Finite Impulse Response (FIR) Filters

→ Digital filters are classified as : FIR & IIR.

→ Depends on the form of unit pulse response of system.

→ In FIR System, the impulse response sequence is of finite duration i.e. having finite no. of non-zero terms.

→ IIR system having infinite no. of non-zero term.

→ System with impulse response, $h(n) = \begin{cases} 2 & |n| \le 4 \\ 0 & \text{otherwise} \end{cases}$

has only a finite no. of non-zero term. System is FIR System

→ System with imp. Response $h(n) = a^n u(n)$ is non-zero for $n \ge 0$ .. IIR.

→ IIR filters are implemented using structures having feedback. (poles & zeros)

→ FIR filters are implemented using structures with no feedback. (all zeros)

FIR having advantages over IIR filters :

- They can have an exact linear phase.
- Stable
- Design methods are generally linear.
- Realised efficiently in hardware
- filter start up transients having finite duration.

Introduction: Design of FIR filter :-

→ filter → Impulse Response is finite

→ Output → depends only on +nt & (or) past values

→ Application → where linear phase is important.

ex:- Data transmission, Speech processing, correlation processing, Interpol

→ characteristics :-

- Impulse Response - finite
- non-recursive FIR filter - stable.
- Phase distortion of freqⁿ Response can be ellimmated by FIR filter
- Implement recursive fir filter, effect of Start up Transient have small duration.
- Quantization noise can be made negligible

1

Advantages

→ Stable

Can be realised in both recursive & non-recursive (non-repeating values)

Exact linear phase

Flexible

Low sensitive to Quantization noise

Efficiently Realized in H/w.

Disadvantages

→ Complex → Costly

→ Requires more filter coefficient to be stored

→ Long duration impulse response require large amount of processing.

→ Narrow transition band FIR filter requires more arithmetic operations & H/W. components

Linear Phase is used to preserve the shape of i/p signal.

Impulse Response of Linear phase FIR Filter :-

Symmetric:

$h(n) = h(M-1-n)$; $0 \le n \le M-1$

Let $M = 8$.

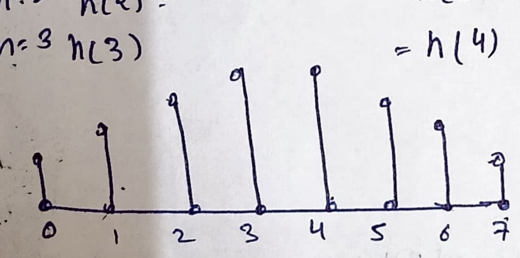
$\therefore h(n) = h(8-1-n)$; $0 \le n \le 7$

$n=0 \quad h(0) = h(8-1-0) = h(7)$

$n=1 \quad h(1) = h(8-1-1) = h(6)$

$n=2 \quad h(2) = h(5)$

$n=3 \quad h(3) = h(4)$



Anti-Symmetric:

$h(n) = -h(M-1-n)$; $0 \le n \le M-1$

Let $M = 8$

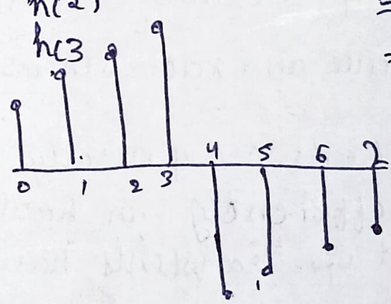
$\therefore h(n) = -h(8-1-n)$; $0 \le n \le 7$

$n=0, \; h(0) = -h(8-1-0) = -h(7)$

$n=1 \quad h(1) = -h(6)$

$n=2 \quad h(2) = -h(5)$

$n=3 \quad h(3) = -h(4)$



Frequency Response of Linear phase FIR filter :-

$h(n) \xleftrightarrow{DTFT} H(\omega)$

Freq Response of FIR filter:

$H(\omega) = \underbrace{H_r(\omega)}_{\text{real part}} e^{j\theta(\omega)}$

(i) Symmetric Impulse Response with 'M' even :-

$$H_r(\omega) = \sum_{n=0}^{M/2-1} 2h(n) \cos\left[\omega\left(n - \frac{M-1}{2}\right)\right]$$

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & ; \; H_r(\omega) > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & ; \; H_r(\omega) < 0 \end{cases}$$

Magnitude & phase Responses

(ii) Symmetric Imp. Response with 'M' odd :-

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} 2h(n) \cos\left[\omega\left[n - \frac{M-1}{2}\right.\right.$$

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & H_r(\omega) > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & H_r(\omega) < 0 \end{cases}$$

(iii) Antisymmetric Impulse Response with even 'M' :-

$$H_r(\omega) = 2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \sin\left[\omega\left(\frac{M-1}{2} - n\right)\right]$$

$$\theta(\omega) = \begin{cases} \frac{\pi}{2} - \omega\left(\frac{M-1}{2}\right) ; & H_r(\omega) > 0 \\ \frac{3\pi}{2} - \omega\left(\frac{M-1}{2}\right) ; & H_r(\omega) < 0. \end{cases}$$

Anti-symmetric Impulse Response with 'M' odd:

$$H_r(\omega) = 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\,\sin\left[\omega\left(\frac{M-1}{2}-n\right)\right]$$

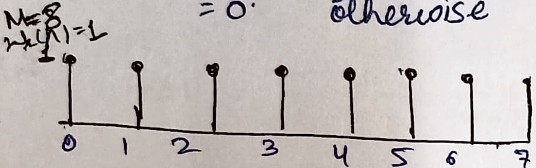
$$\theta(\omega) = \begin{cases} \frac{\pi}{2} - \omega\left(\frac{M-1}{2}\right) & ; H_r(\omega) > 0 \\ \frac{3\pi}{2} - \omega\left(\frac{M-1}{2}\right) & ; H_r(\omega) < 0 \end{cases}$$

Design of Linear Phase FIR filter using Window Method:-

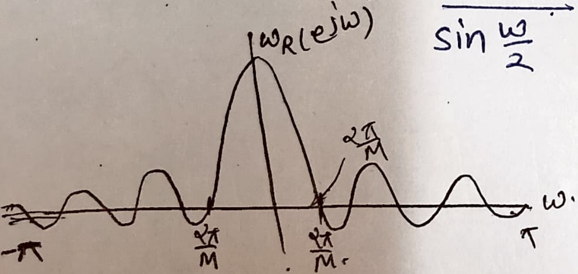
Different types of Windows:

