MATHS - 1

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M-1 Calculus - I

Rolle's Mean Value theorems

Rolle's theorem

If a function $f(x)$ is

(i) $f(x)$ is continuous on the closed interval $[a, b]$

(ii) $f(x)$ is differentiable on the open interval (a, b)

why not use closed interval

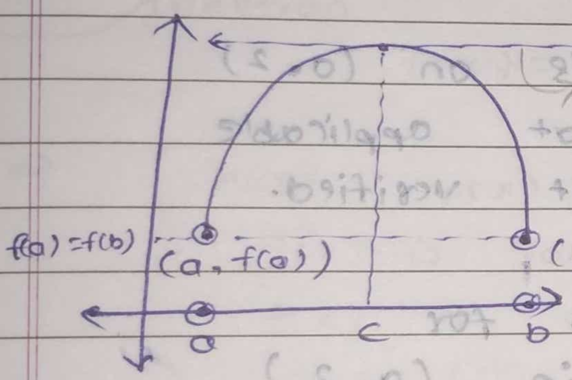
because ... doesn't exist

(iii) $\boxed{f(a) = f(b)}$

then there exists at least one point c in (a, b) (i.e $a < c < b$) such that $\boxed{f'(c) = 0}$

Such that the tangent at this point is parallel to x-axis

$\frac{dy}{dx} = 0$, slope $= 0$



why not take slope in interval / corner

because slope of tangent at corner $\boxed{\text{slope} = \infty}$ or is parallel to y-axis

Same as

① verify RMVT for $f(x) = 1 - 3(x-1)^{2/3}$ $0 \le x \le 2$

=> $f(x) = 1 - 3(x-1)^{2/3}$

$f'(x) = 0 - \cancel{3} \times \frac{2}{\cancel{3}} (x-1)^{-1/3}$

$\boxed{f'(x) = \frac{-2}{(x-1)^{1/3}}}$ .... { rational algebraic function }

∴ which doesn't exist at $x=1$ — because at point $x=1$, are function is not defined

which lies inside in interval $(0, 2)$ — $f'(x) = \infty$ / not def

$f(x)$ is not differentiable on $(0, 2)$

Hence RMVT is not applicable

RMVT is not verified.

② verify Rolle's theorem for

$f(x) = x(x-2)e^{3x/4}$ in $(0, 2)$

=> $f(x)$ is polynomial function

① $f(x)$ is continuous in $[0, 2]$

② $f(x)$ is differentiable in $(0, 2)$

$f(0) = 0$ $\qquad f(2) = 2(2-2)e^{3/2}$

$= 0$

③ $f(0) = f(2)$

∃ at least one point c b/w 0 and 2

such that $f'(c) = 0$

$\frac{3}{4}(c^2 - 2c)e^{3c/4} + (2c-2)e^{3c/4} = 0$

$\left[\frac{3}{4}(c^2 - 2c) + (2c-2)\right]e^{3x/4} = 0$

$\frac{3c^2}{4} - \frac{3}{2}c + 2c - 2 = 0$

$3c^2 + c - 2 = 0$

simply $f(x)$

$f(x) = (x^2 - 2x)e^{3x/4}$

$f'(x) = (x^2 - 2x)e^{3x/4} \cdot \left(\frac{3}{4}\right) + e^{3x/4}(2x-2)$

$f'(x) = \frac{3}{4}(x^2 - 2x)e^{3x/4} + (2x-2)e^{3x/4}$

$$3c^2 + 2c - 8 = 0$$

$$3c^2 + 6c - 4c - 8 = 0$$

$\cancel{3c(c+8)} -$

$$3c(c+2) - 4(c+2) = 0$$

$$(3c-4) = 0 \qquad \boxed{c = -2}$$

$$\boxed{c = \frac{4}{3}}$$

But $c = -2$ does not lies in $(0,2)$ thus $c = 4/3 \in (0,2)$ Hence RMVT is verified

③ verify Rolle's theorem for the following function

$$f(x) = x^{2m-1}(a-x)^{2n} \text{ in } [0,a]$$

⇒ Here $f(x)$ is polynomial

$f(x)$ is continuous on $[a,b]$

$f(x)$ is differentiable on (a,b)

$$f(0) = (0)^{2m-1}(a-0)^{2n} = 0$$

$$f(a) = (a)^{2m-1}(a-a)^{2n} = 0$$

$$f(0) = f(a)$$

~~∃ at least one point~~

$$\therefore f(x) = x^{2m-1}(a-x)^{2n}$$

$$f'(x) = x^{2m-1}\,2n(a-x)^{2n-1}(-1) + (a-x)^{2n}(2m-1)x^{(2m-2)}$$

$$= -2n\,x^{2m-1}(a-x)^{2n}(a-x)^{-1} + (a-x)^{2n}(2m-1)x^{2m-1}x^{-1}$$

$$f'(x) = x^{2m-1}(a-x)^{2n}\left[-2n(a-x)^{-1} + \frac{2m-1}{x}\right]$$

$\exists\, c \in (0,a)$ such that $f'(c) = 0$

$$c^{2m-1}(a-c)^{2n}\left[-2n(a-c)^{-1} + \frac{2m-1}{c}\right] = 0$$

$$\frac{-2n}{(a-c)} + \frac{2m-1}{c} = 0$$

$$\frac{-2nc + (2m-1)(a-c)}{(a-c)c} = 0$$

$$-2nc + 2ma - 2mc - a + c = 0$$

$$(1 - 2n - 2m)c + 2ma = 0$$

? $\boxed{c = \frac{-2ma}{(1-2n-2m)}}$

A) verify Rolle's theorem in $[0, \pi]$

$$f(x) = \frac{\sin x}{e^x}$$

=> $f(x) = \frac{\sin x}{e^x}$

$$f'(x) = \frac{e^x \cos x - \sin x \, e^x}{(e^x)^2}$$

$$= \frac{e^x (\cos x - \sin x)}{(e^x)^2}$$

$$\boxed{f'(x) = \frac{\cos x - \sin x}{e^x}}$$ ...

Note: if when e^x is zero $x = -ve$

when exist every where

Here

$f(x)$ is continuous on $(0, \pi)$

$f(x)$ is differentiable on $[0, \pi]$

$f(0) = \frac{\sin 0}{e^0} = 0$ $\qquad f(\pi) = \frac{\sin \pi}{e^\pi} = 0$

$f(0) = f(\pi)$

$\exists\ c \in (0, \pi)$ such that

$f'(c) = 0$

$\frac{\cos c - \sin c}{e^c} = 0$

$\cos c - \sin c = 0$

$\cos c = \sin c$ $\qquad \frac{\sin c}{\cos c} = 1$ $\qquad \tan c = 1$

$c = \tan^{-1}(1)$

$c = \tan^{-1}(\tan \pi/4)$

$\boxed{c = \pi/4}$

c is belong b/w $(0, \pi)$

$\therefore$ RMVT is verified.

Note :- when function differentiable is always continuous

(5) verify RMVT for $f(x) = \begin{cases} x^2+1 & ; 0 \le x \le 1 \\ 3-x & ; 1 \le x \le 2 \end{cases}$

=> To check differentiability of $f(x)$ on $(0,2)$ enough to chek differentiability at $x=1$

L.H.D

Left Hand derivative at $x=1$ is given by

$f(x) = x^2+1 \quad f'(x) = 2x$

$$f'(x)\Big|_{x=1} = 2x\Big|_{x=1} = 2(1) = \underline{\underline{2}}$$

R.H.D at $x=1$ is given by

$f(x) = 3-x \qquad f(x) = -1$

$$f'(x)\Big|_{x=1} = -1\Big|_{x=1} = \underline{\underline{-1}}$$

L.H.D $\neq$ R.H.D

$\therefore f(x)$ is not differentiable at $x=1$

and Hence not differentiable on $(0,2)$

RMVT is not applicable.

H.W

(6) $x^3 - 12x$ in $[0, 2\sqrt{3}]$ Ans $c=2$

=> $f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12$ ... {polynomial function}

$f(x)$ is continuous in $[0, 2\sqrt{3}]$

$f(x)$ is discontinuous in $(0, 2\sqrt{3})$

$f(0) = 0 - 12(0) = 0 \qquad f(2\sqrt{3}) = 8 \times 3\sqrt{3} - 12 \times 2\sqrt{3}$

$= 24\sqrt{3} - 24\sqrt{3} = 0$

$f(0) = f(2\sqrt{3})$

$\exists\, c \in (0, 2\sqrt{3})$ such that

$f'(c) = 0 \qquad 3c^2 - 12 = 0 \qquad c^2 = \frac{12}{3} = 4 \qquad c = \sqrt{4}$

$\boxed{c = \pm 2}$

$c = +2 \qquad c \neq -2$

$\boxed{c=2}$ is belong in $(0, 2\sqrt{3})$.

