Soundwaves in electron solids with Berry curvature

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What did we do?

- Learned some quantum mechanics
- Initiated investigation into propagation of sound waves in topological electron solids.

Dispersion relations for soundwaves in solids

We are looking at $\omega(q)$ at $q \to 0$ (wavelength is much bigger than lattice parameter a).
In ordinary solids:

$$\omega(q) = v_s q \qquad v_s = \text{sound speed.}$$

Known fact: for a 2D lattice of electrons interacting with a Coulomb potential:

$$\omega_L \sim \sqrt{q} \tag{1}$$

$$\omega_T \sim q \tag{2}$$

Add a magnetic field $\vec{B} = B\hat{z}$:

$$\omega_1 = \omega_c = \frac{eB}{mc} \tag{3}$$

$$\omega_2 \sim q^{\frac{3}{2}} \tag{4}$$

Berry phase, connection and curvature

Idea: add something else and see how $\omega(q)$ changes.
Consider a hamiltonian $\hat{H}(\mathbf{a})$ dependent on n parameters a_μ and its eigenstates $|\Psi_n(\mathbf{a})\rangle$ with energies $E_n(\mathbf{a})$
Let a_μ change in time slowly:

$$t = Ts, \quad s \in [0,1], \quad T \to \infty \tag{5}$$

We want to construct a state $|\Psi_n(t)\rangle$ which solves Schrodinger's equation:

$$i\hbar\partial_t |\Psi_n(t)\rangle = \hat{H} |\Psi_n(t)\rangle \tag{6}$$

Such state can be constructed as:

$$|\Psi_n(t)\rangle = e^{i\int_\Gamma A_\mu da^\mu} \times \exp\left(-\frac{i}{\hbar}T\int_0^T E_n(s)ds\right)|\Psi_n(\mathbf{a})\rangle + O(\frac{1}{T}) \tag{7}$$

Berry phase, connection and curvature, part 2

The Berry phase is

$$\gamma(s) = \int_{\Gamma} A_\mu da^\mu, \tag{8}$$

where

$$A_\mu = \langle \Psi_n(\mathbf{a}) | \, \partial_\mu \, | \Psi_n(\mathbf{a}) \rangle \tag{9}$$

is the Berry connection
For more rigorous consideration see "Киселев В.В. Квантовая механика. Часть 2"
Note: Berry phase is geometric.
Now let Γ be a loop and $\Gamma = \partial M$. Stokes theorem:

$$\gamma(1) = \oint_{\Gamma} A_\mu da^\mu = \int_M F_{\mu\nu} da^\mu \wedge da^\nu \tag{10}$$

Berry phase, connection and curvature, part 3

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is called Berry's curvature and is gauge invariant.

In 3D:

$$\gamma(1) = \int_M d\mathbf{s}\, rot\mathbf{A}; \tag{11}$$

Berry phase in crystals, Bloch's bands

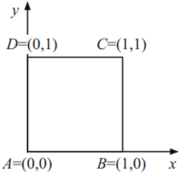
Now let's explore some effects in electrons crystals. Let $\hat{H}(\mathbf{r})$ be periodic, $\hat{H}(\mathbf{r}+\mathbf{a}) = \hat{H}(\mathbf{r})$. According to Bloch's theorem, eigenstates are given by:

$$\Psi_{nq}(\mathbf{r}) = e^{i\mathbf{qr}} u_{nq}(\mathbf{r}), \quad u_{nq}(\mathbf{r}+\mathbf{a}) = u_{nq}(\mathbf{r}) \tag{12}$$

Let's view $\mathbf{q}$ as a parameter, than the Berry connection is

$$A_\mu = \langle u_n(\mathbf{q})| \, \partial_\mu \, |u_n(\mathbf{q})\rangle \tag{13}$$

Let's integrate $\Omega = dA$ over the Brillouin zone in 2D. BZ is a torus:



Chern number

$$c = \int_{BZ} \Omega = \int_0^1 [A_x(x,0) - A_x(x,1)]\, dx - \int_0^1 [A_y(0,y) - A_y(1,y)]\, dy \tag{14}$$