(a) Rectangular window:

$$w_R(n) = 1 \quad 0 \le n \le M-1$$
$$= 0 \quad \text{otherwise}$$

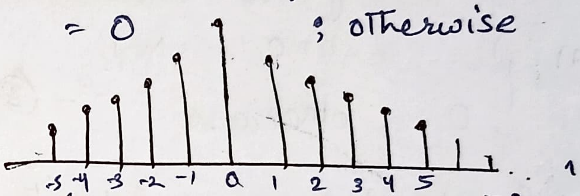


spectrum: $W_R(e^{j\omega}) = \dfrac{\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}}$

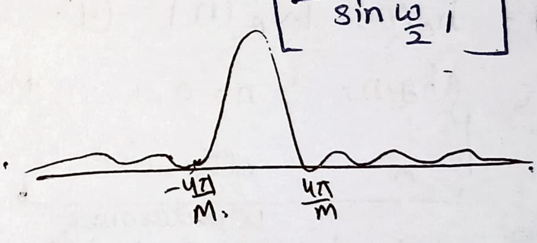


(b) Bartlett Window [Triangular].

$$w_T(n) = 1 - \frac{2\left|n - \left(\frac{M-1}{2}\right)\right|}{M-1} ; \quad 0 \le n \le M-1$$
$$= 0 \quad ; \text{otherwise}$$

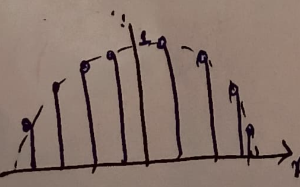


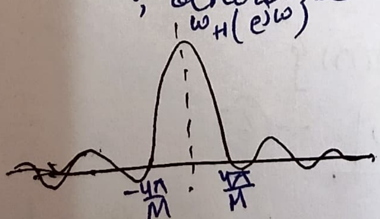
spectrum: $W_R(e^{j\omega}) = \left[\dfrac{\sin\omega\left(\frac{M-1}{2}\right)}{\sin\frac{\omega}{2}}\right]^2$



(c) Hamming Window:

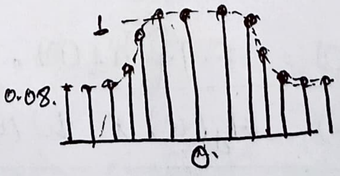
$$W_H(n) = 0.5 - 0.5\cos\frac{2\pi n}{M-1} ; \quad 0 \le n \le M-1$$
$$= 0 \quad ; \text{otherwise}$$

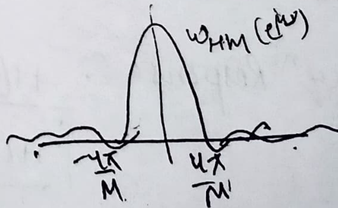




(d) Hamming Window:-

$$W_{HM}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right) ; \quad 0 \le n \le M-1$$
$$= 0 \quad \text{otherwise}$$





Procedure to design linear phase FIR filter using windows:-

$H_d(\omega) \rightarrow$ desired freq^n Response $\quad h_d(n) \rightarrow$ desired sample Response

$H_d(\omega) \rightarrow$ F.T of $h_d(n)$

$$\therefore \quad H_d(\omega) = \sum_{n=0}^{\infty} h_d(n)\, e^{-j\omega n} \quad -(1)$$

$h_d(n) \rightarrow$ inverse F.T of $H_d(\omega)$

$$\therefore \quad \underbrace{h_d(n)}_{\infty \text{ duration}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega \quad (2)$$

we make ∞ duration to finite $\xRightarrow{\text{multiply}}$ $h_d(n)$ by window sequence 'M'.
↳ Reduced to length 'M'.

Examples: Rectangular Window:-

$$W_R(n) = \begin{cases} 1 & ; n = 0, 1, \cdots M-1 \\ 0 & ; \text{otherwise} \end{cases} \quad -(3)$$

$h_d(n) \rightarrow$ Sample Response $\rightarrow$ Infinite

$$\therefore h(n) = h_d(n) \cdot W_R(n) \quad -(4)$$

$$\therefore h(n) = \underbrace{\begin{cases} h_d(n) & ; n = 0, 1, \cdots M-1 \\ 0 & ; \text{otherwise} \end{cases}}_{\text{windowing}} \quad -(5)$$

we have made $h_d(n)$, a finite value.

from (4), $h(n) = h_d(n) \cdot W_R(n)$.

generally, $\boxed{h(n) = h_d(n) \cdot \underbrace{\omega(n)}_{\text{generalized window}}}$ $\Rightarrow$ unit sample response FIR filter

freq^n Response: $H(\omega) = F.T\{h_d(n) \cdot \omega(n)\}$

$$\boxed{H(\omega) = H_d(\omega) * W(\omega)} \quad -(6).$$

Design the symmetric FIR low pass filter whose

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & ; |\omega| \le \omega_c \\ 0 & ; \text{otherwise} \end{cases}$$ with $M=7$ & $\omega_c = 1$ rad/sam. [use Rectangular window]

Step (1). obtain $h_d(n)$: desired input Response

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega \quad -(1)$$

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & ; -1 \le \omega \le 1 \\ 0 & ; \text{otherwise} \end{cases} \quad -(2)$$

(2) in (1).

$$h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} e^{-j\omega\tau} \cdot e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega(n-\tau)} \cdot d\omega$$

$$h_d(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-1}^{1} = \frac{1}{2\pi}\left[\frac{e^{j(n-\tau)} - e^{-j(n-\tau)}}{j(n-\tau)}\right]$$

$$h_d(n) = \frac{1}{\pi(n-\tau)}\left[\frac{e^{j(n-\tau)} - e^{-j(n-\tau)}}{2j}\right]$$

$$\boxed{\because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$h_d(n) = \frac{\sin(n-\tau)}{\pi(n-\tau)} \quad n \ne \tau \quad -(3)$$

If $n = \tau$,

$$h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} 1 \cdot d\omega = \frac{1}{2\pi}[2] = \frac{1}{\pi} \quad -(4)$$

$$h_d(n) = \begin{cases} \dfrac{\sin(n-\tau)}{\pi(n-\tau)} & ; n \ne \tau \\ \dfrac{1}{\pi} & ; n = \tau \end{cases} \quad -(5)$$

Determine value of τ.

