1. Understand the Rolle's mean value theorem and apply it to solve the problems
2. Understand the Lagrange's theorem and apply it to solve the problems.
3. Understand the Cauchy's mean value theorem and apply it to solve the problems
4. Expand the function in the Taylor's and McLaurin series.
5. Find the limits of algebraic function which gives the Indeterminate forms.
6. Test the convergence and divergence of Infinite series.

Rolle's Mean value theorems

Lecture : 01

1. Learning Objective:
Student shall be able to Understand and apply Role's mean value theorem.

2. Introduction: The mean value theorem tells us (roughly) that if we know the slope of the secant line of a function whose derivative is continuous, then there must be a tangent line nearby with that same slope. This lets us draw conclusions about the behavior of a function based on knowledge of its derivative.

3. Key Definitions:

Role's Theorem: If a function f(x) is

(i) f (x) is continuous on the closed interval [a, b]

(ii) f (x) is differentiable on the open interval (a, b)

(iii) f(a)=f(b)

then there exists at least one point c in (a, b) (i.e. a < c < b) such that f '(c)=0.

Alternative or Another Statement of Rolle's Theorem: If a function $f(x)$ is

(i) continuous in $[\, a, a+h \,]$

(ii) differentiable in $(a, a+h)$

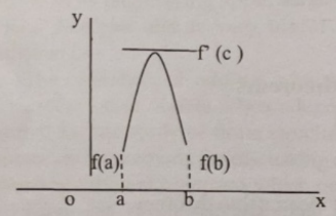
(iii) $f(a) = f(a+h)$,

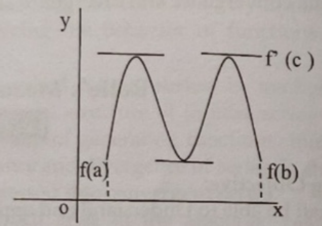
then there exists at least one real number θ such that

$$f'(a+\theta h)=0, \qquad for\ 0<\theta<1.$$

Geometrical Interpretation of Rolle's Theorem:

If graph of the function is a continuous curve between $x = a$ and $x = b$ having a unique tangent at all points in (a, b) and $f(a) = f(b)$ then there exists at least one point P between $x = a$ and $x = b$ on the curve, such that the tangent at this point is parallel to X - axis i.e. $f'(c) = 0$.





Algebraic Interpretation of Rolle's Theorem:

Let f(x) be a polynomial in x and let the two roots of f(x)=0 be x=a and x=b. Then according to Rolle's Theorem, at least one root of $f'(x) = 0$ lies between a and b.

3. Sample Problems

(1) Verify the Rolle's theorem for

$$f(x) = (x-a)^m (x-b)^n \ \ in\ (a,b),\ \text{where m,n are +ve integers}$$

Solution: (i) Since m, n are positive integers,
$f(x) = (x-a)^m(x-b)^n$, being a polynomial, thus
$f(x)$ is continuous in [a, b].
(ii) $f(x)$ is differentiable in (a, b) as it is a polynomial

$$f'(x) = m\,(x-a)^{m-1}(x-b)^n + n\,(x-a)^m\,(x-b)^{n-1}$$

$$f'(x) = (x-a)^{m-1}(x-b)^{n-1}[\,m\,(x-b) + n\,(x-a)]$$

$$f'(x) = (x-a)^{m-1}(x-b)^{n-1}[\,(m+n)x - (mb+na)]$$

f ′(x) exists for every value of x in (a, b). Therefore $f(x)$ is differentiable in (a, b).

(iii) $f(a) = f(b) = 0$

Thus, f(x) satisfied all the conditions of Roll's theorem. Hence, Rolle 's Theorem is applicable.
Therefore, there exists at least one point c in (a, b) such that $f'(c) = 0$

$$f'(c) = m\,(c-a)^{m-1}(c-b)^n + n\,(c-a)^m\,(c-b)^{n-1} = 0$$

$$f'(c) = (c-a)^{m-1}(c-b)^{n-1}[\,m\,(c-b) + n\,(c-a)] = 0$$

$$\therefore mc - mb + nc - na = 0 \Rightarrow (m+n)c = mb + na$$

$$\therefore \quad c = \frac{m\,b + n\,a}{m+n}$$

which represents a point c that divides the interval [a, b] in the ratio of m: n,
Thus, c lies between a and b, i.e. $a < c < b$. Hence Rolle's theorem is verified.

(2) Prove that the equation $2x^3 - 3x^2 - x + 1 = 0$ has at least one root between 1 and 2.

Solution: Consider the function $f'(x) = 2x^3 - 3x^2 - x + 1 = 0$

$f(x) = \int (2x^3 - 3x^2 - x + 1)dx = \frac{x^4}{2} - x^3 - \frac{x^2}{2} + x$

Deliberately ignoring the integral constant

$f(1) = \frac{1}{2} - 1 - \frac{1}{2} + 1 = 0$ and

$f(2) = \frac{2^4}{2} - 2^3 - \frac{2^2}{2} + 2 = 8 - 8 - 2 + 2 = 0$

Hence two roots of f(x) are 1 and 2

Therefore, By Algebraic interpretation of Rolle's Theorem, we can say that f′(x) has at least one root between the roots of f (x).

(3) Verify Rolle's theorem for $f(x) = x\,(x-2)\,e^{\frac{3x}{4}}$ in (0,2)

Solution: f(x) is continuous in [0,2]

f(x) is differentiable in (0,2)

f (0) = 0, f(2) = 0

let f′ (c) = 0 ⇒ $f'(c) = \left\{\left[(x-2) + x + \frac{3}{4}x(x-2)\right]e^{\frac{3}{4}x}\right\}|_{x=c} = 0$

$$\Rightarrow (3c)^2 + 2c - 8 = 0 \Rightarrow c = -2 \text{ or } \frac{8}{6}$$

But c = - 2 does not lie in (0,2) thus c = 8/6 ∈ (0,2)

Hence Rolle's theorem is verified.

Exercise 1

1. Verify Rolle's Theorem

(i) $f(x)=\frac{\sin x}{e^x}\, in[0,\pi]$

(ii) $f(x)=1-3(x-1)^{\frac{2}{3}}\, in 0\le x\le 2$

(iii) $f(x)=e^{-x}(\sin x-\cos x)$ in $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$

(iv) $f(x)=\begin{cases} x^2+1, & 0\le x\le 1 \\ 3-x, & 1\le x\le 2. \end{cases}$

2. Use Rolle's Theorem to prove that the equation $ax^2+bx=\frac{a}{3}+\frac{b}{2}$ *has a root between* 0 *and* 1.

Let's Check away from lecture

1. *If* $y=(x-1)^3(x-2)^2$ *in* [1,2] *then c by R.M.V.T. theorem is*

(a) $\frac{8}{5}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) None

2. *If* $y=\cos x$ *in* $[-\pi,\pi]$ *then c by R.M.V.T. theorem is*

(a) π (b) 0 (c) $\frac{\pi}{4}$ (d) None

3. If $f(x)=(x+2)^3(x-3)^4$ in [-2, 3] then c of R.M.V.T. is

(a) 1/5 (b) 1/6 (c) 1/7 (d) 1/8

4. If $f(x)=\ln\left\{\frac{x^2+6}{5x}\right\}$ in [2, 3] then c of R.M.V.T. is

(a) $\sqrt{5}$ (b) $\sqrt{6}$ (c) $\sqrt{7}$ (d) $\sqrt{3}$

5. Is Rolle's theorem is applicable for $f(x)=x^2$ in [1,2]

(a) Applicable (b) Not Applicable (c) Cannot be said

Homework Problems for the day

1. Verify Rolle's Theorem

(i) $f(x)=\tan x,\quad 0\le x\le \pi$ (ii) $f(x)=\log\left[\frac{x^2+ab}{(a+b)x}\right] in[a,b],\quad a>0, b>0.$

(iii) $f(x)=|x|\ in[-1,1]$

2. *If* $f(x)=x(x+1)(x+2)(x+3)$, *then show that* $f'(x)$ *has three real roots.*

3. Show that between any two roots of $e^x\cos x-1=0$ there exist at least one root of $e^x\sin x-1=0$.

4. Apply the Rolle's theorem and find the value of c for $f(x)=x^3-4x$ in the interval [-2,2].

Learning from the topic: Learner will be able to apply Rolle's mean value theorem.

Lagrange's mean value theorem

Lecture : 02

1. Learning Objective:
Student shall be able to Understand and apply Lagrange's mean value theorem.

2. Key Definitions:

Lagrange's mean value theorem

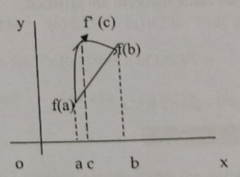
Suppose f(x) is a function that satisfies the following.

(i) f(x) is continuous on the closed interval [a, b]

(ii) f(x) is differentiable on the open interval (a, b)

Then there is a number c such that a < c < b and $f'(c)=\frac{f(b)-f(a)}{b-a}$

Geometrical Interpretation of L.M.V.T.: If curve is continuous from a point A to a point B and has tangent at every point on it then there exists at least one-point C between A and B, such that tangent at this point is parallel to the chord AB.



3. Sample Problems

(1) Verify Lagrange's mean value theorem for the function $f(x) = x^3 - 2x^2 - 3x - 6$ in $[-1, 4]$.

Solution: We have $f(x) = x^3 - 2x^2 - 3x - 6$ in $[-1, 4]$, $f(-1) = -6$ and $f(4) = 14$

Since f(x) is polynomial, therefore continuous hence f(x) is continuous in $[-1, 4]$.

$f'(x) = 3x^2 - 4x - 3$ hence f(x) is differentiable in $(-1, 4)$.

All the conditions of Lagrange's mean value theorem are satisfied.

Therefore, there exists $c \in (-1,4)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(-1)}{4-1}$$

$$\Rightarrow 3c^2 - 4c - 3 = \frac{f(4) - f(-1)}{4-(-1)} = \frac{14-(-6)}{5} = 4$$

$$\Rightarrow 3c^2 - 4c - 7 = 0$$
$$\Rightarrow (3c-7)(c+1) = 0$$
$$\Rightarrow c = \frac{7}{3}, -1$$

The LMVT guarantees that $c \in (-1,4)$, and since $7/3 \in (-1,4)$
Hence, Lagrange's mean value theorem is verified. Here, $x=-1$ is excluded since $-1 \notin (-1,4)$.

(2) Using Lagrange's mean value theorem, show that $\sin x \le x$ for $x \ge 0$.

Solution: We have f(x) = x-sin x defined in [0, x].

f (0) = 0 and f(x)=x-sin x

clearly, f(x) is everywhere continuous and differentiable. so

(i) f(x) is continuous in [0, x].

(ii) f'(x)=1- cos x

= f(x) is differentiable in (0, x).

Thus, both the conditions of Lagrange's mean value theorem are satisfied.

$c \in (0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0}, \quad 1-\cos c = \frac{x-\sin x}{x-0} \quad \text{since } 1-\cos c \ge 0 \ \forall x$$

Thus, $\frac{x-\sin x}{x-0} \ge 0$

$= x \ge \sin x$ Hence $\sin x \le x$ for all $x \ge 0$

(3). Show that for any $x \ge 0$, $1 + x < e^x < 1 + x e^x$

Solution: Take $f(x) = e^x - (1 + x)$

$$f'(x) = e^x - 1 \ge 0 \ for \ x \ge 0$$

so f(x) is increasing function and f(0) = 0

$f(x) > 0$ for any $x \ge 0$

$e^x - (1 + x) > 0 \ or \ 1 + x < e^x$ ...(1)

Now consider the function
$g(x) = 1 + xe^x - e^x \Rightarrow g(0) = 0 \ and \ g'(x) = e^x + xe^x - e^x = xe^x \ge 0$ for any $x \ge 0$.
Thus, g(x) is increasing function and therefor g(x) > 0
$\Rightarrow 1 + xe^x - e^x > 0 \Rightarrow e^x < 1 + xe^x$ ... (2)

From equation (1) and (2) we can say that $1 + x < e^x < 1 + x e^x$.

Exercise 2

1. Verify Lagrange's mean value theorem for the function $f(x) = x^2+x-1$ in [0, 4].

2. Using Lagrange's mean value prove that $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$,

& hence deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$.

Let's check away from lecture

1. *If* $y = (x-1)^{\frac{3}{2}}$ *in* $[1,2]$ *then c by L.M.V.T. theorem is*

   (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) *None.*

2. *If* $y = \cos x$ *in* $[0, 2\pi]$ *then c by L.M.V.T. theorem is*

   (a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) *None*

3. If $f(x) = x(x-1)(x-2)$ in [0,1/2] then c of LMVT is.

   (a) 0.256 (b) 0.156 (c) 0.236 (d) 0.276

4. If $f(x) = sin^{-1}x$ [0,1] then c of LMVT is

   (a) 0.567 (b) 0.432 (c) 0.756 (d) 0.771

Homework Problems for the day

1. Verify Lagrange's mean value theorem for the function

(i) $f(x)=x^{\frac{2}{3}}$ in $[-8, 8]$.

(ii) $f(x) = 2x^2 - 7x - 10$ over $[2,5]$ and find c using LMVT

2. Apply Lagrange's mean value theorem for the function log(x) in [a, a+h] & determine θ in terms of a and h and deduce that $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$.

3. If $0 < a < b$, prove that $\left(1-\frac{a}{b}\right) < \log\frac{b}{a} < \left(\frac{b}{a}-1\right)$ Hence, prove that $\frac{1}{4} < \log\frac{4}{3} < \frac{1}{3}$

Learning from the topic: Learner will be able to apply Lagrange's mean value theorem.

Cauchy's mean value theorem

Lecture : 03

1. Learning Objective:
Student shall be able to understand and apply Cauchy's mean value theorem.

2. Key Definitions:

Cauchy's mean value theorem

Let functions f(x) and g(x) be

(i) Continuous on [a,b]

(ii) Differentiable on the interior of (a,b).

further that $g'(x) \neq 0$, $\forall\ x \in (a,b)$.

There exists a point $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

4. Sample Problems

(1). If $f(x) = e^x$ and $g(x) = e^{-x}$, prove that **c** of Cauchy's mean value theorem is the Arithmetic mean between a and b, a > 0, b > 0.

Solution: f(x) and g(x), being exponential function, are continuous on [a, b] and

$f'(x) = e^x, g'(x) = e^{-x}$ *exists for all x in (a, b)*

therefore $f(x) = e^x$ and $g(x) = e^{-x}$, differentiable on (a, b) and $g'(x) \neq 0$ for any c in (a, b)

Thus, f(x) and g(x) satisfies all the conditions of Cauchy's Mean Value theorem, there exists at least one point c in (a,b).

By Cauchy's Theorem $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, a < c < b$

$\therefore f(x) = e^x, f'(x) = e^x$ and $\therefore g(x) = e^{-x}, f'(x) = -e^{-x}$

$$\frac{e^b-e^a}{e^{-b}-e^{-a}} = \frac{e^c}{-e^{-c}}, a < c < b \Rightarrow \frac{e^b-e^a}{\frac{1}{e^b}-\frac{1}{e^a}} = -e^{2c}, a < c < b,$$

$\therefore -e^{a+b} = -e^{2c}$.

$\Rightarrow \therefore a+b = 2c \Rightarrow \therefore c = \frac{a+b}{2}$

Hence, c is the A.M. between a and b.

(2). Using Cauchy's mean value theorem,

show that $\frac{\sin b - \sin a}{\cos a - \cos b} = \cot c$ w□ere $a < c < b$, a>0, b>0

Solution: We have to prove that $\frac{\sin b - \sin a}{\cos a - \cos b} = \cot c$

Let f(x)=sin x and g(x)=cos x are defined in the interval (a, b) such that f(x) and g(x), being trigonometric functions, be continuous on [a,b] and differentiable on (a,b) with $g'(x) \neq 0$ for any c in (a,b).

By Cauchy's Theorem $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$, $a < c < b$

$\frac{\sin b - \sin a}{\cos b - \cos a} = -\frac{\cos c}{\sin c} = -\cot c$ where $a < c < b$,

Hence, $\frac{\sin b - \sin a}{\cos a - \cos b} = \cot c$ where $a < c < b$,

Exercise 3

1. If $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$,
then prove that "c" of CMVT is geometric mean between a & b

2. Using appropriate MVT prove that $\frac{\sin b - \sin a}{e^b - e^a} = \frac{\cos c}{e^c}$ for a<c<b.
Hence deduce that $e^c \sin x = (e^x - 1)\cos c$.

3. If $U = e^x$ and $V = e^{-x}$ in [1,2] then c by C.M.V.T. is
(a) $\frac{3}{2}$ (b) $\frac{7}{4}$ (c) $\frac{5}{3}$ (d) none

2. $If\ U = \log x\ and\ V = \cos x\ in\ [1,2]\ then\ can\ C.M.V.T.theorem\ is\ applicable?$

$(a)\ applicable\quad (b) not\ applicable\quad (c) cannot\ be\ said$

4. If $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{\sqrt{x}}$ in [a,b] then c of C.M. V.T. is

(a) $\sqrt{ab}$ (b) $\sqrt{\frac{a}{b}}$ (c) ab (d) $\frac{a}{b}$

5. If $f(x)=x^2$ and $g(x)=\sqrt{x}$ in [1,4] then c of C.M. V.T. is

(a) $\left(\frac{15}{4}\right)^{\frac{3}{2}}$ (b) $\left(\frac{15}{4}\right)^{\frac{1}{2}}$ (c) $\left(\frac{15}{4}\right)^{\frac{2}{3}}$ (d) $\left(\frac{15}{4}\right)^{\frac{1}{3}}$

6. Lagrange's Mean Value Theorem for $f(x)=\log_e x$ in the interval [1,e] is

(a) False (b) Not always false (c) True (d) Not always true

Homework Problems for the day

1. Verify Cauchy's mean value theorem for

(i) $x^2\ \&\ x^4\ in\ [a,b]\ where\ a>0, b>0,$ (ii) $\sin x\ \&\ \cos x\ in \left[0,\frac{\pi}{2}\right]$

(iii) $x^2\ \&\ x^3\ in\ [1,2]$

2. If $f(x)=\frac{1}{x^2}$ and $g(x)=\frac{1}{x}$, prove that c of CMVT is the harmonic mean between a and b.

Learning from the topic: Learner will be able to apply Cauchy's mean value theorem.

Taylor's and Maclaurin's series

Lecture: 04

1. Learning Objective:

Student shall be able to find the Taylor's series expansion of a function.

2. Introduction:

The Taylor series can be used to calculate the value of an entire function at every point, if the value of the function, and of all its derivatives, are known at a single point. In some engineering problems of integral calculus, it is difficult to find the value of integral by usual means. In such cases by expanding the function into convergent series of infinite terms one can overcome that problem of integration. In similar way expansion of the function plays an important role in finding Laplace Transform of the function.

3. Important Formulae / Theorems / Properties/Definitions:

(1)Taylor's Series: The Taylor series of a real or complex-valued function f (x) that is infinitely differentiable at 'a' real or complex number a is the power series

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots+\frac{(x-a)^n}{n!}f^n(a)+\ldots$$

(2) Alternative form of Taylor's Series: If $x-a=h$ then

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\ldots\ldots\ldots+\frac{h^n}{n!}f^n(a)+\ldots\ldots$$

4. Important Steps to be followed for solving the problems:

- While finding the Taylor's Series expansion of the function one should identify first that expansion is required in powers of which factor i.e. to identify the center of the power series.
- According to the center of power series one should use appropriate form of the Taylor's Series to expand the function.

5. Sample problems

1. Express $f(x) = 2x^3 + 3x^2 - 8x + 7$ in powers of (x-2).

Solution: Let $f(x)=2x^3+3x^2-8x+7$ and $a=2, f(2)=34$

$$\therefore f'(x) = 6x^2 + 6x - 8, f'(2) = 28$$

$$f''(x) = 12x + 6, f''(2) = 30$$

$$f'''(x)=12,\quad f'''(2)=12$$

Hence,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2)+\ldots\ldots\ldots$$

$\therefore f(x) = 2x^3 + 3x^2 - 8x + 7 = 34 + (x-2) \times 28 + (x-2)^2 \times 30 + (x-2)^3 \times 12 = 34 + 28(x-2) + 30(x-2)^2 + 12(x-2)^3$

2. Using Taylor's theorem evaluate up to 4 places of decimals $\sqrt{25.15}$

Solution: By Taylor's series expansion we have,

$$f(x+h) = f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots \tag{1}$$

$Here\ f(x)=\sqrt{x} \qquad \therefore f(x+h)=\sqrt{x+h}$

$$Now\ f(x)=\sqrt{x} \Rightarrow f'(x)=\frac{1}{2\sqrt{x}}\ ;\ f''(x)=-\frac{1}{4}x^{-\left(\frac{3}{2}\right)}\ ;\ f'''(x)=\frac{3}{8}x^{-\left(\frac{5}{2}\right)}$$

therefore, equation (1) becomes:

$$\sqrt{x+h}=\sqrt{x}+h\frac{1}{2\sqrt{x}}+\frac{h^2}{2!}\left(-\frac{1}{4}x^{-\left(\frac{3}{2}\right)}\right)+\frac{h^3}{3!}\frac{3}{8}x^{-\left(\frac{5}{2}\right)}+\dots$$

Now set x = 25 and h = 0.15 which gives

$$\sqrt{25+0.15}=\sqrt{25}+0.15\frac{1}{2\sqrt{25}}+\frac{(0.15)^2}{2!}\left(-\frac{1}{4}25^{-\left(\frac{3}{2}\right)}\right)+\frac{(0.15)^3}{3!}\frac{3}{8}25^{-\left(\frac{5}{2}\right)}+\dots$$

$$\sqrt{25+0.15}=5.0150$$

(3) Expand sin x in powers of $\left(x-\frac{\pi}{2}\right)$.

Solution: Let $f(x)=\sin x$ and $a=\frac{\pi}{2}, f(a)=\sin\left(\frac{\pi}{2}\right)=1$

$\therefore f'(x)=\cos x, f'\left(\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)=0,$

$f''(x)=-\sin x, f''\left(\frac{\pi}{2}\right)=-1$

$f'''(x)=-\cos x, f'''\left(\frac{\pi}{2}\right)=0,\ \ f^{iv}(x)=\sin x, f^{iv}\left(\frac{\pi}{2}\right)=1$

Hence by Taylor's Series we get,

$$f(x)=f(2)+(x-2)f'(2)+\frac{(x-2)^2}{2!}f''(2)+\frac{(x-2)^3}{3!}f'''(2)+\dots\dots\dots$$

$$f(x)=f\left(\frac{\pi}{2}\right)+\left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}f'''\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}f^{iv}\left(\frac{\pi}{2}\right)+\dots\dots\dots$$

$$\therefore \sin x=1+\left(x-\frac{\pi}{2}\right)(0)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}(-1)+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}(0)+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}(1)+\dots\dots$$

$$\therefore \sin x=1+-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}+\dots\dots$$

Exercise 4

1. Expand $f(x)=x^5-x^4+x^3-x^2+x-1$ in powers of (x-1) and find f(0.99).

Ans.: $f(x)=3(x-1)+6(x-1)^2+7(x-1)^3+4(x-1)^4+(x-1)^5, -0\cdot0294$

2. By using Taylor's series expand $\tan^{-1}(x)$ in positive powers of (x-1) up to first four nonzero terms.

Ans : $\tan^{-1}(x)=\frac{\pi}{4}+\frac{(x-1)}{2}-\frac{(x-1)^2}{4}+.\frac{(x-1)^3}{12}\dots\dots\dots$

3. Arrange in powers of x, by Taylor's theorem $f(x+2)=7+(x+2)+3(x+2)^3+(x+2)^4$

Ans.: $49+69x+42x^2+11x^3+x^4$.

Choose the correct option from the following:

1. *By Taylors series* $f(x+h)$

(a) $f(x)+hf'(x)+\frac{h^2}{2!}f''(x)\dots.$ (b) $1+hf(x)+\frac{h^2}{2!}f'(x)+\frac{h^3}{3!}f''(x)\dots.$

(c) $f(h)+hf'(x)+\frac{h^2}{2!}f''(h)\dots.$ (d) *none*

2. In Taylors Series expansion of function in powers of (x-a) centre of power series is:

(a) x=1 (b) x=0 (c) x=a (d) none of these

3. Expand $f(x)=x^2+x-1$ in powers of (x-2) using Taylors Series.

(a) $(x-2)^2-5(x-2)+5$ (b) $(x-2)^2+5(x-2)-5$ (c) $(x-2)^2+5(x-2)+5$ (d) none of these

4. If $f(x)=x^3+3x^2+15x-10$ then the value of $f\left(\frac{11}{10}\right)$ is

(a) 11.461 (b) 10.327 (c) 11.321 (d) 8.724

5. The Taylor's series expansion of $f(x)=e^x$ around x=3 is

(a) $e^3\sum_{n=0}^{\infty}\frac{(x-3)^n}{n!}$ (b) $e^x\sum_{n=0}^{\infty}\frac{(x-3)^n}{n!}$ (c) $e^3\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$ (d) $e^3\sum_{n=0}^{\infty}(x-3)^n$

Homework Problems for the day

1. Expand $\tan^{-1}(x+h)$ in powers of h and hence find the value of $\tan^{-1}(1.003)$ upto 5 places of decimal.

Ans.: 0.78690.

2. Using Taylor's theorem evaluate upto 4 places of decimals.

(i) $\sqrt{1.02}$ (ii) $\sqrt{10}$

Ans.: i) 1.0099 ii) 3.16227

3. Expand log (cosx) about $\frac{\pi}{3}$ using Taylor's expansion.

4. Expand $(2x^3+7x^2+x-1)$ in powers of (x-2)

5. Find the Taylor's series of $f(x) = \log \cos x$, around $x = \frac{\pi}{3}$.

6. If $y = \sin \log(x^2 + 2x + 1)$, Expand y in ascending powers of x upto x^6.

Learning from the topic: Student will be able to expand a function of single variable into positive ascending integral powers (x-a) using Taylor's Series.

McLaurin's Series

Lecture:05

1. **Learning Objective:**
Student shall be able to find the Maclaurin's series expansion of any type of function.

2. **Introduction:** The Taylor's Series expansion which are done around the x=0 is called Maclaurin's series.

3. Key Definitions:

(1) **Maclaurin's Series:** If f(x) be a given function of x which can be expanded into a convergent series of positive ascending integral powers of x then,

$$f(x) = f(0)+xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots\dots + \frac{x^n}{n!}f^n(0) + \dots\dots$$

4. Important Formulae / Theorems / Properties: Standard Expansions:

$(i)\ e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\dots\dots\dots$ $\qquad (ii)\ \sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\dots\dots$

$(iii)\ \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\dots\dots$ $\qquad (iv)\ \tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\dots\dots\dots$

$(v)\ \sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots\dots\dots$ $\qquad (vi)\ \cosh x = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\dots\dots$

$(vii)\ \tanh x = x-\frac{x^3}{3}+\frac{2x^5}{15}-\dots\dots\dots$ $\qquad (viii)\ \tan^{-1} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots\dots$

$(ix)\ \sin^{-1} x = x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7}+\dots\dots$ $(x)\ \cos^{-1} x = \frac{\pi}{2}-\left\{x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\frac{x^5}{5}+\dots\dots\right\}$.

$(xi)\ \log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\dots\dots\dots$ $(xii)\ \log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\dots\dots\dots$

$(xiii)\ (1+x)^m = 1+mx+\frac{m(m-1)}{2!}x^2+\dots\dots\dots$ *In particular if* $m=-1$ *then*

$(a)\ (1+x)^{-1} = 1-x+x^2-x^3+\dots\dots\dots$ $\qquad (b)\ (1-x)^{-1} = 1+x+x^2+x^3+\dots\dots\dots$

5. Important Steps to be followed for solving the problems:

i. If the required expansion is in powers of x then it is identified as problem of Maclaurin's Series.

ii. While finding the Maclaurin's Series expansion of the function one should identify first that which standard expansion can be used to get the desire series.

iii. Secondly if it is found difficult to use appropriate standard expansion then one can apply Maclaurin's Series to the given function which is to be expanded.

6. Sample problems

(1). Prove that $\log(1+e^x) = \log 2+\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{192}x^4+\dots$

Solution: Let $f(x) = \log(1+e^x)$

$$f'(x) = \frac{e^x}{(1+e^x)} \qquad f''(x) = \frac{e^x}{(1+e^x)^2} \qquad f'''(x) = \frac{e^x-e^{2x}}{(1+e^x)^3}$$

$$f^{iv}(x) = \frac{(1+e^x)(e^x-2e^{2x})-3e^x(e^x-e^{2x})}{(1+e^x)^4}$$

$$f(0) = \log 2,\ f'(0) = \frac{1}{2},\ f''(0) = \frac{1}{4},\ f'''(0) = 0,\ f^{iv}(0) = \frac{-1}{8}.$$

Using the Maclaurin's series

$$f(x) = f(0)+xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots\dots + \frac{x^n}{n!}f^n(0) + \dots\dots$$

$$\log(1+e^x) = \log 2+\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{192}x^4+\dots$$

(2). Show that $x \operatorname{cosec} x = 1+\frac{x^2}{6}+\frac{7}{360}x^4+\dots\dots\dots$

Solution: $x \operatorname{cosec} x = \frac{x}{\sin x} = \frac{x}{x-\frac{x^3}{3!}-\frac{x^5}{5!}\dots}$ (using std. expansion of sin x)

$$= \frac{1}{1-\frac{x^2}{6}+\frac{x^4}{120}\dots} = \left[1-\left(\frac{x^2}{6}-\frac{x^4}{120}\right)\dots\right]^{-1}$$

$$= 1+\frac{x^2}{6}-\frac{x^4}{120}+\ldots\left(\frac{x^2}{6}\ldots\right)^2 \text{ (Neglecting higher power)}$$

$$=1+\frac{x^2}{6}+\left(\frac{1}{36}-\frac{1}{120}\right)x^4\ldots \quad =1+\frac{x^2}{6}+\left(\frac{10-3}{360}\right)x^4\ldots$$

$$=1+\frac{x^2}{6}+\frac{7x^4}{360}\ldots$$

(3). Show that $e^x cosx = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} \ldots\ldots$

Solution: Given function $f(x) = e^x \cos x \Rightarrow f(0) = 1$

$$f'(x) = e^x \cos x - e^x \sin x \Rightarrow f'(0) = 1$$

$$f''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x \Rightarrow f''(0) = 0$$

$$f'''(x) = -2e^x \cos x - 2e^x sinx \Rightarrow f'''(0) = -2$$

$$f^{iv}(x) = -4e^x \cos x \Rightarrow f^{iv}(0) = -4$$

By Taylor's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0)\ldots$$

$$e^x cosx = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!}\ldots$$

Exercise 5

1. *Expand* 5^x *upto the first three nonzero terms of the series.*

2. *Expand* $\sqrt{1+\sin x}$. *Ans*: $1+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}+\frac{x^4}{384}\ldots\ldots$

3. *Prove that* $\sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots\ldots$

Choose the correct option from the following:

1. Using Maclaurin's theorem we can expand a^x as:

(a) $1-x\log a-\frac{(x\log a)^2}{2!}-\ldots$ (b) $1-x\log a+\frac{(x\log a)^2}{2!}-\ldots$

(c) $1+x\log a+\frac{(x\log a)^2}{2!}+\ldots$ (d) $1-x\log a+\frac{(x\log a)^2}{2!}-\ldots$

2. *Expansion of* $x\cos x$ *is*:

(a) $x-\frac{x^3}{2!}+\frac{x^5}{4!}-\ldots$ (b) $x+\frac{x^3}{2!}+\frac{x^5}{4!}+\ldots$ (c) $x-\frac{x^3}{2!}-\frac{x^5}{4!}-\ldots$ (d) *none of these*

Homework Problems for the day

1. *Prove that* $\sin\left(e^x-1\right) = x+\frac{x^2}{2}-\frac{5x^4}{24}+\ldots\ldots$

2. *Prove that* $\log(1+\tan x) = x-\frac{x^2}{2}+\frac{2x^3}{3}+\ldots\ldots$

3. *Obtain the series for* $\log(1+x)$ *and hence find the series of* $\log_e\left(\frac{1+x}{1-x}\right)$ *and hence find the value of* $\log_e\left(\frac{11}{9}\right)$.

Learning from the topic: Learning from the topic: Student will be able to expand a function of single variable into positive ascending integral powers 'x' using Maclaurin's Series.

McLaurin's Series continue

Lecture: 06

1. Sample problems

(1). Prove that $sec^2 x = 1 + x^2 + \frac{2x^4}{3} + \ldots\ldots$

Solution: $\sec^2 x = \left[\frac{1}{\cos x}\right]^2 = \left[\frac{1}{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\right)}\right]^2$

Using $[1-x]^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots$ We get

$$\sec^2 x = 1 + 2\left(\frac{x^2}{2!} - \frac{x^4}{4!}+\ldots\right) + 3\left(\frac{x^2}{2!}-\ldots\right)^2 + \ldots \quad \text{considering terms up to } x^4$$

$$\sec^2 x = 1 + 2\left(\frac{x^2}{2!} - \frac{x^4}{4!}+\ldots\right) + 3\left(\frac{x^4}{4}\right) + \ldots = 1 + x^2 + \left(\frac{3}{4}-\frac{2}{4!}\right)x^4 + \ldots$$

$$= 1 + x^2 + \frac{2}{3}x^4 + \ldots$$

(2). Prove that $(1+x)^{\frac{1}{x}} = e - \frac{e}{2}x + \frac{11}{24}x^2 + \ldots$

Solution: Let $y = (1+x)^{\frac{1}{x}}$

$$\log y = \frac{1}{x}\log(1+x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right)$$

$$= 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$$

$$= 1 + z \quad \text{where } z = -\frac{x}{2} + \frac{x^2}{3} - \ldots$$

$$y = e^{1+z} = e.e^{z}$$

$$= e.\left(1 + \left(-\frac{x}{2} + \frac{x}{3} + \ldots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x}{3} + \ldots\right)^2 + \ldots\right)$$

$$= e.\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{x^2}{8} \ldots\right]$$

$$= e - \frac{e}{2}x + \frac{11}{24}x^2 + \ldots$$

(3). Show that $\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots\ldots$

Solution: i) $y = \log(\sec x)$ $\quad y(0) = 0$

$y_1 = \tan x$ $\quad y_1(0) = 0$

$y_2 = 1 + y_1^2$ $\quad y_2(0) = 1$

$y_3 = 2y_1y_2$ $\quad y_3(0) = 0$

$y_4 = 2y_2^2 + 2y_1y_3$ $\quad y_4(0) = 2$

$$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots$$

$$\therefore \log(\sec x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \ldots\ldots$$

(4) Show that $\log[\sec x] = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} \ldots\ldots\ldots$

Solution: $\log(\sec x) = \log\left(\frac{1}{\cos x}\right) = \log\left(\frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots}\right)$

Let $y = \log(\sec x)$

$\therefore \quad \frac{dy}{dx} = \frac{1}{(\sec x)}.(\sec x)\tan x \Rightarrow \tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots$

$\therefore \quad$ an integrating we get

$$y = \int\left\{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right\}dx + C = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots + C$$

To find c put $x = 0$

$\Rightarrow \quad y(0) = \log(\sec 0) = 0 + C$

$\therefore \quad 0 = C$

$\therefore \quad \log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots$

(5). Pr o ve that $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots$

Solution: $y = \tan^{-1}x$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2} = 1 - x^2 + (x^2)^2 - (x^2)^3 + (x^2)^4 \ldots\ldots$$

$$\therefore \frac{dy}{dx} = 1 - x^2 + x^4 - x^6 + \ldots\ldots$$

Integrating we get $\int_0^x \frac{dy}{dx}dx = \int_0^x (1 - x^2 + x^4 - x^6 + \ldots\ldots)dx$

$$\therefore y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$$

(6) Show that $f(x) = e^{-\frac{1}{x^2}}$ can be expanded by Maclaurin's series, but the series does not converge to f(x).

Solution: Since $f(x) = e^{-\frac{1}{x^2}}, f(0) = 0$

$f'(x) = e^{-\frac{1}{x^2}}.\left(\frac{2}{x^3}\right)$

$f''(x) = e^{-\frac{1}{x^2}}.\left(\frac{-6}{x^4}\right) + e^{-\frac{1}{x^2}}.\left(\frac{2}{x^3}\right)^2$

In general,

$f^n(x) = e^{-\frac{1}{x^2}}[\left(\frac{2}{x^n}\right)^n + \text{lower powers of } x]$

$\therefore f^n(0) = 0$

Thus f(x) is infinitely differentiable and by Maclurin's series

$f(x) = f(0) + x f'(0) + \cdots . + R_n$

$\therefore \quad e^{-\frac{1}{x^2}} = 0 + 0 + \cdots + 0 + R_n$

$\therefore \quad e^{-\frac{1}{x^2}} = R_n$

As $n \to \infty$, $Rn = e^{-\frac{1}{x^2}} \neq 0$ for $x \neq 0$

Thus, $f(x) = 0+0+0+\ldots +\infty$

Therefore, f(x) converges to zero but not to $e^{-\frac{1}{x^2}}$.

Exercise 6

1. Show that $\log[1+\sin x] = x - \frac{x^2}{2} + \frac{x^3}{6} + \ldots\ldots\ldots\ldots$

2. Expand $\log(1+x+x^2+x^3)$ upto a term in x^8.

Ans.: $x+\frac{x^2}{2}+\frac{x^3}{3}-\frac{3x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\frac{x^7}{7}-\frac{3x^8}{8}+\dots\dots\dots$

4. Prove that $x = y-\frac{y^2}{2!}+\frac{y^3}{3!}-\frac{y^4}{4!}+\dots\dots\dots$ *if* $y = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}\dots\dots\dots$

5. Prove that $\sin^{-1}x = x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7}+\dots\dots$

6. Expand $\sin e^x$ as a power series of x establishing its validity.

Let's check away from lecture

Choose the correct option from the following:

1. By Maclaurin's series f(x)

   (a) $f(a)+hf'(a)+\frac{h^2}{2!}f''(a)\dots$ (b) $1+hf(a)+\frac{h^2}{2!}f'(a)+\frac{h^3}{3!}f''(a)\dots$

   (c) $f(0)+hf'(0)+\frac{h^2}{2!}f''(0)\dots$ (d) *none*

2. By Maclaurin's series function can be expanded into positive ascending integral powers of

   (a) h (b) x (c) a (d) $f(x)$

3. In the Maclaurin series expansion of $y = e^x\log(1+x)$ The coefficients of x^5

   (a) $\frac{9}{5!}$ (b) $\frac{5}{5!}$ (c) $\frac{4}{5!}$ (d) $\frac{3}{5!}$

Homework Problems for the day

1. *Show that* $(1+x)^x = 1+x^2-\frac{x^3}{2}+\frac{5x^4}{6}+\dots\dots\dots$

2. *Prove that* $e^{e^x} = e\left[1+x+x^2+\frac{5x^3}{6}+\dots\dots\dots\right]$

3. *Prove that* $x\operatorname{cosec}x = 1+\frac{x^2}{6}+\frac{7}{360}x^4+\dots\dots\dots$

4. *Prove that* $\log\left(\frac{\sinh x}{x}\right) = \frac{x^2}{6}-\frac{x^4}{180}+\dots\dots$

5. *Prove that* $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots\dots\dots$

6. . Find the expansion of $\log\tan\left(\frac{\pi}{4}+x\right)$ upto x^5.

Learning from the topic: Student will be able to expand a function of single variable into positive ascending integral powers 'x' using Maclaurin's Series.

Indeterminate forms

Lecture: 07

1. Learning Objective:
Student shall be able to identify indeterminate forms and apply L'Hospital rules to evaluate the given limit.

2. Introduction: In calculus or in other branch of mathematics it is often occur that the value of an algebraic expression involving the Independent variable cannot be evaluated or its limiting value cannot be evaluated by putting the value of independent variable these are called the Indeterminate forms. To find the limiting value of an indeterminate form L'Hopital has suggested a method.

3. Key Definitions:

(1) L'Hospital Rule : If $f(x)$ and $g(x)$ are two functions of x which can be expanded by Taylor's series in the neighborhood of $x=a$ and if $f(a) = g(a) = 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

4. Important steps to be followed to solve the problem

a) Those limits which cannot be evaluated by using certain rules of limits are known as indeterminate forms.

b) There are seven types of indeterminate forms given as follows:

   (*i*) $\frac{0}{0}$ (*ii*) $\frac{\infty}{\infty}$ (*iii*) $0\times\infty$ (*iv*) $\infty-\infty$

   (*v*) 1^∞ (*vi*) 0^0 (*vii*) ∞^0

   these limits can be evaluated by using L'Hospital Rule.

c) To apply L'Hospital rule $\lim_{x\to a}\frac{f(x)}{g(x)}$ should be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, if not then first convert it in this form and then apply L'Hospital rule.

5. Sample Problems

(1) *Evaluate* $\lim_{x\to1}\frac{x^x-x}{x-1-\log x}$.

Solution : $\lim_{x\to1}\frac{x^x-x}{x-1-\log x}$ is of the form $\left[\frac{0}{0}\right]$ $= \lim_{x\to1}\frac{x^x(1+\log x)-1}{1-\frac{1}{x}}$ $\left[\frac{0}{0}\right]$

$$= \lim_{x \to 1} \frac{x^x\left(\frac{1}{x}\right) + x^x(1+\log x)^2}{\frac{1}{x^2}} = \frac{1+1}{1} = 2$$

(2) Evaluate $\lim_{x \to 0}\left[\frac{a}{x} - \cot\frac{x}{a}\right]$

Solution: Put $\frac{x}{a} = y$ $\quad \therefore$ *as* $x \to 0,\ y \to 0$

$$\lim_{x \to 0}\left[\frac{a}{x} - \cot\frac{x}{a}\right] = \lim_{y \to 0}\left[\frac{1}{y} - \cot y\right] \ [\infty - \infty]$$

$$= \lim_{y \to 0}\left[\frac{1}{y} - \frac{1}{\tan y}\right] = \lim_{y \to 0}\left[\frac{\tan y - y}{y \tan y}\right] \ \left[\frac{0}{0}\right]$$

$$= \lim_{y \to 0} \frac{\tan y - y}{y^2}\frac{y}{\tan y} = \lim_{y \to 0} \frac{\sec^2 y - 1}{2y} = 0$$

(3) Evaluate $\lim_{x \to a}\left(2 - \frac{x}{a}\right)^{\tan\left(\pi x/2a\right)}$

Solution: Let $\frac{x}{a} = t$, $\quad \therefore$ *as* $x \to a,\ t \to 1$

$$\lim_{x \to a}\left(2 - \frac{x}{a}\right)^{\tan\left(\pi x/2a\right)} = \lim_{t \to 1}(2-t)^{\tan\left(\pi t/2\right)}$$

Let $L = \lim_{t \to 1}(2-t)^{\tan\left(\pi t/2\right)}$

$$\therefore \log L = \lim_{t \to 1} \tan\left(\frac{\pi t}{2}\right)\log(2-t) \ [\infty \times 0]$$

Exercise 7

1. *Evaluate* $\lim_{x \to \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2xe}$. **Ans.:** $\frac{\pi^2}{2e}$

2. *Show that* $\lim_{x \to 0} \log_{\tan x} \tan 2x = 1.$

3. *Show that* $\lim_{x \to \infty}\left[\frac{1^{\frac{1}{x}} + 2^{\frac{1}{x}} + 3^{\frac{1}{x}} + 4^{\frac{1}{x}}}{4}\right]^{4x} = 24.$

4. Evaluate $\lim_{x \to \frac{\pi}{2}}(\cos x)^{\cos^2 x}$ Ans: 1

5. $\lim_{x \to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ Ans: $e^{\frac{-1}{6}}$

Let's check take away from the lecture

1. $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$ *is*
   (a) e^{-1} (b) e (c) 1 (d) *none*
2. $\lim_{x \to 0} x^{\sin x}$ *is*
   (a) 1 (b) 2 (c) 3 (d) *none*
3. $\lim_{x \to \frac{\pi}{2}} (\sec x)^{\cot x}$ *is*
   (a) 0 (b) 1 (c) 2 (d) *none*

Homework Problems for the day

1. Show that $\lim_{x \to 0} \log_x \sin x = 1.$
2. *Evaluate* $\lim_{x \to 0} \tan x \log x$. **Ans.:** 0
3. *Evaluate* $\lim_{x \to 0}\left[\frac{1}{x^2} - \cot^2 x\right]$. **Ans.:** $\frac{2}{3}$
4. Prove that $\lim_{x \to \infty}\left(\frac{a^{\frac{1}{x}} + b^{\frac{1}{x}} + c^{\frac{1}{x}} + d^{\frac{1}{x}}}{4}\right)^x = (abcd)^{\frac{1}{4}}$

Learning from the topic: Learner will be able to evaluate limiting value of indeterminate forms.

Indeterminate forms continued………

Lecture: 08

(1) Evaluate $\lim_{x \to 0} \frac{e^{2x} - (1+x)^2}{x \log(1+x)}$

Solution: $\lim_{x \to 0} \frac{e^{2x} - (1+x)^2}{x \log(1+x)} = \lim_{x \to 0} \frac{\left(1 + 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \ldots\right) - \left(1 + 2x + x^2\right)}{x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)}$

$$= \lim_{x\to 0} \frac{x^2\left(1+\frac{4x}{3}+\frac{2x^2}{3}+\dots\right)}{x^2\left(1-\frac{x}{2}+\frac{x^2}{3}-\dots\right)} = 1$$

(2) Evaluate $\lim_{x\to\infty} \frac{e^{1/x}+e^{2/x}+\dots\dots+e^{x/x}}{x}$ using series expansion.

Solution: $\lim_{x\to\infty} \frac{e^{1/x}\left[\left(e^{1/x}\right)^x-1\right]}{(e^{1/x}-1)}\frac{1}{x}$ (Numerator is a G.P. having x terms with $r = e^{1/x}$)

$$= \lim_{x\to\infty} \frac{e^{1/x}(e-1)}{x\left(e^{1/x}-1\right)} = \lim_{x\to\infty} \frac{e^{1/x}(e-1)}{x\left(1+\frac{1}{x}+\frac{1}{2x^2}+\dots\dots\dots-1\right)}$$

$$= \lim_{x\to\infty} \frac{e^{1/x}(e-1)}{\left(1+\frac{1}{x}+\frac{1}{2x^2}+\dots\dots\dots-1\right)} = \lim_{x\to\infty} \frac{e^{1/x}(e-1)}{\left(1+\frac{1}{2x}+\dots\dots\dots\right)} = e-1$$

(3) Expanding in the term of Maclaurin's series Evaluate $\left(\frac{sinhx}{x}\right)^{\frac{1}{x^2}}$.

