Assignment-2

Q.1 Explain how DFT can be used as linear transformation tool in DSP?

Ans-1 $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$; $0 \le k \le N-1$

$W_N \rightarrow$ Twiddle factor or phase factor.

$W_N = e^{-j2\pi/N}$; $W_N^K \rightarrow$ periodic 'N'.

put $n = 0, 1, 2, \ldots, N-1$

$x(k) = x(0) + x(1)\, W_N^{k(1)} + x(2)\, W_N^{k \cdot 2} + \ldots + x(N-1)\, W_N^{k \cdot (N-1)}$

$\boxed{K=0}$

$X(0) = x(0) + x(1).1 + x(2).1 + \ldots + x(N-1).1$

for $k=1$

$X(1) = x(0) + x(1)\, W_N^1 + x(2)\, W_N^2 + \ldots + x(N-1)\, W_N^{(N-1)}$

$\vdots$

$K = N-1$

$X(N-1) = x(0) + x(1)\, W_N^{(N-1).1} + x(2).\, W_N^{(N-1)2} + \ldots + x(N-1)\, W_N^{(N-1)(N-1)}$

We can write in matrix form

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N\times N} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ \vdots \\ x(N-1) \end{bmatrix}_{N\times 1}$$

$$\boxed{X_N = W_N \cdot x_N} \longrightarrow (1) \quad \text{for DFT}$$

Pre multiply above eq.^{n} by W_N^{-1}

$$W_N^{-1} X_N = \cancel{W_N^{-1}} \cdot \cancel{W_N}\, x_N$$

$$\boxed{W_N^{-1} X_N = x_N} \longrightarrow (2)$$

IDFT of the eq.^{n} (2) is given by

$$x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k)\, W_N^{-kn}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} X(k) \left(W_N^{kn}\right)^* \quad [\because W_N^* \rightarrow \text{complex conjugate } W_N]$$

$$\boxed{x_N = \frac{1}{N} X_N W_N^*} \longrightarrow (3) \quad \text{for IDFT}$$

Compare eq.^{n} ② & ③

$$\boxed{W_N^{-1} = \frac{1}{N} W_N^{*}}$$ Ans.

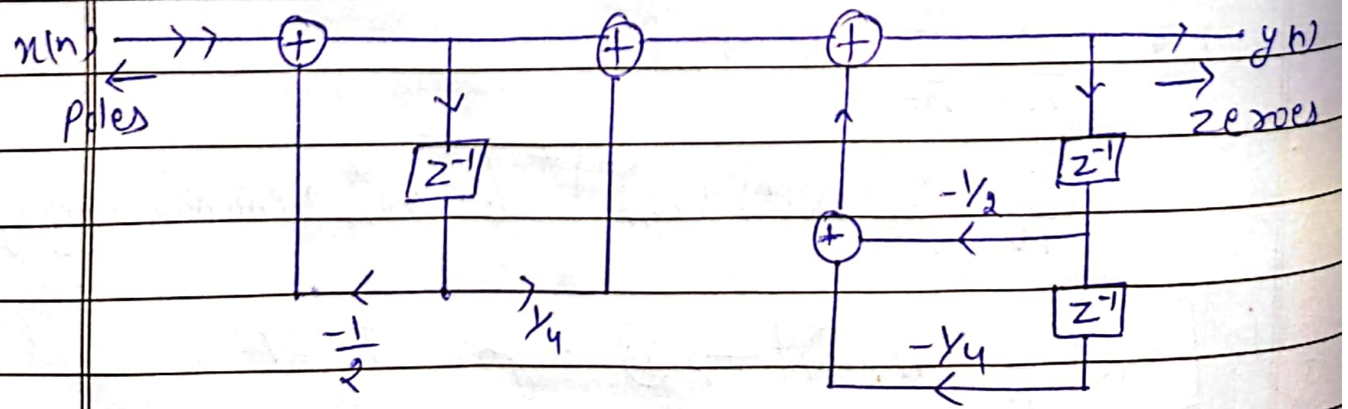
Q. 2 Determine Cascade & Parallel Realization of following system function?

