Vertex algebras & Integrable Systems

Based on the course by Mikhail Bershtein

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Lecture 1

Problem 1.0

Define new variables for the quiver vertices

$$X_i = e^{x_i}.$$

From relations

$$X_i X_j = q^{2\varepsilon_{ij}} X_j X_i \tag{1}$$

in quantum case derive the commutation relations and the Poisson bracket for x_i.

Solution. Let's begin with the quantum case. Assume that $[x_i, x_j]$ commutes with every x_k. Then by BCH formula

$$X_i X_j = e^{x_i + x_j + \frac{1}{2}[x_i, x_j]}, \qquad X_j X_i = e^{x_i + x_j + \frac{1}{2}[x_j, x_i]}.$$

Clearly from (1)

$$[x_i, x_j] = 2\log q \varepsilon_{ij} = \hbar \varepsilon_{ij}, \tag{2}$$

where $\hbar = 2\log q$. By the definition of the Poisson bracket we have

$$[x_i, x_j] = \hbar\{x_i, x_j\} + O(\hbar^2),$$

therefore

$$\{x_i, x_j\} = \varepsilon_{ij}.$$

Problem 1.1

Mutation in vertex k is a map $\mu_k : X_i \mapsto \tilde{X}_i, \ \ \varepsilon_{ij} \mapsto \tilde{\varepsilon}_{ij}$,

$$\hat{\tilde{X}}_i = \begin{cases} \hat{X}_k^{-1}, & i = k \\ \hat{X}_i, & \varepsilon_{ik} = 0 \\ \hat{X}_i \prod\limits_{n=1}^{\varepsilon_{ik}} (1 + q^{1-2n} \hat{X}_k), & \varepsilon_{ik} > 0 \\ \hat{X}_i \prod\limits_{n=1}^{-\varepsilon_{ik}} (1 + q^{1-2n} \hat{X}_k^{-1})^{-1}, & \varepsilon_{ik} < 0 \end{cases} \tag{3}$$

$$\tilde{\varepsilon}_{ij} = \begin{cases} -\varepsilon_{ij}, & i = k \text{ or } j = k, \\ \varepsilon_{ij} + \frac{\varepsilon_{ik}|\varepsilon_{jk}| - \varepsilon_{jk}|\varepsilon_{ki}|}{2} & \text{otherwise.} \end{cases} \tag{4}$$

show that $\mu_k \circ \mu_k = \mathrm{id}$.

Solution. First let's analyse how the combinatorial data mutates:

$$\tilde{\tilde{\varepsilon}}_{ij} = \begin{cases} \tilde{\varepsilon}_{ij} = \varepsilon_{ij}, & \text{if } i = j \text{ or } j = k, \\ \tilde{\varepsilon}_{ij} + \frac{1}{2}(\tilde{\varepsilon}_{ik}|\tilde{\varepsilon}_{kj}| - \tilde{\varepsilon}_{jk}|\tilde{\varepsilon}_{ki}|) & \text{otherwise} \end{cases}$$

Note that

$$\tilde{\varepsilon}_{ij}+\frac{1}{2}(\tilde{\varepsilon}_{ik}|\tilde{\varepsilon}_{kj}|-\tilde{\varepsilon}_{jk}|\tilde{\varepsilon}_{ki}|) = \varepsilon_{ij}+\frac{1}{2}(\varepsilon_{ik}|\varepsilon_{jk}|-\varepsilon_{jk}|\varepsilon_{ik}|)+\frac{1}{2}(-\varepsilon_{ik}|\varepsilon_{jk}|+\varepsilon_{jk}|\varepsilon_{ik}|) = \varepsilon_{ij},$$

where we used $\tilde{\varepsilon}_{ik} = -\varepsilon_{ik}$.

(a) If $i = k$,

$$\hat{\tilde{\tilde{X}}}_i = \hat{\tilde{X}}_i^{-1} = \hat{X}_i.$$

(b) If $\varepsilon_{ik} = 0$,

$$\hat{\tilde{\tilde{X}}}_i = \hat{\tilde{X}}_i = \hat{X}_i.$$

(c) If $\varepsilon_{ik} > 0$, then $\tilde{\varepsilon}_{ik} < 0$:

$$\begin{aligned} \hat{\tilde{\tilde{X}}}_i = \hat{\tilde{X}}_i \prod_{n=1}^{-\tilde{\varepsilon}_{ik}} \left(1 + q^{1-2n}\hat{\tilde{X}}_k^{-1}\right)^{-1} \\ = \hat{X}_i \prod_{m=1}^{\varepsilon_{ik}} \left(1 + q^{1-2m}\hat{X}_k\right) \prod_{n=1}^{\varepsilon_{ik}} \left(1 + q^{1-2n}\hat{X}_k\right)^{-1} = \hat{X}_i, \end{aligned}$$

where we used $\hat{\tilde{X}}_k = \hat{X}_k^{-1}$

(d) the analysis is the same as for (c).

Problem 1.2

Consider a quiver

$$\begin{tikzcd} \hat{X}_1 \arrow[r] & \hat{X}_2 \arrow[r] & \hat{X}_3 \end{tikzcd}$$

(a) Prove that mutatuion in vertex 2 μ_2 is well-defined, i.e $\hat{\tilde{X}}_i\hat{\tilde{X}}_j = q^{2\tilde{\varepsilon}_{ij}}\hat{\tilde{X}}_j\hat{\tilde{X}}_i$.

