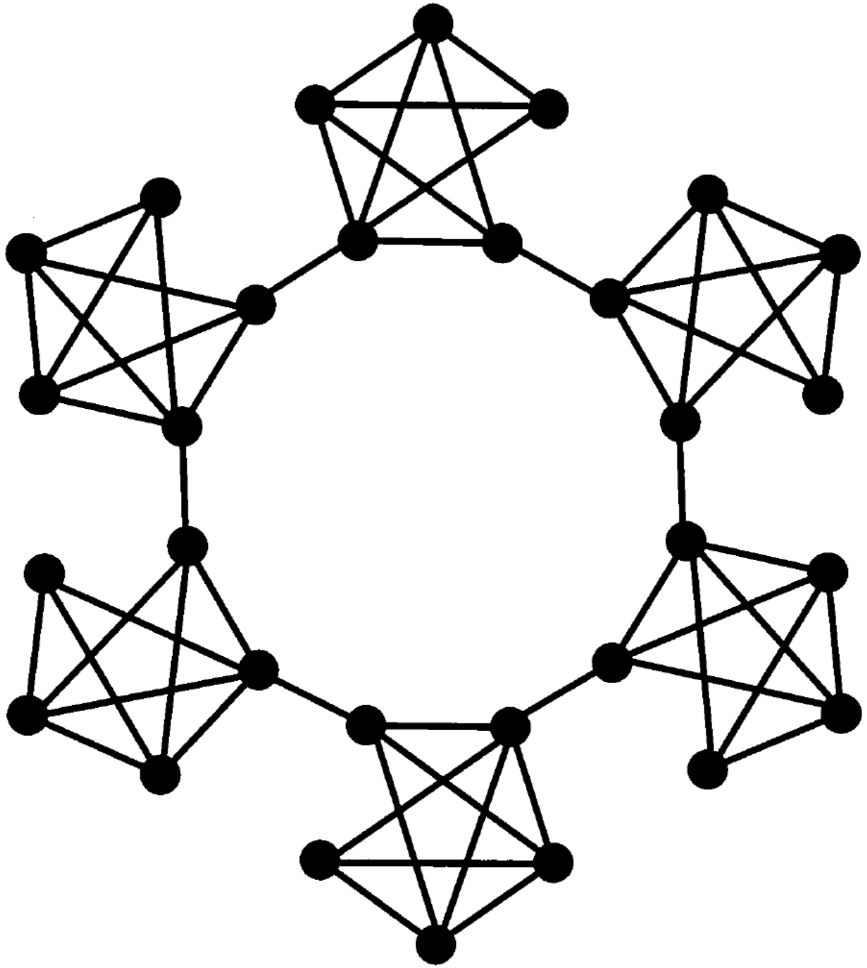
Networks, Dynamics, and the Small‐World Phenomenon. Duncan J. Watts. American Journal of Sociology (1999)