Now remember that $|u_n(x,1)\rangle$ and $|u_n(x,0)\rangle$ are physically equivalent states, therefore

$$|u_n(x,1)\rangle = e^{i\theta_x(x)} |u_n(x,0)\rangle \tag{15}$$

After putting this in (14) it's easy to get

$$c = \theta_x(1) - \theta_x(0) + \theta_y(0) - \theta_y(1) \tag{16}$$

From (15) it is also easy to observe, that Chern number c is an integer, since

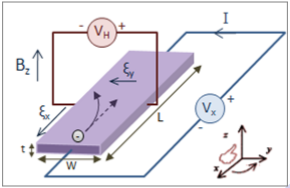
$$e^{2\pi i c} |u_n(0,0)\rangle = |u_n(0,0)\rangle \tag{17}$$

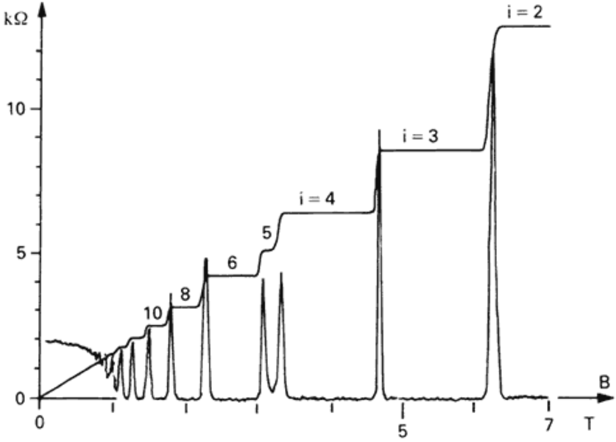
Quantum Hall effect

Important note: Chern number is a topological invariant, i.e. systems with different Chern numbers cannot be adiabatically transformed one into another.

Direct consequnce of the previous analysis is the Quantization of Hall conductivity (to derive that, one has to write other terms in (7) and in the expectation value of velocity operator Ω will pop up)

$$\sigma_{xy} = \frac{e^2}{\hbar} \int_{BZ} \frac{1}{(2\pi)^2} \Omega = \frac{e^2}{\hbar} c \tag{18}$$




kΩ
10
5
0
i = 2
i = 3
i = 4
5
6
8
10
0
5
7
T
B

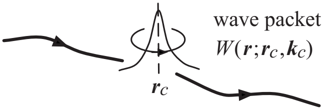
Question: What is the interplay between the sound waves and wave function topology in an electron solid?

Semiclassical EOM for Bloch electrons (very roughly speaking)

We will consider the dynamics of a wave packet.

$$|W_0\rangle = \int d\mathbf{q} w(\mathbf{q}, t) |\Psi_n(\mathbf{q})\rangle \tag{19}$$

Since wave packet now has a finite size, it gains a magnetization $\mathbf{m}(\mathbf{q})$ Semiclassical equations of motion for a Bloch electron (== center of a wave packet)



Semiclasscal equations for Bloch electrons

$$\dot{\mathbf{r}} = \frac{\mathbf{p}}{m} - \mathbf{B} \cdot \frac{\partial \mathbf{m}}{\partial \mathbf{p}} - \frac{1}{\hbar}\dot{\mathbf{p}} \times \mathbf{\Omega}(\mathbf{p}) \tag{20}$$

$$\dot{\mathbf{p}} = -e\mathbf{E} - e\dot{\mathbf{r}} \times \mathbf{B} \tag{21}$$

For a EOM of an electron on 2D lattice one should add $grad\, U$ in (21) For case, when $\Omega = const, \;\; m(q) = a + bq^2$ we didnt't observe any qualitative changes in dispersion relations, in fact, both modes are rescaled by a parameter $\frac{m}{A\mu}$,

$$A = 1 + \frac{eB\Omega}{\hbar}, \tag{22}$$

$$1/\mu = 1/m - \frac{2bB}{\hbar} \tag{23}$$

What's next?

In the right circumstances, we anticipate there should be *qualitative* change to dispersion of sound waves in a topological electron crystal

Ex: For a "Hall crystal" – 2D electron crystal with quantized Hall conductance[1] – what is the dispersion?

In an electon crystal with quantum *anomalous* Hall effect (Chern insulator), what is the dispersion?

[1] Tesanovic (1989)

Спасибо за внимание!