Basic network models

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Computational Modelling of Social Systems

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So far

- **Block 1: Fundamentals of agent-based modelling**
- **Block 2: Opinion dynamics**
- **Block 3: Fundamentals of agent-based modelling**
    - $\boxed{\textbf{Today: Basic network models}}$
    - Modelling small worlds
    - Scale-free networks

Overview

1. Random graphs
2. Poisson random graphs
3. Giant component
4. Null models

Random graphs

1. Random graphs

2. Poisson random graphs

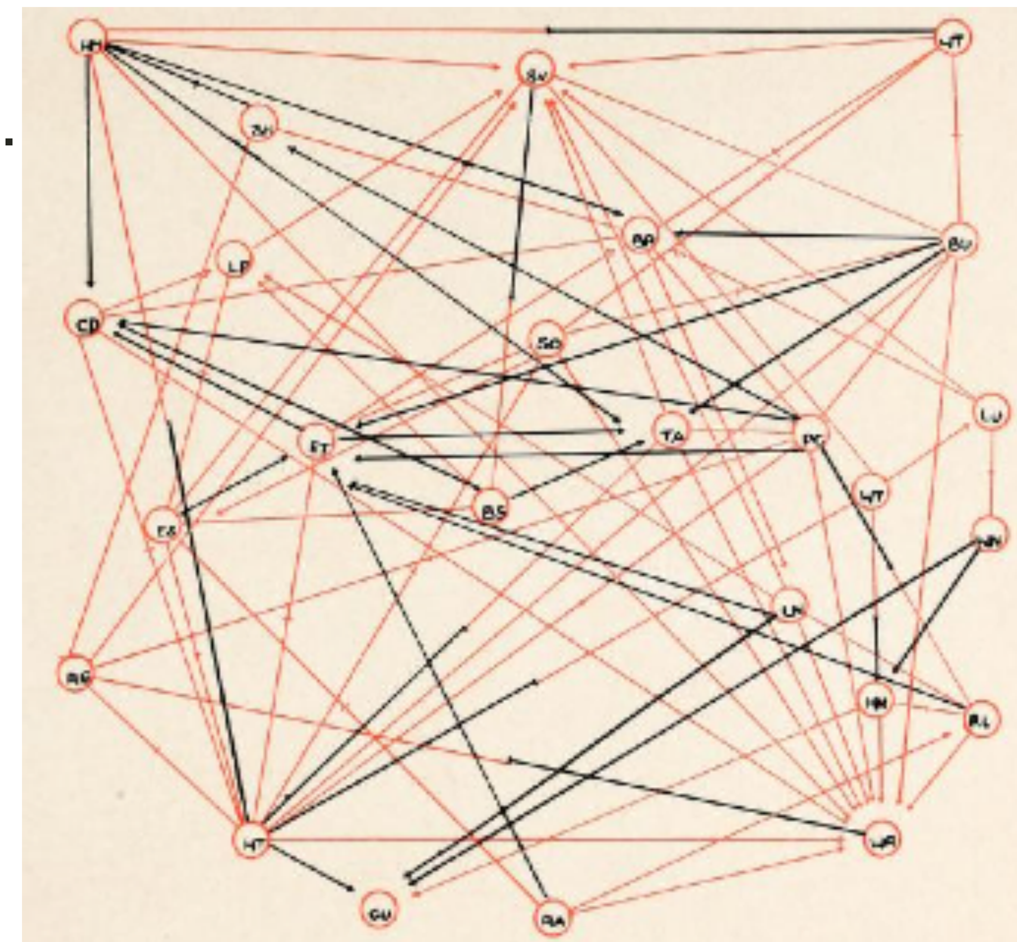
3. Giant component

4. Null models

Refresher: Social networks

Social Networks are models to represent individuals and the relationships between them. The minimal components of a social network are:

- **Nodes** (or vertices), which represent individuals. These individuals are social actors, for example, humans, animals, fictional characters...
- **Links** (or edges) are relationships between individuals, for example, friendship, family ties, interaction, communication...



Jacob Moreno's sociogram

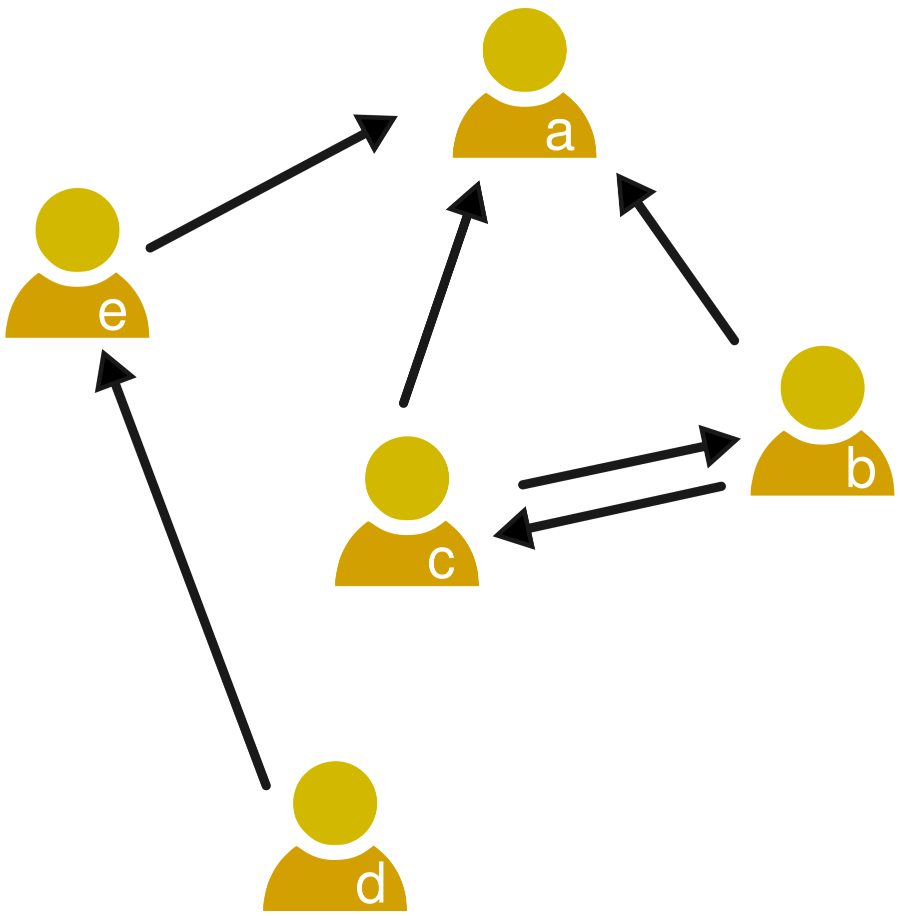
Refresher: Representing social networks

A graph or network is a tuple $G = (n, m)$

- n is a set of vertices or nodes
- m ⊆ n × n is a set of edges or links
- n × n is the Cartesian product (i.e. the set of all possible links)

Edges are denoted as ordered pairs (i, j), which means that a link points from node i to node j.

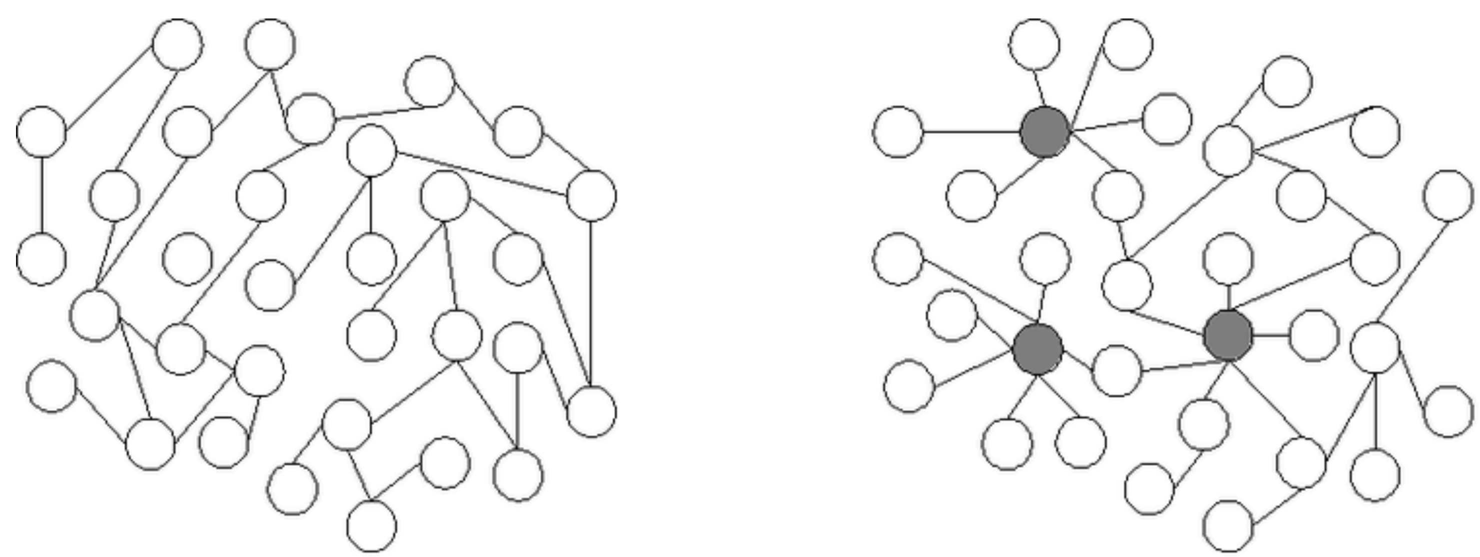
The example of the picture can be written as:
$n = a, b, c, d, e$
$m = (b, a), (c, a), (e, a), (d, e), (c, b), (b, c)$



Random graphs

- **Networks** that possess a particular property
  - Which are otherwise random (e.g., fixed degree distribution)

- **Analysis** performed by comparing (two) different model network structures
  - E.g., non-scale-free networks vs. scale-free networks



