Homework 3

Theorem 1. Let F be a monotone function over a complete lattice $(A, \sqsubseteq)$. Then for every non-decreasing sequence of elements $x_0, x_1, \ldots$, we have that $F(\bigsqcup\{x_0, x_1, \ldots\}) \sqsupseteq \bigsqcup\{F(x_i) \mid i \in \mathbb{N}\}$.

Proof. We write $X = F(\bigsqcup\{x_0, x_1, \ldots\})$ and $Y = \bigsqcup\{F(x_i) \mid i \in \mathbb{N}\}$. Let us assume the contrary, i.e., that $X \sqsubset Y$. Therefore, there has to be $i \in \mathbb{N}$ such that $F(x_i) \not\sqsubseteq X$. Otherwise X would be an upper bound of $\{F(x_i) \mid i \in \mathbb{N}\}$ which would violate the definition of a least upper bound of Y. From the definition of an upper bound, we know that $x_i \sqsubseteq \bigsqcup\{x_0, x_1, \ldots\}$. If we apply monotonicity to the last inequality, we get $F(x_i) \sqsubseteq F(\bigsqcup\{x_0, x_1, \ldots\}) = X$ which is a contradiction. $\square$

Remark 2. It is not true that for any monotone function F over a complete lattice $(A, \sqsubseteq)$ and every non-decreasing sequence of elements $x_0, x_1, \ldots$, we have $F(\bigsqcup\{x_0, x_1, \ldots\}) \sqsubseteq \bigsqcup\{F(x_i) \mid i \in \mathbb{N}\}$.

Proof. Let us take the set $A = \mathbb{N} \cup \infty_1 \cup \infty_2$ and $\sqsubseteq$ defined naturally for $\mathbb{N}$ and $\mathbb{N} \sqsubset \infty_1 \sqsubset \infty_2$. This is a complete lattice. We define a monotone function F as: $F(x) = \infty_1$ for every $x \in \mathbb{N}$, $F(\infty_1) = F(\infty_2) = \infty_2$.

If we define a nondecreasing sequence $x_0, x_1, \ldots = \mathbb{N}$, then

$$F\left(\bigsqcup\{x_0, x_1, \ldots\}\right) = F(\infty_1) = \infty_2 \not\sqsubseteq \infty_1 = \bigsqcup\{\infty_1\} = \bigsqcup\{F(x_i) \mid i \in \mathbb{N}\}.$$

$\square$