Ans-

$$H(z) = \frac{1+\frac{1}{4}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)}$$

Cascade Realization :- $H(z) = H_1(z)\, H_2(z)$

$$H_1(z) = \underbrace{\frac{1+\frac{1}{4}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)}}_{\text{Direct-II}} \quad \& \quad H_2(z) = \underbrace{\frac{1}{1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}}_{\text{Direct-II}}$$

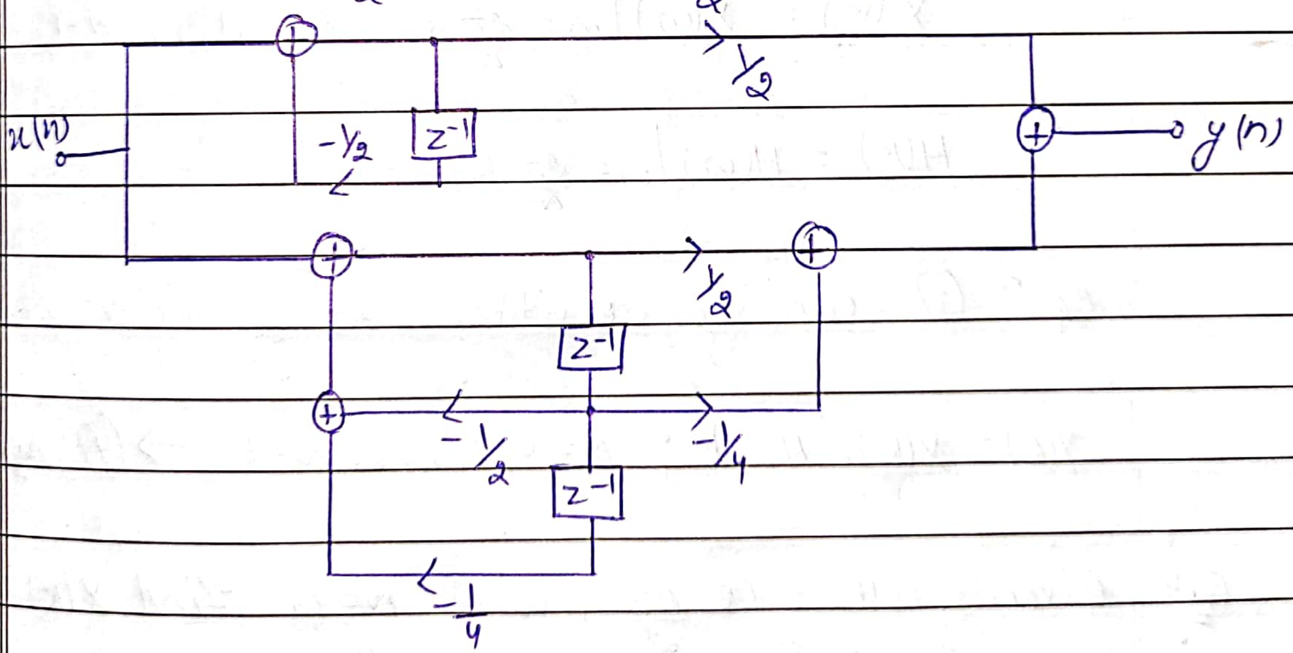


→ Parallel Realization :- partial fraction expansion of $H(z)$.

$$H(z) = \frac{1+\frac{1}{4}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)} = \frac{A_1}{\left(1+\frac{1}{2}z^{-1}\right)} + \frac{A_2 z^{-1}+A_3}{\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)}$$

$A_1 = \frac{1}{2}$, $A_2 = \frac{-1}{4}$ & $A_3 = \frac{1}{2}$

$$H(z) = \frac{1/2}{1+\frac{1}{2}z^{-1}} + \frac{\frac{1}{2}-\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$$



Q.3 Explain application of DFT in linear filtering & spectrum Alogrithm?