Solution: Given that $y = \left(\frac{sinhx}{x}\right)^{\frac{1}{x^2}}$

Taking log on both sides $\log y = \left(\frac{sinhx}{x}\right)$

$$\Rightarrow \frac{1}{x^2}\log\left(\left(1+x^2\left(\frac{1}{3!}+\frac{x}{4!}+\frac{x^2}{5!}+\dots\right)\right)\right)$$

$$= \frac{1}{x^2}\left(\left(\frac{x^2}{3!}+\frac{x^3}{4!}\dots\right)-\frac{\left(\frac{x^2}{3!}+\frac{x^3}{4!}\dots\right)^2}{2!}+\frac{\left(\frac{x^2}{3!}+\frac{x^3}{4!}\dots\right)^3}{3!}\dots\right)$$

$$\Rightarrow \frac{1}{x^2}\left(\frac{x^2}{3!}+\frac{x^4}{4!}+\dots\dots\right)=\left(\frac{1}{3!}+\frac{x^2}{4!}+\dots\dots\right)=\frac{1}{6}$$

$$\Rightarrow \log y = \frac{1}{6} \Rightarrow y = e^{\frac{1}{6}}$$

Exercise 8

1 Evaluate $\lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^x$. Ans: $e^{\frac{2}{a}}$

2. *Prove that* $\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}}-e}{x} = -\frac{e}{2}$.

3. Prove that $\lim_{x\to 0}\frac{\log_{\sin x}\cos x}{\log_{\sin\frac{x}{2}}\cos\frac{x}{2}} = 4$

4. If $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3} = 1$. Find a and b. **Ans.:** $a = \frac{-5}{2}$, $b = \frac{-3}{2}$

Let's check take away from the lecture

1. $\underset{x\to\frac{\pi}{2}}{Lim}\ (\sin x)^{\tan x}$ *is*

(*a*) 1 (*b*) $\frac{1}{2}$ (*c*) $\frac{2}{3}$ (*d*) *none*

2. Value of $\frac{tanx}{tan3x}$

(a) 1 (b) 2 (c) 3 (d) 4

3. Value of $\frac{x^2}{e^{x^2}}$

(a) 1 (0) (c) -1 (d) 2

Homework Problems for the day

1. *Evaluate* $\lim_{x\to 0}\frac{\sin^{-1}x-x}{x^3}$. **Ans.:** $\frac{1}{6}$.

2. *Evaluate* $\lim_{x\to 0}\frac{\sin x\sin^{-1}x-x^2}{x^6}$. **Ans.:** 1/18

3 Evaluate $\lim_{x\to 0}\frac{1-x^x}{x\log x}$. Ans: -1

Learning from the topic: Learner will be able to expand the function and evaluate limiting value of indeterminate forms.

Convergence of sequence and Series

Lecture: 09

De 'Alembert's Ratio Test

1. Learning Objective:

1. Learners shall be able to test the convergence and divergence of Infinite series.
2. Learners shall be able to identify the infinite series and apply the De 'Alembert ratio test

2. Introduction:

A series is the sum of the terms of an infinite sequence of numbers. Given an infinite sequence (a_1, a_2, a_3...), the n th partial sum S_n is the sum of the first n terms of the sequence. That is, $S_n = \sum_{k=1}^{n} a_k$. A series is convergent if the sequence of its partial sums (S_1,S_2,S_3,...) tends to a finite limit , that means that the partial sums become closer and closer to a given number when the number of their terms increases.

3. Important Formulae:

De' Alembert's ratio test:

For the series $\sum an$ *find* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

1. if L < 1 the series is absolutely convergent (and hence convergent)
2. if L > 1 the series is divergent.
3. if L=1 the test is inconclusive. The series may be divergent, conditionally convergent or absolutely convergent.

4.Sample Problems:

1). Determine if the following series is convergent or divergent $\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)}$

Solution:

$$L = \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}\frac{4^{2n+1}(n+1)}{(-10)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-10)(n+1)}{4^2(n+2)}\right| = \lim_{n\to\infty}\left|\frac{(-10)(1+1/n)}{16(1+2/n)}\right|$$

$$= \frac{10}{16} = \frac{5}{8} < 1$$

So, by ratio test this series is convergent

2.Determine whether the series is convergent or divergent $\sum_{n=1}^{\infty}\frac{n!}{5^n}$.

Solution: $L = \lim_{n\to\infty}\left|\frac{(n+1)!5^n}{5^{n+1}\; n!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{5\; n!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{5}\right| = \infty > 1$

So by the ratio test series is divergent.

3). Determine whether the series is convergent or divergent $\sum_{n=0}^{\infty}\left(\frac{5n-3n^3}{7n^3+2}\right)^n$.

Solution:

$$L = \lim_{n\to\infty}\left|\left(\frac{5n-3n^3}{7n^3+2}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\left(\frac{5n-3n^3}{7n^3+2}\right)\right| = \lim_{n\to\infty}\left|\left(\frac{5/n^2-3}{7+2/n^3}\right)\right| = \frac{3}{7} < 1$$

Hence by root test series is convergent

Exercise 9

1) Test for convergence of the series whose nth term is $\frac{n^2}{2^n}$
2) Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n.3^n}$

   Ans: convergent if x>3, divergent if x<3
3) Test for convergence $\frac{1}{2}x + x^2 + \frac{9}{8}x^3 + x^4 + \frac{25}{32}x^5 + \ldots\infty$
4) Test the convergence of the series $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots.\infty$

Let's check take away from the lecture

1. In the ratio test if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ then series is

   (a) convergent (b) divergent (c)Oscillatory (d) cannot be said

2. Investigate the series whose nth term is $\sum_{n=1}^{\infty}\frac{3^n}{n^2}$

   (a) convergent (b) divergent (c)Oscillatory (d) cannot be said

3. Investigate the series whose nth term is $\sum_{n=1}^{\infty}\frac{n^3}{(\log 3)^n}$

   (a) convergent (b) divergent (c)Oscillatory (d) cannot be said

Homework Problems for the day

1) Discuss the convergence and divergence of the series $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \ldots\infty$
2) Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n \quad (x > 0)$

3) Test for convergence of the series whose nth term is $\sum_{n=1}^{\infty}\frac{n!2^n}{n^n}$

Learning from the topic: Learner will be able to test the convergence and divergence of an Infinite series using Ratio test and nth root test.

Lecture: 10

Cauchy's n^{th} Root Test

1. Learning Objective:

1. Learners shall be able to check whether the given series is convergent or divergent of Infinite series.
2. Learners shall be able to identify the infinite series and apply the Cauchy's nth root test.

2. Introduction:

A series is the sum of the terms of an infinite sequence of numbers. Given an infinite sequence $(a_1, a_2, a_3 \ldots)$, the nth partial sum S_n is the sum of the first n terms of the sequence. That is, $S_n = \sum_{k=1}^{n} a_k$. A series is convergent if the sequence of its partial sums $(S_1, S_2, S_3, \ldots)$ tends to a finite limit ; that means that the partial sums become closer and closer to a given number when the number of their terms increases.

3. Important Formulae:

Root Test:

Suppose we have the series $\sum a_n$ Define $L=\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}|a_n|^{\frac{1}{n}}$

1. If L<1 the series is absolutely convergent
2. If L>1 the series is divergent
3. If L=1 then test fails

Sample Problems:

1). Determine if the following series is convergent or divergent $\sum_{n=1}^{\infty}\frac{n^n}{3^{1+2n}}$

Solution:

$L=\lim_{n\to\infty}\left|\frac{n^n}{3^{1+2n}}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{3^{\frac{1}{n}+2}}=\frac{\infty}{3^2}=\infty>1$ so by root test this series is divergent

2). Determine whether the series is convergent or divergent $\sum_{n=0}^{\infty}\left(\frac{5n-3n^3}{7n^3+2}\right)^n$.

Solution:

$$L=\lim_{n\to\infty}\left|\left(\frac{5n-3n^3}{7n^3+2}\right)^n\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\left(\frac{5n-3n^3}{7n^3+2}\right)\right|=\lim_{n\to\infty}\left|\left(\frac{5/n^2-3}{7+2/n^3}\right)\right|=\frac{3}{7}<1$$

Hence by root test series is convergent

Exercise 10

1.Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{n^n}$

2.Discuss the convergence of the series $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots\infty$

3. Test the convergence of the series $\frac{1}{3}+(\frac{2}{5})^2+(\frac{3}{7})^3+\ldots.+(\frac{n}{2n+1})^n+\ldots$

Let's check take away from the lecture

1.In the root test if $\lim_{n\to\infty}|a_n|^{\frac{1}{n}}<1$ then series is

(a) convergent (b) divergent (c) Oscillatory (d) cannot be said

2.Investigate the series whose nth term is $\sum_{n=1}^{\infty}\frac{n^n}{2^{3n-1}}$

(a) convergent (b) divergent (c) Oscillatory (d) cannot be said

3.Investigate the series whose nth term is $\sum_{n=1}^{\infty}(\frac{n^2}{\log 2)^n}$

(a) convergent (b) divergent (c) Oscillatory (d) cannot be said

Homework Problems for the day

1. Examine the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

2.Examine the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{(\log n)^n}$

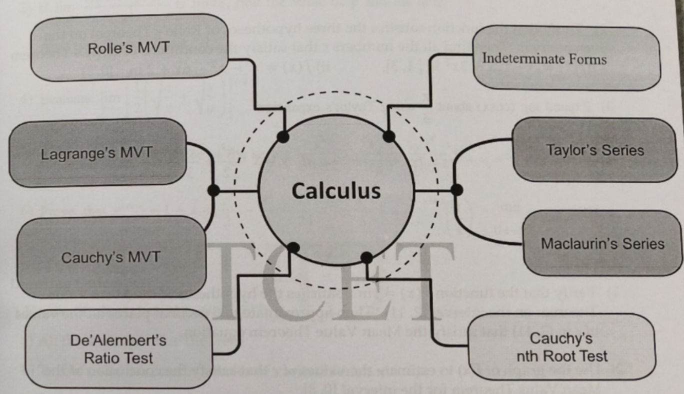
3. Test the convergence of the series $1+(\frac{1}{2})^2+(\frac{1}{3})^3+\ldots.$

Learning from the topic: Learner will be able to test the convergence and divergence of an Infinite series using n^{th} root test.

Tutorial Questions

1) Verify Roll's Theorem for the function f(x) = x (x+3) $e^{-x/2}$ in $-3 \le x \le 0$.
2) Verify Lagrange's mean value theorem for the function $f(x) = \log x \; in \; [1.e]$.
3) Verify Cauchy's mean value theorem for the function $f(x) = x^2 + 2 \; and g(x) = x^3 - 1$ in [1,2].
4) Expand $\sqrt{1 + \sin x}$ by standard expansion.
5) *Evaluate* $\lim_{x \to 0} \frac{\log(1+x^3)}{\sin^3 x}$
6) *Prove that* $\lim_{n \to 0} \sin x \log x = 0$
7) Prove that $\lim_{x \to \infty} [\frac{1}{x}]^{\frac{1}{x}} = 1$
8) Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$
9) Test for convergence of the series $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$.
10) Test for convergence of the series $\frac{1}{2} + \frac{2}{3} x + (\frac{3}{4})^2 x^2 + (\frac{4}{5})^3 x^3 + \cdots$

Concept Map



Problems for Self-assessment:

Level 1

1) Consider the function $f(x)=\sqrt{x-2}$. On what intervals are the hypotheses of the Mean Value Theorem satisfied?
   (A) [0, 2] (B) [1, 5] (C) [2, 7] (D) None of these

2) Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all the numbers c that satisfy the conditions of Rolle's Theorem
   i) $f(x) = 5 - 12x + 3x^2$ in [1, 3], ii) $f(x) = x^3 - x^2 - 6x + 2$ in [0, 3]

3) Expand log (cosx) about $\frac{\pi}{3}$ using Taylor's expansion

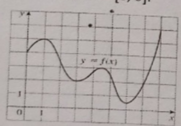
4) Prove that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots\ldots\ldots$ *if* $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\ldots\ldots\ldots$

5) Evaluate $\lim_{x\to 0}\left(\frac{1}{x}\right)^{2\sin x}$.

Level 2

1) Verify that the function $f(x) = \sin x$ satisfies the hypotheses of the Mean Value Theorem on the interval [2, 11]. Then approximate to 3 decimal places all the values of c in (2, 11) that satisfy the Mean Value Theorem equation.

2) Use the graph of **f(x)** to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval [0, 8].



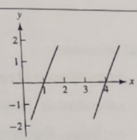
3) Show that $\log[1+\sin x] = x - \frac{x^2}{2} + \frac{x^3}{6} + \ldots\ldots\ldots\ldots$

4) Using Taylor's theorem evaluate upto 4 places of decimals $\sqrt{9\cdot 12}$

5) Prove that $\lim_{x\to 1}\left[\frac{1}{\log x} - \frac{x}{x-1}\right] = -\frac{1}{2}$

Level 3

1) f is a continuous function. A portion of the graph of f is shown to the right. Explain why f' must have a root in the interval (1, 4).



2) Prove that $\sec^{-1}\left(\frac{1}{1-2x^2}\right) = 2\left\{x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots\right\}$..

3) If $\lim_{x\to 0}\frac{\sin 2x + p\sin x}{x^3}$ is finite, find the value of p and the limit

   Ans: p = -2 and limit -1

4) Evaluate $\lim_{x\to a}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}}$

5) Discuss the convergence of the series $\frac{x^2}{2\log 2} + \frac{x^3}{3\log 3} + \frac{x^4}{4\log 4} + \ldots$

6) Prove that $e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 - \frac{x^5}{5} + \ldots\ldots$

Learning Outcomes:

1.Know: Student should be able to

a) All three mean value theorem

b) Taylor's and McLaurin's series of various function

c) Define limits of various types of standard function, indeterminate forms, L'Hospital rules.

2. Comprehend: Student should be able to comprehend all three mean value theorem and expansion of function in the form of Taylor's and McLaurin's series and L'Hsopitals rule.

3. Apply, analyze and synthesize: Student should be able to

Apply mean value theorem to find the roots of an equation in an interval and apply Taylor's series to find the approximate value of function and Solve the problems of indeterminate forms with help of L' Hospital rule.

Digital references:

1. https://math.libretexts.org/Bookshelves/Calculus/Map%3A_Calculus__Early_Transcendentals_(Stewart)/04%3A_Applications_of_Differentiation/4.02%3A_The_Mean_Value_Theorem
2. https://openstax.org/books/calculus-volume-1/pages/4-4-the-mean-value-theorem
3. https://en.wikipedia.org/wiki/Mean_value_theorem

Add to Knowledge: The application of mean value theorem is in the calculation of average value of any function or the root calculation of any polynomial. Whereas the expansion of any function gives the quantize factorization of any function and Indeterminate forms arise in many areas of Mathematics and Engineering where to calculate the value of the function we use the technique of Indeterminate forms.

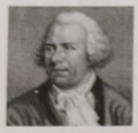
Self-Evaluation

Name of student: **Class & Div.:** **Roll No:**

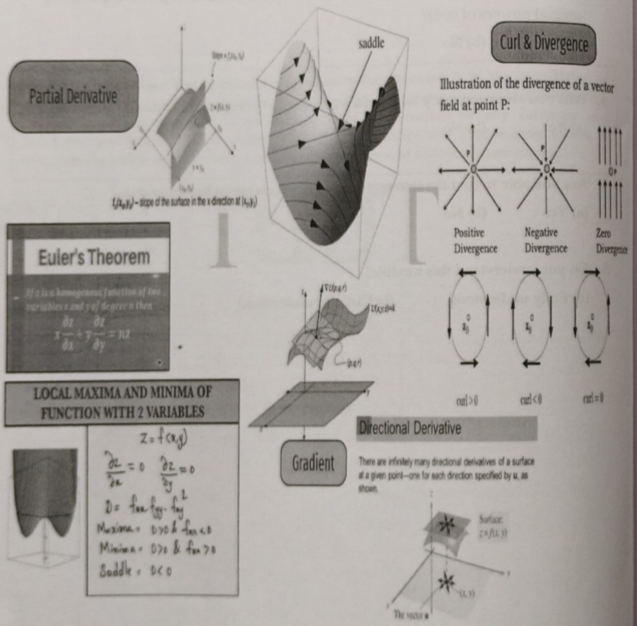
1. Are you able to evaluate the different forms of limit using L' Hospital Rule?
   (a) Yes (b) No
2. Do you understand how to expand function of one variable in positive ascending integral powers of (x-a)?
   (a) Yes (b) No
3. Will you able to identify how to fit a curve for a given bivariate data?
   (a) Yes (b) No
4. Are you able to find the two regression lines for a given bivariate data?
   (a) Yes (b) No
5. Do you understand this module?
   (a) Fully understood (b) Partially understood (c) Not at all

Module 2 : Multivariable Calculus (Differentiation)

Leonhard Euler



Jean le Rond d'Alembert



Module 2 : Multivariable Calculus (Differentiation)

1. Motivation:

Multivariable calculus is taken as extension of calculus of one variable to the calculus of function of several variables where we need to differentiate or integrate the function of several variable with respect to one of the variables. Differentiating the function of several variable with respect to one of the variables is called the partial differentiation. The Partial differentiation emerges as a need whenever we want to determine the rate of change of function of several variable with respect to one of the variables. Many important Physical processes in nature are governed by partial differentiation specially in fluid dynamics, optical and digital communication and signal processing one of the direct applications of partial differentiation is evaluation of maxima and minima of the function of several variables and in the evaluation of gradient divergence and curl.

2. Syllabus:

Lecture No.	Title	Duration (Hrs.)	Self-study (Hrs.)
11	Partial derivatives (first order)	1	2
12	Partial derivatives (higher order)	1	2
13	Composite function (Total Differentiation)	1	2
14	Composite function (Total Differentiation)	1	2
15	Euler's theorem on homogeneous functions in two variables	1	2
16	Maxima, minima and saddle point	1	2
17	Maxima, minima and saddle point continued	1	2
18	Gradient	1	2
19	Gradient continues (Directional Derivatives)	1	2
20	Divergence	1	2
21	Curl	1	2

3. Prerequisite:

Students are expected to know the concept of Function, the concept of vector algebra, limits, continuity, and derivative of function of one independent variable, rules and formulae of differentiation of function of one independent variable which are introduced to them in 12th standard.

4. Learning objectives: Learners shall be able to

1. Evaluate the partial derivative of function of Multiple variable
2. Evaluate the gradient of a scalar valued function to find tangent and Normal to any curve
3. Evaluate the problems on homogeneous functions in two variables using Euler's Theorem
4. Evaluate the Curl of a vector valued function and identify conservative fields.
5. Evaluate divergence of a vector field and find the source, sink, solenoidal fields.
6. Evaluate the local maxima and minima of function of two variable.

Partial Differentiation (First order)

Lecture: 11

1. Learning Objective:

Student shall be able to find partial derivative of standard functions with respect to a variable.

2. Introduction of Partial Derivative:

- If y=f(x) is a function where x is independent variable and y is dependent variable. when we change x, correspondingly y will change then to find the rate of change of y with respect to x we calculate derivative of y with respect to x, which is called ordinary derivative.
- If we have a function like z=f(x, y) where one particular variable z which is called dependent variable is depending on more than one variable x and y then to find the rate of change of z with respect to either x or y is called partial derivative of z with respect to that variable.
- e.g. consider the example of area of a rectangle since area (A)=length(x)*Breadth (y)$\Rightarrow$A=x y i.e. area is a function of two variables x, y Now area of the rectangle can vary three ways (1) by changing the length and breadth both (2) by changing the length only and keeping the breadth fixed (3) by changing the breadth only and keeping the length fixed. In the case (2) and (3) if we want to evaluate the rate of change of area with respect to length only that is called partial derivative of A with respect to x whereas rate of change of area with respect to breadth is called the partial derivative of A with respect to y represented respectively by $\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}$.
- The rate of change of Area "A" with respect to the change in length and breadth both is called total derivative A with respect to x and y.

3. Key Notations:

Partial derivative of z w.r.t. to x of first order: $\frac{\partial z}{\partial x}$ *or* z_x

4. Key Definitions:

(1) Functions of two or more variables : If z has one definite value for each pair of values of x and y then z is called a function of two variables x and y. We denote it by $z = f(x, y)$. Here z is the dependent variable and x and y are independent variables.

(2) Partial Derivative: The partial derivative of $z = f(x, y)$ w.r.t. x is the ordinary derivative of z w.r.t. x treating y as constant.

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \right)$$

Similarly, $$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \left(\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} \right)$$

The partial derivative is also known as directional derivatives it is the derivative with respect to one variable so it can be understood as rate of change of variable in one direction.

5. Important Steps to be followed for solving the problems:

1. While finding the partial derivative of z with respect to x in the function z=f (x, y) we treat x, x as variable and another variable y as constant.
2. If you are differentiating with one variable, then partial derivative with respect to that variable is evaluated using the formula of ordinary derivative.

6. Sample Problems:

(1) If $u = x^y$ then, find $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$.

Solution: Given that $u = x^y$

Now $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^y\right) = y\, x^{y-1}$ (following x^n formula for differentiation)

Again $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^y\right) = x^y \log x$ (following a^x formula for differentiation)

(2) If $u = \log(\tan x + \tan y)$ then, prove that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2$.

Solution: Given that $u = \log(\tan x + \tan y)$

Now $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\log(\tan x + \tan y)) = \frac{1}{\tan x + \tan y} \frac{\partial}{\partial x}(\tan x + \tan y)$$

$$= \frac{1}{\tan x + \tan y}\left(\frac{\partial}{\partial x}\tan x + \frac{\partial}{\partial x}\tan y \right)$$

$$= \frac{1}{\tan x + \tan y}\left(\sec^2 x + 0\right)$$

$$\therefore \ \sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x \times \sec^2 x}{\tan x + \tan y} = \frac{2 \tan x}{\tan x + \tan y}$$

$$\text{similarly} \quad \sin 2y \frac{\partial u}{\partial y} = \frac{2 \tan y}{\tan x + \tan y}$$

On adding we get,

$$\text{LHS} = \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = \frac{2 \tan x}{\tan x + \tan y} + \frac{2 \tan y}{\tan x + \tan y} = 2 = \text{RHS}$$

(3) If $z(x+y) = (x-y)$ find $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$.

Solution: Given that $z(x+y) = (x-y) \Rightarrow z = \frac{(x-y)}{(x+y)}$

$$\therefore \frac{\partial z}{\partial x} = \frac{(x-y)-(x+y)}{(x+y)^2} = \frac{-2y}{(x+y)^2} \ \& \ \frac{\partial z}{\partial y} = \frac{(x-y)+(x+y)}{(x+y)^2} = \frac{2x}{(x+y)^2}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = \left[\frac{-2y-2x}{(x+y)^2}\right]^2 = \frac{4}{(x+y)^2}.$$

Exercise 11

1. Evaluate $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$ *for* $u = e^{x^y}$. **Ans:** $\frac{\partial u}{\partial x} = e^{x^y} yx^{y-1}, \frac{\partial u}{\partial y} = e^{x^y} x^y \log x$

2. If $u = (1-2xy+y^2)^{-\frac{1}{2}}$, then show that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$.

3. If $u = \log(\tan x + \tan y + \tan z)$, then show that $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2$.

Let's check take away from lecture

1. *If* $u = \frac{z}{x} + \frac{y}{z}$ *then find* $\frac{\partial u}{\partial x}$.

(a) $\frac{1}{x}$ (b) $\frac{-z}{x^2}$ (c) $\frac{-y}{z^2}$ (d) $\frac{1}{z}$

2. *If* $u = x^2y + 2xy + xy^2$ *then find* $\frac{\partial u}{\partial z}$.

(a) $2xy + 2y + y^2$ (b) 0 (c) $x^2 + 2x + y^2$ (d) *None*

3. *If* $u = x^y$ *then find* $\frac{\partial u}{\partial x}$.

(a) yx^{y-1} (b) $x^y \log x$ (c) $y^x \log y$ (d) $x y^{x-1}$

4. *If* $u = \log\left(\frac{x+y}{x-y}\right)$ *then find* $\frac{\partial u}{\partial y}$.

(a) $\frac{2y}{x^2-y^2}$ (b) $\frac{2y}{y^2-x^2}$ (c) $\frac{2x}{x^2-y^2}$ (d) $\frac{2x}{y^2-x^2}$

5. *If* $u = e^{\frac{y}{x}}$ *then find* $\frac{\partial u}{\partial y}$.

(a) $e^{\frac{y}{x}}\frac{y}{x^2}$ (b) $e^{\frac{y}{x}}\frac{x}{y^2}$ (c) $-e^{\frac{y}{x}}\frac{y}{x^2}$ (d) $-e^{\frac{y}{x}}\frac{x}{y^2}$

Homework Problems for the day

1. If $u = x^3y + e^{xy^2}$, determine $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ **Ans:** $\frac{\partial u}{\partial x} = 3x^2y + y^2e^{xy^2}, \frac{\partial u}{\partial y} = x^3 + 2xye^{xy^2}$

2. If $u(x+y) = x^2 + y^2$, then show that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.

3. *If* $u = \log\left(\frac{x}{y}\right)$ *then find* $u_x + u_y$ **Ans:** $u_x + u_y = \frac{y-x}{xy}$

Learning Outcome: Students shall be able to find the partial derivative of the function

Lecture :12

1.Learning Objective:

Student shall be able to find partial derivative of higher order standard functions with respect to a variable.

2.Introduction:

- To differentiate a function of multiple variables with any one variable successively more than once is called higher order partial derivative of the function with respect to any one variable
- If a function z=f(x,y) is a function of two variable x, y and if it is differentiable successively twice with respect to x or y then its second order derivative is representing the similar things as second order ordinary derivative represents acceleration if first order derivative represents the speed in case of ordinary derivative

3.Key Notations : Partial derivatives of second order: If $z = f(x, y)$ then $\frac{\partial^2 f}{\partial x^2}$ *or* f_{xx}, $\frac{\partial^2 f}{\partial x \partial y}$ *or* f_{xy}, $\frac{\partial^2 f}{\partial y \partial x}$ *or* f_{yx}, $\frac{\partial^2 f}{\partial y^2}$ *or* f_{yy} are called 2nd order partial derivatives.

4.Sample Problems:

(1) If $u = x^2y + e^{xy^2}$ then, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Solution: Given that $u = x^2 y + e^{xy^2}$

Now

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^2 y + e^{xy^2}\right) = \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial y}(e^{xy^2}) = x^2(1) + e^{xy^2}\frac{\partial}{\partial y}(x y^2) = x^2 + e^{xy^2} x(2y)$$

Again, $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 + 2xy\, e^{xy^2}\right) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(2xy\, e^{xy^2})$$

$$= (2x) + 2y(1)e^{xy^2} + 2xy e^{xy^2}\frac{\partial}{\partial x}(xy^2) = 2x + 2y\, e^{xy^2} + e^{xy^2}\, 2xy\,(y^2) \ldots\ldots(1)$$

Now $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2 y + e^{xy^2}\right) = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial x}(e^{xy^2}) = (2x)y + e^{xy^2}\frac{\partial}{\partial x}(xy^2)$$

$$= 2xy + e^{xy^2}(y^2)$$

Again, $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy + y^2 e^{xy^2}\right) = \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial y}(y^2 e^{xy^2})$$

$$= 2x(1) + (2y)e^{xy^2} + e^{xy^2}\frac{\partial}{\partial y}(xy^2) = 2x + 2y\, e^{xy^2} + y^2 e^{xy^2} x(2y) \ldots\ldots(2)$$

From (1) and (2) we get $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

(2) If $z = \log(e^x + e^y)$ then show that rt-s²=0 where $r = \dfrac{\partial^2 z}{\partial x^2}, t = \dfrac{\partial^2 z}{\partial y^2}, s = \dfrac{\partial^2 z}{\partial x \partial y}$.

Solution: Given $z = \log\left(e^x + e^y\right)$ ....(1) differentiate (1) partially w.r.to x,

$\dfrac{\partial z}{\partial x} = \dfrac{1}{e^x + e^y} e^x$ ....(2) differentiate (2) partially w.r.to x, we get

$$\frac{\partial^2 z}{\partial x^2} = r = \frac{e^{x+y}}{\left(e^x + e^y\right)^2}$$

differentiate (2) partially w.r.to y $\dfrac{\partial^2 z}{\partial x \partial y} = s = \dfrac{-e^{x+y}}{\left(e^x + e^y\right)^2}$ differentiate (1) partially w.r.to y,

we get, $\dfrac{\partial z}{\partial y} = \dfrac{1}{e^x + e^y} e^y$ ....(3) differentiate (3) partially w.r.to y, we get

$$\frac{\partial^2 z}{\partial y^2} = t = \frac{e^{x+y}}{\left(e^x + e^y\right)^2}$$

$$\therefore rt - s^2 = \frac{e^{x+y}}{\left(e^x + e^y\right)^2} \cdot \frac{e^{x+y}}{\left(e^x + e^y\right)^2} - \left(-\frac{e^{x+y}}{\left(e^x + e^y\right)^2}\right)^2 \Rightarrow rt - s^2 = 0$$

(3) If $u = x^y$ then evaluate $\dfrac{\partial^3 u}{\partial x^2 \partial y}$.

Solution: Given that $u = x^y$ differentiate it with respect to y $\Rightarrow \dfrac{\partial u}{\partial y} = x^y \log x$ differentiate it with respect to x $\Rightarrow \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{x^y}{x} + y x^{y-1}\log x$ differentiate it with respect to x $\Rightarrow$

$$\frac{\partial^2 u}{\partial x^2 \partial y} = (y-1)x^{y-2} + y x^{y-2} + y(y-1)x^{y-2}\log x$$

(4) If $u = \tan^{-1}\dfrac{y}{x}$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

Solution: Given that $u = \tan^{-1}\dfrac{y}{x} \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{1}{1 + y^2/x^2}\left(\dfrac{-y}{x^2}\right) = -\dfrac{y}{x^2 + y^2}$

$\dfrac{\partial u}{\partial y} = \dfrac{1}{1 + y^2/x^2}\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2 + y^2}$ now $\dfrac{\partial^2 u}{\partial x^2} = -y\dfrac{-1}{(x^2 + y^2)^2}2x$ and $\dfrac{\partial^2 u}{\partial y^2} = x\dfrac{-1}{(x^2 + y^2)^2}2y$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(5) If $z = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right)$ then show that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x^2 - y^2}{x^2 + y^2}$.

Solution: Given that $z = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right)$ differentiate it with respect y

$$\frac{\partial z}{\partial y} = \frac{x^4}{x^2 + y^2}\left(\frac{1}{x}\right) - 2y \tan^{-1}\left(\frac{x}{y}\right) - \frac{y^4}{x^2 + y^2}\left(-\frac{x}{y^2}\right) \Rightarrow = \frac{x^3}{x^2 + y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2 x}{x^2 + y^2}$$

$$= \frac{x^3 + y^2 x}{x^2 + y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right) \Rightarrow \frac{x(x^2 + y^2)}{(x^2 + y^2)} - 2y\tan^{-1}\left(\frac{x}{y}\right) \Rightarrow x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

differentiate it with respect to x once again

$$\frac{\partial^2 z}{\partial x \partial y} = 1 - \frac{2y^3}{x^2 + y^2}\left(\frac{1}{y}\right) \Rightarrow 1 - \frac{2y^2}{x^2 + y^2} \Rightarrow \frac{x^2 + y^2 - 2y^2}{x^2 + y^2} \Rightarrow \frac{x^2 - y^2}{x^2 + y^2}$$

(6) Show that $z = f(x + at) + \phi(x - at)$ is the solution of $a^2 \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial t^2}$ for all f and ϕ.

Solution: Given that $z = f(x + at) + \phi(x - at) \Rightarrow \dfrac{\partial z}{\partial x} = f'(x + at) + \phi'(x - at)$

$\Rightarrow \dfrac{\partial^2 z}{\partial x^2} = f''(x + at) + \phi''(x - at)$ ...(1) further $\dfrac{\partial z}{\partial x} = af'(x + at) - a\phi'(x - at)$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = a^2 f'(x + at) + a^2\phi'(x - at) \ldots\ldots(2)$$

From (1) and (2) we get $a^2 \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial t^2}$.

Exercise 12

1. If $z = x^y + y^x$, evaluate $\dfrac{\partial^2 z}{\partial y \partial x}$ **Ans:** $\dfrac{\partial^2 z}{\partial y \partial x} = yx^{y-1}\log x + x^{y-1} + y^{x-1} + xy^{x-1}\log y$

2. If $z = \tan(y+ax) + (y-ax)^{\frac{3}{2}}$, then show that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$.

3. If $z^3 - zx - y = 0$, then show that
$\dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{(3z^2+x)}{(3z^2-x)^3}$

4. If $u = \log\left(x^3+y^3-x^2y-xy^2\right)$, then show that
$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}.$$

Let's check take away from lecture

1. If $u = \log(x+y)$ *then find* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

   (a) 0 (b) $\dfrac{2}{(x+y)^2}$ (c) $\dfrac{-4}{(x+y)^2}$ (d) *None*

2. If $u = \log(x^2+y^2)$, evaluate $\dfrac{\partial^2 u}{\partial x \partial y}$.

   (a) 1 (b) $\dfrac{2}{\left(x^2+y^2\right)}$ (c) $\dfrac{-4xy}{\left(x^2+y^2\right)^2}$ (d) *None*

3. If $u = 3(ax+by+cz)^2 - (x^2+y^2+z^2)$ and $a^2+b^2+c^2 = 1$ then evaluate the value of
$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

   (a) 0 (b) $ax^2+by^2+cz^2$ (c) $x^2+y^2+z^2$ (d) *None*

4. *If* $u = x^3y + e^{xy^2}$ evaluate $\dfrac{\partial^2 u}{\partial x \partial y}$.

   (a) $3x^2 + 2ye^{xy^2}(1+xy^2)$ (b) $2ye^{xy^2}(1+xy^2)$ (c) $2ye^{xy^2}(1-xy^2)$ (d) $3x^2 - 2ye^{xy^2}(1+xy^2)$

5 If $u = e^{xyz}$, *evaluate* $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$

   (a) $\left(1+3xyz+x^2y^2z^2\right)e^{xyz}$ (b) $\left(1-3xyz+x^2y^2z^2\right)e^{xyz}$ (c) $\left(1+3xyz-x^2y^2z^2\right)e^{xyz}$ (d) *None*

Homework Problems for the day

1. If $u = x f(x+y) + y\phi(x+y)$, then show that $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

2. If $a^2x^2 + b^2y^2 = c^2z^2$, then evaluate $\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2} + \dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2}$. **Ans.:** $\dfrac{1}{c^2 z}$.

3. Find the value of n so that $v = r^n\left(3\cos^2\theta - 1\right)$ satisfies the equation
$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = 0 \qquad Ans: n = 2 \text{ or } -3$$

4. If u = $\log\left(x^3+y^3+z^3-3xyz\right)$, prove that $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = -\dfrac{9}{(x+y+z)^2}$

5. If $\theta = t^n e^{-r^2/4t}$ then, find the value of n so that
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t} \qquad Ans: n = -3/2$$

Learning Outcome: Students shall be able to find the partial derivative of the function

Composite function - Total Differentiation

Lecture: 13

1. **Learning Objective:** Student shall be able to find partial derivative of composite functions and total derivative of Implicit function with respect to a variable.
2. **Introduction:**

- Composite function is the composition of two function. In this concept one function is defined over other function.
- To find the partial derivative of a composite function is not always simple but follows the chain rule in an extended form.

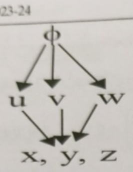
3. **Key Definitions:**

- **(1) Chain Rule**: If $z = f(u)$ where u is again a function of two variables x and y i.e. $u = \phi(x,y)$, then z becomes composite function of x and y.

$$\begin{tikzcd}[row sep=small, column sep=tiny] & z \arrow[d] & \\ & u \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd}$$

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} \quad or \quad \frac{df}{du}\frac{\partial u}{\partial x} = f'(u)\frac{\partial u}{\partial x}$$
$$\frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} \quad or \quad \frac{df}{du}\frac{\partial u}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

(2) **Composite Function:** Consider the function $\phi = f(u,v,w)$ where u, v, w are the functions of x, y, z. i.e. $\phi \to u,v,w \to x,y,z$. Then partial derivatives of ϕ w.r.t. x, y and z are given as,



$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial x}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial y}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w}\frac{\partial w}{\partial z}$$

(3)Total Differential : If $u = f(x,y)$ and $x = \phi(t)$ and $y = \psi(t)$ then total derivative of u is given as $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.

4. Important Steps to be followed for solving the problems:

- While finding partial derivative of composite function it is required that to decide first what are dependent variable and what are independent variables.
- Draw dependency cart before finding the derivative and then use chain rule.

5.Sample problems:

(1) If $u = x^2 + y^2 + z^2$, where $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$ than show that $\frac{du}{dt} = 4e^{2t}$.

Solution: Given $u = x^2 + y^2 + z^2$ *Put* $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$

$$\therefore u = \left(e^t\right)^2 + \left(e^t \sin t\right)^2 + \left(e^t \cos t\right)^2 = e^{2t}\left(1 + \sin^2 t + \cos^2 t\right)$$

$$\therefore u = 2e^{2t}$$

Diff. above equation w.r.to t, we get $\frac{du}{dt} = 4e^{2t}$

(2) If z = f (x, y) and $x = e^u + e^{-v}$ *and* $y = e^{-u} - e^v$, then show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

Solution: Here $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$ ....(1)

Now $x = e^u + e^{-v} \Rightarrow \frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(e^u + e^{-v}\right) = e^u + 0$

and $\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(e^u + e^{-v}\right) = 0 + e^{-v}$

Similarly $\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(e^{-u} - e^{-v}\right) = -e^u + 0$

and $\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(e^{-u} - e^{-v}\right) = 0 - (-e^{-v})$

On substituting in (1) we get $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}(-e^{-u})$ ....(2)

Similarly $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$ ....(3)

Hence subtracting (2) from (1),

$$\text{LHS} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial x} + e^v\frac{\partial z}{\partial y}$$

$$= \left(e^u + e^{-v}\right)\frac{\partial z}{\partial x} - \left(e^{-u} - e^v\right)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \text{RHS}$$

(3) If $x = u + v + w, y = uv + vw + uw, z = uvw$ and ϕ is a function of x, y and z prove that $x\frac{\partial \phi}{\partial x} + 2y\frac{\partial \phi}{\partial y} + 3z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$

Solution: Given $x = u + v + w \Rightarrow \frac{\partial x}{\partial u} = 1, \frac{\partial x}{\partial v} = 1, \frac{\partial x}{\partial w} = 1$ ....(1)

Given $y = uv + vw + uw \Rightarrow \frac{\partial y}{\partial u} = v, \frac{\partial y}{\partial v} = w, \frac{\partial y}{\partial w} = u$ ....(2)

Given $z = uvw \Rightarrow \frac{\partial z}{\partial u} = vw, \frac{\partial z}{\partial v} = uw, \frac{\partial z}{\partial w} = uv$ ....(3)

Here $\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial u}$

$\therefore \frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}.1 + \frac{\partial \phi}{\partial y}.(v + w) + \frac{\partial \phi}{\partial z}.vw$ ....(4)

$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial v}$

$\therefore \frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}.1 + \frac{\partial \phi}{\partial y}.(u + w) + \frac{\partial \phi}{\partial z}uw$ ....(5)

$\frac{\partial \phi}{\partial w} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial w}$

$\therefore \frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}.1 + \frac{\partial \phi}{\partial y}.(u + v) + \frac{\partial \phi}{\partial z}uv$ ....(6)

$RHS = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$

From equations (4), (5), (6)

$$RHS = u\left(\frac{\partial \phi}{\partial x}.1 + \frac{\partial \phi}{\partial y}.(v + w) + \frac{\partial \phi}{\partial z}.vw\right) + v\left(\frac{\partial \phi}{\partial x}.1 + \frac{\partial \phi}{\partial y}.(u + w) + \frac{\partial \phi}{\partial z}uw\right)$$

$$+ w\left(\frac{\partial \phi}{\partial x}.1 + \frac{\partial \phi}{\partial y}.(u + v) + \frac{\partial \phi}{\partial z}uv\right)$$

$$= (u + v + w)\frac{\partial \phi}{\partial x} + 2(uv + vw + uw)\frac{\partial \phi}{\partial y} + 3uvw\frac{\partial \phi}{\partial z}$$

$$=x\frac{\partial\phi}{\partial x}+2y\frac{\partial\phi}{\partial y}+3z\frac{\partial\phi}{\partial z}.$$

Exercise 13

1. If $x=\sqrt{vw},\ y=\sqrt{wu},\ z=\sqrt{uv}$ and φ is a function of x, y & z, then show that
$$x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}+z\frac{\partial\phi}{\partial z}=u\frac{\partial\phi}{\partial u}+v\frac{\partial\phi}{\partial v}+w\frac{\partial\phi}{\partial w}.$$

2. If $z=f(u,v),\ u=\log(x^2+y^2),\ v=\frac{y}{x}$, than show that
$$x\frac{\partial z}{\partial y}-y\frac{\partial z}{\partial x}=(1+v^2)\frac{\partial z}{\partial v}.$$

3. If $z=f(u,v),\ u=lx+my, v=ly-mx$, then show that:
$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$$

4. If u = f(r) and $r^2 = x^2 + y^2 + z^2$ prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=f''(r)+\frac{2}{r}f'(r)$

Let's check take away from lecture

1. If $u=xyz,\ x=at^2, y=2at, z=t$ Evaluate $\frac{du}{dt}$.

   (a) $16at^2$ (b) $8a^2t^3$ (c) $16a^2t^3$ (d) None

2. If $z=u^2+v^2,\ u=at^2, v=2at$, determine $\frac{dz}{dt}$.

   (a) $4a^2t^3+8a^2t$ (b) $8a^2t^3+6a^2t$ (c) $8a^2t^3+4a^2t$ (d) None

3. If $z=u+v,\ u=at, v=2at$, determine $\frac{dz}{dt}$.

   (a) $3a$ (b) $4a$ (c) $6a$ (d) $8a$

4. If $z=e^{xy}, x=t\cos t,\ y=t\sin t$, find $\frac{dz}{dt}$ at $t=\frac{\pi}{2}$.

   (a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{2}$ (c) $-\frac{\pi^2}{2}$ (d) $-\frac{\pi^2}{4}$

5. If $z=\sin^{-1}(x-y), x=3t,\ y=4t^3$, determine $\frac{dz}{dt}$

   (a) $\frac{3}{\sqrt{1+t^2}}$ (b) $\frac{12}{\sqrt{1-t^2}}$ (c) $\frac{12}{\sqrt{1+t^2}}$ (d) $\frac{3}{\sqrt{1-t^2}}$

Homework Problems for the day

1. If z = f (x, y), x = u coshv, y = u sinhv, then show that
$$\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial u}\right)^2-\frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2$$

Hint: Draw the dependency chart and then find the derivative accordingly

2. If $w=\phi(u,v)$, $u=x^2-y^2-2xy,\ v=y$, then show that
$$(x+y)\frac{\partial w}{\partial x}+(x-y)\frac{\partial w}{\partial y}=(x-y)\frac{\partial w}{\partial v}.$$

3. If z = f(x, y) and $x=u\cos\alpha-t\sin\alpha,\ \ y=u\sin\alpha+t\cos\alpha$, then show that
$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$

Hint: Draw the dependency chart and then find the derivative accordingly

4. If z = f (x, y), $x=e^u\cos v,\ y=e^u\sin v$, show that

   (i) $x\frac{\partial z}{\partial v}+y\frac{\partial z}{\partial u}=e^{2u}\frac{\partial z}{\partial y}$

   (ii) $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right].$

5. If $u=x^2-y^2, v=2xy$ *and* $z=f(u,v)$, *prove that*
$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4(u^2+v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]$$

Learning Outcome: Students shall be able to find the partial derivative of composite function

Composite function -Total Differentiation continued....

Lecture :14

Sample problems:

(1) If $u=f\left[e^{y-z},e^{z-x},e^{x-y}\right]$, *then show that* $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$

Solution: *Let* $r=e^{x-y}, s=e^{y-z}, t=e^{z-x}$ *then* $u=f(r,s,t)$

$$\therefore\ \frac{\partial r}{\partial x}=e^{x-y}(1)\qquad \frac{\partial s}{\partial x}=0\qquad \frac{\partial t}{\partial x}=e^{z-x}(-1)$$

$$\frac{\partial r}{\partial y}=e^{x-y}(-1)\qquad \frac{\partial s}{\partial y}=e^{y-z}(1)\qquad \frac{\partial t}{\partial y}=0$$

$$\frac{\partial r}{\partial z}=0\qquad \frac{\partial s}{\partial z}=e^{y-z}(-1)\qquad \frac{\partial t}{\partial z}=e^{z-x}(1)$$

$$\therefore\ \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$
$$=\frac{\partial u}{\partial r}e^{x-y}+\frac{\partial u}{\partial s}0+\frac{\partial u}{\partial t}\left(-e^{z-x}\right)=\frac{\partial u}{\partial r}e^{x-y}-\frac{\partial u}{\partial t}e^{z-x}\ \ldots\ldots\ldots\ldots\ (1)$$

$$Ily \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}(-e^{x-y}) + \frac{\partial u}{\partial s}e^{y-z} + \frac{\partial u}{\partial t}(0) = -\frac{\partial u}{\partial r}e^{x-y} + \frac{\partial u}{\partial s}e^{y-z} \text{ .... (2)}$$

$$Ily \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-e^{y-z}) + \frac{\partial u}{\partial t}e^{z-x} = -\frac{\partial u}{\partial s}e^{y-z} + \frac{\partial u}{\partial t}e^{z-x} \text{ .... (3)}$$

Adding (1),(2) & (3) *we get* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Exercise 14

1. If $u = f(2x-3y, 3y-4z, 4z-2x)$ than show that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0.$

2. If u = $f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, than show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

3. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Let's check take away from lecture

1. If $z = u^2 - v^2$, $u = at^2$, $v = 2at$, determine $\frac{dz}{dt}$.

   (a) $4a^2t^3 - 8a^2t$ (b) $8a^2t^3 - 6a^2t$ (c) $8a^2t^3 - 4a^2t$ (d) None

2. If $z = e^{x+y}$, $x = t\cos t$, $y = t\sin t$ than $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$.

   (a) $e^{\frac{\pi}{2}}\left(1-\frac{\pi}{2}\right)$ (b) $e^{\frac{\pi}{2}}\left(1+\frac{\pi}{2}\right)$ (c) $e^{\frac{\pi}{2}}\left(\frac{\pi}{2}-1\right)$ (d) None

Homework Problems for the day

1. If u = $f(x^2-y^2, y^2-z^2, z^2-x^2)$, then show that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0.$
2. If u = f(x -y, y - z, z - x), then show that $u_x + u_y + u_z = 0$.
3. If $f(lx+my+nz, x^2+y^2+z^2) = 0$, then show that $(ly-mx)+(ny-mz)\frac{\partial z}{\partial x} + (lz-nx)\frac{\partial z}{\partial y} = 0$

Learning Outcome: Students shall be able to find the partial derivative of composite function

Euler's Theorem on Homogeneous Function in two variables

Lecture :15

1. **Learning Objective:** Student shall be able to evaluate the problems on homogeneous function using Euler's theorem.
2. **Introduction:**

   Sometimes to evaluate first order standard expression of the type $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ and second order standard expression of the type $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2}$ where z is a function of two variables x and y, the direct differentiation of the function becomes tedious and is depending upon the complexity of the function. Euler's theorem on homogeneous function is a mathematical tool to evaluate such expressions without performing actual differentiation. To implement Euler's theorem enough to check homogeneity of the function. This module deals with Euler's theorem on homogeneous functions in two variables.
3. **Key Definitions:**

   Homogeneous function: A function $f(x,y)$ is called a homogeneous function in two variables of degree n if by putting $X = xt$, $Y = yt$; function $f(x,y)$ reduce to $t^n f(x,y)$. i.e. $f(X,Y) = t^n f(x,y)$
4. **Important Formulae / Theorems / Properties:**

   (i) Euler's Theorem: (For function of two variables)
   If z is a homogeneous function of two variables x and y of degree n then

   $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

   (ii) Deduction from Euler's Theorem :
   If z is a homogeneous function of two variables x, y of degree n then

   $$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

   (iii) Corollary 1: If z is homogeneous function of degree n in x and y and $z = f(u)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$.

