2 Marks ke question joada gaya hai

Module - 3 : Complex Number

For a pair of real No. a, b
$z = a+ib$ is complex number.

where

$a = Re(z) \rightarrow$ real part of z
$b = Im(z) \rightarrow$ Imaginary part of z

$\bar{z} = a - ib$ ------ conjugate of z

$\underbrace{|z|}_{\text{modulus of } z} = |a+ib| = \sqrt{a^2+b^2}$

|z| θ, r → modulus

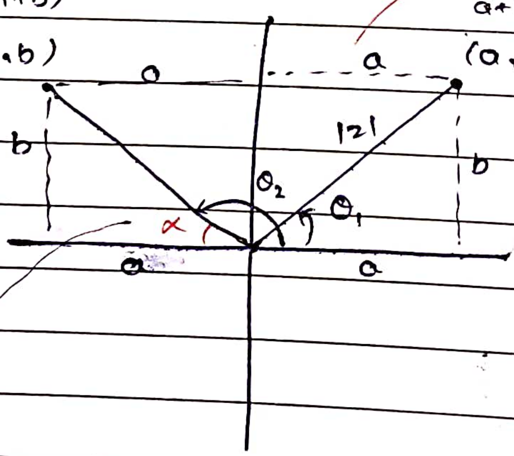
~~argument~~ argument

For $z = a+ib$
$|z|^2 = a^2+b^2$
$\boxed{|z| = \sqrt{a^2+b^2}}$

For $z = (-a+ib)$ (-ai+b)

$\boxed{|z| = \sqrt{a^2+b^2}}$

a+ib



$\tan\theta = y/x$

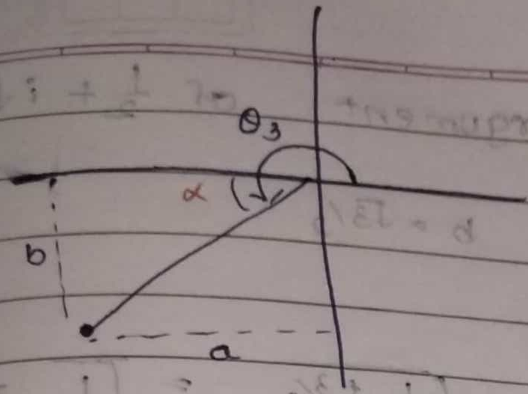
$\boxed{\theta_1 = \tan^{-1}(y/x)}$

$\theta_2 + \alpha = 180 = \pi$

$\boxed{\alpha = \tan^{-1}(b/a)}$

$\theta_2 + \tan^{-1}(b/a) = \pi$

$\boxed{\theta_2 = \pi - \tan^{-1}(b/a)}$



For $= -a - ib$

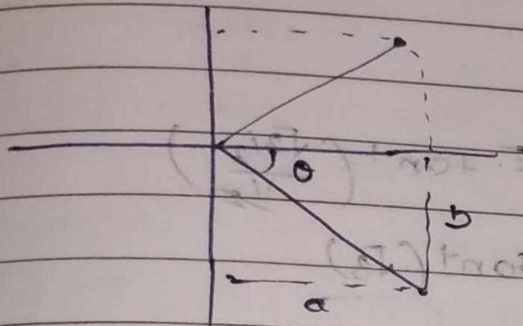
$$\boxed{|z| = \sqrt{a^2+b^2}}$$

$\alpha = \tan^{-1}(b/a)$

$\theta_3 = 180 + \alpha$

$= \pi + \alpha$

$$\boxed{\theta_3 = \pi + \tan^{-1}(b/a)}$$



for $= a - ib$

$$\boxed{|z| = \sqrt{a^2+b^2}}$$

$$\boxed{\theta = -\tan^{-1}(b/a)}$$

↳ reflection of 1st quadrant

Note: ① If $\boxed{0 \le \theta \le 2\pi}$ then θ known as general value of argument.

② If $\boxed{-\pi \le \theta \le \pi}$ then θ is known as principal value of argument.

$x^5 - i = 0$ using De'moivre's theorem

$x^5 = i$

$x = (0+i)^{1/5}$ $\quad \because \{z = a+bi\}$

$r = 1 \quad \theta = 90° = \tan^{-1}(\frac{1}{0})$

$(0+i) = (re^{i\theta})$

$(0+i)^{1/5} = (1e^{i\pi/2})^{1/5}$

$= (\cos\frac{\pi}{2} + \sin\frac{\pi}{2} i)^{1/5}$ X

$= (\cos\frac{5\pi}{2} + \sin\frac{5\pi}{2} i)$

$= [\cos(3\pi - \pi/2) + \sin(3\pi - \pi/2) i]$

$= (-\cos\pi/2 + \sin\pi/2\, i)$

$= (-0 + 1(i))$

$\boxed{(0+i)^{1/5} = i}$

$\boxed{x = i}$

? $= (e^{i\pi/10})$

$= (\cos\frac{\pi}{10} + i\sin\frac{\pi}{10})$

E.g ① Find modulus and argument of $\frac{1}{2} + i\sqrt{3}/2$

$\Rightarrow$ $z = a + ib$ $\quad a = 1/2 \quad b = \sqrt{3}/2$

$|z| = \sqrt{a^2 + b^2}$

$= \sqrt{(1/2)^2 + (\sqrt{3}/2)^2} = \sqrt{\frac{1}{4} + 3/4} = \sqrt{1} = 1$

$\boxed{|z| = r = 1}$

$\theta = \tan^{-1}(b/a) = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$

$= \tan^{-1}(\sqrt{3})$

$\boxed{\theta = \pi/3}$

(SM)

Form of Complex Number :-

(1) Cartesian form :

$$\boxed{z = a + ib}$$

(2) Polar form :

Put

$a = r\cos\theta$ ...(1)

$b = r\sin\theta$ ...(2)

$z = r\cos\theta + ir\sin\theta$

$$\boxed{z = r[\cos\theta + i\sin\theta]}$$

Note :

(i) we want r in terms of a and b

Taking square eq(1), (2) and adding.

$r^2\cos^2\theta + r^2\sin^2\theta = a^2+b^2$

$r^2(\cos^2\theta + \sin^2\theta) = a^2+b^2$

$r = \sqrt{a^2+b^2}$

(ii) we want θ in terms of a and b

~~Taking~~ divided eq (2) and (1)

$\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = b/a$

$\tan\theta = b/a$

$\theta = \tan^{-1}(b/a)$

(3) exponential form :

$$\boxed{z = re^{i\theta}}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

(4) $\boxed{\text{De' Moivre's theorem}}$:

For any real value of $\boxed{n}$, the one of the value $(\cos\theta + i\sin\theta)^n$ is given by $\cos(n\theta) + i\sin(n\theta)$

Note : $\boxed{\text{~~Law of~~ indices}}$ and $\boxed{\text{De' Moivre's theorem}}$

Both are Not same

→ applicable for $(\cos\theta + i\sin\theta)^n$

- $(\cos\theta - i\sin\theta)^n$ → not applicable

then 1st simplify $\cos(-\theta) + i\sin(-n\theta)$

→ then now applicable to ...

e.g (1) Find $(1+i)^{100}$

$\Rightarrow$ $1+i = re^{i\theta}$ .....By exponential form

where $r = |1+i|$ $\quad \theta = \arg(1+i)$

$r = \sqrt{2}$ $\quad \theta = \pi/4$

$1+i = \sqrt{2}\, e^{i\pi/4}$

$(1+i)^{100} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{100}$

$= \sqrt{2}\; e^{i\,100\times\frac{\pi}{4}}$

$= (\sqrt{2})^{100} e^{i25\pi}$ $\quad \{e^{i\theta} = \cos\theta + i\sin\theta\}$

$= 2^{50}\left[\cos(25\pi) + i\sin(25\pi)\right]$

$= 2^{50}\left[-1 + (0)i\right]$

$= -2^{50}$

(2) Find the value of $(\sqrt{3} - i)^4$ used De Moivre's theorem → Imp

$\Rightarrow$ $(\sqrt{3} - i) = re^{i\theta}$ ....By exponential form

$r = \sqrt{3+1} = 2$ $\quad \theta = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta = -\frac{\pi}{6}$

$(\sqrt{3} - i) = 2e^{-i\pi/6}$

$(\sqrt{3} - i)^4 = \left(2e^{-i\pi/6}\right)^4$

$= 2^4\, e^{\left(-i4\pi/6\right)}$

$= 2^4\left[\cos(-2\pi/3) + i\sin(-2\pi/3)\right]$

$= 2^4\left[\cos 2\pi/3 - i\sin 2\pi/3\right]$

$= 2^4\left[\cos(\pi - \pi/3) - i\sin(\pi - \pi/3)\right]$

$= 2^4\left(-\cos\pi/3 - i\sin\pi/3\right)$

$= 2^4\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= \frac{2^4}{2}(-1 - i\sqrt{3})$ $\quad = 8(-1 - \sqrt{3}i)$

$= -8(1 + \sqrt{3}i)$

sin 0 1 0 -1 0
cos 1 0 -1 0 1

a+ib)

(2) Find the value of $(1+i)^6$ using De' Moivre's theorem

$\Rightarrow$ $(1+i) = re^{i\theta}$ $\qquad r = \sqrt{1+1} = \sqrt{2}$

$\theta = \pi/4$

$(1+i) = \sqrt{2}\, e^{i\pi/4}$

$r = |z| = \sqrt{a^2+b^2}$

$(1+i)^6 = \left(\sqrt{2}\, e^{i\pi/4}\right)^6$

$= (\sqrt{2})^6 \left(e^{i\pi/4}\right)^6$ ---- expotential

$= 2^3 \left(\cos \pi/4 + i\sin \pi/4\right)^6 \Rightarrow$ polar form

$= 8\left(\cos 6\pi/4 + i\sin 6\pi/4\right)$

$= 8\left(\cos \frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$ //

$= \cancel{8\left[\cos(\pi+\pi/2) + i\sin(\pi/2+\pi)\right]}$

$= \cancel{8}\left[\cancel{0+0}i\right.$

(3) Prove using De' Moivre's theorem that $(4n)^{th}$ power of $\frac{1+7i}{(2-i)^2}$ is equal to $(-4)^n$.
where n is the integer

$\Rightarrow \frac{1+7i}{(2-i)^2} = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{(3-4i)} = \frac{(1+7i)(3+4i)}{(3-4i)(3-4i)} = \frac{3+4i+21i+28i^2}{9-16i^2}$

$= \frac{3+25i-28}{9-16(-1)} = \frac{-25+25i}{25}$

$\Rightarrow (-1+i)$

$(-1+i) = re^{i\theta}$ $\qquad r = \sqrt{2}$ $\qquad \theta = \pi - \pi/4$

$\theta = 3\pi/4$

	0	90	180	270
sin	0	1	0	-1 0
cos	1	0	-1	0 1

$(-1+i) = \sqrt{2}\, e^{i3\pi/4}$

$(-1+i)^{4n} = \left(\sqrt{2}\, e^{i3\pi/4}\right)^{4n}$

$= \left((\sqrt{2})^4 e^{i3\pi \cancel{4}/\cancel{4}}\right)^n = \left(2^2 e^{i3\pi}\right)^n$

$= 4^n\left(\cos 3\pi + i\sin 3\pi\right)^n$

$= 4^n(-1+0i)^n$

$= 4^n(-1)^n = (-4)^n$ // Hence proved.

only for +

De' Moivre's theorem :

cor.