(a) Random network

On average all nodes have two edges with low variability

(b) Scale-free network

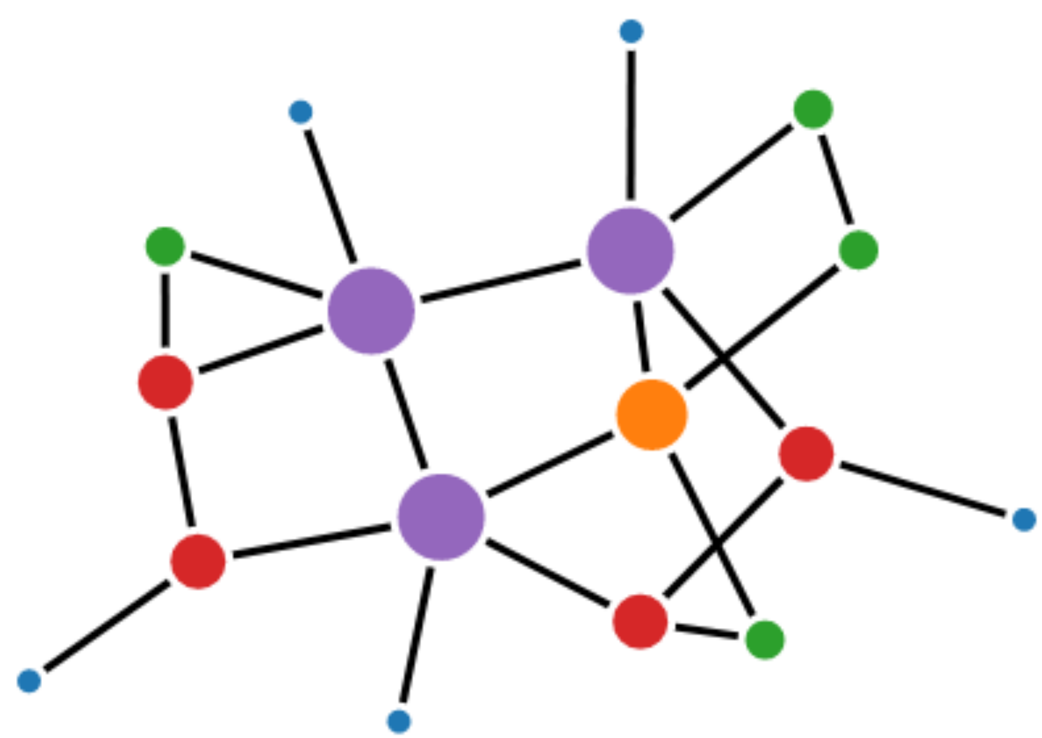
There are hubs (highly connected nodes) to nodes with low connectivity

Random graphs

- **Measurements** of network data for (two) different structures
  - Mathematically
  - Computationally
  - **Statistically**

- **Behavior** of such mathematical models of (two) different network structures
  - E.g., dynamic processes on networks - interaction between nodes in a social (connected) network
  - Consensus, opinion formation models ...

Random graph ensamble



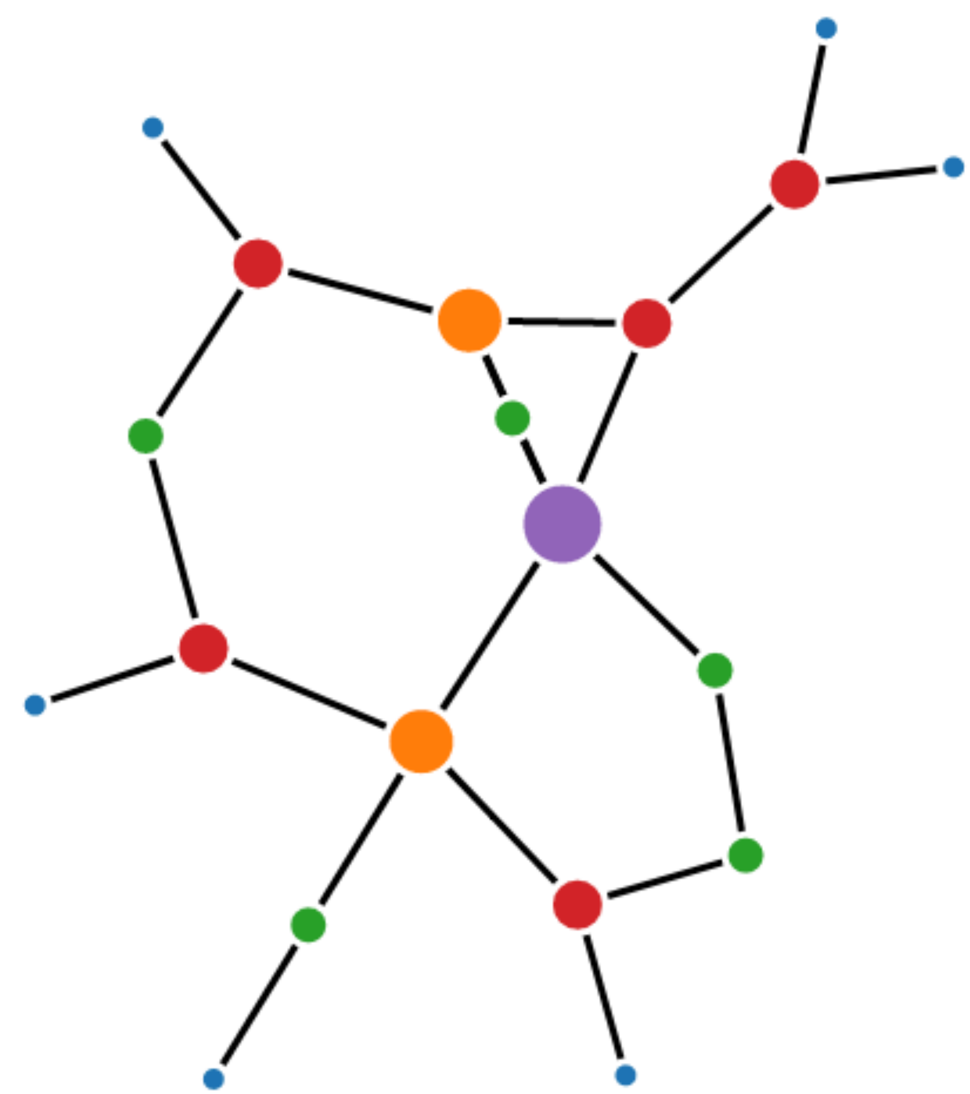
- Consider a **simple graph** with the following fixed parameters:
  - Number of vertices (n)
  - Number of edges (m)

- For such **Random graph G(n,m)** created in the following method:
  - Place edge (m) between (uniformly) random chosen pair
  - Prevent multiedge/self-edge by: distinct pairing, only to already not connected vertices

Random graph ensamble

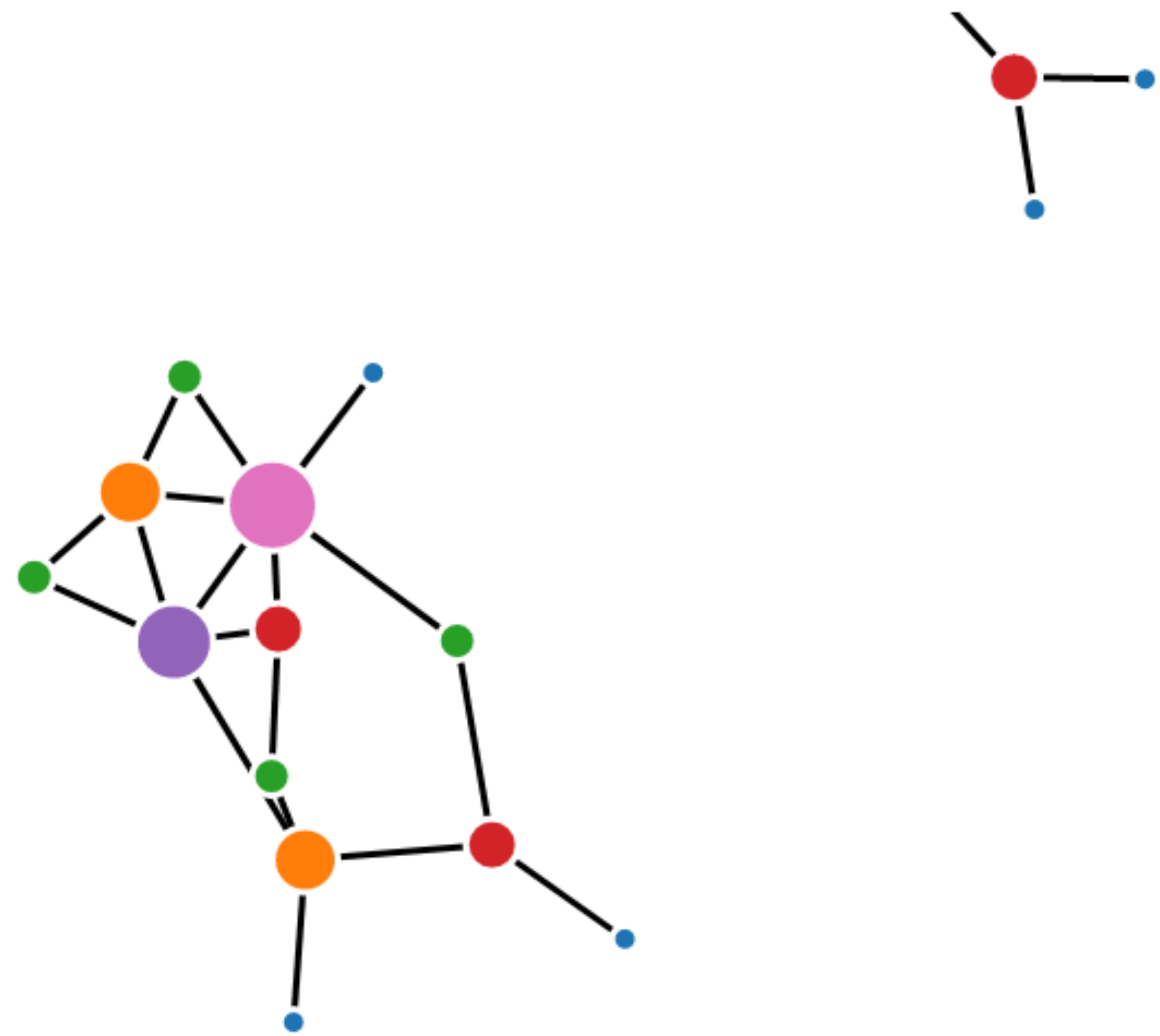
- There is not one but an ensamble of networks that satisfy the fixed constraints
  - E.g., for n=20 and m=22

- Probability of such networks $P(G)$
  - $P(G) = \frac{1}{\omega}$, for such networks where ω is the number of such networks
  - $P(G) = 0$, for all other networks

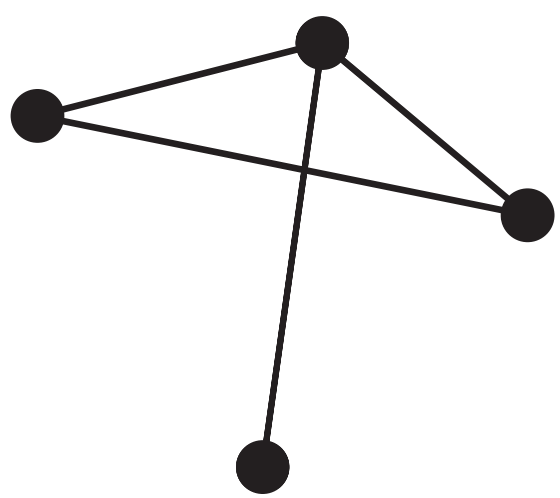


Random graph ensamble

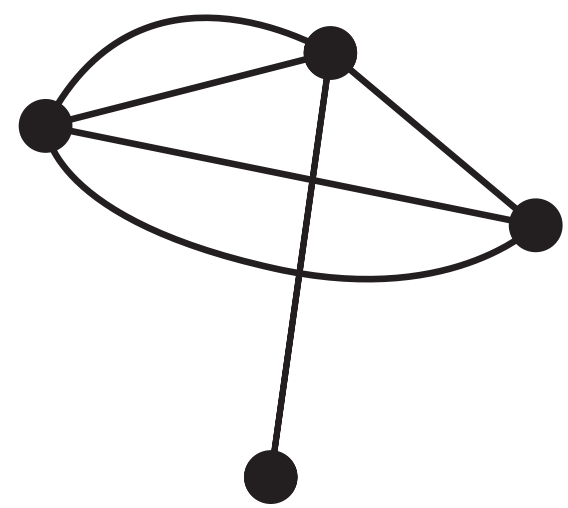
- Segregated components are also possible



