Formalisms Every Computer Scientist Should Know: Homework Solutions

1 Homework 1

Theorem 1.1. *If there is a one-to-one function on a set A to a subset of a set B and there is also a one-to-one function on B to a subset of A, then A and B are equipollent.*

Proof. We need to show that there exists a bijection between A and B.

Assume such two functions exist. Fix the two functions $\widehat{f} : A \to B$ and $\widehat{g} : B \to A$ be one-to-one functions.

Then there exist bijections $\widehat{f}^{-1} : \widehat{f}(A) \to A$ and $\widehat{g}^{-1} : g(B) \to B$.

Define $A_C \subseteq A$ as the set of such elements $a \in A$ that $(\widehat{g} \circ \widehat{f})^n a = a$ for some integer $n \in \mathbb{N}$ (zero is not in $\mathbb{N}$).

In the same way, define $B_C = \{b \in B \mid \exists n \in \mathbb{N} : (\widehat{f} \circ \widehat{g})^n b = b\}$.

For a given element $a \in A$, define a sequence of ancestors $X(a) = (a, \widehat{g}^{-1}(a), \widehat{f}^{-1}(\widehat{g}^{-1}(a)), \ldots)$. The sequence is finite if the last element is in $A \setminus \widehat{g}(B)$ or in $B \setminus \widehat{f}(A)$, and infinite otherwise. In the same way, for $b \in B$, define $X(b) = (b, \widehat{f}^{-1}(b), g^{-1}(\widehat{f}^{-1}(b)), \ldots)$.

Define $A_I = \{a \in A \mid X(a) \text{ is infinite}\}$, $A_E = \{a \in A \mid |X(a)| \text{ is even}\}$, $A_O = \{a \in A \mid |X(a)| \text{ is odd}\}$. Observe that A_I, A_E, A_O are disjoint and cover A. Similarly, define B_I, B_E, B_O.

Notice that $A_C \subseteq A_I$, since for any $a \in A_C$, $X(a)$ is periodic and infinite.

I claim that $\widehat{f}$ maps A_I bijectively onto B_I. To see this, we need to show that for any $b \in B_I$, there exists a preimage $a \in A_I$ such that $\widehat{f}(a) = b$. Such a is the second element of $X(b)$, $a = \widehat{f}^{-1}(b)$. Then $\widehat{f}(a) = b$. a is indeed in A_I, because $X(a) = (a = \widehat{f}^{-1}(b), \widehat{g}^{-1}(\widehat{f}^{-1}(b)), \ldots)$ is also infinite.

Next, show that f maps A_E bijectively onto B_O. To see this, we need to show that for any $b \in B_O$, there exists a preimage $a \in A_E$ such that $\widehat{f}(a) = b$. Such a is the second element of $X(b)$, $a = \widehat{f}^{-1}(b)$. Then $\widehat{f}(a) = b$. a is indeed in A_E, because $X(a) = (a = \widehat{f}^{-1}(b), \widehat{g}^{-1}(\widehat{f}^{-1}(b)), \ldots)$. Then $|X(a)| = |X(b)| - 1$ is even.

In the same way, g maps B_E bijectively onto A_O. Therefore, g^{-1} maps A_O bijectively onto B_E.

Define a map $h : A \to B$:

$$h(a) = \begin{cases} \widehat{f}(a), & a \in A_I \\ \widehat{f}(a), & a \in A_E \\ \widehat{g}^{-1}(a), & a \in A_O \end{cases}, \tag{1}$$

which is the desired bijection between A and B. $\square$

2 Homework 2

Theorem 2.1 (Knaster-Tarski)**.** *Let $(A, \sqsubseteq)$ be a complete lattice and let F be a monotone function on A. Further, let $\hat{y} = \bigsqcup\{x \mid x \sqsubseteq F(x)\}$ and $\hat{z} = \bigsqcap\{x \mid F(x) \sqsubseteq x\}$. It holds, that:*

1. $\hat{y}$ and $\hat{z}$ are fixpoints of F,

2. for all fixpoints x of F, $\hat{z} \sqsubseteq x \sqsubseteq \hat{y}$

Proof. Suppose $(A, \sqsubseteq)$ is a complete lattice and let F be a monotone function on A. Let U be the set of elements $x \in A$ for which $x \sqsubseteq F(x)$ and let D be the set of elements $x \in A$ for which $x \sqsupseteq F(x)$. Then as described in the lecture let us denote the join and meet of these two sets respectively as $\hat{y} = \bigsqcup U$ and $\hat{z} = \bigsqcap D$ (this meet and join exist because the considered lattice is complete). We shall prove:

1. $\hat{y}$ and $\hat{z}$ are fixpoints of F.