Social networks aren't lattices



Extreme case: the caveman graph



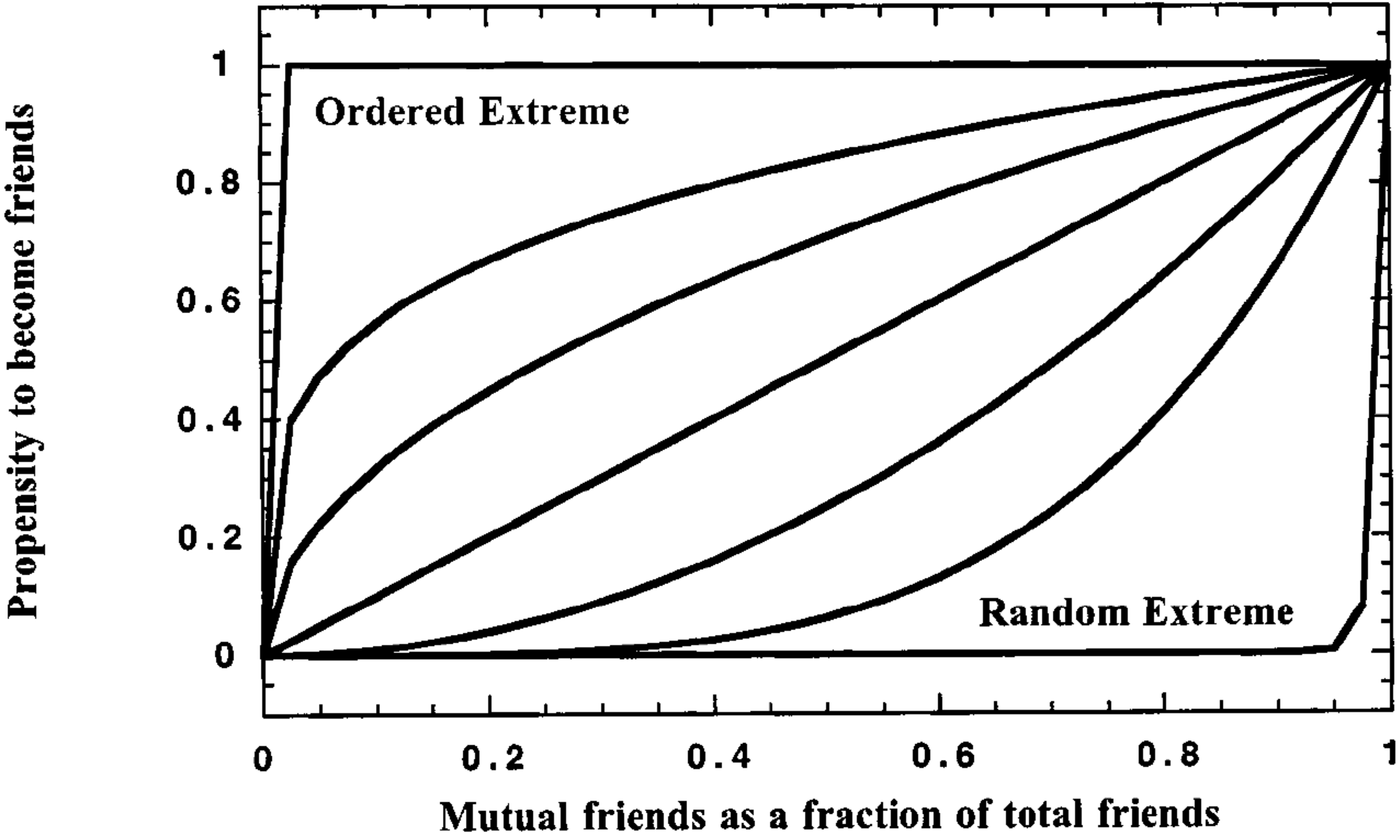
Modeling social small worlds

- Start with a ring of n nodes
- For each pair of nodes (in random order):
  - Calculate number of shared friends $m_{i,j}$
  - Calculate propensity to connect $R_{i,j}$ based on $m_{i,j}$
  - Connect them with probability $R_{i,j}$

$R_{i,j}$ has a non-zero but low base probability p of any two nodes connecting regardless of their number of shared friends

The dependence of $R_{i,j}$ on $m_{i,j}$ interpolates between a regular (caveman network) and a random network. The reality should be somewhere in between