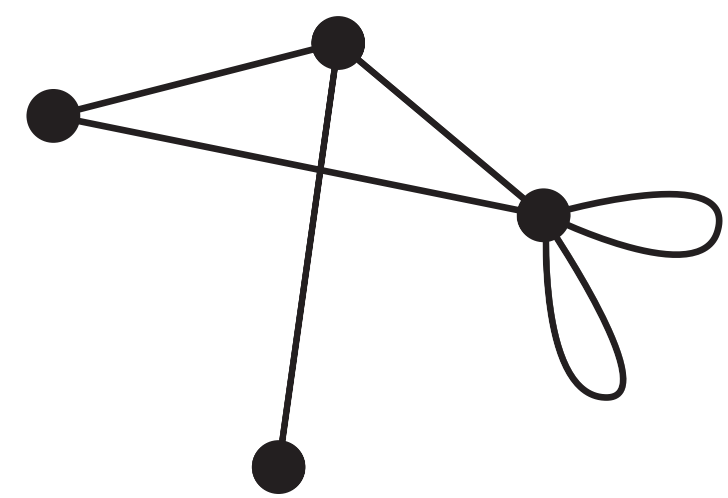
Simple graph



simple graph



nonsimple graph with multiple edges



nonsimple graph with loops

Random graph ensamble

- Ensambles have *typical* behavior based on **properties (averages)**
  - Number of edges is the number of connections between all vertices

$$\langle m \rangle$$

  - Diameter of a graph is the largest finite distance between any two vertices

$$\langle l \rangle = \sum_G P(G) l(G) = \frac{1}{\omega} \sum_G l(G)$$

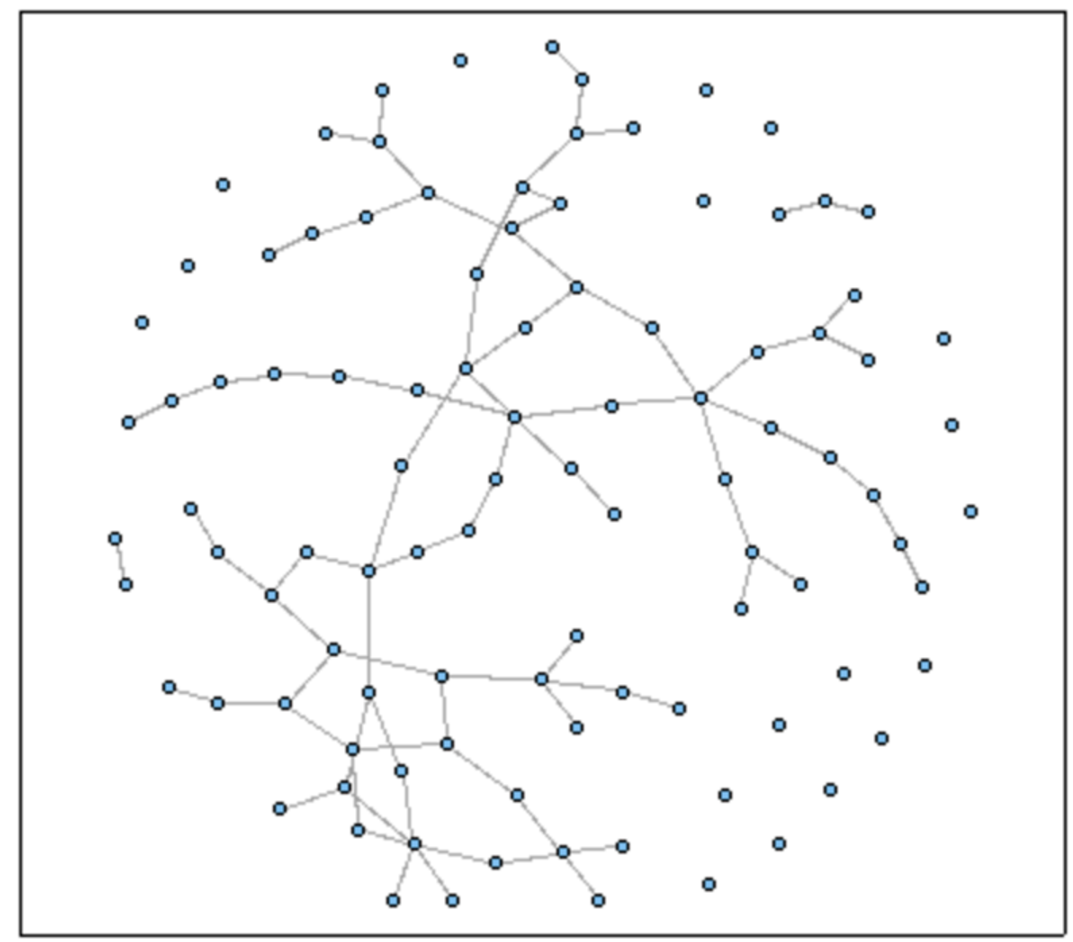
  - Degree of a vertex is the number of adjacent edges, **also known as** c

$$\langle k \rangle = c = 2\frac{m}{n}$$

Two interesting random graphs ensembles

1. G(n,m) - fixed number of edges m
2. G(n,p) - fixed **probability** of edges $p(m)$
   - The edges m are not-fixed
   - The vertices n are fixed

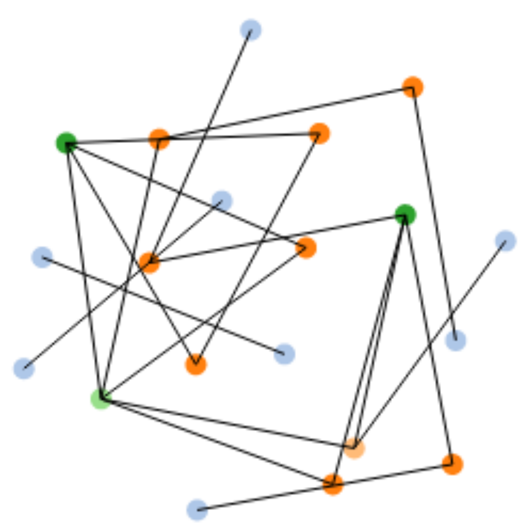
- E.g., Erdos - Renyi model



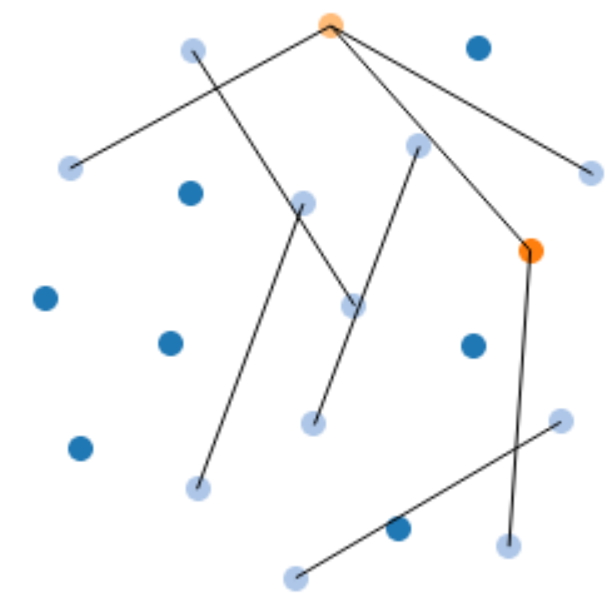
G(n,p) ensamble

- Probability of such G(n,p) networks $P(G)$:
  - $P(G) = p^m (1-p)^{\binom{n}{2}-m}$, for such networks
  - $P(G) = 0$, for non-simple graphs

- p = 0.1



- p = 0.05



Analytical behavior analysis of G(n,p) ensemble

- Mean number of edges $\langle m \rangle$
  - **Distinct vertex pairs** $\binom{n}{2}$ * probability of an edge p;

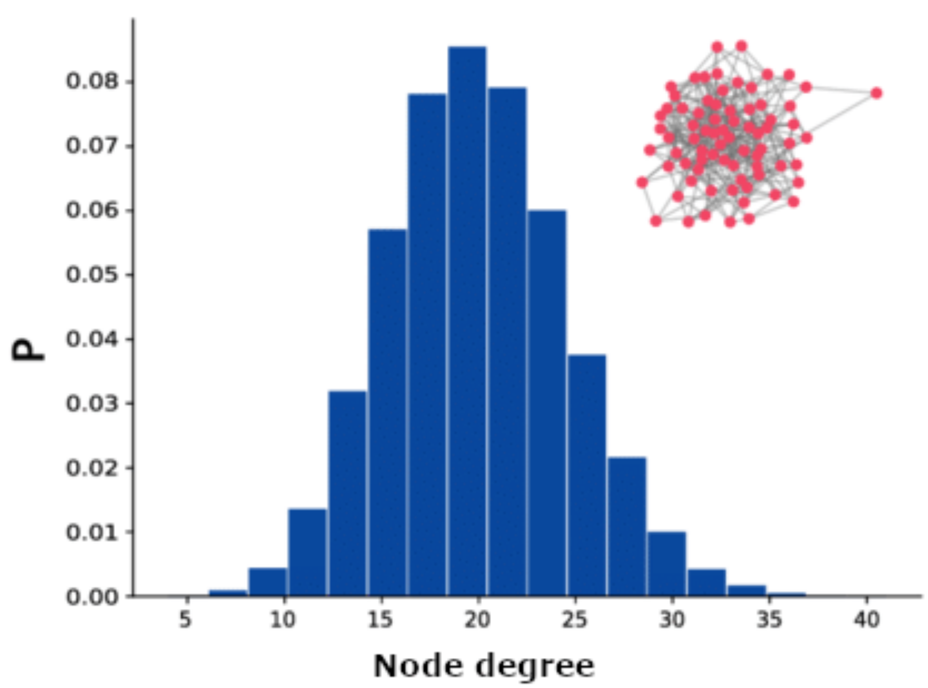
$$\langle m \rangle = \binom{n}{2} p$$

- Mean degree of vertex c
  - Number of other vertices $(n-1)$ * probability of an edge p

$$c = (n-1)p$$

- Degree distribution p_k (for a vertex connected with k others)
  - **Distinct vertex pairs** $\binom{n-1}{k}$ * probability of connection with k vertices, but not with others