Ans:- Linear filtering using DFT :-

$$y(n) = \sum_{n=-\infty}^{\infty} h(k)\, x(n-k) \longrightarrow (1)$$

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F.T

$$Y(\omega) = F\left\{ \sum_{K=\infty}^{\infty} h(k)\, x(n-K) \right\} \longrightarrow (2)$$

Convolution property of F.T

$$F\{x_1(n) * x_2(n)\} = X_1(\omega) \cdot X_2(\omega)$$

eq.^{n} (2) can be written as

$$Y(\omega) = H(\omega) \cdot X(\omega) \quad \text{——} (3)$$

We know that

$$Y(K) = Y(\omega)\Big|_{\omega = \frac{2\pi K}{N}}$$

$$X(K) = X(\omega)\Big|_{\omega = \frac{2\pi}{N}K} \quad K = 0, 1, \dots N-1$$

$$H(K) = H(\omega)\Big|_{\omega = \frac{2\pi}{N}K}$$

eq.^{n} (3) can be written as

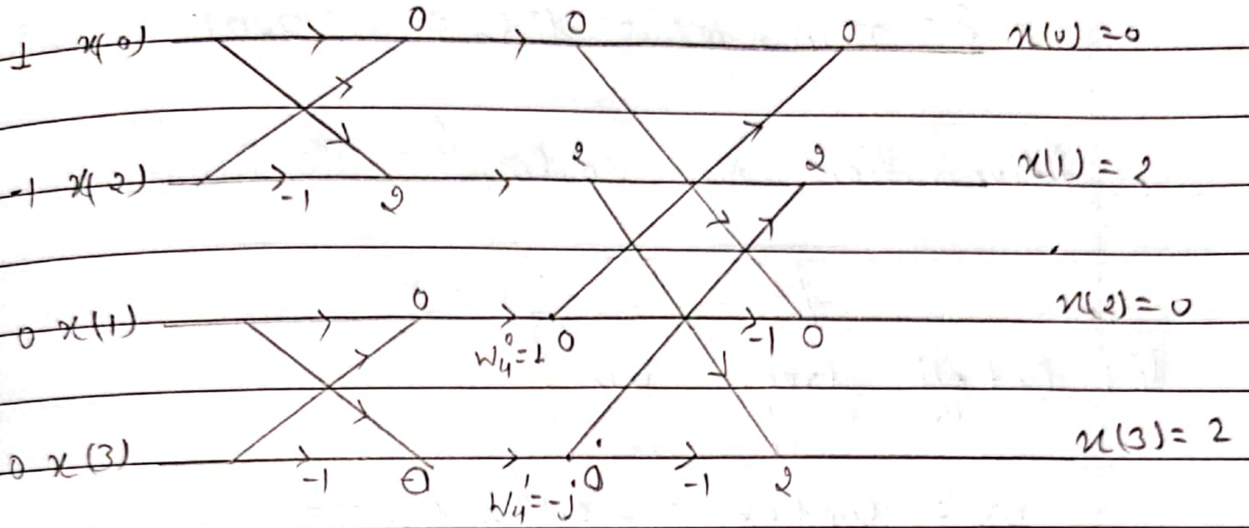
$$Y(K) = X(K) \cdot H(K) \; ; \; K = 0, 1, \dots N-1 \rightarrow (4) \quad \text{Ans}$$

Q.4 Given $x(n) = \cos\frac{n\pi}{2}$, with $N=4$. Find $X(K)$ using DIT-FFT Algorithm?

Ans-4 given $N = 4$ and $x(n) = \{1, 0, -1, 0\}$

$$W_N^K = e^{-j\left(\frac{2\pi}{N}\right)K}$$

$$W_4^0 = 1 \quad \& \quad W_4^1 = e^{-j\pi/2} = -j$$



$X(K) = \{0, 2, 0, 2\}$ Ans.

Q.5 Design an FIR linear phase filter using Kaiser window to meet the following specification:-

$$0.99 \le |H(e^{j\omega})| \le 1.01, \text{ for } 0 \le |\omega| \le 0.19\pi$$
$$|H(e^{j\omega})| \le 0.01, \text{ for } 0.21\pi \le |\omega| \le \pi$$

Sol:-

$$1 - 0.01 \le |H(e^{j\omega})| \le 1 + 0.01 \; ; \; 0 \le |\omega| \le 0.19\pi$$
$$|H(e^{j\omega})| \le 0.01 \; ; \; 0.21\pi \le |\omega| \le \pi$$

$\delta_1 = 0.01$ & $\delta_2 = 0.01$ $\quad W_p = 0.19\pi \quad W_s = 0.21\pi$

$$\therefore \Delta\omega = \omega_s - \omega_p = 0.21\pi - 0.19\pi$$
$$= 0.02\pi$$