Modeling propensity to triadic closure

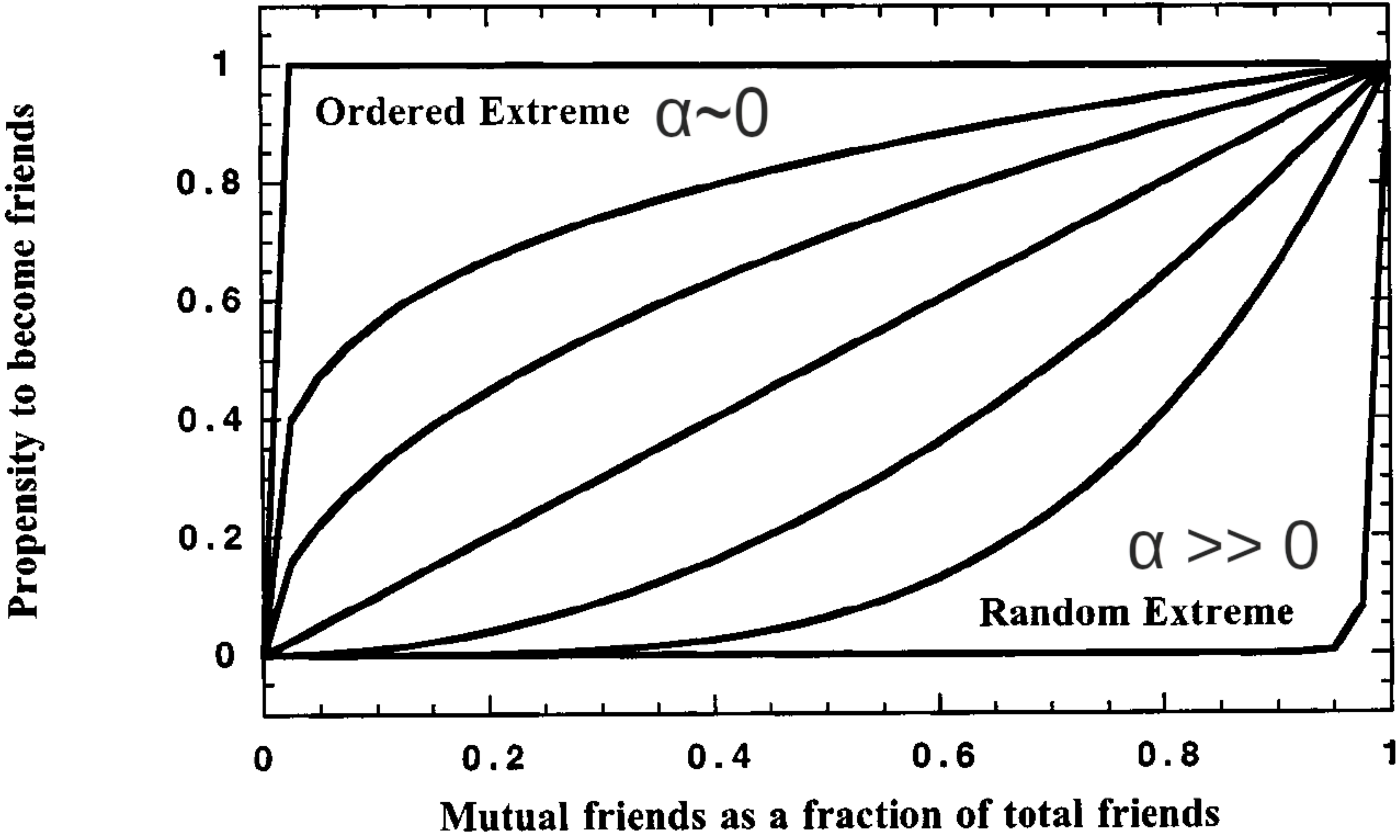


Formalizing propensity to triadic closure

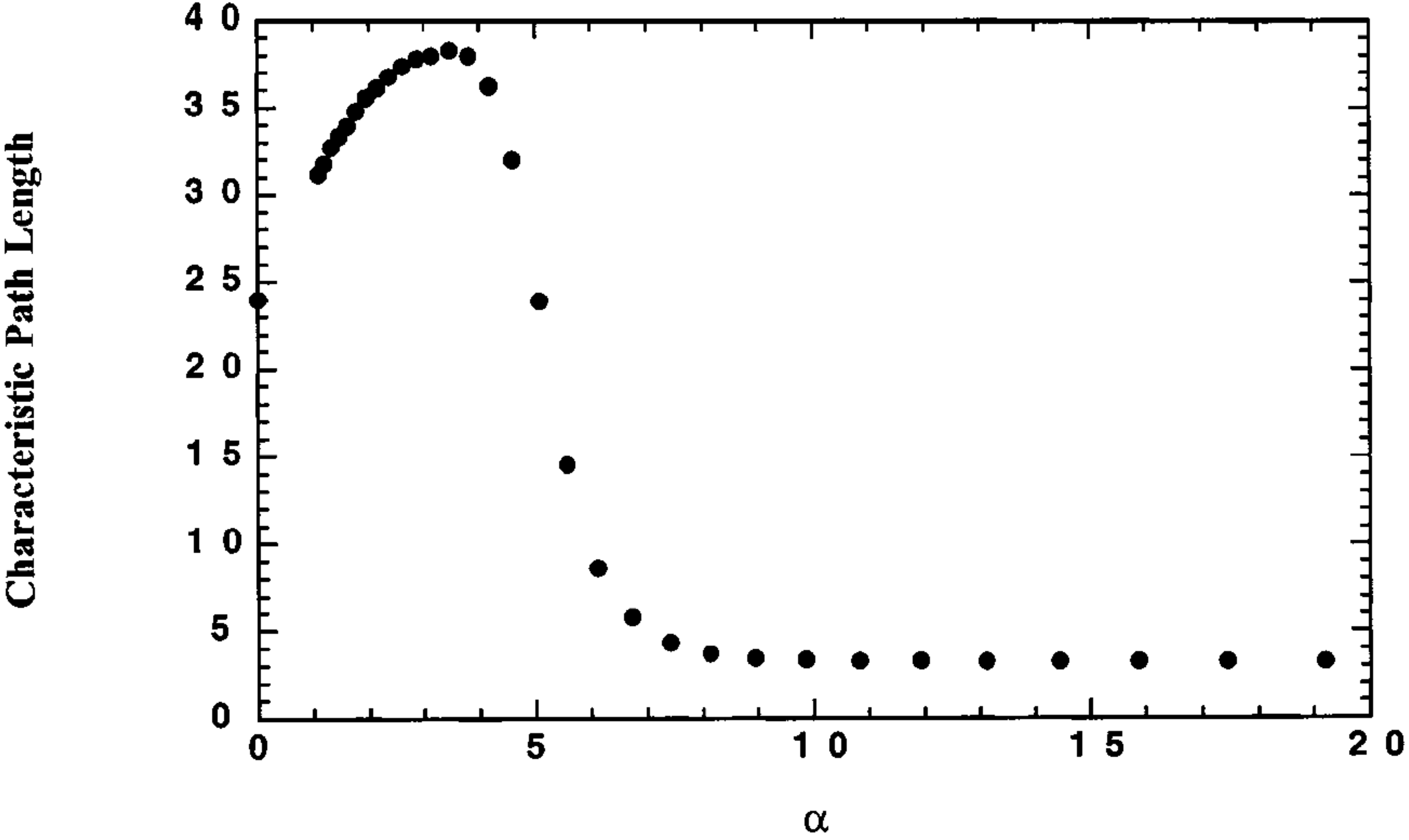
$$R_{i,j} = \begin{cases} 1 & m_{i,j} \geq k \\ \left[\dfrac{m_{i,j}}{k}\right]^{\alpha} (1 - p) + p & k > m_{i,j} > 0 \\ p & m_{i,j} = 0 \end{cases}$$

- $R_{i,j}$: propensity to connect of agents i and j
- $m_{i,j}$: mutual friends between i and j
- p: base probability to connect when no common friends
- k: average degree of network
- α: exponent that defines curvature of $R_{i,j}$ versus $m_{i,j}$
    - $0 \leq \alpha \leq \infty$