H.W

(7) $\sin x$ in $[0, 2\pi]$ Ans: $c = \pi/2$ and $3\pi/2$

$\Rightarrow$ $f(x) = \sin x$

$f'(x) = \cos x$ ... { trigonometry function

(1) $f(x)$ is continuous on $[0, 2\pi]$

(2) $f(x)$ is differentiable on $(0, 2\pi)$

$f(0) = \cancel{\cos}\ \sin x = 0$ $\quad f(2\pi) = \sin(2\pi) = 0$

(3) $f(0) = f(2\pi)$

$\exists$ $c \in$ belong in $(0, 2\pi)$

$f'(c) = 0$

$\cos c = 0$ $\quad \cos c = \cos \pi/2$ $\quad \cos c = \cos(\text{[illegible]}$

$\boxed{c = \pi/2}$ $\quad \boxed{c = \frac{3\pi}{2}}$

most ask for (2m)
(5m) } also (RMVT) Imp

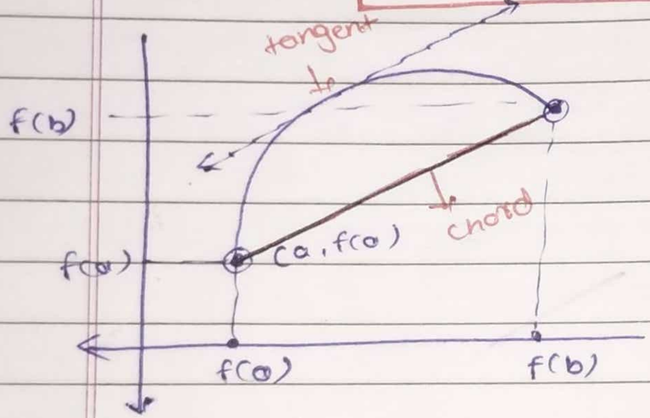
Lagrange's mean value theorem :

If f(x) is defined on [a, b] such that

(1) f(x) is continuous on [a, b]
(2) f(x) is differentiable on (a, b)

then ∃ at least one point c b/w a and b

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$



Note :- slope of tangent is parallel to chord

(1) verify lagrange's mean value theorem

$f(x) = x^3 - 2x^2 - 3x - 6$ in $[\underset{a}{-1}, \underset{b}{4}]$

=> $f(x)$ is a polynomial function

$f(x)$ is continuous on $[-1, 4]$

$f(x)$ is differentiable on $(-1, 4)$

All the condition of LMVT are satified

$\exists\ c \in (-1, 4)$ such that

$f(b) = 64 - 32 - 12 - 6$
$= 32 - 18 = 14$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f'(c) = 3x^2 - 4x - 3$

$f(a) = -1 - 2 + 3 - 6$
$= -6$

$$= \frac{14 - (-6)}{4 - (-1)} = \frac{20}{5} = 4$$

$\cancel{c^3 - 2c^2 - 3c - 6 = 4}$

$3c^2 - 4c - 3 = 4$

$3c^2 - 4c - 7 = 0$

$3c^2 + 3c - 7c - 7 = 0$

$3c(c+1) - 7(c+1) = 0$

-21

$-7 \quad +3$

$\boxed{c = 7/3}$ $\boxed{c = -1}$

$c \neq -1$ is not belong $(-1, 4)$

$c = 7/3$ is belong in $(-1, 4)$

Hence LMVT is verified

(2) verify LMVT $f(x) = e^{-x}$ in $[-1, 1]$

=> $f(x)$ is exponential function

$f(x)$ is continuous on $[-1, 1]$

$f(x)$ is differentiable on $(-1, 1)$

All condition of LMVT is satisfied

$\exists\, c \in (-1, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f'(c) = e^{-x}(-1)$ $\qquad f(1) = e^{-x} = e^{-1} = \frac{1}{e}$

$= -e^{-x}$ $\qquad f(-1) = e^{-(-1)} = e^{1} = e$

$$-e^{-c} = \frac{\frac{1}{e} - e}{2}$$

$$-e^{-c} = \frac{1 - e^2}{2e}$$

$$\frac{-1}{e^c} = \frac{(1 - e^2)}{2e}$$

$$e^c = \frac{2e}{(e^2 - 1)}$$

Taking log on both side

$$\log(e^c) = \log\left(\frac{2e}{e^2 - 1}\right)$$

$$c \log e = \log(2e) - \log(e^2 - 1)$$

$$\boxed{c = \log\left(\frac{2e}{e^2 - 1}\right)} \in (-1, 1)$$

Hence, LMVT is verified.

(3) If $f(x) = \sin^{-1}x$ $[0,1]$ then c of LMVT is

=> $f(x)$ = not defined nhi dega because put $[0,1]$ in function give value

Here, $f(x)$ is inverse trigonometric function.

$f(x)$ is continuous on $[0,1]$

$f(x)$ is differentiable on $(0,1)$

All condition of LMVT are satisfied

$\exists c \in (0,1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\pi/2}{1}$$

$$f'(c) = \frac{1}{\sqrt{1-c^2}}$$

$f(0) = 0$

$f(1) = \pi/2$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\pi}{2}$$

squaring both side

$$\frac{1}{1-c^2} = \frac{\pi^2}{4}$$

$$\boxed{c^2 = 1 - \frac{4}{\pi^2}}$$

$$c = \sqrt{0.5947}$$

$$\boxed{c = 0.771}$$

Note :- algebric mean $\Rightarrow \boxed{C = \frac{a+b}{2}}$

most ask for (2M)/(5M) geometric mean $\Rightarrow \boxed{C = \sqrt{ab}}$

H.M $\Rightarrow \boxed{C = \frac{\frac{1}{a}+\frac{1}{b}}{2}} \rightarrow$ (assume Both a and b +ve)

Cauchy's mean value theorem :

↳ It is the ratio of two function is applicable for LMVT

If $f(x)$ and $g(x)$ are two function defined on $[a, b]$

Such that,

(1) Both $f(x)$ and $g(x)$ are continuous on $[a, b]$

(2) Both $f(x)$ and $g(x)$ are differentiable on (a, b)

(3) $\boxed{g'(x) \neq 0}$ for all x in (a, b)

then there exist atleast one point c between a and b such that,

$$\frac{f'(c)}{g'(c)} = \frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}}$$

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

e.g (3) third

Imp

In Cauchy's theorem

↓

If question have

$f'(x) \neq 0$

$g'(x) \neq 0$

to take $f(x)$, $g(x)$ as per situation to prove that

① If $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ then prove that 'c' of CMVT is geometric mean b/w a and b

=> Here $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ on $[a, b]$

$f'(x) = \frac{1}{2\sqrt{x}}$ which exist on (a, b)

and

$g'(x) = -\frac{1}{2} x^{-3/2} = -\frac{1}{2x^{3/2}} = \frac{-1}{2x\sqrt{x}}$ which exist on (a, b)

$f(x)$ and $g(x)$ both are differentiable on (a, b) and Hence both are continuous on $[a, b]$

$g'(x) = -\frac{1}{2x\sqrt{x}} \neq 0$ for all x in (a, b)

∴ By CMVT ∃ c b/w a and b such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2c\sqrt{c}}} = \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}$$

$$-c = \frac{(\sqrt{b} - \sqrt{a})}{\frac{(\sqrt{a} - \sqrt{b})}{\sqrt{ab}}} \qquad c = \frac{\cancel{(\sqrt{a} - \sqrt{b})} \times \sqrt{ab}}{\cancel{(\sqrt{a} - \sqrt{b})}}$$

$$\boxed{c = \sqrt{ab}}$$

$$H.M = \frac{1/a + 1/b}{2} = \frac{b+a}{2ab}$$

$$A.M = \frac{a+b}{2}$$

(2) If $f(x) = e^x$ and $g(x) = e^{-x}$, prove that c of Cauchy's mean value therom is the Arithmetic mean b/w a and b, $a>0$, $b>0$

$\Rightarrow$ Here $f(x) = e^x$ and $g(x) = e^{-x}$ on $[a,b]$

$f'(x) = e^x$ which is exist on (a,b)

and

$g'(x) = -e^{-x}$ which is exist on (a,b)

$f(x)$ and $g(x)$ Both are continuous on $[a,b]$

and Here Both are differential on $[a,b)$

$g'(x) = -e^{-x} \neq 0$ for all value of x in (a,b)

$\therefore$ By ~~...~~ CMVT $\exists$ c b/w a and b such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}} \qquad \frac{-e^c}{1/e^c} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$-e^{2c} = \frac{(e^b - e^a)}{\frac{(e^a - e^b)}{e^a e^b}}$$

$$-e^{2c} = \frac{(e^b - e^a)}{-(e^b - e^a)} \times (e^a e^b)$$

$$\cancel{-} e^{2c} = \cancel{-} (e^a e^b)$$

Taking log on both side

$$2c \log e = \log(e^a + e^b)$$

$$= \log e^a + \log e^b$$

$$2c = a + b$$

$$\boxed{c = \frac{a+b}{2}}$$

why take $f(x) = \cos x$
$g(x) = e^x$

because e^x is not equal to zero
$g'(x) \neq 0$

(3) using appropriate CMVT prove that
$(\sin b - \sin a) e^c = \cos c (e^b - e^a)$ for $a \leq c < b$