(i) $z = \cos\theta + i\sin\theta$

$$\boxed{\frac{1}{z} = \cos\theta - i\sin\theta}$$

(ii) $$\boxed{(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta}$$

(iii) $$\boxed{(\cos\theta - i\sin\theta)^n = [(\cos\theta + i\sin\theta)]^{-n}}$$

(iv) $$\boxed{\frac{1}{(\cos\theta + i\sin\theta)^n} = (\cos n\theta - i\cos n\theta)}$$

(v) $z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos n\theta - i\sin n\theta$

$$\boxed{\cos n\theta = \frac{1}{2}\left[z^n + z^{-n}\right]}$$

and

$$\boxed{\sin\theta = \frac{1}{2i}\left[z^n + z^{-n}\right]}$$

(vi) $z_1 = r_1 e^{i\theta_1}$ $\qquad z_2 = r_2 e^{i\theta_2}$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{e^{i\theta_1}}{e^{i\theta_2}} = \frac{r_1}{r_2} e^{(i\theta_1 - i\theta_2)}$$

e.g

(1) Evaluate $(1+i\sqrt{3})^8 + (1-i\sqrt{3})^8 = -2^8$

=> let $z = 1+i\sqrt{3}$

$r = \sqrt{(1)^2+(\sqrt{3})^2}$

$r = \sqrt{1+3} = \sqrt{4}$

$\boxed{r=2}$

$\theta = \tan^{-1}(\sqrt{3})$

$\theta = \tan^{-1}(\tan \pi/3)$

$\boxed{\theta = \pi/3}$

let $\bar{z} = 1-i\sqrt{3}$

$\boxed{r=2}$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$

$\theta = -\tan^{-1}(\sqrt{3})$

$\boxed{\theta = -\pi/3}$

$z = r(\cos\theta + i\sin\theta)$

$z = 2(\cos \pi/3 + i\sin \pi/3)$

$1+i\sqrt{3} = 2(\cos \pi/3 + i\sin \pi/3)$

$1-i\sqrt{3} = 2(\cos \pi/3 - i\sin \pi/3)$

$L.H.S = (1+i\sqrt{3})^8 + (1-i\sqrt{3})^8$

$= 2^8(\cos \pi/3 + i\sin \pi/3)^8 + 2^8(\cos \pi/3 - i\sin \pi/3)^8$

$= 2^8\left[(\cos \pi/3 + i\sin \pi/3)^8 + (\cos \pi/3 - i\sin \pi/3)^8\right]$

$= 2^8\left[\cos 8\pi/3 + \cancel{i\sin 8\pi/3} + \cos 8\pi/3 - \cancel{i\sin 8\pi/3}\right]$

$= 2^8\left[2(\cos 8\pi/3)\right]$

$= 2^9 \cos\left(3\pi - \frac{\pi}{3}\right)$

$= -2^9 \cos \pi/3$

$= 2^9 \times \frac{1}{2}$

$= -2^8$

(2) Evaluate $(1+i)^{100} + (1-i)^{100}$

$\Rightarrow$ let $z = (1+i)$ $\qquad$ $z = (1-i)$

$r = \sqrt{2}$ $\qquad$ $r = \sqrt{2}$

$\theta = \tan^{-1}(1)$

$\theta = \pi/4$ $\qquad$ $\theta = -\pi/4$

$z = r(\cos\theta + i\sin\theta)$

$z = \sqrt{2}\,(\cos\pi/4 + i\sin\pi/4)$

$(1+i)^{100} = \left(\sqrt{2}\,(\cos\pi/4 + i\sin\pi/4)\right)^{100} = (\sqrt{2})^{100}\left(\cos\frac{100\pi}{4} + i\sin\frac{100\pi}{4}\right)$

$= (2)^{50}\,(\cos 25\pi + i\sin 25\pi)$

$(1-i)^{100} = (2)^{50}\,(\cos 25\pi - i\sin 25\pi)$

$(1+i)^{100} + (1-i)^{100} = (2)^{50}(\cos 25\pi + i\sin 25\pi) +$
$(2)^{50}(\cos 25\pi - i\sin 25\pi)$

$= (2)^{50}\left[\cos 25\pi + \cancel{i\sin 25\pi} + \cos 25\pi - \cancel{i\sin 25\pi}\right]$

$= (2)^{50}\,(2\cos 25\pi)$

$= (2)^{51}(-1)$

$= -(2)^{51}$

(3) $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ and $\bar{z}$ is the conjugate of z,
Prove that $(z)^{10} + (\bar{z})^{10} = 0$

$\Rightarrow$ let $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ $\qquad \bar{z} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$ $\qquad r = 1$

$= \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1}$

$r = 1$ $\qquad \boxed{\theta = -\pi/4}$

$\theta = \tan^{-1}\left(\frac{\cancel{1/\sqrt{2}}}{\cancel{1/\sqrt{2}}}\right)$

$\boxed{\theta = \pi/4}$

$z = r(\cos\theta + i\sin\theta)$

$\bar{z} = r(\cos\theta - i\sin\theta)$

$z = 1(\cos\pi/4 + i\sin\pi/4)$

$\bar{z} = 1(\cos\pi/4 - i\sin\pi/4)$

$L.H.S = (z)^{10} + (\bar{z})^{10}$

$= (\cos\pi/4 + i\sin\pi/4)^{10} + (\cos\pi/4 - i\sin\pi/4)^{10}$

$= \cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2} + \cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2}$

$= 2\cos\frac{5\pi}{2} = 2\cos\left(3\pi - \frac{\pi}{2}\right) = -2\cos\pi/2$

$= -2(0) = 0 = R.H.S$

(4) Find the modulus and the principal value of argument of $\frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}}$

$\Rightarrow \frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}}$

$(1+i\sqrt{3}) \Rightarrow \quad r = \sqrt{(1)^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = \underline{2}$ $\quad \theta = \tan^{-1}(\frac{\sqrt{3}}{1})$ $\boxed{\theta = \pi/3}$

$(\sqrt{3}-i) \Rightarrow \quad r = \sqrt{(\sqrt{3})^2+(-1)^2} = \sqrt{4} = \underline{2}$ $\quad \theta = \tan^{-1}(-\frac{1}{\sqrt{3}})$

$\theta = -\tan^{-1}(\frac{1}{\sqrt{3}})$

$\boxed{\theta = -\frac{\pi}{6}}$

$(1+i\sqrt{3}) = (re^{i\theta})$

$(1+i\sqrt{3})^{16} = (re^{i\theta})^{16}$

$= (2e^{i\pi/3})^{16}$

$(1+i\sqrt{3})^{16} = 2^{16}(e^{i\pi/3})^{16} = 2^{16}[\cos \pi/3 + i\sin \pi/3]^{16}$

$(\sqrt{3}-i) = (re^{i\theta})$

$(\sqrt{3}-i)^{17} = (2e^{-i\pi/6})^{17}$

$(\sqrt{3}-i)^{17} = 2^{17}(e^{-i\pi/6})^{17} = 2^{17}[\cos\pi/6 - i\sin\pi/6]^{17}$

$\frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}} = \frac{2^{16}[\cos\pi/3 + i\sin\pi/3]^{16}}{2^{17}[\cos\pi/6 - i\sin\pi/6]^{17}}$

$= \frac{1}{2}\frac{[\cos\frac{16\pi}{3} + i\sin\frac{16\pi}{3}]}{[\cos\frac{17\pi}{6} - i\sin\frac{17\pi}{6}]}$

① 1^{st} hume (+) kar padega ② - ko angle me dal do

$= \frac{1}{2}\frac{[\cos\frac{16\pi}{3} + i\sin\frac{16\pi}{3}]}{[\cos(-\frac{17\pi}{6}) + i\sin(-\frac{17\pi}{6})]}$

$= \frac{1}{2}\{\cos(\frac{16\pi}{3} - (-\frac{17\pi}{6})) + i\sin(\frac{16\pi}{3} - (\frac{-17\pi}{6}))]$

$= \frac{1}{2}\{\cos(\frac{32\pi + 17\pi}{6}) + i\sin(\frac{32\pi+17\pi}{6})]$

$= \frac{1}{2}[\cos(\frac{49\pi}{6}) + i\sin(\frac{49\pi}{6})] = \frac{1}{2}(\cos\frac{\pi}{6} + i\sin\pi/6]$

$= \frac{1}{2}\cos(8\pi + \frac{\pi}{6}) + i\sin(8\pi + \pi/6]$

$\left\{ \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{(i\theta_1 - i\theta_2)} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \right\}$

$\frac{32}{17}$ $\frac{12}{15}$ $\frac{16}{}$ $\frac{12}{6}$

$= \frac{1}{2}\left[\cos \pi/6 + i \sin \pi/6\right]$

$= \frac{1}{2}\left[\right.$ comparing with

$z = r(\cos\theta + i\sin\theta)$

amplitude/argument $= \pi/6$

modulus $= 1/2$

(5) Express in the form $a+ib$. $\dfrac{(1+i)^6 \, (1-i\sqrt{3})^4}{(1-i)^8 \, (1+i\sqrt{3})^5}$

$\Rightarrow$

$(1+i) \Rightarrow r = \sqrt{2} \qquad \theta = \tan^{-1}(1) = \pi/4$

$(1-i) \Rightarrow r = \sqrt{2} \qquad \theta = -\pi/4$

$(1-i\sqrt{3}) \Rightarrow r = 2 \qquad \theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) \qquad \theta = -\frac{\pi}{3}$

$(1+i\sqrt{3}) \Rightarrow r = 2 \qquad \theta = \pi/3$

(1) $(1+i)^6 = (re^{i\theta})^6$

$= (\sqrt{2}\, e^{i\pi/4})^6 = (\sqrt{2})^6 \, e^{i6\pi/4} = [(\sqrt{2})^2]^3 \, e^{i3\pi/2}$

$= (2)^3 \, e^{i3\pi/2}$

(2) $(1-i)^8 = (re^{i\theta})^8$

$= (\sqrt{2}\, e^{-i\pi/4})^8 = (\sqrt{2})^8 \, e^{-i8\pi/4} = (2)^4 \, e^{-i2\pi}$

(3) $(1-i\sqrt{3})^4 = (re^{i\theta})^4$

$= (2)^4 (e^{i4\theta}) = (2)^4 \, e^{-i4\pi/3}$

(4) $(1+i\sqrt{3})^5 = (re^{i\theta})^5$

$= (2)^5 \, e^{i5\theta} = (2)^5 \, e^{i5\pi/3}$

$$\frac{(1+i)^6}{(1-i)^8}\frac{(1-i\sqrt{3})^4}{(1+i\sqrt{3})^5} = \frac{(2)^3 e^{i3\pi/2}}{(2)^4 e^{-i2\pi}} \cdot \frac{(2)^4 e^{-i4\pi/6}}{(2)^5 e^{i5\pi/3}}$$

$$\left\{ \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \right\} = \frac{1}{4}\left[e^{i\left[\frac{3\pi}{2} - \left(-\frac{4\pi}{2}\right)\right]} \cdot e^{i\left(-\frac{4\pi}{6} - \frac{5\pi}{6}\right)} \right]$$