$\boxed{h(n) = h(M-1-n)}$

$\boxed{\because h(n) = h_d(n) \cdot \omega(n)}$

$h_d(n) \cdot \omega(n) = h_d(M-1-n) \cdot \omega(n)$

$h_d(n) = h_d(M-1-n)$

$$\frac{\sin(n-\tau)}{\pi(n-\tau)} = \frac{\sin(M-1-n-\tau)}{\pi(M-1-n-\tau)}$$

$$\frac{-\sin(n-\tau)}{-\pi(n-\tau)} = \frac{\sin(M-1-n-\tau)}{\pi(M-1-n-\tau)}$$

[Multiply (-) sign in nr. & dr. else value will not be get]

[$\because -\sin\theta = \sin(-\theta)$]

$$\frac{\sin[-(n-\tau)]}{\pi[-(n-\tau)]} = \frac{\sin(M-1-n-\tau)}{\pi(M-1-n-\tau)}$$

$$-(n-\tau) = M-1-n-\tau$$

$$-n+\tau = M-1-n-\tau$$

$$2\tau = M-1$$

$$\boxed{\tau = \frac{M-1}{2}}$$ in eqn (5)

(5) $\Rightarrow$ $h_d(n) = \begin{cases} \dfrac{\sin\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} & ; n \neq \frac{M-1}{2} \\ \dfrac{1}{\pi} & ; n = \frac{M-1}{2} \end{cases}$

$\therefore$ $M=7$, $h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & n\neq 3 \\ \dfrac{1}{\pi} & n=3 \end{cases}$

Put $n = 0$ to 6,

$n=0$, $h_d(0) = 0.01496$

$h_d(1) = 0.14472$

$h_d(2) = 0.26785$

$h_d(3) = \frac{1}{\pi}$

$h_d(4) = 0.26785$

$h_d(5) = 0.14472$

$h_d(6) = 0.01497$

upto this same for all windows

$\therefore$ $h(n) = h_d(n) \cdot w(n)$

$w_R(n) = \begin{cases} 1 ; 0 \le n \le 6 \\ 0 ; \text{other.} \end{cases}$

$h(n) = \begin{cases} h_d(n) & ; 0 \le n \le 6. \\ 0 & ; \text{otherwise} \end{cases}$

Coefficients of FIR filter:

$h(0) = 0.01496$

$h(1) = 0.14472$

$h(2) = 0.26785$

$h(3) = \frac{1}{\pi}$

$h(4) = 0.26785$

$h(5) = 0.14472$

$h(6) = 0.01497$

Symmetric filter.

Hence, $h(n) = h(6-n)$

Designing FIR filter using hanning window :-

Num Design the symmetric FIR low pass filter whose

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & ; |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

with $M = 7$ & $\omega_c = 1$ rad/sam

use hanning window

Procedure is same like, we solved on Rectangular window & we get

$n=0$. $h_d(0) = 0.01496$.

$n=1$, $h_d(1) = 0.14472$ -

$n=2$, $h_d(2) = 0.26785$

$n=3$, $h_d(3) = 1/\pi$

$n=4$, $h_d(4) = 0.26785$.

$n=5$, $h_d(5) = 0.14472$

$n=6$, $h_d(6) = 0.01497$

~~n=7, h_d(7)=~~

Hanning window $w(n) = 0.5\left[1 - \cos\left(\frac{2\pi n}{M-1}\right)\right]$

$\because M=7$, $w(n) = 0.5\left[1 - \cos\left(\frac{2\pi n}{6}\right)\right]$

$w(n) = 0.5\left[1 - \cos\left(\frac{\pi n}{3}\right)\right]$ $n = 0$ to 6. & $M=7$

Rad. Mode $0.5 \times (1 - \cos(\pi A \div 3))$

For,

$n=0$, $w(0) = 0$

$n=1$, $w(1) = 0.25$

$n=2$, $w(2) = 0.75$

$n=3$, $w(3) = 1$

$n=4$, $w(4) = 0.75$.

$n=5$, $w(5) = 0.25$

$n=6$, $w(6) = 0$.

~~Here~~ ,

$h(n) = h_d(n) \cdot w(n)$

$\therefore h(0) = \underline{0}$

$h(1) = 0.03618$

$h(2) = 0.20089$

$h(3) = 1/\pi$

$h(4) = 0.20089$

$h(5) = 0.03618$

$h(6) = \underline{0}$.

Symmetric FIR filter

Hamming Window :-

Num. Determine the filter coefficient $h_d(n)$ for desired freq^n Response of a LPF given by,

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega} & ; -\frac{\pi}{4} < \omega \le \frac{\pi}{4} \\ 0 & ; \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

If we define new filter coefficients by $h_d(n) \cdot \omega(n)$, where ω

$$\omega(n) = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Determine $h(n)$ & also freq^n Response $H(e^{j\omega})$ & compare with $H_d(e^{j\omega})$

Determine $H(e^{j\omega})$ using Hamming window.

step 1: $h_d(n)$
step 2: $h(n)$
Step 3: $H(e^{j\omega})$
Step 4: compare $H(e^{j\omega})$ with $H_d(e^{j\omega})$
Step 5: $H(e^{j\omega})$ by Hamming window.

(i) $h_d(n)$:-

$$\boxed{h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega\tau}\, d\omega} = \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j2\omega} \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j\omega(n-2)} \cdot d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-2)}}{j(n-2)}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2\pi}\left[\frac{e^{j\frac{\pi}{4}(n-2)} - e^{-j(\frac{\pi}{4})(n-2)}}{j(n-2)}\right]$$

$$= \frac{1}{\pi(n-2)}\left[\frac{e^{j\frac{\pi}{4}(n-2)} - e^{-j(\frac{\pi}{4})(n-2)}}{2j}\right] \Rightarrow \boxed{h_d(n) = \frac{\sin\frac{\pi}{4}(n-2)}{\pi(n-2)}} \quad n \neq 2.$$

for $n = 2$,

$$\frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j\omega(n-2)} \cdot d\omega = \frac{1}{2\pi}\left[\omega\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2\pi}\left[\frac{2\pi}{4}\right] = \boxed{\frac{1}{4}}$$

$$\therefore \boxed{h_d(n) = \begin{cases} \frac{\sin\frac{\pi}{4}(n-2)}{\pi(n-2)} & ; n \neq 2 \\ \frac{1}{4} & ; n = 2 \end{cases}}$$