(b) Define

$$\Phi(X) = \prod_{j=1}^{\infty}(1 + q^{2j-1}X^{-1}), \qquad \tilde{\mu}_k(\hat{X}_i) = \Phi^{-1}(\hat{X}_k^{-1})\hat{X}_i\Phi(\hat{X}_k^{-1})$$

and

$$\mu'_k(\hat{X}_i) = \begin{cases} \hat{X}_i^{-1} & j = k, \\ :\hat{X}_i\hat{X}_k^{\max(\varepsilon_{ik},0)}: & j \neq k, \end{cases}.$$

Check that $\mu_2 = \tilde{\mu}_2 \circ \mu'_2$

Solution.

(a) Combinatorial data changes as follows:

$$\begin{cases} \varepsilon_{12} = \varepsilon_{23} = 1, & \text{the rest of } \varepsilon_{ij} = 0, \\ \tilde{\varepsilon}_{12} = \tilde{\varepsilon}_{23} = \tilde{\varepsilon}_{31} = -1, & \text{the rest of } \tilde{\varepsilon}_{ij} = 0. \end{cases} \tag{5}$$

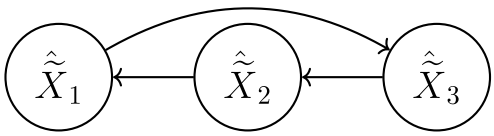

Figure 1. Quiver after the mutation.

From now on let us drop the hats over the X_i for the sake of saving time typing this. Quiver variables after mutation look as follows:

$$\begin{cases} \widetilde{X}_1 = X_1(1 + q^{-1}X_2), \\ \widetilde{X}_2 = X_2^{-1}, \\ \widetilde{X}_3 = X_3(1 + q^{-1}X_2^{-1})^{-1}, \end{cases}$$

so

$$\widetilde{X}_1\widetilde{X}_2 = X_1(1 + q^{-1}X_2)X_2^{-1} = X_1X_2^{-1} + q^{-1}X_1,$$

and

$$\widetilde{X}_2\widetilde{X}_1 = X_2^{-1}X_1 + qX_1. \tag{6}$$

Now we have to figure out how X_1 commutes with X_2^{-1}. One can write

$$X_iX_jX_i^{-1}X_j^{-1} = (X_iX_j)(X_jX_i)^{-1} = q^{2\varepsilon_{ij}}(X_iX_j)(X_iX_j)^{-1} = q^{2\varepsilon_{ij}},$$

$$X_iX_j^{-1} = q^{-2\varepsilon_{ij}}X_j^{-1}X_i. \tag{7}$$

Therefore (6) becomes

$$q^2(X_1X_2^{-1} + q^{-1}X_1) = q^2\widetilde{X}_1\widetilde{X}_2 = q^{2\tilde{\varepsilon}_{21}}\widetilde{X}_1\widetilde{X}_2,$$

so far it is OK. $\widetilde{X}_2\widetilde{X}_3 = q^{2\tilde{\varepsilon}_{23}}\widetilde{X}_3\widetilde{X}_2$ is obtained the same way.

$$\begin{aligned}\widetilde{X}_1\widetilde{X}_3 &= X_1(1+q^{-1}X_2)X_3(1+q^{-1}X_2^{-1})^{-1} \\ &= X_1X_2X_3\Big(q^{-2}(X_2+q^{-1})^{-1}+(q+X_2^{-1})^{-1}\Big) \\ &= X_1X_2X_3q^{-2}\Big((X_2+q)^{-1}+q^{-2}(X_2^{-1}+q^{-1})^{-1}\Big).\end{aligned}$$

A simple analysis of the terms in brackets gives

$$q^{-2}(X_2+q^{-1})^{-1}+(q+X_2^{-1})^{-1} = (X_2+q)^{-1}+q^{-2}(X_2^{-1}+q^{-1})^{-1} = q^{-1}. \quad (8)$$

(b) Case of $i=2$ is trivial since $\hat{X}_2$ commutes with $\Phi(\hat{X}_2)$, let's have a look at case of $i=1$;

$$\mu_2'(\hat{X}_1) =: \hat{X}_1\hat{X}_2 := q^{-1}\hat{X}_1\hat{X}_2,$$

$$\Phi^{-1}(\hat{X}_2^{-1})q^{-1}\hat{X}_1\hat{X}_2\Phi(\hat{X}_2^{-1}) = \Phi^{-1}(\hat{X}_2^{-1})q^{-1}X_1\prod_{j=1}^{\infty}(1+q^{2j-1}\hat{X}_2^{-1})\hat{X}_2.$$

With the help of (7) we obtain

$$\begin{aligned}&\Phi^{-1}(\hat{X}_2^{-1})\prod_{j=1}^{\infty}(1+q^{2j-3}\hat{X}_2^{-1})\cdot q^{-1}\hat{X}_1\hat{X}_2 \\ &\qquad = \Phi^{-1}(\hat{X}_2^{-1})(1+q^{-1}\hat{X}_2^{-1})\Phi(\hat{X}_2^{-1})\cdot q^{-1}\hat{X}_1\hat{X}_2 \\ &\qquad = \hat{X}_1q^{-1}(1+q\hat{X}_2^{-1})\hat{X}_2 = \hat{X}_1(1+q\hat{X}_2).\end{aligned}$$