$\delta = \min$ of δ_1 & $\delta_2 \Rightarrow \delta = 0.01$

Attenuation $A = -20 \log_{10} \delta = -20 \log_{10} 0.01$

$$\boxed{A = 40}$$

(i) Cut off freq^n ω_c :-

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.19\pi + 0.21\pi}{2}$$

$$\omega_c = 0.2\pi$$

(ii) To obtain β & M :-

$$\beta = \begin{cases} 0.1102\,(A - 8.7) \; ; & A > 50 \\ 0.5842\,(A-21)^{0.4} + 0.07886\,(A-21) \; ; & 21 \le A \le 50 \\ 0 \; ; & A < 21 \end{cases}$$

$\therefore \beta = 0.5842\,(40-21)^{0.4} + 0.07886\,(40-21)$

$\beta = 3.395$

$\therefore M = \dfrac{A - 8}{2.285\,\Delta\omega} = \dfrac{40 - 8}{2.285\,(0.02\pi)} = 223$

(iii) eq.^{n} for Kasier window :-

$$\alpha = \frac{M}{2} = \frac{223}{2} = 111.5$$

$$w_K(n) = \begin{cases} \dfrac{I_0\left\{\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{1/2}\right\}}{I_0(\beta)} & ; \; 0 \le n \le M \\ 0 & ; \; \text{otherwise} \end{cases}$$

So,

$$w(n) = \begin{cases} \dfrac{I_0\left\{3.395\left[1-\left(\frac{n-111.5}{111.5}\right)^{1/2}\right]\right\}}{I_0 \; (3.395)} & ; \; 0 \le n \le 223 \\ 0 & ; \; \text{otherwise} \end{cases}$$

(iv) Obtain $h_d(n)$;

Ideal freq. Response

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & ; \; -\omega_c \le \omega \le \omega_c \\ 0 & ; \; \text{otherwise} \end{cases}$$

$$h(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left[n-\frac{M-1}{2}\right]\right]}{\pi\left(n-\frac{M-1}{2}\right)} & ; \; n \ne \frac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & ; \; n = \frac{M-1}{2} \end{cases}$$

$M = 223$; $\frac{M}{2} = 111.5$

$\therefore h_d(n) = \frac{\sin\left[0.2\pi\left(n - \frac{223}{2}\right)\right]}{\pi\left(n - \frac{223}{2}\right)}$

$h_d(n) = \frac{\sin\left[0.2\pi(n-111.5)\right]}{\pi(n-111.5)}, \quad 0 \le n \le 223$

(v) obtain $h(n)$:

$h(n) = h_d(n) \cdot \omega(n)$

$$\therefore h(n) = \begin{cases} \frac{\sin\left[0.2\pi(n-111.5)\right]}{\pi(n-111.5)} \cdot \frac{I_0\left[3.395\left[1-\left(\frac{n-111.5}{111.5}\right)^2\right]^{1/2}\right]}{I_0(3.395)} & ; 0 \le n \le 223 \\ 0 & ; \text{otherwise} \end{cases}$$

Q.6 The frequency response of low pass filter is

$$H(e^{j\omega}) = \begin{cases} e^{-j2\omega} & ; -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & ; -\frac{\pi}{4} \le \omega \le \pi \end{cases}$$

window function is defined as $\omega(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

Sol:- Given

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & -\pi/4 \le \omega \le \pi/4 \\ 0, & \pi/4 \le |\omega| \le \pi \end{cases}$$

$$\therefore h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j2\omega} \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\pi/4} e^{j\omega(n-2)}\, d\omega$$

$$= \frac{1}{2\pi(n-2)} \left[\frac{e^{j(n-2)\pi/4} - e^{-j(n-2)\pi/4}}{2j} \right]$$

$$= \frac{1}{\pi(n-2)} \sin \frac{\pi}{4}(n-2), \quad n \neq 2$$

For $n = 2$, using L' Hospital Rule

$$h_d(2) = \frac{1}{4}$$

The other filter coefficients are given by

$h_d(0) = \frac{1}{2\pi} = h_d(4)$ and $h_d(1) = \frac{1}{\sqrt{2}\pi} = h_d(3)$

The filter coefficients of the filter would be then

$$h(n) = h_d(n) \cdot \omega(n)$$

$\therefore$ $h(0) = \frac{1}{2\pi} = h(4)$, $h(1) = \frac{1}{\sqrt{2}\pi} = h(3)$ & $h(2) = \frac{1}{4}$