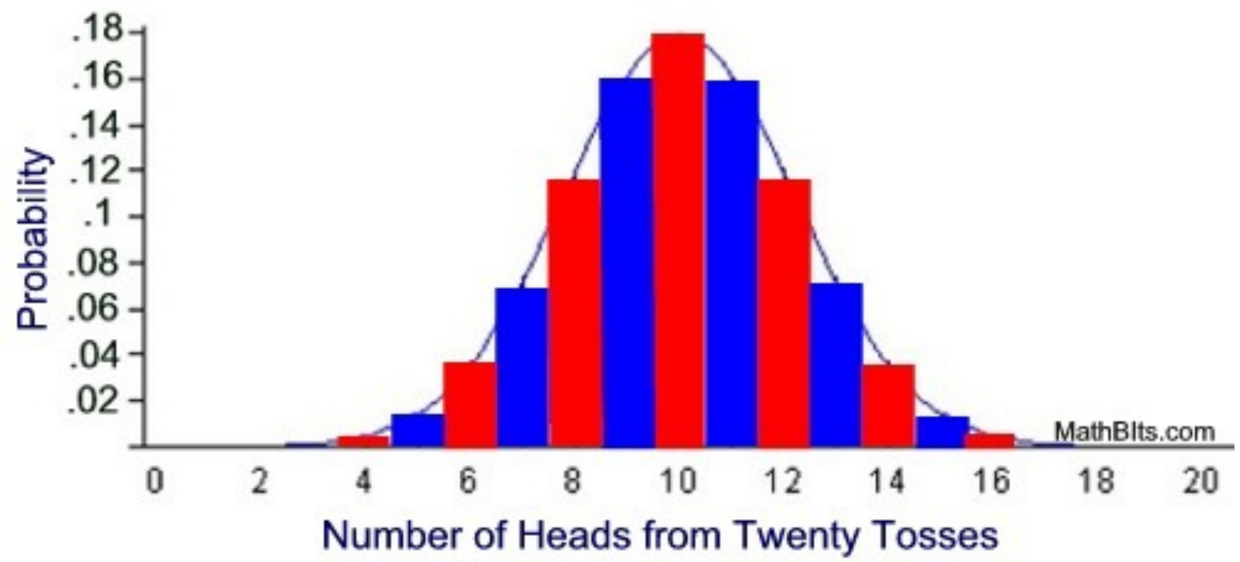
$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$



Degree distribution with $n = 10^5, c = 20$ is a **binomial distribution**

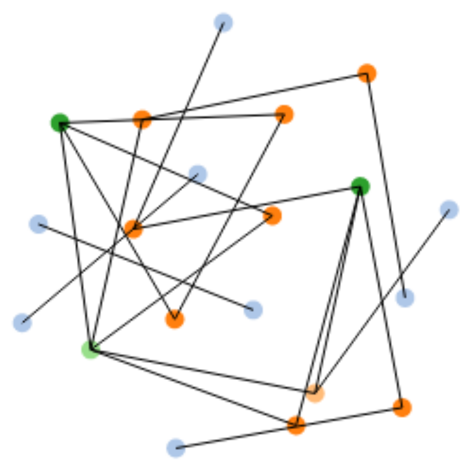
Recap basics: Binomial distribution

- Binomial distribution with parameters n and p
  - The discrete probability distribution of the number of successes in a sequence of n independent experiments
  - Each experiment asks a yes-no question (boolean-valued outcome)
  - Success (with probability p)
  - Failure (with probability q = 1 − p)

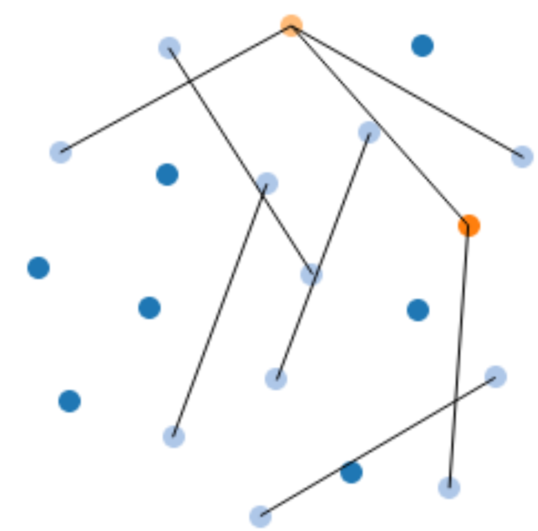


Example G(n,p) ensemble

- $n = 20, p = 0.1$
- $\langle m \rangle = 190 * 0.1 = 19$
- $c = (20 - 1) * 0.1 = 1.9$
- $p_3 = 969 * 0.001 * 0.185 = 0.18$



- $n = 20, p = 0.05$
- $\langle m \rangle = 190 * 0.05 = 9.5$
- $c = (20 - 1) * 0.05 = 0.95$
- $p_3 = 969 * .000125 * .44 = .05$



Poisson random graphs

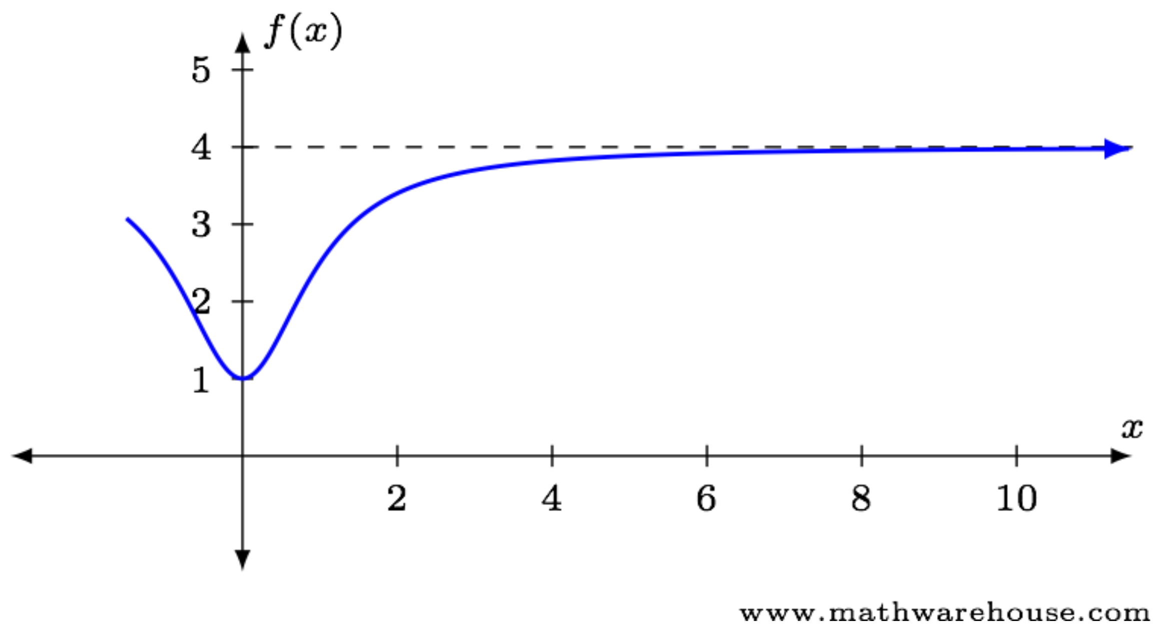
1. Random graphs

2. *Poisson random graphs*

3. Giant component

4. Null models

Function asymptotically approaches some value



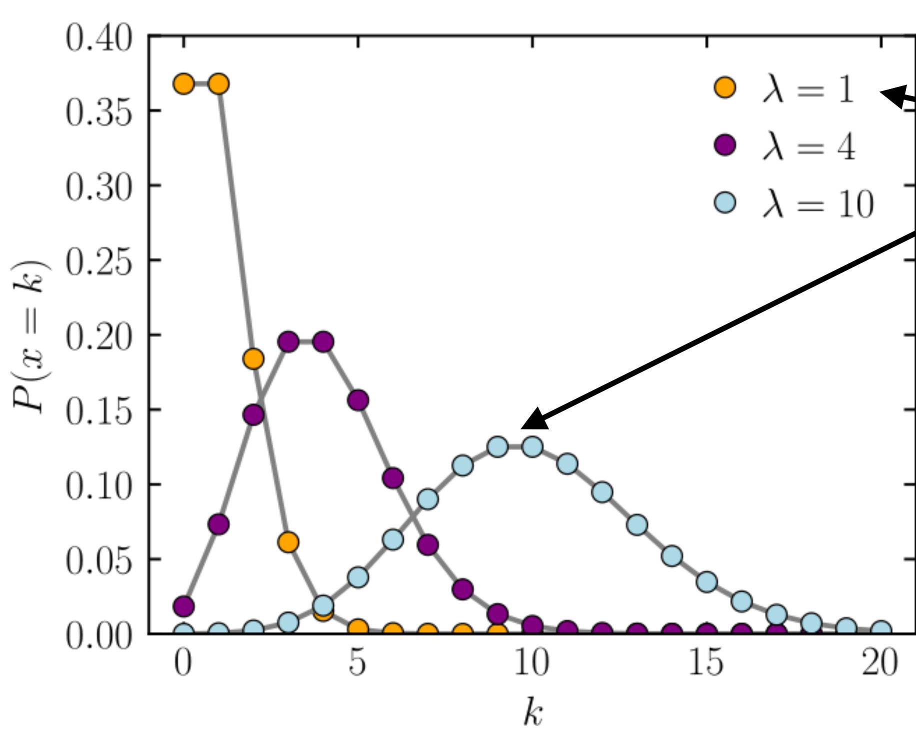
- Asymptotically equal-to/approaches some value
  - The function approaches a particular finite value
  - Horizontal asymptotes are defined as limits at infinity

Large networks $n \to \infty$

- What happens to the degree distribution as n goes to infinity $n \to \infty$

$$\lim_{n\to\infty} \binom{n}{k} p_n^k (1-p_n)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

- The binomial distribution approaches the **Poisson distribution**



λ occurs at the pick of the distribution

Which turns to be c (average degree of the nodes)