   (iv) Corollary 2 : If z is homogeneous function of degree n in x and y and $z = f(u)$ then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u)-1]$ *where* $g(u) = n\frac{f(u)}{f'(u)}$.

   Note: Corollary 1 and 2 are also known as 2nd form of Euler's theorem.
5. **Sample Problems:**

1) If z is a homogeneous function of two variables x and y of degree n then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$$

Proof : Let $z=f(x,y)$ is a homogeneous function of order n.

$\therefore\quad z=f(X,Y)=t^n f(x,y)$ Where $X=xt$ and $Y=yt$

Differentiating the above w.r.t. to t ,

$$\frac{\partial z}{\partial t}=\frac{\partial f}{\partial X}\frac{\partial X}{\partial t}+\frac{\partial f}{\partial Y}\frac{\partial Y}{\partial t}=n\,t^{n-1}f(x,y)$$

$$\frac{\partial z}{\partial t}=x\frac{\partial f}{\partial X}+y\frac{\partial f}{\partial Y}=n\,t^{n-1}f(x,y)\text{ as }\frac{\partial X}{\partial t}=x\text{ and }\frac{\partial Y}{\partial t}=y$$

Taking $t=1$ *we get* $X=x$ and $Y=y$

$$\therefore\ \frac{\partial z}{\partial t}=x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=n\,1^{n-1}f(x,y)$$

$$\therefore\ x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=n\,f(x,y)\quad\text{or}\quad x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz\qquad[\because f(x,y)=z].$$

2) If z is a homogeneous function of two variables x, y of degree n then

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z.$$

Proof : Since z is a homogeneous function of x, y of degree n ; therefore by

Euler's theorem $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ ------ (1)

Differentiating (1) partially w.r.t. x,

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$$

$$\Rightarrow\quad x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}\qquad\text{-----(2)}$$

Similarly, differentiating (1) partially w.r.t. y,

$$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}=n\frac{\partial z}{\partial y}$$

$$\Rightarrow\quad x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}\qquad\text{-----(3)}$$

Multiplying equation (2) by x and (3) by y and adding,

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=(n-1)nz\qquad(u\sin g\ (1))$$

$$\Rightarrow\ x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$

3) If z is homogeneous function of degree n in x and y and $z=f(u)$ then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}.$$

Proof : By Euler's theorem,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz=n\,f(u)$$

$$\Rightarrow\ x\frac{\partial f(u)}{\partial x}+y\frac{\partial f(u)}{\partial y}=n\,f(u)$$

$$\Rightarrow\ x\,f'(u)\frac{\partial u}{\partial x}+y\,f'(u)\frac{\partial u}{\partial y}=n\,f(u)$$

$$\Rightarrow\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

4) If z is homogeneous function of degree n in x and y and $z=f(u)$

then $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\,[g'(u)-1]$ *where* $g(u)=n\frac{f(u)}{f'(u)}$.

Proof : By Euler's theorem if $z=f(u)$ then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=g(u)\qquad\text{----- (1)}$$

Differentiating (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=g'(u)\frac{\partial u}{\partial x}$$

$$\Rightarrow\ x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(g'(u)-1)\frac{\partial u}{\partial x}\qquad\text{-----(2)}$$

Similarly, differentiating (1) partiallly w.r.t. y,

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=g'(u)\frac{\partial u}{\partial y}$$

$$\Rightarrow\ x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(g'(u)-1)\frac{\partial u}{\partial y}\qquad\text{-----(3)}$$

Multiplying equation (2) by x and equation (3) by y and adding

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(g'(u)-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(g'(u)-1)\,g(u)\qquad[u\sin g\ (1)]$$

Where $g(u)=n\frac{f(u)}{f'(u)}$

5) *If* $u = \log\frac{x+y}{\sqrt{x^2+y^2}} + \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, *prove that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin w\cos 2w}{4\cos^3 w} \text{ where } w = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right).$$

Solu : *Let* $u = v + w$ *where* $v = \log\frac{x+y}{\sqrt{x^2+y^2}}$ *and* $w = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

putting $X = xt, Y = yt, Z = zt$ *in* v

$$v(X,Y,Z) = \log\frac{xt+yt}{\sqrt{x^2t^2+y^2t^2}} = \log\frac{t(x+y)}{t\sqrt{x^2+y^2}} = t^0\, v(x,y,z)$$

$\therefore$ *v is a homogeneous function of degree* 0, *using deduction on Euler's theorem*

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 0 \quad \text{--------- (1)}$$

putting $X = xt, Y = yt, Z = zt$ *in* w

$$w(X,Y,Z) = \sin^{-1}\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}} = \sin^{-1}\frac{t(x+y)}{t^{1/2}(\sqrt{x}+\sqrt{y})}$$

Here w is not a homogeneous function of x, y, z.

However, $f(w) = \sin w = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ *is a homogeneous function of degree* $1/2$.

Using corollary (2) *of Euler's theorem*,

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = g(w)[g'(w)-1]$$

where $g(w) = n\frac{f(w)}{f'(w)} = \frac{1}{2}\frac{\sin w}{\frac{d}{dw}(\sin w)} = \frac{1}{2}\frac{\sin w}{\cos w} = \frac{1}{2}\tan w$

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = g(w)[g'(w)-1] = \frac{1}{2}\tan w\left(\frac{1}{2}\sec^2 w - 1\right)$$

$$= -\frac{\sin w\cos 2w}{4\cos^3 w}$$

From (1) *and* (2) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin w\cos 2w}{4\cos^3 w}$

Exercise 15

1. *If* $u = x^3\sin^{-1}\frac{y}{x} + x^4\tan^{-1}\frac{y}{x}$, *find the value of*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \text{ at } x = 1, y = 1$$

Ans.: $\frac{17\pi}{2}$.

2. *If* $z = x^n f\left(\frac{y}{x}\right) + y^{-n} f\left(\frac{x}{y}\right)$, *prove that* $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$

3. If $u = \tan^{-1}\left(\frac{x^2+y^2}{x-y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2\sin^3 u\cos u$$

4. *If* $u = \sin^{-1}\left\{\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$, *prove that* (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(\tan^2 u + 13)$

Let's check take away from lecture

1. $u = \log x^2 - 2\log y$ *is homogeneous of degree* ?

(a) 2 (b) 0 (c) 1 (d) None

2. *If* $u = \frac{x^3-y^3}{x^{\frac{2}{3}}+y^{\frac{3}{2}}}$ *then u is homogeneous of degree*?

(a) $\frac{7}{3}$ (b) $\frac{3}{2}$ (c) Not homogeneous (d) None

3. *If* $u = \log\left(\frac{x^4+y^4}{x+y}\right)$ *then* $xu_x + yu_y = ?$

(a) -3 (b) 3 (c) 0 (d) None

4. *If* $u = x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$ *then* $x^2u_{xx} + y^2u_{yy} + 2xyu_{xy} = ?$

(a) $5u$ (b) $4u$ (c) $20u$ (d) None

5. *If* $u = \sin^{-1}(xyz)$ *then* $xu_x + yu_y + zu_z = ?$

(a) $3\tan u$ (b) $3\cot u$ (c) $3\sin u$ (d) $3\cos u$

Homework Problems for the day

1. *If* $z = \frac{(x^2+y^2)^n}{2n(2n-1)} + x f\left(\frac{y}{x}\right) + \phi\left(\frac{x}{y}\right)$, *then show that* $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (x^2+y^2)^n$

2. *If* $v = \frac{1}{r}f(\theta)$ *where* $x = r\cos\theta$, $y = r\sin\theta$, *show that* $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + v = 0$

3. If u = f (v) where v is a homogenous. function of x, y of degree n,

prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nv f'(v)$. *Hence deduce that if* $u = \log v$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n$

4. *If* $u = \frac{x^3y^3}{x^3+y^3} + \log\left(\frac{xy}{x^2+y^2}\right)$, *then prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{6x^3y^3}{x^3+y^3}$

5. *If* $u = \frac{1}{3}\log\left(\frac{x^3+y^3}{x^2+y^2}\right)$, *find the value of (i)* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ *(ii)* $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy}$

Ans.(i) $\frac{1}{3}$ (ii) $-\frac{1}{3}$.

6. *If* $x = e^u \tan v$, $y = e^u \sec v$, *prove that* $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 0$

7. *If* $u = \sin^{-1}\left(x^2+y^2\right)^{\frac{1}{5}}$, *show that* $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = \frac{2}{25}\tan u\,(2\tan^2 u - 3)$.

8. *If* $u = \log\frac{x+y}{\sqrt{x^2+y^2}} + \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, *prove that* $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin w\cos 2w}{4\cos^3 w}$

where $w = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Learning Outcome: Students shall be able to evaluate problems based on homogeneous functions in two variables using Euler's theorem.

Maxima, Minima, and saddle points

Lecture: 16

1. **Learning Objective:** Student shall be able to find the Maxima and minima of a function of two variable.
2. **Introduction:**

   A function f(x, y) of two variables is said to have a relative maximum or minimum at a point (a, b) if there is a disc centered at (a, b) such that f(a, b) ≥ f(x, y) or f(a ,b) ≤ f(x, y)) for all points (x, y) that lie inside the disc. A function f is said to have an absolute maximum or minimum at (a, b) if f(a, b) ≥ f(x, y) or f(a ,b) ≤ f(x, y) for all points (x, y) that lie inside in the domain of f(x, y) .
3. **Working rule for finding the maxima and minima of function** $f(x,y)$ **is:**

Step 1: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate them to zero.

Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ for x and y.

Let $(a_1,b_1),(a_2,b_2)$.......... be the pairs of values (roots).

Step 2: Calculate the values of $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x\partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ and substitute in them by turns $(a_1,b_1),(a_2,b_2)$.......... for x, y.

Step 3: (i) If $rt - s^2 > 0$ and $r < 0$ or $t < 0$ at (a_1,b_1), $f(x,y)$ is maximum at (a_1,b_1) and $f(a_1,b_1)$ is maximum value.

(ii) If $rt - s^2 > 0$ and $r > 0$ or $t > 0$ at (a_1,b_1), $f(x,y)$ is minimum at (a_1,b_1) and $f(a_1,b_1)$ is minimum value.

(iii) If $rt - s^2 < 0$ at (a_1,b_1) then $f(x,y)$ is neither maximum nor minimum at (a_1,b_1). Such point is known as a saddle point.

(iv) If $rt - s^2 = 0$ at (a_1,b_1), no conclusion can be drawn about a maximum or minimum value.

4. **Sample Problem**

(1) Show that the minimum value of $f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ *is* $3a^2$.

Solution:

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = 0 \Rightarrow y - \frac{a^3}{x^2} = 0 \Rightarrow x^2y = a^3 \quad ...(1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow x - \frac{a^3}{y^2} = 0 \Rightarrow xy^2 = a^3 \quad ...(2)$$

Equating equations (1) and (2),

$$x^2y = xy^2 \Rightarrow x = y$$

Substituting in equation (1),

$x^3 = a^3$ $x = a$

\ $y = a$

Stationary point is (a, a).

Step II: $r = \frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}$, $s = \frac{\partial^2 f}{\partial x\partial y} = 1$, $t = \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$

Step III: At (a, a),

$r = 2, s = 1, t = 2$

$rt - s^2 = (2)(2) - (1)^2 = 3 > 0$ and $r > 2$

Hence, $f(x, y)$ is minimum at (a, a), $f_{\min} = f(a, a) = a\,a + \frac{a^3}{a} + \frac{a^3}{a} = 3a^2$.

(2) Examine the function $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ for extreme values.

Solution: We have $f(x,y)=x^3+3xy^2-3x^2-3y^2+4$

$\therefore \quad f_x = 3x^2+3y^2-6x \qquad f_y = 6xy-6y$

$\& \quad r=f_{xx}=6x-6 \qquad s=f_{xy}=6y \qquad t=f_{yy}=6x-6$

Now solving $f_x=0=f_y$ simultaneously,

$3x^2+3y^2-6x=0 \;\; \& \;\; 6xy-6y=0 \;\; i.e. \;\; 3(x^2+y^2-2x)=0 \;\; \& \;\; 6y(x-1)=0$

Now, $y(x-1)=0$ *gives* $y=0$ *and* $x=1$

When $y=0$ *we get* $x^2-2x=0 \Rightarrow x=0$ *or* $x=2$ *i.e.* (0,0) & (2,0)

and *When* $x=1$ *we get* $y^2-1=0 \Rightarrow y=1$ *or* $y=-1$ *i.e.* (1,1) & (1,–1)

Thus, the stationary points are (0,0) , (2,0) , (1,1) & (1,–1)

Sr. No.	Stationary point	R	s	t	$rt-s^2$	conclusion
1	(0,0)	-6	0	-6	36>0	Accept the point
2	(2,0)	6	0	6	36>0	Accept the point
3	(1,1)	0	6	0	-36<0	Reject the point
4	(1,-1)	0	-6	0	-36<0	Reject the point

Thus $f(x,y)$ has stationary pt. at (0,0) and (2,0).

Also , r = -6 < 0 at (0,0) $f(x,y)$ is maximum at (0,0) and $f_{max}(x,y)=f(0,0)=4$

And , r = 6 > 0 at (2,0) $f(x,y)$ is minimum at (2,0) and $f_{min}(x,y)=f(2,0)=0$.

(3) Examine the function $u = x^3y^2(12-3x-4y)$ for extreme values.

Solution: $u(x,y) = 12x^3y^2 - 3x^4y^2 - 4x^3y^3$

Step I:

$$\frac{\partial u}{\partial x}=36x^2y^2-12x^3y^2-12x^2y^3=12x^2y^2(3-x-y)$$

$$\frac{\partial u}{\partial y}=24x^3y-6x^4y-12x^3y^2=6x^3y(4-x-2y)$$

For extreme values,

$$\frac{\partial u}{\partial x}=0 \Rightarrow 12x^2y^2(3-x-y)=0 \Rightarrow x=0, y=0, x+y=3$$

$$\frac{\partial u}{\partial y}=0 \Rightarrow 6x^3y(4-x-2y)=0 \Rightarrow x=0, y=0, x+2y=4$$

Considering six pairs of equations,

(i) $x=0 \quad y=0$ (ii) $x=0 \quad x+2y=4$ (iii) $y=0 \quad x+2y=4$

(iv) $x=0 \quad x+y=3$ (v) $y=0 \quad x+y=3$ (vi) $x+y=3 \;\; x+2y=4$

Solving these equations, following pairs of stationary points are obtained as

(0, 0), (0, 2), (4, 0), (0, 3), (3, 0), (2, 1)

Step II: $r=\frac{\partial^2 u}{\partial x^2}=72xy^2-36x^2y^2-24xy^3=12xy^2(6-3x-2y)$

$$s=\frac{\partial^2 u}{\partial x\,\partial y}=72x^2y-24x^3y-36x^2y^2=12x^2y(6-2x-3y)$$

$$t=\frac{\partial^2 u}{\partial y^2}=24x^3-6x^4-24x^3y=6x^3(4-x-4y)$$

Step III:

(x, y)	R	s	t	$rt-s^2$	Conclusion
(0, 0)	0	0	0	0	no conclusion
(0, 2)	0	0	0	0	no conclusion
(4, 0)	0	0	0	0	no conclusion
(0, 3)	0	0	0	0	no conclusion
(3, 0)	0	0	0	0	no conclusion
(2, 1)	−48	−48	−96	2304 > 0 and r < 0	maximum

Hence, function is maximum at (2, 1)

$u_{max} = 2^3 1^2 (12 - 3.2 - 4.1) = 16$

(4) Find the stationary value of x^3+y^3-3axy, $a>0$

Solution: $f(x,y)=x^3+y^3-3axy$

Step I : For extreme values,

$$\frac{\partial f}{\partial x}=3x^2-3ay=0, \quad \frac{\partial f}{\partial y}=3y^2-3ax=0$$

From equation (1),

$$y=\frac{x^2}{a}$$

Putting in equation (2),

$x^4 - a^3x = 0$

$x(x-a)(x^2+ax+a^2)=0$

$\therefore x=0, x=a$

Then $y=0, y=a$

Hence, stationary points are (0, 0) and (a, a).

Step II:

$r=\frac{\partial^2 f}{\partial x^2}=6x,\quad s=\frac{\partial^2 f}{\partial x\,\partial y}=-3a,\quad t=\frac{\partial^2 f}{\partial y^2}=6y$

Step III: At (0, 0)

$rt-s^2=(0)(0)-(-3a)^2=-9a^2<0$

Hence, function $f(x, y)$ is neither maximum nor minimum at (0, 0)

At (a, a)

$rt-s^2=(6a)(6a)-(-3a)^2=27a^2<0$

$r=6a>0$

Hence, function $f(x, y)$ is minimum at (a, a).

$f_{\min}=a^3+a^3-3a^3=-a^3$

Exercise16

Test the following functions for maxima & minima

1. $xy(3a-x-y)$ Ans. (0, 0), (3a, 0), (0, 3a) undecided $f_{\max}=a^3$ at (a, a)
2. $x^3+3xy^2-15x^2-15y^2+72x$ **Ans:** (5,1), (5,−1), (4,0), (6,0), $f_{\max}=112$, $f_{\min}=108$
3. $x^3+y^3-63x-63y+12xy$ Ans: Stationary points are (−1, 5), (5, −1) (−7, −7), (3, 3). $u_{\max}=2156$ and $u_{\min}=-216$.

Let's check take away from lecture

11. f(x, y) = sin(x).cos(y) Which of the following is a critical point?

(a) (Π/4, Π/2) (b) (- Π/4, Π/4) (c) (0, Π/2) (d) (0, 0)

2. Consider the vertical cone. The minimum value of the function in the region f(x,y) = c is

(a) Constant (b) 1 (c) 0 (d) -1

3. Which if the following is a critical point of 2xy – x²y – xy² ?

(a) (2/3, 2/3) (b) (1/3 , 1/3) (c) (-2/3, 2/3) (d) (-1/3, 1/3)

Homework problems for the day

Find all the stationary points of the functions and test whether the function is maximum or minimum at those points:

1. $f(x,y)=x^3y^2(1-x-y)$
2. $f(x,y)=xy(a-x-y)$
3. $f(x,y)=x^3+xy^2+21x-12x^2-2y^2$
4. $f(x,y)=y^2+4xy+3x^2+x^3$

Ans.: (1) $(0,0),\left(\frac{1}{2},\frac{1}{3}\right)$, $f_{\max}=\frac{1}{432}$ *at* (0,0) *undecided*

(2) $(0,0),(0,a),(a,0)\rightarrow$ *undecided*; $f_{\max}=\frac{a^3}{27}$ *at* $\left(\frac{a}{3},\frac{a}{3}\right)$

(3) (1, 2), (1, -2), (-1, 2), (-1, -2) $f_{\max}=38$ at (-1, -2) , $f_{\min}=2$ at (1, 2)

(4) (0,0) and $\left(\frac{2}{3},\frac{-4}{3}\right)$, $f_{\min}=\frac{-4}{27}$

Learning Outcome: Students shall be able to find Maxima and Minima of function of two variable.

Maxima, Minima, and saddle points continued………

Lecture: 17

1.Sample Problem

1. Find the stationary values of $\sin x.\sin y.\sin(x+y)$.

Solution

$f(x,y)=\sin x.\sin y.\sin(x+y)$

$\frac{\partial f}{\partial x}=\sin y[\sin x.\cos(x+y)+\cos x.\sin(x+y)]$

$=\sin y.\sin(2x+y),$

$\frac{\partial f}{\partial y}=\sin x[\sin y.\cos(x+y)+\cos y.\sin(x+y)]$

$=\sin x.\sin(x+2y),$

$r=\frac{\partial^2 f}{\partial x^2}=2\sin y.\cos(2x+y)$

$s=\frac{\partial^2 f}{\partial x\partial y}$

$=\sin x.\cos(x+2y)+\cos x.\sin(x+2y)$

$=\sin(2x+2y)$

$t=\frac{\partial^2 f}{\partial y^2}=2\sin x.\cos(x+2y)$

Now $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

Gives $\sin y.\sin(2x+y)=0$

$\sin x.\sin(x+2y)=0$

i.e. $\sin y=0$ or $\sin(2x+y)=0$

and $\sin x=0$ and $\sin(x+2y)=0$

i.e. $y=0$ and $2x+y=\pm\pi$

and $x=0$ and $x+2y=\pm\pi$

i.e. when $y=0$ and $x=0$ gives point $(0,0)$

when $y=0$ and $x+2y=\pm\pi$ gives point $(\pm\pi,0)$

when $2x+y=\pm\pi$ and $x=0$ gives points $(0,\pm\pi)$.

when $2x+y=\pm\pi$ and $x+2y=\pm\pi$ gives points $\left(\pm\frac{\pi}{3},\pm\frac{\pi}{3}\right)$

$(0,0),(\pm\pi,0),(0,\pm\pi)$ and $\left(\pm\frac{\pi}{3},\pm\frac{\pi}{3}\right)$ are stationary point of $f(x,y)$

(x, y)	r	s	t	rt – s²	Conclusion
(0, 0)	0	0	0	0	no conclusion
$(\pm\pi,0)$	0	0	0	0	no conclusion
$(0,\pm\pi)$	0	0	0	0	no conclusion
$\left(\frac{\pi}{3},\frac{\pi}{3}\right)$	$-\sqrt{3}<0$	$\frac{-\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4}>0$	Maxima
$\left(\frac{-\pi}{3},\frac{-\pi}{3}\right)$	$\sqrt{3}>0$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{9}{4}>0$	Minima

$$f_{\max}\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{3}\right).\sin\left(\frac{\pi}{3}\right).\sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)$$

$$=\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}$$

$$=\frac{3\sqrt{3}}{8}.$$

and

$$f_{\min}\left(\frac{-\pi}{3},\frac{-\pi}{3}\right)=\sin\left(\frac{-\pi}{3}\right).\sin\left(\frac{-\pi}{3}\right).\sin\left(-\frac{\pi}{3}-\frac{\pi}{3}\right)$$

$$=-\frac{\sqrt{3}}{2}.\left(-\frac{\sqrt{3}}{2}\right).\left(-\frac{\sqrt{3}}{2}\right)$$

$$=\frac{-3\sqrt{3}}{8}.$$

Exercise17

1. Examine the function $f(x,y)=x^4+y^4-x^2-y^2+1$ for extreme values and also find maximum and minimum values of $f(x,y)$. Ans. $f_{\max}(0,0)=1$ and $f_{\min}\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)=\frac{1}{2}$

2. Find the maximum or minimum value of the function $f(x,y)=x^3y^2(12-3x-4y)$ if exists. Ans. $f_{\max}\left(\frac{1}{3},\frac{1}{2}\right)=\frac{1}{432}$

Let's check take away from lecture

1. Consider the $f(x,y)=x^2+y^2-a$. For what value of a , do we have critical points of the function.
   (a) independent of a (b) for any real number except 0
   (c) $a\in(0,\infty)$ (d) $a\in(-1,1)$
2. The function $x^2\,y^3\,(2-x-y)$ can have at the most how many critical points?
   (a) 6 (b) 9 (c) 4 (d) 3
3. Discuss the minimum value of $f(x,y)=x^2+y^2+6x+12$
   (a) 3 (b) -3 (c) 9 (d) -9

Homework problems for the day

Find the maximum or minimum value of the following functions if exists:

1. $f(x,y)=x^4+y^4-4a^2xy$ Ans. $f_{\min}=-8a^4$ at (a,a) and (-a,-a).
2. $f(x,y)=x^3+y^3-63x-63y+12xy$ Ans. $f_{\max}=2156$, $f_{\min}=-216$, stationary points are (-1,5), (5, -1), (-7,-7), (3,3)

Gradient

Lecture :18

1. **Learning Objective:** Student shall be able to find the gradient of a scaler function $\phi(x,y,z)$.
2. **Introduction:**

- The gradient of an scaler function $\phi(x,y,z)$ is obtained by applying the vector differential operator ∇ on $\phi(x,y,z)$ as $\nabla\phi$ .this is vector quantity which represents an outgoing Normal vector to the surface $\phi(x,y,z)=c$.
- If $\phi(x,y,z)$ is a scaler function then it's directional derivative in the three standard directions will be $\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}$ then gradient of $\phi(x,y,z)$ is the vector sum of all these directional derivative as $\nabla\phi=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial x}$ hence gradient can be utilized to find the derivative of any function in a particular direction .

3. **Formulae:**

To find the gradient of any scaler valued function at a point (x_1,y_1,z_1) of the function $\phi(x,y,z)$ evaluate $\nabla\phi=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial x}\Big|_{(x_1,y_1,z_1)}$

4. **Sample Problems:**

1). Find a unit vector normal to the surface $x^3+y^3+3xy=3$ at (1, 2, -1).

Sol: Given $\phi=x^3+y^3+3xy-3$

The normal to the surface is $\nabla\phi=(3x^2+3y)\hat{i}+(3y^2+3x)\hat{j}+0k$

At point $(1,2,-1)$, $\nabla\phi=9\hat{i}+6\hat{j}+0\hat{k}$.

Therefore, a unit vector normal to the surface is: $\frac{\nabla\phi}{\|\nabla\phi\|}=\frac{9\hat{i}+6\hat{j}+0k}{\sqrt{117}}$.

Exercise 18

1. If $\phi=3x^2y-y^3z^2$ find $\nabla\phi$ at the point P (1,-2,-1). [Ans: $-12\hat{i}-9\hat{j}-16\hat{k}$]
2. Find a unit vector normal to the surface $xy^3z^2=4$ at (-1, -1, 2).
3. Show that grad $(1/r)=-\bar{r}/r^3$.
4. Find the directional derivative of $div(\overline{u})$ at the point (1,2,2) in the direction of the outer normal of the sphere $x^2+y^2+z^2=9$ for $\bar{u}=x^4\hat{i}+y^4\hat{j}+z^4\hat{k}$. Ans: 68

Let's check take away from lecture

1. The gradient of r is
   (a) $\bar{r}/r$ (b) $-\bar{r}/r^3$ (c) 0 (d) 1
2. Gradient of a constant is
   (a) 0 (b) Not defined (c) 1 (d) 2
3. *What is the value of* $\nabla(r^2)$ *is*
   (a) $\bar{r}$ (b) $2\bar{r}$ (c) $-2\bar{r}$ (d) None
4. *What is the value of u if* $\nabla u=2r^4\bar{r}$
   (a) $\frac{2r^5}{5}+k$ (b) $8r^3+k$ (c) $\frac{r^6}{3}+k$ (d) $\frac{r^3}{3}+k$
5. *Find* $\nabla\phi$ *if* $\phi=xyz$ *at* (1,2,3)
   (a) $6i-3j+2k$ (b) $6i+3j-2k$ (c) $6i+3j+2k$ (d) None

Homework Problems for the day

1. If $\phi=\log(x^2+y^2+z^2)$, find $\nabla\phi$ at (2,1,1).
2. Find $\nabla\phi$, when $\phi=xyz$ at (1,2,3). [Ans: $6\hat{i}+3\hat{j}+2\hat{k}$]
3. What is the greatest rate of increase of $u=x^2+yz^2$ at the point (1, -1, 3)?
4. Find a unit normal to the surface $x^2y+2xz=4$ at the point (2, -2, 2).
   [Ans: $\frac{-1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$]
5. Find a unit vector normal to the surface $x^3+y^3+3xy=3$ at (1, 2, -1).

Learning Outcome: Students shall be able to find the Gradient of scaler valued function

Gradient Continued (Directional derivative)

Lecture :19

1. Sample Problem:

(1) Find is directional derivative of $\phi=xy^2+yz^3$ at the point (2,-1, 1) in the direction of the vector $\hat{i}+2\hat{j}+2\hat{k}$.

Solution: Here

$$\nabla\phi=\frac{\partial}{\partial x}(xy^2+yz^3)\hat{i}+\frac{\partial}{\partial y}(xy^2+yz^3)\hat{j}+\frac{\partial}{\partial z}(xy^2+yz^3)\hat{k}$$

$$=y^2\hat{i}+(2xy+z^3)\hat{j}+(3yz^2)\hat{k}$$

$$\Rightarrow \nabla\phi=\hat{i}-3\hat{j}-3\hat{k} \text{ at the point } (2,-1,1).$$

Directional derivative of ϕ in the direction $\hat{i}+2\hat{j}+2\hat{k}=(\hat{i}-3\hat{j}-3\hat{k})\frac{\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{1^2+2^2+3^2}}$

$$=\frac{(1\cdot1-3\cdot2-3\cdot2)}{3}=-\frac{11}{3}$$

Exercise 19

1. Find the directional derivative of $\phi=4xz^3-3x^2y^2z$ at (2, -1, 2) in the direction of 2i+3j+6k.
2. (a) In what direction from the point (2,1,-1) is the directional derivative of $\phi=x^2yz^3$ a maximum? [Ans: Direction of $\nabla\phi$ i.e. $-4\hat{i}-4\hat{j}+12\hat{k}$]
   (b) What is the magnitude of this maximum? [Ans: $\sqrt{176}$]
3. What is directional derivative of $\phi=xy^2+yz^3$ at the point (2,-1, 1) in the direction of the normal to the surface $x\log z-y^2=-4$ at (-1, 2, 1). [Ans: $\frac{15}{\sqrt{17}}$]

Let's check take away from lecture

1. The maximum directional derivative of $\phi=xyz$ at point (1, -1, 1) is
   (a) $\sqrt{3}$ (b) $\sqrt{6}$ (c) $3\sqrt{3}$ (d) $6\sqrt{6}$
2. Directional derivative is
   (a) scalar quantity (b) vector quantity c) distance quantity (d) none

   (a) $Grad.\phi=0$ (b) $Div.\phi=0$ (c) $Curl\ \phi=0$ (d) None
4. *What is the maximum directional derivative of* $\phi=x^2yz^3$ *at* $(2,1,-1)$?
   (a) $\sqrt{171}$ (b) $\sqrt{176}$ (c) $\sqrt{129}$ (d) None

Homework Problems for the day

1. Find directional derivative of $\phi = x^4 + y^4 + z^4$ at the point A (1,-2, 1) in the direction of line AB where B = (2, 6,-1).

   $\left[\text{Ans: } \frac{-260}{\sqrt{69}}\right]$

2. Find the directional derivative of $\phi = e^{2x-y+z}$ at the point (1, 1,-1) in a direction towards the point (-3, 5, 6).

   $\left[\text{Ans: } -\frac{20}{9}\right]$

3. In what direction is the directional derivative of $\phi = x^2y^2z^4$ at (3,-1,-2) maximum? Find its magnitude.
4. Find the directional derivative of $x^3 + y^3 + z^3 - xyz$ at P (1, 1, 1) in the direction of normal to surface $x \log z + y^2 = 4$ at Q (1,-2, 1).

   $\left[\text{Ans: } \frac{-6}{\sqrt{17}}\right]$

Learning Outcome: Students shall be able to find the directional derivative of the scaler valued function

(Self-Study Topic)
Divergence
Lecture: 20

1. **Learning Objective:** Student shall be able to find the divergence of a vector valued function
2. **Introduction:**

   In vector calculus, divergence is the vector operator of vector differential operator on a vector field. It produces a scalar field giving the quantity of the vector field's source at each point. While air is heated in a region, it expands in all directions, and thus the velocity field points outward from that region. If the divergence of a vector field is positive then the vector field is said to be source field and if divergence is negative then it is known as sink field where as if divergence of any vector field is zero then it is called solenoidal field.

3. **Formulae:** If $\bar{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ then $\nabla \bullet \bar{F} = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}).\bar{F}$

   $$= (\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z})$$

If $\nabla.\bar{F} > 0$ *then* $\bar{F}$ *is known a source field*

If $\nabla.\bar{F} < 0$ *then* $\bar{F}$ *is known a sink field*

If $\nabla.\bar{F} = 0$ *then* $\bar{F}$ *is known a solenoidal field*

4. **Sample Problem:**

   1) If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ *then show that* $\nabla \bullet \bar{r} = 3$.

   Solution: Since $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

   and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

   $$\nabla \bullet \bar{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \bullet \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

   $$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

Exercise20

1. Find $div F$ *where* $F = grad\,(x^3 + y^3 + z^3 - 3xyz)$.
2. $\text{div}\,(\bar{a} \bullet \bar{r})\,\bar{a} = a^2$.
3. Show that $\bar{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ is solenoidal.

Let's check take away from lecture

a. A vector field which has a vanishing divergence is called as
   (a) Solenoidal field (b) Rotational field (c) Hemispheroidal field (d) Irrotational field

b. What is the divergence of the vector field $\bar{f} = 3x^2 i + 5xy^2 j + xyz^3 k$ at the point (1, 2, 3)
   (a) 89 (b) 80 (c) 124 (d) 100

3. *What is the value of* $\nabla \cdot \bar{r} = ?$
   (a) 1 (b) 0 (c) 3 (d) 2

4. *If* $\bar{a}$ *is a constant vector then find* $\nabla \cdot \bar{a}$
   (a) 1 (b) a (c) 0 (d) None

5. *If* $\bar{F}$ *is solenoidal vector field; then ....*
   (a) $Curl\,\bar{F} = 0$ (b) $Div.\bar{F} = 0$ (c) $Grad.\bar{F} = 0$ (d) None

Homework Problems for the day

1. If $\bar{F} = xye^{2x}\,\hat{i} + xy^2\cos z\,\hat{j} + x^2\cos xy\,\hat{k}$, find $\text{div}\,\bar{F}$.
2. If $\bar{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$. Find $\nabla.\bar{A}$ at (1, -1, 1). [Ans: −3]
3. Evaluate $div(3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$ *at the point* (1,2,3).
4. If $\bar{F} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$ prove that $\nabla \bullet \bar{F} = \frac{2}{r}$.

Learning Outcome: Students shall be able to evaluate the divergence of the vector field

(Self-Study Topic)
Curl
Lecture :21

1. **Learning Objective:** Student shall be able to find the Curl of a vector valued function
2. **Introduction:**

   In vector calculus, curl is the cross operation of vector differential operator on a vector field. It produces a vector field. Curl of a vector field describe the Infinitesimal rotation of the vector field. If the curl of a vector field is zero, then the vector field is said to be conservative or irrotational.
3. **Formulae:** If

$$\bar{F}=f_1\hat{i}+f_2\hat{j}+f_3\hat{k} \text{ then } \nabla\times\bar{F}=(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z})\times f_1\hat{i}+f_2\hat{j}+f_3\hat{k}$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}=\hat{i}(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z})+\hat{j}(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x})+\hat{k}(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y})$$

If $\nabla\times\bar{F}=0$ *then* $\bar{F}$ *is known a conservative or irrotational field*

4. **Sample Problem:**

1) If $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$ *then show that* $\nabla\times\bar{r}=0$.

Solution: Since $\nabla=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$

and $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\nabla\times\bar{r}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}$$

$$=\hat{i}\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)-\hat{j}\left(\frac{\partial z}{\partial x}-\frac{\partial x}{\partial z}\right)+\hat{k}\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)$$

$$=\hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(0-0)=0.$$

Exercise 21

1. If $\bar{F}=xye^{2z}\,\hat{i}+xy^2\cos z\,\hat{j}+x^2\cos xy\,\hat{k}$, find curl $\overline{F}$.
2. Show that div $(\bar{a}\times\bar{r})=0$.
3. Show that $\bar{F}=(z^2+2x+3y)\,\hat{i}+(3x+2y+z)\hat{j}+(y+2xz)\hat{k}$ is irrotational.

Let's check take away from lecture

1. Curl operation result will always be
   (a) Vector (b) Scalar (c) Scalar or Vector (d) None of These
2. Laplacian operator ∇^2 v
   (a) Grad (Div V) (b) Div (Grad V) (c) Curl (Div V) (d) Div (Curl V)
3. *What is the value of* $\nabla\times\bar{r}=?$
   (a) 1 (b) 0 (c) $\bar{r}$ (d) $\bar{0}$
4. *If* $\bar{a}$ *is a constant vector then find* $\nabla\times\bar{a}$
   (a) $\bar{a}$ (b) a (c) 0 (d) $\bar{0}$
5. *If* $\bar{F}$ *is irrotational vector field; then ....*
   (a) *Curl* $\bar{F}=0$ (b) *Div.*$\bar{F}=0$ (c) *Grad.*$\bar{F}=0$ (d) *None*

Homework Problems for the day

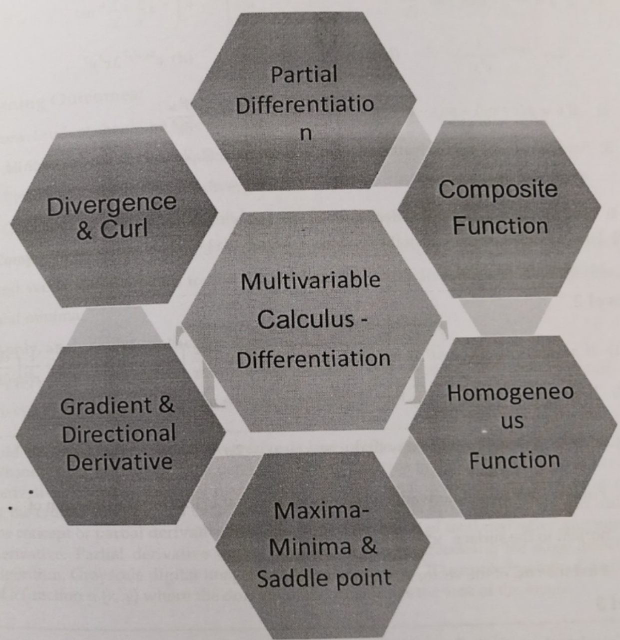
1. If $\vec{A}=x^2z\hat{i}-2y^3z^2\hat{j}+xy^2z\hat{k}$, find $\nabla\times\vec{A}$ at (1,-1,1). $\left[\text{Ans}: -6\hat{i}+0\hat{j}+0\hat{k}\right]$
2. Find curl $\bar{F}$, where $\bar{F}=\text{grad}(x^3+y^3+z^3-3xyz)$.
3. Evaluate *curl* $\left[e^{xyz}(\hat{i}+\hat{j}+\hat{k})\right]$.
4. Show that curl $(\bar{a}\times\bar{r})=2\bar{a}$.
5. Show that $\bar{F}=(z^2+2x+3y)\,\hat{i}+(3x+2y+z)\hat{j}+(y+2xz)\hat{k}$ is irrotational.
6. Show that $\bar{F}=ye^{xy}\cos z\,\hat{i}+xe^{xy}\cos z\,\hat{j}-e^{xy}\sin z\,\hat{k}$ is irrotational vector field.

Learning Outcome: Students shall be able to evaluate the curl of the vector field

Tutorial Questions

1. If $u(x+y) = x^2 + y^2$, then show that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.
2. If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$, prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$
3. If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, then show that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$.
4. *If* $x = e^u \tan v$, $y = e^u \sec v$, *prove that* $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 0$
5. Find the stationary value of $x^3 + y^3 - 3axy$, $a > 0$
6. Find a unit normal to the surface $x^2y + 2xz = 4$ at the point (2, -2, 2).
7. Find the directional derivative of $x^3 + y^3 + z^3 - xyz$ at P (1, 1, 1) in the direction of normal to surface $x \log z + y^2 = 4$ at Q (1,-2, 1).
8. If $\overline{F} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$ prove that $\nabla \bullet \overline{F} = \frac{2}{r}$.
9. Show that $\overline{F} = \left(z^2 + 2x + 3y\right)\hat{i} + \left(3x + 2y + z\right)\hat{j} + \left(y + 2xz\right)\hat{k}$ is irrotational.

Concept Map



Problems for Self-assessment:

Level 1

1) *If* $u = e^{\log x^2 y^3}$ *then find* $\frac{\partial u}{\partial y}$

   (a) $e^{\log x^2 y^3} \frac{1}{x^2 y^3} 3x^2 y^3$ (b) $3x^2 y^2$ (c) $2xy^3$ (d) $e^{\log x^2 y^3} 3x^2 y^2$

2) If $u = \phi(x+\kappa y) + \psi(x-\kappa y)$, then show that $\frac{\partial^2 u}{\partial y^2} = \kappa^2 \frac{\partial^2 u}{\partial x^2}$.

3) Find the directional derivative of $\phi = x^2 y \cos z$ at $(1, 2, \pi/2)$ in the direction of $\bar{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$.

4) Find maximum directional derivative of: $\phi = (4x - y + 2z)^2$ at P(1,2,1).

5) Find all stationary points of the function $x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$ and evaluate its Maxima and Minima

Level 2

1) If $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, then show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

2) If $u = f\left[e^{y-z}, e^{z-x}, e^{x-y}\right]$, then show that $u_x + u_y + u_z = 0$.

3) If $u = \frac{e^{x+y+z}}{e^x + e^y + e^z}$, then show that $u_x + u_y + u_z = 2u$.

4) Find the directional derivative of $\phi = xy(x - y + z)$ at P(1,2,1) in the direction of normal to the surface $x^2 + y^2 + az^2 = 4$ at Q(1,1,1)

5) Find the curl of the vector A = yz i + 4xy j + y k

Level 3

1) Calculate the curl of $\vec{F} = x^3 y^2 \hat{i} + x^2 y^3 z^4 \hat{j} + x^2 z^2 \hat{k}$.

2) If $x^2 \sin(y^3) + xe^{3z} = 0$ then evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

3) *If* $u = \log r$ & $r^2 = x^2 + y,^2$ *prove that* $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + 1 = 0$

4) *If* $u = \operatorname{cosec}^{-1}\left\{\sqrt{\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}}\right\}$, *then prove that* $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u)$

5) *Show that* $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + xu_x + yu_y = 9u$,

   if $u = x^3\left(\tan^{-1}\frac{y}{x} + \frac{y}{x} e^{-\frac{y}{x}}\right) + y^{-3}\left(\sin^{-1}\frac{x}{y} + \frac{x}{y}\log\frac{x}{y}\right)$

Learning Outcomes:

1. Know: Student should be able to

a) All Partial derivative of explicit function

b) Evaluate the gradient, curl, divergence

c) Calculate the maxima and Minima of function of two variables.

2. Comprehend: Student should be able to comprehend the concepts of partial derivative and vector differentiation to find the gradient, curl, and divergence along with maxima and minima.

3. Apply, analyze and synthesize: Student should be able to

Apply partial derivative to find the maxima and minima and gradient curl and divergence of a vector function and scaler function.

Add to Knowledge: Partial derivative is one of the important concepts of calculus. Whenever we want to find the derivative of any function in the direction, we use partial derivative. To study the local maxima or minima, partial derivative contributes as one of the crucial concepts. To evaluate the error or to do the error analysis we can utilize the concept of partial derivative in the form of the relation of total derivative and partial derivative. Partial derivative can be applied in image processing for edge detection algorithm. Grayscale digital images can be considered as 2D sampled points of a graph of a function u (x, y) where the domain of the function is the area of the image.

Digital references:

1. https://www.intmath.com/differentiation/10-partial-derivatives.php
2. https://en.wikipedia.org/wiki/Mean_value_theorem

Self-Evaluation

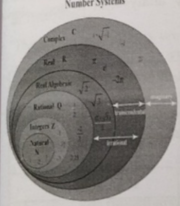
Name of student: **Class & Div:** **Roll No:**

1. Are you able to evaluate the Partial derivative?
   (a) Yes (b) No
2. Do you understand how to find the Gradient, Divergence and Curl and its physical meaning?
   (a) Yes (b) No
3. Will you able to identify how to find the maxima and minima of a function of two variable?
   (a) Yes (b) No
4. Are you able to find maxima minima of function of three variable with the constraint?
   (a) Yes (b) No
5. Do you understood this module?
   (a) Fully understood (b) Partially understood (c) Not at all

Module 3: Complex Number

Infographics

Number Systems



Abraham De Moivre (1667-1754)

De Moivre's Theorem

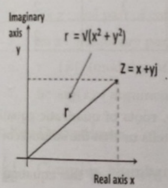
- De Moivre's Theorem states that, for all real values of n,

$$\boxed{(\cos\theta + i\sin\theta)^n \equiv \cos n\theta + i\sin n\theta}$$

- So for any complex number,

$$\boxed{z^n = r^n(\cos\theta + i\sin\theta)^n \equiv r^n(\cos n\theta + i\sin n\theta)}$$

Argand Diagrams

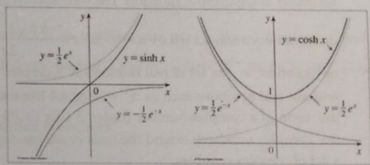


Complex Number representation

HYPERBOLIC FUNCTIONS

The graphs of hyperbolic sine and cosine can be sketched using graphical addition, as in these figures.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$



Module 3
Complex Number

1. Motivation:

This topic deals with introduction to complex number and their algebra and its application to fi[nd] the roots of equation and introduction of hyperbolic function and their properties as it is used [in] many engineering problems, widely used for complex frequency analysis of system, sig[nal] processing etc.