The frequency response $H(e^{j\omega})$ is given by

$$H(e^{j\omega}) = \sum_{n=0}^{4} h(n)\, e^{-j\omega n}$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega}$$

$$= e^{-j2\omega}\left[h(0)\,e^{j2\omega} + h(1)e^{j\omega} + h(2) + h(3)e^{-j\omega} + h(4)e^{-j2\omega}\right]$$

$$= e^{-j2\omega}\left[h(2) + h(0)\left[e^{j2\omega} + e^{-j2\omega}\right] + h(1)\left[e^{j\omega} + e^{-j\omega}\right]\right]$$

$$= e^{-j2\omega}\left\{\frac{1}{4} + \frac{1}{2\pi}\left[e^{j2\omega} + e^{-j2\omega}\right] + \frac{1}{\sqrt{2}\pi}\left[e^{j\omega} + e^{-j\omega}\right]\right\}$$

Ans.

Q.7 Discuss Rounding off & Truncation errors in sign magnitude representation.

Ans- Rounding off & truncation introduced an error whose magnitude depends on the number of bits truncated or rounded-off. The sign magnitude & the two's complement representation of fixed point binary numbers are :-

Number	Sign Magnitude	Two's Complement
7	0111	0111
6	0110	0110
5	0101	0101
4	0100	0100
3	0011	0011
2	0010	0010
1	0001	0001
0	0000	0000
-0	1000	0000
-1	1001	1111
-2	1010	1110
-3	1011	1101
-4	1100	1100
-5	1101	1011
-6	1110	1010
-7	1111	1001

(i) Truncation error for sign magnitude :-

When the input number x is positive, truncation results in reducing the magnitude of the number. Thus, the truncation error is negative and the range is given by

$$-(2^{-B} - 2^{-L}) \leq \varepsilon_T \leq 0$$

Thus, the truncation error is positive and its range is

$$0 \leq \varepsilon_T \leq (2^{-B} - 2^{-L})$$

The overall range of the truncation error for the sign magnitude representation :-

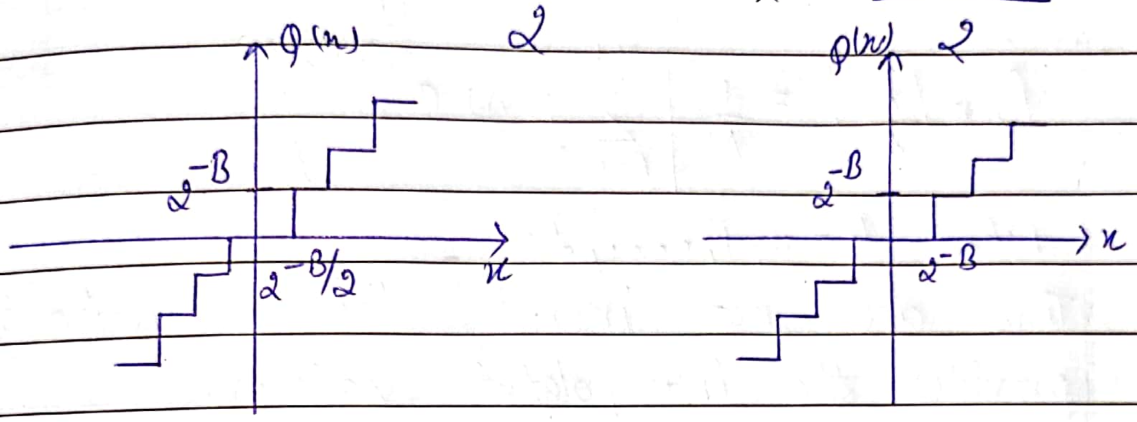
$$-2^{-B} \leq \varepsilon_T \leq 2^{-B}$$

(ii) Round-off error for a sign magnitude :-

The rounding of a binary number involves only the magnitude of the number and its independent of the type of fixed-point binary representation. The error due to rounding may be either positive or negative and the peak value is $\left(\frac{2^{-B} - 2^{-L}}{2}\right)$.