Poisson random graph

- Special case of G(n,p) ensemble for large networks $n \to \infty$
- Mean degree c - constant over large networks
  - Number of friends does not depend on the number of people in the world

$$c = (n-1)p \Rightarrow p = \frac{c}{n-1}, n \to \infty$$

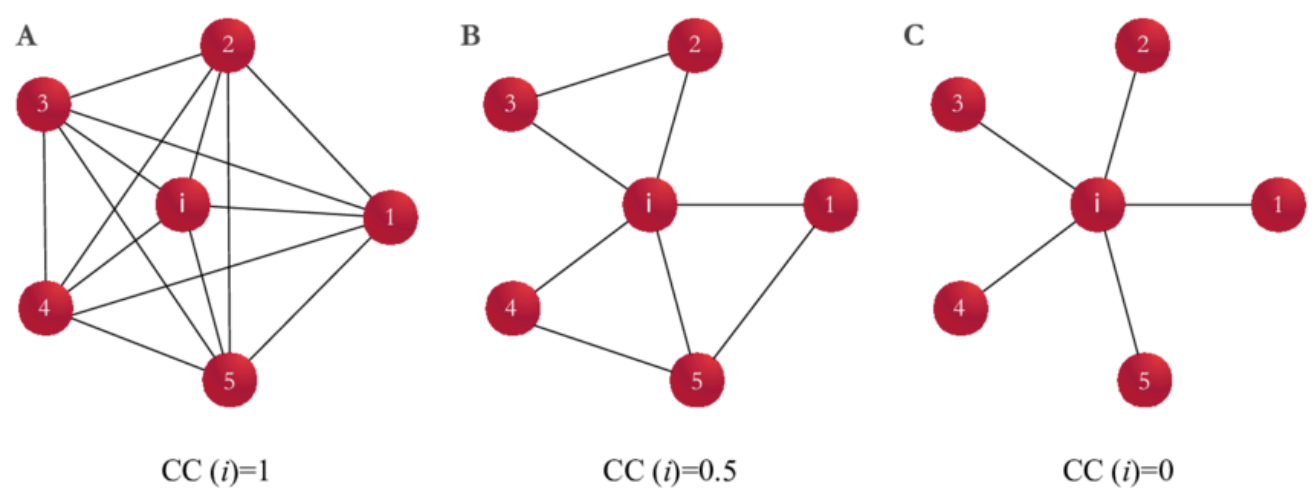
- Vanishingly small probability p of an edge
- Degree distribution (for a vertex) p_k

$$p_k = e^{-c}\frac{c^k}{k!}, n \to \infty$$

Analytical behavior analysis of G(n,p) ensemble

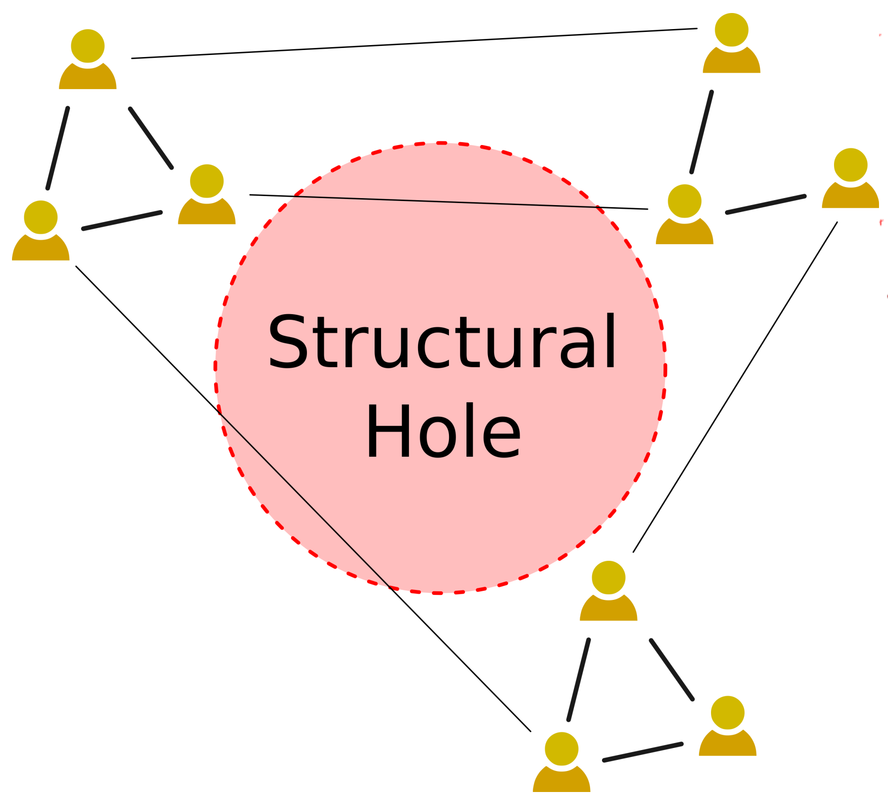
- Clustering coefficient C
  - Transitive behavior (property) in a network
  - What is the probability that two vertex neighbors are also neighbors of each other?

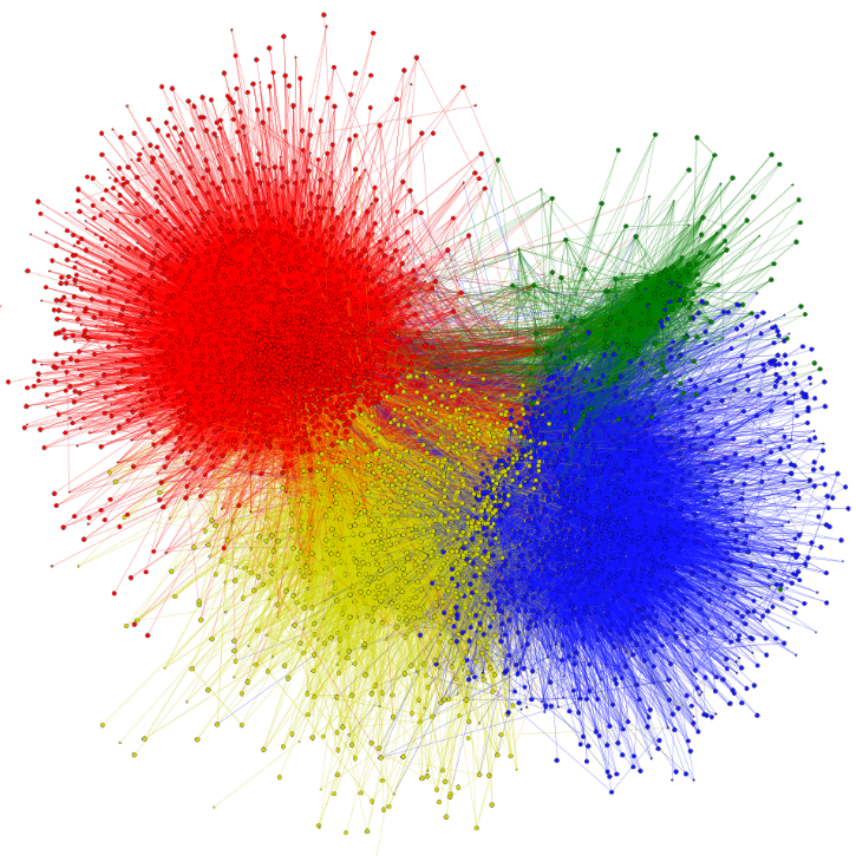
$$C = \frac{c}{n-1}$$



- Poisson random graphs for large networks
  - Clustering coefficient $C \simeq 0, n \to \infty$, if mean degree c is fixed

- Difference between the RNG and real-world networks: real-world networks have a high clustering coefficient C





Giant component

1. Random graphs

2. Poisson random graphs

3. $\boxed{\textit{Giant component}}$

4. Null models

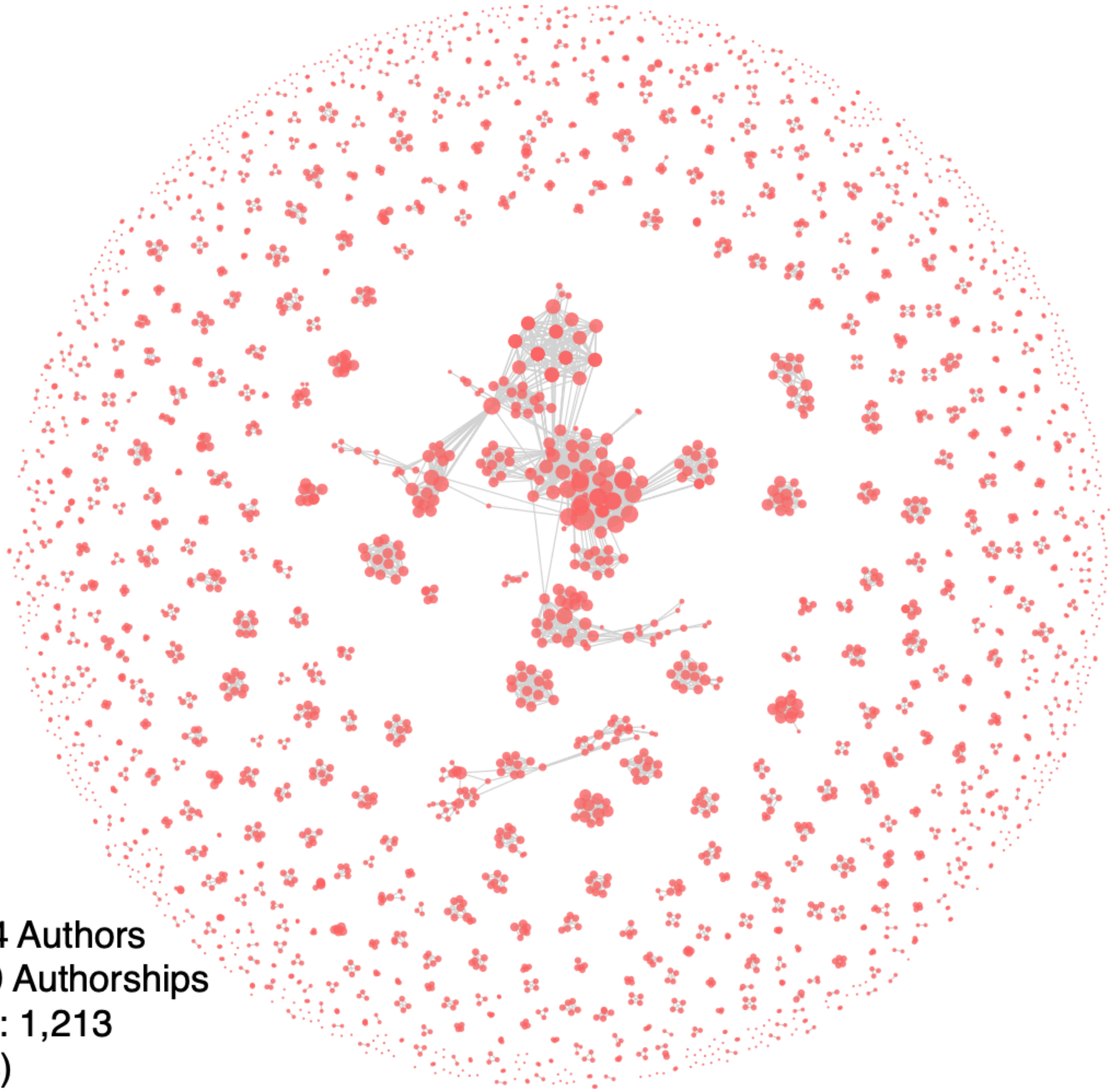
Largest component for large networks $n \to \infty$

- If $p = 0$; independent of size n
  - Disconnected network
  - n separate components - vertices isolated

- If $p = 1$; dependent of size n
  - 1-single component
  - Each vertex is connected to each other
  - **Giant component** - connected component of a network that contains most of the entire nodes in the network

- Most networks (should) have a large component that fills the network
  - To be able to perform the intended role

