Angular momentum in quantum mechanics and representations of SO(3) and SU(2)

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1 Визитка

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2 Introduction

The purpose of this talk is to establish the role of representation theory of $SO(3)$ and $SU(2)$ in quantum mechanics. We start by giving some definitions of the representations on Hilbert spaces and define angular momentum as a generator of the rotational symmetry. Then we give a classification of the irreducible representations of the Lie algebra $so(3)$ and the Lie Group $SO(3)$. Then we consider a basic example of an $SO(3)$ action on a Hilbert space $L_2(\mathbb{R}^{\nVDash})$ and obtain some famous results by constructing an orthonormal basis of spherical functions in $L_2(S^2)$ and determining the spectrum of the $\hat{L}^2$. Then we discuss the role of irreducible projective representations of $SO(3)$, discuss their 1-to-1 correspondence with the irreducible representations of $so(3)$ and define the particles with a spin with their help.

3 Angular momentum in QM and its relation to SO(3) symmetry

In classical hamiltonian mechanics angular momentum appears as the generator of the rotational symmetry. We know that if $SO(3)$ acts on a phase space and hamiltonian $H(q,p)$ remains invariant under such action, than $\{H,L_i\} = 0$ for $L_i = \varepsilon_{ijk} q_i p_k$ and L_i is an integral of motion. To establish a similar connection between the symmetry under $SO(3)$ and angular momentum, we have to define angular momentum the same way - as the generator of the $SO(3)$ action on the Hilbert space $\mathbb{H}$ of our system. Moreover, due to the two specifics of quantum mechanics, first being the fact that two states are physically equivalent if they differ by a constant, second being the fact that quantum observables correspond to hermitian operators, and all the expectation values are measured by the inner product, it is natural to consider projective and unitary representations, defined below.
Definition:
Suppose $\mathbb{H}$ is a Hilbert space, G is a matrix Lie group. A unitary representation of G is a strongly continous group homomorphism

$$\rho:\ G\ \to\ U(\mathbb{H}), \tag{1}$$

where $U(\mathbb{H})$ is the group of the unitary operators on $\mathbb{H}$.
All the necessary all the detail arising from the infidite-dimensionality of $\mathbb{H}$ will be neglected from now on.
Definition: A projective representation of a matrix Lie group G is a group homomorphism:

$$\rho : G\ \to PU(\mathbb{H}), \tag{2}$$

where $PU(\mathbb{H}) = U(\mathbb{H})/e^{i\theta}I$. Projective representation is just a homomorphism "up to a phase". For a finite representation ρ of a Lie group G, $d\rho$ is a corresponding representation of the Lie algebra g of G. Moreover, for any Lie group homomorphism

$$\Phi : G_1\ \to\ G_2, \tag{3}$$

there exists a Lie algebras homomorphism:

$$\phi : g_1\ \to g_2, \tag{4}$$

such that

$$\Phi(\exp(tX)) = \exp(t\phi(x)),\ \ \phi(x) = \frac{d}{dt}\bigg|_{t=0} \Phi(\exp(tX)) \tag{5}$$

The similar construction can be built for unitary representations as well. One can proof the following
Proposition:

If $\Pi : G \to U(\mathbb{H})$ is a unitary representation , than for any $X \in g$ there exists a unique skew-self-adjoint operator $\pi(X)$, such that

$$\Pi(\exp(tX)) = \exp(t\pi(X)), \tag{6}$$

$$\pi([X,Y]) = [\pi(X), \pi(Y)] \tag{7}$$

Now we are ready to give the definition of the angular momentum in quantum mechanics.
Definition: Let Π be a unitary representation of $SO(3)$ of a Hilbert space $\mathbb{H}$. Let $\{F_1, F_2, F_3\}$ be a basis in $so(3)$ Lie algebra. The angular momentum operator is defined as:

$$\hat{L}_i = i\hbar\pi(F_i) = i\hbar \frac{d}{dt}\bigg|_{t=0} \Pi(\exp(tF_i)) \tag{8}$$

Let's consider an example. Let $\mathbb{H} = L_2(\mathbb{R}^{\not{\not 3}})$, Π be a regular representation:

$$\Pi(R)\psi = \psi(R^{-1})x. \tag{9}$$

by a direct computation we can find the angular momentum for this case:

$$\hat{L}_i\psi(x) = i\frac{d}{dt}\bigg|_{t=0} \psi(e^{-tX}x) = i(\varepsilon_{ijk}x_i\partial_k)\psi(x), \tag{10}$$

which gives the famous formula for the angular momentum obtained after canonical quantization.