Case of $i=3$ is similar

$$\widetilde{\mu}_2\circ\mu'(\hat{X}_3) = \Phi^{-1}(\hat{X}_2^{-1})\hat{X}_3\Phi(\hat{X}_2^{-1}).$$

Take a look at the identity:

$$\begin{aligned}\hat{X}_3\Phi(\hat{X}_2^{-1}) = \hat{X}_3\prod_{j=1}^{\infty}(1+q^{2j-1}\hat{X}_2^{-1}) &= \prod_{j=1}^{\infty}(1+q^{2j+1}\hat{X}_2^{-1})\hat{X}_3 \\ &= \prod_{j=1}^{\infty}(1+q^{2j-1}\hat{X}_2^{-1})(1+q\hat{X}_2{}^{-1})^{-1}\hat{X}_3.\end{aligned}$$

Therefore, after simplification, we obtain:

$$\mu_2(\hat{X}_3) = \Phi^{-1}(\hat{X}_2^{-1})\Phi(\hat{X}_2^{-1})\left(1+q\hat{X}_2^{-1}\right)^{-1}\hat{X}_3 = \hat{X}_3\left(1+q^{-1}\hat{X}_2^{-1}\right)^{-1}.$$

Summarizing what we found we get exactly the formula for $\mu_2(\hat{X}_i)$:

$$\widetilde{\mu}_2\circ\mu_2'(\hat{X}_i) = \begin{cases}\hat{X}_i\left(1+q^{-1}\hat{X}_2\right) & i=1, \\ \hat{X}_i^{-1} & i=2, \\ \hat{X}_3\left(1+q^{-1}\hat{X}_2^{-1}\right)^{-1} & i=3.\end{cases}$$

Lecture 2

Problem 2.0

Check that[a]

$$(q-q^{-1})\,\mathcal{E} = X_1 + X_{12}, \qquad K' = X_{124}$$
$$(q-q^{-1})\,\mathcal{F} = X_4 + X_{43}, \qquad K = X_{431}.$$

generate $\mathcal{D}_q(\mathfrak{b})$, i.e

$$K\mathcal{E} = q^2\mathcal{E}K, \quad K\mathcal{F} = q^{-2}\mathcal{F}K,$$

$$K'\mathcal{E} = q^{-2}\mathcal{E}K', \quad K'\mathcal{F} = q^2\mathcal{F}K',$$

and $[E,F] = \frac{K'-K}{q-q^{-1}}$.

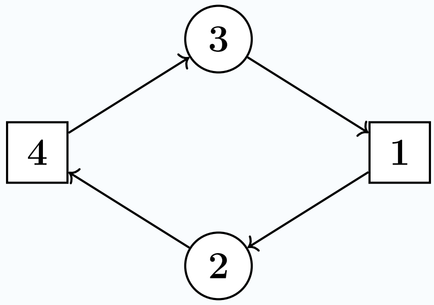

Figure 2. Quiver for $\mathcal{U}_q(\mathfrak{sl}_2)$

[a] Here $X_{ij} :=: X_iX_j := q^{\varepsilon_{ji}}X_iX_j$ and the same for X_{ijk}.

Solution. The formula for the ordering of three operators

$$X_{ijk} = e^{\varepsilon_{ji}+\varepsilon_{kj}+\varepsilon_{ki}}X_iX_jX_k, \tag{9}$$

follows trivially from BCH formula:

$$X_iX_{jk} = X_{ijk}e^{\frac{1}{2}[x_i,x_j+x_k]} = X_{ijk}q^{\varepsilon_{ij}+\varepsilon_{ik}} = q^{\varepsilon_{kj}}X_iX_jX_k.$$

Everything else is straightforward, for example:

$$K = X_{431} = q^{-2}X_4X_3X_1,$$

so

$$(q-q^{-1})KE = q^{-2}X_4X_3X_1(X_1+q^{-1}X_1X_2) = q^{-2}X_4X_3X_1^2 + q^{-3}X_4X_3X_1^2X_2.$$

At the same time

$$\begin{aligned}(q-q^{-1})EK &= q^{-2}(X_1+q^{-1}X_1X_2)X_4X_3X_1 \\ &= q^{-2}(X_1X_4X_3X_1 + q^{-1}X_1X_2X_4X_3X_1) \\ &= q^{-2}(q-q^{-1})KE,\end{aligned}$$

using (1) many times. The same way another 3 relations can be derived. Finally,

$$[E,F] = \frac{q^{-1}}{(q-q^{-1})^2}\Big(K'(q^2-1)+K(1-q^2)\Big) = \frac{K'-K}{q-q^{-1}}. \tag{10}$$

Problem 2.1

Derive the formula for Weyl ordering:

$$q^{\varepsilon_{ji}}e^{\hat{x}_i}e^{\hat{x}_j} = q^{\varepsilon_{ij}}e^{\hat{x}_j}e^{\hat{x}_i} = \hat{x}_{ij},$$

Solution. In (2) we derived $[\hat{x}_i,\hat{x}_j] = 2\log q\varepsilon_{ij}$, so BCH formula gives

$$q^{\varepsilon_{ji}}e^{\hat{x}_i}e^{\hat{x}_j} = q^{\varepsilon_{ji}}e^{\hat{x}_i+\hat{x}_j+\frac{1}{2}[\hat{x}_i,\hat{x}_j]} = e^{\hat{x}_i+\hat{x}_j} = q^{\varepsilon_{ij}}e^{\hat{x}_j}e^{\hat{x}_i}.$$

Lecture 3 & examples session

Problem 3.0

Perform a mutation of the quiver that corresponds to the triangulation of a rectangle with two triangles in the unfrozen vertex.