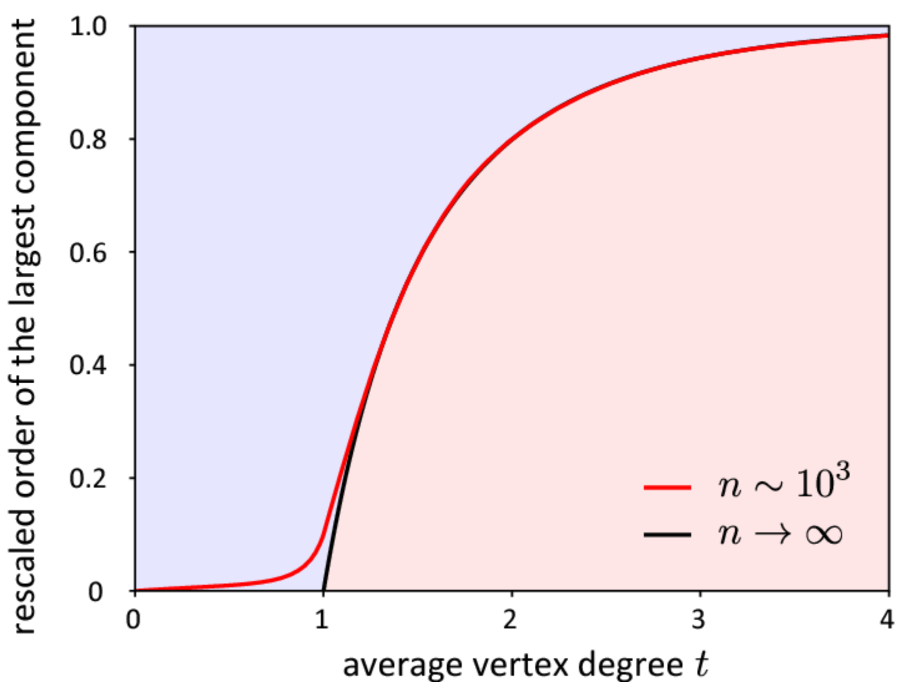
LARGEST CONNECTED COMPONENT OF A CO-AUTHORSHIP NETWORK



Nodes: 3,184 Authors
Edges: 5,110 Authorships
Components: 1,213
LCC: 55 (2%)
Node size: Degree

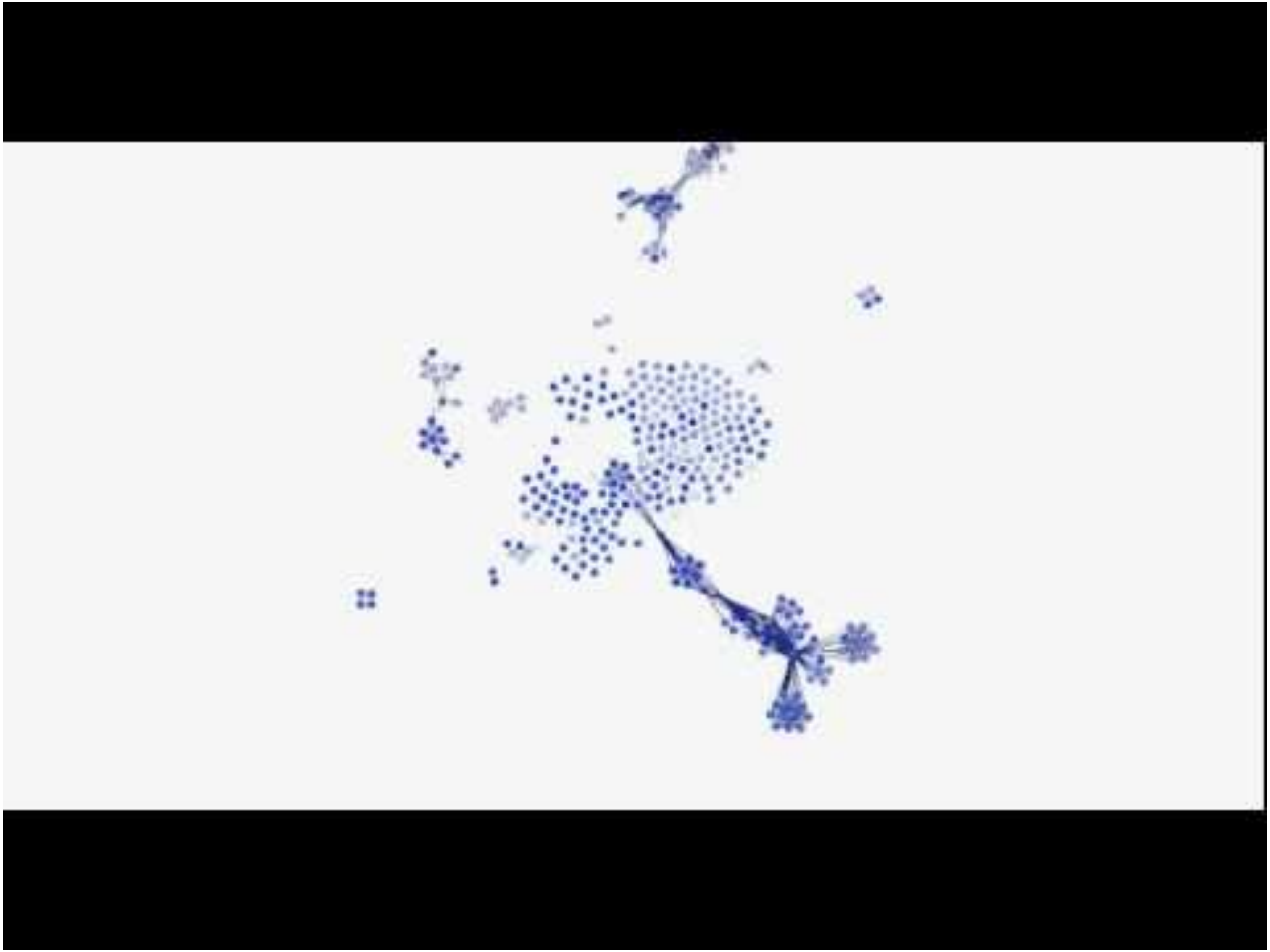
Sensitivity of parameter $p[0, 1]$ vs. largest component

- How the size to the largest component changes in respect to p?
  - Expected: The size increases gradually and it is **extensive** around $p = 1$
  - Reality: Largest component suddenly changes for one particular value of p
  - **Phase transition**

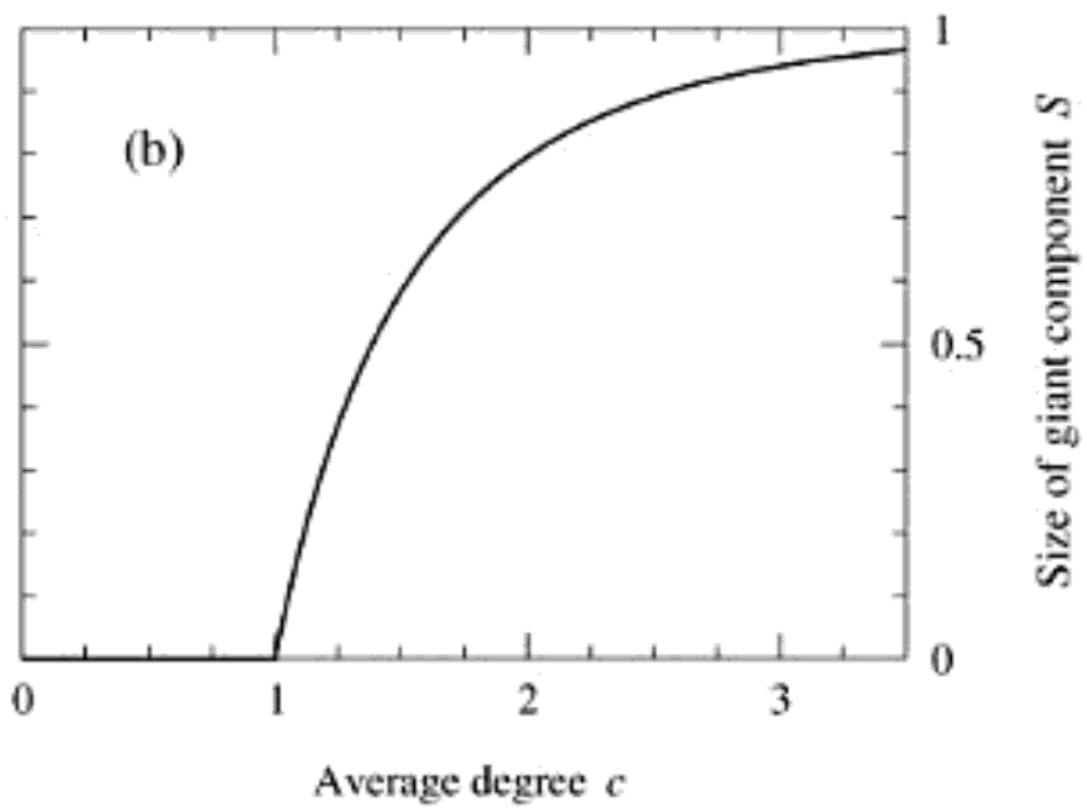


Kang, M., & Petrášek, Z. (2014). Random graphs: Theory and applications from nature to society to the brain

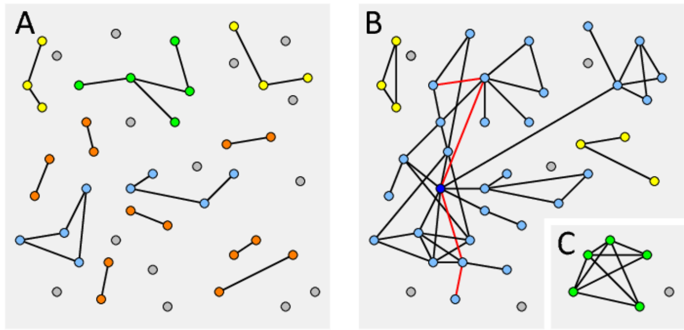
GROWTH OF THE LARGEST CONNECTED COMPONENT AND CONNECTIVITY OF THE NETWORK



Size of the giant component

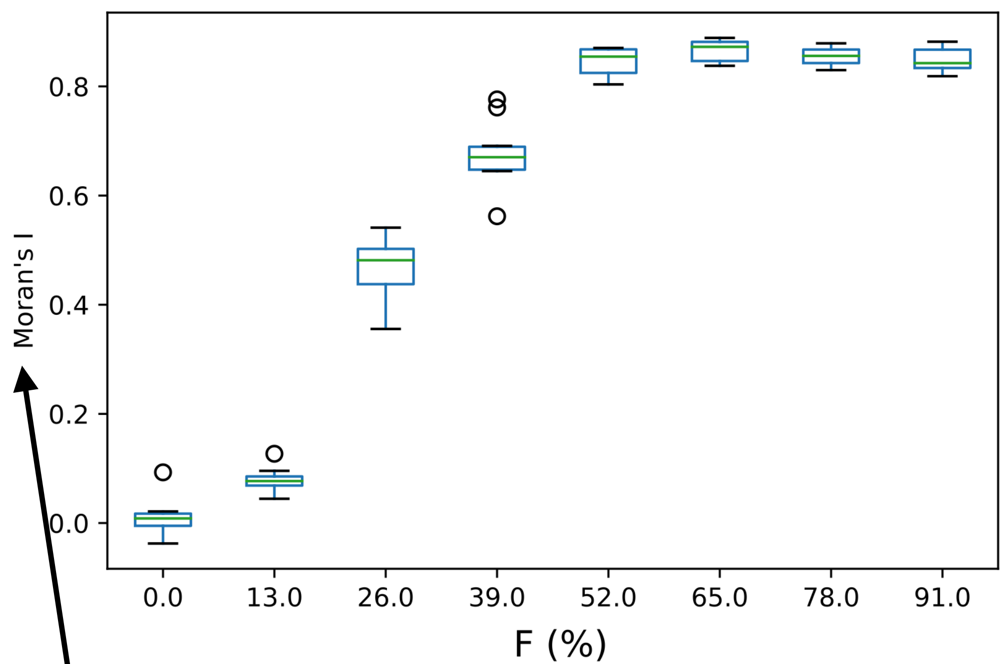


- Size of the giant component for $c > 1$ is larger than one of the possible solutions $S = 0$
  - This is illustrated on the figure for all values of c, with a limit at $c = 1$ for two given cases
  - Phase transition