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{(i\theta_1 - i\theta_2)} \qquad = \frac{1}{4}\left[e^{i\left(\frac{3\pi}{2} + \frac{4\pi}{2}\right)} \cdot e^{i\left(-\frac{9}{3}\pi\right)} \right]$$

$$\left\{ z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = (r_1 \cdot r_2)\, e^{(i\theta_1 - i\theta_2)} \right\} = \frac{1}{4}\left[e^{i\left(\frac{7\pi}{2}\right)} \cdot e^{(-3\pi)i} \right]$$

$$= \frac{1}{4}\left[e^{i\left[\frac{7\pi}{2} - 3\pi\right]} \right]$$

$$= \frac{1}{4}\left[e^{i\left(\frac{7\pi - 6\pi}{2}\right)} \right]$$

$$= \frac{1}{4} e^{(-i\pi/2)}$$

$$= \frac{1}{4}\left[\cos(-\pi/4) + i\sin(-\pi/4) \right]$$

$$= \frac{1}{4}\left[\cos\pi/4 - i\sin(\pi/4) \right]$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right]$$

$r = \frac{1}{4}$ $\qquad$ $\theta = \pi/4$

×

$$\boxed{r = \frac{1}{4} \qquad \theta = \frac{\pi}{2}}$$

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(6) Show that $(4n)^{th}$ Power of $\frac{1+7i}{(2-i)^2}$ is equal to $(-4)^n$ where n is positive

$\Rightarrow$ $\frac{1+7i}{(2-i)^2} = \frac{1+7i}{(2)^2-2i+i^2} = \frac{1+7i}{4-4i+(-1)} = \frac{1+7i}{(3-4i)}$

$= \frac{(1+7i)(3+4i)}{(3-4i)(3+4i)}$

$= \frac{3+4i+21i+28i^2}{9-16(i)^2} = \frac{3+25i-28}{9+16}$

$= \frac{-25+25i}{25}$

$= (-1+i) \quad ---(1)$

$(-1+i) \Rightarrow r = \sqrt{2} \qquad \theta = \tan^{-1}\left(-\frac{1}{1}\right)$

$\theta = \pi - \frac{\pi}{4}$

$(-1+i) = (re^{i\theta})$

$(-1+i)^{4n} = \left[\sqrt{2}\ e^{i3\pi/4}\right]^{4n} \qquad \theta = 3\pi/4$

$= (\sqrt{2})^{4n} \left(e^{i3n\pi}\right)$

$= ((\sqrt{2})^2)^{2n}\ e^{i3n\pi}$

$= (2)^{2n} \left(e^{i3\pi}\right)^n$

$= (2)^{2n} (\cos 3\pi + i\sin 3\pi)^n$

$= 4^n (\cos(2\pi+\pi) + i\sin(2\pi+\pi))^n$

$= 4^n (\cos\pi + i\sin\pi)^n$

	0	$\pi/2$	180
sin	0	1	0
cos	1	0	-1

$= 4^n (-1+0i)^n$

$= 4^n (-1)^n$

$= (-4)^n$

(10m) for ESE

★

On root of the equation

(Q) If α, β are the root of equation $x^2 - 2\sqrt{3}x + 4 = 0$, Prove that $\alpha^3 + \beta^3 = 0$

$\Rightarrow$ $x^2 - 2\sqrt{3}x + 4 = 0$

compare with quadratic equation

$ax^2 + bx + c = 0$

$a = 1 \quad b = -2\sqrt{3} \quad c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2\sqrt{3} \pm \sqrt{(4\times3) - 4\times1\times4}}{2\times1}$$

$$x = \frac{2\sqrt{3} \pm \sqrt{12-16}}{2}$$

$$x = \frac{2\sqrt{3} \pm \sqrt{-4}}{2} = \frac{2\sqrt{3} \pm \sqrt{-1}\sqrt{4}}{2} = \frac{2\sqrt{3} \pm i2}{2}$$

$$x = \frac{\cancel{2}}{\cancel{2}}(\sqrt{3} \pm i) = (\sqrt{3} \pm i)$$

$\boxed{x = \sqrt{3} + i}$ or $\boxed{x = \sqrt{3} - i}$

let

$\alpha = (\sqrt{3} + i)$ and $\beta = (\sqrt{3} - i)$

$\downarrow$

$(\sqrt{3} + i) \Rightarrow r = 2 \quad \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta = \tan^{-1}(\tan \pi/6)$

$\theta = \pi/6$

$(\sqrt{3} - i) \Rightarrow r = 2 \quad \theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$\theta = \tan^{-1}(-\pi/6)$

$\theta = -\frac{\pi}{6}$

$(\alpha)^3 = (\sqrt{3} + i)^3 = (re^{i\theta})^3$

$= r^3 (e^{i3\pi/6})$

$= (2)^3 (e^{i\cancel{3}\pi/\cancel{6}_2})$

$= 8 \, e^{(i\pi/2)}$

$= 8(\cos\pi/2 + i\sin\pi/2)$

$(\beta)^3 = (\sqrt{3} + i)^3 = (re^{i\theta})^3$

$= r^3 (e^{-i3\pi/6})$

$= (2)^3 (e^{-i\cancel{3}\pi/\cancel{6}_2})$

$= 8 \, e^{(-i\pi/2)}$

$= 8(\cos(-\pi/2) + i\sin(-\pi/2))$

$= 8[\cos\pi/2 - i\sin\pi/2]$

L.H.S $= \alpha^3 + \beta^3$

$= 8e^{(i\pi/2)} + 8e^{(-i\pi/2)}$

$= 8[e^{(i\pi/2)} + e^{(i\pi/2)}]$

$= 8[e^{(i\pi/2 - i\pi/2)}]$

$= 8[e^0] = 0$

(10 M) For ESE ★

on root of the equation

(8) If α, β are the root of Equation $x^2 - 2\sqrt{3}x + 4 = 0$, Prove that $\alpha^3 + \beta^3 = 0$

$\Rightarrow$ $x^2 - 2\sqrt{3}x + 4 = 0$

compare with quadratic equation

$ax^2 + bx + c = 0$

$a = 1 \quad b = -2\sqrt{3} \quad c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2\sqrt{3} \pm \sqrt{(4\times3) - 4\times1\times4}}{2\times1}$$

$$x = \frac{2\sqrt{3} \pm \sqrt{12-16}}{2}$$

$$x = \frac{2\sqrt{3} \pm \sqrt{-4}}{2} = \frac{2\sqrt{3} \pm \sqrt{-1}\sqrt{4}}{2} = \frac{2\sqrt{3} \pm i\sqrt{2}}{2}$$

$$x = \frac{\cancel{2}}{\cancel{2}}(\sqrt{3} \pm i) = (\sqrt{3} \pm i)$$

$\boxed{x = \sqrt{3} + i}$ or $\boxed{x = \sqrt{3} - i}$

let

$\alpha = (\sqrt{3} + i)$ and $\beta = (\sqrt{3} - i)$

$(\sqrt{3} + i) \Rightarrow r = 2 \quad \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta = \tan^{-1}(\tan \pi/6)$

$\theta = \pi/6$

$(\sqrt{3} - i) \Rightarrow r = 2 \quad \theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$\theta = \tan^{-1}(-\pi/6)$

$\theta = -\frac{\pi}{6}$

$(\alpha)^3 = (\sqrt{3} + i)^3 = (re^{i\theta})^3$

$= r^3 (e^{i3\theta/6})$

$= (2)^3 (e^{i\cancel{3}\pi/\cancel{6}_2})$

$= 8 e^{(i\pi/2)}$

$= 8(\cos \pi/2 + i \sin \pi/2)$

$(\beta)^3 = (\sqrt{3} + i)^3 = (re^{i\theta})^3$

$= r^3 (e^{-i3\pi/6})$

$= (2)^3 (e^{-i\cancel{3}\pi/\cancel{6}_2})$

$= 8 e^{(-i\pi/2)}$

$= 8 (\cos(-\pi/2) + i\sin(-\pi/2))$

$= 8[\cos \pi/2 - i \sin \pi/2]$

~~L.H.S = $\alpha^3 + \beta^3$~~

$= 8 e^{(i\pi/2)} + 8 e^{(-i\pi/2)}$

$= 8[\cancel{e^{(i\pi/2)}} + e^{(i\pi/2)}]$

$= 8[\cancel{e^{(i\pi/2 - i\pi/2)}}]$

$= 8[e^0] = 0$

$L.H.S = \alpha^3 + \beta^3$

$= 8\left[\cos\pi/2 + i\sin\pi/2\right] + 8\left[\cos\pi/2 - i\sin\pi/2\right]$

$= 8\left[\cos\pi/2 + \cancel{i\sin\pi/2} + \cos\pi/2 - \cancel{i\sin\pi/2}\right]$

$= 8\left[2\cos\pi/2\right]$

$= 16\cos\pi/2 = 16\times 0$

$= 0$ $= R.H.S$

Hence proved.

***

(Q) If α, β are the roots of the equation $x^2 + 2x + 2 = 0$, prove that $\alpha^n \cdot \beta^n = 2^n$

$\Rightarrow x^2 + 2x + 2 = 0$

$ax^2 + bx + c = 0$

$a = 1 \quad b = 2 \quad c = 2$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-2 \pm \sqrt{4 - 4\times 2\times 1}}{2}$

$= \dfrac{-2 \pm \sqrt{-4}}{2} = \dfrac{-2 \pm \sqrt{-1}\,2}{2}$

$= \dfrac{-2 \pm i2}{2} = \dfrac{2(-1 \pm i)}{2}$

$\boxed{x = (-1 + i)}$ or $x = (-1 - i)$

let

$\boxed{\alpha = (-1 + i)}$ $\boxed{\beta = (-1 - i)}$

$(-1+i) \Rightarrow r = \sqrt{2} \quad \theta = \pi - \tan^{-1}(\pi/4)$

$\theta = 3\pi/4$

$(-1-i) \Rightarrow r = \sqrt{2} \quad \theta = \pi + \tan^{-1}(\pi/4)$

$\theta = -\frac{3\pi}{4}$

$\alpha^n = (-1+i)^n = (re^{i\theta})^n$

$= (\sqrt{2}\, e^{i3\pi/4})^n$

$= (\sqrt{2})^n \left[\cos 3\pi/4 + i\sin 3\pi/4\right]^n$

$\beta^n = (-1+i)^n = (re^{i\theta})^n$

$= (\sqrt{2}\, e^{i5\pi/4})^n$

$= (\sqrt{2})^n \left[\cos\frac{5\pi}{4} + i\sin 5\pi/4\right]^n$

~~$\alpha^n \cdot \beta^n = (\sqrt{2})^n \left[\cos(\pi - \frac{\pi}{4}) + i\sin(\pi - \pi/4)\right]^n \cdot (\sqrt{2})^n \left[\cos(\pi + \frac{\pi}{4}) + i\sin(\pi + \pi/4)\right]$~~

~~$= (\sqrt{2})^{2n} (-\cos n\pi/4 + i\sin n\pi/4)(-\cos n\pi/4 - i\sin n\pi/4)$~~

~~$= (\sqrt{2})^{2n} \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)^n$~~

~~$= ((\sqrt{2})^n)^2 [$~~

$\alpha^n \cdot \beta^n = (\sqrt{2})^n \left[\cos 3n\pi/4 + i\sin 3n\pi/4\right] \cdot (\sqrt{2})^n \left[\cos 3n\pi/4 - i\sin 3n\pi/4\right]$

$= \left[(\sqrt{2})^n\right]^2 \left\{(\cos 3n\pi/4 + i\sin 3n\pi/4)\right\} \left\{(\cos(-3n\pi/4) + i\sin(-3n\pi/4)\right\}$

$= \left[(\sqrt{2})^2\right]^n \left(\cos\left(\frac{3n\pi}{4} - \frac{3n\pi}{4}\right) + i\sin\left(\frac{3n\pi}{4} - \frac{3n\pi}{4}\right)\right)$

$= (2)^n \left[\cos 0 + i\sin 0\right] = (2)^n (1 + 0i)$

$= (2)^n$ Hence proved.