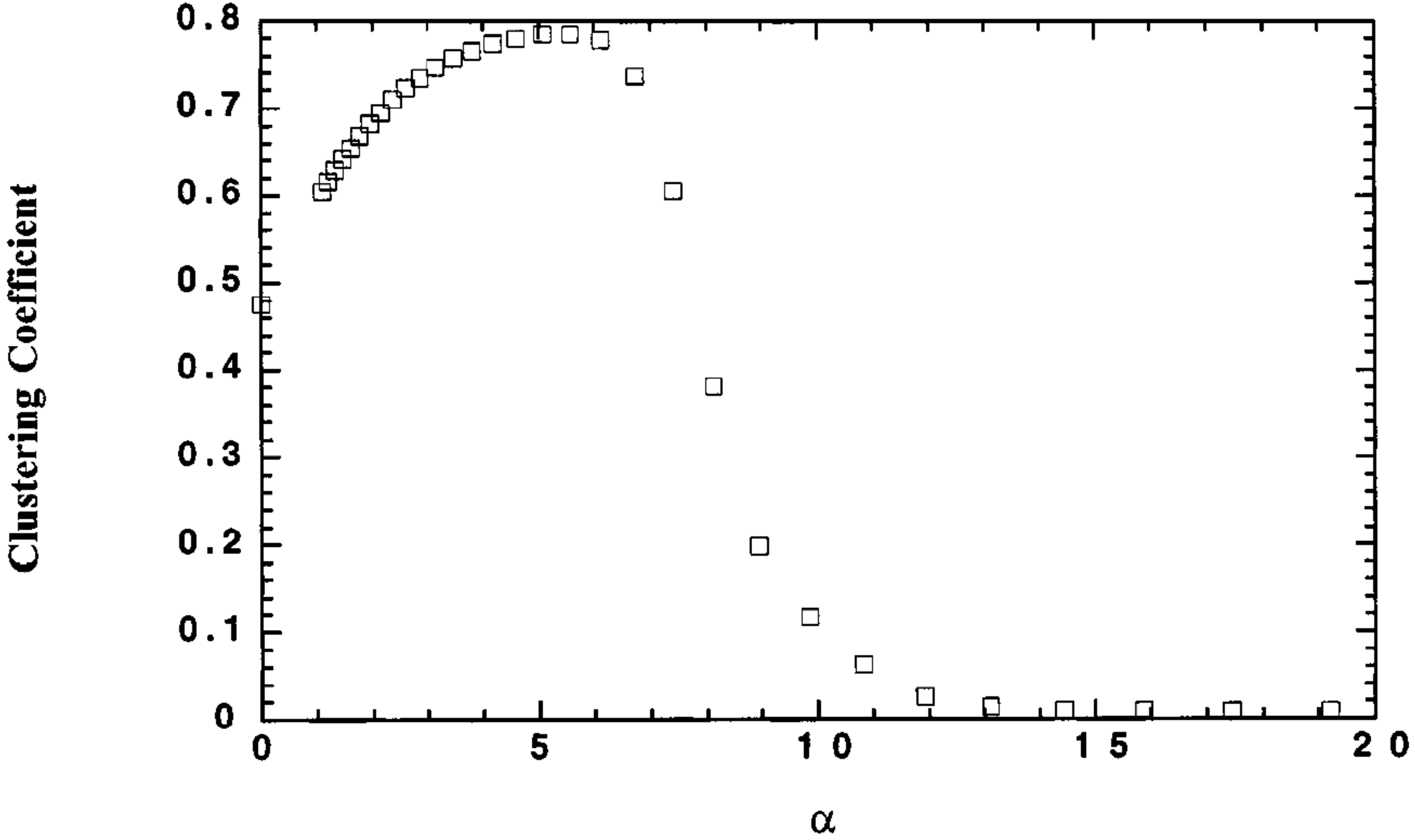
α in propensity



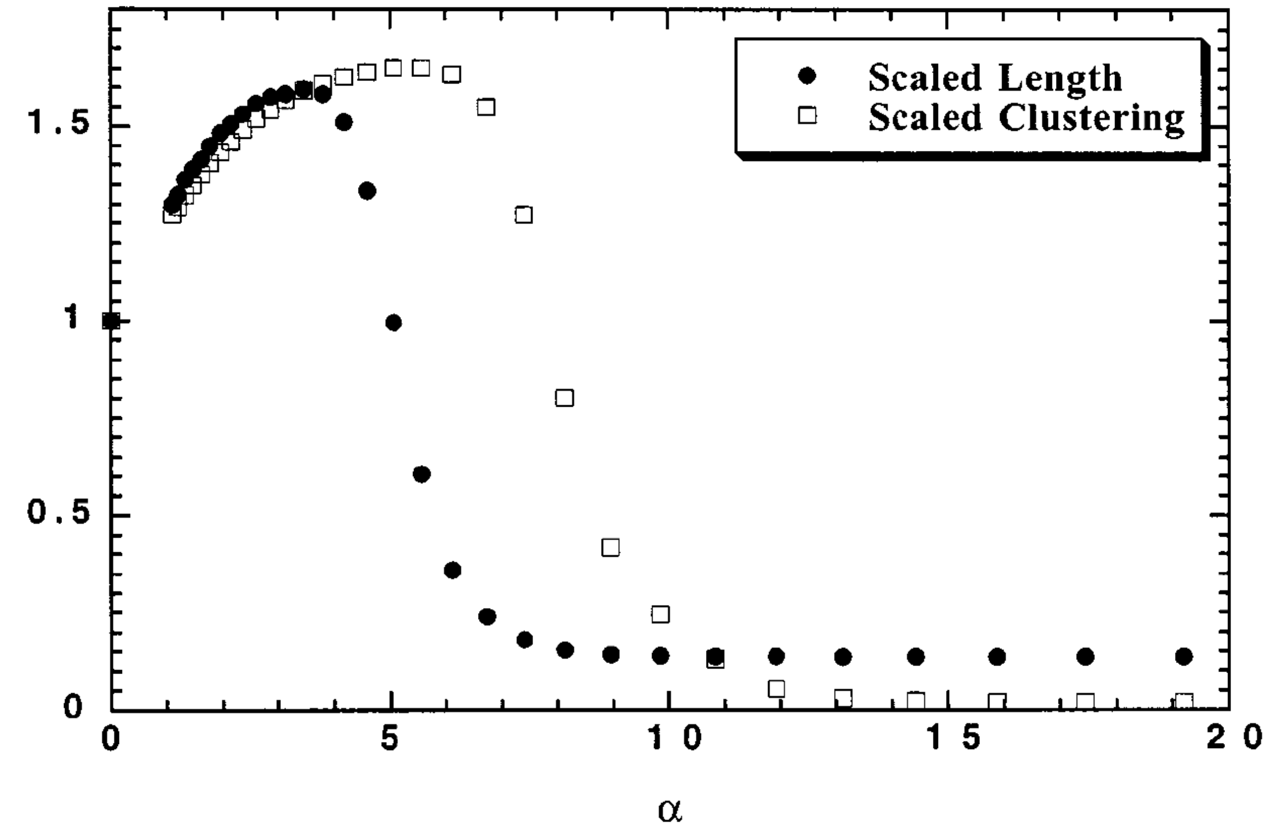
Avg path length vs α



Clustering coefficient vs α



"Small-worldness" vs α



For an intermediate range of values of α, average path length is low while the clustering coefficient is high.

Summary

- The small world phenomenon
  - Anecdotes and literature suggest that the distance between two random people in the world is not so large
  - Refresher: average path length and average clustering coefficient

- The Watts-Strogatz model
  - From a regular lattice to a random network rewiring with probability p
  - For a wide range of middle values of p, clustering is high and average path lengths are low
  - Empirical networks show this when compared to random networks

- Social small world model
  - Cavemen rather than lattices as regular network
  - α modulates shape of propensity to connect vs friendships share
  - Shows small-world behavior for an intermediate range of α

Quiz

- What fraction of letters in Milgram's experiment reached their destination?
- What is the average path length of a network with an isolated node?
- What is the average clustering coefficient of a tree?
- What is the null model to compare small-world measurements?
- What is a better model of a social network, a lattice or a cavemen network?
- if α is zero, do you connect to the friends of your friends very often?