Hence deduce that $\boxed{\dfrac{\sin x}{(e^x - 1)} = \dfrac{\cos c}{e^c}}$

=>

$$\frac{\sin b - \sin a}{(e^b - e^a)} = \frac{\cos c}{e^c}$$

on comparing with $\dfrac{f(b) - f(a)}{g(b) - g(a)}$

we get

$f(x) = \sin x$ and $g(x) = e^x$ on $[a, b]$
for $0 < a < b$

Note: let assume a, b +ve

now
$f'(x) = \cos x$ }
$g'(x) = e^x$ } which is exist on (a, b)

Here, $f(x)$ and $g(x)$ are continuous on $[a, b]$
$f(x)$ and $g(x)$ are differentiable (a, b)

$g'(x) = e^x \neq 0$ for all x in (a, b)

$\exists$ c b/w a and b such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

$$\boxed{\frac{\cos c}{e^c} = \frac{\sin b - \sin a}{e^b - e^a}}$$

To prove deduction
let assume put $a = 0$ $b = x$

$$\frac{\cos c}{e^c} = \frac{\sin x - \sin 0}{e^x - e^0}$$

$$\boxed{\frac{\cos c}{e^c} = \frac{\sin x}{e^x - 1}}$$

(4) verify Cauchy's mean value theorem for

(i) $f(x) = x^2$ and $g(x) = x^4$ in the interval $[a,b]$

(ii) $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $[1, e]$

(i)

$\Rightarrow$ $f(x) = x^2$ $g(x) = x^4$

f(x) and g(x) Both are polynomial

f(x), g(x) are continuous on $[a,b]$

f(x) and g(x) are differentiable on (a,b)

$\exists c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{2c}{4c^3} = \frac{b^2 - a^2}{b^4 - a^4}$$

$$\frac{1}{2c^2} = \frac{b^2 - a^2}{(b^2)^2 - (a^2)^2}$$

$$\frac{1}{2c^2} = \frac{(b^2 - a^2)}{\cancel{(b^2 - a^2)}\,(b^2 + a^2)}$$

$$2c^2 = b^2 + a^2$$

$$c = \sqrt{\frac{a^2 + b^2}{2}} \in (a, b)$$

Cauchy's mean value verified

(ii) $f(x) = \log x$ $g(x) = \frac{1}{x}$ in $[1, e]$

$f(x)$ and $g(x)$

$f'(x) = \frac{1}{x}$
$g'(x) = -\frac{1}{x^2}$ } which exist in $[1, e]$

Here Both $f(x)$ and $g(x)$ is continuous on $[a, b]$
$f(x)$ and $g(x)$ is differentiable on (a, b)

$\exists c \in (1, e)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{1/c}{-1/c^2} = \frac{\log e - \log 1}{\frac{1}{e} - 1}$$

$$-c = \frac{\log e}{\frac{1}{e} - 1}$$

$$-c = \frac{1}{\frac{1}{e} - 1} \qquad -c = \left(\frac{e}{1-e}\right)$$

$$\boxed{c = \left(\frac{e}{e-1}\right)} \in (1, e)$$

(MVT) is verified

(5) verify Cauchy's mean value (CMV) theorem (i)
$x^3 - 3x^2 + 2x$, $x^3 - 5x^2 + 6x$ in $(0, 1/2)$

$\Rightarrow$ let $f(x) = x^3 - 3x^2 + 2x$ $\quad g(x) = x^3 - 5x^2 + 6x$

Both $f(x)$ and $g(x)$ are polynomial

$g(x)$ and $f(x)$ ~~is~~ are continuous on $[a, b]$
$f(x)$ and $g(x)$ are differentiable on (a, b)

$\exists\, c \in (0, 1/2)$ such that

$f'(x) = 3x^2 - 6x + 2$
$g'(x) = 3x^2 - 10x + 6$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{3x^2 - 6x + 2}{3x^2 - 10x + 6} = \frac{(\frac{1}{2})^3 - 3(\frac{1}{2})^2 + 1 - 0}{(\frac{1}{2})^3 - 5\frac{1}{4} + 3 - 0}$$

$$= \frac{\frac{1}{8} - \frac{3}{4} + 1}{\frac{1}{8} - \frac{5}{4} + 3}$$

$\frac{1-6}{8} + 1 \quad -\frac{5}{8}+1 \quad \frac{3}{8}$

$-\frac{9}{8} + 3$

$$= \frac{3/8}{15/8}$$

$$\frac{3x^2 - 6x + 2}{3x^2 - 10x + 6} = \frac{1}{5}$$

$15x^2 - 30x + 10 = 3x^2 - 10x + 6$
$12x^2 - 20x + 4 = 0$
$6x^2 - 10x + 2 = 0$
$3x^2 - 5x + 1 = 0$

?

te :- Finding derivative of is [drawbacle]

5 # Taylor's Series

small to High like 0, 1, 2, 3, 4 ....

If f(x) can be expanded into convergent series of positive ascending integral powers of (x-a)

then

$$\boxed{f(x) = f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \cdots \infty}$$

ex :-

(1)

ex :-

(1) $f(x) = 2x^3 + 3x^2 - 8x + 7$ in power of $(x-2)$

$\Rightarrow$ let $f(x) = 2x^3 + 3x^2 - 8x + 7$

and $a = 2$

$f(2) = 2(8) + 3(4) - 8\times2 + 7$
$= 16 + 12 - 9$
$= 16 + 3 = \underline{19}$

$f'(x) = 6x^2 + 6x - 8$

$f'(2) = 6\times4 + 6\times2 - 8$
$= 24 + 12 - 8 = \underline{28}$

$f''(x) = 12x + 6$

$f''(2) = 12\times2 + 6$
$= 24 + 6 = \underline{30}$

$f'''(x) = 12$

$f'''(2) = 12$

$$f(x) = f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \cdots$$

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2 f''(2)}{2!} + \frac{(x-2)^3 f'''(2)}{3!} + \cdots$$

$$f(x) = 19 + (x-2)28 + \frac{\cancel{30}^{15}(x-2)^3}{\cancel{2}\times1} + \frac{(x-2)^3\,\cancel{12}^{2}}{3\times\cancel{2}}$$

$$f(x) = 19 + 28(x-2) + 15(x-2)^3 + 2(x-2)^3$$

$$\boxed{2x^3 + 3x^2 - 8x + 7 = 19 + 28(x-2) + 15(x-2)^3 + 2(x-2)^3}$$

(2) Expand $\sin x$ in power of $(x - \pi/2)$

$\Rightarrow$ let $f(x) = \sin x$ and $a = \frac{\pi}{2}$ $\quad f(\frac{\pi}{2}) = \sin \pi/2 = 1$

$f'(x) = \cos x$ and $f'(\pi/2) = \cos \pi/2 = 0$

$f''(x) = -\sin x$ $\quad f''(\pi/2) = -\sin(\pi/2) = -1$

$f'''(x) = -\cos x$ $\quad f'''(\pi/2) = -\cos \pi/2 = 0$

$f''''(x) = \sin x$ $\quad f''''(\pi/2) = \sin \pi/2 = 1$

$f'''''(x) = \cos x$ $\quad f'''''(\pi/2) = \cos \pi/2 = 0$

by Taylor's series we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^2 f'''(a)}{3!} + \dots$$

$$f(x) = f(\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2}) f'(\pi/2)}{1!} + \frac{(x-\pi/2)^2 f''(\pi/2)}{2!} + \frac{(x-\frac{\pi}{2})^3 f'''(\pi/2)}{3!}$$
$$+ \frac{(x-\pi/2)^4 f''''(\pi/2)}{4!} + \dots$$

$$f(x) = 1 + \frac{(x-\pi/2) \times 0}{1!} + \frac{(x-\pi/2)^2 (-1)}{2} + \frac{(x-\frac{\pi}{2})^3 \times 0}{3 \times 2}$$
$$+ \frac{(x-\pi/2)^4\, 1}{4 \times 3 \times 2} + \dots$$

$$f(x) = 1 - \frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{24} + \dots$$

$$\boxed{\sin x = 1 - \left(\frac{x-\pi/2}{2}\right)^2 + \left(\frac{x-\pi/2}{24}\right)^4 + \dots}$$

(3) using Taylor's series, expand $\tan^{-1}(x)$ in positive power of $(x-1)$ upto first four Non-zero terms.