(10) If α, β are the roots of the equation $z^2 \sin^2\theta - z \cdot \sin 2\theta + 1 = 0$. Prove that

$$\alpha^n + \beta^n = 2\cos n\theta \operatorname{cosec}^n \theta.$$

$\Rightarrow$

$$z^2 \sin^2\theta - z\sin 2\theta + 1 = 0$$

$$ax^2 + bx + c = 0$$

$a = \sin^2\theta \qquad b = -\sin 2\theta \qquad c = 1$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{+\sin 2\theta \pm \sqrt{(\sin 2\theta)^2 - 4 \times \sin^2\theta}}{2\sin^2\theta}$$

$$z = \frac{\sin 2\theta \pm \sqrt{(2\sin\theta\cos\theta)^2 - 4\sin^2\theta}}{2\sin^2\theta}$$

$$z = \frac{2\sin\theta\cos\theta \pm \sqrt{4\sin^2\theta\cos^2\theta - 4\sin^2\theta}}{2\sin^2\theta}$$

$$z = \frac{2\sin\theta\cos\theta \pm 2\sin\theta\sqrt{(\cos^2\theta - 1)}}{2\sin^2\theta}$$

$$z = \cancel{2}\sin\theta \left(\frac{\cos\theta \pm \sqrt{-\sin^2\theta}}{\cancel{2}\sin^2\theta} \right)$$

$$z = \frac{\cos\theta \pm \sqrt{-1}\,\sin\theta}{\sin\theta} = \frac{\cos\theta}{\sin\theta} \pm \frac{i\sin\theta}{\cos\theta}$$

$$=$$

$$z = ((\cos\theta \pm i\sin\theta)\operatorname{cosec}\theta$$

$\boxed{z = \operatorname{cosec}\theta\,[\cos\theta + i\sin\theta]}$ or $\boxed{z = \operatorname{cosec}\theta\,[\cos\theta - i\sin\theta]}$

$\alpha = \operatorname{cosec}\theta\,[\cos\theta + i\sin\theta]$ $\beta = \operatorname{cosec}\theta\,[\cos\theta - i\sin\theta]$

$\alpha^n + \beta^n = [\operatorname{cosec}\theta\,(\cos\theta + i\sin\theta)]^n + [\operatorname{cosec}\theta\,\{\cos\theta - i\sin\theta\}]^n$

$\alpha^n + \beta^n = \operatorname{cosec}^n\theta\,[\cos n\theta + i\cancel{\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}]$

$= \operatorname{cosec}^n\theta\,(2\cos n\theta)$

$= \operatorname{cosec}^n\theta\,(2\cos n\theta)$

$= 2\cos n\theta\,\operatorname{cosec}^n\theta$

Hence proved

(11) If α and β are the root of equation $x^2-2x+4=0$, then using De' moivre's Theorem show that $\alpha^n+\beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$ and hence find the value of $\alpha^{10}+\beta^{10}$.

$\Rightarrow$ $x^2-2x+4=0$

$ax^2+bx+c=0$

$a=1 \quad b=-2 \quad c=4$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2\pm\sqrt{4-16}}{2\times1} = \frac{2\pm\sqrt{-12}}{2}$$

$$= \frac{2\pm\sqrt{-3\times4}}{2} = \frac{2\pm2\sqrt{-1}\sqrt{3}}{2} = \frac{\cancel{2}\pm\cancel{2}\sqrt{3}i}{\cancel{2}}$$

$$=(1\pm\sqrt{3}i)$$

$x = 1+\sqrt{3}i$ or $x = 1-\sqrt{3}i$

let

$\boxed{\alpha = 1+\sqrt{3}i}$ $\qquad$ $\boxed{\beta = 1-\sqrt{3}i}$

$\Rightarrow r=2$ $\qquad$ $\Rightarrow r=2$

$\theta = \tan^{-1}(\sqrt{3})$ $\qquad$ $\theta = -\tan^{-1}(\sqrt{3})$

$\boxed{\theta = \frac{\pi}{3}}$ $\qquad$ $\boxed{\theta = -\pi/3}$

$(1+\sqrt{3}i) = re^{i\theta}$

$(1+\sqrt{3}i)^n = (2e^{i\pi/3})^n$

$= 2^n e^{(in\pi/3)}$

$\alpha^n = (1+\sqrt{3}i)^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$

$(1-\sqrt{3}i) = re^{i\theta}$

$(1+\sqrt{3}i)^n = (2e^{-i\pi/3})^n$

$= 2^n e^{(-ni\pi/3)}$

$\beta^n = 2^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$

$$\alpha^n+\beta^n = 2^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right) + 2^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)$$

$$=2^n\left[\cos\frac{n\pi}{3} + \cancel{i\sin\frac{n\pi}{3}} + \cos\frac{n\pi}{3} - \cancel{i\sin\frac{n\pi}{3}}\right]$$

$$=2^n\left[2\cos n\pi/3\right] = 2^n\,2\cos n\pi/3 = 2^{n+1}\cos\frac{n\pi}{3}$$

$$\alpha^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) \Longrightarrow \alpha^{10} = 2^{10}\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$$

$$\beta^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin n\pi/3\right) \Longrightarrow \beta^{10} = 2^{10}\left(\cos\frac{10\pi}{3} - i\sin\frac{10\pi}{3}\right)$$

$$\alpha^{10} + \beta^{10} = 2^{10}\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right) + 2^{10}\left(\cos\frac{10\pi}{3} - i\sin\frac{10\pi}{3}\right)$$

$$= 2^{10}\left[\cos\frac{10\pi}{3} + \cancel{i\sin\frac{10\pi}{3}} + \cos\frac{10\pi}{3} - \cancel{i\sin\frac{10\pi}{3}}\right]$$

$$= 2^{10}\, 2^{1}\left(\cos\frac{10\pi}{3}\right) = 2^{(11)}\cos\frac{10\pi}{3}$$

Root of Equation

π 0

$\sin\pi = 0$ $\sin 0 = 0$

$\cos\pi - 1$ $\cos 0 = 1$

$$\cos\theta = \cos(2k\pi + \theta)$$
$$\sin\theta = \sin(2k\pi + \theta)$$

$$(\cos\theta + i\sin\theta)^{1/n} = \left[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\right]^{1/n}$$

$$\boxed{(\cos\theta + i\sin\theta)^{1/n} = \left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right]}$$

putting $k = 0, 1, 2, \ldots (n-1)$, all n roots of the equation obtained

or

we get n roots of the complex number

Imp

1) $1 = \cos 0 + i\sin 0$
2) $-1 = \cos\pi + i\sin\pi$
3) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
4) $-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$
5) $1+i = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

(1) Find the cube root of unity. if ω is a complex cube roots of unity. prove $(1-\omega)^6 = -27$

=> $\sqrt[3]{1}$

let $x = \sqrt[3]{1} = (1)^{1/3}$

$= (\cos 0 + i \sin 0)^{1/3}$

$= \left[\cos\left(\frac{2k\pi + 0}{3}\right) + i \sin\left(\frac{2k\pi + 0}{3}\right)\right]$

$x = \cos\left(\frac{2k\pi}{3}\right) + i \sin\left(\frac{2k\pi}{3}\right)$

$\therefore$ cube root of complex number are

~~k=0, 1, 2, 3~~ $k = 0, 1, (3-1)$

$k = 0, 1, 2.$

$k=0$

$x_0 = \cos\left(\frac{2\times 0\pi}{3}\right) + i \sin\left(\frac{2\times 0\pi}{3}\right)$

$x_0 = \cos 0 + i \sin 0 = 1$

$k=1$

$x_1 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = \omega$

$k=2$

$x_2 = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)$

$= \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right]^2 = (\omega)^2$

$1 + \omega + \omega^2 = \cos 0 + i \sin 0 + \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) + \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}$

$= 1 + \cos\left(\pi - \frac{\pi}{3}\right) + i \sin\left(\pi - \frac{\pi}{3}\right) + \cos\left(\pi + \frac{\pi}{3}\right) + i \sin\left(\frac{\pi + \pi}{3}\right)$

$= 1 - \cos(\pi/3) + \cancel{i \sin \pi/3} - \cos(\pi/3) - \cancel{i \sin \pi/3}$

$= 1 - \frac{1}{2} - \frac{1}{2} = 1 - 1 = 0$

$1 + \omega + \omega^2 = 0$

$\cancel{1 + \omega +}$

$\boxed{1 + \omega^2 = -\omega}$

$L.H.S = (1-\omega)^6$

$= [(1-\omega)^2]^3$

$= [1 - 2\omega + \omega^2]^3$

$= [1 + \omega^2 - 2\omega]^3$

$= (-\omega - 2\omega)^3$

$= (-3\omega)^3$

$= -27\omega^3$

$= -27(1)$

$= -27$

(2) Solve $x^6 + 1 = 0$ using De'moivre's theorem

$\Rightarrow$

$x = -1$

$x = (-1)^{1/6}$

$x = (\cos\pi + i\sin\pi)^{1/6}$

$x = [\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]^{1/6}$

$x = \left[\cos\left(\frac{2k\pi + \pi}{6}\right) + i\sin\left(\frac{2k\pi + \pi}{6}\right)\right]$

putting $k = 0, 1, 2, 3, 4, 5$ we get all the 6 roots of the given equation

$k = 0$

$x = \left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$k = 1$

$x = \left[\cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6}\right] = (0 + i)$

$k = 2$

$x = \left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right] = \cos(\pi - \pi/6) + i\sin(\pi - \pi/6)$

$= -\cos\pi/6 + i\sin\pi/6$

$= \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$k = 3$

$x = \left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right] = \cos\left(\pi + \frac{\pi}{6}\right) + i\sin\left(\pi + \frac{\pi}{6}\right)$

$= -\cos\frac{\pi}{6} - i\sin\pi/6$

$= \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$k=4$

$$x = \left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right] = \cos\left(\frac{3\pi}{2}\right) + i\sin\frac{3\pi}{2}$$

$$= \cos\left(\pi + \frac{\pi}{2}\right) + i\sin(\pi + \pi/2)$$

$$= -\cos\pi/2 - i\sin\pi/2$$

$$= (0 - i)$$

$k=5$

$$x = \left[\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right] = \cos\left(2\pi - \frac{\pi}{6}\right) + i\sin\left(2\pi - \frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{6} - i\sin\pi/6$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$\therefore$ roots are, $(0+i)$, $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$, $\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$, $\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$, $(0-i)$, $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

(3) Solve $x^6 - i = 0$ using De'moivre's theorem

$\Rightarrow \quad x^6 = i$

$x = (i)^{1/6}$

$x = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/6}$

$x = \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{1/6}$

$x = \left[\cos\left(\frac{4k\pi + \pi}{2}\right) + i\sin\left(\frac{4k\pi + \pi}{2}\right)\right]^{1/6}$

$x = \cos\left(\frac{4k\pi + \pi}{12}\right) + i\sin\left(\frac{4k\pi + \pi}{12}\right)$ .. using De moivre's theorem

putting $k = 0, 1, 2, 3, 4, 5$, we get all the 6 roots is given equation

$k = 0$

$\boxed{x_0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}}$

$k = 2$

$\boxed{x_2 = \cos\left(\frac{9\pi}{12}\right) + i\sin\left(\frac{9\pi}{12}\right)}$

