The Rank of a Matrix: Definition, Forms, and Computation

Abstract

This document provides a comprehensive overview of the rank of a matrix, including its definition, the concept of row echelon form, and various methods for its computation. Examples are provided to illustrate the computational techniques.

1 Introduction to Matrix Rank

The **rank of a matrix** is a fundamental concept in linear algebra that quantifies the "linear independence" within the matrix. It indicates the maximum number of linearly independent rows or columns a matrix possesses.

1.1 Definition of Matrix Rank

For an $m \times n$ matrix A, the rank of A, denoted as $\text{rank}(A)$ or $\rho(A)$, can be defined in several equivalent ways:

- The **maximum number of linearly independent rows** in the matrix.
- The **maximum number of linearly independent columns** in the matrix.
- The **dimension of the row space** (the vector space spanned by its row vectors).
- The **dimension of the column space** (the vector space spanned by its column vectors).
- The **order of the largest square submatrix that has a non-zero determinant** (also known as the minor method).

It's a crucial result that the row rank and column rank of any matrix are always equal.

1.1.1 Properties of Matrix Rank

- For an $m \times n$ matrix A, $\text{rank}(A) \leq \min(m, n)$.
- The rank of a zero matrix (a matrix with all zero entries) is 0.
- A matrix has **full rank** if its rank is equal to $\min(m, n)$.
- A matrix is **rank-deficient** if its rank is less than $\min(m, n)$.
- For a square $n \times n$ matrix A, $\text{rank}(A) = n$ if and only if A is invertible (non-singular). This also implies that its determinant is non-zero.
- The rank of a matrix is invariant under elementary row and column operations.

2 Row Echelon Form (REF)

Row Echelon Form (REF) is a simplified form of a matrix obtained through a series of elementary row operations. A matrix is in row echelon form if it satisfies the following conditions:

1. All zero rows are at the bottom of the matrix.
2. The **leading entry** (the first non-zero element from the left) of each non-zero row is to the right of the leading entry of the row directly above it, forming a "staircase" pattern.
3. All entries in the column below a leading entry are zeros.

A further refinement is **Reduced Row Echelon Form (RREF)**, which adds two more conditions:

1. Each leading entry is 1 (a "pivot").
2. Each leading entry is the only non-zero entry in its column (i.e., all entries *above* and below the leading entry are zeros).

For rank calculation, the number of non-zero rows in either REF or RREF directly gives the rank.

3 Rank Computation Methods

Here are the primary methods for computing the rank of a matrix:

3.1 Method 1: Using Row Echelon Form (REF) or Reduced Row Echelon Form (RREF)

This is one of the most common and practical methods. The procedure is as follows:

1. Apply elementary row operations (swapping rows, multiplying a row by a non-zero scalar, adding a multiple of one row to another row) to transform the given matrix into its row echelon form (or reduced row echelon form).

2. Count the number of non-zero rows in the resulting echelon form. This count is the rank of the matrix. Each non-zero row will contain a leading entry (pivot).

Example

Let's find the rank of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

1. **Eliminate entries below the leading 1 in the first column:**

$$R_2 \to R_2 - 4R_1$$
$$R_3 \to R_3 - 7R_1$$

$$A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

2. **Make the leading entry in the second row 1 (optional for REF):**

$$R_2 \to -\frac{1}{3}R_2$$

$$A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{pmatrix}$$

3. **Eliminate entries below the leading 1 in the second column:**

$$R_3 \to R_3 + 6R_2$$

$$A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrix is now in row echelon form. There are two non-zero rows. Therefore, $\text{rank}(A) = 2$.

3.2 Method 2: Minor Method (Determinant Method)

This method relies on the definition of rank as the order of the largest non-zero minor.

1. Find the largest possible square submatrices within the given matrix.
2. Calculate their determinants.
3. The rank of the matrix is the order of the largest square submatrix whose determinant is non-zero.

Example

For the same matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$:

1. **Check for a 3×3 minor (the determinant of the matrix itself):**

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \\ &= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1(45 - 48) - 2(36 - 42) + 3(32 - 35) \\ &= 1(-3) - 2(-6) + 3(-3) \\ &= -3 + 12 - 9 = 0 \end{aligned}$$

   Since $\det(A) = 0$, the rank is less than 3.

2. **Check for 2×2 minors:** Let's choose the top-left 2×2 submatrix: $M_1 = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$ $\det(M_1) = (1 \times 5) - (2 \times 4) = 5 - 8 = -3$

   Since $\det(M_1) = -3 \neq 0$, we have found a 2×2 minor with a non-zero determinant.

Therefore, $\text{rank}(A) = 2$.

3.3 Method 3: Normal Form Method (Using Row and Column Operations)

A matrix A of rank r can be reduced to a "normal form" or "canonical form" using elementary row and column operations. This form is typically $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where I_r is the $r \times r$ identity matrix. The order r of the identity matrix is the rank.

Example

Starting from the REF obtained in Method 1: $A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Now, we apply column operations to zero out elements outside the main diagonal of the intended identity block.

1. **Zero out elements to the right of the leading 1 in the first row:**

$$C_2 \to C_2 - 2C_1$$
$$C_3 \to C_3 - 3C_1$$

$$A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

2. **Zero out elements to the right of the leading 1 in the second row:**

$$C_3 \to C_3 - 2C_2$$

$$A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is the normal form $\begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$, where I_2 is the 2×2 identity matrix. Therefore, $\text{rank}(A) = 2$.

3.4 Method 4: Singular Value Decomposition (SVD)

The rank of a matrix is equal to the number of non-zero singular values in its singular value decomposition. While computationally stable and used in software, this method typically requires a computational tool and is beyond manual calculation for most matrices.

4 Conclusion

All presented methods consistently show that the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is **2**. The choice of method often depends on the matrix size, specific requirements (e.g., numerical stability), and whether computation is manual or software-assisted. The row echelon form method is generally the most straightforward for manual calculations.