$\Rightarrow$ let $f(x) = \tan^{-1}(x)$ and $a=1$

$f(1) = \tan^{-1}(1)$
$= \tan^{-1}(\tan \pi/4)$
$= \pi/4$

$f'(x) = \frac{1}{1+x^2}$ $\quad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1 \cdot 2x}{(1+x^2)^2}$ $\quad f''(1) = -\frac{\cancel{2}}{(\cancel{2})^2} = \frac{-1}{2}$

$f'''(x) = -2(1+x^2)^{-2} \cdot x$

$f'''(x) = -2\left[(1+x^2)^{-2} + x(-2)(1+x^2)^{-3} \times 2x\right]$

$f'''(1) = -2\left[\frac{1}{(1+x^2)^2} + -(4x\frac{1}{(1+x^2)^3}\right]$

$= -2\left[\frac{1}{4} - \frac{4\times 1}{8}\right] = -2\left[\frac{1}{4} - \frac{2}{4}\right]$

$= -2\left(\frac{-1}{4}\right) = \frac{1}{2}$

By Taylor's series

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \cdots$

$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2 f''(1)}{2!} + \frac{(x-1)^3 f'''(1)}{3!} + \cdots$

$f(x) = \frac{\pi}{4} + (x-1)\left(+\frac{1}{2}\right) + \frac{(x-1)^2}{2}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{3\times 2}\left(\frac{1}{2}\right) + \cdots$

$f(x) = \pi/4 + \left(+\frac{(x-1)}{2}\right) + \left(-\frac{(x-1)^2}{4}\right) + \frac{(x-1)^3}{12} + \cdots$

$$\boxed{\tan^{-1}(x) = \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} + \cdots}$$

comparing $(x-a)$ $a=-1$

(4) Expand $f(x) = (x^4 - x^3 + x^2 - x + 1)$ in power $(x+1)$ By using Taylor's Series.

$\Rightarrow$ let $f(x) = (x^4 - x^3 + x^2 - x + 1)$ and $a=-1$

$f(-1) = 1+1+1+1+1 = 5$

$f'(x) = 4x^3 - 3x^2 + 2x - 1$ $\qquad f'(-1) = -4 - 3 - 2 - 1 = -10$

$f''(x) = 12x^2 - 6x + 2$ $\qquad f''(-1) = 12 + 6 + 2 = 20$

$f'''(x) = 24x - 6$ $\qquad f'''(-1) = -24 - 6 = -30$

$f''''(x) = 24$ $\qquad f''''(-1) = 24$

and further all derivative are zero

By Taylor's Series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \cdots$$

$$f(x) = f(-1) + (x+1) f'(-1) + \frac{(x+1)^2 f''(-1)}{2} + \frac{(x+1)^3 f'''(-1)}{3\times 2} + \frac{(x+1)^4 f''''(-1)}{24} + \cdots$$

$$f(x) = 5 + (x+1)(-10) + \frac{(x+1)^2 \, 20}{2} + \frac{(x+1)^3(-30)}{6} + \frac{(x+1)^4 (24)}{24}$$

$$\boxed{x^4 - x^3 + x^2 - x + 1 = 5 - 10(x+1) + 10(x+1)^2 - 5(x+1)^3 + (x+1)^4.}$$

The Taylor's series expansion which are done around the $\boxed{x=0}$ $\rightarrow (x=a)$

Maclaurin's Series :

At ~~x~~=0

put in Taylor's we get maclaurin series

a=0

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0)}$$

Note : Maclaurin's series is a special case of Taylor's series when the value of $\boxed{a=0}$

Standard Expansions :

~~$\sin x = x$~~

(1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ — integer power continuous and +ve value

(2) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$ — integer power continuous and Alternate $\left(\begin{smallmatrix}+ve\\-ve\end{smallmatrix}\right)$

$\frac{x^1}{1!}$

(3) $\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ — odd power and Alternate $\left(\begin{smallmatrix}+ve\\-ve\end{smallmatrix}\right)$

! wala nai

(4) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ — even power and Alternate $\left(\begin{smallmatrix}+ve\\-ve\end{smallmatrix}\right)$

$\frac{1 \times x^0}{0!}$

$\tan^{-1} x$ → same But factorial not in $\tan^{-1}x$, total Same as $\sin x$

Sine hyperbolic of x

(5) $\sinh x = \left(\frac{e^x - e^{-x}}{2}\right)$

$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$ — odd power and (+ve) value

(6) $\cosh x = \left(\frac{e^x + e^{-x}}{2}\right)$

$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$ — even power and (+ve) value

$\frac{e^x - e^{-x}}{e^x + e^{-x}}$

(7) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$

(8) $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \cdots$ — alternate $\left(\begin{smallmatrix}+ve\\-ve\end{smallmatrix}\right)$

(9) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ — integer ~~odd~~ power always (+ve)

(10) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$ — integer ~~odd~~ power always (−ve)

(11) ~~log~~ $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$

(12) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

(13) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$$\boxed{x^3+y^3+z^3 = (x+y+z)\,(x^2+y^2+z^2-xy-yz-xz)}$$

(14) $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{x^5}{5} + \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)\frac{x^7}{7} + \cdots$

(15) $\cos^{-1}x = \frac{\pi}{2} - \left[x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \cdots\right]$

(16) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

ex :-

(1) prove that $\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \dots$

$\Rightarrow$ let $f(x) = \log(1+e^x)$ $\quad f(0) = \log(1+e^0) = \log 2$

$f'(x) = \frac{1 \times e^x}{(1+e^x)}$ $\quad f'(0) = \frac{1}{2}$

$\cancel{f''(x) = \frac{e^x}{(1+e}}$

$f''(x) = \frac{(1+e^x)e^x - (e^x)^2}{(1+e^x)^2}$

$= \frac{e^x + \cancel{e^{2x}} - \cancel{e^{2x}}}{(1+e^x)^2}$

$= \frac{e^x}{(1+e^x)^2}$ $\quad f''(0) = \frac{1}{(2)^2} = \frac{1}{4}$

$f'''(x) = \frac{(1+e^x)^2 e^x - e^x 2(1+e^x)e^x}{(1+e^x)^4}$

$= \frac{e^x(1+e^x) - 2e^{2x}}{(1+e^x)^3} = \frac{e^x - e^{2x}}{(1+e^x)^3}$ $\quad f'''(0) = \frac{1(2)-2}{(2)^3} = \frac{1 \times 0}{8} = 0$

$f''''(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x})3(1+e^x)^2 e^x}{(1+e^x)^6}$

$= \frac{\cancel{(1+e^x)^2}[(1+e^x)(e^x - 2e^{2x}) - (e^x - e^{2x})3(1+e^x)^2 e^x}{(1+e^x)^6}$

$= \frac{(1+e^x)(e^x - 2e^{2x}) - (e^x - e^{2x})3(1+e^x)^2 e^x}{(1+e^x)^4}$

$f^{iv}(0) = \frac{2(1-2) - (1-1)3(2)^2 1}{(2)^4}$

$= \frac{-\cancel{2}}{8 \times \cancel{2}} = -\frac{1}{8}$

using By maclaurin's series

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \dots$

$f(x) = \log 2 + \frac{1}{2}x + \frac{x^2}{2} \times \frac{1}{4} + \frac{x^3}{3 \times 2} \times 0 - \frac{x^4}{4 \times 3 \times 2} \times \frac{1}{8}$

$\boxed{\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192}}$

(2) Show that $e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \dots$

$\Rightarrow$ let $f(x) = e^x \cos x$ $\qquad f(0) = 1$

$f'(x) = e^x(-\sin x) + \cos x \, e^x$ $\qquad f'(0) = 1$
$= e^x(\cos x - \sin x)$

$f''(x) = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x)$ $\qquad f''(0) = 0$
$= \cancel{e^x \cos x} - e^x \sin x - e^x \sin x - \cancel{e^x \cos x}$
$= -2e^x \sin x$

$f'''(x) = -2(e^x \sin x + \cos x \, e^x)$ $\qquad f'''(0) = -2$
$= -2e^x(\sin x + \cos x)$

$f^{iv}(x) = -2[e^x(\sin x + \cos x) + e^x(\cos x - \sin x)]$
$= -2[e^x \cancel{\sin x} + e^x \cos x + e^x \cos x - e^x \cancel{\sin x}]$
$= -4 e^x \cos x$ $\qquad f^{iv}(0) = -4$

By using Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{iv}(0)}{4!} + \dots$$

$$= 1 + 1 \times x + \frac{x^2 \times 0}{2!} - \frac{2x^3}{3!} - \frac{4x^4}{4!} - \dots$$

$$\boxed{e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \dots}$$

Hence proved

(3) expand $\sqrt{1+\sin x}$

$\Rightarrow$ $\sqrt{1+\sin x} = \sqrt{\sin^2 x/_2 + \cos^2 x/_2 + 2\sin x/_2 \cos x/_2}$

$= \sqrt{(\sin x/_2 + \cos x/_2)^2} = \sin x/_2 + \cos x/_2$

$f(x) = \sin x/_2 + \cos x/_2$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \sin x/_2 = \frac{x}{2} - \frac{(x/_2)^3}{3!} + \frac{(x/_2)^5}{5!} \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \cos x/_2 = 1 - \frac{(x/_2)^2}{2!} + \frac{(x/_2)^4}{4!}$

using $\sin x$ and $\cos x$ series we get

$f(x) = \frac{x}{2} - \frac{x^3}{8 \times 6} + \frac{x^5}{32 \times 120} - \cdots + 1 - \frac{x^2}{8} + \frac{x^4}{16 \times 24}$

$$\boxed{\sqrt{1+\sin x} = 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + \cdots}$$