2. Syllabus:

Lecture No.	Title	Duration (Hrs.)	Self-study (Hrs.)
22	Power of Complex Expressions	1	2
23	Roots of an Equation	1	2
24	Hyperbolic Functions	1	2
25	Hyperbolic Functions continued ……	1	2
26	Inverse Hyperbolic Functions	1	2
27	Separation of real and imaginary part	1	2
28	Logarithmic of complex number	1	2

3. Prerequisite:

Numbers, natural numbers, real numbers, formulae of trigonometry, roots of quadratic equatio[n], formulae of logarithmic function. The well-known quadratic formula tells us that the solution of th[e] quadratic equation $ax^2+bx+c=0$. is $z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and when $b^2-4ac<0$ this equation ha[s] no solution in the set of real numbers R. Complex numbers are introduced to solve the equation[s] without real solutions such as $x^2=-1$ or $x^2-10x+40=0$. Every real number can be written i[n] the form of a complex number {ordered pair $(x,0)$} by taking its imaginary part as zero. Hen[ce] the complex numbers extend the reals or we can say that $R\subset C$, where R = set of reals and [C] = set of complex numbers.

4. Learning Objectives: Learners shall be

1. State complex numbers and list all the properties of it.
2. Calculate roots of an equation using De'Moivre's theorem.
3 Define hyperbolic functions and list standard formulae for them.
4. Define inverse hyperbolic functions and state standard formulae for them.
5. Identify the real and imaginary parts of complex expressions.

Power of Complex Expressions

Lecture: 22

1. Learning Objective: Student shall be able to find all the roots of an equation using De'moivre's theorem.

2. Introduction:

The complex numbers are an extension of the real numbers obtained by introducing an imaginary unit i, where $i=\sqrt{-1}$. The operations of addition, subtraction, multiplication, and division are applicable on complex numbers. A negative real number can be obtained by squaring a complex number. With a complex number, it is always possible to find solutions to polynomial equations of degree more than one. Complex numbers are used in many applications, such as control theory, signal analysis, quantum mechanics, relativity, etc.

3. Key Notations:

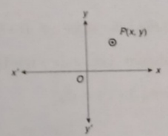
1. i: iota (imaginary unit)
2. z: complex number
3. $\bar{z}$: conjugate of complex number
4. Re z : real part of complex number z
5. Im z : imaginary part of z
6. r : $|z|$ (modulus of z)
7. θ : arg(z) (argument of z)

4. Key Definitions:

I. **Complex Number:** A complex number z is an ordered pair (x, y) of real numbers x and y,written as $z=(x, y)$. Here, x is called real part of z and is written as " Re (z)" and y is called Imaginary part of z and is written as " Im (z)".

II. **Geometrical Representation of Complex Numbers (Argand's Diagram)**

Any complex number $z = x + iy$ can be represented as a point P (x, y) in the xy-plane .



Plot of a given complex number $z = x + iy$, as the point P (x, y) in xy-plane is known as **Argand's diagram**. The x-axis is called the real axis, y-axis is called the imaginary axis and xy-plane is called complex plane.

III. **Complex Conjugate :** The complex conjugate $\bar{z}$ of a complex number

$z = x + iy$ is defined by $\bar{z} = x - iy$.

IV. Polar form of a complex number : The polar form of $z = x + iy$ is

$z = r\ (\cos\theta + i\sin\theta)$ where $x = r\cos\theta,\ y = r\sin\theta$.

V. Exponential form of a complex number : If $z \neq 0$, then $z = r\ (\cos\theta + i\sin\theta) = re^{i\theta}$ is called exponential form of complex number.

VI. Modulus and Amplitude or Argument of a complex number : The absolute value or modulus of z is denoted by $|z|$ or mod(z) and is given as

$$|z| = r = \sqrt{x^2 + y^2}$$

The argument or amplitude of z is denoted by arg(z) or amp(z) and is given as

$$\arg(z) = \theta = \tan^{-1}\frac{y}{x}.$$

VII. De- Moivre's theorem:

Statement: For any real number n, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.
i.e. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

5. Method: De Moivre's theorem can be used to find the roots of an algebraic equation. It is conside[red] as applications of De Moivre's theorem.

To solve the equation $z^n = x+iy$ write polar form of $x+iy$ as $z^n = r\,(\cos\theta + i\sin\theta)$

$$z = r^{\frac{1}{n}}\left[\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\right]^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$$

Putting $k = 0, 1, 2, \ldots\ldots (n-1)$, all n roots of the equation are obtained.

Note : (i) Complex roots always occur in conjugate pair if coefficients of different powers of x including constant terms in the equation are real.

(ii) Continued product means product of all the roots of the equation

5. Sample Problems:

1) *Find the value of* $(1+i)^6$ using De'Moivre's theorem

Solution: Let $z = (x + iy) = 1 + i$

$r = |1 + i| = (2)^{\frac{1}{2}}$

$$\theta = \arg(1 + i) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$z = 1 + i = (2)^{\frac{1}{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$z^6 = (1+i)^6 = \{(2)^{\frac{1}{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\}^6 = 2^3\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) = 8\left((\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

2) Prove that $(x + iy)^{\frac{m}{n}} + (x - iy)^{\frac{m}{n}} = 2(x^2 + y^2)^{\frac{m}{2n}}\cos\left(\frac{m}{n}\tan^{-1}\frac{y}{x}\right)$ using De'Moivre's theorem.

Solution: Let $x + iy = r(\cos\theta + i\sin\theta)$

$$r = |x + iy| = \sqrt{x^2 + y^2} = (x^2 + y^2)^{\frac{1}{2}}$$

$$\theta = \arg(x + iy) = \tan^{-1}\frac{y}{x}$$

$$(x + iy)^{\frac{m}{n}} + (x - iy)^{\frac{m}{n}} = \{r(\cos\theta + i\sin\theta)\}^{\frac{m}{n}} + \{r(\cos\theta - i\sin\theta)\}^{\frac{m}{n}}$$

$$= r^{\frac{m}{n}}\left\{\cos\frac{m\theta}{n} + i\sin\frac{m\theta}{n} + \cos\frac{m\theta}{n} - i\sin\frac{m\theta}{n}\right\} = r^{\frac{m}{n}}\,.\,2\cos\frac{m\theta}{n}$$

Substituting values of r and θ ,

$$(x + iy)^{\frac{m}{n}} + (x - iy)^{\frac{m}{n}} = 2(x^2 + y^2)^{\frac{m}{2n}}\cos\left(\frac{m}{n}\tan^{-1}\frac{y}{x}\right)$$

3) Prove that $\dfrac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$ using De'Moivre's theorem and hence

show that $\left(1 + \sin\frac{\pi}{5} + i\cos\frac{\pi}{5}\right)^5 + i\left(1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right) = 0$

Solution:
$$\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} = \frac{1 + \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)}{1 + \cos\left(\frac{\pi}{2} - \theta\right) - i\sin\left(\frac{\pi}{2} - \theta\right)}$$

$$= \frac{2\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + 2i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) - 2i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} = \frac{\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}$$

$$= \frac{e^{i\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}}{e^{-i\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}} = e^{2i\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} = e^{i\left(\frac{\pi}{2} - \theta\right)} = \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right) = \sin\theta + i\cos\theta$$

Putting $\theta = \frac{\pi}{5}$ in the above expression,

$$\frac{1 + \sin\frac{\pi}{5} + i\cos\frac{\pi}{5}}{1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}} = \sin\frac{\pi}{5} + i\cos\frac{\pi}{5}$$

$$\left(\frac{1 + \sin\frac{\pi}{5} + i\cos\frac{\pi}{5}}{1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}}\right)^5 = \left(\sin\frac{\pi}{5} + i\cos\frac{\pi}{5}\right)^5 = \left\{\cos\left(\frac{\pi}{2} - \frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right)\right\}^5$$

$$= \cos 5\left(\frac{3\pi}{10}\right) + i\sin 5\left(\frac{3\pi}{10}\right) \qquad \text{[Using De Moivre's theorem]}$$

$$= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i(-1) = -i$$

$$\left(1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5=-i\left(1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}\right)^5$$

$$\left(1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5+i\left(1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}\right)^5=0$$

4) If $\alpha = i+1$, $\beta = 1-i$ and $\tan\phi = \frac{1}{x+1}$, then prove that

$\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta} = \sin n\phi \operatorname{cosec}^n\phi$ using De'Moivre's theorem.

Solution : $\alpha = i+1,\ \beta = 1-i,\ \tan\phi = \frac{1}{x+1}$

$\cot\phi = x+1,\ x = \cot\phi - 1$

$$\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}=\frac{(\cot\phi-1+i+1)^n-(\cot\phi-1+1-i)^n}{i+1-1+i}=\frac{\left(\frac{\cos\phi}{\sin\phi}+i\right)^n-\left(\frac{\cos\phi}{\sin\phi}-i\right)^n}{2i}$$

$$=\frac{(\cos\phi+i\sin\phi)^n-(\cos\phi-i\sin\phi)^n}{2i\sin^n\phi}=\frac{\left(e^{i\phi}\right)^n-\left(e^{-i\phi}\right)^n}{2i\sin^n\phi}=\frac{e^{in\phi}-e^{-in\phi}}{2i\sin^n\phi}=\frac{2i\sin n\phi}{2i\sin^n\phi}=\sin n\phi\operatorname{cosec}^n\phi$$

5) Prove using De'Moivre's theorem $\left(\frac{-1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n$ has the value –1, if $n = 3k \pm 1$ (not a multiple of 3) and 2 if $n = 3k$ (multiple of 3) where k is an integer.

Solution: Let $\frac{-1+i\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$, then $\frac{-1-i\sqrt{3}}{2} = r(\cos\theta - i\sin\theta)$

$$r=\left|-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

$$\theta=\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)=\tan^{-1}\left(-\sqrt{3}\right)=\frac{2\pi}{3}$$

$$\frac{-1+i\sqrt{3}}{2}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=e^{\frac{2i\pi}{3}} \quad\text{and}\quad \frac{-1-i\sqrt{3}}{2}=\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}=e^{-\frac{2i\pi}{3}}$$

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n=\left(e^{\frac{2i\pi}{3}}\right)^n+\left(e^{-\frac{2i\pi}{3}}\right)^n=e^{\frac{2in\pi}{3}}+e^{-\frac{2in\pi}{3}}=2\cos\frac{2n\pi}{3}$$

If $n = 3k \pm 1$

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n=2\cos\frac{2\pi}{3}(3k\pm1)=2\cos\left(2k\pi\pm\frac{2\pi}{3}\right)$$

$$=2\cos\left(\pm\frac{2\pi}{3}\right)\ [\because \cos(2k+\theta)=\cos\theta]=2\cos\frac{2\pi}{3}=2\left(-\frac{1}{2}\right)=-1,\ \text{if } n=3k\pm1$$

If $n = 3k$ $\left(\frac{-1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n=2\cos\frac{2\pi}{3}(3k)=2\cos 2k\pi=2,$ if $n = 3k\ [\because \cos 2k\pi = 1]$

6) If α and β are the roots of the equation $x^2-2x+2=0$, then using De'Moivre's Theorem *show* that $\alpha^n+\beta^n=2.2^{\frac{n}{2}}\cos(\frac{n\pi}{4})$ *and hence find the value of* $\alpha^8+\beta^8$

Sol: *Given* equation $x^2-2x+2=0$ *where* $a=1, b=-2, c=2$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4-8}}{2}$$

$$=\frac{2\pm\sqrt{-4}}{2}=\frac{2\pm2i}{2}=1\pm i$$

$$\therefore \alpha=1+i \quad and \quad \beta=1-i$$

$1+i = r(\cos\theta + i\sin\theta)$

$$r=|1+i|=(2)^{\frac{1}{2}}$$

$$\theta=\arg(1+i)=\tan^{-1}(1)=\frac{\pi}{4}$$

$$\alpha=1+i=(2)^{\frac{1}{2}}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$$

$$\alpha^n=(1+i)^n=\{(2)^{\frac{1}{2}}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})\}$$

$$\alpha^n=\{\left(2^{\frac{1}{2}}\right)(cos(\frac{\pi}{4})+i\,sin((\frac{\pi}{4}))\}^n=2^{\frac{n}{2}}(cos(\frac{n\pi}{4})+i\,sin((\frac{n\pi}{4})$$

$$\beta^n=\{\left(2^{\frac{1}{2}}\right)(cos(\frac{\pi}{4})-i\,sin((\frac{\pi}{4}))\}^n=2^{\frac{n}{2}}(cos(\frac{n\pi}{4})-i\,sin((\frac{n\pi}{4})$$

$$\therefore \alpha^n+\beta^n=2.2^{\frac{n}{2}}cos(\frac{n\pi}{4})$$

$$\therefore \alpha^8+\beta^8=2.2^{\frac{8}{2}}cos(\frac{8\pi}{4})=2^5\cos(2\pi)=32$$

Exercise 22

1. *Find the value of* $(\sqrt{3}-i)^4$ using De'Moivre's theorem

2. Prove using De'Moivre's theorem that $(4n)^{\text{th}}$ power of $\frac{1+7i}{(2-i)^2}$ is equal to $(-4)^n$, where n is a positive integer.

3. If α and β are the roots of the equation $z^2 \sin^2\theta - z\sin 2\theta + 1 = 0$, prove that using De'Moivre's theorem $\alpha^n + \beta^n = 2\cos n\theta \operatorname{cosec}^n\theta$ (n is +ve integer).

4. If α and β are the roots of the equation $x^2 - 2x + 4 = 0$, then using De'Moivre's Theorem *show* that $\alpha^n + \beta^n = 2^{n+1}\cos(\frac{n\pi}{3})$ *and hence find the value of* $\alpha^{10} + \beta^{10}$

Let's check take away from lecture

Choose the correct option from the following.

1. *Find the value of* i^{76}

   (a) i (b) 1 (c) -1 (d) $-i$

2. *Argument of* $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$ *is*

   (a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) None

3. *Complex conjugate of* $\frac{1+i}{1-i}$ *is*

   (a) i (b) $-i$ (c) doesn't exists (d) none

4. *Polar form of* $-i$ *is*

   (a) $e^{\frac{3\pi}{2}i}$ (b) $e^{-\frac{3\pi}{2}i}$ (c) $-e^{\frac{\pi}{2}i}$ (d) None

5. n^{th} *roots of unity are in*

   (a) G.P. (b) A.P. (c) H.P. (d) None

Homework Problems for the day

1. *Find the value of* $(\sqrt{3} + i)^4$ using De'Moivre's theorem

2. *If* $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, simplify $(z)^{10} + (\bar{z})^{10}$. **Ans:** 0

3. If α and β are the roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$, then using De'Moivre's Theorem prove that $\alpha^3 + \beta^3 = 0$.

Learning from the topic: Learner will be able to apply De'Moivre's theorem to find roots of a equation and find the power.

Roots of an Equation

Lecture: 23

Sample Problems:

1. Solve $x^6 - i = 0$ using De'Moivre's theorem.

Solution : $x^6 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, $= \cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)$

$x = \left\{\cos(4k+1)\frac{\pi}{2} + i\sin(4k+1)\frac{\pi}{2}\right\}^{\frac{1}{6}} = \cos(4k+1)\frac{\pi}{12} + i\sin(4k+1)\frac{\pi}{12}$ [using De Moivre's theorem]

Putting $k = 0, 1, 2, 3, 4, 5$, we get all the 6 roots of the given equation.

2. Calculate the roots of $x^{10} + 11x^5 + 10 = 0$.

Solution : $x^{10} + 11x^5 + 10 = 0$

$$x^{10} + 10x^5 + x^5 + 10 = 0$$

$$x^5(x^5 + 10) + 1(x^5 + 10) = 0$$

$$(x^5 + 1)(x^5 + 10) = 0$$

All the roots of $x^{10} + 11x^5 + 10 = 0$ are the roots of

$$x^5 + 1 = 0 \text{ and } x^5 + 10 = 0$$

$$x^5 + 1 = 0$$

$x^5 = -1 = \cos\pi + i\sin\pi = \cos(2k_1\pi + \pi) + i\sin(2k_1\pi + \pi)$

$x = \{\cos(2k_1 + 1)\pi + i\sin(2k_1 + 1)\pi\}^{\frac{1}{5}} = \cos(2k_1 + 1)\frac{\pi}{5} + i\sin(2k_1 + 1)\frac{\pi}{5} = e^{i(2k_1+1)\frac{\pi}{5}}$

Putting $k_1 = 0, 1, 2, 3, 4$ we get all the 5 roots of $x^5 + 1 = 0$

$$x^5 + 10 = 0$$

$x^5 = -10 = 10(\cos\pi + i\sin\pi) = 10\{\cos(2k_2\pi + \pi) + i\sin(2k_2\pi + \pi)\}$

$x = \{10\cos(2k_2 + 1)\pi + i\sin(2k_2 + 1)\pi\}^{\frac{1}{5}}$

$= (10)^{\frac{1}{5}}\left\{\cos(2k_2 + 1)\frac{\pi}{5} + i\sin(2k_2 + 1)\frac{\pi}{5}\right\} = (10)^{\frac{1}{5}} e^{i(2k_2+1)\frac{\pi}{5}}$

Putting $k_2 = 0, 1, 2, 3, 4$ we get all 5 roots of $x^5 + 10 = 0$

All the 10 roots of the equation $x^{10} + 11x^5 + 10 = 0$ are given by $e^{i(2k_1+1)\frac{\pi}{5}}$ and $(10)^{\frac{1}{5}} e^{i(2k_2+1)\frac{\pi}{5}}$ where $k_1 = k_2 = 0, 1\ 2, 3, 4$

3. If $\alpha, \alpha^2, \alpha^3, \alpha^4$ are roots of $x^5 - 1 = 0$, then show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$ using De'Moivre's theorem.

Solution: One root of $x^5 - 1 = 0$ is obviously 1, remaining roots are given as $\alpha, \alpha^2, \alpha^3, \alpha^4$

$\therefore$

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$(x-1)(x^4 + x^3 + x^2 + x + 1) = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$[\because x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \ldots + 1)]$

$$x^4 + x^3 + x^2 + x^2 + 1 = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

Putting $x = 1$ on both the sides,

$$1+1+1+1+1 = (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)$$

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$$

4. Show using De'Moivre's theorem

$$x^5 - 1 = (x-1)\left[x^2 + 2x\cos\frac{\pi}{5} + 1\right]\left[x^2 + 2x\cos\frac{3\pi}{5} + 1\right]$$

Solution: $x^5 - 1 = 0$

$$x^5 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$x = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{5}} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5} = e^{i\frac{2k\pi}{5}}$$

Putting $k = 0, 1, 2, \ldots 4$, all the roots are obtained.

$$x_0 = 1, \quad x_1 = e^{i\frac{2\pi}{5}} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = e^{i\frac{2\pi}{5}}, \quad x_2 = e^{i\frac{4\pi}{5}} = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = e^{i\frac{4\pi}{5}}$$

$$x_3 = e^{i\frac{6\pi}{5}} = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \cos\left(2\pi - \frac{4\pi}{5}\right) + i\sin\left(2\pi - \frac{4\pi}{5}\right)$$

$$= \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} = e^{-i\frac{4\pi}{5}} \qquad \left[\because \begin{matrix}\cos(2\pi-\theta) = \cos\theta \\ \sin(2\pi-\theta) = -\sin\theta\end{matrix}\right]$$

$= \overline{x_2}$, (conjugate of x_2)

$$x_4 = e^{i\frac{8\pi}{5}} = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos\left(2\pi - \frac{2\pi}{5}\right) + i\sin\left(2\pi - \frac{2\pi}{5}\right) = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} = e^{-i\frac{2\pi}{5}}$$

$= \overline{x_1}$, (conjugate of x_1)

$\because$ x_0, x_1, x_2, x_3, x_4 are roots of $x^5 - 1 = 0$

$$x^5 - 1 = (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

$$= (x-1)(x - e^{\frac{2i\pi}{5}})(x - e^{\frac{4i\pi}{5}})(x - e^{-\frac{4i\pi}{5}})(x - e^{-\frac{2i\pi}{5}})$$

$$= (x-1)(x - e^{\frac{2i\pi}{5}}).(x - e^{-\frac{4i\pi}{5}}).(x - e^{-\frac{4i\pi}{5}}).(x - e^{-\frac{2i\pi}{5}})$$

$$= (x-1)\left\{x^2 - x\left(e^{\frac{2i\pi}{5}} + e^{-\frac{2i\pi}{5}}\right) + 1\right\}\left\{x^2 - x\left(e^{\frac{4i\pi}{5}} + e^{-\frac{4i\pi}{5}}\right) + 1\right\}$$

$$= (x-1)\left\{x^2 - x.2\cos\frac{2\pi}{5} + 1\right\}\left\{x^2 - x.2\cos\frac{4\pi}{5} + 1\right\}$$

$$= (x-1)\left\{x^2 - 2x\cos\left(\pi - \frac{3\pi}{5}\right) + 1\right\} \qquad [\because \cos(\pi-\theta) = -\cos\theta]$$

$$= (x-1)\left\{x^2 + 2x\cos\frac{3\pi}{5} + 1\right\}\left\{x^2 + 2x\cos\frac{\pi}{5} + 1\right\}$$

$$\therefore \quad (x^5 - 1) = (x-1)\left\{x^2 + 2x\cos\frac{3\pi}{5} + 1\right\}\left\{x^2 + 2x\cos\frac{\pi}{5} + 1\right\}$$

5. Find all the roots of $x^{12} - 1 = 0$ using De'Moivre's theorem and identify the roots which are also the roots of $x^4 - x^2 + 1 = 0$

Solution: $x^{12} - 1 = 0 \Rightarrow (x^6)^2 - 1 = 0 \Rightarrow (x^6+1)(x^6-1) = 0$

All the roots of $x^6 + 1 = 0$ and $x^6 - 1 = 0$ are the roots of $x^{12} - 1 = 0$

$$x^6 + 1 = 0$$

$$x^6 = -1 = \cos\pi + i\sin\pi = \cos(2k_1\pi + \pi) + i\sin(2k_1\pi + \pi)$$

$$x = \{\cos(2k_1+1)\pi + i\sin(2k_1+1)\pi\}^{\frac{1}{6}} = \cos(2k_1+1)\frac{\pi}{6} + i\sin(2k_1+1)\frac{\pi}{6} = e^{i(2k_1+1)\frac{\pi}{6}}$$

Putting $k_1 = 0, 1, 2, 3, 4, 5$ we get all roots of $x^6 + 1 = 0$

$$x^6 - 1 = 0$$

$$x^6 = 1 = \cos 0 + i\sin 0 = \cos 2k_2\pi + i\sin 2k_2\pi$$

$$x = (\cos 2k_2\pi + i\sin 2k_2\pi)^{\frac{1}{6}} = \cos\frac{2k_2\pi}{6} + i\sin\frac{2k_2\pi}{6} = \cos\frac{k_2\pi}{3} + i\sin\frac{k_2\pi}{3} = e^{i\frac{k_2\pi}{3}}$$

Putting $k_2 = 0, 1, 2, 3, 4, 5$ we get all roots of $x^6 - 1 = 0$

All the roots of $x^{12} - 1 = 0$ are given by $e^{i(2k_1+1)\frac{\pi}{6}}$ and $e^{i\frac{k_2\pi}{3}}$ for $k_1 = k_2 = 0, 1, 2, 3, 4, 5$

Now
$$x^6 + 1 = 0$$

$$(x^2)^3 + 1 = 0$$

$$(x^2+1)\{(x^2)^2 - x^2 + 1\} = 0 \qquad [\because a^3 - b^3 = (a-b)(a^2 - ab + b^2)]$$

$$(x^2+1)(x^4 - x^2 + 1) = 0$$

This shows that all the roots of $x^6 + 1 = 0$ except $x = \pm i$ which corresponds to $x^2 + 1 = 0$ are the roots of $x^4 - x^2 + 1 = 0$.Roots of $x^6 + 1 = 0$ are

$$x_0 = e^{\frac{i\pi}{6}}, \quad x_1 = e^{i\frac{3\pi}{6}} = e^{\frac{i\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = i, \quad x_2 = e^{i\frac{5\pi}{6}}, \quad x_3 = e^{i\frac{7\pi}{6}}$$

$$x_4 = e^{i\frac{9\pi}{6}} = e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 - i = -i, \quad x_5 = e^{i\frac{11\pi}{6}}$$

Except $x_1 = i$ and $x_4 = -i$ remaining roots x_0, x_2, x_3 and x_5 are the roots of the equation $x^4 - x^2 + 1 = 0$.

6. Find all the values of $(1-i)^{\frac{2}{3}}$ using De'Moivre's theorem.

Solution : Let $z = (1-i)^{\frac{2}{3}}$

$$(1-i)^{\frac{2}{3}} = \left[\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right\}\right]^{\frac{2}{3}} = \left\{\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right\}^{\frac{2}{3}}$$

$$=\left\{(\sqrt{2})^2\left(\cos\frac{2\pi}{4}-i\sin\frac{2\pi}{4}\right)\right\}^{\frac{1}{3}}=\left\{2\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)\right\}^{\frac{1}{3}}=\left[2\left\{\cos\left(2k\pi+\frac{\pi}{2}\right)-i\sin\left(2k\pi+\frac{\pi}{2}\right)\right\}\right]^{\frac{1}{3}}$$

$$=2^{\frac{1}{3}}\left\{\cos(4k+1)\frac{\pi}{6}-i\sin(4k+1)\frac{\pi}{6}\right\}$$ [using De Moivre's theorem]

Putting $k=0,1,2,$

$$z_0=2^{\frac{1}{3}}\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)=2^{\frac{1}{3}}\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right), \qquad z_1=2^{\frac{1}{3}}\left(\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}\right)=2^{\frac{1}{3}}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)$$

$$z_2=2^{\frac{1}{3}}\left(\cos\frac{9\pi}{6}-i\sin\frac{9\pi}{6}\right)=2^{\frac{1}{3}}(i)$$

7.Calculate continued product of all the values of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$ using De'Moivre's theorem.

Solution : $\frac{1}{2}+i\frac{\sqrt{3}}{2} = r(\cos\theta + i\sin\theta)$

$$r=\left|\frac{1}{2}+i\frac{\sqrt{3}}{2}\right|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$$

$$\theta=\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$$

$$\frac{1}{2}+i\frac{\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}=\left(\cos3.\frac{\pi}{3}+i\sin3.\frac{\pi}{3}\right)^{\frac{1}{4}}=(\cos\pi+i\sin\pi)^{\frac{1}{4}}$$

$$=\{\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)\}^{\frac{1}{4}}=\left\{\cos(2k+1)\frac{\pi}{4}+i\sin(2k+1)\frac{\pi}{4}\right\}$$

Putting $k=0,1,2,3$

$$x_0=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=e^{\frac{i\pi}{4}}, \quad x_1=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=e^{i\frac{3\pi}{4}}$$

$$x_2=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=e^{i\frac{5\pi}{4}}, \quad x_3=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=e^{i\frac{7\pi}{4}}$$

Continued product is

$$x_0x_1x_2x_3=e^{\frac{i\pi}{4}}.e^{i\frac{3\pi}{4}}.e^{i\frac{5\pi}{4}}.e^{i\frac{7\pi}{4}}=e^{i\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)}=e^{i\frac{16\pi}{4}}=e^{i4\pi}=\cos4\pi+i\sin4\pi=1+i.0$$

Hence, continued product of all the values of $\frac{1}{2}+i\frac{\sqrt{3}}{2}$ is 1.

Exercise 23

1. Find the value of $\{i\}^{\frac{1}{5}}$ using De'Moivre's theorem

2. Solve $x^6+1=0$ using De'Moivre's theorem
3. Calculate the roots of following equations

   (i) $x^4-x^3+x^2-x+1=0$ (ii) $x^7+x^4+i(x^3+1)=0$

   Ans : (i) $x=e^{i(2k+1)\frac{\pi}{5}}$; $k=0,1,3,4$

   (ii) $x=e^{i(2k_1+1)\frac{\pi}{3}}$; $k_1=0,1,2$ and $x=e^{-i(4k_2+1)\frac{\pi}{8}}$; $k_2=0,1,2,3$.

4. Show using De'Moivre's theorem that all the roots of $(x+1)^7=(x-1)^7$ are given by $\pm i\cot\frac{k\pi}{7}$, $k=1,2,3$. Also check whether roots are in conjugate pairs or not.

5. Find continued product of $\{1+i\}^{\frac{4}{5}}$ using De'Moivre's theorem. *Ans.*: -4

Let's check take away from lecture

Choose the correct option from the following.

1. *The product of n^{th} roots of unity is*

   (a) 1 (b) $(-1)^{n-1}$ (c) -1 (d) *None*

2. *How many roots exists for the equation* $(x+1)^n=(x-1)^n$

   (a) $n-1$ (b) n (c) $n+1$ (d) *None*

3. *For the equation* $x^4-6x^3+15x^2-18x+10=0$ *if one root is* $2-i$ *then one of the other root is*

   (a) $-2+i$ (b) $2+i$ (c) $-2-i$ (d) *None*

Homework Problems for the day

1. Solve $x^6-1=0$ using De'Moivre's theorem
2. Solve the equation $x^2+x^{-2}=i$ using De'Moivre's theorem

   [**Hint** : $x^4-ix^2+1=0$, $x^2=\frac{i(1\pm\sqrt{5})}{2}$, $\quad x=\frac{(1\pm\sqrt{5})}{2}\left\{\cos\left(2k\pi+\frac{\pi}{2}\right)\frac{1}{2}+i\sin\left(2k\pi+\frac{\pi}{2}\right)\frac{1}{2}\right\}$

3. Solve $x^6+1=0$ using De'Moivre's theorem.

   Ans.: $x=e^{i(2k+1)\frac{\pi}{6}}$; $k=0,1,2,3,4,5$.

4. Calculate the roots of following equations

   (i) $x^4+x^3+x^2+x+1=0$ (ii) $(x+1)^8+x^8=0$ (iii) $(2x-1)^5=32x^5$

   Ans : (i) $x=e^{i\frac{2k\pi}{5}}$; $k=1,2,3,4$.

   (ii) $x=-\frac{1}{2}-\frac{i}{2}\cot(2k+1)\frac{\pi}{16}$; $k=0$ *to* 7.

(iii) **Hint :** $\left(\frac{2x-1}{2x}\right)^5=1,\quad \frac{2x-1}{2x}=\cos\frac{2k\pi}{5}+i\sin\frac{2k\pi}{5},\quad k=1,2,3,4$

5. Show that the roots of the equation $(x+1)^6+(x-1)^6=0$ are given by $-i\cot\left[\frac{(2k+1)\pi}{12}\right]$ *for* $k=0,1,2,3,4,5$ using De'Moivre's theorem .

6. Prove that n^{th} root of unity are in geometric progression. Also find sum of n^{th} roots of unity. Hence prove that $(1-w)^6=-27$.

7. If α, β are complex cube roots of unity, then prove that $\alpha^{3m}+\beta^{3m}=2$, where m is any integer.

8. Find all the values of $\left(\frac{2+3i}{1+i}\right)^{\frac{1}{4}}$ using De'Moivre's theorem.

Ans. : $\left(\frac{13}{2}\right)^{\frac{1}{8}}e^{i\left(2k\pi+\tan^{-1}\frac{1}{15}\right)}\quad k=0,1,2,3]$

9. Find continued product of (i) $\{i\}^{\frac{2}{3}}$ (ii) $(-1)^{\frac{1}{n}}$

Ans : (i) -1 (ii) $(-1)^n$

10. Calculate the roots of following equations

(i) $x^5+\sqrt{3}=i$. Ans. $x=2^{\frac{1}{5}}e^{\left(\frac{12k+5}{30}\right)\pi i}$; $k=0,1,2,3,4$

11. *If* $1+2i$ *is one root of the equation* $x^4-3x^3+8x^2-7x+5=0$, find other roots using De'Moivre's theorem. *Ans* : $1-2i,\ \frac{1}{2}\pm i\frac{\sqrt{3}}{2}$

Learning from the topic: Learner will be able to apply De'Moivre's theorem to find roots of equation and find the power, sum and Product of it.

Hyperbolic Functions

Lecture: 24

1. Learning Objectives:

Learners will be able to simplify the expressions involving circular, inverse circular and hyperbolic functions.

2. Introduction:

In mathematics, hyperbolic functions are analogues of the ordinary trigonometric functions defined for the hyperbola rather than on the circle: just as the points (cos t, sin t) form a circle with a unit radius, the points (cosh t, sinh t) form the right half of the equilateral hyperbola.

3. Key Notations: sinh$x$: Sine hyperbolic x

cosh$x$: Cosine hyperbolic x

4. Key Definitions: Hyperbolic functions: Hyperbolic functions are defined as

$\sinh z=\frac{e^z-e^{-z}}{2}$, $\cosh z=\frac{e^z+e^{-z}}{2}$ and $\tanh z=\frac{e^z-e^{-z}}{e^z+e^{-z}}$

where z can be real or complex.

From the above definitions following values of hyperbolic function are obtained.

Z	$-\infty$	0	∞
sinh z	$-\infty$	0	∞
cosh z	∞	1	∞
tanh z	-1	0	1

Note : sinh (-z) = -sinh z , cosh (-z) = cosh z

5. Important Formulae

(1) Relation between Circular and Hyperbolic functions

(i) $\sin iz=i\sinh z$ and $\sinh z=-i\sin iz$

Proof : By Euler's formula, $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$

Replacing z by iz ,

$$\sin iz=\frac{e^{i^2z}-e^{-i^2z}}{2i}=-i\frac{\left(e^{-z}-e^{z}\right)}{2}=i\frac{\left(e^{z}-e^{-z}\right)}{2}=i\sinh z$$

$\boxed{\sin iz=i\sinh z}$

$\sinh z=\frac{1}{i}\sin iz=-i\sin iz$

$\boxed{\sinh z=-i\sin iz}$

(ii) $\cos iz=\cosh z$

Proof : By Euler's formula, $\cos z=\frac{e^{iz}+e^{-iz}}{2}$

Replacing z by iz ,

$$\cos iz=\frac{e^{i^2z}+e^{-i^2z}}{2}=\frac{e^{-z}+e^{z}}{2}=\cosh z$$

$\boxed{\cos iz=\cosh z}$

(iii) $\tan iz=i\tanh z$ and $\tanh z=-i\tan iz$

Proof : $\tan iz=\frac{\sin iz}{\cos iz}=\frac{i\sinh z}{\cosh z}=-i\tanh z$

$\boxed{\tan iz=i\tanh z}$

$\tanh z=\frac{1}{i}\tan iz=-i\tan iz$

$\boxed{\tanh z=-i\tan iz}$

(2) Formulae on Hyperbolic Functions

A. (i) $\cosh^2 z-\sinh^2 z=1$

(ii) $1-\coth^2 z = -\operatorname{cosech}^2 z$

(iii) $1-\tanh^2 z = \operatorname{sech}^2 z$

B. (i) $\sinh 2z = 2\sinh z \cosh z$

(ii) $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 1 + 2\sinh^2 z$

(iii) $\tanh 2z = \dfrac{2\tanh z}{1+\tanh^2 z}$

C. (i) $\sinh 3z = 3\sinh z + 4\sinh^3 z$

(ii) $\cosh 3z = 4\cosh^3 z - 3\cosh z$

(iii) $\tanh 3z = \dfrac{3\tanh z + \tanh^3 z}{1+3\tanh^2 z}$

D. (i) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(ii) $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$

(iii) $\tanh(z_1 \pm z_2) = \dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

D. (i) $\sinh z_1 + \sinh z_2 = -2\sinh\left(\frac{z_1+z_2}{2}\right)\cosh\left(\frac{z_1-z_2}{2}\right)$

(ii) $\sinh z_1 - \sinh z_2 = 2\cosh\left(\frac{z_1+z_2}{2}\right)\sinh\left(\frac{z_1-z_2}{2}\right)$

(iii) $\cosh z_1 + \cosh z_2 = 2\cosh\left(\frac{z_1+z_2}{2}\right)\cosh\left(\frac{z_1-z_2}{2}\right)$

(iv) $\cosh z_1 - \cosh z_2 = 2\sinh\left(\frac{z_1+z_2}{2}\right)\sinh\left(\frac{z_1-z_2}{2}\right)$

F. (i) $2\sinh z_1 \cosh z_2 = \sinh(z_1+z_2) + \sinh(z_1-z_2)$

(ii) $2\cosh z_1 \sinh z_2 = \sinh(z_1+z_2) - \sinh(z_1-z_2)$

(iii) $2\cosh z_1 \cosh z_2 = \cosh(z_1+z_2) + \cosh(z_1-z_2)$

(iv) $2\sinh z_1 \sinh z_2 = \cosh(z_1+z_2) - \cosh(z_1-z_2)$

6. Sample problems :

1) Prove that $\tanh(\log\sqrt{3}) = 0.5$

Solution : $\tanh(\log\sqrt{3}) = \dfrac{e^{\log\sqrt{3}} - e^{-\log\sqrt{3}}}{e^{\log\sqrt{3}} + e^{-\log\sqrt{3}}} = \dfrac{\sqrt{3}-\frac{1}{\sqrt{3}}}{\sqrt{3}+\frac{1}{\sqrt{3}}} = \dfrac{3-1}{3+1} = 0.5$

2) Solve the equation $7\cosh x + 8\sinh x = 1$ for real values of x.

Solution: $7\cosh x + 8\sinh x = 1$

$$7\left(\frac{e^x+e^{-x}}{2}\right)+8\left(\frac{e^x-e^{-x}}{2}\right)=1$$

$$15e^x - \frac{1}{e^x} = 2$$

$$15e^{2x} - 2e^x - 1 = 0$$

$$e^x = \frac{2\pm\sqrt{4+60}}{30} = \frac{2\pm 8}{30} = \frac{1}{3}, -\frac{1}{5}$$

For real values of x, e^x cannot be negative.

$e^x = \frac{1}{3} \Rightarrow x = \log\frac{1}{3} = -\log 3.$

Exercise 24

1. *If* $\tanh x = \frac{2}{3}$, *find the value of* x *and* $\cosh 2x$.

   Ans.: $x = \log\sqrt{5}$, $\cosh 2x = \frac{13}{5}$

2. Prove that $(\cosh x - \sinh x)^n = \cosh nx - \sinh nx$

3. If $5\sinh x - \cosh x = 5$, find tan hx. Ans: $\frac{4}{5}$ or $\frac{-3}{5}$

4. *If* $\tanh x = \frac{1}{2}$, *prove that* cos h 2x = 5/3

Let's check take away from lecture

Choose the correct option from the following:

1. $\cosh x + \sinh x$ *is*

   (a) $-e^x$ (b) $-e^{-x}$ (c) $e^x + e^{-x}$ (d) e^x

2. $\dfrac{d(\cosh 3t)}{dt}$ *is*

   (a) $\sinh 3t$ (b) $-\sinh 3t$ (c) $3\sinh 3t$ (d) *None*

3. $\tanh(\log\sqrt{5})$ *is*

   (a) 3 (b) 2 (c) $\frac{2}{3}$ (d) *None*

4. $2\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right)$ *is*

   (a) $\cosh A + \cosh B$ (b) $\cosh\frac{A}{2} + \cosh\frac{B}{2}$ (c) $\cosh A - \cosh B$ (d) $\cos B - \cos A$

5. *If* $\tanh x = \frac{1}{2}$ *then value of* $\sinh 4x$ *is*

(*a*) $\frac{42}{9}$ (*b*) $\frac{40}{9}$ (*c*) $\frac{40}{3}$ (*d*) $-\frac{40}{9}$

Homework Problems for the day

1. *Find the value of* $\tanh \log\sqrt{5}$.

   Ans. $\frac{2}{3}$

2. *If* $\tanh x = \frac{1}{2}$, *prove that* $\sinh 2x = \frac{4}{3}$

3. *Solve the equation for real values of* x, $17\cosh x + 18\sinh x = 1$.

   Ans.: $x = -\log 5$.

4. *Find* $\tanh x$ *if* $5\sinh x - \cosh x = 5$.

   Ans : $4/5$

5. *Prove that* $\cosh^5 x = \frac{1}{16}[\cosh 5x + 5\cosh 3x + 10\cosh x]$

6. *Find* $\tanh x$ *if* $6\sinh x + 2\cosh x + 7 = 0$.

   Ans : $-15/17$

Learning from the topic: Learner will be able to apply Hyperbolic function to simplify the expression.

Hyperbolic Functions Continued ……

Lecture: 25

Sample problems :

1.If $u = \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$, prove that

(i) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$ (ii) $\cosh u = \sec\theta$

Solution :

(i) $u = \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$

$$e^u = \frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{\theta}{2}} \Rightarrow \frac{e^u}{1} = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} \Rightarrow u = \log\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right) = 2\tanh^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\Rightarrow \frac{u}{2} = \tanh^{-1}\left(\tan\frac{\theta}{2}\right) \Rightarrow \tanh\frac{u}{2} = \tan\frac{\theta}{2}$$

(ii) From (i) $\tanh\frac{u}{2} = \tan\frac{\theta}{2} \Rightarrow \tanh^2\frac{u}{2} = \tan^2\frac{\theta}{2}$

Using componendo-dividendo,

$$\frac{1+\tanh^2\frac{u}{2}}{1-\tanh^2\frac{u}{2}} = \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \Rightarrow \cos hu = \sec\theta$$

2.Prove that $\cosh^5 x = \frac{1}{16}[\cosh 5x + 5\cosh 3x + 10\cosh x]$

Solution :

$$\cosh^5 x = \left(\frac{e^x+e^{-x}}{2}\right)^5 = \frac{1}{2^5}(e^{5x} + 5e^{4x}\cdot e^{-x} + 10e^{3x}\cdot e^{-2x} + 10e^{2x}\cdot e^{-3x} + 5e^x\cdot e^{-4x} + e^{-5x})$$

$$= \frac{1}{2^5}\{(e^{5x}+e^{-5x}) + 5(e^{3x}+e^{-3x}) + 10(e^x+e^{-x})\}$$

$$= \frac{1}{2^5}(2\cosh 5x + 10\cosh 3x + 20\cosh x) = \frac{1}{16}(\cosh 5x + 5\cosh 3x + 10\cosh x)$$

3) If $\log(\tan x) = y$, prove that

(i) $\cosh ny = \frac{1}{2}(\tan^n x + \cot^n x)$ (ii) $\sinh ny = \frac{1}{2}(\tan^n x - \cot^n x)$

(iii) $\sinh(n+1)y + \sinh(n-1)y = 2\sinh ny \operatorname{cosec} 2x$

(iv) $\cosh(n+1)y + \cosh(n-1)y = 2\cosh ny \operatorname{cosec} 2x$

Solution : $\log(\tan x) = y$

$$\left.\begin{aligned} e^y &= \tan x \\ e^{-y} &= \cot x \end{aligned}\right\} \quad \ldots (1)$$

(i) $\cosh ny = \frac{e^{ny}+e^{-ny}}{2} = \frac{1}{2}\{(e^y)^n + (e^{-y})^n\} = \frac{1}{2}(\tan^n x + \cot^n x)$ [using (1)]

(ii) $\sinh ny = \frac{1}{2}(\tan^n x - \cot^n x)$

$\sinh ny = \frac{e^{ny}-e^{-ny}}{2} = \frac{1}{2}(\tan^n x - \cot^n x)$ [using (1)]

(iii) $\sinh(n+1)y + \sinh(n-1)y = 2\sinh ny \operatorname{cosec} 2x$

$$\sinh(n+1)y + \sinh(n-1)y = 2\sinh\frac{(n+1+n-1)y}{2}\cosh\frac{(n+1-n+1)y}{2}$$

$$= 2\sinh ny\cosh y = 2\sinh ny\cdot\frac{e^y+e^{-y}}{2} = 2\sinh ny\left(\frac{\tan x+\cot x}{2}\right)$$

$$= 2\sinh ny\left(\frac{\sin^2 x+\cos^2 x}{2\sin x\cos x}\right) = 2\sinh ny\left(\frac{1}{\sin 2x}\right) = 2\sinh ny \operatorname{cosec} 2x$$

(iv) $\cosh(n+1)y + \cosh(n-1)y = 2\cosh ny \operatorname{cosec} 2x$

$$\cosh(n+1)y + \cosh(n-1)y = 2\cosh\frac{(n+1+n-1)y}{2}\cosh\frac{(n+1-n+1)y}{2}$$

$= 2\cosh ny \cosh y = 2\cosh ny\left(\frac{e^{y}+e^{-y}}{2}\right) = 2\cosh ny\left(\frac{\tan x+\cot x}{2}\right)$

$= 2\cosh ny\left(\frac{\sin^2 x+\cos^2 x}{2\sin x\cos x}\right) = 2\cosh ny \operatorname{cosec} 2x$

Exercise 25

1. *Prove that* $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1+\sinh^2 x}}} = -\sinh^2 x.$

2. *Prove that* $\left(\dfrac{1+\tanh x}{1-\tanh x}\right)^3 = \cosh 6x + \sinh 6x.$

3. If $\tan\left(\dfrac{x}{2}\right) = \tanh\left(\dfrac{u}{2}\right)$ then prove that

(i) $\sin h\, u = \tan x$ (ii) $\cos h\, u = \sec x$ (iii) $u = \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$

Let's check take away from lecture

Choose the correct option from the following:

1. What is the value of $(\cosh x + \sin h\, x)^2$ is
(a) $\cosh 2x + \sin h 2x$ (b) $\cosh 2x - \sin h 2x$ (c) $\cos h 2x$ (d) $\sin h 2x$

2. What is the value of $(\cosh x - \sin h\, x)^2$ is
(a) e^{-4x} (b) e^{4x} (c) e^{-i4x} (d) e^{i4x}

3. If $\log(\tan x) = y$ then $\sinh 2y$ is
(a) $\frac{1}{2}(\tan^2 x - \cot^2 x)$ (b) $(\tan^2 x - \cot^2 x)$ (c) $(\tan^2 x + \cot^2 x)$ (d) $\frac{1}{2}(\tan^2 x + \cot^2 x)$

4. If $\cosh^6 x = a\cos h\, 6x + b\cos h\, 4x + c\cos h\, 2x + d$ then what is the value of $5a-5b+3c-2d$
(a) 1 (b) 0 (c) 2 (d) 3

5. What is the coefficient of $\sin h\, 5x$ in expansion of $\sin^5 x$
(a) 2/16 (b) 1/16 (c) 5/16 (d) 3/16

Homework Problems for the day

1. *Prove that* $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\cosh^2 x}}} = \cosh^2 x.$

2. Prove that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

3. If $\tan\left(\dfrac{x}{2}\right) = \tanh\left(\dfrac{u}{2}\right)$ then prove that

(i) $\sin h\, u = \tan x$ (ii) $\cos h\, u = \sec x$ (iii) $u = \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$

4. If $u = \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$, prove that $\tan h\, u = \sin\theta$

Learning from the topic: Learner will be able to apply Hyperbolic function to simplify the expression.