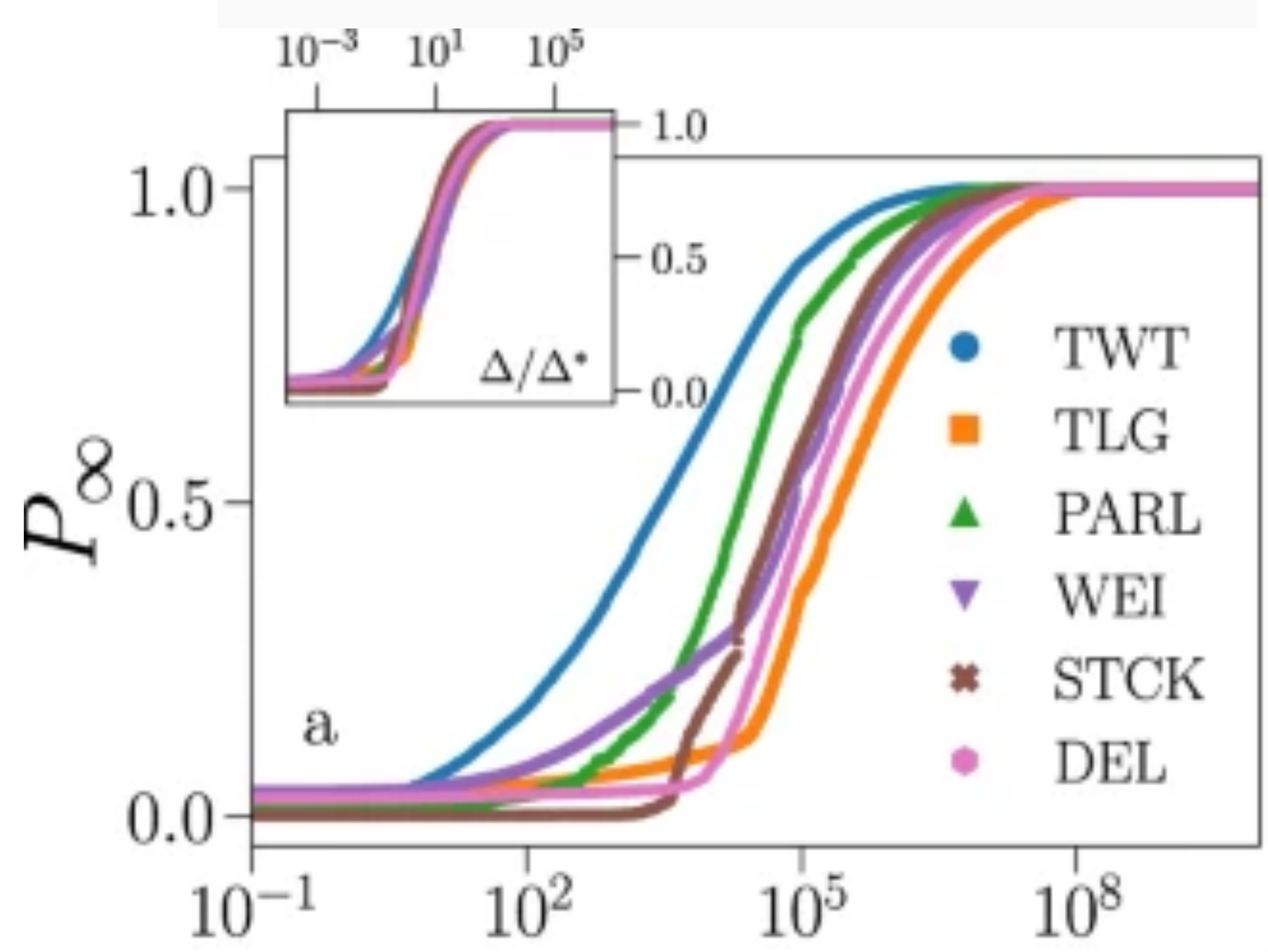
Phase transition examples

- Schelling



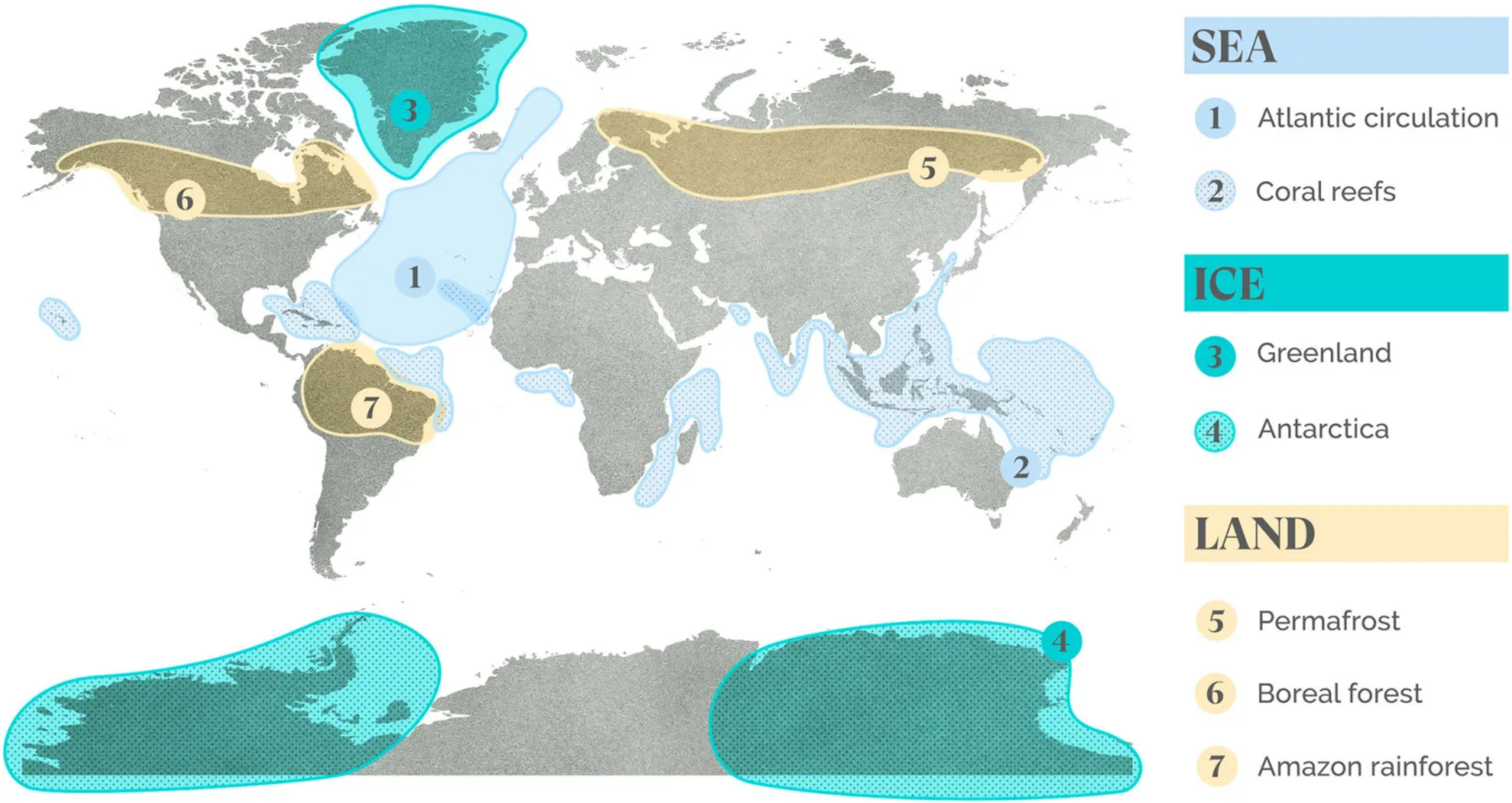
$$I = \frac{N}{W} \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij}(x_i - \bar{x})(x_j - \bar{x})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$$

- Spreading models



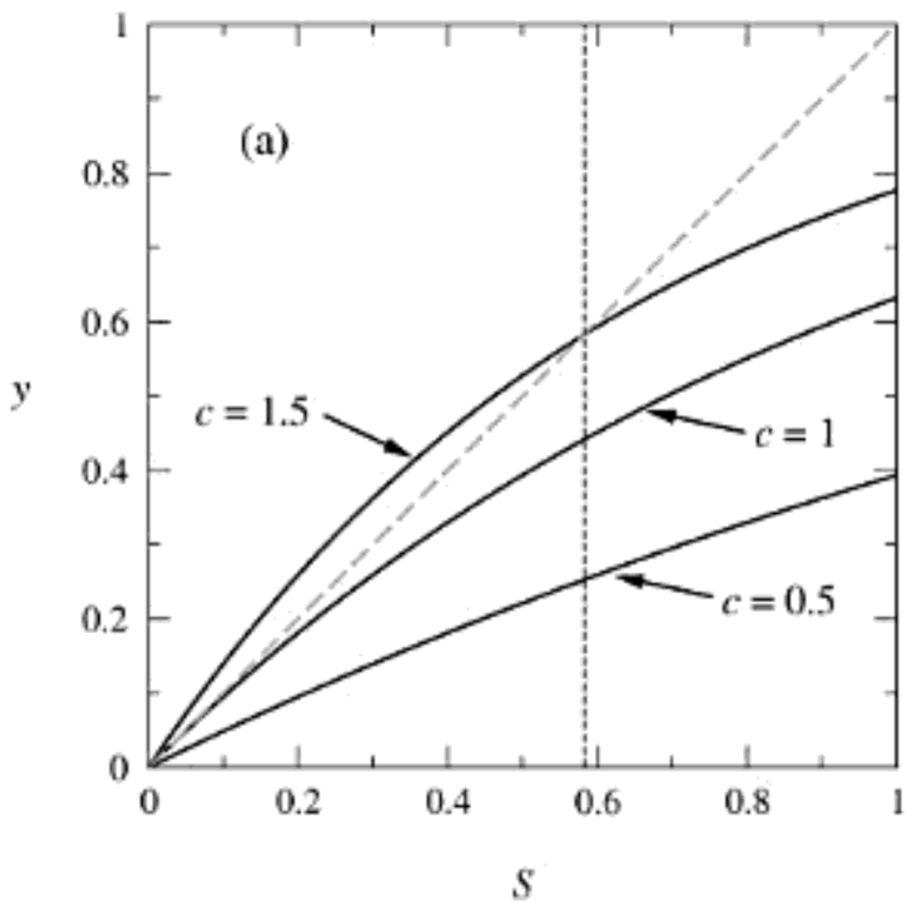
[1] Notarmuzi, D., Castellano, C., Flammini, A. *et al.* Universality, criticality and complexity of information propagation in social media. *Nat Commun* 13, 1308 (2022). https://doi.org/10.1038/s41467-022-28964-8