(4) expand 5^x upto 1^{st} three non-zero terms of the series

$\Rightarrow$ $f(x) = 5^x$ $\qquad f(0) = 5^0 = 1$

$f'(x) = 5^x \log 5$ $\qquad f'(0) = 5^0 \log 5 = \log 5$

$f''(x) = 5^x (\log 5)^2$ $\qquad f''(0) = 5^0 (\log 5)^2 = (\log 5)^2$

$f'''(x) = 5^x (\log 5)^3$ $\qquad f'''(0) = (\log 5)^3 =$

$f^{iv}(x) = 5^x (\log 5)^4$ $\qquad f^{iv}(0) = (\log 5)^4$

By maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)$$

$$+ \frac{x^4 f^{iv}(0)}{4!} + \cdots$$

$$\boxed{5^x = 1 + (\log 5) x + \frac{(\log 5)^2 x^2}{2!} + \frac{(\log 5)^3 x^3}{3!} + \cdots}$$

(5) expand $\log(1+x+x^2+x^3)$ upto x^8

$\Rightarrow f(x) = \log[(1+x) + x^2(1+x)]$

$= \log[(1+x)(1+x^2)]$

$= \log(1+x) + \log(1+x^2)$

series of $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8}$

$\log(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots \right.$ $\left. - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8}\right) + \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots\right)$

$-\frac{1}{4} - \frac{1}{3}$

$= \cancel{x + \frac{x^2}{2} + \frac{x^3}{3}}$

$\frac{7}{12}$

$$\boxed{\log(1+x+x^2+x^3) = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3}{4}x^4 + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3}{8}x^8 + \cdots}$$

$-\left(\frac{1}{4} + \frac{2}{4}\right)$

(4) Expansions of function By using Maclaurin series

$$f(x) = \log(1+\sin x)$$

$\Rightarrow$ $f(x) = \log(1+\sin x)$ $\quad \therefore f(0) = 0$

$f'(x) = \dfrac{\cos x}{(1+\sin x)}$ $\quad f'(0) = 1$

$f''(x) = \dfrac{(1+\sin x)(-\sin x) - \cos^2 x}{(1+\sin x)^2}$ $\quad f''(0) = -1$

$= \dfrac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$

$f''(x) = \dfrac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$

$= \dfrac{-\sin x - 1}{(1+\sin x)^2}$

$= \dfrac{-\cancel{(1+\sin x)}}{(1+\sin x)^2}$

$= -\dfrac{1}{(1+\sin x)}$

$f'''(x) = \dfrac{\cos x}{(1+\sin x)^2}$ $\quad f'''(0) = 1$

$f^{iv}(x) = \dfrac{-2}{(1+\sin x)^3}$ $\quad f^{iv}(0) = -2$

By maclaurin series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$$= 0 + x(1) + \frac{x^2(-1)}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$$

$$\boxed{\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots}$$

(5) $\log(\sec x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \cdots$

$\Rightarrow$ $f(x) = \log(\sec x)$ $\quad f(0) = 0$

$f'(x) = \frac{\sec x \cdot \tan x}{\sec x}$ $\quad f'(0) = 0$

$f''(x) = \sec^2 x = 1 + \tan^2 x$ $\quad f''(0) = 1$

$f'''(x) = 2\tan x \sec^2 x$ $\quad f'''(0) = 0$

$= 2\tan x(1 + \tan^2 x)$

$= 2\tan x + 2\tan^3 x$

$f^{iv}(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x$

$= 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x)$

$= 2 + 2\tan^2 x + 6\tan^2 x + 6\tan^4 x$

$= 2 + 8\tan^2 x + 6\tan^4 x$ $\quad f^{iv}(0) = 2$

$f^{v}(x) = 16\tan x \sec^2 x + 24\tan^3 x \sec^2 x$ $\quad f^{v}(0) = 0$

$= 16\tan x + 16\tan^3 x + 24\tan^3 x + 24\tan^5 x$

$f^{vi}(x) = 16\sec^2 x + 48\tan x \sec^2 x + 72\tan^2 x \sec^2 x + 144\tan^5 x \sec^2 x$, $f^{vi}(0) = 16$

By using Maclaurin Series

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0)$

$+ \frac{x^5}{5!} f^{v}(0) + \frac{x^6}{6!} f^{vi}(0) + \cdots$

$= 0 + 0 + \frac{x^2}{2} + \frac{x^3}{3!} \times 0 + \frac{x^4}{4!} \times 2 + \frac{x^5}{5!} \times 0$

$\frac{x^6}{6!} \times 16$

$\frac{16 \times 2}{6 \times 5 \times 4 \times 3 \times 2}$ 3

$$\boxed{\log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

(6) Expand in power of x, $e^{x\sin x}$

$\Rightarrow$ let $f(x) = e^{(x\sin x)}$

let $x\sin x = y$

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots$$

$$e^{x\sin x} = 1 + (x\sin x) + \frac{(x\sin x)^2}{2!} + \frac{(x\sin x)^3}{3!} + \cdots$$

$$e^{x\sin x} = 1 + x\sin x + \frac{x^2(\sin x)^2}{2!} + \frac{x^3(\sin x)^3}{3!} + \cdots$$

~~not write~~ It will be okay even if you don't write

$$= 1 + x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + \frac{x^2}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \frac{x^3}{3!}\left(x - \cancel{\frac{x^3}{3!}} \cdots\right)^3$$

Square

$$= 1 + x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \frac{x^2}{2!}\left(x^2 - \frac{2x^4}{3!} + \frac{x^6}{3!} + \cdots\right) + \frac{x^3}{3!}(x^3 + \cdots)$$

$$= 1 + x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \left(\frac{x^4}{2!} - \frac{2x^6}{2!\,3!} + \frac{x^8}{2!\,3!} + \cdots\right) + \left(\frac{x^9}{3!} + \cdots\right)$$

$$\cancel{= 1 + x + \frac{x^3}{2!} - \frac{x^4}{3!} - \frac{2x^5}{3!\,2!} + \frac{x^6}{5!} + \frac{x^7}{3!\,2!} + \frac{x^9}{3!}}$$

$$= 1 + x - \frac{x^4}{6} + \frac{x^4}{2} + \frac{x^6}{120} - \frac{x^6}{6} + \cdots$$

$$\boxed{e^{(x\sin x)} = 1 + x^2 + \frac{x^4}{3} + \frac{x^6}{120} - + \cdots}$$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

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(7) prove that $x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots$

$L.H.S = x \operatorname{cosec} x$

$= \frac{x}{\sin x}$

$= \frac{x}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)}$

$= \frac{x/x}{\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}{x}} = \frac{1}{\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right]}$

$= \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right]^{-1}$

$= \left[1 - \underbrace{\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \dots\right)}_{}\right]^{-1}$

$(1 - x)^{-1}$

comparing with expand of $(1-x)^{-1}$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 \dots$

$= 1 + \left\{\frac{x^2}{3!} - \frac{x^4}{5!} + \dots\right\} + \left\{\frac{x^2}{3!} - \frac{x^4}{5!} + \dots\right\}^2 + \left\{\frac{x^2}{3!} - \frac{x^4}{5!} - \dots\right\}^3 + \dots$

$= 1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^4}{(3!)^2} + \dots$

$= 1 + \frac{x^2}{3!} - \frac{x^4}{120} + \frac{x^4}{36}$

$= 1 + \frac{x^2}{3!} - \frac{36x^4 + 120x^4}{120 \times 36}$

36
×2

$= 1 + \frac{x^2}{3!} - \frac{\cancel{84}\, x^4}{\cancel{120 \times 36}}$ 42 21 7 / 60 360

$\boxed{x \operatorname{cosec} x = 1 + \frac{x^2}{6} - \frac{7x^4}{360} + \dots}$ Hence proved

(8) show that $e^{x}\cdot\cos x = 1 + x - \frac{2x^3}{3!} - \frac{4x^4}{4!} \dots$

$\Rightarrow$ $f(x) = e^{x}\cos x$ $\quad f(0) = 1$

$f'(x) = e^{x}(\sin x) + \cos x\, e^{x}$

$= e^{x}(\cos x - \sin x)$ $\quad f'(0) = 1$

$f''(x) = e^{x}(\cos x - \sin x) + e^{x}(-\sin x - \cos x)$ $\quad f''(0) = 0$

$= e^{x}[\cancel{\cos x} - \sin x - \sin x - \cancel{\cos x}]$

$= -2e^{x}\sin x$

$f'''(x) = -2[e^{x}\cos x + \sin x\, e^{x}]$ $\quad f'''(0) = -2$

$= -2[e^{x}(\cos x + \sin x)]$

$f''''(x) = -2[e^{x}(\cos x + \sin x) + e^{x}(-\sin x + \cos x)]$ $\quad f''''(0) = -4$