(ii) $h(n)$: $\boxed{h(n) = h_d(n) \cdot \omega(n)}$; ~~asasa~~ $\Rightarrow h(n) = h_d(n) \cdot 1 \;\; ; 0 \le n \le 4$

$\boxed{h(n) = h_d(n)}$ Rectangular window

$n=0$, $h(0) = h_d(0) = 0.159091$
$n=1$, $h(1) = h_d(1) = 0.224989$
$n=2$, $h(2) = h_d(2) = 1/4$
$n=3$, $h(3) = h_d(3) = 0.224989$
$n=4$, $h(4) = h_d(4) = 0.159091$

(iii) $H(e^{j\omega})$:- freq^n Response of the filter

$M=5$ as $n=0-4$
↳ odd

$$\therefore H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \left\{ h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n).\cos\omega\left(n-\frac{M-1}{2}\right) \right\}$$

$$M=5,\ H(\omega) = e^{-2j\omega}\left[h(2) + 2\sum_{n=0}^{1} h(n).\cos\omega(n-2)\right]$$

$$= e^{-2j\omega}\left[h(2) + 2h(0).\cos\omega(-2) + 2h(1).\cos\omega(-1)\right] \quad [\cos(-\theta)=\cos\theta]$$

$$= e^{-2j\omega}\left[0.25 + 2\times 0.159091\ \cos 2\omega + 2\times 0.224989\ \cos\omega\right]$$

$$H(\omega) = e^{-j2\omega}\left[0.25 + 0.318\cos 2\omega + 0.45\cos\omega\right]$$

$$\boxed{|H(\omega)| = 0.25 + 0.318\cos 2\omega + 0.45\cos\omega}$$ Magnitude Response

$$\boxed{\angle H(\omega) = \begin{cases} -2\omega ; & |H(\omega)| > 0 \\ -2\omega + \pi ; & |H(\omega)| < 0 \end{cases}}$$ Phase Response

(iv) Compare $H(e^{j\omega})$ with $H_d(e^{j\omega})$:

Both f^n are different. $H_d(e^{j\omega})$ window f^n is rect.

(v) $H(e^{j\omega})$ using Hamming window:

Hamming window: $w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$; $0 \le n \le M-1$, $M=5$

$w(n) = 0.5 - 0.46\cos\left(\frac{\pi n}{2}\right)$; $0 \le n \le 4$.

$h(n) = h_d(n).w(n) = ?$

$h_d(0) = 0.15901$
$h_d(1) = 0.22498$
$h_d(2) = 0.25$
$h_d(3) = 0.224984$
$h_d(4) = 0.159091$

$n=0,\ w(0) = 0.08$	$h(0) = 0.01273$
$n=1,\ w(1) = 0.54$	$h(1) = 0.12149$
$n=2,\ w(2) = 1$	$h(2) = 0.25$
$n=3,\ w(3) = 0.54$	$h(3) = 0.12149$
$n=4,\ w(4) = 0.08$	$h(4) = 0.01273$

$\boxed{M=5}$ $$H(e^{j\omega}) = H(\omega) = e^{-j2\omega}\left[h(2) + 2\sum_{n=0}^{1} h(n)\cos\omega(n-2)\right]$$

$$= e^{-j2\omega}\left[0.25 + 2\times 0.01273\cos 2\omega + 2\times 0.12149\cos\omega\right]$$

So, $$\boxed{H(e^{j\omega}) = e^{-j2\omega}\left[0.25 + 0.02546\cos 2\omega + 0.243\cos\omega\right]}$$

Kaiser Window:-

Num: Design FIR filter (linear phase filter) using Kaiser window to meet following specifications:

$$0.99 \le |H(e^{j\omega})| \le 1.01 \quad ; \quad 0 \le |\omega| \le 0.19\pi$$

$$|H(e^{j\omega})| \le 0.01 \quad ; \quad 0.21\pi \le |\omega| \le \pi$$

$$1-0.01 \le |H(e^{j\omega})| \le 1+0.01 \quad ; \quad 0 \le |\omega| \le \underbrace{0.19\pi}_{\omega p = \text{Pass band edge freq}}$$

$$|H(e^{j\omega})| \le 0.01 \quad ; \quad \underbrace{0.21\pi}_{\omega s - \text{Stop band edge freq}} \le |\omega| \le \pi$$

Ripple factors

- $\delta_1 = 0.01$, $\delta_2 = 0.01$, $\omega p = 0.19\pi$, $\omega s = 0.21\pi$
- $\Delta\omega = \omega s - \omega p = 0.21\pi - 0.19\pi = 0.02\pi$
- $\delta = \min$ of δ_1 & $\delta_2 \Rightarrow \delta = 0.01$
- Attenuation $A = -20\log_{10}\delta = -20\log_{10}(0.01) = 40$

(i) Cutoff freqn ω_c:
$$\omega_c = \frac{\omega p + \omega s}{2} = \frac{0.19\pi + 0.21\pi}{2} = 0.2\pi$$

(ii) To obtain β (shape parameter) & M (length of filter)

$$\beta = \begin{cases} 0.1102(A-8.7) & ; \quad A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & ; \quad 21 \le A \le 50 \\ 0 & ; \quad A \le 21 \end{cases}$$

$$\therefore \beta = 0.5848(40-21)^{0.4} + 0.07886(40-21)$$

$$\boxed{\beta = 3.395}$$

- $$M = \frac{A-8}{2.285\,\Delta\omega} = \frac{40-8}{2.285 \times 0.02\pi} = \boxed{M = 223}$$

) Equation for Kaiser Window:- $\quad \alpha = \frac{M}{2} = \frac{223}{2} = 111.5$

$$W_K(n) = \begin{cases} \dfrac{I_0\left[\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{\frac{1}{2}}\right]}{I_0\beta} & ; \quad 0 \le n \le M \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\boxed{\omega_k(n) = \begin{cases} \dfrac{I_0\left[3.395\left(1-\left(\frac{n-111.5}{111.5}\right)^2\right)^{1/2}\right]}{I_0(3.395)} & ; \quad 0 \le n \le 223 \\ 0 & ; \quad \text{otherwise} \end{cases}}$$

(iv) Obtain $h_d(n)$:

Ideal freq^n Response: $H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & ; -\omega_c \le \omega \le \omega_c \\ 0 & ; \text{otherwise} \end{cases}$