Solution.

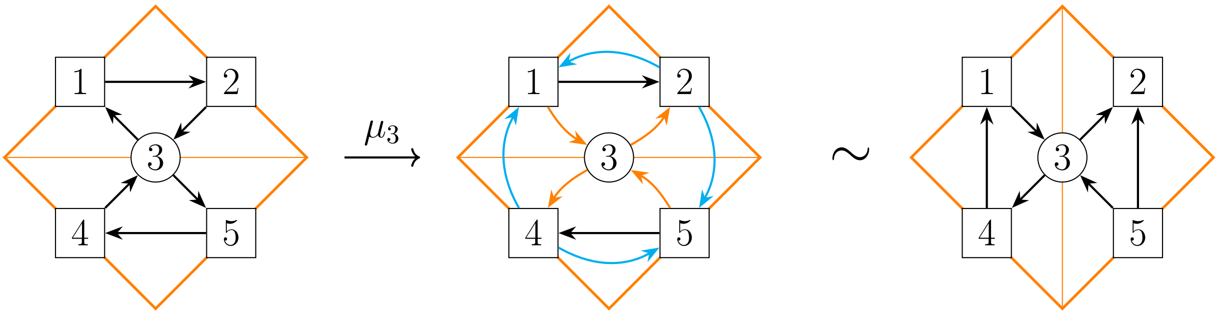

Figure 3. The first diagram is a quiver corresponding to the triangulation of a rectangle. In the second diagram, orange arrows indicate reversed directions, and blue arrows are added to complete every cycle of length 3 with vertex 3 in the middle. The third diagram is obtained after erasing all cycles of length 2, which preserves the data encoded by the quiver.

Problem 3.1

1. For the following quiver on a punctured disk find L^+ and L^- corresponding to parralel transportы shown on fig.3. Find the generators $\mathcal{E}$, $\mathcal{F}$, K, and K', knowing that

$$L^+ = \begin{pmatrix} 1 & (q-q^{-1})\mathcal{F} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K^{1/2} & 0 \\ 0 & K^{-1/2} \end{pmatrix}, \tag{11}$$

$$L^- = \begin{pmatrix} (K')^{1/2} & 0 \\ 0 & (K')^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (q-q^{-1})\mathcal{E} & 1 \end{pmatrix}. \tag{12}$$

2. Suppose:

$$\Delta(L^\pm) = L^\pm \otimes L^\pm,$$

find the corresponding co-products: $\Delta\mathcal{E}, \Delta\mathcal{F}, \Delta K$, and $\Delta K'$.

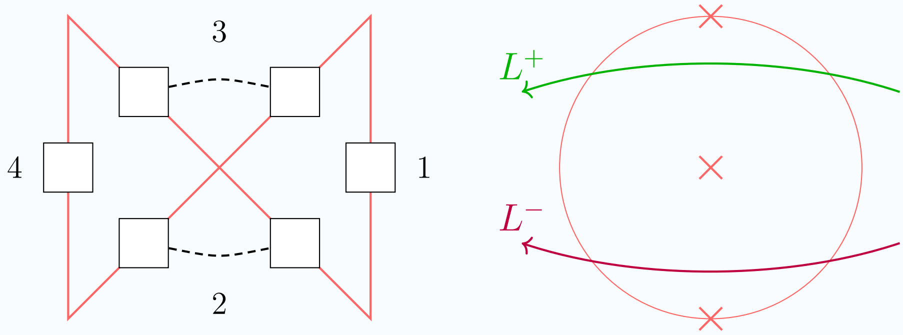

Figure 4. Quiver and a diagram for writing $L^{\pm}$ operators. Dashed lines show gluing of vertexes.

Solution. **(a)** After looking at the picture, one can write

$$L^- = H(X_4)FH(X_2)FH(X_1), \qquad L^+ = H(X_4)EH(X_3)EH(X_1). \tag{13}$$

A simple calculation shows that (13) becomes:

$$L^+ = \begin{pmatrix} (X_4X_3X_1)^{1/2} & X_4^{1/2}\left(X_3^{-1/2}+X_3^{1/2}\right)X_1^{-1/2} \\ 0 & (X_4X_3X_1)^{-1/2} \end{pmatrix},$$

$$L^- = \begin{pmatrix} (X_4X_2X_1)^{1/2} & 0 \\ X_4^{-1/2}\left(X_1^{1/2}+X_2^{-1/2}\right)X_3^{1/2} & (X_4X_2X_3)^{-1/2} \end{pmatrix},$$

so comparing components of $L^{\pm}$ in this representation with (11) we obtain

$$k = X_4X_3X_1, \qquad \mathcal{F} = \frac{X_4(1+qX_3)}{q-q^{-1}}, \qquad k' = X_4X_2X_1, \qquad \mathcal{E} = \frac{(1+X_2)}{q-q^{-1}}.$$