Phase transitions - Tipping points - Points of no return



***https://grist.org/climate-tipping-points-amazon-greenland-boreal-forest/

Analytical behavor analysis of G(n,p) ensamble for large networks $n \to \infty$



- S - size proportion of a large component
- c - average degree of a vertex

1. Size $S = 0$
   - Small c - no giant component
2. Size $S = 0$ and $S > 0$
   - Large c - there is giant component
   - Transition phase happendes in between small/large c, where gradient of the diagonal matches the gradient of the curve for $S = 0$

Null models

1. Random graphs

2. Poisson random graphs

3. Giant component

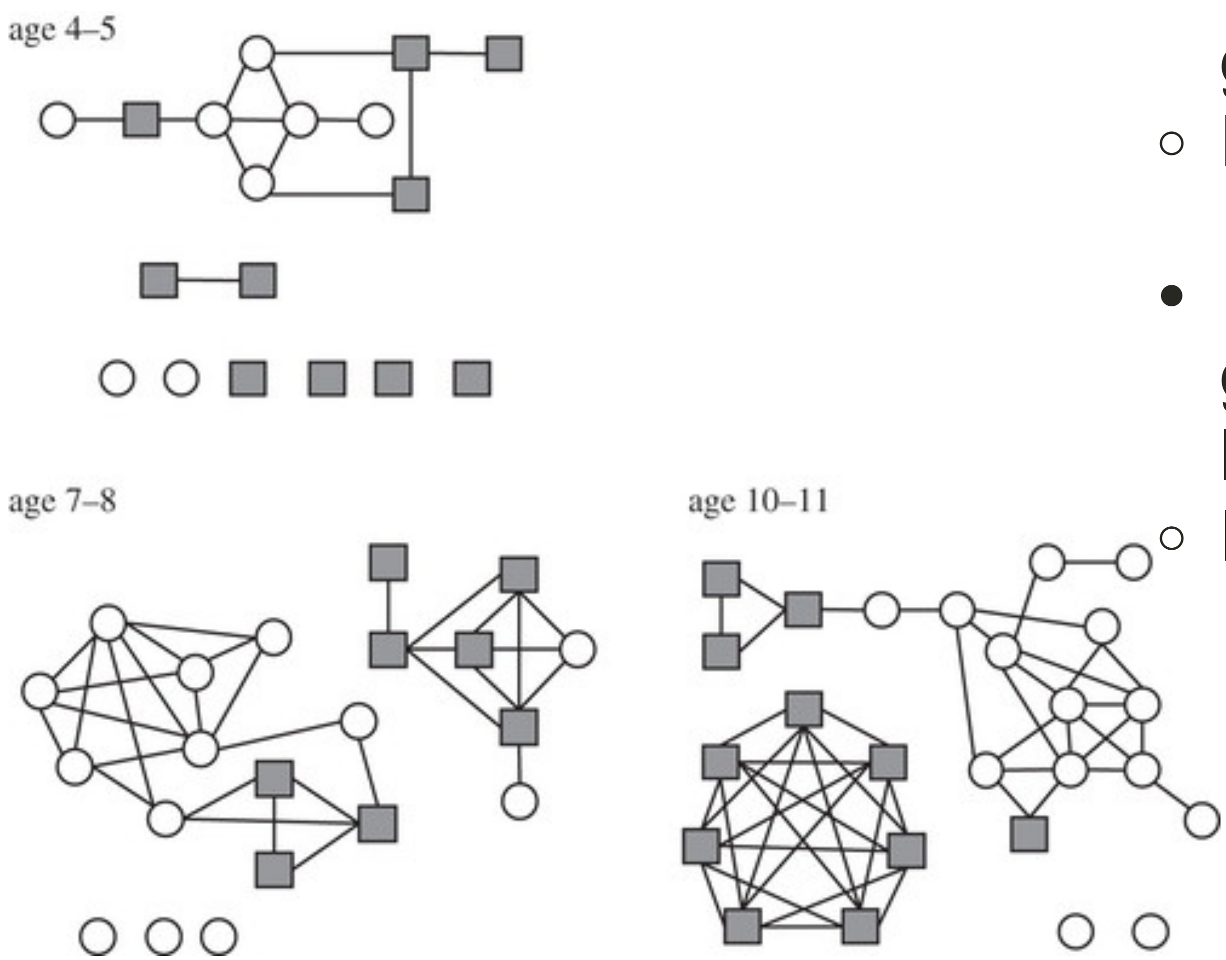
4. *Null models*

Random graphs as null models

- Most real-world networks show an average distance that is comparable to that of the corresponding random graph
  - But their average clustering coefficient is much higher

- Exploratory data analysis
  - Given the **observed** value of a structural measure in a real-world network it can be compared with the **expected** value of that measure in a graph in which the network’s structure is randomized
  - Any structure is tested to whether it deviates from the structure of some pre-defined model

Social network example



- Increasing complexity, cliquishness, and gender segregation with age
  - Filled squares, girls; open circles, boys.
- It is well known that, in general, same-gender friendships are more likely than boy-girl friendships
  - Especially in primary school.

Social network example

- Conlan et al. compared the number of these edges with the expected one in a model where each child maintains the number of declared and received friendship declarations concerning the gender of the other child
  - The mutuality is still statistically significant in almost all cases
  - The effect is less pronounced than if compared with a simple directed random graph in which the gender is not regarded.

- *Note that this can be modeled structurally by defining two different in-degrees and two different out-degrees, each differentiated by the gender of the other child*
  - The random graph model then maintains both types of in- and outdegrees

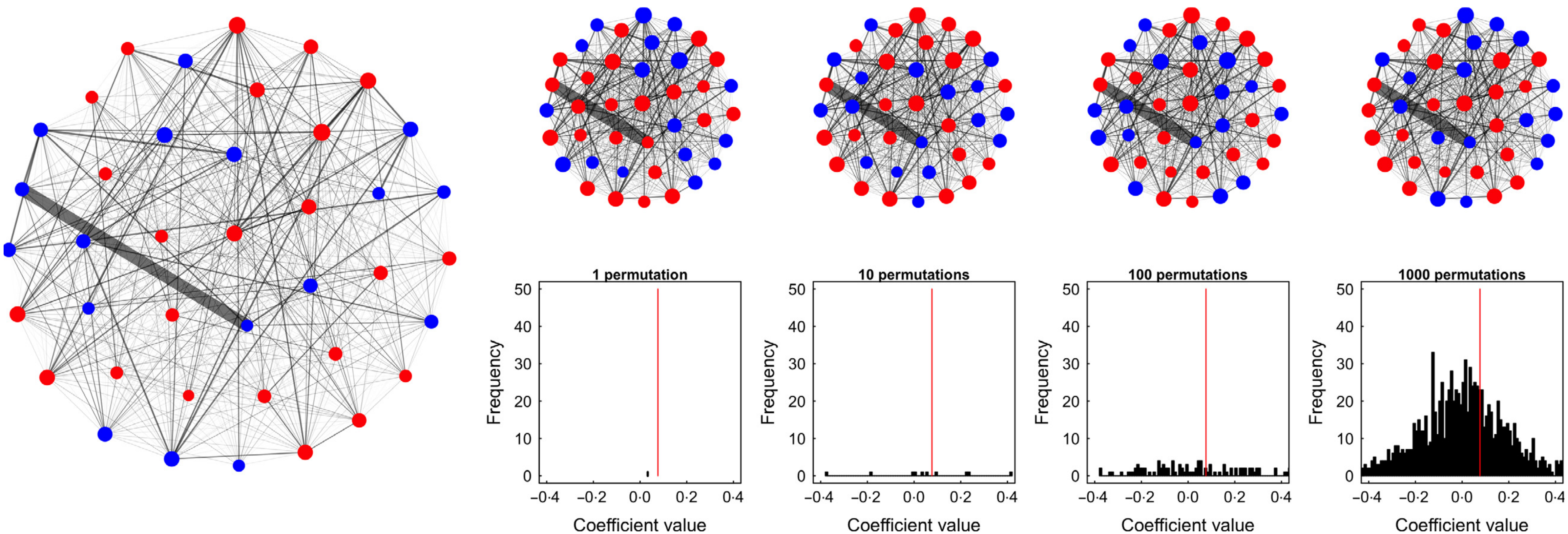
Conlan et al. (2011) Measuring social networks in British primary schools through scientific engagement

Random graphs as null models

- In terms of statistics, the null-hypothesis assumes that graph $\mathcal{G}$ was produced by the random graph model
  - This null hypothesis is rejected if it is very unlikely that the model is true
  - The random graph model is also often called **the null model**

- The **p-value** $P(G|\mathcal{G})$ itself, i.e., the probability that the observed graph G was produced by some random graph model $\mathcal{G}$, does not directly give you the probability $P(\mathcal{G}|G)$, i.e., the probability that the model is true when G is observed
  - Read more: Permutation test lecture in Foundations of CSS

Comparison between an observed and null model



Example of null model with node permutation [1]

Configuration models: a family of random graph models designed to generate networks from a given degree sequence.

[1] Farine, Damien R. "A guide to null models for animal social network analysis." *Methods in Ecology and Evolution* 8.10 (2017): 1309-1320.

Summary

- Random graphs
  - Simple graphs with fixed parameters (vertices, edges)
  - There is an ensemble of networks that satisfy the fixed constraints

- Large networks where $n \to \infty$
  - Degree distribution of a vertex approaches the Poisson distribution
  - Clustering coefficient $C \simeq 0$ if mean degree c is fixed

- Giant component contains most of the entire nodes in the network
  - Largest component suddenly changes for one particular value of p
  - Phase transition happens at $c = 1$

- The null hypothesis assumes that graph $\mathcal{G}$ was produced by the RNG
  - Real-world networks' average clustering coefficient is high
  - This null hypothesis is rejected if the assumption is unlikely

Quiz

- If we flip a coin 20 times, what probability is for 10 heads?
- Do all graphs in the ensemble G(n,m) have the same number of edges?
- What is the density of a network?
- What is the largest possible diameter l of a network with 20 nodes?