Inverse forier transform,

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n - \frac{M-1}{2}\right)\right]}{\pi\left(n - \frac{M-1}{2}\right)} & ; n \ne \frac{M-1}{2} \\[2ex] \dfrac{\omega_c}{\pi} & ; n = \frac{M-1}{2} \end{cases}$$

$M = 223$, $\frac{M}{2} = 111.5$

Substituting ω_c,

$$\boxed{h_d(n) = \frac{\sin\left[0.2\pi(n-111.5)\right]}{\pi(n-111.5)} \quad ; 0 \le n \le 223.}$$

(V) Obtain $h(n)$:

$h(n) = h_d(n) \cdot w(n)$

$$\boxed{h(n) = \begin{cases} \dfrac{\sin\left(0.2\pi(n-111.5)\right)}{\pi(n-111.5)} \cdot \dfrac{I_0\left\{3.395\left[1 - \left(\frac{n-111.5}{111.5}\right)^2\right]^{1/2}\right\}}{I_0(3.395)} & ; 0 \le n \le 223 \\[2ex] 0 & ; \text{otherwise} \end{cases}}$$

Infinite Impulse Response Filter

→ y(n) depends → x(n) & x(n-1) also y(n-1)
→ O/P of IIR filter y(n) will depend on i/n/t & past inputs & past O/P.

or

→ Difference Eqn :-

$y(n) = b_0 x(n) + b_1 x(n-1) + \dots + b_M(n-M) - a_1 y(n-1) - \dots - a_N y(n-N)$

→ Transfer function :-

$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} \dots b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$; $b_i \rightarrow (M+1)$ numerator; $a_i \rightarrow N$ denominator

Y(z) & X(z) → z-transform of x(n) & y(n)

→ Poles (S) → inside the unit circle → STABLE

→ Smaller filter size. → linear phase is not easy to obtain

→ Objective → Determine the filter numerator & denominator
→ Satisfy filter specification (Pass band, SB, cutoff freqn for LP, HP, BP, BR filter)

→ Advantage: Easy to design & implement.

→ Disadvantage: → Non linear → non-stable. → ~~IIR~~ FIR

Bilinear Transformation design Method of IIR filter :-

$\boxed{\text{Digital filter specification.}}$

↓ 1. Transformation with freq^n wrapping

$\boxed{\text{Analog filter Specification.}}$

↓ 2. Transform^n by Lowpass prototype filter

$\boxed{\text{Analog filter Transfer function.}}$

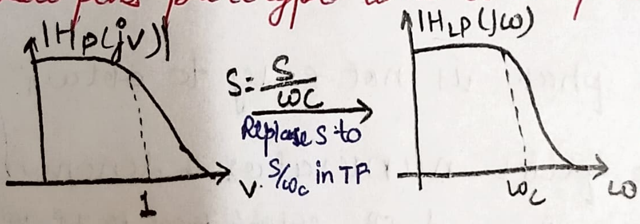
↓ 3. Bilinear Transformation.

$\boxed{\text{Digital filter transfer function \& frequency response verification.}}$

General Procedure

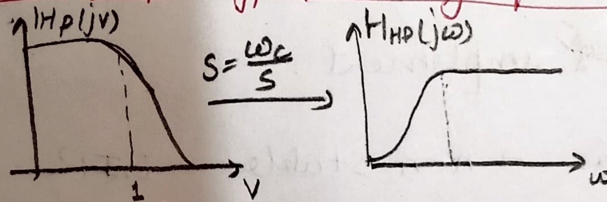
Analog Filters using Lowpass Prototype Transformation :-

① Low pass prototype into a lowpass filter :



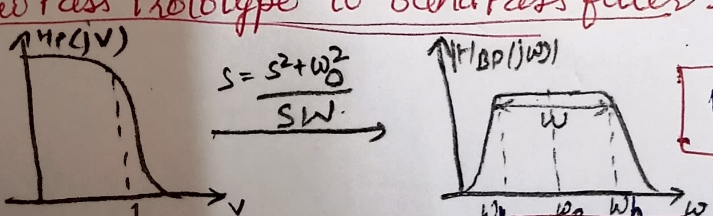
$$\boxed{H_{LP}(S) = [H_p(S)]_{S = \frac{S}{\omega_c}}}$$

② Low Pass Prototype to High pass filter :-



$$\boxed{H_{HP}(S) = H_p(S)\Big|_{S = \frac{\omega_c}{S}}}$$

③ Low Pass Prototype to Band Pass filter :-

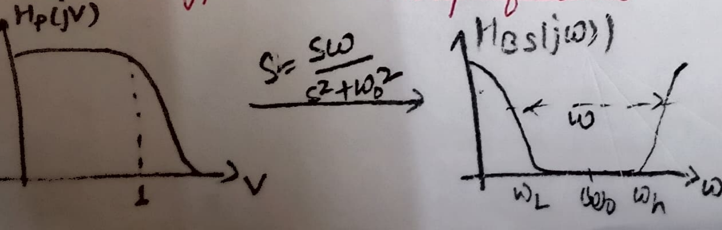


$$\boxed{H_{BP}(S) = H_p(S)\Big|_{S = \frac{S^2+\omega_0^2}{SW}}}$$

$W = \omega_h - \omega_L \rightarrow$ Passband BW

$\boxed{\omega_0 = \sqrt{\omega_L \cdot \omega_h}}$

④ Low Pass Prototype to Bandstop filter :-



$$H_{BS} = H_p(S)\Big/_{S = \frac{SW}{(S^2+\omega_0^2)}}$$

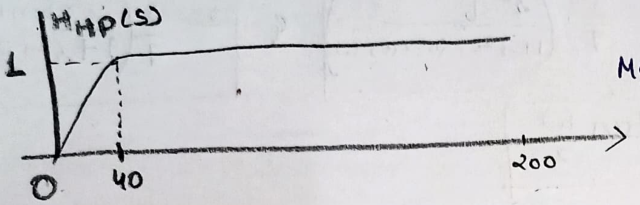
Given a lowpass prototype $H_p(S) = \frac{1}{S+1}$. Determine each of ... 200 rad/S. analog filter & plot their magnitude response from 0 to 200 rad/S.

(i) A HP Filter with $\omega_c = 40$ rad/sec.

(ii) A BP filter with $\omega_0 = 100$ rad/sec & BW of 20 rad/sec.