(b) Consider L^-:

$$\Delta(L^-) = \begin{pmatrix} k'^{1/2} & 0 \\ (q-q^{-1})\mathcal{E}\,k'^{-1/2} & k'^{-1/2} \end{pmatrix} \otimes \begin{pmatrix} k'^{1/2} & 0 \\ (q-q^{-1})\mathcal{E}\,k'^{-1/2} & k'^{-1/2} \end{pmatrix} =$$

$$= \begin{pmatrix} k'^{1/2}\otimes k'^{1/2} & 0\otimes 0 \\ (q-q^{-1})\mathcal{E}\,k'^{-1/2}\otimes k'^{1/2} + k'^{-1/2}\otimes (q-q^{-1})\mathcal{E}\,k'^{-1/2} & k'^{-1/2}\otimes k'^{-1/2} \end{pmatrix}.$$

At the same time,

$$\Delta(L^-) = \begin{pmatrix} \Delta(k'^{1/2}) & 0 \\ (q-q^{-1})\Delta(\mathcal{E})\Delta(k'^{-1/2}) & \Delta(k'^{-1/2}) \end{pmatrix},$$

therefore

$$\Delta(k') = k'\otimes k'.$$

Comparing upper right corner:

$$(q-q^{-1})\Delta(\mathcal{E})\Delta(k'^{-1/2}) = (q-q^{-1})(\mathcal{E}\,k'^{-1/2})\otimes k'^{1/2} + k'^{-1/2}\otimes(q-q^{-1})(\mathcal{E}k'^{-1/2}),$$

multiplying both parts by $(q-q^{-1})^{-1}k'^{1/2}\otimes k'^{1/2}$ we get

$$\Delta(\mathcal{E}) = \mathcal{E}\otimes k + I\otimes\mathcal{E}.$$

For L^+ after multiplication of we have the following result:

$$\Delta(L^+) = \begin{pmatrix} k^{1/2}\otimes k^{1/2} & (q-q^{-1})(k^{1/2}\otimes(\mathcal{F}k^{-1/2}) + (\mathcal{F}k^{-1/2})\otimes k^{-1/2}) \\ 0\otimes 0 & k^{-1/2}\otimes k^{-1/2} \end{pmatrix}.$$

Which is equal to another representation:

$$\Delta(L^+) = \begin{pmatrix} \Delta(k^{1/2}) & (q-q^{-1})\,\Delta(\mathcal{F})\,\Delta(k^{-1/2}) \\ 0 & \Delta(k^{-1/2}) \end{pmatrix}.$$

From here it is easy to find

$$\Delta(k) = k\otimes k.$$

Comparing the upper-right entries of both matrices:

$$(q-q^{-1})(k^{1/2}\otimes(\mathcal{F}k^{-1/2}) + (\mathcal{F}k^{-1/2})\otimes k^{-1/2}) = (q-q^{-1})\,\Delta(\mathcal{F})\,(k^{-1/2}\otimes k^{-1/2}),$$

we get

$$\Delta(\mathcal{F}) = k\otimes\mathcal{F} + \mathcal{F}\otimes I.$$

Problem 3.2

For the quiver on Fig.2 check that in variables $\hat{\hat{X}}_i = \mu_2(X_i)$ expressions for $\mathcal{E},\mathcal{F},K,K'$:

$$(q-q^{-1})\,\mathcal{E} = \hat{X}_1 + \hat{X}_{12}, \qquad K' = \hat{X}_{124},$$
$$(q-q^{-1})\,\mathcal{F} = \hat{X}_4 + \hat{X}_{43}, \qquad K = \hat{X}_{431}.$$

become

$$(q-q^{-1})\,\mathcal{E} = \hat{\hat{X}}_1, \qquad k' = \hat{\hat{X}}_{14},$$
$$(q-q^{-1})\,\mathcal{F} = \hat{\hat{X}}_4 + \hat{\hat{X}}_{34} + \hat{\hat{X}}_{24} + \hat{\hat{X}}_{234}, \qquad k = \hat{\hat{X}}_{1234}.$$

Solution. First of all, let's write out all the quiver data after the mutation:

$$\begin{cases} \widetilde{\varepsilon}_{21} = \widetilde{\varepsilon}_{42} = -1, \\ \widetilde{\varepsilon}_{14} = \widetilde{\varepsilon}_{43} = \widetilde{\varepsilon}_{31} = 1. \end{cases} \tag{14}$$

$$\hat{\hat{X}}_i = \mu_2(\hat{X}_i) = \begin{cases} \hat{X}_1(1+q^{-1}\hat{X}_2), & i=1; \\ \hat{X}_2^{-1}, & i=2; \\ \hat{X}_3, & i=3; \\ \hat{X}_4(1+q^{-1}\hat{X}_2^{-1})^{-1}, & i=4. \end{cases},$$

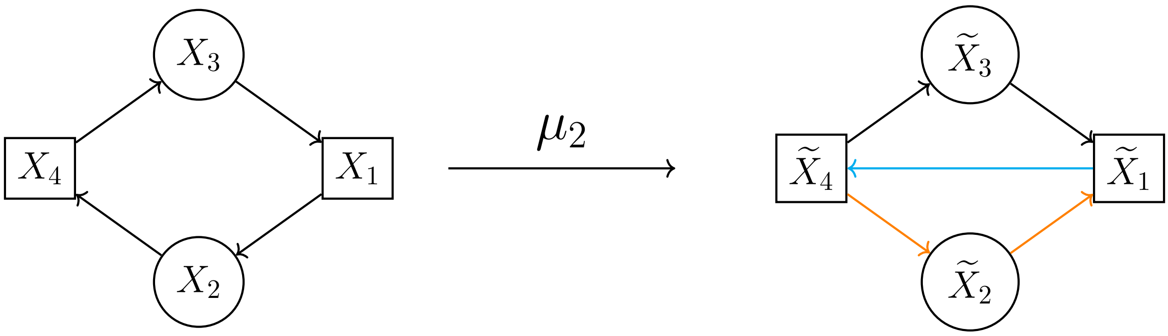

Figure 5. Mutation of Fig.2 in vertex 2.