Inverse Hyperbolic Functions

Lecture: 26

1. Learning Objectives:

Learners will be able to simplify the expressions involving inverse circular and inverse hyperbolic functions.

2. Introduction: The inverse hyperbolic functions are the inverse functions of the hyperbolic functions. For a given value of a hyperbolic function, the corresponding inverse hyperbolic function provides the corresponding hyperbolic angle.

3. Key Notations:

$\sinh^{-1} x$: Inverse Sine hyperbolic x

$\cosh^{-1} x$: Inverse Cosine hyperbolic x

4. Key Definitions:

Inverse Hyperbolic functions: Let $x = \sinh y$ then $y = \sinh^{-1} x$ is defined as inverse hyperbolic function of x. Where x can be complex or real.

Similarly, we can define $\cosh^{-1} x, \tanh^{-1} x, \coth^{-1} x, \operatorname{sech}^{-1} x, \operatorname{cosech}^{-1} x$.

5. Important Formulae:

1. $\sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)$
2. $\cosh^{-1} x = \log\left\{x+\sqrt{x^2-1}\right\}$

3. $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

6. Sample Problems:

1) Prove that $\sinh^{-1}(\tan\theta) = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$.

Solution : $\sinh^{-1}(\tan\theta) = \log\left(\tan\theta + \sqrt{\tan^2\theta+1}\right) = \log(\tan\theta + \sec\theta)$

$$= \log\left(\frac{\sin\theta+1}{\cos\theta}\right) = \log\left\{\frac{\cos\left(\frac{\pi}{2}-\theta\right)+1}{\sin\left(\frac{\pi}{2}-\theta\right)}\right\}$$

$$= \log\left\{\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right\} = \log\left\{\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

$$= \log\left[\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}\right] = \log\left\{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}.$$

2) Prove that $\operatorname{cosech}^{-1} z = \log\left[\frac{1+\sqrt{1+z^2}}{z}\right]$. Is it defined for all values of z?

Solution : Let $y = \operatorname{cosech}^{-1} z \Rightarrow \operatorname{cosech} y = z \Rightarrow \frac{2}{e^y - e^{-y}} = z \Rightarrow e^y - \frac{1}{e^y} = \frac{2}{z} \Rightarrow e^{2y} - \frac{2}{z}e^y - 1 = 0$

$$e^y = \frac{\frac{2}{z} \pm \sqrt{\frac{4}{z^2}+4}}{2} = \frac{1+\sqrt{1+z^2}}{z}\left[\because 1-\sqrt{1+z^2} < 0 \text{ and } e^y \text{ is always positive}\right] y = \log\left[\frac{1+\sqrt{1+z^2}}{z}\right]$$

Exercise 26

1. Prove that (i) $\tanh^{-1} x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

   (ii) $\operatorname{sech}^{-1}(\sin\theta) = \log\cot\left(\frac{\theta}{2}\right)$

2. *If* $\cosh x = \sec\theta$, *prove that*

   (*i*) $x = \log(\sec\theta + \tan\theta)$ (*ii*) $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$

   (*iii*) $\sinh x = \tan\theta$ (*iv*) $\tanh x = \sin\theta$

   (*v*) $\tanh\frac{x}{2} = \pm\tan\frac{\theta}{2}$

3. *Prove that* $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = -\frac{i}{2}\log\left(\frac{a}{x}\right)$

Choose the correct option from the following:

1. $\sinh^{-1} x$ is
   (a) $\log\left(x+\sqrt{x^2+1}\right)$ (b) $\log\left(x+\sqrt{x^2-1}\right)$ (c) $\log\left(x-\sqrt{x^2+1}\right)$ (d) $\log\left(x-\sqrt{x^2-1}\right)$

2. The value of $\operatorname{sech}^{-1}(\sin\theta)$ is
   (a) $\log\cot\left(\frac{\theta}{2}\right)$ (b) $\log\cos\left(\frac{\theta}{2}\right)$ (c) $\log\sin\left(\frac{\theta}{2}\right)$ (d) $\log\tan\left(\frac{\theta}{2}\right)$

3. The value of $\operatorname{sech}^{-1}(x)$ is
   (a) $\log\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ (b) $\log\left(\frac{1-\sqrt{1-x^2}}{x}\right)$ (c) $\log\left(\frac{1+\sqrt{1+x^2}}{x}\right)$ (d) $\log\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

4. The value of $\sinh^{-1} 5$ is
   (a) $\log\left(5+\sqrt{26}\right)$ (b) $\log\left(5+\sqrt{24}\right)$ (c) $\log\left(5-\sqrt{26}\right)$ (d) $\log\left(5-\sqrt{24}\right)$

Homework Problems for the day

1. Prove that

   (i) $\sinh^{-1} x = \cosh^{-1}\left(\sqrt{1+x^2}\right)$

   (ii) $\tanh^{-1}(\cos\theta) = \cosh^{-1}(\operatorname{cosec}\theta)$

   (iii) $\sinh^{-1}(\tan\theta) = \log(\sec\theta + \tan\theta)$

   (iv) $\tanh^{-1}(\sin\theta) = \cosh^{-1}(\sec\theta)$

2. If $\tan\frac{x}{2} = \tanh\frac{u}{2}$, show that

   (i) $\sinh u = \tan x$

   (ii) $\cosh u = \sec x$

   (iii) $u = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

3. *If* $u = \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$, *prove that*

   (*i*) $\sinh u = \tan\theta$ (*ii*) $\tanh u = \sin\theta$

Learning from the topic: Learner will be able to apply Inverse Hyperbolic function

Separation of real and imaginary part of Hyperbolic Function

Lecture: 27

1. Learning Objectives:

Learners will be able to identify the real and imaginary parts of complex expressions of sine and cosine ,tangent of circular and Hyperbolic functions.

2. Introduction:

Most of the problems of complex quantities can simplified involving circular, Hyperbolic, and logarithmic functions by separating them into real and imaginary parts. To separate into real and imaginary parts first we write complex conjugate of the given equation e.g. If $\tan(\alpha + i\beta) = x + iy$ is given then write its complex conjugate $\tan(\alpha - i\beta) = x - iy$ and then add, subtract, or take inverse according to the required result.

3. Important Formulae:

(i) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

(ii) $\sin(x - iy) = \sin x \cosh y - i \cos x \sinh y$

(iii) $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

(iv) $\cos(x - iy) = \cos x \cosh y + i \sin x \sinh y$

(v) $\tan(x + iy) = \dfrac{\sin 2x}{\cos 2x + \cosh 2y} + i\dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$

(vi) $\tan(x - iy) = \dfrac{\sin 2x}{\cos 2x + \cosh 2y} + i\dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$

(vii) $\tanh(x + iy) = \dfrac{\sinh 2x}{\cosh 2x + \cos 2y} + i\dfrac{\sin 2y}{\cosh 2x + \cos 2y}$

(viii) $\tanh(x - iy) = \dfrac{\sinh 2x}{\cosh 2x + \cos 2y} + i\dfrac{\sin 2y}{\cosh 2x + \cos 2y}$

(ix) $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$

(x) $\sinh(x - iy) = \sinh x \cos y - i \cosh x \sin y$

(xi) $\cosh(x + iy) = \cosh x \cos y - i \sinh x \sin y$

(xii) $\cosh(x - iy) = \cosh x \cos y + i \sinh x \sin y$

4. Sample Problems:

1. If $\tan(x + iy) = 2 + 3i$, find value of x

Solution : $\because \tan(x + iy) = 2 + 3i$

$\therefore \tan(x - iy) = 2 - 3i$

$$\tan 2x = \tan[(x+iy)+(x-iy)] = \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$= \frac{2+3i+2-3i}{1-(2+3i)(2-3i)} = \frac{4}{1-13} = \frac{4}{-12} = -1/3$$

$$2x = \tan^{-1}(-1/3)$$

$$x = \frac{1}{2}\tan^{-1}(-1/3)$$

2. If $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$, show that $\cos 2\theta \cosh 2\phi = 3$.

Solution : $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$

$\sin\theta\cos i\phi + \cos\theta\sin i\phi = \tan\alpha + i\sec\alpha$

$\sin\theta\cosh\phi + i\cos\theta\sinh\phi = \tan\alpha + i\sec\alpha$

Comparing real and imaginary parts,

$\sin\theta\cosh\phi = \tan\alpha$ ... (1)

$\cos\theta\sinh\phi = \sec\alpha$ ... (2)

Eliminating α from equation (1) and equation (2),

$\sec^2\alpha - \tan^2\alpha = \cos^2\theta\sinh^2\phi - \sin^2\theta\cosh^2\phi$

$$1 = \frac{(1+\cos 2\theta)}{2}\frac{(\cosh 2\phi - 1)}{2} - \frac{(1-\cos 2\theta)}{2}\frac{(1+\cosh 2\phi)}{2}$$

$4 = \cosh 2\phi - 1 + \cos 2\theta\cosh 2\phi - \cos 2\theta - 1 - \cosh 2\phi + \cos 2\theta + \cos 2\theta\cosh 2\phi$

$= -2 + 2\cos 2\theta\cosh 2\phi$

$3 = \cos 2\theta\cosh 2\phi$.

3. If $\cos(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$ prove that $\phi = \dfrac{1}{2}\log\left[\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right]$.

Solution : $r(\cos\alpha + i\sin\alpha) = \cos(\theta + i\phi)$

$= \cos\theta\cos i\phi - \sin\theta\sin i\phi = \cos\theta\cosh\phi - i\sin\theta\sinh\phi$

Comparing real and imaginary parts,

$r\cos\alpha = \cos\theta\cosh\theta$ ... (1) and $r\sin\alpha = -\sin\theta\sinh\theta$ ... (2)

Dividing equation (2) by equation (1) ,

$$\frac{r\sin\alpha}{r\cos\alpha} = \frac{-\sin\theta\sinh\phi}{\cos\theta\cosh\phi}$$

$$\tan\alpha = -\tanh\phi\tan\theta \Rightarrow \tanh\phi = -\tan\alpha\cdot\cot\theta$$

$$\phi = \tanh^{-1}(-\tan\alpha\cot\theta) = \frac{1}{2}\log\frac{1-\tan\alpha\cot\theta}{1+\tan\alpha\cot\theta}$$

$$= \frac{1}{2}\log\frac{\cos\alpha\sin\theta - \sin\alpha\cos\theta}{\cos\alpha\sin\theta + \sin\alpha\cos\theta} = \frac{1}{2}\log\left[\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right].$$

4. Prove that $\operatorname{cosech}^{-1} z = \log\left[\frac{1+\sqrt{1+z^2}}{z}\right]$. Is it defined for all values of z?

Solution : Let

$$y = \operatorname{cosech}^{-1} z$$

$$\operatorname{cosech} y = z$$

$$\frac{2}{e^y - e^{-y}} = z$$

$$e^y - \frac{1}{e^y} = \frac{2}{z}$$

$$e^{2y} - \frac{2}{z}e^y - 1 = 0$$

$$e^y = \frac{\frac{2}{z} \pm \sqrt{\frac{4}{z^2}+4}}{2}$$

$$= \frac{1+\sqrt{1+z^2}}{z} \quad \left[\because\ 1-\sqrt{1+z^2} < 0 \text{ and } e^y \text{ is always positive}\right]$$

$$y = \log\left[\frac{1+\sqrt{1+z^2}}{z}\right]$$

It is not defined for $z < 0$.

5). Separate into real and imaginary parts and identify them.

(i) $\tan(x+iy)$

(ii) $\tan^{-1}(x+iy)$

Solution :

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\text{Real part} = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

$$\text{Imaginary part} = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

(ii) Let $a + ib = \tan^{-1}(x+iy)$ ... (1)

$a - ib = \tan^{-1}(x-iy)$ ... (2)

Adding equations (1) and (2),

$$2a = \tan^{-1}(x+iy) + \tan^{-1}(x-iy) = \tan^{-1}\frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$a = \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}$$

Subtracting equation (2) from equation (1),

$$2ib = \tan^{-1}(x+iy) - \tan^{-1}(x-iy) = \tan^{-1}\frac{x+iy-x+iy}{1+(x+iy)(x-iy)}$$

$$\tan 2ib = \frac{2iy}{1+x^2+y^2} \implies i\tanh 2b = \frac{2iy}{1+x^2+y^2}$$

$$2b = \tanh^{-1}\frac{2y}{1+x^2+y^2} = \frac{1}{2}\log\frac{1+\frac{2y}{1+x^2+y^2}}{1-\frac{2y}{1+x^2+y^2}} = \frac{1}{2}\log\frac{1+x^2+y^2+2y}{1+x^2+y^2-2y}$$

$$b = \frac{1}{4}\log\frac{(1+y)^2+x^2}{(1-y)^2+x^2}$$

Hence $\tan^{-1}(x+iy) = \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2} + \frac{i}{4}\log\frac{(1+y)^2+x^2}{(1-y)^2+x^2}$

6). Separate real and imaginary parts of $(1+i)^i$

Solution: Let $x + iy = (1+i)^i$ Taking log on both the sides,

$$\log(x+iy) = i\log(1+i) = i\left\{\frac{1}{2}\log 2 + i\tan^{-1}1\right\} = \frac{i}{2}\log 2 - \frac{\pi}{4}$$

$$x + iy = e^{\frac{i}{2}\log 2}\, e^{-\frac{\pi}{4}} = e^{i\log\sqrt{2}}\, e^{-\frac{\pi}{4}} = e^{-\frac{\pi}{4}}\left\{\cos\left(\log\sqrt{2}\right) + \sin\left(\log\sqrt{2}\right)\right\}$$

Comparing real and imaginary part on both the sides,

$$x = e^{-\frac{\pi}{4}}\cos\left(\log\sqrt{2}\right), \quad y = e^{-\frac{\pi}{4}}\sin\left(\log\sqrt{2}\right)$$

Exercise 27

1. Identify real and imaginary parts of $\cos^{-1}\frac{3i}{4}$

Ans.: $(i)\ \frac{\pi}{2} + i\log\left(\frac{1}{2}\right)$

2. If $\cos(u+iv) = x+iy$, prove that $(1+x)^2 + y^2 = (\cosh v + \cos u)^2$
3. If $\tan(\theta + i\phi) = e^{i\alpha}$, show that $\phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$
4. If $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that that $\cos^4\theta = \sin^2\alpha = \sinh^4\phi$

Let's check take away from lecture

1. If $\tan\left(\frac{\pi}{6}+i\alpha\right) = (x+iy)$ then $x^2+y^2+\frac{2x}{\sqrt{3}}\cos x\cosh y$ is

(a) 1 (b) -1 (c) 2 (d) 0

2. The value of $\sin^{-1}(\operatorname{cosec}\theta)$ is

(a) $\frac{\pi}{2} + i\log\cot\left(\frac{\theta}{2}\right)$ (b) $\frac{\pi}{2} + i\log\tan\left(\frac{\theta}{2}\right)$ (c) $\frac{\pi}{2} + i\log\cos\left(\frac{\theta}{2}\right)$ (d) $\frac{\pi}{2} + i\log\sin\left(\frac{\theta}{2}\right)$

3. The real part of $\sin^{-1}(\operatorname{cosec}\theta)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

4. The imaginary part of $\sin^{-1}(\operatorname{cosec}\theta)$ is

(a) $\log\cot\left(\frac{\theta}{2}\right)$ (b) $\log\tan\left(\frac{\theta}{2}\right)$ (c) $\log\cos\left(\frac{\theta}{2}\right)$ (d) $\log\sin\left(\frac{\theta}{2}\right)$

5. The imaginary part of $\cos(x+iy)$ is

(a) sin x sinhy (b) sinh x sinhy (c) -sin x sinhy (d) sinh x siny

Homework Problems for the day

1. If $\cos(\alpha \pm i\beta) = x + iy$, prove that

   (i) $\dfrac{x^2}{\cosh^2\beta} + \dfrac{y^2}{\sinh^2\beta} = 1$ (ii) $\dfrac{x^2}{\cos^2\alpha} - \dfrac{y^2}{\sin^2\alpha} = 1$

2. *If* $\tan\left(\dfrac{\pi}{6} + i\alpha\right) = x + iy$, *prove that* $x^2 + y^2 + \dfrac{2x}{\sqrt{3}} = 1$.

3. If $\operatorname{cosec}\left(\dfrac{\pi}{4} \pm ix\right) = u + iv$, prove that $(u^2+v^2)^2 = 2(u^2 - v^2)$

4. If $\cos(\theta \pm i\phi) = \cos\alpha + i\sin\alpha$, prove that $\theta = \dfrac{1}{2}\log\left[\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right]$

Learning from the topic: Learner will be able to separate the function of complex variable i real and Imaginary part Including logarithmic function.

Logarithmic Function

Lecture: 28

1. Learning Objectives:

Learners will be able to identify the concept of Logarithmic functions of complex variable and comp number.

2. Introduction:

In this topic log is introduced as complex valued function so it is defined for negative r numbers also. In this topic log function is separated in to real and imaginary part.

3. Key Notations:

1. log z: Principle value of logarithm 2. Log z: General value of logarithm

4. Key Definitions:

Logarithm of a complex number: The natural logarithm of $z = x + iy$ is denoted by log z or ln z and is defined as the inverse of the exponential function; i.e. w = log z defined for $z \neq 0$ as $e^w = z$. If $z = x + iy$, then

$\log z = \log|z| + i\arg z = \log\sqrt{x^2+y^2} + i\tan^{-1}\dfrac{y}{x}$ (Cartesian form) and

$\log z = \log r + i\theta$ (Polar form)

5. Important Formulae:

(1) $\log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\dfrac{y}{x}$ (2) $\log\left(re^{i\theta}\right) = \log r + i\theta$

(3) $Log(x+iy) = \log\sqrt{x^2+y^2} + i(2n\pi + \tan^{-1}\dfrac{y}{x})$

6. Sample Problems:

1). Simplify $\log(e^{i\alpha} + e^{i\beta})$.

Solution : $\log(e^{i\alpha} + e^{i\beta}) = \log[\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta)]$

$= \log[(\cos\alpha + \cos\beta) + i(\sin\alpha + \sin)\beta]$

$$= \log\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2i\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]$$

$$= \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\left\{\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right\}\right]$$

$$= \log 2\cos\left(\frac{\alpha-\beta}{2}\right) + \log\left[\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right] = \log 2\cos\left(\frac{\alpha-\beta}{2}\right) + \log e^{i\left(\frac{\alpha+\beta}{2}\right)}$$

$$= \log 2\cos\left(\frac{\alpha-\beta}{2}\right) + i\left(\frac{\alpha+\beta}{2}\right)$$

2). Prove that $\log\left[\dfrac{\sin(x+iy)}{\sin(x-iy)}\right] = 2i\tan^{-1}(\cot x\tanh y)$.

Solution:

$$\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = \log\{\sin(x+iy)\} - \log\{\sin(x-iy)\}$$

$$= \log\{\sin x\cosh y + i\cos x\sinh y\} - \log\{\sin x\cosh y - i\cos x\sinh y\}$$

$$= \frac{1}{2}\log\{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y\} + i\tan^{-1}\left(\frac{\cos x\sinh y}{\sin x\cosh y}\right)$$

$$= -\frac{1}{2}\log\{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y\} - i\tan^{-1}\left(\frac{-\cos x\sinh y}{\sin x\cosh y}\right)$$

$$= i\tan^{-1}(\cot x\tanh y) + i\tan^{-1}(\cot x\tanh y)$$

$$= 2i\tan^{-1}(\cot x\tanh y).$$

3). If $i^{i^{\cdot^{\cdot^{\infty}}}} = z$, where $z = x + iy$, then prove that

(i) $|\bar{z}|^2 = e^{-(4n+1)\pi y}, n \in I$

(ii) $\tan\dfrac{\pi x}{2} = \dfrac{y}{x}$ and $x^2 + y^2 = e^{-\pi y}$

Solution : $i^{i^{\cdot^{\cdot^{\infty}}}} = x + iy \Rightarrow i^{x+iy} = x + iy$ Taking log on both sides,

$$(x+iy)\log i = \log(x+iy) \Rightarrow \log(x+iy) = (x+iy)\, i\left(2n\pi + \frac{\pi}{2}\right), n \in I$$

$$= i\left(2n\pi + \frac{\pi}{2}\right)x - \left(\frac{4n+1}{2}\right)\pi y$$

$$x + iy = e^{i(4n+1)\frac{\pi x}{2}}\, e^{-\left(\frac{4n+1}{2}\right)\pi y}$$

Comparing with $re^{i\theta}$,

$$r= |x+iy| = |x-iy| = \left[e^{-\left(\frac{4n+1}{2}\right)\pi y}\right]$$

$$|x-iy|^2 = \left[e^{-\left(\frac{4n+1}{2}\right)\pi y}\right]^2 \Rightarrow |\bar{z}|^2 = e^{-(4n+1)\pi y}. \text{ and } \theta = \tan^{-1}\frac{y}{x} = \left(\frac{4n+1}{2}\right)\pi x$$

For $n=0$, $\tan^{-1}\frac{y}{x} = \frac{\pi x}{2}$, $\frac{y}{x} = \tan\frac{\pi x}{2}$ and $|\bar{z}|^2 = e^{-\pi y}$, $x^2+y^2 = e^{-\pi y}$.

4). Prove that real part of principle value of $(1+i)^{\log i}$ is $e^{-\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right)$

Solution : Let $x+iy = (1+i)^{\log i}$ Taking log on both the sides,

$$\log(x+iy) = \log i \cdot \log(1+i) = \frac{i\pi}{2}\left\{\frac{1}{2}\log(2) + i\tan^{-1} 1\right\}$$

$$= \frac{i\pi}{2}\cdot\frac{1}{2}\log 2 + i^2\frac{\pi}{2}\cdot\frac{\pi}{4} = \frac{i\pi}{4}\log 2 - \frac{\pi^2}{8} = \frac{-\pi^2}{8} + \frac{i\pi}{4}\log 2$$

$$x+iy = e^{-\frac{\pi^2}{8}}\cdot e^{\frac{i\pi}{4}\log 2} = e^{-\frac{\pi^2}{8}}\left\{\cos\left(\frac{\pi}{4}\log 2\right) + i\sin\left(\frac{\pi}{4}\log 2\right)\right\}$$

Comparing real part on both the sides,

$$\text{Real part} = e^{-\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right).$$

5). Show that $(1+i\tan\alpha)^{-i} = e^{2m\pi+\alpha}[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)]$

Solution : Let $x+iy = (1+i\tan\alpha)^{-i}$ Taking log on both the sides,

$$\log(x+iy) = -i\log(1+i\tan\alpha) = -i\left\{\frac{1}{2}\log(1+\tan^2\alpha) + i(2m\pi + \tan^{-1}\tan\alpha)\right\}$$

$$= -i\left\{\frac{1}{2}\log\sec^2\alpha + i(2m\pi+\alpha)\right\} = i\log(\sec^2\alpha)^{-\frac{1}{2}} + (2m\pi+\alpha)$$

$$= i\log(\cos\alpha) + (2m\pi+\alpha)$$

$$x+iy = e^{i\log\cos\alpha}e^{(2m\pi+\alpha)} = e^{2m\pi+\alpha}[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)]$$

Hence, $(1+i\tan\alpha)^{-i} = e^{2m\pi+\alpha}[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)]$

6). Prove that $i^{i^i} = \cos\theta + i\sin\theta$ where $\theta = (4n+1)\frac{\pi}{2}e^{-\left(2m+\frac{1}{2}\right)\pi}$

Solution : Let $i^i = x+iy$ Taking log on both the sides,

$$i\log i = \log(x+iy) \Rightarrow i\cdot i\left(2m\pi + \frac{\pi}{2}\right) = \log(x+iy)$$

$$x+iy = e^{-\left(2m+\frac{1}{2}\right)\pi} \Rightarrow i^i = e^{-\left(2m+\frac{1}{2}\right)\pi}$$

$$\cos\theta + i\sin\theta = i^{i^i} = i^{e^{-\left(2m+\frac{1}{2}\right)\pi}}$$

Taking log on both the sides,

$$\log(\cos\theta + i\sin\theta) = e^{-\left(2m+\frac{1}{2}\right)\pi}\log i = e^{-\left(2m+\frac{1}{2}\right)\pi}\, i\left(2n\pi + \frac{\pi}{2}\right)$$

$$= e^{-\left(2m+\frac{1}{2}\right)\pi}\cdot i(4n+1)\frac{\pi}{2} = i\phi, \text{ say}$$

$$\cos\theta + i\sin\theta = e^{i\phi} \Rightarrow e^{i\theta} = e^{i\phi}$$

Comparing both the sides, $\theta = i\phi = e^{-\left(2m+\frac{1}{2}\right)\pi}\cdot(4n+1)\frac{\pi}{2}$

Exercise 28

1. Evaluate the value of $\log_2(-3)$ **Ans.** $\log_2(-3) = \frac{\log 3 + i\pi}{\log 2}$
2. Prove that i^i *is real and hence find the value of* $\sin\log_e i^i$ **Ans** -1
3. Show that $\log\left\{\frac{\cos(x-iy)}{\cos(x+iy)}\right\} = 2i\tan^{-1}(\tan x\tanh y)$.
4. Show that $\tan\left[i\log\left\{\frac{(a-ib)}{(a+ib)}\right\}\right] = \frac{2ab}{a^2-b^2}$.
5. If $\tan(\log(x+iy)) = a+ib$ and $a^2+b^2 \neq 1$, then prove that
   $$\tan\left[\log(x^2+y^2)\right] = \frac{2a}{1-(a^2+b^2)}.$$
6. Separate the real & imaginary part of principle value of $(1+i\sqrt{3})^{1+i\sqrt{3}}$

Let's check take away from lecture

1. Find the value of log(-6)

   (a) $\log 6 + 2i\pi$ (b) $\log 6 + i\pi$ (c) $\log 6 - 2i\pi$ (d) $\log 6 - i\pi$

2. Find the value of $\log_3(-2)$ is

   (a) $\frac{\log 2 + i\pi}{\log 3}$ (b) $\frac{\log 2 - i\pi}{\log 3}$ (c) $\frac{\log 2 + 2i\pi}{\log 3}$ (d) $\frac{\log 2 - 2i\pi}{\log 3}$

3. Value of i^i *in term of* e is

   (a) $e^{-\frac{\pi}{3}}$ (b) $e^{-\frac{\pi}{2}}$ (c) $e^{-\frac{\pi}{6}}$ (d) $e^{-\frac{3\pi}{2}}$

4. The value of $\tan i\theta$ is

   (a) $i\tan\theta$ (b) $-i\tanh\theta$ (c) $-i\coth\theta$ (d) $i\tanh\theta$

5. Real part of log (1+i) is

   (a) $log 2$ (b) $log\sqrt{2}$ (c) $log\sqrt[3]{2}$ (d) $log 3$

Homework Problems for the day

1. Determine the values of (*i*) $\log i$ (*ii*) $\log(-5)$ (*iii*) $\log(1+i)$.

   Ans. (i) $\log i = \frac{i\pi}{2}$ (ii) $\log(-5) = \log 5 + i\pi$ (iii) $\log(1+i) = \log\sqrt{2} + \frac{i\pi}{4}$

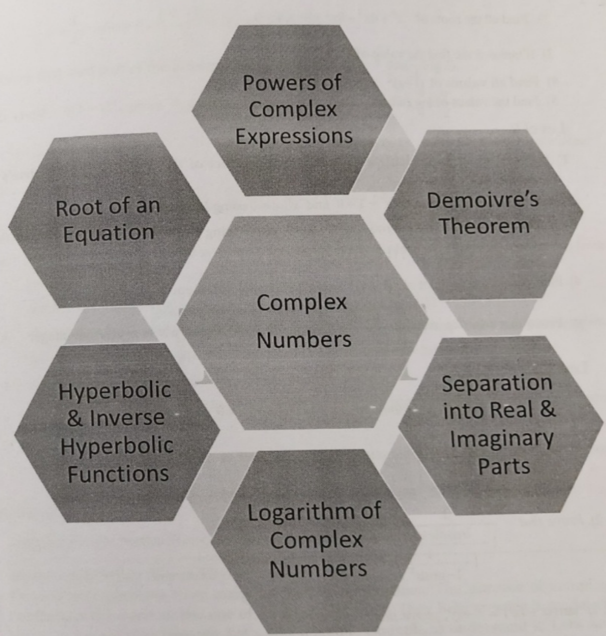
2. Show that $i\log\left(\frac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x$.

3. If $i^{\log(1+i)} = A + iB$, then show that one of the value of A is $e^{-\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right)$.

4. Prove that i^i is real and hence find the value of $\cos\log_e i^i$

Learning from the topic: Learner will be able to separate the function of complex variable real and Imaginary part Including logarithmic function.

Concept Map



Problems for Self-assessment:

Level 1

1) Find all the roots of the equation $x^4 = (1+i)$
2) Find all the roots of $x^4+4x^3+8x^2+8x+4=0$
3) If tanhx=3 the find the value of x.
4) Find all values of $(1-i)^{\frac{1}{4}}$.
5) Find the values of log (-7)

Level 2

1). Solve $x^5 = 1+i$ and calculate the continued product of the roots using De'Moivre's theorem.

2) Find the roots common to $x^4+1=0$ and $x^6 - i = 0$ using De'Moivre's theorem

3). If $\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$ are the roots of $x^7 - 1 = 0$, prove using De'Moivre's theorem
$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)(1-\alpha^5)(1-\alpha^6)=7$

4). Find continued product of $\{1+i\}^{\frac{4}{5}}$ using De'Moivre's theorem

5). Prove that $(4n)^{th}$ power of $\dfrac{1+7i}{(2-i)^2}$ is equal to $(-4)^n$, where n is a positive integer

Level 3

1). *Show that the roots of the equation* $(x+1)^6+(x-1)^6=0$ *are given by*

$$-i\cot\left[\frac{(2k+1)\pi}{12}\right] \quad for\ k=0,1,2,3,4,5.$$

2). *Prove that* $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\cosh^2 x}}} = \cosh^2 x.$

3). *If* $\tan(\alpha+i\beta) = x+iy$, *prove that*

(i) $x^2+y^2+2x\cot 2\alpha = 1$ (ii) $x^2+y^2-2y\coth 2\beta = -1.$

4). *If* $\cos\alpha\cosh\beta = \dfrac{x}{2}$, $\sin\alpha\sinh\beta = \dfrac{y}{2}$, *show that*

(i) $\sec(\alpha-i\beta)+\sec(\alpha+i\beta) = \dfrac{4x}{(x^2+y^2)}$ (ii) $\sec(\alpha-i\beta)-\sec(\alpha+i\beta) = -\dfrac{4iy}{(x^2+y^2)}$.

Tutorial Questions

1. *Show that the roots of the equation* $(x+1)^6+x^6=0$ *are given by*
$-\frac{1}{2}-\frac{i}{2}\cot(\frac{\theta}{2})$ *where* $\theta = \frac{(2k+1)\pi}{6}$ *for* $k=0,1,2,3,4,5.$
2. Prove that real part of the principal value of $i^{\log(1+i)}$ *is* $e^{-\frac{\pi^2}{8}}\cos(\frac{\pi}{4}\log 2)$
3. If $\tan(\theta+i\phi) = e^{i\alpha}$, *prove that* (i) $\vartheta = \frac{n\pi}{2}+\frac{\pi}{4}$, (ii) $\phi = \frac{1}{2}\log\tan(\frac{\pi}{4}+\frac{\alpha}{2})$
4. Calculate all the values of $\{1+i\}^{\frac{1}{5}}$, *show that their product is* $1+i$ using De'Moivre's theorem.
5. Show that
$$Log\left\{\frac{(a-b)+i(a+b)}{(a+b)+i(a-b)}\right\} = i\left\{2n\pi i+\tan^{-1}\frac{2ab}{a^2-b^2}\right\}.$$

Learning Outcome:

1. **Know:** (a) Learners should be able to express complex numbers in polar and exponential form and test the convergence of Infinite series
(b)Learners should be able to write expressions of circular and hyperbolic functions.
2. **Comprehend:** (a) Learners should be able to construct the different forms of complex number.
(b) Learners should be able to expand powers of trigonometric functions.
3. **Apply, Analyze, and synthesize:**
(a) Learners should be able to calculate the powers and roots of exponential and trigonometric functions.
(b) Learners should be able to find the real and imaginary parts of circular and hyperbolic functions.

Digital references:

1.https://www.mathsisfun.com/numbers/complex-numbers.html

Add to Knowledge (Beyond Syllabus):
The complex numbers have many interesting applications. The purpose of complex numbers is the same as the one of Mathematics itself. They help creating a model of our world simpler to manage. Let's take a simple example to understand it. Let's say you wanted to go to college, which is like 1 km to the east and 2 km to the north. This distance is nothing but "2+3i". In the study of electromagnetic wave propagation, the complex numbers are incredibly useful and allows us to make predictions about wave behaviors. So, if you wrote your question using a smartphone, you were using complex numbers. Spring motions, electric circuits, and vibration and resonance are daily life phenomena that we all deal with. If we require to perform any calculations about them, we need second order differential equations, and these equations are going to have complex roots.

Self- Evaluation

Name of learners: **Class & Div:** **Roll No:**

1. Do you understand how to convert complex no. from Cartesian form to polar/exponential form?
   (a) Yes (b) No

2. Will you able to find the root of an algebraic equation using D'Moivre's theorem?
   (a) Yes (b) No

3. Are you able to find the trigonometric expansion in form of power and multiple angle
   (a) Yes (b) No

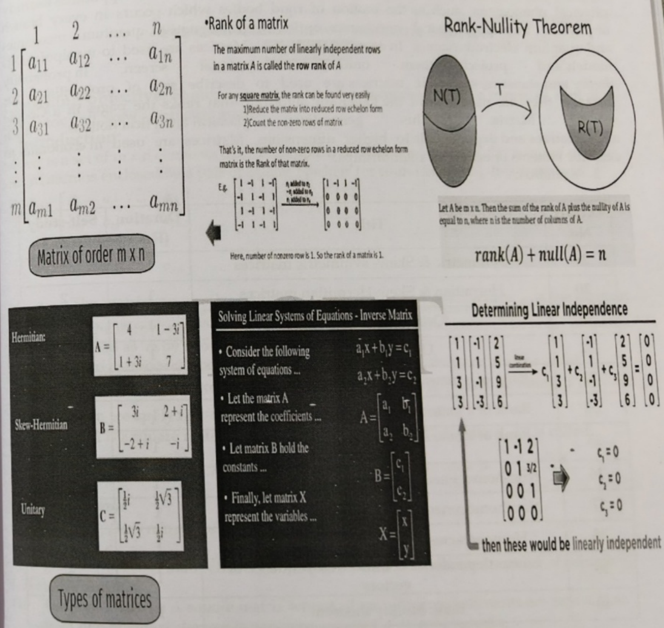
4. Do you understand how to separate real and imaginary part of complex expressions?
   (a) Yes (b) No

5. Will you able to identify the relation between hyperbolic functions and trigonometric (circular) functions?
   (a) Yes (b) No

6. Do you understand this module?
   (a) Fully understood (b) Partially understood (c) Not at all

Module 4: Matrices- I

Infographics



Module 4: Matrices- I

1. Motivation:
Applications of matrices are found in most scientific fields. Matrices are applicable to study physical phenomena, such as the motion of rigid bodies which occurs in every branch of physics including classical mechanics, optics, electromagnetism, quantum mechanics and quantum electrodynamics. In computer graphics, Matrices are used to manipulate 3D models and project them onto a 2-dimensional screen. In probability theory and statistics, stochastic matrices are used to describe sets of probabilities; for instance, they are used within the PageRank algorithm that ranks the pages in a Google search. Matrix calculus generalizes classical analytical notions such as derivatives and exponentials to higher dimensions. Matrices are used in economics to describe systems of economic relationships.

2. Syllabus:

Lecture No.	Title	Duration (Hrs.)	Self-study (Hrs.)
29	Symmetric & Skew- symmetric matrices	1	2
30	Hermitian & Skew-Hermitian matrices	1	2
31	Orthogonal matrices	1	2
32	Unitary matrices	1	2
33	Rank of a matrix, Reduction to Echelon form	1	2
34	Rank of a matrix, Reduction to Normal form	1	2
35	Rank of a matrix, Reduction to PAQ Normal form	1	2
36	Non-homogeneous linear system of equations	1	2
37	Non-homogeneous linear system of equations	1	2
38	Homogeneous linear system of equations	1	2
39	Linear dependence & linear independence of vectors	1	2
40	Rank Nullity theorem	1	2

3. Prerequisite:
Learners are expected to know the concept of vectors, determinant, addition and multiplication of matrices, transpose, and inverse of a matrix.

4. Learning Objective:
1. Learner shall be able to know different type of matrices.
2. Learner shall be able to know theorems on matrices.
3. Learner shall be able to know how to find rank of matrices using normal form.

4. Learner shall be able to solve the problems on Rank Nullity Theorem.
5. Learner shall be able to know application of matrices to calculate the solution of system of homogeneous and non –homogeneous equations.
6. Learner shall be able to solve the problems on Linear dependence, and independence of vectors.

Symmetric and Skew-Symmetric Matrices

Lecture: 29

1.Learning objective: Students shall be able to decompose a square matrix as sum of a symmetric and Skew symmetric matrix.

2. Introduction:
Matrix is a set of m x n elements (real or complex) arranged in a rectangular array of m rows and n columns enclosed by a pair of square brackets (or round brackets). It is written as

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The matrix can also be expressed in the form $A=\left[a_{ij}\right]_{m\times n}$ where a_{ij} is the element in the i^{th} row and j^{th} column, written as element of the matrix. The order of matrix is read as m by n or written as m × n.
- Two matrices can be added or subtracted provided they are of same order.
- The product of two matrices $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{p\times q}$ can be found if n=p and then we apply row-column method to multiply them.
- If A is a square matrix and $|A|\neq 0$ then $AA^{-1}=I=A^{-1}A$, where A^{-1} is called inverse of a matrix A. The formula for finding the inverse is $A^{-1}=\frac{adj.A}{|A|}$.

3. Key Notations:
1. $|A|$: *Determinant of A*
2. $A=\left[a_{ij}\right]_{m\times n}$: Matrix having i number of rows and j denotes number of columns.

4. Key Definitions:
(1) Diagonal Matrix: A square matrix whose all non -diagonal elements are zero and at least one diagonal element is non-zero, is called a diagonal matrix.

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

(2) **Scalar Matrix**: A square matrix, whose all diagonal elements are equal, is called a scalar matrix. e.g. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(3) **Upper Triangular Matrix:** A square matrix, in which all the elements below the diagonal are zero, is called an upper triangular matrix. $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$

(4) **Lower Triangular Matrix:** A square matrix, in which all the elements above the diagonal are zero, is called a lower triangular matrix. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 3 & 5 & 2 \end{bmatrix}$

(5) **Trace of Matrix:** The sum of all the principal diagonal elements of a square matrix is called the trace of a matrix. e.g. $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 6 & -2 \\ -1 & 0 & 3 \end{bmatrix}$

Trace of A=2+6+3=11.

(6) **Transpose of a Matrix:** A matrix obtained by interchanging rows and columns of a matrix is called transpose of a matrix and is denoted by A' or A^T.

e.g. $A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 6 \\ -4 & 1 & 5 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 0 & -4 \\ -1 & 2 & 1 \\ 3 & 6 & 5 \end{bmatrix}$

i.e. *If* $A = \left[a_{ij}\right]_{m\times n}$, *then* $A^T = \left[a_{ji}\right]_{n\times m}$

(7) **Symmetric Matrix:** A square matrix $A = \left[a_{ij}\right]_{m\times n}$ is called symmetric if $a_{ij} = a_{ji}$ for all i and j i.e. $A = A^T$ e.g. $\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & i & -3i \\ i & -2 & 4 \\ -3i & 4 & 3 \end{bmatrix}$

(8) **Skew Symmetric Matrix:** A square matrix $A = \left[a_{ij}\right]_{m\times n}$ is called skew symmetric if $a_{ij} = -a_{ji}$ for all i and j. i.e. $A = -A^T$. Thus, the diagonal elements of a skew symmetric matrix are all zero. e.g. $\begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$

(9) **Conjugate of a matrix:** A matrix obtained from any given matrix A, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and it is denoted by $\bar{A}$.

e.g. $A = \begin{bmatrix} 1+3i & 2+5i & 8 \\ -i & 6 & 9-i \end{bmatrix}$, $\quad \bar{A} = \begin{bmatrix} 1-3i & 2-5i & 8 \\ i & 6 & 9+i \end{bmatrix}$

(10) **Transpose Conjugate of a matrix:** The transpose of the conjugate of a matrix A is called the conjugate transpose of A and is denoted by A^{θ}. e.g. $A^{\theta} = \left(\bar{A}\right)^T = \overline{\left(A^T\right)}$.

5. Sample Problems:

(1) Show that every square matrix can be uniquely expressed as the sum of a symmetric and skew symmetric matrix.

Proof: Let A is a square matrix.

Every matrix A can be expressed as follows .. $A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$ (1)

i.e. A can be expressed as sum of $\frac{1}{2}\left(A + A^T\right)$ and $\frac{1}{2}\left(A - A^T\right)$....

Let $P = \frac{1}{2}\left(A + A^T\right)$ and $Q = \frac{1}{2}\left(A - A^T\right)$

To prove that P is symmetric and Q is skew symmetric.

Now, $P^T = \left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A^T + A) = P$

And $Q^T = \left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T) = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -Q$

Hence P is symmetric and Q is skew symmetric.

Uniqueness: To check the uniqueness of P and Q. Therefore, there exist two different matrices B and C such that

$B^T = B$ *and* $C^T = -C$ *and* $A = B + C$ (2)

$\therefore A^T = (B+C)^T = B^T + C^T = B - C$ (as $B^T = B$ and $C^T = -C$) (3)

From (2) and(3), $A + A^T = 2B \Rightarrow B = \frac{1}{2}(A + A^T) = P$

$A - A^T = 2C \Rightarrow C = \frac{1}{2}(A - A^T) = Q$

Therefore, P and Q are unique.

(2) Express the matrix A as the sum of symmetric and skew-symmetric matrices.

$$A=\begin{bmatrix}1&0&5&3\\-2&1&6&1\\3&2&7&1\\4&-4&2&0\end{bmatrix}$$

Solution:

$$A=\begin{bmatrix}1&0&5&3\\-2&1&6&1\\3&2&7&1\\4&-4&2&0\end{bmatrix} \Rightarrow A^T=\begin{bmatrix}1&-2&3&4\\0&1&2&-4\\5&6&7&2\\3&1&1&0\end{bmatrix}$$

Let $P=\frac{1}{2}(A+A^T) = \frac{1}{2}\begin{bmatrix}2&-2&8&7\\-2&2&8&-3\\8&8&14&3\\7&-3&3&0\end{bmatrix}$ As $P_{ij}=P_{ji} \Rightarrow P$ *is Symmetric.*

Let $Q=\frac{1}{2}(A-A^T) = \frac{1}{2}\begin{bmatrix}0&2&2&-1\\-2&0&4&5\\2&-4&0&-1\\1&-5&1&0\end{bmatrix}$ As $Q_{ij}=-Q_{ji} \Rightarrow Q$ *is Skew–Symmetric.*

Now $$P+Q=\frac{1}{2}\begin{bmatrix}2&-2&8&7\\-2&2&8&-3\\8&8&14&3\\7&-3&3&0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0&2&2&-1\\-2&0&4&5\\2&-4&0&-1\\1&-5&1&0\end{bmatrix}=\begin{bmatrix}1&0&5&3\\-2&1&6&1\\3&2&7&1\\4&-4&2&0\end{bmatrix}=A$$

Exercise 29

1) Show that the diagonal element of Hermitian Matrix is purely real and diagonal element of a skew Hermitian Matrix are either purely imaginary or zero.

2) Express Matrix A as the sum of symmetric and skew symmetric matrix where

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$

3) Express Matrix A as the sum of symmetric and skew symmetric matrix where

$$\begin{bmatrix}3&-2&6\\2&7&-1\\5&4&0\end{bmatrix}$$

Let's check take away from lecture

1. *If* $D=diag(d_1,d_2,d_3,....,d_n)$ where $d_i \neq 0$ for all $i=1,2,.....n$ then D^{-1} is equal to
   (a) D (b) $diag(d_1^{-1},d_2^{-1},...,d_n^{-1})$ (c) I_n (d) *none of these*

2. If $A=\begin{bmatrix}1&-5&7\\0&7&9\\11&8&9\end{bmatrix}$ then trace of the matrix A is
   (a) 17 (b) 25 (c) 3 (d) 12

3. The matrix A satisfying the equation $\begin{bmatrix}1&3\\0&1\end{bmatrix}A=\begin{bmatrix}1&1\\0&-1\end{bmatrix}$ is
   (a) $\begin{bmatrix}1&4\\-1&0\end{bmatrix}$ (b) $\begin{bmatrix}1&-4\\1&0\end{bmatrix}$ (c) $\begin{bmatrix}1&4\\0&-1\end{bmatrix}$ (d) *none of these*

4. For which value of X will be the matrix $\begin{bmatrix}8&x&0\\4&0&2\\12&6&0\end{bmatrix}$ become singular
   (a) 4 (b) 6 (c) 8 (d) 12

5. Let A and B be real symmetric matrices of size n x n.
   Then which one of the following is true
   (a) $AA'=I$ (b) $A=A^{-1}$ (c) $AB=BA$ (d) $(AB)'=BA$

6. If A is a symmetric matrix and $n \in N$ then A^n is
   (a) symmetric (b) skew symmetric (c) diagonal matrix (d) none of these

Homework Problems for the day

1) Define symmetric and skew symmetric matrix.

2) Express the matrix $\begin{bmatrix}2&-4&9\\14&7&13\\3&5&11\end{bmatrix}$ as a sum of symmetric and skew symmetric matrix.

Learning from the topic: Learners will remember the Symmetric and skew-Symmetric matrices and results based on them.

Hermitian & Skew-Hermitian Matrices

Lecture: 30

1.Learning objective: Students shall be able to decompose a square matrix as sum of a Hermitian and Skew Hermitian matrix.

2. Key Definitions:

(1) Hermitian Matrix: A square matrix $A=\left[a_{ij}\right]$ is called Hermitian if

$[a_{ij}]=[\overline{a_{ji}}]$ for all i and j. i.e. $A=A^{\theta}$. e.g. $\begin{bmatrix} 1 & 2+3i & 3-4i \\ 2-3i & 0 & 2-7i \\ 3+4i & 2+7i & 2 \end{bmatrix}$

(2) **Skew Hermitian Matrix:** A square matrix $A=\left[a_{ij}\right]$ is called skew Hermitian if

$[a_{ij}]=-[\overline{a_{ji}}]$ for all and j. i.e. $A=-A^{\theta}$. Hence diagonal elements of a skew-Hermitian matrix must be either purely imaginary or zero. $\begin{bmatrix} i & 2+3i \\ 2-3i & 0 \end{bmatrix}$

3. Sample Problems:

(1) Show that every square matrix can be uniquely expressed as the sum of a Hermitian and skew-Hermitian matrix.