$k = 1$

$\boxed{x_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}}$

$k = 3$

$x_3 = \cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)$

$x_3 = \cos\left(\pi + \frac{\pi}{12}\right) + i\sin\left(\pi + \frac{\pi}{12}\right)$

$\boxed{x_3 = -\cos\pi/12 - i\sin\pi/12}$

$k = 4$

$x_4 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$

$= \cos\left(\pi + \frac{5\pi}{12}\right) + i\sin\left(\pi + \frac{5\pi}{12}\right)$

$\boxed{x_4 = -\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}}$

$k = 5$

$x_5 = \cos\left(\frac{21\pi}{12}\right) + i\sin\left(\frac{21\pi}{12}\right)$

$\cancel{= \cos\left(2\pi - \frac{3\pi}{12}\right) + i\sin\left(2\pi - \frac{3\pi}{12}\right)}$

$\cancel{= \cos\left(\frac{3\pi}{12}\right) - i\sin\left(\frac{3\pi}{12}\right)}$

$= \cos\left(\pi + \frac{9\pi}{12}\right) + i\sin\left(\pi + \frac{9\pi}{12}\right)$

$\boxed{x_5 = -\cos 9\pi/12 - i\sin(9\pi/12)}$

(4) calculate the root of $x^{10} + 11x^5 + 10 = 0$

$\Rightarrow$ $x^{10} + 11x^5 + 10 = 0$

$x^{10} + 10x^5 + 1x^5 + 10 = 0$

$x^5(x^5 + 10) + (x^5 + 10) = 0$

$(x^5 + 1) = 0$ $(x^5 + 10) = 0$

$\boxed{x^5 + 1 = 0}$

$x^5 = (-1)$

$x = (-1)^{1/5}$

$= (\cos\pi + i\sin\pi)^{1/5}$

$x = \cos\left(\frac{2k\pi + \pi}{5}\right) + i\sin\left(\frac{2k\pi + \pi}{5}\right)$ $\left\{ \cos\theta + i\sin\theta = \cos(2k\pi + \theta) + i\sin(2k\pi + \theta) \right.$

$x = \cos(2k_1 + 1)\frac{\pi}{5} + i\sin(2k_1 + 1)\pi/5$

$\boxed{x = e^{i(2k_1+1)\pi/5}}$

putting $k_1 = 0, 1, 2, 3, 4$ we get all 5 roots of $x^5 + 1 = 0$

$(x^5 + 10) = 0$

$x^5 = (-10)$

$x = (-10)^{1/5} = +10^{1/5}(-1)^{1/5}$

$x = 10^{1/5}(\cos\pi + i\sin\pi)^{1/5}$

$x = 10^{1/5}\left[\cos\left(\frac{2k_2 + \pi}{5}\right) + i\sin\left(\frac{2k_2 + \pi}{5}\right)\right]$

$\boxed{x = 10^{1/5} e^{i(2k_2+1)\pi/5}}$

Putting $k_2 = 0, 1, 2, 3, 4$ we get all 5 roots of $x^5 + 10 = 0$

$\therefore$ All the root of equation $x^{10} + 11x^5 + 10 = 0$ are give by $e^{i(2k_1+1)\pi/5}$ and $(10)^{1/5} e^{i(2k_2+1)\pi/5}$

where $k_1 = k_2 = 0, 1, 2, 3, 4$

(5) Find all the value of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1

=> let

$x = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$

$x = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{3/4}$

$x = \left[\left(\cos\frac{\pi}{3} + i\sin\pi/3\right)^{3}\right]^{1/4}$

$x = \left(\cos\pi + i\sin\pi\right)^{1/4}$

$x = \left[\cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right)\right]$

putting $k=0,1,2,3$ we get all the 4 root is given equation

$\therefore \cos\left(\frac{\pi}{4}\right) + i\sin(\pi/4)$, $\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

$\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)$, $\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)$

Required continued product

$= \cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$

$= \cos\left(\frac{\cancel{16}^{4}\pi}{\cancel{4}}\right) + i\sin\left(\frac{\cancel{16}^{4}\pi}{\cancel{4}}\right)$

$= \cos(4\pi) + i\sin(4\pi)$

$= 1 + i0$

$= 1$

Hence proved

$3\pi, \pi$
$\cos(\pi) = -1$
$\sin(\pi) = 0$

$0, 2\pi, 4\pi$
$\cos(0) = 1$
$\sin(0) = 0$

(7) Solve $x^5 = 1+i$ and calculate the continued product of the roots using De'moivre's theorem

→ $x^5 = (1+i)$ $\quad x = (1+i)^{1/5}$

$= \left(\sqrt{2}\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)\right)^{1/5}$

$= (\sqrt{2})^{1/5}\left(\cos\frac{\pi}{4} + \sin\pi/4\right)^{1/5}$

$= 2^{1/10} \left(\cos 2k+\pi/4 + \sin(2k+\pi/4)\right)^{1/5}$

$= 2^{1/10}\left(\cos(8k+1)\pi/4 + \sin(8k+1)\pi/4\right)^{1/5}$

$= 2^{1/10}\left(e^{i(8k+1)\pi/4}\right)^{1/5}$

$\boxed{x = 2^{1/10} e^{i(8k+1)\frac{\pi}{20}}}$; $k = 0, 1, 2, 3, 4 \ldots$ by D'M theorem

$k = 0, 1, 2, 3, 4$ represent 5 distinct root of $x^5 = 1+i$

$k=0$: $x = 2^{\frac{1}{10}} e^{i\pi/20}$

$k=1$ $\quad x = 2^{\frac{1}{10}} e^{i9\pi/20}$

$k=2$: $x = 2^{\frac{1}{10}} e^{i17\pi/20}$

$k=3$ $\quad x = 2^{\frac{1}{10}} e^{i25\pi/20}$

$k=4$: $x = 2^{\frac{1}{10}} e^{i33\pi/20}$

∴ The continued product $= 2^{\frac{1}{10}} e^{\frac{i\pi}{20}} \cdot 2^{\frac{1}{10}} e^{\frac{i9\pi}{20}} \cdot 2^{\frac{1}{10}} e^{\frac{i17\pi}{20}} \cdot 2^{\frac{1}{10}} e^{\frac{i25\pi}{20}} \cdot 2^{\frac{1}{10}} e^{i33\pi/20}$

$= \left((2)^{\frac{1}{10}}\right)^5 e^{i\left(\frac{\pi}{20} + \frac{9\pi}{20} + \frac{17\pi}{20} + \frac{25\pi}{20} + \frac{33\pi}{20}\right)}$

$= \sqrt{2}\, e^{i\left(\frac{85\pi}{20}\right)} = \sqrt{2}\, e^{\frac{i17\pi}{4}}$

$= \sqrt{2}\, e^{i(4\pi + \pi/4)}$

$= \sqrt{2}\left[\cos\left(4\pi + \frac{\pi}{4}\right) + i\sin(4\pi + \pi/4)\right]$

$= \sqrt{2}\left(\cos\pi/4 + \sin\pi/4\, i\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = (1+i)$

(9)

Find all the value of $(1-i)^{2/3}$ using De'moivre's theorem

let

$x = (1-i)^{2/3}$ $\qquad r = \sqrt{2}$

$= (\sqrt{2})^{2/3}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(-\pi/4\right)\right]^{2/3}$ $\qquad \theta = -\tan^{-1}(b/a)$

$\because$ { θ lies in 4th quadrant

$= (2)^{1/3}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]^{2/3}$ $\qquad \theta = -\tan^{-1}\left(\frac{1}{1}\right)$

$= (2)^{1/3}\left(\cos\frac{2\pi}{4} - i\sin\frac{2\pi}{4}\right)^{1/3}$ $\qquad \theta = -\pi/4$

$= (2)^{1/3}\left(\cos\frac{\pi}{2} - i\sin\pi/2\right)^{1/3}$

$= (2)^{1/3}\left[\cos\left(\frac{2k\pi+\pi/2}{3}\right) - i\sin\left(\frac{2k\pi+\pi/2}{3}\right)\right]$

$= (2)^{1/3}\left[\cos\left(\frac{4k\pi+\pi}{6}\right) - i\sin\left(\frac{4k\pi+\pi}{6}\right)\right]$

putting $k = 0, 1, 2$, we get all 3 root of equati

$k=0$ ...... $\boxed{x_0 = (2)^{1/3}\left[\cos\pi/6 - i\sin\pi/6\right]}$ $= (2)^{1/3}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$k=1$ ... $\boxed{x_1 = (2)^{1/3}\left[\cos 5\pi/6 - i\sin 5\pi/6\right]}$ $= (2)^{1/3}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$k=2$ ... $\boxed{x_2 = (2)^{1/3}\left[\cos 9\pi/6 - i\sin 9\pi/6\right]}$ $= (2)^{1/3}(i)$

(10) Find the value of $\{i\}^{1/5}$ using De'moivre's theorem

$\Rightarrow$ let $x = i^{1/5}$

$x^5 = i$ which is degree 5 eqn. $\therefore$ value of $(i)^{1/5}$ are same as roots of $x^5 = i$

$= \left(\cos \pi/2 + \sin \pi/2\right)^{1/5}$

$= \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + \sin\left(2k\pi + \pi/2\right)\right]^{1/5}$

$= \left[\cos \frac{(4k+1)}{5}\frac{\pi}{2} + \sin \frac{(4k+1)}{5}\pi/2\right]$ $\because$ De moivre's theorem

$x = e^{i(4k+1)\frac{\pi}{10}}$; $k = 0, 1, 2, 3, 4$

$x = \left[\cos \frac{(4k+1)\pi}{10} + i\sin \frac{(4k+1)\pi}{10}\right]$ give 5 distinct roots of $x^5 = i$

i.e 5 distinct value of $(i)^{1/5}$

$k=0$ --- $\boxed{x_0 = \cos \pi/10 + i\sin \pi/10}$

$k=1$ ---- $\boxed{x_1 = \cos \frac{5\pi}{10} + i\sin \frac{5\pi}{10}}$

$k=2$ --- $\boxed{x_2 = \cos \frac{9\pi}{10} + i\sin \frac{9\pi}{10}}$

$k=3$ --- $\boxed{x_3 = \cos \frac{13\pi}{10} + i\sin \frac{13\pi}{10}}$

$k=4$ --- $\boxed{x_4 = \cos \frac{17\pi}{10} + i\sin \frac{17\pi}{10}}$

10 marks

(5) Show that the roots of equation $(x+1)^6 + (x-1)^6 = 0$ are given by $-i\cot\left[\frac{(2k+1)\pi}{12}\right]$ for $k = 0, 1, 2, 3, 4, 5$ using De' Moivre's theorem.