$\mathcal{E}$ can be rewritten as

$$(q-q^{-1})\,\mathcal{E}=\hat{X}_1+\hat{X}_{12}=\hat{X}_1+q^{-1}\hat{X}_1\hat{X}_2=\hat{\widetilde{X}}_1.$$

The expected expression for $\mathcal{F}$ is

$$(q-q^{-1})\mathcal{F}=\hat{\widetilde{X}}_4+\hat{\widetilde{X}}_{34}+\hat{\widetilde{X}}_{24}+\hat{\widetilde{X}}_{234}.$$

With the help of (9) we can rewrite it as follows:

$$\begin{aligned}\hat{X}_4(1+q^{-1}\hat{X}_2^{-1})^{-1}+q\,\hat{X}_3\hat{X}_4(1+q^{-1}\hat{X}_2^{-1})^{-1}+&\\ +q\,\hat{X}_2^{-1}\hat{X}_4(1+q^{-1}\hat{X}_2^{-1})^{-1}+q^2\hat{X}_2^{-1}\hat{X}_3\hat{X}_4(1+q^{-1}\hat{X}_2^{-1})^{-1}&\\ =\Big(\hat{X}_4(1+q^{-1}\hat{X}_2^{-1})+q(1+q\,\hat{X}_2^{-1})\hat{X}_3\hat{X}_4\Big)\,(1+q^{-1}\hat{X}_2^{-1})^{-1}&.\end{aligned}$$

Using (7) we get what we need:

$$\begin{aligned}\Big(\hat{X}_4(1+q^{-1}\hat{X}_2^{-1})+q\hat{X}_3\hat{X}_4(1+q^{-1}\,\hat{X}_2^{-1})\Big)\,(1+q^{-1}\hat{X}_2^{-1})^{-1}&\\ =\hat{X}_4+q\hat{X}_3\hat{X}_4=\hat{X}_4+\hat{X}_{34}=(q-q^{-1})\,\mathcal{F}&\end{aligned}\tag{15}$$

For k and k' the procedure is the same.

Problem 3.3

(a) Let $L' \in \mathrm{Mat}_{n\times n} \otimes A'$, $L'' \in \mathrm{Mat}_{n\times n} \otimes A''$ satisfy $RLL = LLR$ relation:

$$RL'_1L'_2 = L'_2L'_1R, \qquad RL''_1L''_2 = L''_2L''_1R, \tag{16}$$

where

$$L_1 = L \otimes I, \qquad L_2 = I \otimes L.$$

Show that

$$L = L'L'' = \sum_{i,j,k} E_{ij} \otimes L'_{ik} \otimes L''_{kj} \in \mathrm{Mat}_{n\times n} \otimes A' \otimes A''$$

also satisfies the $RLL = LLR$ relation:

$$RL_1L_2 = L_2L_1R.$$

b) Let $R = \hbar r + O(\hbar^2)$, $\hbar = 2\log q$.
Show that
$$\{L_1, L_2\} = [r, L_1L_2].$$

Solution.

(a) Let's begin by computing the matrix elements of L_1 and L_2:

$$(L^{'}_1)_{ijkl} = L^{'}_{ij}\delta_{kl}, \quad (L^{'}_2)_{ijkl} = \delta_{ij}L^{'}_{kl}$$

$$RL_1L_2 = R(L'L'' \otimes I)(I \otimes L'L'')$$

$$L_1 = L'L'' \otimes I = \sum E_{ij} \otimes E_{mn} \otimes (L'_{ik} \otimes L''_{kj}\delta_{mn}).$$

Note that

$$L'_1L''_1 = \sum E_{ij} \otimes E_{kl} \otimes (L'_1)_{iqkp} \otimes (L''_1)_{qjpl} =$$
$$= \sum E_{ij} \otimes E_{kl} \otimes (L'_{iq}\delta_{kp}) \otimes (L''_{qj} \otimes \delta_{pl}) =$$
$$= \sum E_{ij} \otimes E_{kl} \otimes L'_{iq} \otimes L''_{qj}\delta_{kl} = L_1,$$

therefore
$$RL_1L_2 = RL'_1L''_1L'_2L''_2.$$

Now note that $L'_1L''_2 = L''_2L'_1$ and $L'_2L''_1 = L''_1L'_2$. For example,

$$L'_1L''_2 = \sum E_{ij} \otimes E_{kl} \otimes (L'_1)_{ipkq} \otimes (L''_2)_{pjql} = \sum E_{ij} \otimes E_{kl} \otimes L'_{ij} \otimes L''_{kl}. \tag{17}$$