(i) High Pass Filter :-

$$H_P(S) = \frac{1}{S+1}\Bigg|_{S = \frac{\omega_c}{S} = \frac{40}{S}} \Rightarrow H_{HP}(S) = \frac{1}{\frac{40}{S}+1} = \frac{S}{40+S} \leftarrow \text{TF for HP.}$$



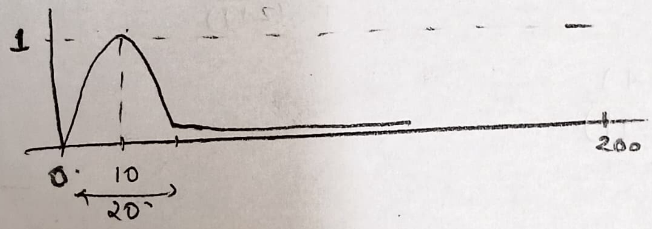
Magnitude Response

(ii) Band Pass Filter :-

$\omega_0 = \sqrt{\omega_L \cdot \omega_H} = 100. \Rightarrow \omega_0^2 = 100 \Rightarrow \omega_0 = 10$ rad/sec

$W = \omega_H - \omega_L = 20$ rad/sec.

$$H_P(S) = \frac{1}{S+1}\Bigg|_{S = \frac{S^2+\omega_0^2}{SW}} \Rightarrow H_{BP}(S) = \frac{1}{\frac{S^2+100}{20S}+1} = \frac{20S}{S^2+20S+100}$$



Bilinear Transformation & freqn Warping :-

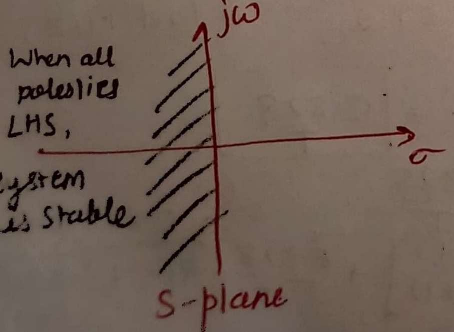
→ In freqn Warping, one special representation is transformed to another representation.

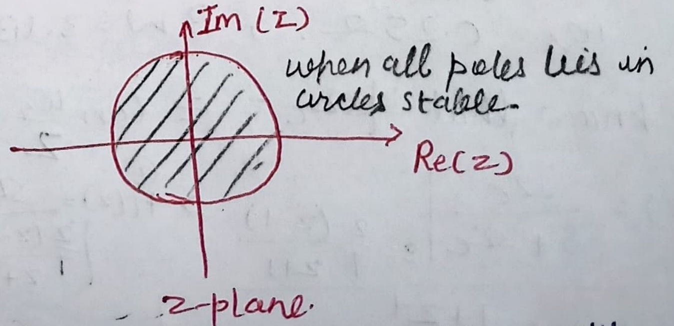
→ Bilinear transformation is used to transform analog filter TF into digital filter transformation function. $\boxed{\text{Analog Filter TF} \rightarrow \text{Digital filter TF}}$

$H(S) \rightarrow H(Z)$

S domain to z domain

In Bilinear transformation, $\boxed{H(z) = H(S)\Big|_{S = \frac{2}{T} \cdot \frac{(z-1)}{(z+1)}}}$ T → Sampling Period





In Bilinear transformation, we need to map splane into z-plane. A Filter → D. Filter

8

Mapping Properties:

1. LHS of s-plane is mapped into inside the circle of z-plane.
2. RHS of s-plane is mapped into outside the circle of z-plane
3. +ve $j\omega$ axis of s-plane is mapped into +ve half of unit circle & -ve half of $-j\omega$ axis is mapped into -ve half of unit circle.

The General characteristic of mapping $z = e^{ST}$:

$s = \sigma + j\Omega$ & $z = re^{j\omega}$

$S = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$ we get, $\boxed{\sigma = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega}\right)}$ & $\boxed{\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right)}$

with $r = 1$, $\sigma = 0$. & $\boxed{\Omega = \frac{2}{T}\tan\frac{\omega}{2}}$

$\boxed{\omega = 2\tan^{-1}\frac{\Omega T}{2}}$

Num: Apply bilinear transformation to $H(S) = \frac{2}{(S+1)(S+3)}$ with $T = 0.1S$

For bilinear transformation, $H(z) = H(S)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$

$$H(z) = \frac{2}{(s+1)(s+3)}\Bigg|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{2}{\left[\frac{2}{T}\left(\frac{z-1}{z+1}\right)+1\right]\left[\frac{2}{T}\left(\frac{z-1}{z+1}\right)+3\right]}$$

using $T = 0.1$ Sec.

$$H(z) = \frac{2}{\left[20\cdot\frac{(z-1)}{(z+1)}+1\right]\left[20\cdot\frac{(z-1)}{(z+1)}+3\right]} = \frac{2(z+1)^2}{(21z-19)(23z-17)}$$

$$\boxed{H(z) = \frac{0.0041(1+z^{-1})^2}{1-1.644z^{-1}+0.668z^{-2}}}$$

Num. A Digital filter with 3dB bandwidth of 0.25π is to be designed from analog filter whose system response $H(S) = \frac{\Omega_c}{S+\Omega_c}$ use bilinear Trans & find $H(z)$.

$\omega_r = 0.25\pi$, $BW = 3dB$.

we know that $\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2} = \frac{2}{T}\tan 0.125\pi = \boxed{\frac{0.828}{T}}$

① $H(z) = \frac{\Omega_c}{S+\Omega_c}\Big|_{s=\frac{2}{T}\frac{(z-1)}{z+1}}$ ⇒ ② $H(z) = \frac{\Omega_c}{\left[\frac{2}{T}\frac{(z-1)}{z+1}\right]+\Omega_c}$ ⇒ ③ $\frac{\frac{0.828}{T}}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)}\right]+\frac{0.828}{T}}$ ⇒ ④ $\frac{0.828(z+1)}{2(z-1)+0.828(z+1)}$

⑤ ⟹ $\boxed{H(z) = \frac{1+z^{-1}}{3.414-1.414z^{-1}}}$

IIR Filter design by Impulse Invarient Method

Steps to design :-

- For given specifications, find H(S), TF of analog filter.
- Select sampling rate of digital filter, T secondspersample.
- Express analog filter TF as a sum of single pole filters

$$H(S) = \sum_{K=1}^{N} \frac{C_K}{S - P_K}$$

- Compute z-transform of digital filter by using formula.