Proof: Let A is a square matrix.

Every matrix A can be expressed as follows $A=\frac{1}{2}\left(A+A^{\theta}\right)+\frac{1}{2}\left(A-A^{\theta}\right)$ (1)

i.e. A can be expressed as sum of $\frac{1}{2}\left(A+A^{\theta}\right)$ and $\frac{1}{2}\left(A-A^{\theta}\right)$

Let $P=\frac{1}{2}\left(A+A^{\theta}\right)$ and $Q=\frac{1}{2}\left(A-A^{\theta}\right)$

To prove that P is Hermitian and Q is skew Hermitian.

Consider $P^{\theta}=\left(\frac{1}{2}(A+A^{\theta})\right)^{\theta}=\frac{1}{2}(A+A^{\theta})^{\theta}=\frac{1}{2}(A^{\theta}+(A^{\theta})^{\theta})=\frac{1}{2}(A^{\theta}+A)=P$

Consider $Q^{\theta}=\left(\frac{1}{2}(A-A^{\theta})\right)^{\theta}=\frac{1}{2}(A-A^{\theta})^{\theta}=\frac{1}{2}(A^{\theta}-(A^{\theta})^{\theta})=\frac{1}{2}(A^{\theta}-A)=-\frac{1}{2}(A-A^{\theta})=-Q$

Hence P is Hermitian and Q is Skew-Hermitian.

Uniqueness: To verify the uniqueness of P and Q.

Suppose there exist two different matrices B and C such that

$B^{\theta}=B$, $C^{\theta}=-C$ and $A=B+C$ (2)

$\therefore A^{\theta}=(B+C)^{\theta}=B^{\theta}+C^{\theta}=B-C$ (as $B^{\theta}=B$ and $C^{\theta}=-C$) (3)

From (2) and (3), $A+A^{\theta}=2B \Rightarrow B=\frac{1}{2}(A+A^{\theta})=P$

$A-A^{\theta}=2C \Rightarrow C=\frac{1}{2}(A-A^{\theta})=Q$

Therefore, P and Q are unique.

2. Express Matrix A as the sum of a Hermitian and skew-Hermitian matrix where

$$A=\begin{bmatrix} 2-i & 3+i & 2i \\ 3 & 0 & 4-i \\ -5 & 2-i & 3+i \end{bmatrix}$$

Solution:

$$A'=\begin{bmatrix} 2-i & 3 & -5 \\ 3+i & 0 & 2-i \\ 2i & 4-i & 3+i \end{bmatrix}$$

$$A^{\theta}=(\overline{A})'=\begin{bmatrix} 2+i & 3 & -5 \\ 3-i & 0 & 2+i \\ -2i & 4+i & 3-i \end{bmatrix}$$

$$P=\frac{1}{2}(A+A^{\theta})=\frac{1}{2}\begin{bmatrix} 4 & 6+i & -5+2i \\ 6-i & 0 & 6 \\ -5-2i & 6 & 6 \end{bmatrix}$$

$$=\begin{bmatrix} 4 & \frac{(6+i)}{2} & \frac{(-5+2i)}{2} \\ \frac{(6-i)}{2} & 0 & 3 \\ \frac{(-5-2i)}{2} & 3 & 3 \end{bmatrix}$$

$$Q=\frac{1}{2}(A-A^{\theta})=\frac{1}{2}\begin{bmatrix} -2i & i & 5+2i \\ i & 0 & 2-2i \\ -5+2i & -2-2i & 2i \end{bmatrix}$$

$$=\begin{bmatrix} -i & \frac{i}{2} & \frac{(5+2i)}{2} \\ \frac{i}{2} & 0 & 1-i \\ \frac{(-5+2i)}{2} & -1-i & i \end{bmatrix}$$

$$A=P+Q=\begin{bmatrix} 4 & \frac{(6+i)}{2} & \frac{(-5+2i)}{2} \\ \frac{(6-i)}{2} & 0 & 3 \\ \frac{(-5-2i)}{2} & 3 & 3 \end{bmatrix}+\begin{bmatrix} -i & \frac{i}{2} & \frac{(5+2i)}{2} \\ \frac{i}{2} & 0 & 1-i \\ \frac{(-5+2i)}{2} & -1-i & i \end{bmatrix}$$

Exercise 30

1) Show that the diagonal element of Hermitian Matrix is purely real and diagonal element of a skew Hermitian Matrix are either purely imaginary or zero.
2) Express Matrix A as the sum of a Hermitian and skew-Hermitian matrix where

$$A=\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$$

3) Express Matrix A as the sum of a Hermitian and skew-Hermitian matrix where

$$A=\begin{bmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+4i & -i \end{bmatrix}$$

Let's check take away from lecture

1. The matrix $\begin{bmatrix} 0 & -4+i \\ 4+i & 0 \end{bmatrix}$ is

(a) Symmetric (b) Skew Symmetric (c) Hermitian matrix (d) Skew Hermitian

2. Let A and B are symmetric matrices, then AB - BA is

(a) Symmetric (b) Skew Symmetric (c) Hermitian matrix (d) Skew Hermitian

3. If A is skew Hermitian matrix, then A^{θ}

(a) A (b) – A (c) 0 (d) diagonal matrix

4. A square matrix A is Hermitian if

(a) $A^T=A$ (b) $A^T=-A$ (c) $A^{\theta}=-A$ (d) $A^{\theta}=A$

5. The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is

(a) Symmetric (b) Skew Symmetric (c) Hermitian matrix (d) Skew Hermitian

Homework Problems for the day

1) Define Hermitian and skew-Hermitian matrix.

2) Express A = $\begin{bmatrix} 2 & 4+i & 4i \\ 3i & 6-i & 2 \\ 6 & 4-2i & 1-i \end{bmatrix}$ as the sum of a Hermitian and skew-Hermitian matrix.

Learning from the topic: Learners will remember the Hermitian and skew-Hermitian matrices and results based on them.

Orthogonal Matrix

Lecture: 31

1. Learning Objective:

Learners shall be able to identify the special type of matrices and their inverses.

2. Introduction:

A matrix is said to be orthogonal if the inverse of the matrix is its inverse. $A^{-1}=A^T$

3.Key Definitions:

Orthogonal Matrix: A square matrix A is called orthogonal if $AA^T=I=A^TA$.

4.Sample Problems

1) Determine l,m,n and find A^{-1} if $A=\begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}$ is orthogonal.

Solution: Given A is orthogonal from definition, $AA'=I$

$$\begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}\begin{bmatrix} 0 & l & l \\ 2m & m & -m \\ n & -n & n \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ 4m^2+n^2=1,\ 2m^2-n^2=0 \qquad \therefore\ 6m^2=1 \qquad \therefore\ m=\pm\frac{1}{\sqrt{6}},\ n=\pm\frac{1}{\sqrt{3}}$$

Also, $l^2+m^2+n^2=1 \Rightarrow l=\pm\frac{1}{\sqrt{2}} \therefore A^{-1}=A'=\begin{bmatrix} 0 & \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{2}{\sqrt{6}} & \pm\frac{1}{\sqrt{6}} & \pm\frac{1}{\sqrt{6}} \\ \pm\frac{1}{\sqrt{3}} & \pm\frac{1}{\sqrt{3}} & \pm\frac{1}{\sqrt{3}} \end{bmatrix}$

Exercise 31

1) Show that if A is orthogonal then $|A|=\pm 1$.

(Hint : $|A|=|A^T|$)

2) Verify the following matrices orthogonal and hence find its inverse.

$A=\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

3) Find a, b, c & A^{-1} if $A=\frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ is orthogonal.

Ans: (2 ,-2 ,1) & (-2 , 2 , -1).

Let's check take away from

1. If A is an orthogonal matrix, then A^{-1} equals
   (a) A (b) A^T (c) A^2 (d) none of these

2. Find a & b if $A=\frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$ is orthogonal.
   (a) 1 & 8 (b) 1 & −8 (c) −1 & 4 (d) −1 & −4

3. If $A=\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$,then A + 2 A^T equals
   (a) A (b) $-A^T$ (c) $2A^2$ (d) A^T

4. **If** $A=\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$, then A A^T is

(a) Zero Matrix (b) I_2 (c) A^T (d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Homework Problems for the day

1) Verify the following matrices are orthogonal and hence find its inverse.

i) $A=\frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$ ii) $A=\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

iii) $A=\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ iv) $A=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

Learning from the topic: Learners will be able to remember the Orthogonal Matrices.

Unitary Matrix

Lecture: 32

1. **Learning Objective:**

Learners shall be able to identify the special type of matrices and their inverses.

2. **Introduction:**

A matrix is said to be orthogonal if the inverse of the matrix is its transpose whereas a matrix will be unitary if its inverse is its conjugate transpose.

3.Key Definitions:

(1) Unitary Matrix: A square matrix A is called unitary if $AA^\theta = I = A^\theta A$.

4.Sample Problems

1) Show that the matrix $\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is unitary.

Solution: Let $A=\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ then $A'=\begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{-1+i}{2} & \frac{1-i}{2} \end{bmatrix} \therefore A^\theta=\left(\overline{A'}\right)=\begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$

$$\therefore A^{\theta}A = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is unitary.

2) Show that the matrix A is unitary where $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$.

Solution: $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$A^{\theta} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^{\theta}A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Exercise 32

1.Show that A is unitary and hence find A^{-1} where $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$.

2. Given that $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, *show that* $(I-A)(I+A)^{-1}$ *is unitary matrix.*.

Let's check take away from lecture

1. If A is an unitary matrix, then A^{-1} equals
   (a) A (b) A^{T} (c) A^{θ} (d) I

2. $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary matrix if and only if
   (a) $a^2+b^2+c^2=0$ (b) $b^2+c^2+d^2=0$ (c) $a^2+b^2+c^2+d^2=0$ (d) $a^2+b^2+c^2+d^2=1$

Homework Problems for the day

1) Show that A is unitary and hence find A^{-1} where $A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

2) Is the matrix A is unitary where $A = \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$.

Learning from the topic: Learners will be able to remember the Unitary matrices.

Rank of Matrix, Reduction to Echelon

Lecture: 33

1.Learning Objective:

Learners shall be able to calculate the rank of matrices by reducing them to Echelon form.

2.Introduction:

Here with the help of elementary transformations (row and column) we will learn to find rank of matrix which will give idea about number of linearly independent rows and columns in a matrix.

3. Key Definitions:

(1) **Rank of the Matrix:** The rank of the matrix is said to be r if it possesses the following properties:

(i) there is at least one non- zero minor of order r.

(ii) every minor of order greater than r is zero.

Rank of the matrix is denoted by $\rho(A)$.

(2) **Normal form or Canonical form:** Every $m \times n$ matrix of rank r can be reduced to the normal form $\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$ by a finite sequence of elementary transformations. This form is called the 'normal form' or the first Canonical form of the matrix A. By reducing the matrix to a normal form through an elementary transformation. We can determine the rank of the matrix.

(3) **Echelon Form of a matrix:** A matrix A is said to be in echelon form if

(i) Every zero row of matrix A occurs below a non-zero row.

(ii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

4. Key Notations:

$\rho(A)$: Rank of a matrix A

5. Sample Problems

1) Express the following matrix to echelon form and find its rank A= $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix} \xrightarrow{R_3 - 5R_1} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix}$

$$\xrightarrow{R_3 - 8R_2} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{12}R_3} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$\therefore$ Rank of A=3.

Exercise 33

1) Evaluate the rank of the following matrices by reducing them to Echelon form:

i) $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Ans: (i) ρ (A) = 2 (ii) ρ (A) = 2

Let's check take away from lecture

1. The rank of the matrix, A= $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

2. To reduce a matrix to normal form row and column transformation both can be applied
(a) true (b) false (c) none of these

3. If Rank of (A) = 2 and Rank of (B) = 3 then Rank of (AB) is:
(a) 6 (b) 3 (c) 2 (d) Data inadequate

4. To reduce a matrix to Echelon form only row transformation is allowed
(a) true (b) false (c) none of these

Homework Problems for the day

1) Evaluate the rank of the following matrices by reducing them to Echelon form:

i) $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Ans: (i) ρ (A) = 3 (ii) ρ (A) = 2

Learning from the topic: Learners will be able to calculate the rank of the matrices by reducing them to Echelon form.

Rank of Matrix, Reduction to normal form

Lecture: 34

1.Learning Objective:

Learners shall be able to calculate the rank of matrices by reducing them to normal form.

2. Key Definitions:

(1) **Normal form or Canonical form:** Every $m \times n$ matrix of rank r can be reduced to the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by a finite sequence of elementary transformations. This form is called the 'normal form' or the first Canonical form of the matrix A. By reducing the matrix to a normal form through an elementary transformation. We can determine the rank of the matrix.

3. Key Notations:

$\rho(A)$: Rank of a matrix A

4. Sample Problems

1) Express the following matrix to normal form and find its rank $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Solution: Let

$$A=\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$By\ \ R_2 \leftrightarrow R_1$ $\qquad$ $By\ \ R_2-2R_1, R_3-3R_1, R_4+6R_1$

$$=\begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$By\ \ C_2+C_1, C_3+2C_1, C_4+4C_1$ $\qquad$ $By\ \ R_2-R_3$ $\qquad$ $By\ \ C_3+6C_2, C_4+3C_2$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 33 & 22 \\ 0 & 9 & 66 & 44 \end{bmatrix}$$

$By\ \ R_3-4R_2, R_4-9R_2$ $\qquad$ $By\ \ R_4-2R_3$ $\qquad$ $By\ \ \frac{1}{11}R_3$ $\qquad$ $By\ \ C_3-C_4$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$By\ \ C_4-2C_3$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $\therefore$ Rank of A=3.

2) Evaluate the rank of the following matrix by reducing to normal form:

$$A=\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

Solution:

R_2-R_1, R_3-3R_1 $\qquad$ R_3-R_2

$$A=\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2-C_1, C_3+C_1, C_4-C_1$ $\qquad$ $\left(-\frac{1}{2}\right)C_2, \left(\frac{1}{3}\right)C_3, \left(-\frac{1}{2}\right)C_4$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C_3-C_2, C_4-C_2

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A)=2$

3) Reduce the following matrix to its normal form and hence find its rank.

$$A=\begin{bmatrix} 3 & -2 & 0 & 1 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Solution:

Given $A=\begin{bmatrix} 3 & -2 & 0 & 1 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ R_{13} $\quad A \square \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 7 \\ 3 & -2 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

$R_3-3R_1 \quad A \square \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 7 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

$C_2-2C_1,\ C_3+3C_1,\ C_4-2C_1 \quad A \square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 7 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ $R_2-R_4 \quad A \square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

$$R_3 - 4R_2\ ,\ R_4 - R_2 \quad A \square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 9 & -29 \\ 0 & 0 & 2 & -5 \end{bmatrix} \quad C_4 - 6C_2 \quad A \square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & -29 \\ 0 & 0 & 2 & -5 \end{bmatrix}$$

$$R_3 - 4R_4 \quad A \square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 2 & -5 \end{bmatrix} \quad R_4 - 2R_3 \quad A \square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

$$C_4 + 9C_3 \quad A \square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix} \quad \frac{1}{13}C_4 \quad A \square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, Rank of A = Rank of non-zero rows = 4.

Exercise 34

1) Evaluate the rank of the following matrices by reducing them to Normal form:

i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

iii) $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & -1 & 2 & 2 \\ 4 & 1 & 0 & -1 \\ 9 & 1 & 5 & 6 \end{bmatrix}$

Ans: (i) $\rho(A) = 2$ (ii) $\rho(A) = 3$ (iii) $\rho(A) = 4$

1. Which of the following form is a normal form

(a) $\begin{pmatrix} I_r & o \\ o & o \end{pmatrix}$ (b) $\begin{pmatrix} I_r \\ o \end{pmatrix}$ (c) $(I_r \quad o)$ (d) (I_r) (e) all

2. To reduce a matrix to normal form row and column transformation both can be applied

(a) true (b) false (c) none of these

3. The rank of the matrix, $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$ where $a = b \neq c$ is

(a) 0 (b) 1 (c) 2 (d) 3

4. The rank of the matrix, $A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$ is

(a) 1 (b) 2 (c) 3 (d) 4

Homework Problems for the day

1) Evaluate the rank of the following matrices by reducing them to Normal form :

i) $A = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \\ 0 & -4 & -1 & 2 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ iii) $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

Ans: (i) $\rho(A) = 3$ (ii) $\rho(A) = 2$ (iii) $\rho(A) = 3$

Learning from the topic: Learners will be able to calculate the rank of the matrices by reducing them to normal form.

Rank of Matrix, Reduction to PAQ normal

Lecture: 35

1.Learning Objective:

Learners shall be able to calculate the rank of matrices by reducing them to PAQ form.

2.Introduction:

This is another method to find the rank of the matrix. In this method we write given matrix A as A= PAQ ------------(1)

Then apply row as well as column transformations to reduce matrix A on the left side to normal form and row transformation on P on right side and column transformation on Q.

3. Sample Problems

1) Find non-singular matrices P and Q such that PAQ is in the normal form and hence, find $\rho(A)$ for the matrix $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$ *Since* $A = I_3 A I_3$

$$\therefore \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore -\frac{1}{2}C_3 \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{2} \end{bmatrix}$$

$$\therefore C_3 - C_1 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{2} \end{bmatrix}$$

$$\therefore R_2 - 2R_1,\ R_3 - R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{2} \end{bmatrix}$$

$$\therefore \frac{1}{3}R_2,\ R_3 - R_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{2} \end{bmatrix} \therefore \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

$$Where\ P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & \frac{1}{3} & 0 \\ -1 & -1 & 1 \end{bmatrix},\quad Q = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{2} \end{bmatrix} \qquad \therefore \rho(A) = 2$$

2) Find non-singular matrices P&Q such that PAQ is in normal form where

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

Solution: Since $A = I_3 A I_3$

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = I_3 A I_3$$

R_{13}

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A\, I_3$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -3 \\ 1 & 0 & -2 \end{bmatrix} A\, I_3$$

$C_2 - 2C_1,\ C_3 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(-C_2)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 - 6R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 7 & -6 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{1}{5}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ \frac{7}{5} & \frac{-6}{5} & \frac{4}{5} \end{bmatrix} A \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 = PAQ \text{ where } P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ \frac{7}{5} & \frac{-6}{5} & \frac{4}{5} \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 35

1) Find nonsingular matrices P and Q such that PAQ is in the normal form and hence find ρ (A) and ρ (PAQ) and find inverse of matrix A if exists for the following:

i) $A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 1 & 4 & -5 \\ -1 & -5 & -5 & 7 \end{bmatrix}$ ii) $A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

Ans: (i) $\rho(A) = 2$ (ii) $\rho(A) = 3$

2) Find A^{-1} and $\rho(A)$ after converting to PAQ form where $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Let's check take away from lecture

1. A 5x7 matrix has all its entries equal to -1. The rank of the matrix is
   (a) 7 (b) 5 (c) 1 (d) 0

2. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is
   (a) 0 (b) 1 (c) 2 (d) 3

3. The rank of the matrix $\begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$ is
   (a) 0 (b) 1 (c) 2 (d) 3

Homework Problems for the day

1) Find non -singular matrices P and Q such that PAQ is in the normal form and hence find $\rho(A)$ and $\rho(PAQ)$ for the following:

i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ iv) $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

Ans: (i) $\rho(A) = 2$ (ii) $\rho(A) = 3$ (iii) $\rho(A) = 3$ (iv) $\rho(A) = 3$

2. Find non -singular matrices P and Q such that PAQ is in the normal form where

$$A = \begin{bmatrix} 3 & -2 & 0 & 1 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

And hence find $\rho(A)$ and $\rho(PAQ)$. Check whether A^{-1} can be expressed in terms of Q and P and hence find A^{-1}

Non-homogeneous Linear system of Equations

Lecture: 36

1. Learning Objective:
Learners shall be able to learn the application of matrices to solve the non-homogeneous system of equations.

2. Introduction:

We will learn one of the applications of rank of matrices to solve the non-homogeneous system of equations. For this purpose, we will write the given system in matrix notation as AX=B. Then we consider the augmented matrix [A : B] and find the rank of augmented matrix.

3. Key Definitions:
Consistency of an equation: The system of equations is consistent if and only if the coefficient matrix A and the augmented matrix [A : B] are of the same rank. i.e. $\rho(A) = \rho[A:B]$. There are two cases:

Case I: If $\rho(A) = \rho[A:B] = n$, no. of unknowns, the system has unique solution.

Case II: If $\rho(A) = \rho[A:B] < n$, no. of unknowns, the system has infinite solutions.

When $\rho(A) \neq \rho[A:B]$, the system is said to be inconsistent and has no solution.

4. Sample Problems

1) Discuss the consistency of the system and if consistent, solve the equations:
$4x-2y+6z=8,\quad x+y-3z=-1,\quad 15x-3y+9z=21$

Solution: In matrix form, AX=B $\Rightarrow \begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$

Augmented matrix, $[A:B] = \begin{bmatrix} 4 & -2 & 6 & \vdots 8 \\ 1 & 1 & -3 & \vdots -1 \\ 15 & -3 & 9 & \vdots 21 \end{bmatrix} \overset{R_{12}}{=} \begin{bmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{bmatrix}$

$$\overset{\frac{1}{2}R_2, \frac{1}{3}R_3}{=} \begin{bmatrix} 1 & 1 & -3 & -1 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 3 & 7 \end{bmatrix} \overset{R_3 - 2R_2}{=} \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -3 & 9 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = \rho[A:B] = 2 < 3$ (No. of unknowns)

Hence the system is consistent and has infinite solutions.

$\begin{matrix} x+y-3z=-1 \\ -3y+9z=6 \end{matrix} \Rightarrow$ Number of parameters = 3 – 2=1

Let $z=t$ Then $y=3t-2$ and $x=-1(3t-2)+3t=1$.

Hence $x=1, y=3t-2, z=t$ is the solution of the system where t is a parameter.

Exercise 36

1) Is the following system of equation consistent

$x + y + z = 6$
$x - y + 2z = 5$
$3x + y + z = 8$
$2x - 2y + 3z = 7$

2) Discuss the consistency of the system and if consistent solve the equation:

i) $x + y + z = 6$
$x + 2y + 3z = 14$
$2x + 4y + 7z = 30$

Ans : Consistent : (x, y, z) = (0, 4, 2).

ii) $x + y + z = 5$
$x + 2y + 3z = 10$
$x + 2y + 3z = 8$

Ans : Inconsistent.

iii) $x - 2y + z - w = 2$
$x + 2y + 4w = 1$
$4x - z + 3w = -1$

Ans : Consistent : $(x, y, z, w) = (\frac{1}{3}-t, \frac{1}{3}-\frac{3}{2}t, \frac{7}{3}-t, t)$.

Let's check away from lecture

1. The system of linear equations $4x+2y=7, 2x+y=6$ has
(a) unique solution (b) no solution
(c) an infinite number of solutions (d) exactly two distinct solutions

2. The system of linear equations $x+y=2, 2x+2y=5$ has
(a) unique solution (b) no solution
(c) an infinite number of solutions (d) exactly two distinct solutions

3. A is a 3x4 real matrix and Ax=b is an inconsistent system of equations. The highest possible rank of A is
(a) 1 (b) 2 (c) 3 (d) 4

Homework Problems for the day

1) Discuss the consistency of the system and if consistent solve the equation:

i) $2x - 3y + 7z = 5$
$3x + y - 3z = 13$
$2x + 19y - 47z = 32$

Ans: Inconsistent.

ii) $x + y + z = -3$
$3x + y - 2z = -2$
$2x + 4y + 7z = 7$

Ans: Inconsistent.

2) Discuss the consistency of the system and if consistent solve the equation:
$2x - y - z = 2$
$x + 2y + z = 2$
$4x - 7y - 5z = 2$

Ans: consistent

Learning from the topic: Learners will be able to solve non-homogeneous system of linear algebraic equations.

Non-homogeneous Linear system of Equations continued.........

Lecture: 37

Sample Problems

1) Investigate for what values of λ and μ the equations

$x+2y+z=8$
$2x+2y+2z=13$
$3x+4y+\lambda z=\mu$ have (i) no solution (ii) unique solution (iii) many solutions.

Solution: In matrix form, AX=B$\Rightarrow \begin{bmatrix}1&2&1\\2&2&3\\3&4&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}8\\13\\\mu\end{bmatrix}$

Augmented matrix, $[A:B]=\begin{bmatrix}1&2&1&\vdots 8\\2&2&2&\vdots 13\\3&4&\lambda&\vdots \mu\end{bmatrix} \Rightarrow R_2-2R_1, R_3-3R_1 \begin{bmatrix}1&2&1&8\\0&-2&0&-3\\0&-2&\lambda-3&\mu-24\end{bmatrix}$

$\Rightarrow R_3-R_2 \begin{bmatrix}1&2&1&8\\0&-2&0&-3\\0&0&\lambda-3&\mu-21\end{bmatrix}$

(i) If $\lambda=3, \mu\neq 21, then\ \rho(A)\neq\rho[A:B] \Rightarrow$ Hence, system is inconsistent and has no solution.

(ii) If $\lambda\neq 3, \mu\ have\ any\ value, then\ \rho(A)=\rho[A:B]=3$ (No. of unknowns)
Hence, system is consistent and has unique solution.

(iii) If $\lambda=3, \mu=21, then\ \rho(A)=\rho[A:B]=2<$Number of unknowns
Hence, system is consistent and has infinite solutions.

2) Investigate for what values of 'λ' and 'μ' the system of equations

$x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$

has (i) no solution (ii) unique solution (iii) an infinite no. of solutions.

Solution: In matrix form, AX=B

$$\begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$$

Augmented matrix $[A:B]=\begin{bmatrix}1 & 1 & 1 & \vdots 6\\ 1 & 2 & 3 & \vdots 10\\ 1 & 2 & \lambda & \vdots \mu\end{bmatrix}$

$R_2-R_1,\ R_3-R_2$ □ $\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 4\\ 0 & 0 & \lambda-3 & \mu-10\end{bmatrix}$

(i) If $\lambda=3, \mu\neq 10$, *then* $\rho(A)\neq\rho(A:B)$

Hence, system is inconsistent and has no solution.

(ii) If $\lambda\neq 3$, μ *have any value then* $\rho[A:B]=3$ (No. of unknowns)

Hence, system is consistent and has unique solution.

(iii) If $\lambda=3$, $\mu=10$, *then* $\rho(A)=\rho[A:B]=2<$ Number of unknowns

Hence, system is consistent and has infinite solutions.

Exercise 37

1. Investigate for what values of λ and μ the system of equations

$2x + 3y + 5z = 9;\ 7x + 3y - 2z = 8;\ 2x + 3y + \lambda z = \mu$

have (i) no solution (ii)unique solution (iii) many solutions.

Ans: (i) $\lambda = 5, \mu \neq 9$ (ii) $\lambda \neq 5, \forall\ \mu$ (iii) $\lambda = 5, \mu = 9$

2 Determine the values of λ so that the equations :

$x + y + z = 1$

$x + 2y + 4z = \lambda$

$x + 4y + 10z = \lambda^2$

have a solution and solve them completely in each case.

Ans :i) $\lambda = 2, z = t, y = 1-3t, x = 2t$ ii) $\lambda = 1, z = t, y = -3t, x = 1+2t$

Let's check away from lecture

1. The value of λ in the equations have no solution is:

$2x-3y+6z-5t=3$

$y-4z+t=1$

$4x-5y+8-9t=k$

(a) 1 (b) 7 (c) -7 (d) 4

2. The value of λ in the equations have unique solution is:

$3x - y + \lambda z = 0$

$2x - y + z = 2$

$x - 2y - \lambda z = -1$

(a) -2/7 (b) 7/2 (c) -7/2 (d) 2/7

Homework Problems for the day

1) Discuss for all values of λ, the nature of solution of the system of equations

$\lambda x+2y-2z=1$

$4x+2\lambda y-z=2$

$6x+6y+\lambda z=3$

Ans: (i) unique solution $\lambda \neq -2$

(iii) infinitely many solutions $\lambda = 2$

2) Show that the system of equations

$3x + 4y + 5z = \alpha$

$4x + 5y + 6z = \beta$

$5x + 6y + 7z = \gamma$

are consistent only if α, β & γ are in A. P.

[Hint : Reduce the augmented matrix [A | B] into row echelon form and then for consistency we derive $\gamma - 2\beta + \alpha = 0$]

Learning from the topic: Learners will be able to solve non-homogeneous system of linear algebraic equations.

Homogeneous Linear system of Equations

Lecture: 38

1.**Learning Objective:** Learners shall be able to learn the application of matrices to solve the homogeneous system of equations.

2.**Introduction:**

We will learn one more application of rank of matrices to solve the homogeneous system of equations. For this purpose, we will write the given system in matrix notation as AX=O. Then we consider the augmented matrix [A]and find the rank of the matrix.

3.**Key Definitions:**

(1) Consistency of an equation: The homogeneous system of equations is always consistent. There are two cases:

Case I: If $\rho(A)=r=n$ no. of unknowns, the system has unique trivial solution.

Case II: If $\rho(A)=r<n$ no. of unknowns, the system has infinite solutions.

4. **Sample Problems**

1) Solve the following system of equations:

$x+2y+3z=0\ ;\ 2x+3y+z=0\ ;\ 4x+5y+4z=0\ ;\ x+2y-2z=0$

Solution: In matrix form, $AX=0 \quad \Rightarrow \begin{bmatrix}1&2&3\\2&3&1\\4&5&4\\1&2&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$$\Rightarrow R_2-2R_1, R_3-4R_1, R_4-R_1 \begin{bmatrix}1&2&3\\0&-1&-5\\0&-3&-8\\0&-1&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\Rightarrow R_3-3R_2, R_4-R_2 \begin{bmatrix}1&2&3\\0&-1&-5\\0&0&7\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$\rho(A)=3$ (No. of unknowns) $\Rightarrow$ Hence the system has a trivial solution $x=0, y=0, z=0$.

Exercise 38

1) Solve the following system of equations completely.

$2x-2y-5z=0$
$4x-y+z=0$
$3x-2y+3z=0$
$x-3y+7z=0$

Ans: $(x, y, z) = (0, 0, 0)$

2) Solve $3x - y - z = 0,\ x + y + 2z = 0,\ 5x + y + 3z = 0$

Ans: $(x, y, z) = \left(-k/4, -7k/4, k\right)$

3) Find the value of λ for which following system of equations has nonzero solutions

$x+2y+3z=\lambda x$
$3x+y+2z=\lambda y$
$2x+3y+z=\lambda z$

Ans: $\lambda = 6, x = y = z = t$

3) The following system of equations is given below.

$ax+by+cz=0$
$bx+cy+az=0$
$cx+ay+bz=0$

(i) Is equation is homogeneous or not
(ii) Is system has a trivial or non -trivial solution
(iii) find the non -trivial conditions and
(iv) find non- trivial solution for condition satisfies.

Let's check take away from lecture

1. Which of the following system of equations is homogeneous

(a) $AX=O$ (b) $AX=B$ (c) none of these

2. The homogeneous system of equations is always consistent

(a) true (b) false (c) none of these

Homework Problems for the day

1) Solve $x+y-z+w=0,\ x-y+2z-w=0,\ 3x+y+w=0$.

Ans: Consistent, $(x,y,z,w) = \left(-\frac{t_1}{2}, \frac{3t_1}{2} - t_2, t_1, t_2\right)$

2) Discuss for what values λ the system of equations has non-trivial solution? Obtain the solution for real value of λ where

$3x+y-\lambda z=0,\ 4x-2y-3z=0,\ 2\lambda x+4y+\lambda z=0.$

Ans: i) When $\lambda \neq -9$ and $\lambda \neq 1$ then $(x, y, z) = (0, 0, 0)$
ii) When $\lambda = -9$ and $\lambda \neq 1$ then $(x, y, z) = (-3t/2, -9t/2, t)$
iii) When $\lambda \neq -9$ and $\lambda = 1$ then $(x, y, z) = (t/2, -t/2, t)$

Learning from the topic: Learners will be able to solve homogeneous system of linear algebraic equations.

Linear dependence, Linear independence of vectors

Lecture: 39

1. **Learning Objective:** Learners shall be able to distinguish between linear dependent and independent vectors.

2. **Key Definitions:**

1) **Linear Dependence:** A set of r vectors $X_1, X_2, \ldots X_r$ is said to be linearly dependent if there exist r scalars (numbers) $k_1, k_2, \ldots k_r$ not all zero, such that

$k_1X_1 + k_2X_2 + \ldots + k_rX_r = 0$.

2) **Linear Independence:** A set of r vectors $X_1, X_2, \ldots X_r$ is said to be linearly independent if there exist r scalars (numbers) $k_1, k_2, \ldots k_r$, such that if $k_1X_1 + k_2X_2 + \ldots + k_rX_r = 0$ then $k_1 = k_2 = \ldots = k_r = 0$.

3. **Sample Problems**

1) Examine whether the vectors $[1,1,-1], [2,3,-5], [2,-1,4]$ are linearly independent or dependent.

Solution: Let $X_1 = [1,1,-1], X_2 = [2,3,-5], X_3 = [2,-1,4]$

Let $k_1X_1 + k_2X_2 + k_3X_3 = 0$

$\therefore k_1[1,1,-1] + k_2[2,3,-5] + k_3[2,-1,4] = [0,0,0]$

On multiplying and adding,

$$\begin{aligned} k_1+2k_2+2k_3&=0\\ k_1+3k_2-k_3&=0\\ -k_1-5k_2+4k_3&=0 \end{aligned} \quad \Rightarrow \text{In matrix form, } \begin{bmatrix}1&2&2\\1&3&-1\\-1&-5&4\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow R_2-R_1, R_3+R_1 \begin{bmatrix}1&2&2\\0&1&-3\\0&-3&6\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow R_3+3R_2 \begin{bmatrix}1&2&2\\0&1&-3\\0&0&-3\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow \rho(\text{coefficient matrix}) = 3$

$\Rightarrow$ System has trivial solution as r = n namely $k_1=0$, $k_2=0, k_3=0$.

Thus, the vectors are linearly independent.

2) Examine whether the vectors $[2,-1,3,2],[1,3,4,2],[3,-5,2,2]$ are linearly independent or dependent. If dependent find the relation between them.

Solution: Let $X_1=[2,-1,3,2]$, $X_2=[1,3,4,2]$, $X_3=[3,-5,2,2]$

Let $k_1X_1+k_2X_2+k_3X_3=0$ ....(1)

$\therefore k_1[2,-1,3,2]+k_2[1,3,4,2]+k_3[3,-5,2,2]=[0,0,0,0]$

On multiplying and adding,

$$\begin{aligned} 2k_1+k_2+3k_3&=0\\ -k_1+3k_2-5k_3&=0\\ 3k_1+4k_2+2k_3&=0\\ 2k_1+2k_2+2k_3&=0 \end{aligned} \quad \Rightarrow \text{In matrix form, } \begin{bmatrix}2&1&3\\-1&3&-5\\3&4&2\\2&2&2\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\Rightarrow R_1+R_2 \begin{bmatrix}1&4&-2\\-1&3&-5\\3&4&2\\2&2&2\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Rightarrow \begin{matrix}R_2+R_1\\R_3-3R_1\\R_4-2R_1\end{matrix}$$

$$\begin{bmatrix}1&4&-2\\0&7&-7\\0&-8&8\\0&-6&6\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{matrix}R_2\left(\frac{1}{7}\right)\\R_3\left(\frac{1}{8}\right)\\R_4\left(\frac{1}{6}\right)\end{matrix} \begin{bmatrix}1&4&-2\\0&1&-1\\0&-1&1\\0&-1&1\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Rightarrow \begin{matrix}R_3+R_2\\R_4+R_2\end{matrix} \begin{bmatrix}1&4&-2\\0&1&-1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \text{..........(2)}$$

$\rho(\text{coefficient matrix}) = 2$ and as r < n, the vectors are linearly dependent.

$\Rightarrow$ System has infinitely many solution.

Here number of unknowns = 3.

Number of parameters to be introduced = 3-2 =1.

Let $k_3=t$

Rewriting (2) as system of equations,

$$\begin{aligned} k_1+4k_2-2k_3&=0\\ k_2-k_3&=0 \end{aligned}$$

Solving these equations we get, $k_3=t,\quad k_2=t,\quad k_1=-2t$

On substituting these values in (1) we get

$-2t\,X_1+t\,X_2+t\,X_3=0 \Rightarrow 2X_1=X_2+X_3$

Exercise 39

1. Examine whether the following vectors are linearly independent or dependent:
   (i) [1,1,1,3], [1,2,3,4], [2,3,4,7] **Ans:** Dependent
   (ii) (1,2,-1,0), (1,3,1,2), (4,2,1,0), (6,1,0,1) **Ans:** Independent
   (iii) X_1 = (a, b, c), X_2 = (b, c, a), X_3 = (c, a, b) where a+ b+ c $\neq$ 0. **Ans:** Independent

Let's Check away from lecture

1. Among the following the pair of vectors orthogonal to each other is
   (a) $[3,4,7][3,4,7]$ (b) $[1,0,0][1,1,0]$ (c) $[1,0,2][0,5,0]$ (d) $[1,1,1][-1,-1,-1]$
2. Are the following vectors linearly dependent where
   $X_1=[3,2,7]$, $X_2=[2,4,1]$ *and* $X_3=[1,-2,6]$
   (a) Dependent (b) Independent (c) Can't say (d) None of these

Home Work Problems for the day

1. Examine whether the following vectors are linearly independent or dependent:
   (i) [1,-1,1], [2,1,1], [3,0,2] **Ans:** Dependent, $x_1+x_2-x_3=0$
   (ii) [3,1,-4], [2,2,-3], [3,0,2] **Ans:** Independent
   (iii) [1,1,-1], [2,-3,5], [2,-1,4] **Ans:** Independent
   (iv) X_1=[1,3,4,2], X_2=[3,-5,2,6], X_3=[2,-1,3,4] **Ans:** Dependent $x_1+x_2+2x_3=0$

Learning from the topic: Learners will be able to distinguish linearly independent and dependent vectors.

Rank Nullity Theorem (Self Study)

Lecture: 40

1.Learning Objective:
Student shall be able to verify rank nullity theorem.

2. Introduction:

We will learn to verify Rank Nullity Theorem with the help of solution of the homogeneous system of equations. For this purpose, we will write the given matrix as AX=O. Then we consider the augmented matrix [A]and find the rank of the matrix.

3.Key Definitions:

i) The Null Space of a Matrix: The null space of a real m×n matrix A to be the set of all real solutions to the associated homogeneous linear system Ax = 0.
Thus, null space(A) = $\{x \in R^n : Ax = 0\}$.

ii) Nullity: The dimension of null space (A) is referred to as the nullity of A and is denoted nullity(A).

4. Important formulae / Theorem:

i) In order to find nullity(A), we need to determine a basis for null space (A). Recall that if rank(A) = r, then any row-echelon form of A contains r leading ones, which correspond to the bound variables in the linear system. Thus, there are n−r columns without leading ones, which correspond to free variables in the solution of the system Ax = 0. Hence, there are n - r free variables in the solution of the system Ax = 0. We might therefore suspect that nullity(A) = n - r.

ii) Rank Nullity Theorem: For any mxn matrix A, rank (A) + nullity (A) = n.

5.Sample Problems

1) If $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{bmatrix}$ find a basis for null space (A) and verify Rank-Nullity Theorem

Solution: We must find all solutions to Ax = 0. Reducing the augmented matrix of this system yields Given $R_2 - 3R_1$ $A \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & -7 & -7 & 0 \\ 0 & -1 & 7 & 7 & 0 \end{bmatrix}$ $R_3 + R_1$ $\sim \begin{bmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & -7 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Consequently, there are two free variables, $x3 = t1$ and $x4 = t2$,
so that $x_2 = 7t_1 + 7t_2$, $x_1 = -9t_1 - 10t_2$.

Hence, null space (A) $= \{(-9t_1 - 10t_2, 7t_1 + 7t_2, t_1, t_2) : t_1, t_2 \in R\}$
$= \{t_1(-9, 7, 1, 0) + t_2(-10, 7, 0, 1) : t_1, t_2 \in R\}$
$= \text{span}\{(-9, 7, 1, 0), (-10, 7, 0, 1)\}$.

Since the two vectors in this spanning set are not proportional, they are linearly independent.
Consequently, a basis for null space (A) is $\{(-9, 7, 1, 0), (-10, 7, 0, 1)\}$, so that nullity(A) = 2.
In this problem, A is a 3×4 matrix, and so, in the Rank-Nullity Theorem, n = 4.
Further, from the foregoing row-echelon form of the augmented matrix of the system Ax = 0, we see that rank(A) = 2. Hence, rank(A) + nullity(A) = 2 + 2 = 4 = n,
and the Rank-Nullity Theorem is verified.

Exercise 40

1. Find nullity of $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

2. Find nullity of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

Let's check away from lecture

1. The set $\{ e_1 = (1, 0), e_2 = (0, 1)\}$ is a basis of R^2. Then dimension of R^2 is
(a) 0 (b) 1 (c) 2 (d) 3

2. If A is a matrix with rank 4, and if Ax=b is a consistent linear system, then the general solution of the system contains How many parameters.
(a) 4 (b) 1 (c) 2 (d) 3

Homework Problems for the day

1. Find nullity of $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

2. Find nullity of $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Learning from the topic: Learners will be able to verify rank nullity theorem

Tutorial Questions

1. Find P and Q Such that the Normal form of $A=\begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}$ is PAQ. Hence find the rank of A

2. Examine whether the following vectors are linearly independent or dependent:

   $X_1 = [1, 2, 4]$, $X_2 = [2, -1, 3]$, $X_3 = [0, 1, 2]$ $X_4 = [-3, 7, 2]$

3. Determine the values of λ so that the equations :

   $x + 2y + z = 3 ; x + y + z = \lambda ; 3x + y + 3z = \lambda^2$

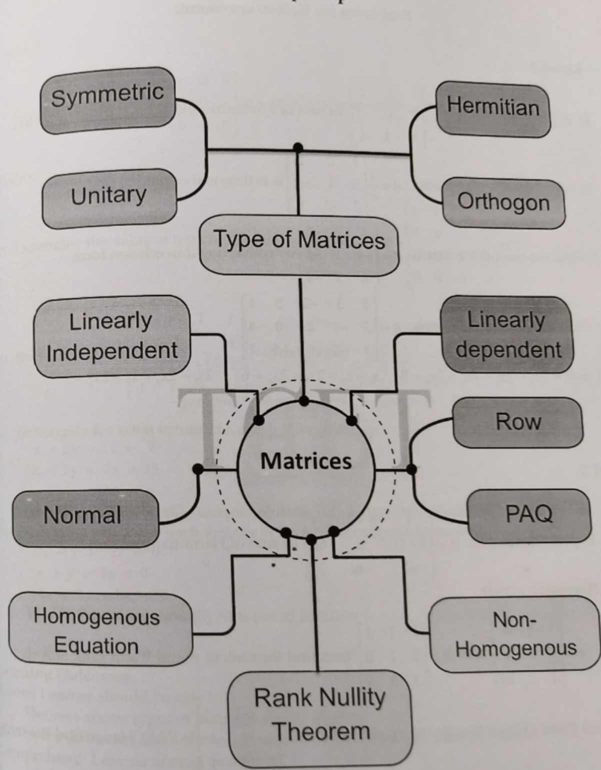
   have a solution and solve them completely in each case.

4. Evaluate the rank of the following matrix by reducing to normal form:

   $$A=\begin{bmatrix} 2 & -1 & 1 & 5 \\ 2 & 4 & -1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

5. Find nullity of $A=\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Concept Map



Problems for Self-assessment:

Level 1

1) Express the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & -1\\ 3 & -1 & 4\end{bmatrix}$ as sum of symmetric and skew symmetric matrix

2) Verify whether the matrix $A=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$ is orthogonal or not.

3) Find the rank of the matrix $A=\begin{bmatrix}1 & 2 & -5\\ -4 & 1 & -6\\ 6 & 3 & -4\end{bmatrix}$ by converting it in echelon form

4) Find the rank of the matrix $A=\begin{bmatrix}2 & 3 & -2 & 5 & 1\\ 3 & -1 & 2 & 0 & 4\\ 4 & -5 & 6 & -5 & 7\end{bmatrix}$

5) Solve $2x_1-2x_2+4x_3+3x_4=9$ $x_1-x_2+2x_3+2x_4=6$ $2x_1-2x_2+x_3+2x_4=3$
$x_1-x_2+x_4=2$

Level 2

1) Express the matrix $A=\begin{bmatrix}1 & 3+2i & -4\\ 3-2i & 5 & 6i\\ -4 & -6i & 3\end{bmatrix}$ as sum of Hermitian and skew Hermitian matrix

2) If $A=\begin{bmatrix}1 & 5 & 4\\ 0 & 3 & 2\\ 2 & 3 & 10\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 3 & 3 & 3\end{bmatrix}$ then find the rank of A and B and rank of A+B

3) Find P and Q Such that the Normal form of $A=\begin{bmatrix}1 & -1 & -1\\ 1 & 1 & 1\\ 3 & 1 & 1\end{bmatrix}$ is PAQ. Hence find the rank of A

4) Solve $x_1+x_2-4x_3=0$ $2x_1-x_2+x_3=3$ $4x_1+2x_2-2x_3=2$

5) Solve $2x+y-z=0$, $2x+5y+7z=52$, $x+y+z=9$

Level 3

1). Express the Matrix $A=\begin{bmatrix}i & 0 & 0\\ 0 & 0 & i\\ 0 & i & 0\end{bmatrix}$ as sum of real symmetric and Real skew symmetric matrix.

2) Determine the value of b such that rank of the matrix $A=\begin{bmatrix}1 & 1 & -1 & 0\\ 4 & 4 & -3 & 1\\ b & 2 & 2 & 2\\ 9 & 9 & b & 3\end{bmatrix}$ is 3.

3) Show that $A=\frac{1}{2}\begin{bmatrix}-1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1\end{bmatrix}$ is orthogonal.

4) Investigate for what values of λ and μ the equations:
$x + 2y + z = 8$
$2x + 2y + 2z = 13$
$3x + 4y + \lambda z = \mu$
Have (i) no solution (ii) unique solution (iii) infinite no. of solutions

5) Determine value of b such that the system of Homogeneous equations
$2x + y + 2z = 0$
$x + y + 3z = 0$
$4x + 3y + bz = 0$
Has (i) Trivial solution (ii) Non-trivial solution

Learning Outcomes:

Know: Learner should be able to

1. Define various types of matrices and its algebra.
2. Solve different forms of homogeneous and non-homogeneous equations.