Clearly, similar formula will be obtained for $L_2''L_1'$.
Finally,

$$RL_1L_2 = RL_1'L_1''L_2'L_2'' = L_2'L_2''L_1'L_2''R = L_2L_1R \tag{18}$$

(b) Now let

$$R \approx I + \hbar r.$$

By definition,

$$[L_1, L_2] = \hbar\{L_1, L_2\} + O(\hbar^2),$$

so let us write the $RLL = LLR$ relation up to $O(\hbar^2)$:

$$(I + \hbar r)L_1L_2 = L_2L_1(I + \hbar r), \qquad \Longrightarrow \qquad L_1L_2 - L_2L_1 = \hbar[r, L_1L_2],$$

therefore

$$\{L_1, L_2\} = [r, L_2L_2].$$

Problem 3.4

(a) For $\mathfrak{sl}_3$ two quivers from Q_Δ (Fig.8) construct Q for $\mathbb{C}_q[G]$ and $\mathcal{U}_q((\mathfrak{g})$

(b) For $\mathcal{U}_q(\mathfrak{sl}_3)$ compute L^+, L^- for the parallel transports shown on Fig.6.

(c) Find $\mathcal{F}_1$,$\mathcal{F}_2$, k_1, k_2 from

$$L^+ = \begin{pmatrix} 1 & \left(q - q^{-1}\right)\mathcal{F}_1 & * \\ & 1 & \left(q - q^{-1}\right)\mathcal{F}_2 \\ & & 1 \end{pmatrix} \begin{pmatrix} k_1^{2/3}k_2^{1/3} & 0 & 0 \\ 0 & k_1^{-1/3}k_2^{1/2} & 0 \\ 0 & 0 & k_1^{-1/3}k_2^{-2/3} \end{pmatrix} \tag{19}$$

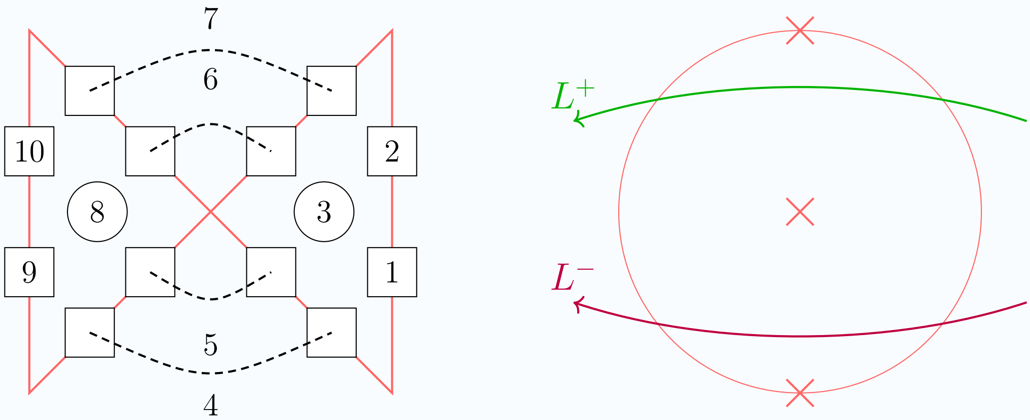

Figure 6

Solution. **(a)** From two Q_Δ (Fig.8) we construct the left side of Fig.7. Gluing the vertices, we get quiver for $\mathcal{U}_q(g)$

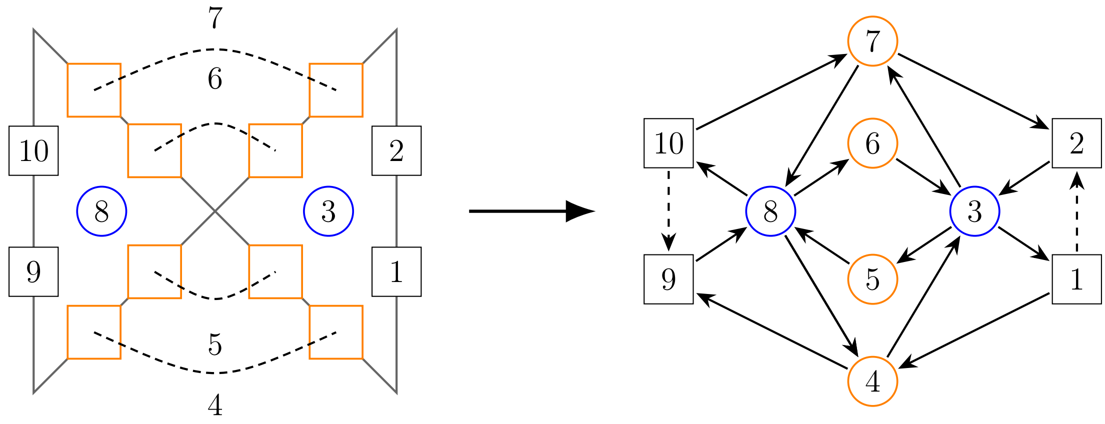

Figure 7. Process of gluing of frozen vertices, marked with orange color. After the gluing this vertices became unfrozen (second picture).