$\Rightarrow$

$$(x+1)^6 + (x-1)^6 = 0$$

$$(x+1)^6 = -(x-1)^6$$

$$\left(\frac{(x+1)}{(x-1)}\right)^6 = -1$$

$$\frac{(x+1)}{(x-1)} = (-1)^{\frac{1}{6}}$$

$$\frac{x+1}{x-1} = (\cos\pi + i\sin\pi)^{\frac{1}{6}}$$

$$= \left[\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)\right]^{\frac{1}{6}}$$

$$\frac{x+1}{x-1} = \cos\left(\frac{2k\pi+\pi}{6}\right) + i\sin\left(\frac{2k\pi+\pi}{6}\right) \quad \ldots \text{By D.M theorem}$$

let $\frac{2k\pi+\pi}{6} = \theta$

$$\frac{x+1}{x-1} = \frac{\cos\theta + i\sin\theta}{1}$$

By dividendo and componendo

$$\frac{x+1+(x-1)}{x+1-(x-1)} = \frac{\cos\theta + i\sin\theta + 1}{\cos\theta + i\sin\theta - 1}$$

$$\frac{\cancel{2}x}{\cancel{2}} = \frac{1+\cos\theta + i\sin\theta}{-(1-\cos\theta) + i\sin\theta}$$

$$x = \frac{2\cos^2\theta/2 + i\,2\sin\theta/2\cos\theta/2}{-2\sin^2\theta/2 + i2\sin\theta/2\cos\theta/2}$$

$$= \frac{\cancel{2}\cos\theta/2}{\cancel{2}\sin\theta/2}\left(\frac{\cos\theta/2 + i\sin\theta/2}{-\sin\theta/2 + i\cos\theta/2}\right)$$

$$= \cot(\theta/2)\left(\frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}\right)$$

$$= \cot \theta/_2 \left[\frac{\cos \theta/2 + i \sin \theta/2}{i \cos(+\theta/_2) + \sin(+\theta/_2)} \right]$$

$$= \cot \theta/_2 \left\{ \frac{(\cos \theta/_2 + i\sin \theta/_2)}{[\cos(\pi/_2 + \theta/_2) + i\sin(\pi/_2 + \theta/_2)]} \right\}$$

$$= \cot \frac{\theta}{2} \left\{ (\cos \theta/_2 + i \sin \theta/_2) \left[\cos(\pi/_2 + \theta/_2) + i \sin(\pi/_2 + \theta/_2) \right]^{-1} \right\}$$

$$= \cot \theta/_2 \left\{ (\cos \theta/_2 + i \sin \theta/_2) \left[\cos(\pi/_2 + \theta/_2) - i \sin(\pi/_2 + \theta/_2) \right] \right\}$$

$$= \cot \frac{\theta}{2} \left[\cos \theta/_2 \cos(\pi/_2 + \theta/_2) \cancel{+ i \cos \theta/_2} \sin(\pi/_2 + \theta/_2) + i \sin \theta/_2 \cos \pi/_2 + \theta/_2 + \sin(\pi/_2 + \theta/_2) \sin \theta/_2 \right]$$

$$= \cot \theta/_2 \left\{ \left[\cos \theta/_2 \cos(\pi/_2 + \theta/_2) + \sin(\pi/_2 + \theta/_2) \sin \theta/_2 \right] - i \left[\sin(\pi/_2 + \theta/_2) \cos \theta/_2 - \sin \theta/_2 \cos(\pi/_2 + \theta/_2) \right] \right\}$$

$$= \cot \theta/_2 \left\{ \cos(\pi/_2 + \cancel{\theta/_2} - \cancel{\theta/_2}) - i \left[\sin(\pi/_2 + \cancel{\theta/_2} - \cancel{\theta/_2}) \right] \right\}$$

$$= \cot \frac{\theta}{2} \left\{ \cos \pi/_2 - i \sin \pi/_2 \right\}$$

$$= \cot \theta/_2 \, (0 - i)$$

$$= -i \cot \theta/_2 \qquad = -i \cot \left(\frac{(2k+1)\pi/6}{2} \right)$$

$$= -i \cot \left(\frac{(2k+1)\pi}{12} \right)$$

$$k = 0, 1, 2, 3, 4, 5.$$

(2) $(x+1)^8 + x^8 = 0$ $\quad (x+1)^8 = -x^8$

$$\left(\frac{x+1}{x}\right)^8 = -1$$

$$\left(\frac{x+1}{x}\right) = (\cos\pi + i\sin\pi)^{1/8}$$

$$\frac{x+1}{x} = \cos\left(\frac{2k\pi+\pi}{8}\right) + i\sin\left(\frac{2k\pi+\pi}{8}\right)$$

let $\theta = \dfrac{2k\pi+\pi}{8}$

$$\frac{x+1}{x} = \cos\theta + i\sin\theta$$

$$\frac{x+1+x}{\cancel{x}+1-\cancel{x}} = \cos\theta + i\sin\theta$$

$$2x+1 = \frac{\cos\theta + i\sin\theta + 1}{\cos\theta - 1 + i\sin\theta}$$

$$2x+1 = \frac{(1+\cos\theta) + i\sin\theta}{-(1-\cos\theta) + i\sin\theta}$$

$$2x+1 = \frac{2\cos^2\theta/2 + i\,2\sin\theta/2\cos\theta/2}{-2\sin^2\theta/2 + i\,2\sin\theta/2\cos\theta/2} = \frac{\cancel{2}\cos\theta/2}{\cancel{2}\sin\theta/2}\left(\frac{\cos\theta/2 + i\sin\theta/2}{-\cos\theta/2 + i\sin\theta/2}\right)$$

$$= \cot\theta/2\left(\frac{\cos\theta/2 + i\sin\theta/2}{[\sin(\pi/2+\theta/2) + i\sin(\pi/2+\theta/2)]}\right)$$

$$= \cot\theta/2\left[(\cos\theta/2 + i\sin\theta/2)\cdot\left[\sin(\pi/2+\theta/2) + i\cos(\pi/2+\theta/2)\right]^{-1}\right]$$

$$= \cot\theta/2 \quad \left[(\cos\theta/2 + i\sin\theta/2)\right]\left[(\sin\tfrac{\pi}{2}+\theta/2 - i\cos(\pi/2+\theta/2)\right]$$

$$= \cot\theta/2\left[\sin(\pi/2+\theta/2)\cos\theta/2 - i\cos\theta/2\cos(\pi/2+\theta/2) + i\sin\theta/2\sin(\pi/2+\theta/2) - i^2\sin\theta/2\cos(\pi/2+\theta/2)\right]$$

$$= \cot\theta/2\left[\sin(\pi/2+\cancel{\theta/2}-\cancel{\theta/2}) - i\cos(\pi/2+\cancel{\theta/2}-\theta/2)\right] = \cot\theta/2\left[\sin\pi/2 - i\cos\pi/2\right]$$

$$\boxed{x = -\tfrac{1}{2} - \tfrac{1}{2}\cot\tfrac{(2k+1)\pi}{16}}$$

$k = 0, 1, 2, 3, 4, 5, 6, 7$

$$2x+1 = -\cot\theta/2$$

$$2x = -1 - \cot\theta/2$$

$$x = -\tfrac{1}{2} - \tfrac{1}{2}\cot\theta/2$$

(5) $(2x-1)^5 = 32x^5$

$$\left(\frac{2x-1}{x}\right)^5 = (2)^5 \times 1$$

$$\left(\frac{2x-1}{2x}\right)^5 = [\cos 0 + i\sin 0]$$

$$\left(\frac{2x-1}{2x}\right)^5 = \left[\cos\left(\frac{2k\pi+0}{5}\right) + i\sin(2k\pi+0)\right]$$

$$\left(\frac{2x-1}{2x}\right) = \left(\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\right)$$

let $\theta = \frac{2k\pi}{5}$

$$\frac{2x-1}{2x} = [\cos\theta + i\sin\theta]$$

Applying dividendo and componendo

$$\frac{2x-1+2x}{2x-1-2x} = \cot\theta/_2 \left[\frac{\cos\theta/_2 + i\sin\theta/_2}{-\cos\theta/_2 + i\sin\theta/_2}\right]$$

$$\frac{4x-1}{-1} = \cot\theta/_2 \left[\frac{\cos\theta/_2 + i\sin\theta/_2}{-\cos\theta/_2 + i\sin\theta/_2}\right]$$

$$= \cot\theta/_2 \left[\cos\theta/_2 + i\sin\theta/_2\right]$$

$$4x-1 = -\cot\theta/_2\,(0+i) = -i\cot\theta/_2$$

$$4x = 1 - i\cot\theta/_2$$

$$\boxed{x = \frac{1}{4} - \frac{1}{4}\, i\cot\frac{2k\pi}{10}}$$

$k = 0, 1, 2, 3, 4$

Hyperbolic function

- **Definition** for z real or complex

(1) $\sinh z = \frac{e^{z} - e^{-z}}{2}$

(2) $\cosh z = \frac{e^{z} + e^{-z}}{2}$

(3) $\tanh z = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$

(4) $\operatorname{cosech} z = \frac{2}{e^{z} - e^{-z}}$

(5) $\operatorname{sech} z = \frac{2}{e^{z} + e^{-z}}$

(6) $\coth z = \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}$

- **Relation** b/w hyperbolic and trigonometric function

(1) $\sin(i\theta) = i \sinh\theta$

(2) $\cos(i\theta) = \cosh\theta$

(3) $\tan(i\theta) = i \tanh\theta$

(4) $\operatorname{cosec}(i\theta) = -i \operatorname{cosech}\theta$

(5) $\sec(i\theta) = \operatorname{sech}\theta$

(6) $\cot(i\theta) = \frac{1}{i} \frac{1}{\tanh\theta} = -i \coth\theta$

(1) $\sinh(i\theta) = i\sin\theta$

(2) $\cosh(i\theta) = \cos\theta$

(3) $\tanh(i\theta) = i\tan\theta$

(4) $\operatorname{cosech}(i\theta) = -i \operatorname{cosec}\theta$

(5) $\operatorname{sech}(i\theta) = \sec\theta$

(6) $\coth(i\theta) = -i\cot\theta$

Hyperbolic Indentities

$\cos^2\theta + \sin^2\theta = 1$ → $\boxed{\cosh^2\theta - \sinh^2\theta = 1}$

$1 + \tan^2\theta = \sec^2\theta$ → $\boxed{1 - \tanh^2\theta = \mathrm{sech}^2\theta}$

$1 + \cot^2\theta = \mathrm{cosec}^2\theta$ → $\boxed{1 - \coth^2\theta = -\mathrm{cosech}^2\theta}$

$\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$= \frac{2\tan\theta}{1+\tan^2\theta}$

→ $\boxed{\sinh 2\theta = 2\sinh\theta \; \cosh\theta}$ $= \frac{2\tanh\theta}{1-\tanh^2\theta}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

$= \frac{1-\tan^2\theta}{1+\tan^2\theta}$

→ $\boxed{\cosh(2\theta) = \cosh^2\theta + \sinh^2\theta}$

$= 2\cosh^2\theta - 1$

$= 1 + 2\sinh^2\theta$

$= \frac{1+\tanh^2\theta}{1-\tanh^2\theta}$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ → $\boxed{\tanh(2\theta) = \frac{2\tanh\theta}{1+\tanh^2\theta}}$

$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2$

$\sinh(\theta_1 \pm \theta_2) = \sinh\theta_1 \cosh\theta_2 \pm \cosh\theta_1 \sinh\theta_2$

$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$

$\cosh(\theta_1 \pm \theta_2) = \cosh\theta_1 \cosh\theta_2 \mp \sinh\theta_1 \sinh\theta_2$

?