2. For all fixpoints x of F the following holds $\hat{z} \sqsubseteq x \sqsubseteq \hat{y}$.

1. Let us first prove that $\hat{y}$ is a fixpoint of F. Let us pick an arbitrary element $u \in U$, then by definition of $\hat{y}$ we have $u \sqsubseteq \hat{y}$. Because F is monotone this implies $F(u) \sqsubseteq F(\hat{y})$. And because u was picked arbitrarily from U this means that $F(\hat{y})$ is an upper bound on U. But $\hat{y}$ is the least upper bound of U and thus $\hat{y} \sqsubseteq f(\hat{y})$. Again by applying the monotonicity of F we get $F(\hat{y}) \sqsubseteq F(F(\hat{y}))$ and we can conclude $F(\hat{y}) \in U$. But because $\hat{y}$ is the join of U and $F(\hat{y})$ is an element of U it follows $F(\hat{y}) \sqsubseteq \hat{y}$. Together with the inequality $\hat{y} \sqsubseteq F(\hat{y})$ this gives us $F(\hat{y}) = \hat{y}$, hence $\hat{y}$ is a fixpoint of F. The proof that $\hat{z}$ is a fixpoint of F can be done along the same lines by considering $\sqsupseteq$ instead of $\sqsubseteq$ and the set D instead of U.

2. Suppose $\hat{x}$ is a fixpoint of F. Then clearly $\hat{x} \in U$ and hence $\hat{x} \sqsubseteq \hat{y}$ as $\hat{y}$ is the join of U. Similarly $\hat{x} \in D$ and hence $\hat{x} \sqsupseteq \hat{z}$ as $\hat{z}$ is the meet of D. So it follows that $\hat{z} \sqsubseteq \hat{x} \sqsubseteq \hat{y}$ and the proof is done. $\square$

3 Homework 4

3.1 Sorting

Definition 3.1 ($\texttt{ordered}$)**.** A function $\texttt{ordered} : \Sigma^* \to \{true, false\}$ is defined as follows. $\texttt{ordered}(\varepsilon) = true$, $\texttt{ordered}(a) = true$, $\texttt{ordered}(abx) = (a \leq b) \wedge \texttt{sorted}(bx)$, where $a \in \Sigma$, $x \in \Sigma^*$.

Lemma 3.1. $\texttt{sorted}(\texttt{ms}(x)) = true$.

Proof. We will use induction on the length of x. Obviously, $\texttt{sorted}(\texttt{ms}(\varepsilon)) = \texttt{sorted}(\texttt{ms}(a)) = true$.

Recall that $\texttt{ms}(x) = \texttt{merge}(\texttt{ms}(\texttt{odd}(x)), \texttt{ms}(\texttt{even}(x)))$. By the induction assumption, we know that $\texttt{ms}(\texttt{odd}(x)), \texttt{ms}(\texttt{even}(x))$ are sorted, since their length is less than the length of x. Now it is enough to show that $\texttt{sorted}(\texttt{merge}(y, z)) = true$ if y and z are sorted. Consider $\texttt{sorted}(\texttt{merge}(ay, bz))$. Without loss of generality, let $a \leq b$, so $\texttt{sorted}(\texttt{merge}(ay, bz)) = \texttt{sorted}(a\texttt{merge}(y, bz)) = true$, since a is less or equal to both b and the next element of y, and $\texttt{sorted}(\texttt{merge}(y, bz)) = true$ holds by induction. ☐

Definition 3.2 ($\texttt{permutation}$)**.** For all natural n, the function $\texttt{permutation} : \Sigma^n \times \Sigma^n \to \{true, false\}$ is defined as follows. $\texttt{permutation}(\varepsilon, \varepsilon) = true$, $\texttt{permutation}(a, b) = (a == b)$, $\texttt{permutation}(ax, yaz) = \texttt{permutation}(x, yz)$, and if $a \notin y$ then $\texttt{permutation}(ax, y) = false$, where $a, b \in \Sigma$, $x, y, z \in \Sigma^*$.