$$H(z) = \sum_{K=1}^{N} \frac{C_K}{1 - e^{P_K T} z^{-1}}$$

- For high Sampling rate, use $H(z) = \sum_{K=1}^{N} \frac{T C_K}{1 - e^{P_K T} z^{-1}}$

Mapping formula for impulse Invarient transformation :

$$\frac{1}{S - P_K} \rightarrow \frac{1}{1 - e^{P_K T} z^{-1}}$$

It shows that analog pole at $S = P_K$ is mapped into digital pole at $z = e^{P_K T}$

Important Transformations —

$$\frac{1}{(S + S_K)^m} \rightarrow \frac{(-1)^{m-1}}{\lfloor m-1} \cdot \frac{d^{m-1}}{ds^{m-1}} \left[\frac{1}{1 - e^{-ST} z^{-1}} \right] ; \quad S \rightarrow S_K$$

$$\frac{S + a}{(S+a)^2 + b^2} \longrightarrow \frac{1 - e^{aT} (\cos bT) z^{-1}}{1 - 2e^{-aT} (\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(S+a)^2 + b^2} \longrightarrow \frac{e^{-aT} (\sin bT) z^{-1}}{1 - 2e^{-aT} (\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

- Impulse Invarient Method should be used for bandlimited filters

Num. Convert analog filter to digital filter whose system function is

$$H(S) = \frac{S+0.2}{(S+0.2)^2+9}$$ use impulse invarient Method, Assume $T = 1s$

: System Response of analog filter in standard form $H(S) = \frac{S+a}{(S+a)^2+b^2}$

$a = 0.2, \; b = 3.$

So, System Response of digital filter $H(z) = \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$

$$H(z) = \frac{1-e^{-0.2T}(\cos 3T)z^{-1}}{1-2e^{-(0.2T)}(\cos 3T)z^{-1}+e^{-2(0.2)T}z^{-2}}$$ & $T = 1s.$

$$H(z) = \frac{1-(0.8187)(-0.99)z^{-1}}{1-2(0.8187)(-0.99)z^{-1}+0.6703z^{-2}}$$

That is, $\boxed{H(z) = \frac{1+0.8105z^{-1}}{1+1.6210z^{-1}+0.6703z^{-2}}}$

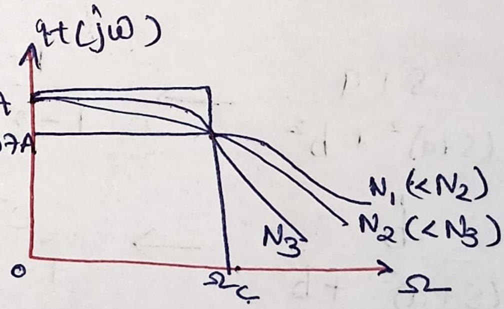
Butterworth filter :-

(i) Butterworth Lowpass filter has magnitude response by.

(ii) $$|H(j\Omega)| = \frac{A}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{1/2}} \quad -(1)$$

A = Gain of filter
Ω_c = 3dB cutoff freqn.
N = order of filter

By increasing filter order 'N', butterworth response approximates ideal response.



Design Parameters :-

Consider Low pass filter with desired specifications.

$$\delta_1 \le |H(e^{j\omega})| \le 1 \quad ; 0 \le \omega \le \omega_1 \quad -(2)$$

$$|H(e^{j\omega})| \le \delta_2 \quad , \omega_2 \le \omega \le \pi \quad -(3)$$

δ_1 = Passband deviati
δ_2 = SB deviation.
ω_1 = Passband freqn.
ω_2 = SB freqn.

Step 1:
The order of filter 'N' is given by:

$$N = \frac{\frac{1}{2}\log\left\{\left[\left(\frac{1}{\delta_2^2}-1\right)\right]\Big/\left(\frac{1}{\delta_1^2}-1\right)\right\}}{\log(\Omega_2/\Omega_1)} \quad -(4)$$

$3.414 - 1.414z^{-1}$

Step 2 :-
Cutoff freqⁿ Ω_c is given by:

$$\Omega_c = \frac{\Omega_1}{\left[\left(\frac{1}{\delta_1^2}\right)-1\right]^{1/2N}} \quad (5)$$

The value Ω_1, Ω_2 are obtained using binear transformation or impulse invarient method.

For Binear transformation: $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$ — (6)

for impulse invarient method: $\Omega = \frac{\omega}{T}$ — (7)

Step 3:- Transfer function H(S):-

Transfer function of butterworth filter:

If N = even, $H(S) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + C_k \Omega_c^2}$; N = even.

If N = odd, $H(S) = \frac{B_0 \Omega_c}{s + C_0 \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + C_k \Omega_c^2}$

where $b_k = 2\sin\left[(2k-1)\pi/2N\right]$ & $C_k = 1$

Parameter B_k :-

If N = even, $A = \prod_{k=1}^{N/2} B_k$

If N = odd, $A = \prod_{k=1}^{(N-1)/2} B_k$.

Steps to design Butterworth filter
1. From given specification, Find 'N' order.
2. Round off N to next integer
3. Calculate 'Ω_c' Cutoff freq.
4. Find transfer function H(S) using appropriate transformation
5. H(Z)

Step 4: System function H(z) :-

Using Bilinear transformation, $H(z) = H(S)\Big|_{s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)}$

using impulse invarient, $\frac{1}{s - P_k} \rightarrow \frac{1}{1 - e^{P_k T} z^{-1}}$

also important transformation can be used as discussed earlier (Page 9)

Num. Design of digital Butterworth filter that satisfies the follo constraints using bilinear transformation. Assume T = 1 s

$$0.9 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le \pi/2$$
$$|H(e^{j\omega})| \le 0.2 \qquad \frac{3\pi}{4} \le \omega \le \pi$$

Given: $\delta_1 = 0.9$, $\delta_2 = 0.2$, $\omega_1 = \frac{\pi}{2}$, $\omega_2 = \frac{3\pi}{4}$

Step 1: Determine analog filter's edge frequencies.

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = \frac{2}{1}\tan\frac{\pi}{4} = 2 \quad \& \quad \Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2\tan\frac{3\pi}{8} = 4.82$$

Therefore, $\frac{\Omega_2}{\Omega_1} = 2.414$.