$= -2\{e^{x}[\cos x + \cancel{\sin x} - \cancel{\sin x} + \cos x]$

$= -4e^{x}\cos x$

By maclaurin series

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \dots$

$f(x) = 1 + x + \frac{x^2}{2!}\times 0 - \frac{2x^3}{3!} - \frac{4x^4}{4!} + \dots$

$$\boxed{e^{x}\cdot\cos x = 1 + x - \frac{2x^3}{3!} - \frac{4x^4}{4!} + \dots}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

(a) prove that $\sin(e^x - 1) = x + \frac{x^2}{2} - \frac{5x^4}{24} + \cdots$

=> let $f(x) = \sin(e^x - 1)$

$$\sin(e^x - 1) = \sin\left[\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - \cancel{1}\right]$$

$$= \sin\left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

$$= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\right) - \frac{1}{3!}\left(x + \frac{x^2}{2!} + \cdots\right)^3 + \cdots$$

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots - \frac{1}{3!}\left(x^3 + \frac{3x^4}{2!} + \frac{x^5}{4} + \frac{x^6}{8} + \cdots\right)$$

$$= x + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} \ldots - \cancel{\frac{x^3}{3!}} - \frac{3x^4}{12} + \cdots$$

$$= x + \frac{x^2}{2!} + \frac{x^4}{24} - \frac{x^4}{4} + \cdots$$

$$= x + \frac{x^2}{2!} - \frac{5x^4}{4 \times 24} + \cdots$$

$$\boxed{\sin(e^x - 1) = x + \frac{x^2}{2!} - \frac{5x^4}{24} + \cdots}$$

Hence proved.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

(10) prove that $(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} - \cdots$

=> let $y = (1+x)^x$

taking log on both sides

$\log y = x \log(1+x)$

$\log y = x\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right]$

$\log y = \left[x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots\right]$

let $z = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots$

$\log_e y = z$

$\therefore y = e^z$

$y = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$y = 1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} + \cdots\right) + \frac{1}{2}\left(x^2 - \frac{x^3}{2} + \cdots\right)^2 + \cdots$

$y = 1 + x^2 - \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^4}{2} + \cdots$

$$\boxed{y = 1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} + \cdots}$$

(11) Expand $(1+x)^{1/x}$ upto the term x^2

$y = (1+x)^{1/x}$

Taking log on both sides

$\log y = \frac{1}{x}\log(1+x)$

$\log y = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)$

$\log y = \left(1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right)$

~~let~~ $z = \left(1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right)$

$\log y = z$

$y = e^z$

$$y = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$(1+x)^{1/x} = 1 + \left(1 - \frac{x}{2} + x^2\right) + \frac{1}{2}\left(1 - \frac{x}{2}\right)^2 + \cdots$$

$$= 1 + 1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{2}\left(1 + \frac{x^2}{2} - x\right) + \cdots$$

$$\boxed{(1+x)^{1/x} = 3 - \frac{3x}{2} + \frac{x^2}{2} + \cdots}$$?

*** some problem are remaining

(11) prove that $(1+x)^{1/x} \doteq e - \frac{e}{2}x + \frac{11}{24}ex^2 + \cdots$

$\Rightarrow$ let $y = (1+x)^{1/x}$

Taking log on both side

$$\log y = \frac{1}{x}\log(1+x)$$

$$\log y = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

$$\log y = \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)$$

$$= 1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)$$

$$\log y = 1 + z \qquad \cdots z = \left(-x/2 + \frac{x^2}{3} \cdots\right)$$

$$y = e^{1+z}$$

$$= e\, e^z$$

$$= e\left(1 + z + \frac{z^2}{2!} + \cdots\right)$$

$$(1+x)^{1/x} = e\left[1 + \left(-x/2 + x^2/3 + \cdots\right) + \frac{1}{2}\left(-x/2 + \cdots\right)^2 + \cdots\right.$$

$$= e\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{x^2}{8} + \cdots\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{11x^2}{24} + \cdots\right]$$

$$\boxed{(1+x)^{1/x} = e - \frac{ex}{2} + \frac{11ex^2}{24} + \cdots}$$

(12) prove that $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$\Rightarrow$ $f(x) = \tan^{-1}x$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $\qquad f''(0) = 0$

$f'''(x) = -2\left[\frac{\quad}{(1+x^2)^2}\right]$

do not this lengthed problem

$f(x) = y = \tan^{-1}x$

differentie with r. x

$$\frac{dy}{dy} = \frac{1}{(1+x^2)}$$

$$(1+x)^{-1} = (1 - x + x^2 - x^3 + \cdots)$$

$$\frac{dy}{dy} = (1 - x^2 + x^4 - x^6 + \cdots)$$

Integerating we get

$$\int \frac{dy}{dx} = \int (1 - x^2 + x^4 - x^6 + \cdots)$$

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\boxed{\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots}$$

DATE

(13) prove that $\log(1+\tan x) = x - \frac{x^2}{2} + \frac{2x^3}{3} + \cdots$

(m-1)

$\Rightarrow$ Here series $\left\{ \log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \right\}$

$$\log(1+\tan x) = (\tan x) - \frac{(\tan^2 x)}{2}^2 + \left(\frac{\tan^3 x}{3}\right) + \cdots$$

Here series $\left\{ \tan x = x + \frac{x^3}{3} + \cdots \right\}$

$$\log(1+\tan x) = \left(x + \frac{x^3}{3} + \cdots\right) - \frac{1}{2}\left(x + \frac{x^3}{3}\right)^2 + \cdots$$

$$= \left(x + \frac{x^3}{3} + \cdots\right) - \frac{1}{2}\left(x^2 + \frac{2x^4}{3} + \frac{x^6}{9} \cdots\right) + \cdots$$

$$= x + \frac{x^3}{3} + \cdots - \frac{1}{2}(x^2 + \cdots)$$

$$= x - \frac{x^2}{2} + \frac{2x^3}{3} //$$

(m-2)

$f(x) = \log(1+\tan x)$ $\quad f(0) = 0$

$f'(x) = \frac{\sec^2 x}{(1+\tan x)}$ $\quad f'(0) = \frac{\sec^2(0)}{(1)} = 1$

$f''(x) = \frac{(1+\tan x)\, 2\sec^2 x \tan x - \sec^2 x \sec^2 x}{(1+\tan x)^2}$ $\quad f''(0) = \frac{(1+0)\, 2\times 0 - 1}{1} = -1$

$f'''(x) = \frac{(1+\tan x)^2 [\qquad - \qquad}{(1+\tan x)^4}$ $\quad f'''(0) = 4$

By using Macluren theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$= 0 + x + \frac{x^2}{2}(-1) + \frac{\cancel{4}^{2} x^3}{3 \times \cancel{2}} + \cdots$$

$$= x - \frac{x^2}{2} + \frac{2x^3}{3} + \cdots //$$

(Imp) For ESE and ISE

Indeterminate forms :

The form of limit which has not yet determin are called indeterminate forms.

There are seven types of indeterminate form

(1) $\frac{0}{0}$ (2) $\frac{\infty}{\infty}$ (3) $0\times\infty$ (4) $\infty-\infty$ (5) $\underbrace{(1)^{\infty}}_{\text{Taking log}}$

(6) $\underbrace{(\infty)^{0} \quad (7)\ 0^{0}}_{\text{Take log}}$

To evaluate indeterminate form we use L' Hospital Rule.

L' Hospital Rule :-

If $f(x)$ and $g(x)$ can be expanded by Taylor's Series abent $\boxed{x=a}$

such that

$\boxed{f(a)=0}$ and $\boxed{g(a)=0}$

then
$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

Note : L' Hospital rule is also applicable for $\frac{\infty}{\infty}$ or $\frac{0}{0}$ form problems :

(1) Evaluate $\lim\limits_{x \to 1} \dfrac{x^x - x}{x - 1 - \log x}$

$\Rightarrow \lim\limits_{x \to 1} \dfrac{x^x - x}{x - 1 - \log x}$ is of the form $\left(\frac{0}{0}\right)$

$\lim\limits_{x \to 1} \dfrac{x^x(1+\log x) - 1}{1 - 0 - \frac{1}{x}}$

$\lim\limits_{x \to 1} \dfrac{x^x(1+\log x) - 1}{1 - \frac{1}{x}}$ is of form $\left(\frac{0}{0}\right)$

$\lim\limits_{x \to 1} \dfrac{x^x\left(\frac{1}{x}\right) + (1+\log x)^2 x^x}{\frac{1}{x^2}}$

$= \dfrac{1+1}{1/1} = \underline{\underline{2}}$

(2) Evaluate $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sin(x\cos x)}{\cos(x\sin x)}$

$\Rightarrow \lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sin(x\cos x)}{\cos(x\sin x)}$ ... $\left(\frac{0}{0}\text{ form}\right)$

$\lim\limits_{x \to \pi/2} \dfrac{\cos(x\cos x)\,[x(-\sin x) + \cos x]}{-\sin(x\sin x)\,[x\cos x + \sin x]}$

$\lim\limits_{x \to \pi/2} = \dfrac{\cos(\pi/2\cos\pi/2)\,[\pi/2(-\sin\pi/2) + \cos\pi/2]}{-\sin(\pi/2\sin\pi/2)\,[\pi/2\cos\pi/2 + \sin\pi/2]}$

$= \dfrac{(1)\,[-\pi/2 + 0]}{(-1)\,[0+1]} = \dfrac{-\pi/2}{-1} = \underline{\underline{\frac{\pi}{2}}}$

(3) Evaluate $\lim_{x \to 1/2} \frac{\cos^2 \pi x}{e^{2x} - 2xe}$

let,

$\Rightarrow L = \lim_{x \to \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2xe}$ ---- $\left(\frac{0}{0} \text{ form}\right)$.