(b) Let's define matrices

$$H_1(x) = \begin{pmatrix} x^{2/3} & & \\ & x^{-1/3} & \\ & & x^{-1/3} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{pmatrix},$$

$$H_2(x) = \begin{pmatrix} x^{1/3} & & \\ & x^{1/3} & \\ & & x^{-2/3} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{pmatrix}.$$

Then (from Fig.8) the translation operators

$$T_1 = H_2(x_6)\, H_1(x_7)\, E_2 H_2(x_3)\, E_1 H_1(x_2)\, E_2 H_2(x_1),$$

$$T_2 = H_2(x_6)\, H_1(x_7)\, F_1 H_1(x_3)\, F_2 H_2(x_5)\, F_1 H_1(x_4).$$

Similarly from right part of Fig.7

$$L^+ = H_1(x_{10})\, H_2(x_8)\, E_2 H_2(x_6)\, E_1 H_1(x_7)\, E_2 H_2(x_3)\, E_1 H_1(x_2)\, E_2 H_2(x_1),$$

$$L^- = H_2(x_9)\, H_1(x_8)\, F_1 H_1(x_5)\, F_2 H_2(x_4)\, F_1 H_1(x_3)\, F_2 H_2(x_1)\, F_1 H_1(x_2).$$

(c) Comparing components of $L^{\pm}$ in this representation with (19) we obtain that

$$k_1 = X_2 X_7 X_{10}, \quad k_2 = X_1 X_3 X_6 X_8 X_9.$$

and

$$\left(q - q^{-1}\right) F_1 = X_2 + X_{27},$$
$$\left(q - q^{-1}\right) F_2 = X_1 + X_8 + X_{368} + X_{3678}.$$

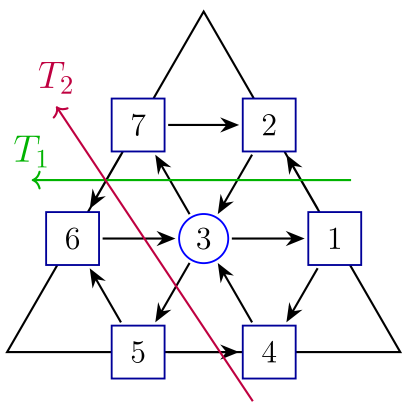

Figure 8. Q_Δ for $\mathfrak{sl}_3$

Problem 3.5

Let $\rho : U_q(\mathfrak{g}) \to End(V_\lambda)$ be a representation of $U_q(\mathfrak{g})$. Let $t^\lambda_{ij} \in U_q(\mathfrak{g})^*$ be the matrix elements of that representation. Define the multiplication of the matrix elements by

$$t^\lambda_{ij} * t^\mu_{kl}(x) = (t^\lambda_{ij} \otimes t^\mu_{kl})\,(\Delta(x))\,.$$

For $t = \sum E_{ij} \otimes t^\lambda_{ji}$, show that

$$Rt_1t_2 = t_2t_1R, \tag{20}$$

where

$$R\colon V_\lambda \otimes V_\lambda \to V_\lambda \otimes V_\lambda$$

is the R-matrix that satisfies the condition

$$\Delta = R^{-1}\Delta^{op}R, \tag{21}$$

where $\Delta^{op} = P\Delta$, P is the permutation operator on $V^\lambda \otimes V^\lambda$

Solution. Similarly to 17

$$t_1t_2 = \sum E_{ij} \otimes E_{mn}\, t^\lambda_{ji} * t^\lambda_{nm},$$

The left side of the 20 becomes

$$(Rt_1t_2)_{ijmn} = R_{iamb}\, t^\lambda_{aj} * t^\lambda_{bn}. \tag{22}$$

Now let's write explicitly the action of $t^\lambda_{aj} * t^\lambda_{bn}$ on $x \in U_q(\mathfrak{g})$

$$t^\lambda_{aj} * t^\lambda_{bn}(x) = t^\lambda_{aj} \otimes t^\lambda_{bn} \circ \Delta(x) = t^\lambda_{aj} \otimes t^\lambda_{bn}\left(\sum E_{pq} \otimes E_{rt} \otimes x_{(1)pq} \otimes x_{(2)rt}\right) =$$
$$= x_{(1)aj}\; x_{(2)rt} = \Delta_{ajrt}(x), \tag{23}$$

where $\Delta_{ajrt} : End(V_\lambda) \otimes End(V_\lambda) \to \mathbb{F}$ is the matrix element of the co-product in representation V_λ. Applying 21 we get

$$\Delta_{ajbn} = (R^{-1}\Delta R)_{ajbn} = (R^{-1})_{aqbp} \cdot \Delta^{\text{op}}_{qrpl} \cdot R_{rjln}.$$

The right side of (22) becomes

$$R_{iamb}(R^{-1})_{aqbp} \cdot \Delta^{\text{op}}_{qrpl} \cdot R_{rjln} = \Delta^{\text{op}}_{irml} \cdot R_{rjln}.$$

Now note that

$$\Delta^{op}_{irml}(x) = t^\lambda_{ir} \otimes t^\lambda_{ml}\Delta^{op}(x) = t^\lambda_{ir} \otimes t^\lambda_{ml}\left(\sum x_{(2)pq} \otimes x_{(1)rt}\right) =$$
$$= t^\lambda_{ml} * t^\lambda_{ir}(x) = (t_2t_1)_{irml}(x). \tag{24}$$

Finally, 22 becomes

$$(Rt_1t_2)_{ijmn} = (t_2t_1)_{irml}R_{rjln} = (t_2t_1R)_{ijmn}$$