$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1 \tan\theta_2}$

$\tanh(\theta_1 \pm \theta_2) = \frac{\tanh\theta_1 \pm \tanh\theta_2}{1 \mp \tanh\theta_1 \tanh\theta_2}$

problems

(1) If $\tanh x = \frac{2}{3}$ find the value of x and $\cosh 2x$

$\Rightarrow$ $\tanh x = \frac{2}{3}$

$$\frac{\sinh x}{\cosh x} = \frac{2}{3}$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{2}{3}$$

$$3(e^x) - 3e^{-x}) = 2e^x + 2e^{-x}$$

$$e^x = 5e^{-x}$$

$$\frac{e^x}{e^{-x}} = 5$$

$$e^{(2x)} = 5$$

$$2x = \log 5$$

$$\boxed{x = \frac{1}{2}\log 5}$$

By hyperbolic identity

$$\cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x} = \frac{1 + (2/3)^2}{1 - (2/3)^2} = \frac{1 + \frac{4}{9}}{1 - \frac{4}{9}}.$$

$$= \frac{13/\cancel{9}}{5/\cancel{9}} = \frac{13}{5}$$

$$\boxed{\cosh 2x = \frac{13}{5}}$$

Note :- we do not use De'moivre's theorem
because - only applicable for trignometry
- only applicable for (+)
$(\cos x + i \sin x)$

(2) prove that $(\cosh x - \sinh x)^n = \cosh(nx) - \sinh(nx)$

$\Rightarrow$ L.H.S $= (\cosh x - \sinh x)^n$

$$= \left(\frac{e^x + e^{-x}}{2} - \left(\frac{e^x - e^{-x}}{2}\right)\right)^n$$

$$= \left(\frac{e^{-x} + \cancel{e^x} - \cancel{e^x} + e^{-x}}{2}\right)^n = \left(\frac{\cancel{2}e^{-x}}{\cancel{2}}\right)^n$$

$$= (e^{-nx}) \ldots \{\text{from law of indices}\}$$

R.H.S $= \cosh(nx) - \sinh(nx)$

$$= \frac{e^{nx} + e^{-nx}}{2} - \left(\frac{e^{nx} - e^{-nx}}{2}\right)$$

$$= \frac{\cancel{e^{nx}} + e^{-nx} - \cancel{e^{nx}} + e^{-nx}}{2} = \frac{\cancel{2}e^{-nx}}{\cancel{2}}$$

$$= (e^{-nx})$$

L.H.S = R.H.S

(3) If $5\sinh x - \cosh x = 5$, Find $\tanh x$

$\Rightarrow$ $\cancel{5\left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{\quad}{2}\right) = 5}$

$5\sinh x - \cos$

(4) If $\tanh x = \frac{1}{2}$, prove that $\cosh 2x = \frac{5}{3}$

L.H.S: $\cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x} = \frac{1 + \frac{1}{4}}{1 - \frac{1}{4}} = \frac{5/\cancel{4}}{3/\cancel{4}} = \frac{5}{3}$

= R.H.S

Hence proved.

(3) If $5\sinh x - \cosh x = 5$, find $\tanh x$

$\Rightarrow$ ~~$5\sinh x - \cosh x = 5$ divided by $\cosh x$ on both side~~

~~$5\tanh x - 1 = \frac{5}{\cosh x}$~~

~~$5\tanh x = \frac{5}{\cosh x} + 1$~~

$$5\sinh x - \cosh x = 5$$

$$5\left(\frac{e^{x}-e^{-x}}{2}\right) - \left(\frac{e^{x}+e^{-x}}{2}\right) = 5$$

$$\frac{5e^{x} - 5e^{-x} - e^{x} - e^{-x}}{2} = 5$$

$$4e^{x} - 6e^{-x} = 10$$

~~$2e^{x} - 3e^{-x} = 5$~~

~~$2e^{x} - 3e^{-x} - 5 = 0$~~

~~$2e^{x} + 2e$~~

$$2e^{x} - 3e^{-x} = 5$$

$$2e^{x} - \frac{3}{e^{x}} = 5$$

$$2(e^{x})^2 - 3 = 5(e^{x})$$

$$2(e^{x})^2 - 5(e^{x}) - 3 = 0$$

$$2(e^{x})^2 - 6e^{x} + e^{x} - 3 = 0$$

$$2e^{x}(e^{x} - 3) + 1(e^{x} - 3) = 0$$

$2e^{x} + 1 = 0$ $\qquad$ $e^{x} - 3 = 0$

$e^{x} = -\frac{1}{2}$ $\qquad$ $\boxed{e^{x} = 3}$

which is not true for x - real

$$\boxed{\frac{1}{e^{x}} = \frac{1}{3}}$$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{e^{x} - \frac{1}{e^{x}}}{e^{x} + \frac{1}{e^{x}}} = \frac{3 - \frac{1}{3}}{3 + \frac{1}{3}} = \frac{8/3}{10/3} = \frac{8}{10}$$

$$= \frac{4}{5}$$

(5) Solve the equation for real value of x,

$17\cosh x + 18\sinh x = 1$

$$17\left(\frac{e^x+e^{-x}}{2}\right) + 18\left(\frac{e^x-e^{-x}}{2}\right) = 1$$

$$\frac{17e^x + 17e^{-x} + 18e^x - 18e^{-x}}{2} = 1$$

$$35e^x - e^{-x} = 2$$

$$35e^x - \frac{1}{e^x} = 2$$

$$35(e^x)^2 - 1 = 2e^x$$

$$35(e^x)^2 - 2e^x - 1 = 0$$

$$35(e^x)^2 - 7e^x + 5e^x - 1 = 0$$

-35

$-7 \quad +5$

$$7e^x(5e^x - 1) + 1(5e^x - 1) = 0$$

$5e^x - 1 = 0$ $\quad$ $7e^x = -1$

$\boxed{e^x = \frac{1}{5}}$ $\quad$ $e^x \neq -1/7$ which is not real for x - real

$$x = \log(1/5) = \log(5)^{-1} = -1\log 5$$

$$\boxed{x = -\log 5}$$

• Find $\tanh x$ if $6\sinh x + 2\cosh x + 7 = 0$

$\Rightarrow$ $6\sinh x + 2\cosh x = -7$

$$6\left(\frac{e^{x}-e^{-x}}{2}\right) + 2\left(\frac{e^{x}+e^{-x}}{2}\right) = -7$$

$$\frac{6e^{x} - 6e^{-x} + 2e^{x} + 2e^{-x}}{2} = -7$$

$$8e^{x} - 4e^{-x} = -14$$

$$4e^{x} - 2e^{-x} = -7$$

$$4e^{x} - \frac{2}{e^{x}} = -7$$

$$4(e^{x})^2 - 2 = -7e^{x}$$

$$4(e^{x})^2 + 7e^{x} - 2 = 0$$

$$4(e^{x})^2 + 8e^{x} - e^{x} - 2 = 0$$

$$4e^{x}(e^{x} + 2) - 1(e^{x} + 2) = 0$$

-8

8 -1

$4e^{x} - 1 = 0$ $\qquad$ $e^{x} \neq -2$ which is not true for x-real

$$\boxed{e^{x} = 1/4}$$

$$\boxed{\frac{1}{e^{x}} = 4}$$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{e^{x} - \frac{1}{e^{x}}}{e^{x} + \frac{1}{e^{x}}} = \frac{\frac{1}{4} - 4}{\frac{1}{4} + 4}$$

$$= \frac{\frac{-15}{\cancel{4}}}{\frac{17}{\cancel{4}}} = \frac{-15}{17}$$

2 → 1 2 1
3 → 1 3 3 1
4 → 1 4 6 4 1
5 → 1 5 10 10 5 1

(6) prove that $\cosh^5 x = \frac{1}{16}\left[\cosh 5x + 5\cosh 3x + 10\cosh x\right]$

$\Rightarrow$ $L.H.S = \cosh^5 x$

$= (\cosh x)^5$

$= \left(\frac{e^x + e^{-x}}{2}\right)^5$

$= \frac{1}{2^5}(e^x + e^{-x})^5$

$= \frac{1}{2^5}\left[e^{5x} + 5 e^{4x} e^{-x} + 10 e^{3x} e^{-2x} + 10 e^{2x} e^{-3x} + 5 e^x e^{-4x} + e^{-5x}\right]$

$= \frac{1}{2^5}\left[(e^{5x} + e^{-5x}) + 5(e^{3x} + e^{-3x}) + 10(e^x + e^{-x})\right]$

$= \frac{1}{2^5}\left[2\cosh 5x + 10\cosh 3x + 20\cosh x\right]$

$\cosh nx = \frac{e^{nx} + e^{-nx}}{2}$

$\boxed{2\cosh nx = e^{nx} + e^{-nx}}$

$= R.H.S$

Hence proved

(7) Find the value of $\tan h(\log\sqrt{5})$

$\Rightarrow$ $\tanh(\log\sqrt{5}) = \left(\frac{e^{\log\sqrt{5}} - e^{-\log\sqrt{5}}}{e^{\log\sqrt{5}} + e^{-\log\sqrt{5}}}\right)$

$\tan hx = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

$\frac{e^{\log_e\sqrt{5}} - e^{\log(\sqrt{5})^{-1}}}{e^{\log_e\sqrt{5}} + e^{\log_e(\sqrt{5})^{-1}}} = \frac{\sqrt{5} - (\sqrt{5})^{-1}}{(\sqrt{5}) + (\sqrt{5})^{-1}} = \frac{\sqrt{5} - \frac{1}{\sqrt{5}}}{\sqrt{5} + \frac{1}{\sqrt{5}}}$

$= \frac{\frac{5-1}{\sqrt{5}}}{\frac{5+1}{\sqrt{5}}} = \frac{4}{6} = \underline{2/3}$

$\cosh^2 x - \sinh^2 x = 1$
$1 + \tanh^2 x = \operatorname{sech}^2 x$
$1 + \coth^2 x = -\operatorname{cosech}^2 x$

Imp

(1) prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1+\sinh^2 x}}} = +\sinh^2 x$

$\Rightarrow$ $L.H.S = \dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1+\sinh^2 x}}} = \dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{\cosh^2 x}}}$

$= \dfrac{1}{1-\dfrac{1}{1-\operatorname{sech}^2 x}} = \dfrac{1}{1-\left(\dfrac{1}{\tanh^2 x}\right)}$

$= \dfrac{1}{1-\coth^2 x} = \dfrac{1}{-\operatorname{cosech}^2 x} = -\sinh^2 x$ = R.H.S

Hence proved

Imp

(2) prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\cosh^2 x}}} = \cosh^2 x$

$\Rightarrow$ $L.H.S = \dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\cosh^2 x}}} = \dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{-\sinh^2 x}}} = \dfrac{1}{1-\dfrac{1}{1+\operatorname{cosech}^2 x}}$

$= \dfrac{1}{1-\dfrac{1}{\coth^2 x}} = \dfrac{1}{1-\tanh^2 x} = \dfrac{1}{\operatorname{sech}^2 x}$

$= \cosh^2 x$ = R.H.S

Hence proved

(13) If that $\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 = \cosh 6x + \sinh 6x$

$\Rightarrow$ $L.H.S = \left(\frac{1+\tanh x}{1-\tanh x}\right)^3$

$$= \left(\frac{1+\frac{e^x - e^{-x}}{e^x+e^{-x}}}{1-\frac{e^x-e^{-x}}{e^x+e^{-x}}}\right)^3 = \left(\frac{e^x+\cancel{e^{-x}}+e^x-\cancel{e^{-x}}}{e^x+e^{-x}-(e^x-e^{-x})}\right)^3$$

$$= \left(\frac{\cancel{2}e^x}{\cancel{2}e^{-x}}\right)^3 = (e^{2x})^3 = e^{6x}$$

$R.H.S = \cosh 6x + \sinh 6x$

$$= \frac{e^{6x}+e^{-6x}}{2} + \frac{e^{6x}-e^{-6x}}{2}$$

$$= \frac{e^{6x}+\cancel{e^{-6x}}+e^{6x}-\cancel{e^{-6x}}}{2} = \frac{\cancel{2}e^{6x}}{\cancel{2}} = e^{6x}$$

$L.H.S = R.H.S$ ~~Hence~~ proved.