4 Classification of the Irreducible representations of $so(3)$ and $SO(3)$

We will give a complete proof of the following
Theorem:
For any $l \in \mathbb{Z} + \frac{1}{2}\mathbb{Z}$ there exists an unique irreducible representation $\pi : so(3) \to gl(V_l)$ of $so(3)$ of dimension $2l+1$, the operator

$$C = \pi(F_1)^2 + \pi(F_2)^2 + \pi(F_3)^2 = l(l+1)Id \tag{11}$$

A question arises, whether every irreducible representation of $so(3)$ is a differential of some representation (and if so, is it irreducible) of $SO(3)$. The annswer is no. We will state and sketch the proof of the following
Theorem:
1) If π_l is an irreducible representation of $so(3)$. If l is an integer, that there exists a representation $\Pi_l : SO(3) \to GL(V_l)$, such that for any $X \in so(3)$:

$$\Pi_l(e^{tX}) = e^{\pi_l(tX)} \tag{12}$$

2) if l is half-integer, no such representation exist.
3) Any irreducible representation of $SO(3)$ has the form of Π_l for some integer l Now we may use this knowledge to decompose the regular representation of $SO(3)$ on $\mathbb{H} = L_2(\mathbb{R})$. We have a decomposition:

$$\mathbb{H} = L_2(\mathbb{R}, r^2dr) \otimes L_2(S^2, dn), \tag{13}$$

where dn is an induced Lebesgue measure. $SO(3)$ acts only on $L_2(S^2)$ leaving the dependance on the radius. For $L_2(S^2)$ we will construct a decomposition

$$L_2(S^2) = \bigoplus_{l \in \mathbb{Z}_{\not +}} V_l, \tag{14}$$

where V_l are orthogonal to each other, V_l has an famous orthonormal basis of spherical functions:

$$Y_{lm}(\theta, \phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} P_l^m(\cos\theta),\ m \in [-l,l], \tag{15}$$

P_l^m are Legendre polynomials. The Casimir operator $\hat{L}^2$ in our case is just the spherical part of the Laplace operator Δ, and has the eigenvalues $\lambda_l = \hbar l(l+1)$, with degeneracy $k(\lambda_l) = 2l+1$

5 Spin

I might prove some of the following statements if I have time. As was stated before, not all irreducible representations can be exponentiated to some action of $SO(3)$. However, any irreducible representation π_l of the Lie algebra can be exponentiated to the representation Π'_l of the universal covering group, which is $SU(2)$ in our case. Unfortunately, this representation will not factor through the covering map, as its kernel does not lie in the kernel of the covering homomorphism, which is $\{\pm I\}$ since $\Pi'_l(-I) = -I$. However, $-I$ lies in the identity class of $PU(V_l)$, therefore π_l defines a projective representation Π_l of $SO(3)$. As was mentioned before, the nature of quantum mechanics allows us to define the rotational symmetry as the existence of a projective unitary representation which commutes with the hamiltonian. The most well-known example of a physical system where a projective representation $SO(3)$ takes place is the particle with the spin l. The Hilbert space of a particle with spin l is

$$\mathbb{H} = V_l \otimes L_2(\mathbb{R}^{\nvDash}), \tag{16}$$

where V_l is the finite dimensional vector space carrying the representation π_l of $so(3)$ of dimension 2l+1. When l is integer, the particle is called a boson, when l is half-integer, it is called a fermion. The angular momentum of such system is following operator:

$$\hat{J}_i = \pi_l(F_i) \otimes \hat{I} + I \otimes \hat{L}_i, \tag{17}$$

$\hat{L}_i$ is defined in (10)

6 Momentum addiition and the Clebsch-Gordan decomposition

TODO

7 Literature

- Тахтаджян Л. А. "Квантовая Механика для Математиков"
- Brian C. Hall "Quantum Theory for Matematicians"
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