Comprehend: Learner should be able to

3. Identify Hermitian, skew-Hermitian, symmetric, skew symmetric matrix
4. Find rank of the matrix by using echelon form, normal form and PAQ form.

Apply, analyze, and synthesize: Learner should be able to

5. Distinguish between linear dependent and independent vectors.
6. Comparing homogeneous and non-homogeneous linear equations.

Digital reference:
https://www.mathsisfun.com/algebra/matrix

Add to Knowledge:

Matrices and matrix multiplication reveal their essential features when related to linear transformations, also known as linear maps. A real m-by-n matrix **A** gives rise to a linear transformation $\mathbf{R}^n \to \mathbf{R}^m$ mapping each vector **x** in $\mathbf{R}^n$ to the (matrix) product Ax, which is a vector in $\mathbf{R}^m$. Conversely, each linear transformation f: $\mathbf{R}^n \to \mathbf{R}^m$ arises from a unique m-by-n matrix **A**: explicitly, the (i, j)-entry of **A** is the i^{th} coordinate of $f(\mathbf{e}_j)$, where $\mathbf{e}_j$ = (0,...,0,1,0,...,0) is the unit vector with 1 in the j^{th} position and 0 elsewhere. The matrix **A** is said to represent the linear map f, and **A** is called the transformation matrix off. Matrices are a key tool in linear algebra.

The rank of a matrix **A** is the maximum number of linearly independent row vectors of the matrix, which is the same as the maximum number of linearly independent column vectors. Equivalently it is the dimension of the image of the linear map represented by **A**. The rank–nullity theorem states that the dimension of the kernel of a matrix plus the rank equals the number of columns of the matrix.

A major branch of numerical analysis is devoted to the development of efficient algorithms for matrix computations, a subject that is centuries old and is today an expanding area of research. Matrix decomposition methods simplify computations, both theoretically and practically. Algorithms that are tailored to particular matrix structures, such as sparse matrices and near-diagonal matrices, expedite computations in finite element method and other computations. Infinite matrices occur in planetary theory and in atomic theory. A simple example of an infinite matrix is the matrix representing the derivative operator, which acts on the Taylor series of a function.

Self-Evaluation

Name of student:

Class & Div: **Roll No:**

1. Are you able to distinguish the different types of matrices?
   (a) Yes (b) No

2. Are you able to find the rank of a matrix using different methods?
   (a) Yes (b) No

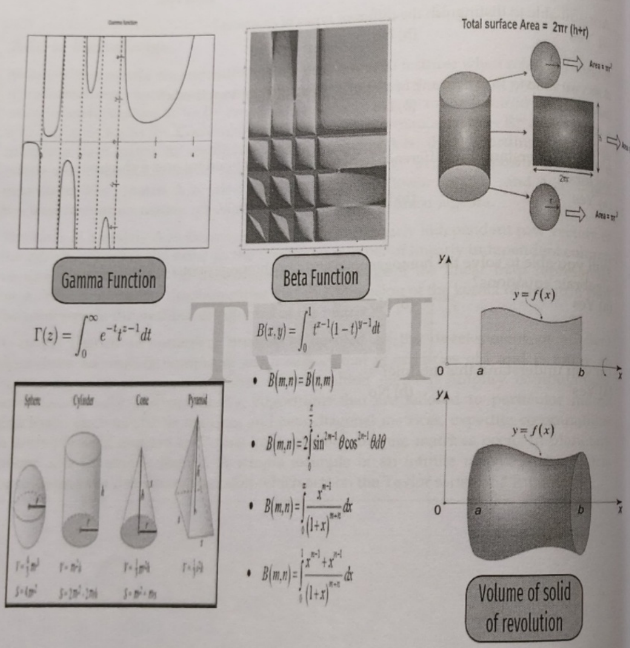
3. Do you understand the difference between linearly dependent and independent vectors?
   (a) Yes (b) No

4. Will you able to solve the homogeneous and non-homogeneous system of linear algebraic equations?
   (a) Yes (b) No

5. Do you understand this module?
   (a) Yes (b) No

Module 5: Calculus-II

Infographics



Module 5: Calculus-II

1. Motivation:
This topic deals with the solving improper integrals using Gamma and Beta functions also it deals with application of single integral in the evaluation of Surface area and volumes of revolutions.

2.Syllabus:

Lecture No.	Title	Duration (Hrs.)	Self-study (Hrs.)
41	Gamma function	1	2
42	Gamma function continued	1	2
43	Beta function	1	2
44	Beta function continued	1	2
45	Beta function continued	1	2
46	Application of single integral in the evaluation of Surface area of revolution.	1	2
47	Surface area of revolution problems continued	1	2
48	Surface area of revolution problems continued	1	2
49	Surface area of revolution problems continued	1	2
50	Application of single integral in the evaluation of volume of revolution.	1	2
51	Volume of revolution problems continued	1	2

3. Prerequisite:
Definite Integrals, Proper Integrals, Improper Integrals

4. Learning objective:

Learners shall be able to
1. Understand improper integral.
2. Apply gamma function to solve improper integrals.
3. Apply Beta function to solve improper integrals.
4. Relate Beta and gamma function.
5. Apply the concept of single integral in the evaluation of surface area.
6. Apply the concept of single integral in the evaluation of volumes of revolutions.

Gamma Function

Lecture: 41

1. Learning objective: Understand Gamma function and Apply gamma function to solve improper integrals.

2. Introduction: The Gamma function (sometimes called the Euler Gamma function) is an advanced mathematical function with some unusual properties. It is an extension of the factorial function. This section emphasizes on the use of gamma functions to evaluate the integral.

3. Key Definitions:

Improper Integral:

(i) If $b=\infty$ or $a=-\infty$, but f is bounded at each point of the interval $\int_a^b f(x)dx$ is called the improper integral of first kind.

(ii)If a and b are real numbers but f is unbounded at some point in $[a,b]$, $\int_a^b f(x)dx$ is called the improper integral of second kind.

Gamma Function: The gamma function is defined as $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}\,dx$

(n is positive integer). This integral is also known as Euler's integral of the second kind.

Another form of Gamma Function is $\Gamma(n)=2\int_0^\infty e^{-x^2}x^{2n-1}dx$

4. Key Notation: Gamma Function: $\Gamma(n)$

5. Important Formulae/ Theorems / Properties:

(i) $\Gamma(1)=1$

Proof: Using definition $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}\,dx$ for $n=1$

$\Gamma(1)=\int_0^\infty e^{-x}x^{1-1}\,dx=\int_0^\infty e^{-x}\,dx=\left|-e^{-x}\right|_0^\infty = 1$ i.e. $\Gamma(1)=1$

(ii) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

(iii) Reduction formula: $\Gamma(n+1)=n\Gamma(n)$

Proof: Using definition $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}\,dx$ for $n=n+1$

$\Gamma(n+1)=\int_0^\infty e^{-x}x^{(n+1)-1}\,dx=\left|-x^n e^{-x}\right|_0^\infty-\int_0^\infty e^{-x}\,n x^{n-1}\,dx==0+n\int_0^\infty e^{-x}x^{n-1}\,dx=n\Gamma(n)$

i.e. $\Gamma(n+1)=n\Gamma(n)$

Remark: Reduction formula is used in three different ways as

(a) $\Gamma(n+1)=(n)!$ if n is +ve integer.

(b) $\Gamma(n+1)=n\Gamma(n)$ if n is +ve number.

(c) $\Gamma(n)=\frac{\Gamma(n+1)}{n}$ if n is -ve fraction.

6.Sample problems:

(1) Prove that $\int_0^\infty e^{-kx}x^{n-1}\,dx=\frac{\Gamma(n)}{k^n}$

Solution: Taking $kx=t$, $dx=\frac{dt}{k}$ When $x=0;\ t=0$ and when $x=\infty;\ t=\infty$

Given integral $=I=\int_0^\infty e^{-kx}x^{n-1}\,dx=\int_0^\infty e^{-t}\left(\frac{t}{k}\right)^{n-1}\frac{dx}{k}$

$=\frac{1}{k^n}\int_0^\infty e^{-t}t^{n-1}\,dx=\frac{\Gamma(n)}{k^n}$

(2) Prove that $\int_0^\infty xe^{-ax}\sin bx\,dx=\frac{2ab}{(a^2+b^2)^2}$.

Solution: $\int_0^\infty xe^{-ax}\sin bx\,dx=\int_0^\infty xe^{-ax}\left[\text{Imaginary part of } e^{ibx}\right]dx$

$=\text{Im. Part}\int_0^\infty xe^{-ax}\cdot e^{ibx}\,dx=\text{Im. Part}\int_0^\infty e^{-(a-ib)x}\cdot x^{2-1}\,dx$

$=\text{Im. Part}\,\frac{\Gamma(2)}{(a-ib)^2}\left[\because\int_0^\infty e^{-kx}x^{n-1}dx=\frac{\Gamma(n)}{k^n}\right]$

$=\text{Im. Part}\,\frac{1}{(a^2-b^2)-2iab}$

$=\text{Im. Part}\,\frac{1}{(a^2-b^2)-2iab}\cdot\left[\frac{(a^2-b^2)+2iab}{(a^2-b^2)+2iab}\right]$

$=\text{Im. Part}\left[\frac{(a^2-b^2)+2iab}{(a^2-b^2)^2+4a^2b^2}\right]$

$=\text{Im. Part}\left[\frac{(a^2-b^2)}{(a^2+b^2)^2}+i\frac{2ab}{(a^2+b^2)^2}\right]=\frac{2ab}{(a^2+b^2)^2}$

(3) Evaluate $\int_0^\infty\frac{e^{-x^3}}{\sqrt{x}}\,dx$

Solution:

Put $x^3 = u \Rightarrow 3x^2 dx = du$

Also $x^3 = u \Rightarrow x = u^{\frac{1}{3}} \quad \therefore dx = \frac{du}{3x^2} = \frac{1}{3}\frac{du}{u^{\frac{2}{3}}}$

$$\therefore I = \int_0^\infty \frac{e^{-u}}{\sqrt{u^{1/3}}}\frac{du}{3u^{2/3}} = \int_0^\infty \frac{e^{-u}u^{-1/6}u^{-2/3}}{3}du = \frac{1}{3}\int_0^\infty e^{-u}u^{-5/6}du = \frac{1}{3}\int_0^\infty u^{(1/6)-1}du = \frac{1}{3}\Gamma\left(\frac{1}{6}\right)$$

Exercise 41

1) Prove that $\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = \left(\frac{1}{4}-x^2\right)\pi\sec\pi x$ provided $-1<2x<1$

2) Prove that $\int_0^\infty \frac{e^{-\sqrt{x}}}{x^{\frac{7}{4}}}dx = \frac{8}{3}\sqrt{\pi}$

3) Prove that $\int_0^1 (x\log x)^4 dx = \frac{4!}{5^5}$

4) Evaluate $\int_0^\infty 7^{-4x^2}dx$

Let us check away from lecture

Choose the correct option from the following

1. $\Gamma(123)$

(a) 122! (b) 124! (c) not defined (d) None

2. $\Gamma(-12)$

(a) 122! (b) 124! (c) not defined (d) None

3. $\Gamma\left(\frac{5}{2}\right)$

(a) $\left(\frac{5}{2}\right)!$ (b) $\frac{3}{4}\sqrt{\pi}$ (c) not defined (d) None

Homework Problems for the day

1) Evaluate $\int_0^\infty (x^2+4)e^{-2x^2}dx$.

Ans: $\frac{17\sqrt{\pi}}{8\sqrt{2}}$

2) Evaluate $\int_0^\infty e^{-\sqrt{x}}x^{\frac{1}{4}}dx$

Ans: $\frac{3}{2}\sqrt{\pi}$.

3) Evaluate $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}dx$

Ans: $\frac{1}{2}\Gamma\left(\frac{1}{4}\right)$.

Gamma Function Continued

Lecture: 42

Solved Problems

1. Evaluate $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}dx$

Solution: $I = \int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}dx$

Put $\sqrt[3]{x} = t$ so $x = t^2$ which gives $dx = 3t^2\,dt$

When $x = 0$ then $t = 0$ and $x = \infty$ then $t = \infty$;

$$I = \int_0^\infty \sqrt{t^3}\, e^{-t}dt = 3\int_0^\infty e^{-t}t^{\frac{3}{2}+2}dt = 3\int_0^\infty e^{-t}t^{\frac{7}{2}}dt = 3\Gamma\frac{7}{2}+1$$

$$= 3.\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2} = \frac{315}{16}\sqrt{\pi}$$

2. Evaluate $\int_0^1 \sqrt[3]{\log\left(\frac{1}{x}\right)}dx$

Solution. Let $I = \int_0^1 \sqrt[3]{\log\left(\frac{1}{x}\right)}dx$

Put $\log\left(\frac{1}{x}\right) = t$ so $\frac{1}{x} = e^t$ i.e. $x = e^{-t}$ $\therefore dx = -e^{-t}dt$

When $x = 0$ then $t = \infty$ and $x = 1$ then $t = 0$

$$\therefore I = \int_\infty^0 \sqrt[3]{t}\,(-e^{-t})dt = -\int_\infty^0 e^{-t}t^{\frac{1}{3}}dt = -\int_\infty^0 e^{-t}t^{\frac{4}{3}-1}dt = \Gamma\frac{4}{3} = \Gamma\frac{1}{3}+1 = \frac{1}{3}\Gamma\frac{1}{3}$$

Exercise: 42

1. Evaluate $\int_0^\infty e^{-x^5}dx$

2. Prove that $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$

3 Evaluate $\int_0^\infty \frac{x^4}{4^x}dx$.

Let us check away from lecture

Choose the correct option from the following

1. $\Gamma n+1 = n!$ can be used when
(a) n *is any integer* (b) n is a negative integer
(c) n is any positive integer (d) n is any real number

2. What is value of $\Gamma\frac{1}{2}$

(a) $\sqrt{\pi}$ (b) $\frac{\sqrt{\pi}}{\sqrt{2}}$ (c) π (d) 2π

Homework problems for the day

1. Evaluate $\int_0^\infty \frac{x^a}{a^x}\,dx$

2. Evaluate $\int_0^\infty e^{-h^2x^2}\,dx$

Learning from the topic: Student will be able to evaluate improper integral with the help of Gamma function.

Beta functions

Lecture 43

1. **Learning objective:** Understand beta function and apply Beta function tosolve improper integrals

2. **Introduction:**Beta function, also called the Euler integral of the first kind, is a special function that is closely related to the gamma function and to binomial coefficients.

3. **Key Definition:**

The Beta Function is defined as $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ where $m,n>0$.

4. **Properties**

(i) B (m, n) = B (n, m)

Proof: By definition $B(n,m)=\int_0^1 x^{n-1}(1-x)^{m-1}\,dx$

when $x=0;\ t=1$ and $x=1;\ t=0$

Taking $1-x=t \Rightarrow dx=-dt$

$$\therefore B(n,m) = \int_1^0 (1-t)^{n-1}t^{m-1}(-dt) = \int_0^1 t^{m-1}(1-t)^{n-1}dt = B(m,n)$$

(ii) **Relation between Beta and Gamma function:** $B(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma m+n}$

(iii) $B(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

Proof: By definition $B(n,m)=\int_0^1 x^{n-1}(1-x)^{m-1}\,dx$

when $x=0;\ t=1$ and $x=0;\ t=\frac{\pi}{2}$

$$B(mn)=\int_0^{\frac{\pi}{2}}(\sin\theta)^{2m-2}(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}(\sin\theta)^{2m-2}(\cos\theta)^{2n-2}\,2\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$$

i.e. $B(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

Remark: The above property can be written as

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

(iv) $\Gamma\frac{1}{2}=\sqrt{\pi}$

Proof: we have $\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta=\frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$

Taking $p=0$ & $q=0$ we get $\int_0^{\frac{\pi}{2}}d\theta=\frac{1}{2}B\left(\frac{1}{2},\frac{1}{2}\right)$

$$\theta\Big|_0^{\frac{\pi}{2}}=\frac{1}{2}\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

$$\frac{\pi}{2}=\frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2 \text{ gives } \Gamma\frac{1}{2}=\sqrt{\pi}$$

5. **Sample problems**

(1) Evaluate $\int_0^{2a}\frac{x^3}{\sqrt{2ax-x^2}}\,dx$

Solution: Given integral $= I = \int_0^{2a}\frac{x^3}{\sqrt{2ax-x^2}}\,dx = \int_0^{2a}(2ax-x^2)^{\frac{-1}{2}}x^3\,dx$

$$= \int_0^{2a} (2a-x)^{\frac{-1}{2}} x^{\frac{-1}{2}} x^3 \, dx = \int_0^{2a} (1-\tfrac{x}{2a})^{\frac{-1}{2}} (2a)^{\frac{-1}{2}} x^{3\frac{-1}{2}} \, dx$$

$$= (2a)^{\frac{-1}{2}} \int_0^{2a} (1-\tfrac{x}{2a})^{\frac{-1}{2}} x^{\frac{5}{2}} \, dx$$

when $x = 0; t = 0$ and $x = 2a; t = 1$

Taking $\frac{x}{2a} = t \Rightarrow x = 2at \Rightarrow dx = 2a\, dt$

On substituting, $I = (2a)^{\frac{-1}{2}} \int_0^1 (1-t)^{\frac{-1}{2}} (2at)^{\frac{5}{2}} (2a dt)$

$$= (2a)^{\frac{-1}{2}+\frac{5}{2}+1} \int_0^1 (1-t)^{\frac{-1}{2}} t^{\frac{5}{2}} \, dt = (2a)^3 \int_0^1 (1-t)^{\frac{-1}{2}} t^{\frac{5}{2}} \, dt = 8a^3 \, B(\tfrac{-1}{2}+1, \tfrac{5}{2}+1)$$

$$= 8a^3 \, B(\tfrac{1}{2}, \tfrac{7}{2})$$

(2) Show that $\int_0^a \frac{dx}{\sqrt[n]{a^n - x^n}} = \frac{\pi}{n} \operatorname{cosec}\left(\frac{\pi}{n}\right)$ where $n > 1$

Solution: Given that $\int_0^a \frac{dx}{\sqrt[n]{a^n - x^n}}$ Put $x^n = a^n t \Rightarrow x = a t^{\frac{1}{n}} \Rightarrow dx = \frac{a}{n} t^{\frac{1}{n}-1}$

when $x = 0; t = 0$ and $x = a; t = 1$

$$\Rightarrow \int_0^1 \frac{\frac{a}{n} t^{\frac{1}{n}-1} dt}{\sqrt[n]{a^n - a^n t}} \Rightarrow \frac{a}{n} \int_0^1 t^{\frac{1}{n}-1} (a^n - a^n t)^{-\frac{1}{n}} dt = \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{-\frac{1}{n}} dt = \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{-\frac{1}{n}+1-1} dt$$

$$= \frac{1}{n} B\left(\frac{1}{n}, 1-\frac{1}{n}\right) = \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{\Gamma(1)} = \frac{1}{n} \frac{\pi}{\sin\frac{\pi}{n}} = \frac{\pi}{n} \operatorname{cosec}\left(\frac{\pi}{n}\right)$$

Exercise 43

1) Evaluate $\int_0^2 x^4 (8-x^3)^{-\frac{1}{3}} dx$ Ans: $\frac{8\left(\Gamma\left(\frac{2}{3}\right)\right)^2}{\Gamma\left(\frac{1}{3}\right)}$

2) Show that $\int_0^1 \sqrt{1-\sqrt{x}} \, dx \int_0^{\frac{1}{2}} \sqrt{2y-(2y)^2} \, dy = \frac{\pi}{30}$

Let us check away from lecture

1) Evaluate the value of B(3,2) – B(2,3)

(a) 12 (b) 0 (c) not defined (d) 5

2) The value of $\beta(\frac{1}{4}, \frac{3}{4})$ is

(a) π (b) $\sqrt{2}\,\pi$ (c) 0 (d) $\sqrt{2}$

Homework Problems for the day

1) Evaluate $\int_0^2 x^2 (2-x)^3 dx$ Ans: $\frac{16}{15}$

2) Prove that $\int_0^3 \frac{x^{\frac{3}{2}}}{\sqrt{3-x}} dx \int_0^1 \frac{dx}{\sqrt{1-x^{\frac{1}{4}}}} = \frac{432\pi}{35}$

3) Prove that $\int_5^9 \sqrt[4]{(9-x)(x-5)} \, dx = \frac{2\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{3\sqrt{\pi}}$

Learning from the topic: Student will be able to evaluate definite integral with the help of Beta function using its basic structure.

Beta function continued...

Lecture: 44

1. **Learning objective:** Relate Beta and gamma function and apply Beta function to solve improper Integrals.
2. **Introduction:** Third structure of Beta function is having rational function in the integrand having the limits 0 to ∞.
3. **Important formulae:**

(i) Another Form of Beta Function: $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m,n)$

Proof: $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Taking $x = \tan^2\theta$, $dx = 2\tan\theta \sec^2\theta \, d\theta$, when $x=0;\ \theta=0$ and $x=\infty;\ \theta=\frac{\pi}{2}$

$$\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}dx=\int_0^{\frac{\pi}{2}}\frac{\left(\tan^2\theta\right)^{m-1}}{(1+\tan^2\theta)^{m+n}}\tan\theta\sec^2\theta\, d\theta$$

$$=2\int_0^{\frac{\pi}{2}}(\tan\theta)^{2m-2+1}(\sec\theta)^{2-2m-2n}\,d\theta$$

$$=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2m-1}(\cos\theta)^{-2m+1-2+2m+2n}\,d\theta$$

$$=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$$

$$=B(m,n)$$

(ii) Duplication formula: Duplication formula is the result in which beta function is represented in the form of beta function and gamma function in gamma function

the formula is as $2^{2m-1}\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)=\sqrt{\pi}\,\Gamma(2m)$

Proof: Since $\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\, d\theta=\frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$

Taking $p=q$, $\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^p\theta\, d\theta=\frac{1}{2}B\left(\frac{p+1}{2},\frac{p+1}{2}\right)$

$$\Rightarrow\quad \frac{1}{2^p}\int_0^{\frac{\pi}{2}}\sin^p 2\theta\, d\theta=\frac{1}{2}\frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{\Gamma(p+1)}\qquad\text{.......(1)}$$

Taking $2\theta=t$, $d\theta=\frac{dt}{2}$ when $\theta=0;\ t=0$ and $\theta=\frac{\pi}{2};\ t=\pi$

$$\therefore\frac{1}{2^p}\int_0^{\frac{\pi}{2}}\sin^p 2\theta\, d\theta=\frac{1}{2^p}\int_0^{\pi}\sin^p t\,\frac{dt}{2}=\frac{1}{2^{p+1}}2\int_0^{\frac{\pi}{2}}\sin^p t\, dt\quad[\because\sin(\pi-t)=\sin t]$$

$$=\frac{1}{2^p}\int_0^{\pi/2}\sin^p t\cos^0 t\, dt=\frac{1}{2^p}\frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}\qquad\text{..........(2)}$$

Equating (1) and (2) we get $\frac{1}{2^p}\frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}=\frac{1}{2}\frac{\left(\Gamma\left(\frac{p+1}{2}\right)\right)^2}{\Gamma(p+1)}$

$$\frac{1}{2^p}\frac{\sqrt{\pi}}{\Gamma\left(\frac{p+2}{2}\right)}=\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(p+1)}\qquad\text{Taking } 2m-1=p,$$

$$\frac{\Gamma(m)}{\Gamma(2m)}=\frac{\sqrt{\pi}}{2^{2m-1}\Gamma\left(m+\frac{1}{2}\right)}\quad\Rightarrow 2^{2m-1}\,\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)=\sqrt{\pi}\,\Gamma(2m)$$

Hence $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}$

4.Sample problems:

1) Evaluate $\int_0^{\infty}\frac{\sqrt{x}}{1+2x+x^2}dx$

Solution: Given integral $=I=\int_0^{\infty}\frac{\sqrt{x}}{1+2x+x^2}dx=\int_0^{\infty}\frac{x^{\frac{1}{2}}}{(1+x)^2}dx$

$$=B\left(\frac{1}{2},\frac{3}{2}\right)=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)}=\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}=\frac{\frac{1}{2}\pi}{1!}=\frac{\pi}{2}$$

Exercise 44

1) Prove that $\int_0^{\infty}\frac{x^5}{(2+3x)^{16}}dx=\frac{5!\,9!}{2^{10}3^6\,15!}$

2) Prove that $B(m,n)=\int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$. Hence find $\int_0^1\frac{x^2+x^3}{(1+x)^7}dx$

3) Prove that $B\left(n+\frac{1}{2},n+\frac{1}{2}\right)=\frac{1}{2^{2n}}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\sqrt{\pi}$

Let us check away from lecture

1) Evaluate $\int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

(a) $B(m,n)$ (b) $B(m+1,n+1)$ (c) $B(m-1,n-1)$ (d) $2B(m,n)$

2) If $B(n,3)=\frac{1}{105}$ and n is a +ve integer, find n.

(a) 3 (b) 5 (c) 1 (d) 2

3) $B(n,n)=\dfrac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\,\Gamma\left(n+\frac{1}{2}\right)}$

(a) *True* (b) *False* (c) *not defined* (d) *None*

Learning from the topic: Student will be able to apply duplication formula.

Beta function continued...

Lecture: 45

Sample Problems

1) *Evaluate* $\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\sin x}}\,dx$

Solution: Given integral $=I=\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\sin x}}\,dx = I_1 I_2$

where $I_1=\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx=\int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}}x\,\cos^0 x\,dx=B\left(\frac{\frac{1}{2}+1}{2},\frac{0+1}{2}\right)=B\left(\frac{3}{4},\frac{1}{2}\right)$

and $I_2=\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\sin x}}\,dx=\int_0^{\frac{\pi}{2}}\sin^{\frac{-1}{2}}x\,\cos^0 x\,dx=B\left(\frac{-\frac{1}{2}+1}{2},\frac{0+1}{2}\right)=B\left(\frac{1}{4},\frac{1}{2}\right)$

Given integral $=I=I_1I_2=B\left(\frac{3}{4},\frac{1}{2}\right)B\left(\frac{1}{4},\frac{1}{2}\right)=\dfrac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}\dfrac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$

$=\dfrac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\dfrac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}=\dfrac{\Gamma\left(\frac{1}{4}\right)\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}=4\pi$

2) Evaluate $\displaystyle\int_0^{\pi/6}\cos^6 3\theta\sin^2 6\theta\,d\theta$

Solution: Let $I=\displaystyle\int_0^{\pi/6}\cos^6 3\theta\sin^2 6\theta\,d\theta$

Put $3\theta=t\Rightarrow\quad d\theta=\dfrac{dt}{3}$

when $\theta=0;\,t=0$ and $\theta=\frac{\pi}{6};\,t=\frac{\pi}{2}$

$I=\displaystyle\int_0^{\pi/2}\cos^6 t.\sin^2(2t)\frac{dt}{3}=\frac{1}{3}\int_0^{\pi/2}\cos^6 t(2\sin t\cos t)^2\,dt\quad=\frac{4}{3}\int_0^{\pi/2}\sin^2 t.\cos^8 t\,dt$

$\therefore$

$=\dfrac{4}{3}\dfrac{1}{2}B\left(\dfrac{2+1}{2},\dfrac{8+1}{2}\right)\quad=\dfrac{2}{3}\dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}\quad=\dfrac{2}{3}\dfrac{\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{\sqrt{2}}=\dfrac{\pi}{3}$

Exercise 45

1. $\displaystyle\int_5^9\sqrt[4]{(9-x)(x-5)}\,dx=\frac{2\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{3\sqrt{\pi}}$

2) Show that $\displaystyle\int_0^a\sqrt{\frac{x^3}{a^3-x^3}}\,dx=\frac{a\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}$

3) Evaluate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\theta+\sin\theta)^{\frac{1}{3}}\,d\theta$ **Ans:** $6\sqrt{\pi}\,\Gamma\left(\frac{2}{3}\right)\Big/2^{\frac{5}{6}}\Gamma\left(\frac{1}{6}\right)$

Learning from the topic: Student should able to solve definite integral by using Beta function.

Application of single integration in the evaluation of surface area of revolutions

Lecture: 46

1. **Learning objective:** Apply the concept of single integral in the evaluation of surface area of revolutions.
2. **Introduction:** In this section we learn to find the surface area of solid of revolution and with the help of single integrals.
3. **Important formulae:**

(i) **Revolution about x-axis:** The surface area of solid generated by the revolution about x-axis of the arc of the curve $y=f(x)$ from $x=a$ *to* $x=b$ is

$$\int_{x=a}^{x=b}2\pi y\,ds.$$

(ii) **Revolution about y-axis:** The surface area of solid generated by the revolution about x-axis of the arc of the curve $x=f(y)$ from $y=a$ *to* $y=b$ is

$$\int_{y=a}^{y=b}2\pi x\,ds.$$

Surface area of revolution:

(a) In Cartesian Coordinates:

For the curve $y=f(x)$, the surface area of revolution about the x-axis from $x=a$ to $x=b$ is given by

$$S=\int_{x=a}^{x=b} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Similarly, the surface area of revolution of the curve about the y-axis from $y=c$ to $y=d$ is given by is given by

$$S=\int_{y=c}^{y=d} 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

(b) In Parametric Form:

For the curve $x=f(t), y=g(t)$ the surface area of revolution about the x-axis is given by

$$S=\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Similarly, the surface area of revolution of the curve about the y-axis is given by

$$S=\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Sample Problem:

1. Find the surface area of the solid revolution generated by revolving the circle $x^2+(y-a)^2=a^2$ about the x-axis

Solution: The equation of circle can be written in parametric form as $x=a\cos t, y=a+a\sin t,\ 0\le t\le 2\pi.$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{(-a\sin t)^2+(a\cos t)^2}=a.$$

Therefore,

$$S=\int 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^{2\pi} 2\pi(a+a\sin t)a\,dt=4\pi^2a^2 \text{ square units.}$$

Exercise 46

1) Find the volume of solid of revolution of region bounded by $y=\sqrt{x}, y=0$ from $x=0$ to $x=4$ about $x-$axis. **Answer:** 8π

(2) Find the volume of solid of revolution of region bounded by

$y=x, y=2$ and $x=0$ about $y-$axis. **Answer:** $\frac{8\pi}{3}$

(3) Find the surface area of the solid generated by revolving $x=e^t\cos t, y=e^t\sin t, 0\le t\le \pi/2$ about the $y-$axis. **Answer:** $\frac{2\sqrt{2}\pi(e^\pi-2)}{5}$

Let's check take away from the lecture

1. The surface area of sphere of radius 4 cm is?
   (a) 16 π (b) 12 π (c) 16 π² (d) 12 π²

2. The part of parabola $y^2=4ax$ cut off by the latus rectum about the tangent at the vertex. Find the curved surface area of the reel thus formed.
   (a) $4\pi a^3/5$ (b) $8\pi a^3/5$ (c) $3\pi a^3/5$ (d) $5\pi a^3/4$

Homework Problems for the day

1. Find the surface area of the solid generated by the revolution of the asteroid $x=a\cos^3t, y=a\sin^3t$ about the $y-axis$.
2. Find the surface area generated by revolving cycloid $x=a(t-\sin t), y=a(1-\cos t)$ about the base.
3. Show that the surface area of the solid generated by the revolution of the asteroid $x=a\cos^3t, y=a\sin^3t$ about the x-axis is $\frac{12\pi^2}{5}$.

Learning from the topic: Student will be able to evaluate surface area of revolution using single integration.

Surface area problems continued...

Lecture: 47

Exercise 47

1. Determine the surface area of the solid obtained by rotating $y=\sqrt{9-x^2}$, $-2\le x\le 2$ about the x-axis.
2. Determine the surface area of the solid obtained by rotating $y=\sqrt[3]{x}, 1\le y\le 2$ about the y-axis.
3. Determine the surface area of the solid obtained by rotating $y=\sin \sin 2x$, $0\le x\le \frac{\pi}{8}$ about x-axis.

Let us check away from lecture

1. The surface of the solid formed by revolving the cardioid $r = (1 + cos\theta)$ about the initial line

(a) $\frac{32\pi}{5}$ (b) $\frac{2\pi}{5}$ (c) $\frac{32\pi a^2}{7}$ (d) None

2. Find the surface area of the solid revolution generated by revolving the circle $x^2 + (y)^2 = 1$ about the x-axis

(a) $4\pi^2$ (b) π^2(c) 4π (d) none

Homework Problems for the day

1. Determine the surface area of the solid obtained by rotating$y = 4 + 3x^2$, $1 \le y \le 2$ about the y-axis.
2. Determine the surface area of the solid obtained by rotating $y = 4 + x^3$, $1 \le x \le 5$ about the x-axis.

Learning from the topic: Student will be able to evaluate surface area by using cartesian form and single integral.

Surface area problems continued...
Lecture: 48

Exercise 48

1. Find the surface area of the object obtained by rotating $y = \frac{1}{4}\sqrt{6x - 2}$, $\frac{\sqrt{2}}{2} \le y \le \frac{\sqrt{5}}{2}$ about the x-axis.
2. Find the surface area of the object obtained by rotating $y = 4 - x, 1 \le x \le 6$ about the y-axis.

1. Integral for surface area of the object by rotating $x = \sqrt{y + 5}$,$5 \le x \le 9$ is

(a) $\int_5^9 2\pi x\, dx$(b) $\int_5^9 4\pi x\, dx$(c) $\int_5^9 2\pi x\sqrt{1 + 4x^2}\, dx$ (d) none

2. The area of the loop of the curve $ay^2=x^2(a-x)$

(a) $8a^2/15$ (b) $16a^2/15$ (c) $5a^2/16$ (d) $3a^2/16$

Homework Problems for the day

1. Find the surface area of the object obtained by rotating $x=2y+5, -1 \le x \le 2$ about the y-axis.
2. Find the surface area of the object obtained by rotating $x = 1 - y^2, 0 \le y \le 3$ about the x-axis.

Learning from the topic: Student will be able to evaluate surface area by using cartesian form and single integral.

Surface area problems continued...
Lecture: 49

Polar form of Surface area [for the curve $r = f(\theta)$]

$$S = \int 2\pi y \frac{ds}{d\theta} d\theta \text{ ,Where}$$

$$y = rsin\theta, \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Sample Problem:

1. Find the surface of the solid formed by revolving the cardioid $r = a(1 + cos\theta)$ about the initial line.

Solution:

The cardioids is symmetric about the initial line and for upper half, θ varies from $0 \text{ to } \pi$.

Also,

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{(a^2 + (1 + cos\theta)^2 + a^2 sin^2\theta)}$$

$$= a\sqrt{[2(1 + cos\theta)]} = a\sqrt{2.2cos^2\left(\frac{\theta}{2}\right)} = 2acos\left(\frac{\theta}{2}\right)$$

$\therefore$ Required surface area $= \int_0^\pi 2\pi y \frac{ds}{d\theta} d\theta = 2\pi \int_0^\pi r\, sin\theta . 2a\, cos\left(\frac{\theta}{2}\right) d\theta$

$$= 4\pi a \int_0^\pi a(1 + cos\theta)\, sin\theta . cos\left(\frac{\theta}{2}\right) d\theta$$

$$= 4\pi a^2 \int_0^\pi 2cos^2\left(\frac{\theta}{2}\right) . 2\, sin\left(\frac{\theta}{2}\right) . cos\left(\frac{\theta}{2}\right) . cos\left(\frac{\theta}{2}\right) d\theta$$

$$= 16\pi a^2 \int_0^\pi cos^4\left(\frac{\theta}{2}\right) sin\left(\frac{\theta}{2}\right) d\theta$$

$$= 16\pi a^2(-2) \int_0^\pi cos^4\left(\frac{\theta}{2}\right)\left(\frac{-1}{2}\right) sin\left(\frac{\theta}{2}\right) d\theta$$

$$= -32\pi a^2 \left[\frac{\cos^5\left(\frac{\theta}{2}\right)}{5}\right]_0^{\pi} = \frac{32\pi a^2}{5}$$

Exercise 49

1. Find the surface of the solid formed by revolving the cardioid $r = a(1 - \cos\theta)$ about the initial line.
2. Find the surface of the solid generated by the revolution of lemniscates $r^2 = a^2\cos 2\theta$ about the initial line.

Let's check away from lecture

1. The surface area of loop of the curve $r = a\sin 3\theta$ is?
(a) $\pi/12$ (b) $\pi a/12$ (c) $\pi a^2/12$ (d) $\pi^2/12$

2. The surface area of loop of the curve $r = 3\sin 2\theta$ is?
(a) $\pi/2$ (b) $\pi/12$ (c) $\pi^2/12$ (d) $\pi^2/2$

Homework Problems for the day

1. Find the surface of the reel formed by revolution round the tangent at the vertex of an arch of the cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$.
2. Find area of loop of the curve $r = a\sin 3\theta$.

Learning from the topic: Student will be able to evaluate surface area by using polar form and single integral.

Application of single integration in the evaluation of volumes of revolutions

Lecture: 50

1. Learning objective: Apply the concept of single integral in the evaluation of volumes of revolutions.

2. Introduction: In this section we learn to find the volume of solid of revolution with the help of single integrals.

3. Important Formulae:

(i) Revolution about x-axis: The volume of solid generated by the revolution about the x-axis, of the area bounded by the curves $y = f(x)$, the x-axis and the coordinates $x = a, x = b$ is

$$\int_a^b \pi y^2 dx.$$

(ii) Revolution about y-axis: The volume of solid generated by the revolution about the y-axis, of the area bounded by the curves $x = f(y)$, the y-axis and the coordinates $y = a, y = b$ is

$$\int_a^b \pi x^2 dy.$$

(iii) If the area bounded by the curves $y = f(x)$, the line y=p and the lines x=a, x=b is revolved about the line y=p(a line parallel to the x-axis),then the volume of the solid of revolution is given by $V=\pi\int_a^b (y-p)^2 dx$.

(iv) If the area bounded by the curves $x = g(y)$, the line x=q and the lines y=c, y=d is revolved about the line x=q(a line parallel to the x-axis),then the volume of the solid of revolution is given by $V=\pi\int_c^d (x-q)^2 dy$.

4. Sample Problem:

1. Find the volume of the solid generated by revolving the region bounded by the curves $y=3-x^2$, $y=-1$ about the line $y=-1$.

Solution:

The volume is given by

$$V=2\pi\int_0^2 (1+y)^2 dx = 2\pi\int_0^2 (1+3-x^2)^2 dx = 2\pi\int_0^2 (16-8x^2+x^4) dx$$

$$= 2\pi\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2 = \frac{512\pi}{15} \text{ cubic units.}$$

2. Find the volume of sphere of radius a.

Solution:

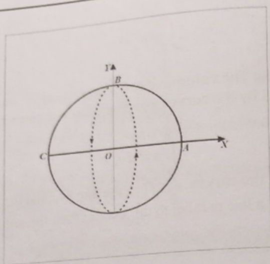
Let, the sphere be generated by the revaluation of the semi-circle ABC, of radius a about its diameter CA.

Taking CA as the x-axis and its mid-point O as the origin, the equation of circle ABC is

$$x^2 + y^2 = a^2.$$

$\therefore$ *Volume of sphere*

$= 2(\textit{volume of solid generated by the revolution about xaxis of the quadrant OAB})$



$$= 2\int_0^a \pi y^2 dx = 2\pi \int_0^a a^2 - x^2 dx = 2\pi \left[a^2 x - \frac{x^3}{3}\right]_0^{\pi}$$
$$= 2\pi \left[a^3 - \frac{a^3}{3}\right] = \frac{4}{3}a^3.$$

This is required volume.

Exercise 50

(1) Find the volume of solid of revolution of region bounded by $y = \sqrt{x}, y = 0$ from $x = 0$ to $x = 4$ about $x-$axis. **Answer**: 8π

(2) Find the volume of solid of revolution of region bounded by $y = x, y = 2$ and $x = 0$ about $y-$axis. **Answer**: $\frac{8\pi}{3}$

3. Find the volume formed by the revolution of loop of the curve $y^2(a + x) = x^2(3a - x)$, about the x-axis.

Let's check take away from the lecture

1. Find the volume of the solid of revolution formed by rotating the curve $y = \sqrt[3]{x}$ along x-axis bounded by lines x=0 and x=1?

(a) $3\pi/5$ (b) $5\pi/3$ (c) $2\pi/3$ (d) $3\pi/2$

2. The volume of the solid of revolution formed by rotating the curve $y = \sqrt{x}$ along x-axis bounded by lines x = 0 and x = 2 is?

(a) 2π (b) $\pi/12$ (c) 12π (d) $\pi/2$

Homework Problems for the day

(1) Find the volume of solid of revolution bounded by $y = \sqrt{x}, y = 0$ from $x = 0$ to $x = 4$ about $y = 2$. **Answer**: $\frac{40\pi}{3}$

(2) Find the volume of the solid generated by revolving the finite region bounded by the curves $y = x^2 + 1$, $y = 5$ about the line $x = 3$. **Answer**: 64π cubic units

3. Find the volume of the solid formed by revolving about the x-axis, the area enclosed by the parabola $y^2 = 4ax$, its evolute $27ay^2 = 4(x - 2a)^3$ and the x-axis.

Learning from the topic: Student will be able to evaluate volume of revolution (cartesian form) using single integration.

Volume of revolutions problems continued...

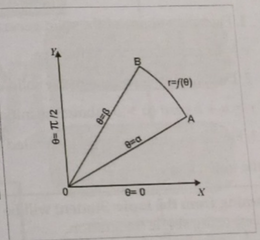
Lecture: 51

Volume of revolution (polar curves):

The volume of solid generated by the revolution of the area bounded by the curve and the radii of vectors

(a) about the initial line OX (is $\int_\alpha^\beta \frac{2\pi}{3}\pi r^3 \sin\theta d\theta$

(b) about the line OY (is $\int_\alpha^\beta \frac{2\pi}{3}\pi r^3 \cos\theta d\theta$



Sample Problem:

1. Find the volume of the solid generated by the revolution of cardioids $r = a(1 + \cos \cos\theta)$ about the initial line.

Solution:

The cardioids is symmetrical about the initial line and for its upper half θ varies from $0 \; to \; \pi$.

$$\therefore Required\ Volume = \int_0^{\pi} \frac{2\pi}{3}\pi r^3 \sin\theta d\theta$$
$$= \frac{2\pi}{3}\int_0^{\pi} a^3(1 + \cos\cos\theta)^3 \sin\theta d\theta$$
$$= -\frac{2\pi a^3}{3}\int_0^{\pi} (1 + \cos\cos\theta)^3(-\sin\theta) d\theta$$
$$= -\frac{2\pi a^3}{3}\left[\frac{(1 + \cos\cos\theta)^4}{4}\right]_0^{\pi} = \frac{8}{3}\pi a^3.$$

Exercise 51

1.Find the volume of solid generated by revolving the lemniscates $r^2 = a^2 cos2\theta$ about the line $\theta = \frac{\pi}{2}$.

2 Find the volume of the solid generated by the revolution of cardioids $r = a(1 - cos\ cos\theta)$ about the initial line.

1. Find the volume of solid generated by revolving curve r= 2a cos θ about the initial line OX.

(a) $4\pi a^2/3$ (b) $3\pi a^2/4$ (c) $\pi a^2/4$ (d) $\pi^2 a/3$

2. The volume of the solid generated by the revolution of the cardioid r = a(1 + cos θ) about the initial line is given by

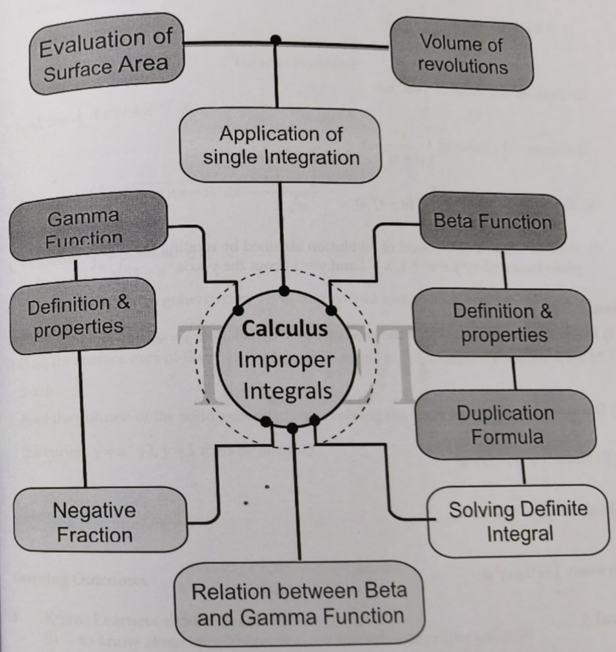
(a) $8\pi a^3/3$ (b) $4\pi a^3/3$ (c) $2\pi a^3/3$ (d) $2\pi a^3$

Homework Problems for the day

1. Find the volume of the solid generated by the revolution of cardioids $r = 2a\ cos\theta$ about the initial line.
2. Determine the volume of the solid obtained by revolving the lemicon $r = a + b\ cos\theta\ (a > b)$ about the initial line.
3. Find the volume of solid generated by revolving the lemniscates $r^2 = a^2 cos2\theta$ about the initial line.

Learning from the topic: Student will be able to evaluate volume of revolution (polar curves) using single integration.

Concept Map



Problems for Self-assessment:

Level 1

1) What is the value of $\int_0^{\infty} e^{-x^2} dx$

2) Evaluate the value of $\int_0^{\frac{\pi}{2}} \sin^5 x dx$

3) Evaluate the value of $\int_0^{\infty} \frac{x^4}{(1+x)^{16}} dx$

4) Evaluate the value of $\int_0^{\infty} x^4 (a-x)^5 dx$

5) Find the volume of the solid of revolution obtained by rotating the region in the xy-plane bounded by $y = x^3 + 1$, $x = 1$ and $y = 1$ about the y-axis

Level 2

1) Determine the surface area of the solid obtained by rotating $y = \sqrt{1-x^2}$ where $-2 \leq x \leq 2$ about the x-axis.

2) Evaluate: $\int_0^{\pi} \sin^2\theta (1-\cos^3\theta) d\theta$

3) Evaluate $\int_0^1 x^4 (1-\sqrt{x})^5 dx$

4) Evaluate $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx$

5)Evaluate $\int_0^1 (x \log x)^3 dx$

Level 3

1) Prove that $\int_0^1 \sqrt{1-\sqrt{x}}\, dx \int_0^{\frac{1}{2}} \sqrt{2y-(2y)^2}\, dy = \frac{\pi}{30}$

2) Evaluate $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx$

3) Find the surface area of the solid generated by revolving $x = a\cos^2 t, y = a\sin^2 t$, $0 \leq t \leq \pi/2$ about the x-axis.