$= \lim_{x \to \frac{1}{2}} \frac{-2\cos \pi x \sin \pi x (\pi)}{2e^{2x} - 2e}$

$= \lim_{x \to 1/2} \frac{-\sin 2\pi x \, \pi}{2e^{2x} - 2e}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 1/2} \frac{-(\cos 2\pi x) 2\pi^2}{4e^{2x}}$

$L = \frac{-(-1) 2\pi^2}{4e} = \frac{\pi^2}{2e}$

(4) Evalute $\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{\tan x - x}$

let $L = \lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{\tan x - x}$ --- $\left(\frac{0}{0} \text{ form}\right)$

$L = \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{\sec^2 x - 1}$

$= \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{\cancel{1} + \tan^2 x - \cancel{1}}$

$e^0 = 1$

$= \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{\tan^2 x}$ --- $\left(\frac{0}{0} \text{ form}\right)$

$= 2\tan x(1 + \tan^2 x)$

$= 2\tan x + 2\tan^3 x$

$= \lim_{x \to 0} \frac{e^x - e^{-x}}{2\tan x \sec^2 x}$ --- $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{e^x + e^{-x}}{2\sec^2 x + 6\tan^2 x \sec^2 x}$

$= \frac{\cancel{2}}{\cancel{2}} = 1$

(5) Find the value of (a, b) if $\lim_{x\to 0} \frac{a\cos x - a + bx^2}{x^4} = \frac{1}{12}$

=> $\lim_{x\to 0} \frac{a\cos x - a + bx^2}{x^4} = \frac{1}{12}$ ($\frac{0}{0}$ form)

$\lim_{x\to 0} \frac{-a\sin x + 2xb}{4x^3} = \frac{1}{12}$

$\lim_{x\to 0} \frac{-a\cos x + 2b}{12x^2} = \frac{1}{12}$ Imp

put limit

$\frac{-a+2b}{0} = \frac{1}{12}$

$-a+2b = \frac{0}{12}$

$\boxed{-a+2b=0}$ ---(1)

$\lim_{x\to 0} \frac{+a\sin x}{24x} = \frac{1}{12}$

$\lim_{x\to 0} \frac{a\cos x}{24} = \frac{1}{12}$

$\frac{a\cos 0^\circ}{24} = \frac{1}{12}$ => $\frac{a}{\cancel{24}\,2} = \frac{1}{\cancel{12}}$ $\boxed{a = 2}$

value of a put in eq (1)

$-a+2b=0$

$-\cancel{2} = -\cancel{2}b$ $\boxed{b=1}$

(6) Show that $\lim_{x\to 0} \log_x \sin x = 1$

L.H.S = $\lim_{x\to 0} \log_x \sin x$

$= \lim_{x\to 0} \frac{\log \sin x}{\log x}$ ... $\frac{\infty}{\infty}$ (form)

$= \lim_{x\to 0} \left[\frac{\frac{\cos x}{\sin x}}{1/x}\right]$

$= \lim_{x\to 0} x\cot x$ ... ($0\times\infty$ form)

$= \lim_{x\to 0} \frac{x}{\frac{\sin x}{\cos x}} = \lim_{x\to 0} \frac{x}{\tan x}$ ... ($\frac{0}{0}$ form)

$= \lim_{x\to 0} \frac{1}{\sec^2 x}$

$= \frac{1}{\sec^2(0)} = \underline{\underline{1}}$

Hence proved.

(7) Evaluate $\lim_{x\to 0} \tan x \cdot \log x$

$\Rightarrow$ let $L = \lim_{x\to 0} \frac{\log x}{\cot x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec}^2 x}$

$= \lim_{x\to 0} \frac{-\sin^2 x}{x}$ --- $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x\to 0} \frac{-2\sin x \cos x}{1}$

$= \frac{0}{1} = 0$

(8) Evaluate $\lim_{x\to 0} \frac{\sin^{-1}x - x}{x^3}$

$\Rightarrow$ let $L = \lim_{x\to 0} \frac{\sin^{-1}x - x}{x^3}$ --- $\left(\frac{0}{0} \text{ form}\right)$

$(1-x^2)^{-1/2}$

$= \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}} - 1}{3x^2}$ --- $\left(\frac{0}{0} \text{ form}\right)$

$-\frac{1}{2}$

$= \lim_{x\to 0} \frac{-\frac{1}{2}(1-x^2)^{-3/2}(-2\cancel{x})}{6\cancel{x}\, 3}$ --- $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 0} \frac{+\frac{1}{2}(1-x^2)^{-3/2}}{3}$

$= \frac{+1/2 \times (1-0)^{-3/2}}{3} = \frac{1}{6}$

(a) Prove that $\lim_{x\to\infty} \left[\frac{1}{x}\right]^{1/x} = 1$

~~Let~~ $L = \lim_{x\to\infty} \left[\frac{1}{x}\right]^{1/x}$ $(0)^0$ form

Taking log both sides

$\log L = \log \lim_{x\to\infty} \left(\frac{1}{x}\right)^{1/x}$

$\log L = \lim_{x\to\infty} \log (1/x)^{1/x}$

$= \lim_{x\to\infty} \frac{1}{x} \log (1/x)$

$= \lim_{x\to\infty} \frac{\log(1/x)}{1/x}$ .....$\left(\frac{0}{0} \text{ form}\right)$ $= \lim_{x\to\infty} \frac{\cancel{x(-1/x^2)}}{-1/x^2}$

$= \lim_{x\to\infty} +x$

$= \lim_{x\to\infty} \frac{\log(1/x)}{x}$ .. $\frac{\infty}{\infty}$ form

$= \lim_{x\to\infty} \frac{\frac{-1/x^2}{1/x}}{1}$

$= \lim_{x\to\infty} (-1/x)$

$= (-1/\infty)$

$\log L = 0$

$L = e^0 = 1$ Hence proved

(10) Show that $\lim_{x \to 0} \frac{\log(\tan 2x)}{\tan x} = 1$

$\Rightarrow$ $L.H.S = \lim_{x \to 0} \frac{\log(\tan 2x)}{\log(\tan x)}$ ... $\frac{\infty}{\infty}$ form

$= \lim_{x \to 0} \frac{\frac{2\sec^2 2x}{\tan 2x}}{\frac{\sec^2 x}{\tan x}}$

$= \lim_{x \to 0} \frac{\frac{\frac{2}{\sin 2x \cos^{\cancel{2}} 2x}}{\cancel{\cos 2x}}}{\frac{\frac{1}{\sin x \cos \cancel{x}}}{\cancel{\cos x}}}$

$\cancel{= \lim_{x \to 0} \frac{2\sin x \cos x}{\sin 2x \cos 2x}}$ $= 2\lim_{x \to 0} \frac{2\sin x \cos x}{2\sin 2x \cos 2x}$

$\cancel{= \lim_{x \to 0} \frac{}{\cos 2x}}$ $= 2\lim_{x \to 0} \frac{\sin 2x}{\sin 4x}$ ... $\frac{0}{0}$ form

$= 2\lim_{x \to 0} \frac{\cos 2x \times \cancel{2}^{1}}{\cos 4x \times \cancel{4}_{2}}$

$= \cancel{\frac{2}{2}} \lim_{x \to 0} \frac{\cos 2x}{\cos 4x}$ $= \lim_{x \to 0} \frac{\cos 2x}{\cos 4x}$ $= \frac{\cos 2(0)}{\cos 4(0)} = 1$

Hence proved

(11) Evaluate $\lim_{x \to \frac{\pi}{2}} (1 - \sin x) \cdot \tan x$

$\Rightarrow L = \lim_{x \to \pi/2} (1 - \sin x) \cdot \tan x$ ..... $(0 \times \infty$ form$)$

$= \lim_{x \to \pi/2} \frac{(1 - \sin x)}{\cot x}$ ... $(\frac{0}{0}$ form$)$

$= \lim_{x \to \pi/2} \frac{\cancel{-}\cos x}{\cancel{-}\operatorname{cosec}^2 x}$

