Теорема Доказательство Пример

Angular momentum and SU(2) representations

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Angular momentum and rotation symmetry

In classical hamiltonian mechanics angular momentum appears as the generator of the rotational symmetry.
If $H(p,q)$ is invariant under some action of $SO(3)$ on the phase space, than than $\{H,L_i\} = 0$ for $L_i = \varepsilon_{ijk} q_i p_k$.
We will define angular momentum the same way - as a generator of the action of $SO(3)$ on the Hilbert space.
In quantum mechanics we should consider unitary representations.

Representations

A unitary representation of a Lie group G is a homomorphism

$$\rho : \ G \ \to \ U(\mathbb{H}), \tag{1}$$

where $U(\mathbb{H})$ is a group of unitary operators on $\mathbb{H}$.
We know that for any Lie group homomorphism $\Phi : G_1 \to G_2$, $\phi = d\Phi$ is a homomorphism of Lie algebras. The similar construction is true for unitary representations.
If Π is a unitary representation of G, than there exists a representation π of $\mathfrak{g} = Lie(G)$ defined by

$$\Pi(\exp(tX)) = \exp(t\pi(X)). \tag{2}$$

Angular Momentum

Let Π be a unitary representation of $SO(3)$ of a Hilbert space $\mathbb{H}$. Let $\{F_1, F_2, F_3\}$ be a basis in $so(3)$ Lie algebra. The angular momentum operator is defined as:

$$\hat{L}_i = i\hbar\pi(F_i) = i\hbar \frac{d}{dt}\bigg|_{t=0} \Pi(\exp(tF_i)) \tag{3}$$

Example

Let's consider an example. Let $\mathbb{H} = L_2(\mathbb{R}^{\nvDash})$, Π be a regular representation:

$$\Pi(R)\psi = \psi(R^{-1}x). \tag{4}$$

by a direct computation we can find the angular momentum for this case:

$$\hat{L}_i\psi(x) = i\hbar \frac{d}{dt}\bigg|_{t=0} \psi(e^{-tX_i}x) = i\hbar(\varepsilon_{ijk}x_i\partial_k)\psi(x), \tag{5}$$

which gives the famous formula for the angular momentum obtained after canonical quantization.

Representation theory of SO(3)

To better understand the systems with angular momentum it is reasonable to find all the irreducible representations of $SO(3)$.
For any $l \in \mathbb{Z} + \frac{1}{2}\mathbb{Z}$ there exists an unique irreducible representation $\pi : so(3) \to gl(V_l)$ of $so(3)$ of dimension $2l+1$, the operator

$$C = \pi(F_1)^2 + \pi(F_2)^2 + \pi(F_3)^2 = l(l+1)Id \qquad (6)$$

The idea of the proof is the following. Construct a basis $\{H,X,Y\}$ in $so(3)$:

$$[H,X] = 2X.\ [H,Y] = -2Y,\ [X,Y] = H \qquad (7)$$

Then X and Y acts as a rising and lowering operators of H's eigenvalues:

$$H(Xv) = (\alpha+2)Xv,\ H(Yv) = (\alpha-2)Xv,\ Hv = \alpha v. \qquad (8)$$

Since the representation is finite-dimensional, we can deduce that there is a highest eigenvalue l, which should be integer, such that the dimension of the representation space is $2l+1$.

A question arises, whether every irreducible representation of $so(3)$ is a differential of some representation (and if so, is it irreducible) of $SO(3)$. The answer is no. We will state and sketch the proof of the following

Theorem:

1) If π_l is an irreducible representation of $so(3)$. If l is an integer, that there exists a representation $\Pi_l : SO(3) \to GL(V_l)$, such that for any $X \in so(3)$:

$$\Pi_l(e^{tX}) = e^{\pi_l(tX)} \tag{9}$$

2) if l is half-integer, no such representation exist.

3) Any irreducible representation of $SO(3)$ has the form of Π_l for some integer l

A decomposition of $L_2(S^2)$

Now we may use this knowledge to decompose the regular representation of $SO(3)$ on $\mathbb{H} = L_2(S^2)$. We have a decomposition.

$$L_2(S^2) = \bigoplus_{l \in \mathbb{Z}_0} V_l, \tag{10}$$

where V_l are orthogonal to each other, V_l has an famous orthonormal basis of spherical functions:

$$Y_{lm}(\theta, \phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} P_l^m(\cos\theta), \ m \in [-l,l], \tag{11}$$

P_l^m are Legendre polynomials. The Casimir operator $\hat{L}^2$ in our case is just the spherical part of the Laplace operator Δ, and has the eigenvalues $\lambda_l = \hbar l(l+1)$, with degeneracy $k(\lambda_l) = 2l + 1$

Spin

Not all irreducible representations can be exponentiated to some action of $SO(3)$. However, any irreducible representation π_l of the Lie algebra can be exponentiated to the representation Π_l' of $SU(2)$. Unfortunately, this representation will not factor through the covering map. However, it gives rise to a projective representation of $SO(3)$, which is just a homomorphism to $PU(\mathbb{H}) = U(\mathbb{H})/S^1$. π_l defines a projective representation Π_l of $SO(3)$. The nature of quantum mechanics allows us to define the rotational symmetry as the existence of a projective unitary representation which commutes with the hamiltonian. The most well-known example of a physical system where a projective representation $SO(3)$ takes place is the particle with the spin l. The Hilbert space of a particle with spin l is

$$\mathbb{H} = V_l \otimes L_2(\mathbb{R}^{\nvDash}), \tag{12}$$

Particle with a spin

The most well-known example of a physical system where a projective representation $SO(3)$ takes place is the particle with the spin l. The Hilbert space of a particle with spin l is

$$\mathbb{H} = V_l \otimes L_2(\mathbb{R}^{\nvDash}), \tag{14}$$

where V_l is the finite dimensional vector space carrying the representation π_l of $so(3)$ of dimension 2l+1. When l is integer, the particle is called a boson, when l is half-integer, it is called a fermion. The angular momentum of such system is following operator:

$$\hat{J}_i = \pi_l(F_i) \otimes \hat{I} + I \otimes \hat{L}_i, \tag{15}$$

$\hat{L}_i$ is defined in (5)