4) Evaluate the integral $\int_{-1}^{1} (1+x)^{p+1} (1-x)^{q-1} dx$

5) Evaluate the Integral $\int_0^{\infty} 3^{-x^2} dx$

Tutorial Problems

1. Evaluate $\int_0^{\infty} (x^2+4) e^{-2x^2} dx$.

2. Prove that $\int_5^9 \sqrt[4]{(9-x)(x-5)}\, dx = \frac{2\left(\lceil\frac{1}{4}\right)^2}{3\sqrt{\pi}}$

3. Evaluate $\int_0^8 \frac{x^3}{\sqrt{8x-x^2}} dx$

4. Find the surface area generated by revolving cycloid $x=a(t + \sin t), y=a(1 + \cos t)$ about the base.

5. Find the surface area of the object obtained by rotating $x = 1 - y^2, 0 \leq y \leq 3$ about the x-axis.

6. Find the volume of the solid generated by revolving the finite region bounded by the curves $y = x^2 + 1$, $y = 5$ about the line $x = 3$.

Learning Outcomes

1. **Know:** Learners should be able
   (i) to know about special function used to solve improper integrals
   (ii) to apply Beta and Gamma functions to evaluate the integral
   (iii) to calculate surface area and volume of revolution using single integration.
2. **Comprehend:** Learners should be able to understand about improper integrals and special function to solve them and use single integration to evaluate the surface area and volume of revolution.
3. **Apply, analyse and synthesize:** Learners should be able to solve the improper integral with the help of special function and use single integration to evaluate the surface area and volume of revolution.

Digital reference:

https://functions.wolfram.com/GammaBetaErf/Gamma/introductions/Gamma/ShowAll.html

Add to knowledge: The Beta function was the known Scattering amplitude in String theory, conjectured by Gabriele Veneziano, an Italian theoretical physicist and a founder of string theory. Gabriele Veneziano, a research fellow at CERN (a European particle accelerator lab) in 1968, observed a strange coincidence - many properties of the strong nuclear force are perfectly described by the Euler beta-function, an obscure formula devised for purely mathematical reasons two hundred years earlier by Leonhard Euler

Self-Evaluation

Name of student: **Class &Div:** **Roll No:**

1. Are you able to differentiate between proper and improper integral?
   (a) Yes (b) No

2. Do you understand importance the improper integral in Gamma and Beta functions?
   (a) Yes (b) No

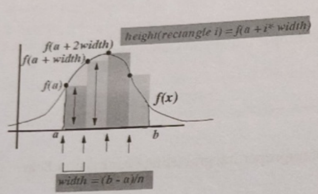
3. Will you able to evaluate the value of improper integral using Gamma and Beta functions?
   (a) Yes (b) No

4. Are you able to find the value of surface area using single integration?
   (a) Yes (b) No

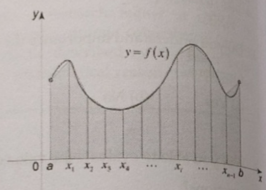
5. Are you able to find the volume of solid of revolution using single integration?
   (a) Yes (b) No

Module 6: Numerical Integration

Infographics

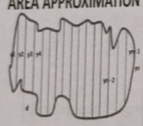


RECTANGLE RULE

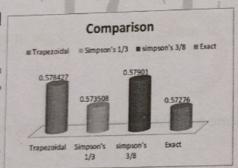


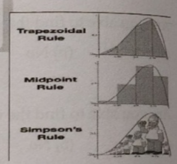
Trapezoidal Rule

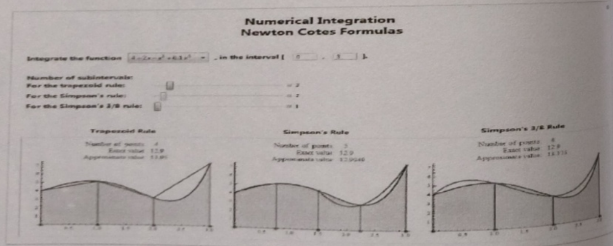
AREA APPROXIMATION



SIMPSON'S 1/3 RULE







Module-6 Numerical Integration

1. **Motivation:**
   Numerical Integration: The process of calculating a definite integral from a set of tabulated value of the integrand f(x) is called Numerical Integration. When numerical integration is applied on single variable function then it is called Quadrature Solving a problem by analytical methods is an ideal way to get an exact result. But in many cases this is not possible, because such methods are not available always. Thus we are required to use numerical methods for differentiation, integration, differential equation and etc.

2. **Syllabus:**

Lecture No.	Title	Duration (Hrs.)	Self-study (Hrs.)
52	Numerical integration - The numerical evaluation of an integral	1	2
53	Rectangle method – based on (piecewise) constant approximation	1	2
54	Trapezoidal rule- based on (piecewise) linear approximation	1	2
55	Trapezoidal rule- based on (piecewise) linear approximation continued	1	2
56	Simpson's 1/3 rd rule	1	2
57	Simpson's 1/3 rd rule continued	1	2
58	Simpson's 3/8th rule	1	2
59	Simpson's 3/8th rule continued	1	2
60	Newton's Cote's quadrature formula	1	2

3. **Prerequisite:**
   The concept of integration on any region and formulae of simple integration.

4 **Learning Objectives:** Learner shall be

1. Able to apply rectangle method, trapezoidal and Simpson's formulae to evaluate the single integrals.
2. Able to apply Newton –Cotes formula evaluate the single integrals.

The Numerical Evaluation of an integral

Lecture 52

1.Learning Objective:
Student shall be able to understand what is Numerical integration, forward andbackward difference operator.

2. Introduction: The present module deal with how to evaluate real integral numerically. So far we are aware about analytical methods to evaluate a real integral of function of single variable. The process of evaluating a definite integral from a set of tabulated values of the integrand is known as Numerical Integration. When integrand is a function of single variable then this process is called as Quadrature, on which Newton-Cotes Quadrature formula is derived and the remaining were the special cases of Newton-Cotes Quadrature formula viz. Trapezoidal Rule, Simpson's 1/3rd Rule, Simpson's 3/8th Rule, Rectangle method etc. depending upon the value of number of subintervals.

3. Key Notations:

(1) Forward difference operator $\Delta : \Delta f(x) = f(x+h) - f(x)$

(2) Backward difference operator $\nabla : \nabla f(x) = f(x) - f(x-h)$

Where h is a constant difference.

4. Key Definitions:

(1) Partition of an Interval: Let $\{x_0 = a, x_1, x_2, \dots, x_n = b\}$ is the partition of an interval $[a, b]$ into n- subintervals. The corresponding values of $y = f(x)$ be $y_0, y_1, y_2, \dots, y_n$ where $y_i = f(x_i)$.

(2) Interpolation : let $y = f(x), x_0 \leq x \leq x_n$ means : corresponding to every value of x in the range $x_0 \leq x \leq x_n$, there exists one or more values of function y. Assuming that f(x) is

single-valued and continuous and that it is known explicitly, then the values of f(x) corresponding to certain given values of x, say $x_0, x_1, \dots, x_n$ can easily be computed and tabulated. The central problem of numerical analysis is the converse one: Given the set of tabular values $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ satisfying the relation $y = f(x)$ where the explicit nature of f(x) is not known, it is required to find a simpler function, say $\varphi(x)$, such that f(x) and $\varphi(x)$ agree at set of tabulated points. Such process is called interpolation. If $\varphi(x)$ is a polynomial then the process is called as polynomial interpolation and $\varphi(x)$ is called interpolating polynomial.

3. **Forward Difference:** If $y_0, y_1, y_2, \dots, y_n$ denote the set of values of y, then $y_1 - y_0$, $y_2 - y_1, y_3 - y_2, \dots, y_n - y_{n-1}$ are called the differences of y . Denoting these differences by $\Delta y_0, \Delta y_1, \Delta y_2, \dots, \Delta y_{n-1}$ respectively, we have $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1, \dots,$ $\Delta y_{n-1} = y_n - y_{n-1}$, where Δ is called the forward difference operator and $\Delta y_0, \Delta y_1, \dots, \Delta y_{n-1}$ are called first forward differences. The differences of first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \dots$, similarly third forward differences, fourth forward differences etc. such that

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_2 + y_0$, similarly $\Delta^3 y_0, \Delta^4 y_0$ etc.

4. Newton's Forward Difference Formula:

The general problem of numerical integration may be stated as follows: Given a set of data points$(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ of a function $y = f(x)$, where f(x) is not known explicitly , it is required to compute the value of the definite integral

$I = \int_a^b y\,dx$ ..... (1)

As in the case of numerical differentiation, one replaces f(x) by an interpolating polynomial $\Phi(x)$ and obtains on integration, an approximate value of the definite integral. Thus different integration formulae can be obtained depending upon the type of the interpolation formulae used. We derive in this section a general formula for numerical integration using Newton's forward difference formula. Let the interval [a, b] is divided into n equal subintervals such that

$a = x_0 < x_1 < x_2 < \dots < x_n = b$, clearly, $x_n = x_0 + nh$. Hence the integral becomes

$I = I = \int_{x_0}^{x_n} y\,dx$ .Approximating y by Newton's forward difference formula, we obtain

$$\int_{x_0}^{x_1} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \quad \cdots (2)$$

Sample Problem:

1) Prove that $\Delta\nabla = \nabla\Delta$

Solution: consider $\Delta\nabla y_r = \Delta(y_r - y_{r-1}) = \Delta y_r - \Delta y_{r-1}$

$$= (y_{r+1} - y_r) - (y_r - y_{r-1})$$

$$= y_{r+1} - 2y_r + y_{r-1} \quad \text{.............}(i)$$

Now consider $\nabla\Delta y_r = \nabla(y_{r+1} - y_r) = \nabla y_{r+1} - \nabla y_r$

$$= (y_{r+1} - y_r) - (y_r - y_{r-1})$$

$$= y_{r+1} - 2y_r + y_{r-1} \quad \text{.............}(ii)$$

From (i) and (ii) we get $\Delta\nabla = \nabla\Delta$

Exercise 52

1. Find values of (i) Δe^{ax} (ii) $\Delta^2 e^x$ (iii) $\Delta = \log x$
2. Show that $(1+\Delta)(1-\nabla)=1$
3. Evaluate $\Delta^2(\frac{1}{x})$ by taking '1' as the interval of differencing.

Choose the correct option from the following.

1. Δx^2 *is*

(a) $(x-h)^2 - x^2$ (b) $(x+h)^2 - x^2$ (c) $x^2 - (x-h)^2$ (d) $x^2 - (x+h)^2$

2. Value of Δe^{ax} is

(a) $e^{ax}[e^h + 1]$ (b) $e^{ax}[e^h - 1]$ (C) $5e^{ax}[e^h + 1]$ (d) $e^{ax}[e^h + 5]$

3. $\Delta^2 x^2$ *with the interval of differencing being unity is*

(a) $2!$ (b) $-2!$ (c) *doesn't exists* (d) *none*

Homework problems for the day

1. Derive the following relations between operators:

i. $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

ii. $\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right)$

2. Obtain $\Delta^2(\cos 2x)$ the interval of differencing being h.

3. Evaluate $\Delta^2(\frac{1}{x})$

Learning from the topic: Learner will be able to apply forward and backward difference operator.

Rectangular Rule

Lecture 53

1. Learning Objective:

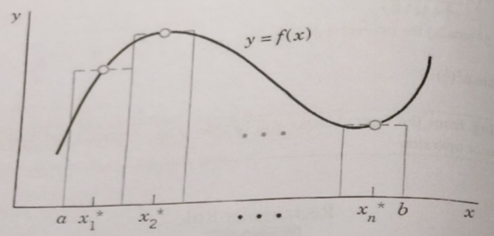
Student shall be able to understand and apply Rectangular rule to evaluate the integral.

2. Key Definitions:

Rectangle Method: Let $f : [a, b] \to R$ .The rectangle method utilizes the Riemann integral definition to calculate an approximate estimate for the area under the curve by drawing many rectangles with very small width adjustment to each other between the graph of function f and X-axis. For simplicity the width of rectangles is chosen to be constant. The rule is obtained by subdividing the interval into n subintervals of equal length $h = \frac{b-a}{n}$ and in each subinterval approximate f by a constant $f(x_j*)$,the value of f at the midpoint $x_j*$of the j^{th} subinterval. Then f is approximated by a step function (piecewise constant function) the 'n' rectangles as show in fig. Have the areas $f(x_1*)h$, $f(x_2*)h, \ldots, f(x_n*)h$, and the rectangular is

$$I = \int_a^b f(x)dx \approx h\,[f(x_1*) + f(x_2*) + f(x_3*) + \ldots + f(x_n*)]$$

$$\approx h \sum_{i=1}^{n} f(x_i^*) \quad \text{where } h = \frac{b-a}{n}$$



Rectangular rule

3. Sample problem

(1). Evaluate $\int_0^4 x^3 dx$ by using rectangle rule. Divide the interval into four equal parts.

Solution.

$\Delta x = h = \frac{b-a}{n} = \frac{4-0}{4} = 1.$

The midpoint values for the four subintervals are 0.5, 1.5, 2.5, 3.5

Here $f(x) = x^3$, $X_1^* = 0.5$, $X_2^* = 1.5$, $X_3^* = 2.5$ $X_4^* = 3.5$

Now applying Rectangle rule we have

$$\int_0^4 x^3 dx = h[\ f(x_1 *) + f(x_2 *) + f(x_3 *) + f(x_4 *)]$$

$$= 1\,[\,(0.5)^3 + (1.5)^3 + (2.5)^3 + (3.5)^3]$$

$$\approx 62.0$$

Exercise 53

1. Evaluate $\int_0^{10} \frac{1}{1+x^2} dx$ by using rectangle rule.
2. Evaluate $\int_0^8 x^3 dx$ by Rectangle rule
3. Evaluate $\int_0^2 x^2 dx$ using rectangle rule with four subintervals.

Let's check away from the lecture

Choose the correct option from the following.

1. Value of $\int_0^4 x^3 dx$ by using Rectangle rule.
(a) 25 (b) 62 (c) 22 (d) 10

2. In Rectangle rule we divide the graph of function into
(a) Ellipse (b) Hyperbola (c) Rectangle (d) circle

3. Value of $\int_0^5 x dx \approx C$ by using Rectangle rule, Value of C is
(a) 25 (b) 62 (c) 22 (d) 10

4. What type of function does the Rectangle rule assume within each subinterval ?
a) Continuous (b) Quadratic (c) Continuous (d) Discontinuous

Homework Problems of the Day

1. Evaluate $\int_1^4 (3x + 1)\, dx$ by using Rectangle rule.
2. Estimate the area under the curve of the function $h(x) = \sin x$ from $x = 0$ to $x = \pi$ using the rectangle method with six rectangles.
3. Approximate the integral of the function $y = e^x$ from $x = 0$ to $x = 1$ using Rectangle method with ten rectangles.

Learning from the topic : Learner will be able to apply Rectangular rule.

Trapezoidal Rule

Lecture 54

1. Learning Objective: Student shall be able to understand and apply Trapezoidal rule to evaluate the integral.

2. Introduction:

The Trapezoidal rule can be obtained by setting n = 1 in Newton's Forward difference formula (eq^n. 2) , all the differences higher than the first will become zero and we obtain

$\int_{x_0}^{x_1} y dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{1}{2}(y_0 + y_1).$ ...(3)

From the next interval $[x_1, x_2]$, we deduce similarly

$$\int_{x_0}^{x_1} y\,dx = \frac{h}{2}[y_1 + y_2] \ldots \ (4)$$

And so on. For the last interval $[x_{n-1}, x_n]$, we have

$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}[y_{n-1} + y_n] \ldots (5)$$

Combining all these expressions (3), (4), (5) we obtain the rule

$$I = \int_a^b f(x)dx = \frac{h}{2}\{[(y_0 + y_n)] + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1})\} \ldots \quad (6(a))$$

The Trapezoidal rule can also have written in other form as

$\int_a^b f(x)\,dx = \frac{h}{2}[X + 2R]$ . . .(6(b)), Where X = sum of extreme ordinates i.e. y_0 and y_n.

R = sum of remaining ordinates i.e. $y_1, y_2, \ldots, y_{n-1}$.

The geometrical significance of this rule is that the curve $y = f(x)$ is replaced by n straight lines joining the points (x_0, y_0) and (x_1, y_1); (x_1, y_1) and (x_2, y_2);. . ., (x_{n-1}, y_{n-1}) and (x_n, y_n). The area bounded by the curve $y = f(x)$, the ordinates $x = x_0$ and $x = x_n$, and the x-axis is then approximately equivalent to the sum of the n trapezium obtained.

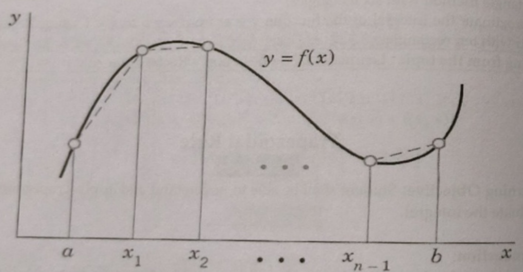


Fig. for Trapezoidal Rule

Sample Examples:

(1) Find the value of the integral $\int_0^1 \frac{x^2}{1+x^3}dx$ by using Trapezoidal rule

Solution: Since to use Simpson's 1/3rd rule we need even number of ordinates we shall divide the interval (0,1) into 4 equal parts by taking h = 0.25 . We prepare the following table.

X	0	0.25	0.50	0.75	1.0
Y	0	0.06153	0.22222	0.36560	0.5
	Y_0	Y_1	Y_2	Y_3	Y_4

By Trapezoidal rule

$$\int_0^1 \frac{x^2}{1+x^3}dx = \frac{h}{2}[X + 2R]$$

$$= \frac{0.25}{2}\left[(0.0+0.5)+2(0.06153+0.22222+00.39560)\right] \ =0.23233$$

(2) Evaluate

$\int_{-3}^{3} x^4 dx$ by using Trapezoidal rule

Solution: Here $y = x^4$ Interval length = (b-a) = 3-(-3) = 6

Wedivide 6 equal interval with $h = \frac{6}{6} = 1$ We form the table below

X	- 3	-2	-1	0	1	2	3
Y	81	16	1	0	1	16	81

By Trapezoidal rule

$$\int_{-3}^{3} x^4 dx = I = \int_{-3}^{3} x^4 dx = \frac{h}{2}\{[(y_0 + y_6)] + 2(y_1 + y_2 + y_3 + y_4 + y_5)\}$$

$$= \frac{1}{2}\{[81 + 81)] + 2(16 + 1 + 0 + 1 + 16)\}$$

$$\approx 161$$

Exercise 54

1. Evaluate $\int_{0.2}^{1.4}\left(\sin x - \log_e x + e^x\right)dx$ by using Trapezoidal rule. Ans: (4.071518)

2. Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ by using Trapezoidal rule

3. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx$ by using Trapezoidal rule

Let's Check away from lecture

Choose the correct option from the following.

1. Trapezoidal rule is :

(a) $\int_a^b f(x)\,dx = \frac{h}{2}[X+2O]$ (b) $\int_a^b f(x)\,dx = \frac{h}{2}[X+2T]$

(c) $\int_a^b f(x)\,dx = \frac{h}{2}[X+2R]$ (d) $\int_a^b f(x)\,dx = \frac{h}{2}[2X+3T]$

2.Which type of functions can the Trapezoidal rule be applied to
(a) Only continuous functions (b) Only linear functions
(c) Only quadratic functions (d) Any function, regardless of type
3. In the Trapezoidal rule, the area under the curve is approximated by the sum of the areas of :
(a) Circles (b) Rectangles (c) Triangles (d) Trapezoids
4. To improve the accuracy of the Trapezoidal rule, you can :
(a) Use a larger step size (b) Use a smaller step size
(c) Only apply it to linear functions (d) Ignore the endpoint corrections

Homework problems of the day

(1)Use the trapezoidal rule to approximate the integral of the function $f(x) = \sqrt{\sin x + \cos x}$ dx by Trapezoidal rule.

Learning from the topic: Student will be able to apply Trapezoidal rule to evaluate the integral

Module 6: Numerical Integration

Trapezoidal Rule Continued...

Lecture 55

1. **Learning objective:** Student shall be able to understand and apply Trapezoidal rule applications in real life problems.
2. **Sample problem**

1) The water under portion of a water tank is divided by horizontal planes one meter apart into the following areas: 472, 398, 302, 198, 116, 60, 34, 12 and 4 sq.m. Use the Trapezoidal rule to find the volume in cubic meters between the two extreme ends.

Solution . We prepare the table as follows

Distance S	1	2	3	4	5	6	7	8	9
Area A	472	398	302	198	116	60	34	12	4

Since V=A x S
By Trapezoidal rule
$V = \frac{h}{2}[X+2R]$
Here h = 1, X = 472 + 4 = 476, R = 398 + 302 + 198 + 116+ 60 + 34 + 12 = 11
$\therefore V = \frac{1}{2}[476 + 2240] = 1358$ cubic meters.

Exercise 55

1.Find the volume of solid of revolution formed by rotating about the x-axis , the area bounded by the lines x = 0, x =1, y = 0, and the curve passing through the points given below.

X	0	0.25	0.50	0.75	1
Y	1	0.9896	0.9589	0.9089	0.8415

2. The velocity of a particle which starts from rest is given by the following table

t(sec)	0	2	4	6	8	10	12	14	16	18	20
v(ft/sec)	0	16	29	40	46	51	32	18	8	3	0

Evaluate using Trapezium rule, the total distance travelled in 20 seconds.

3. The velocity of train which starts from rest is given by the following table, the time being reckoned in minutes from the start and speed in Km/hour

Time	3	6	9	12	15	18
Velocity	22	29	31	20	4	0

Estimate approximately the distance covered in 18 minutes by Trapezoidal rule.

Let's check away from Lecture

Which of the following option is correct?

1. Which order of polynomials can be best be integrated using Trapezoidal rule
(a) 3rd order (b) 4th order (c) 2nd order (d) 1st order

2. Using the following values of x and f(x)

X	0	0.5	1.0	1.5
F(x)	1	A	0	-5/4

The integral $I = \int_0^{1.5} f(x)dx$, evaluated by the Trapezoidal rule, is $5/16$. The value of a is Find the approximate value of $\int_0^4 e^x dx$ by using Trapezoidal Rule
(a) 3/4 (b) 3/2 (c) 7/4 (d) 19/24

Homework problem of the day

1. Find using the Trapezoidal rule from the following table the area bounded by the curves and x-axis from x = 7.47 to x = 7.52

X	7.47	7048	7.49	7.50	7.51	7.52
F(x)	1.93	1.95	1.98	2.01	2.03	2.06

Ans. 0.0996

Learning from the topic: Students will be able to solve the real life problems by using Trapezoidal rule.

Simpson's 1/3rd- Rule

Lecture 56

1. **Learning Objective:** Student shall be able to understand and apply Simpson's 1/3rd rule to evaluate the integral.

2. **Introduction:**
This rule is obtained by putting n = 2, in eq. (1)(i.e. Newton's forward difference formula) i.e. replacing the curve by n/2 arcs of second –degree polynomials or parabolas, we have then

$\int_{x_0}^{x_2} f(x)dx = 2h\,[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0] = \frac{h}{3}(y_0+4y_1+y_2)$

Similarly

$$\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}\,[y_2 + 4y_3 + y_4]$$

Finally $\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$

Summing all the above integrations we obtain,

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\{[(y_0 + y_n)] + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2+y_4 + y_6 + \ldots + y_{n-2})]\}$$

The above formula is Simpson's 1/3 rd rule. It can also be written as

$\int_a^b f(x)\,dx = \frac{h}{3}[X+2E+4O]$,where X = sum of extreme ordinates, E = sum of even Ordinates, O = sum of odd ordinates.

Note : Choice of numbers of sub-intervals 'n' for Simpson's 1/3rd rule , n must be an even positive integer.

3. Sample Problems on Simpson's 1/3rd Rule

1. Find the value of the integral $\int_0^1 \frac{x^2}{1+x^3}dx$ using Simpson's 1/3rd rule. Hence find the value of log 2.

Solution. Since to use Simpson's 1/3rd rule we need even number of ordinates we shall dividethe interval (0,1) into 4 equal parts by taking h = 0.25 . We prepare the following table.

X	0	0.25	0.50	0.75	1.0
Y	0	0.06153	0.22222	0.36560	0.5
	Y_0	Y_1	Y_2	Y_3	Y_4

By Simpson's 1/3rd rule

$$\int_0^1 \frac{x^2}{1+x^3}dx = \frac{h}{3}[X+2E+4O]$$

$$= \frac{0.25}{3}\left[(0.0+0.5)+2(0.22222)+4(0.06153+0.39560)\right]$$

$$= 0.23108$$

Now, $\int_0^1 \frac{x^2}{1+x^3}dx = \left[\frac{1}{3}\log\left(1+x^3\right)\right]_0^1 = \frac{1}{3}\log 2 = 0.23105$

Hence log2 = 0.69315

(2) The velocity of a train which starts from rest is given by following table the time being recorded in minutes from the start and speed in km per hr

Time t	0	2	4	6	8	10	12	14	16	18	20
Velocity v	0	16	28.8	40	46.4	51.2	32.0	17.6	8	8.2	0
	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}

Using Simpson's 1/3 rd rule estimate total distance run in 20 min.

Sol^n : Velocity $v = \frac{ds}{dt}$ so $ds = v\,dt$

Distance run in 20 min $s = \int_0^{20} v\,dt$

$$= \frac{h}{3}[(v_0+v_{10})+4(v_1+v_3+v_5+v_7+v_9)+2(v_2+v_4+v_6+v_8)]$$

Here $v_o = 0$, since the train start from rest Hence following tabular values .
Here time is in minutes in km/hr
So $h = 2\text{min} = \frac{2}{60}$hr i.e. distance run in 20 min

$$= \frac{2}{60} x \frac{1}{3}[(0+0)+4(16+40+51.2+17.6+8.2)+2(28.8+46.4+32+8)]$$

$= 8.2488$ k $\cong 8.25$ km

Exercise 56

1. Evaluate $\int_{0.2}^{1.4}(\sin x - \log_e x + e^x)dx$ by using Simpson's 1/3 rd rule[Ans: 4.05213]
2. Evaluate $\int_4^{5.2} \log x dx$ by Simpson's 1/3rd rule [Ans : 1.8279]
3. Evaluate $\int_{-3}^{3} x^4\,dx$ by using Simpson's 1/3rd rule

Let's check away from lecture

Choose the correct option from the following.

1. *In simpson's 1/3rd rule no.of subintervals should be*
   (a) odd (b) prime (c) proper (d) even
2. What is the degree of precision of the Simpson's 1/3rd rule ?
   (a) First degree (b) Second degree (C) Third degree (D) Fourth degree

Homework problems of the day

(1) Use Simpson's 1/3 rd rule to evaluate $\int_0^{\pi} \frac{\sin x}{x} dx$ with 7 ordinate. (Ans.1.8519)
(2) Use Simpson's 1/3 rd rule to evaluate

$\int_0^2 \frac{2e^{-x^2}}{\sqrt{\pi}} dx$ by taking to equal strips. (Ans. 0.9953)

Learning from the topic: Learner will be able to understand and apply Simpson's 1/3rd rule to evaluate the integral.

Simpson's 1/3rd rule continued....

Lecture 57

1. **Learning Objective:** Student shall be able to understand and apply Simpson's 1/3 rd rule to evaluate the real life problems.

2. **Sample Problems**

1.A river is 80 meter wide. The depth d in meter at a distance x meter from one end of a bank is given in the following table:

X	0	10	20	30	40	50	60	70	80
D	0	4	7	9	12	15	14	8	3

Using Simpson's 1/3rd rule, find approximately the area of c/s of the river.

Solution : c/s area $= \int_0^{80} d\,dx = \frac{h}{3}[d_0 + d_8 + 4(d_1 + d_3 + d_5 + d_7) + 2(d_2 + d_4 + d_6)]$

$$= \frac{10}{3}(213) = 716.66 \text{ sq meter}$$

Area of c/s of river = 710 sq meters.

Exercise 57

1.The velocity of train which starts from rest is given by the following table, the time being reckoned in minutes from the start and speed in Km/hour

Time	3	6	9	12	15	18
Velocity	22	29	31	20	4	0

Estimate approximately the distance covered in 18 minutes by Simpson's 1/3rd rule.

2. The cross section of a tree is sq. Cm at distance x cm from one end corresponding values of A and x are

X	10	30	50	70	90	110	130	150	170
A	120	123	129	131	131	135	142	156	177

Find volume of tree by Simpson's rule in cubic cm between x =10 to x =170 cm.

3.A rocket is launched from ground. Its acceleration is registered during first 80 seconds and is given below:

T(sec)	0	10	20	30	40	50	60	70	80
A(m/s)	30.00	31.63	33.44	35.47	37.75	40.33	43.25	46.69	50.67

Using Simpson's one- third rule, find the velocity of the rocket at time t =80 sec.
[Ans: 3086.1cm/sec]

Let's check away from Lecture

Choose the correct option from the following

1)What is the Simpson's 1/3 rd rule used for ?

(a) Solving linear equation (b) Estimating standard deviation

(c) Calculating probabilities (d) Approximating cubes

2. *Simpson's* $1/3^{rd}$ *rule* is :

(a) $\int_a^b f(x)\,dx = \frac{h}{3}[X+2O+4E]$ (b) $\int_a^b f(x)\,dx = \frac{h}{3}[X+2R+4E]$

(c) $\int_a^b f(x)\,dx = \frac{h}{3}[2X+3E+4O]$ (d) $\int_a^b f(x)\,dx = \frac{h}{3}[X+2E+4O]$

Homework problems of the day

1.In an experiment a quantity of G was measured as follows G(20) = 95.90, G(21) = 96.85, G(22) = 97.77, G(23) = 98.68, G(24) = 99.56,G(25) = 100.41, G(26) =101.24
Compute $\int_{20}^{26} G(x)dx$ by both Simpson's (1/3)rd rule.

Learning from the topic : Student shall be able to understand and apply Simpson's $1/3^{rd}$ rule to evaluate real life problems

Simpson's 3/8th Rule

Lecture 58

1. Learning Objective: Student shall be able to understand and apply Simpson's $3/8^{th}$ rule to evaluate the integral

2. Introduction:
Simpson's $3/8^{th}$ rule can be obtained by keeping n = 3 in eq. (2)(i.e. in Newton's forward difference formula), we observe that all the differences higher than the third will become zero and we obtain

$$\int_{x_0}^{x_1} ydx = 3h(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0$$

$$= 3h\left[y_0 + \tfrac{3}{2}(y_0 - y_1) + \tfrac{3}{4}(y_2 - 2y_1 + y_0) + \tfrac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly

$$\int_{x_0}^{x_1} ydx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$

and summing all these we obtain

$$\int_{x_0}^{x_1} ydx = \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots$$
$$+(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)]$$

$$= \frac{3h}{8}\{[(y_0 + y_n)] + 3(y_1 + y_2 + y_4 \ldots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \ldots + y_{n-3})\}$$

is called Simpson's $3/8^{th}$ rule.
This rule also be written as

$\int_a^b f(x)\,dx = \frac{3h}{8}[X+2T+3R]$, where X = sum of extreme ordinates,

T = sum of multiples of three ordinates, R = sum of the remaining ordinates

Remark : In Simpson's 3/8th rule, the number of subintervals is $n = 3N$. Hence, we have $h = \frac{b-a}{3N}$ or $h = \frac{b-a}{n}$ where n is multiple of 3

3. Sample problems

(1) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Simpson's 3/8th rule.

Solution: We divide the interval into 6 equal parts by taking h = 1 and prepare the following table.

X	0	1	2	3	4	5	6
Y	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6

By Simpson's 3/8th rule:

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[X+2T+3R]$$

$$= \frac{3}{8}[(1+0.027)+2(0.1)+3(0.5+0.2+0.0588+0.0385)]$$

$$= 1.3571$$

Exercise 58

1. Evaluate $\int_{0.2}^{1.4}(\sin x - \log_e x + e^x)dx$ by using Simpson's 3/8th rule **Ans:** 4.05297

2. Evaluate $\int_0^1 \frac{\sin x}{x} dx$ by using Simpson's $3/8^{th}$ rule.**Ans:** 0.9460

3. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx$ by Simpson's $\frac{3}{8}$th rule

Let us check away from lecture

Choose the correct option from the following.

1. *In simpson's $3/8^{th}$ rule no.of subintervals should be*

(a) odd (b) multiples of three (c) multiples of four (d) even

2. *Simpson's $3/8^{th}$ rule is :*

(a) $\int_a^b f(x)dx = \frac{3h}{8}[X+2T+3R]$ (b) $\int_a^b f(x)dx = \frac{3h}{8}[X+2O+3E]$

(c) $\int_a^b f(x)dx = \frac{3h}{8}[X+3T+4R]$ (d) $\int_a^b f(x)dx = \frac{3h}{8}[X+2O+3R]$

Homework problems for the day

1. Find the approximate value of $\int_0^6 e^x dx$ by using Simpson's 3/8th rule.
2. Find $\int_{-1}^{1} \frac{dx}{1+x^2}$ by using Simpson's 3/8th rule

Learning from the topic : Student will be able to understand and apply Simpson's 3/8th rule to evaluate the integral.

Lecture 59

Simpson's 3/8th rule continued…

1.**Learning Objective:** Student shall be able to understand and apply Simpson's 3/8th rule to evaluate the integral.

2.**Introduction:** In this lecture students will learn how to apply the Simpson's 3/8th rule for solving real life problems.

3.**Solved problems**

1. The velocity of train which starts from rest is given by the following table, the time being reckoned in minutes from the start and speed in km/ hr

Time	3	6	9	12	15	18
Velocity	22	29	31	20	4	0
Ordinate	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5

Estimate approximately the distance covered in 18 minutes by Simpson's (3/8)th rule.

Solution.

We know that $\frac{ds}{dt} = v \therefore s = \int v\,dt = \int_0^{18} v dt$

Since to use Simpson's 3/8 th rule we need 3, 6, 9, . . . sub-intervals we take one more ordinate at t = 0

By data at t = 0 , v = 0 ∴ We have

Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
0	22	29	31	20	4	0

By Simpson's (3/8) th rule

$$S = \frac{3h}{8}[X + 2T + 3R] = \frac{3h}{8}[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$$

But $h = 3 \text{ min} = \frac{3}{60} = \frac{1}{20} hrs$

$\therefore \quad S = \frac{3}{8} \cdot \frac{1}{20}[(0+0) + 2(31) + 3(22+29+20+4)] = 5.38125$

Exercise 59

1. Find the volume of solid of revolution formed by rotating about the x-axis the area bounded by the lines x =0, x = 1.5, y =0 and the curve passing through

X	0.00	0.25	0.50	0.75	1.0	1.25	1.50
Y	1.00	0.9655	0.9195	0.8215	0.7081	0.5812	0.5759

2. Evaluate $\int_0^6 e^x \, dx$ by Simpson's 3/8 th rule.

Let's check away from lecture

1. By Simpson's 3/ 8 th rule the value of $\int_0^1 \frac{\sin x}{x} dx$ is (A) 0.52 (b) 0.9460 (c) 0 (d) None of the above

2. In numerical integration to get better result, we select n as
(a) Even (b) odd (c) 1, 2, 3, 4, ... (d) large as possible

Homework of the day

1. Evaluate $\int_{4.0}^{5.2} \log x \, dx$ by Simpson's 3/ 8 th rule

2. Evaluate work done by a force consider a force F(x) = 2x+3 acting on an object within the interval [1,4]

Learning from the topic : Learner will be able to understand and apply Simpson's 3/8th rule to evaluate real life problems.

Lecture 60

1. **Learning Objective:** Student shall be able to understand how Trapezoidal rule, Simpson's 1/3rd and Simpson's 3/8th rule are derived from Newton – Cotes quadrature Formula.

2. **Introduction:**

Newton – Cotes Integration Formula

Let the interpolation points x_i , be equally spaced, i.e. let $x_i = x_0 + ih$, i =0,1,. . .,n and let the end points of the interval of integration be placed such that $x_0 = a$, $x_n = b$, $h = \frac{b-a}{n}$.
Then the definite integral

$I = \int_a^b y\,dx$ .... (a)
is evaluated by an integration formula of the type

$$I_n = \sum_{i=0}^{n} C_i y_i \dots \quad (b)$$

Where the coefficient C_i are determined completely by the abscissa x_i . Integration formulae of the type (b) are called **Newton – Cotes closed integration formulae.** They are 'closed' because since the end.

Let $I = \int_a^b f(x)\,dx$ takes the values $y_0, y_1, y_2, \dots, y_n$ for $x = x_0, x = x_1, \dots, x = x_n$. Let us divide the interval (a, b) into n subintervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \dots,$
$x_n = x_0 + nh = b$

$I = \int_a^b f(x)\,dx = I = \int_{x_0}^{x_0+nh} f(x)\,dx$ where $p = \frac{x-x_0}{h} \therefore ph = x - x_0$

When $x = x_0$ which gives p = 0 and when $x = x_0 + ph$ which gives p = n.

$I = \int_a^b f(x_0 + ph) h\,dp = h \int_0^n f(x_0 + ph)\,dp$

Now by using Newton's forward difference formula i.e. eq^n. (2)

$$\therefore I = h\int_0^n [y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots]dp$$

$$= h\int_0^n [y_0 + p\Delta y_0 + \frac{(p^2-1)}{2!}\Delta^2 y_0 + \left(\frac{p^3 - 3p^2 + 2p}{3!}\right)\Delta^3 y_0 + \dots]dp$$

$$= h\left[\left[y_0 p + \frac{p^2}{2}\Delta^2 y_0 + \frac{1}{2}\left[\frac{p^3}{3} - \frac{p^2}{2}\right]\Delta^2 y_0 + \frac{1}{6}\left[\frac{p^4}{4} - 3\frac{p^3}{3} + \frac{2p^2}{2}\right]\Delta^3 y_0 + \dots\right]_0^n\right]$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)\,dx = h\,[\,y_0 n + \frac{n^2}{2}\Delta^2 y_0 + \frac{1}{2}\left[\frac{n^3}{3} - \frac{n^2}{2}\right]\Delta^2 y_0 + \frac{1}{6}\left[\frac{n^4}{4} - n^3 + n^2\right]\Delta^3 y_0 + \cdots]$$

is called **Newton's Cotes quadrature formula.**

Note : When **n = 1** i.e. when a straight line is assumed the resulting rule of integration is called **Trapezoidal rule.**
When **n =2** i.e. when a parabola is assumed the rule is called **Simpson's one third rule.**
When **n = 3** i.e. when a cubicis assumed the rule is known as **Simpson's 3/8 th rule.**

3. Sample Problem:

1. Find the value of integral $\int_0^1 \frac{x^2}{1+x^3} dx$using (i) Trapezoidal rule (ii) Simpson's 1/3rd rule (iii) Simpson's 3/8th rule

Solution. Let $I = \int_0^1 \frac{x^2}{1+x^3} dx$ we have $h = \frac{1}{6}$ and $y = \frac{x^2}{1+x^3}$. We consider the following table.

X	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
Y	1	0.0276	0.1071	0.2222	0.3428	0.4398	0.5

By Trapezoidal rule, we get

$$I = \frac{h}{2}[x + 2R] = \frac{\frac{1}{6}}{2}[[(y_0 + y_6)] + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{12}[[1 + 0.5)] + 2(0.0276 + 0.1071 + 0.2222 + 0.3428 + 0.4398)$$

$$= 0.314916$$

By Simpson's 1/3 rd rule, we get

$$I = \frac{\frac{1}{6}}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{18}[(1+0.5)+4(0.0276+0.2222+0.4398)+2(0.1071+0.3428)]$$

$$= 0.286566$$

By Simpson's 3/8th rule, we have

$$I = 3\frac{\frac{1}{6}}{8}[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)]$$

$$= \frac{1}{16}[(1+0.5)+2(0.2222)+3(0.0276+0.1071+0.3428+0.4398)]$$

$$I = 0.296518$$

Exercise 60

1. Evaluate $\int_0^6 \frac{dx}{1+3x}$ by using (i) Trapezoidal rule (ii) Simpson's $1/3^{rd}$ rule (iii) Simpson's $3/8^{th}$ rule. Ans. (i) 1.1585 (ii) 1.047333 (iii) 1.06845

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using (i) Trapezoidal rule (ii) Simpson's $1/3^{rd}$ rule (iii) Simpson's $3/8^{th}$ rule. Ans. (i) 0.784216 (ii) 0.785366 (iii) 0.785368

Let's check away from Lecture

1. The Simpson's 1/3 rd rule of integration is exact for all polynomials of degree not exceeding (i) 1 (ii) 2 (iii) 3 (iv) 4

2. Simpson's $3/8^{th}$ rule is applicable only when

(i) n is multiple of 3 (ii) n is multiple of 6

(iii) n is multiple of 8 (iv) n is multiple of 24

3. In Simpson's$1/3^{rd}$ rule the number of intervals must be

(i) a multiple of 3 (ii) a multiple of 6

(iii) odd (iv) even

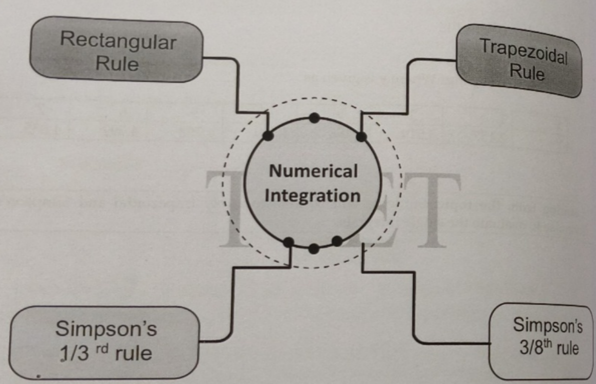
Homework problems for the day

1. Evaluate $\int_1^7 y\,dx$by using
(i) Trapezoidal rule
(ii) Simpson's $1/3^{rd}$ rule
(iii) Simpson's $3/8^{th}$ rule. Where y is given as

X	1	2	3	4	5	6	7
Y	2.157	3.519	4.198	4.539	4.708	4.792	4.835

Learning from the topic: Students will be able to apply trapezoidal and Simpson's formulae to evaluate the single integrals.

Concept Map



Problems for Self-Assessment

Level 1

Q 1.Evaluate $\int_0^4 x^2 dx$ by using Rectangular rule.

Q.2 Evaluate $\int_0^2 e^{x^2} dx$ by Trapezoidal rule with 5 subintervals

Q.3 Evaluate $\int_0^1 \frac{1}{1+3x} dx$ by using Trapezoidal rule and Simpson's $\frac{1}{3} rd$ rule

Level 2

Q.4 Evaluate integral $I = \int_0^1 \frac{1}{1+x} dx$ with h = 1/6 by using Simpson's $1/3^{th}$ rule and $3/8^{th}$ rule.

Q.5 A body in the form of solid of revolution is 9 cm long. The following table gives the diameter D in cms of the section at a distance x from one end. Find volume of the solid

X	0	1.5	3	4.5	6	7.5	9
D	3	3.5	4	4.75	4.25	3.5	3

Level 3

Q. 6 Evaluate value of $\int_0^1 \left(\frac{x^3}{1+x^2}\right) dx$ by using (i) Trapezoidal rule (ii) Simpson's 1/3 rd rule (iii) Simpson's 3/ 8 th rule.

Q.7 Evaluate value of $\int_0^6 \frac{dx}{1+2x}$ by using (i) Trapezoidal rule (ii) Simpson's 1/3 rd rule (iii) Simpson's 3/ 8 th rule.

Tutorial Questions

1. Evaluate $\int_0^4 x^3 dx$ by using Rectangular rule.
2. Evaluate $\int_{-3}^3 x^4 dx$ by Trapezoidal rule and compare it with exact value.
3. Calculate $\int_2^{10} \frac{dx}{1+x}$ upto 4 decimal places by dividing the range into eight places equal parts by Simpson's one third rule. [Ans. 1.2996]
4. Evaluate $\int_0^3 e^{\sqrt{x}} dx$ by Simpson's $(3/8)^{th}$ rule.
5. A body is in the form of a solid of revolution. The diameter d in cm s . Of its sections at various distances x cms. From one end are as given below.

X	0	2.5	5.0	7.5	10.0	12.5	15.0
D	5	5.5	6.0	6.75	6.25	5.5	4.0

Estimate the volume of the solid. (Hint. $\pi d^2/2$) (ans. 407.58 cub.cms.)

Learning Outcomes:

1. **Know:** Students should be able
   a) To compute numerical approximations to the integral of function.
   b) Apply numerical methods to obtain approximate solutions to mathematical problems.
   c) Analyse and evaluate the accuracy of common numerical methods
2. **Comprehend:** Students should be able to obtain the value of definite integral using Numerical methods
3. **Apply, Analyse and synthesize:** Student should be able
   a) to know the option to solve numerical integration if analytic method fails
   b) to compare the result between actual value and approximate value.

Digital references:

1. http://en.wikipedia.org/wiki/numericalintegration
2. https://www.mathsisfun.com/calculus/numericalintegration

Add to knowledge: Numerical integration methods can generally be described as combining evaluations of the integral to get an approximation to the integral. The integral is evaluated at a finite set of points called integration points and a weighted sum of these values is used to approximate the integral. We find definite integral by approximating the areas of integrals. The Trapezoidal rule calls for the approximation of area under a curve by fitting trapezoids under the curve and regularly spaced intervals. Also by using we have shown that the trapezium rule of integration integrates exactly polynomials of degree ≤ 1, that is, the order of the formula is 1. In many science and engineering applications, we require methods which produce more accurate results. Numerical integration has a lot of applications in engineering such as in the computation of area, volumes and surfaces. In Engineering we can approximate the total work done by a varying work done by a varying force acting on a structure or system. In Electrical engineering by using Simpson's $3/8^{th}$ rule we can calculate the power output of a varying electrical signal over a time.

Self-Evaluation

Name of student: **Class & Div:** **Roll No:**

1. Do you understand how to solve numerical integration ?

   (a) Yes (b) No

2. Will you be able to apply Trapezoidal rule, Simpson's rules?

   (a) Yes (b) No

3. Will you be able to apply how to find the velocity of rocket when acceleration and time is given ?

4. Have you understood Rectangular rule?

   (a) Yes (b) No

5. Are you able to evaluate single integral numerically?

   (a) Yes (b) No

6. Can you apply Simpson's 3/8 th rule to find area and volume of revolution ?

   (a) Yes (b) No

7. Do you understand this module?

   (a) Fully understood (b) Partially understood (c) Not at all