The round-off error is symmetric about zero and its range is

$$-\frac{(2^{-B}-2^{-L})}{2} < \varepsilon_R \leq \frac{(2^{-B}-2^{-L})}{2}$$



(a) Rounding (b) Truncation in sign Magnitude

Ans-3 **Spectrum Alogrithm of DFT :**

The periodogram is given by

$$P_{xx}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi f n}\right|^2$$

Let $f = K/N$, where $K = 0, 1, 2, \ldots, N-1$

$$P_{xx}\left(\frac{K}{N}\right) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi k/N}\right|^2$$

where $K = 0, 1, \ldots, N-1$

For frequency-domain, the length of the sequence $x(n)$ can be increased by zero padding. Let the new length be L,

$$P_{xx}\left(\frac{k}{L}\right) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi k/L}\right|^2$$

where $k = 0, 1, \ldots, L-1$

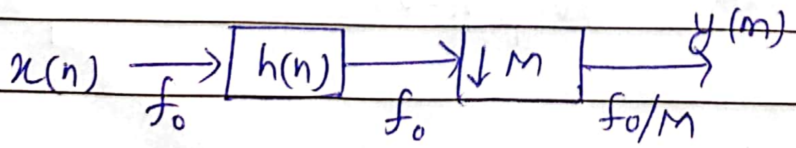
This does not increase the resolution but provides the interpolated values.

Q.8 Discuss briefly Multirate Signal Processing by Decimation & interpolation?

Ans- Consider a system for decimating a signal by an integer factor M. Let the input signal sampling frequency be f_0, then the decimated signal frequency will be f_0/M. The decimation factor can be factorised as :-

$$M = \prod_{i=1}^{I} M_i$$

The resultant network is :-



$$x(n) \xrightarrow{f_0} \boxed{h(n)} \xrightarrow{f_0} \boxed{\downarrow M_1} \xrightarrow{f_0/M_1} \boxed{\downarrow M_2} \xrightarrow{f_0/M_1M_2} \cdots \longrightarrow \boxed{\downarrow M_I} \xrightarrow{f_0/M} y(m)$$

$$x(n) \xrightarrow{f_0} \boxed{h_1(n_1)} \longrightarrow \boxed{\downarrow M_1} \xrightarrow{f_0/M_1} \boxed{h_2(n_2)} \longrightarrow \boxed{\downarrow M_2} \xrightarrow{f_0/M_2} \cdots \longrightarrow \boxed{h_1(n_I)} \longrightarrow \boxed{\downarrow M_I} \xrightarrow{f_0/M} y(m)$$

$$M = M_1 M_2 \ldots M_I = \prod_{i=1}^{I} M_i$$

Fig :- Multistage Decimator

Similarly, the multirate interpolator is shown below. The 1 to L interpolator, has its interpolation factor represented by ...

$$L = \prod_{i=1}^{I} L_i$$

$$x(n) \xrightarrow{f_0} \boxed{\uparrow L} \xrightarrow{Lf_0} \boxed{h(m)} \xrightarrow{Lf_0} y(m)$$

$$x(n) \xrightarrow{f_0} \boxed{\uparrow L_1} \xrightarrow{L_1/f_0} \boxed{\downarrow L_2} \xrightarrow{L_1L_2f_0} \cdots \longrightarrow \boxed{\uparrow L_I} \xrightarrow{Lf_0} \boxed{h(m)} \xrightarrow{Lf_0} y(m)$$

$$x(n) \xrightarrow{f_0} \boxed{\uparrow L_1} \longrightarrow \boxed{H_1(m_1)} \xrightarrow{L_1/f_0} \boxed{\uparrow L_2} \longrightarrow \boxed{H_2(m_2)} \xrightarrow{L_1L_2f_0} \cdots \longrightarrow \boxed{\uparrow L_I} \longrightarrow \boxed{H_I(m_I)} \xrightarrow{Lf_0} y(m)$$

$$L = L_1 L_2 \ldots L_I = \prod_{i=1}^{I} L_i$$

Fig :- Multistage Interpolation