$\cancel{= \lim_{x \to \pi/2}}$ $= \lim_{x \to \pi/2} \frac{\cos x}{\operatorname{cosec}^2 x}$

$= \frac{\cos(\pi/2)}{\frac{1}{\sin^2(\pi/2)}} = \frac{0}{1} = 0$

Evaluate

(12) $\lim_{x \to \pi/2} (\cos x)^{\cos^2 x}$

let $L = \lim_{x \to \pi/2} (\cos x)^{\cos^2 x}$ (0^0 form)

Taking log on both sides

$$\log L = \log \lim_{x \to \pi/2} ((\cos x)^{\cos^2 x}$$

$$\log L = \lim_{x \to \pi/2} \log ((\cos x)^{\cos^2 x}$$

$$\log L = \lim_{x \to \pi/2} \cos^2 x \cdot \log(\cos x) \quad \text{--}(0 \times \infty \text{ form})$$

Note :- $0 \times \infty$ form convert into $\frac{0}{0}$ and $\frac{\infty}{\infty}$ form

$\log(\cos x)$ remain

change $\cos^2 x$ into $\sec^2 x$

$$\log L = \lim_{x \to \pi/2} \frac{\log(\cos x)}{\sec^2 x} \left(\because \frac{\infty}{\infty} \text{ form} \right)$$

$$= \lim_{x \to \pi/2} \frac{-\frac{\sin x}{\cos x}}{2\sec x \cdot \sec x \cdot \tan x}$$

$$= \lim_{x \to \pi/2} \frac{-\frac{\cancel{\sin x}}{\cancel{\cos x}}}{2\sec^2 x \frac{\cancel{\sin x}}{\cancel{\cos x}}}$$

$$= \lim_{x \to \pi/2} \left(-\frac{\cos^2 x}{2} \right)$$

$$\log L = \frac{0}{2}$$

$$\log L = 0$$

$$L = e^0$$

$$\boxed{L = 1}$$

***

(13) Evaluate $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

$\Rightarrow L = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$ —— $(1)^\infty$ form

Taking log on both sides

$$\log L = \log \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$$

$$\log L = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{a^x+b^x+c^x}{3}\right) \quad \cdots \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{\frac{1}{\left(\frac{a^x+b^x+c^x}{3}\right)}\times\frac{1}{3}\left(a^x\log a + b^x\log b + c^x\log c\right)}{1}$$

$$\log L = \lim_{x\to 0}\left(\frac{a^x\log a + b^x\log b + c^x\log c}{a^x+b^x+c^x}\right)$$

$$\cancel{\log L = \frac{3}{a^0+b^0+c^0}}$$

$$\cancel{= \frac{3}{3}}$$

$$\cancel{\log L = 1}$$

$$\cancel{\boxed{L = e}}$$

$$\log L = \frac{a^0\log a + b^0\log b + c^0\log c}{a^0+b^0+c^0}$$

$$= \frac{\log a + \log b + \log c}{3}$$

$$= \frac{\cancel{\log\left(\frac{a+b+c}{3}\right)}}{}$$

$$\log L = \frac{\log(abc)}{3}$$

$$\log L = \log(abc)^{1/3}$$

$$\boxed{L = (abc)^{1/3}}$$

(14) Evaluate $\lim_{x \to 0} (\sec x)^{\cot x}$

$\Rightarrow$ $L = \lim_{x \to 0} (\sec x)^{\cot x}$ $((1)^{\infty}$ Form)

Taking log on both side

$$\log L = \log \lim_{x \to 0} (\sec x)^{\cot x}$$

$$\log L = \lim_{x \to 0} \log (\sec x)^{\cot x}$$

$$\log L = \lim_{x \to 0} \cot x \log(\sec x) \quad (0 \times \infty) \text{ form}$$

$$= \lim_{x \to 0} \frac{\log(\sec x)}{\tan x} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x \to 0} \frac{\frac{\sec x \tan x}{\sec x}}{-\sec^2 x}$$

$$\cancel{= \lim_{x \to 0} \frac{-\sin x \cos^2 x}{-\cos x}}$$

$$\log L = \frac{0}{(1)^2}$$

$$\boxed{L = e^0} \quad \boxed{L = 1}$$

(15) Evaluate $\lim_{x \to y} \frac{x^y - y^x}{x^x - y^y}$

$\Rightarrow$ let $L = \lim_{x \to y} \frac{x^y - y^x}{x^x - y^y}$ ...... $\left(\frac{0}{0} \text{ form}\right)$

Here, y is constant

$$= \lim_{x \to y} \frac{y x^{(y-1)} - y^x \log y}{x^x (1 + \log x) - 0}$$

$$= \frac{y y^{(y-1)} - y^y \log y}{y^y (1 + \log y)}$$

$$\frac{y^y - y^y \log y}{y^y (1 + \log y)}$$

$$L = \frac{(1 - \log y)}{(1 + \log y)}$$

★★★

(16) prove that $\lim_{x\to\infty}\left(\frac{a^{1/x}+b^{1/x}+c^{1/x}+d^{1/x}}{4}\right)^{x} = (abcd)^{\frac{1}{4}}$

$\Rightarrow$

$$L = \lim_{x\to\infty}\left(\frac{a^{1/x}+b^{1/x}+c^{1/x}+d^{1/x}}{4}\right)^{x} \quad \ldots (1^{\infty})\text{ form}$$

~~Taking log on both side~~

$$\cancel{\log L = \lim_{x\to\infty} \log\left(\frac{a^{1/x}+b^{1/x}+c^{1/x}+d^{1/x}}{4}\right)^{x}}$$

$$\cancel{\log L = \lim_{x\to\infty} x \log\left(\frac{a^{1/x}+b^{1/x}+c^{1/x}+d^{1/x}}{4}\right)}$$

$$\cancel{= \lim_{x\to\infty} \frac{\log\left(\frac{a^{1/x}+b^{1/x}+c^{1/x}+d^{1/x}}{4}\right)}{1/x}} \quad \ldots \left(\frac{0}{0}\text{ form}\right)$$

$$\cancel{= \lim_{x\to\infty} \frac{\quad}{-1/x^{2}}}$$

let $\left(\frac{1}{x} = y\right)$ when $x\to\infty$, $y\to 0$

$$L = \lim_{y\to 0}\left(\frac{a^{y}+b^{y}+c^{y}+d^{y}}{4}\right)^{\frac{1}{y}}$$

Taking log on both side

$$\log L = \lim_{y\to 0} \log\left(\frac{a^{y}+b^{y}+c^{y}+d^{y}}{4}\right)^{1/y}$$

$$= \lim_{y\to 0} \frac{1}{y} \log\left(\frac{a^{y}+b^{y}+c^{y}+d^{y}}{4}\right)$$

$$= \lim_{y\to 0} \frac{\log(a^{y}+b^{y}+c^{y}+d^{y}) - \log 4}{y} \quad \ldots \frac{0}{0}\text{ form}$$

$$= \lim_{y\to 0} \frac{\dfrac{(a^{y}\log a + b^{y}\log b + c^{y}\log c + d^{y}\log d)}{(a^{y}+b^{y}+c^{y}+d^{y})}}{1}$$

$$= \frac{a^{0}\log a + b^{0}\log b + c^{0}\log c + d^{0}\log d)}{(a^{0}+b^{0}+c^{0}+d^{0})}$$

$$\log L = \frac{(\log a + \log b + \log c + \log d)}{4}$$

$$\log L = \frac{1}{4}\log(abcd)$$

$$\log L = \log(abcd)^{1/4}$$

$$\boxed{L = (abcd)^{1/4}}$$

(17) If $\lim_{x \to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1$, find a and b

Sol :-

$$\lim_{x \to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1$$

$$\lim_{x \to 0} \frac{x(0 + a(-\sin x)) + (1+a\cos x) - b\cos x}{3x^2} = 1$$

$$\lim_{x \to 0} \frac{-xa\sin x + 1 + a\cos x - b\cos x}{3x^2} = 1$$

∵ limit has finite value

$$\boxed{1 + a - b = 0} \quad \text{---- (1)}$$

$$\lim_{x \to 0} \frac{-(ax\cos x + a\sin x) + a(-\sin x) + b\sin x}{6x} = 1$$

$$\lim_{x \to 0} \frac{-[(ax(-\sin x) + a\cos x) + a\cos x] - a\cos x + b\cos x}{6} = 1$$

$$\frac{-(a+a) - a + b(1)}{6} = 1$$

$\cancel{-2a = 6}$ $\quad -2a - a + b = 6$

$\cancel{\boxed{a = -3}}$ $\quad \boxed{-3a + b = 6}$

$$a - b = -1$$
$$-3a + b = 6$$
$$-2a = 5$$

$\boxed{a = -\frac{5}{2}}$

$-\frac{5}{2} - b = -1$

$-\frac{5}{2} + 1 = b$

$\boxed{b = -\frac{3}{2}}$