Step 2: - Determine order 'N' of filter

$$N \ge \frac{1}{2}\frac{\log\left\{\left[\left(\frac{1}{\delta_2^2}-1\right)/\left(\frac{1}{\delta_1^2}-1\right)\right]\right\}}{\log(\Omega_2/\Omega_1)} = \frac{1}{2}\frac{\log\left\{\frac{24}{0.2346}\right\}}{\log(2.414)} = 2.626 \approx 3$$

Step 3: Determine -3dB cutoff frequency

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{\delta_1^2}-1\right]^{\frac{1}{2N}}} = \frac{2}{\left[\frac{1}{(0.9)^2}-1\right]} = 2.5467$$

Step 4: Determine H(S)

$$H(S) = \frac{B_0\Omega_c}{S + c_0\Omega_c}\prod_{k=1}^{(N-1)/2}\frac{B_k\Omega_c^2}{s^2 + b_k\Omega_c S + C_k\Omega_c^2}$$

$$H(S) = \left(\frac{B_0\Omega_c}{S + c_0\Omega_c}\right)\left(\frac{B_1\Omega_c^2}{s^2 + b_1\Omega_c S + c_1\Omega_c^2}\right)$$

as $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right] \Rightarrow b_1 = 2\sin\frac{\pi}{6} = 1$

$c_k = 1$, $\boxed{c_1 = 1}$, $\boxed{c_0 = 1}$ $\Rightarrow B_0 B_1 = 1$, $B_0 = B_1 = 1$

Therefore,

$$\boxed{H(S) = \left(\frac{2.5467}{S + 2.5467}\right)\left(\frac{6.4857}{s^2 + 2.5467S + 6.4857}\right)}$$

Step 5: Determine H(Z)

$$H(z) = H(S)\Big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}} \Rightarrow H(z) = \left[\frac{2.5467}{S + 2.5467}\right]\left[\frac{6.4857}{\left(\frac{2}{T}\frac{(z-1)}{(z+1)}\right)^2 + 2.5467\left(\frac{z-1}{z+1}\right) + 6.4857}\right]$$

$$\Rightarrow H(z) = \frac{16.5171(z+1)^3}{70.832z^3 + 31.1205z^2 + 27.23z + 2.9}$$

$$\Rightarrow \boxed{H(z) = \frac{0.2332(1+z^{-1})^3}{1 + 0.4394z^{-1} + 0.0416z^{-3}}}$$

N =

Num :- Design a digital Chebyshev filter to satisfy the constraint

$0.707 \le |H(e^{j\omega})| \le 1 \quad 0 \le \omega \le 0.2\pi$ use bilinear transform

$|H(e^{j\omega})| \le 0.1 \quad 0.5\pi \le \omega \le \pi$ and $T = 1s$.

Given: $\delta_1 = 0.707$, $\delta_2 = 0.1$, $\omega_1 = 0.2\pi$, $\omega_2 = 0.5\pi$

Step 1:- Determine analog filter edge frequencies:

$$\Omega_c = \Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = 2\tan 0.1\Omega = 0.6498$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2\tan 0.25\pi = 2$$

& $\frac{\Omega_2}{\Omega_1} = 3.0779$.

Step 2:- Determine Order 'N'

$$N \ge \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{\delta_2^2} - 1\right]^{0.5}\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)} = \frac{\cosh^{-1}\left\{1\left[\frac{1}{(0.1)^2} - 1\right]^{0.5}\right\}}{\cosh^{-1}(3.0799)} = \frac{1.669}{\simeq 2}$$

as $\varepsilon = \left[\frac{1}{\delta_2^2} - 1\right]^{0.5} = \left[\frac{1}{0.707^2} - 1\right]^{0.5} \doteq 1$

Step 3: Determine $H(s)$, $N=2$

$$H(s) = \prod_{K=1}^{N/2} \frac{B_K\Omega_c^2}{s^2 + b_K\Omega_c s + C_K\Omega_c^2} = \frac{B_1\Omega_c^2}{s^2 + b_1\Omega_c s + C_1\Omega_c^2}$$

$$b_K = 2y_N \sin[(2k-1)\pi/2N]$$

$$C_K = y_N^2 + \cos^2\frac{(2k-1)\pi}{2N}$$

$$C_0 = y_N.$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5} + \frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5} + \frac{1}{\varepsilon}\right]^{-\frac{1}{N}}\right\}$$

$$y_N = \frac{1}{2}\left[(2.414)^{\frac{1}{2}} - (2.414)^{-\frac{1}{2}}\right] = \boxed{0.455}$$

$$b_1 = 2y_N \sin[(2k-1)\pi/2N] = \boxed{0.6435}$$

$$C_1 = y_N^2 + \cos^2\frac{(2k-1)\pi}{2N} = \boxed{0.707}$$

For N is even(2).

$$\prod_{k=1}^{N/2} \frac{B_k}{C_k} = \frac{A}{(1+\varepsilon^2)^{1/2}} = 0.707$$

$$\Rightarrow \frac{B_1}{C_1} = 0.707 \Rightarrow \boxed{B_1 = 0.5}$$

System function $H(S) = \dfrac{0.5\,(0.6498)^2}{S^2 + (0.6498)(0.6498)S + (0.707)(0.6498)^2}$

$$\boxed{H(S) = \frac{0.2111}{S^2 + 0.4181S + 0.2985}}$$

Step 4:- Determine H(z) using bilinear transformation.

$$H(z) = H(S)\Big|_{S = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{0.2111}{\left(2\cdot\frac{(z-1)}{(z+1)}\right)^2 + 0.4181\left(2\frac{(z-1)}{(z+1)}\right) + 0.2985}$$

$$H(z) = \frac{0.2111(z+1)^2}{5.1347z^2 - 7.4032 - 3.4623}$$

$$\boxed{H(z) = \frac{0.041(1+z^{-1})^2}{1 - 1.4418z^{-1} + 0.6743z^{-2}}}$$

Elliptical Filter :- (Cauer Filter)

- Filter has equiripple Passband & SB.
- Minimum transaction Bandwidth
- Magnitude squared response:

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 U_N(\Omega/\Omega_c)}$$

$U_N(x)$ = Jacobin elliptical f^n of order N.

ε = constaint related to Passband ripple

Mag^t Response of odd ordered elliptical filter :-