Lemma 3.2. $\texttt{permutation}(x, \texttt{ms}(x)) = true$.

Proof. We will use induction on the length of x. Obviously, $\texttt{permutation}(\varepsilon, \texttt{ms}(\varepsilon)) = \texttt{permutation}(a, \texttt{ms}(a)) = true$.

First, let's show that $\texttt{merge(x,y)}$ is a permutation of xy. Indeed, if, for instance, $a \leq b$, then $\texttt{permutation}(\texttt{merge}(ax, by), axby) = \texttt{permutation}(a\texttt{merge}(x, by), axby) = \texttt{permutation}(\texttt{merge}(x, by), xby) = true$ by induction.

Finally, $\texttt{ms}(x) = \texttt{merge}(\texttt{ms}(\texttt{odd}(x)), \texttt{ms}(\texttt{even}(x)))$. By the induction assumption, we know that $\texttt{ms}(\texttt{odd}(x)), \texttt{ms}(\texttt{even}(x))$ are permutations of $\texttt{odd}(x), \texttt{even}(x)$ respectively, since their length is less than the length of x. Moreover, as we have already shown above, $\texttt{merge}(\texttt{ms}(\texttt{odd}(x)), \texttt{ms}(\texttt{even}(x)))$ is a permutation of $\texttt{ms}(\texttt{odd}(x))\texttt{ms}(\texttt{even}(x))$, which is a permutation of $\texttt{odd}(x)\texttt{even}(x)$. Finally, it is sufficient to prove that $\texttt{odd}(x)\texttt{even}(x)$ is a permutation of x. Using induction, $\texttt{permutation}(\texttt{odd}(ax)\texttt{even}(ax), ax) = \texttt{permutation}(a\texttt{even}(x)\texttt{odd}(x), ax) = \texttt{permutation}(\texttt{even}(x)\texttt{odd}(x), x) = \texttt{permutation}(\texttt{odd}(x)\texttt{even}(x), x) = true$. Here, we used the induction hypothesis and the fact that xy is a permutation of yx (also straightforwardly proved with induction, if needed). ☐

3.2 The human and monkeys theorem

Let's have a relation $<$, defined as follows: $x < y$ if $\texttt{parent}(x, y)$ or there exists z such that $x < z$ and $\texttt{parent}(z, y)$. We will assume that this relation is well-founded. Also, for all x, either $\texttt{human}(x)$ or $\texttt{monkey}(x)$.

Theorem 3.3. *If $x < y$, and* $\texttt{human}(y)$*, and* $\texttt{monkey}(x)$*, then there exist z_1, z_2 such that* $\texttt{parent}(z_1, z_2)$*, and* $\texttt{human}(z_2)$*, and* $\texttt{monkey}(z_1)$*.*

Proof. Define a sequence $w_1, w_2, \ldots$ inductively like this: $w_1 = y$, w_{n+1} is such that $\texttt{parent}(w_{n+1}, w_n)$, and $x < w_{n+1}$. If $w_n = x$, then terminate the sequence. Obviously, $w_1 > w_2 > \ldots$. Since $<$ is well-founded, the sequence has to be finite: $y = w_1 > w_2 > \ldots > w_k = x$.

For the sake of contradiction, suppose, there is no such i, that $\texttt{parent}(w_{i+1}, w_i)$, and $\texttt{human}(w_i)$, and $\texttt{monkey}(w_{i+1})$. Then, by induction, all w_i are humans. This contradicts the fact that w_k is a monkey. Therefore, the desired z_1, z_2 exist among some w_{i+1}, w_i, as requested. □