(4) If $\tan\left(\frac{x}{2}\right) = \tanh\left(\frac{u}{2}\right)$ then proved that

(i) $\sinh u = \tan x$

(ii) $\cosh u = \sec x$

(iii) $u = \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$

(i) $\sinh u = \tan x$

$$L.H.S = \tan x$$

$$= \frac{2\tan x/2}{1-\tan^2 x/2}$$

$$= \frac{2\tanh(u/2)}{1-\tanh^2(u/2)}$$

~~tan(2x) = 1+tan²x~~

$$= \cancel{\frac{2\frac{\sinh(u/2)}{\cosh(u/2)}}{1-\frac{\sinh^2(u/2)}{\cosh^2(u/2)}}} = \cancel{\frac{2\frac{\sinh(u/2)}{\cosh(u/2)}}{\frac{\cosh^2(u/2)-\sinh^2(u/2)}{\cosh^2(u/2)}}}$$

$$= \frac{2\sinh(u/2)}{\cosh(u/2)\operatorname{sech}^2(u/2)} = \frac{2\sinh(u/2)\cosh^2(u/2)}{\cosh(u/2)}$$

$$= 2\sinh(u/2)\cos(u/2)$$

$$= \sinh u = R.H.S$$

(ii) $\cosh u = \sec x$

$$L.H.S = \sec x = \frac{1}{\cos x} = \frac{1}{\frac{1-\tan^2 x/2}{1+\tan^2 x/2}} = \frac{1+\tan^2 x/2}{1-\tan^2 x/2}$$

$$= \frac{1+\tanh^2(u/2)}{1-\tanh^2(u/2)} = \frac{1+\tanh^2(u/2)}{\operatorname{sech}^2(u/2)}$$

$$= \frac{1}{\operatorname{sech}^2(u/2)} + \frac{\tanh^2(u/2)}{\operatorname{sech}^2(u/2)}$$

$$= \cosh^2(u/2) + \sinh^2(u/2)$$

$$= \cosh u \quad \ldots \left[\cosh u = \cosh^2 u/2 + \sinh^2(u/2)\right.$$

$$= R.H.S$$

(iii) $u = \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$

$e^u = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$= \dfrac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}} = \dfrac{1 + \tan x/2}{1 - \tan x/2}$

$e^u = \dfrac{1 + \tanh(u/2)}{1 - \tanh(u/2)}$

$= \dfrac{1+}{\sec}$

$\tan(x/2) = \tanh(u/2)$

$\sinh u = \tan x$

R.H.S $= \tan x$

$= \dfrac{\sin x}{\cos x}$

?

$\sinh u = \tan x$

$u = \sinh^{-1}(\tan x)$

$u = \log\left(\tan x + \sqrt{\tan^2 x + 1}\right)$

$u = \log\left(\frac{\sin x}{\cos x} + \sec x\right)$

$u = \log\left(\frac{\sin x}{\cos x} + \frac{1}{\cos x}\right)$

$= \log\left(\frac{1 + \sin x}{\cos x}\right)$

$= \log\left(\frac{(\cos x/2 + \sin x/2)^2}{(\cos^2 x/2 - \sin^2 x/2)}\right)$

$= \log\left(\frac{(\cos x/2 + \sin x/2)^{\cancel{2}}}{(\cos x/2 - \sin x/2)(\cos x/2 + \sin x/2)}\right)$

$= \log\left(\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right) = \log\left(\frac{1 + \tan x/2}{1 - \tan x/2}\right) = \log$

$\boxed{u = \log\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)}$

Inverse Hyperbolic Function

(1) $\boxed{\sinh^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)}$

(2) $\boxed{\cosh^{-1}(x) = \log\left(x + \sqrt{x^2-1}\right)}$

(3) $\boxed{\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = \log\sqrt{\frac{1+x}{1-x}}}$

let $\sinh^{-1}(x) = y$

$x = \sinh y$

$$x + \sqrt{x^2+1} = \sinh y + \sqrt{\sinh^2 y + 1}$$

$$= \cancel{\cosh}\sinh y + \sqrt{\cosh^2 y} \quad \ldots \{\cosh^2 y - \sinh^2 y = 1\}$$

$$= \cancel{\cos}\sinh y + \cosh y$$

$$= \frac{e^{y} - e^{-y}}{2} + \frac{e^{y} + e^{-y}}{2}$$

$$= \frac{e^{y} - \cancel{e^{-y}} + e^{y} + \cancel{e^{-y}}}{2} = \frac{\cancel{2}e^{y}}{\cancel{2}}$$

$$= e^{y}$$

$$\left(x + \sqrt{x^2+1}\right) = e^{y}$$

$$\log\left(x + \sqrt{x^2+1}\right) = y$$

$$\boxed{\log\left(x + \sqrt{x^2+1}\right) = \sinh^{-1}(x)}$$

For 10 marks

(1) prove that

(i) $\tanh^{-1}(x) = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

(ii) $\operatorname{sech}^{-1}(\sin\theta) = \log\left[\cot\left(\frac{\theta}{2}\right)\right]$

$\Rightarrow$ (i) let

$\tanh^{-1}(x) = y$

$\boxed{x = \tanh(y)}$

(i) $\frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{1-\tanh^2 y}} = \frac{\tanh y}{\sqrt{\operatorname{sech}^2 y}} = \frac{\tanh y}{\operatorname{sech} y}$

$\frac{x}{\sqrt{1-x^2}} = \sinh y$

$\sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = y$

$\boxed{\sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \tanh^{-1}(x)}$

(ii) $\operatorname{sech}^{-1}(\sin\theta) = \log\left[\cot(\theta/2)\right]$

let

$\operatorname{sech}^{-1}(\sin\theta) = y$

$\boxed{\sin\theta = \operatorname{sech} y}$

Taking reciprocal

$\frac{1}{\operatorname{cosec}\theta} = \frac{1}{\cosh y}$

$\cosh y = \operatorname{cosec}\theta$

$y = \cosh^{-1}(\operatorname{cosec}\theta)$

$y = \log\left(\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right)$

$= \log\left(\operatorname{cosec}\theta + \sqrt{\cot^2\theta}\right) = \log(\operatorname{cosec}\theta + \cot\theta)$ $\quad (\cosh^{-1}x = \log(x + \sqrt{x^2-1}))$

$= \log\left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right) = \log\left(\frac{1+\cos\theta}{\sin\theta}\right)$

$= \log\left(\frac{\cancel{2}\cos^{2}\theta/2}{\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}\right) = \log\left(\frac{\cos\theta/2}{\sin\theta/2}\right) = \log(\cot\theta/2)$

$y = \log(\cot\theta/2)$

$\boxed{\operatorname{sech}^{-1}(\sin\theta) = \log(\cot\theta/2)}$

(2) If $\cosh x = \sec\theta$ prove that

(i) $x = \log(\sec\theta + \tan\theta)$

(ii) $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$

(iii) $\sinh x = \tan\theta$ (iv) $\tanh x = \sin\theta$

(v) $\tanh\left(\frac{x}{2}\right) = \pm \tan(\theta/2)$

$\Rightarrow$

(i) $x = \log(\sec\theta + \tan\theta)$

$\rightarrow \cosh x = \sec\theta$

$x = \cosh^{-1}(\sec\theta)$

$= \log(\sec\theta + \sqrt{\sec^2\theta - 1})$

$= \log\left(\frac{1}{\cos\theta} + \sqrt{\tan^2\theta}\right) = \log\left(\frac{1}{\cos\theta} + \tan\theta\right)$

$= \log\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)$

$\boxed{x = \log(\sec\theta + \tan\theta)}$

(ii) $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$

we have $x = \log(\sec\theta + \tan\theta)$

$e^x = \sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$

$e^x = \frac{1+\sin\theta}{\cos\theta}$

$e^{-x} = \frac{\cos\theta}{1+\sin\theta} = \frac{\cos^2\theta/2 - \sin^2\theta/2}{(\cos\theta/2 + \sin\theta/2)^2}$

$e^{-x} = \frac{(\cos\theta/2 - \sin\theta/2)\cancel{(\cos\theta/2 + \sin\theta/2)}}{(\cos\theta/2 + \sin\theta/2)^2}$

$e^{-x} = \frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2 + \sin\theta/2} = \frac{\frac{\cos\theta/2}{\cos\theta/2} - \frac{\sin\theta/2}{\cos\theta/2}}{\frac{\cos\theta/2}{\cos\theta/2} + \frac{\sin\theta/2}{\cos\theta/2}}$

$e^{-x} = \frac{1 - \tan\theta/2}{1 + \tan\theta/2}$ $\boxed{e^{-x} = \tan(\pi/4 - \theta/2)}$

$\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$

$R.H.S = \frac{\pi}{2} - 2\tan^{-1}\left(\tan(\pi/4 - \theta/2)\right)$

$= \pi/2 - 2(\pi/4 - \theta/2)$

$= \cancel{\frac{\pi}{2}} - \cancel{\pi/2} + \theta$

$= \theta$

$= L.H.S$

(iii) $\sinh x = \tan\theta$

we have $\boxed{\cosh^2 x - \sinh^2 x = 1}$

$\cosh^2 x = 1 + \sinh^2 x$

$\sinh^2 x = \cosh^2 x - 1$

$\sinh^2 x = \sec^2\theta - 1$ $\{\cosh x = \sec\theta\}$

$\sinh x = \sqrt{\sec^2\theta - 1}$

$\boxed{\sinh x = \tan\theta}$

(iv) $\tanh x = \sin\theta$

we have $\sinh x = \tan\theta$, $\cosh x = \sec\theta$

$\frac{\sinh x}{\cosh x} = \frac{\sin\theta}{\cos\theta} \times \cos\theta$

$\boxed{\tanh x = \sin\theta}$

$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$

formula

(v) we have $\cosh x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}$

$\sec\theta = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}$ .... $\{\cosh x = \sec\theta\}$

$\cos\theta = \frac{1 - \tanh^2 x}{1 + \tanh^2 x}$

$\frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} = \frac{1 - \tanh^2 x}{1 + \tanh^2 x}$

$\tan^2\theta/2 = \tanh^2 x$

$\tanh^2 x = \tan^2\theta/2$

$\tanh x = \sqrt{(\tan^2\theta/2)}$

$\boxed{\tanh x = \pm \tan\theta/2}$

most ask in ESE For (10M)

(1) Prove that

(i) $\sinh^{-1}(\tan\theta) = \log(\sec\theta + \tan\theta)$

$$L.H.S = \sinh^{-1}(\tan\theta)$$
$$= \log\left(\tan\theta + \sqrt{\tan^2\theta + 1}\right)$$
$$= \log\left(\tan\theta + \sqrt{\sec^2\theta}\right)$$
$$= \log(\tan\theta + \sec\theta)$$
$$= R.H.S$$

(ii) $\sinh^{-1}(x) = \cosh^{-1}\left(\sqrt{1+x^2}\right)$

$$L.H.S = \sinh^{-1}(x)$$
$$= \log\left(x + \sqrt{1+x^2}\right)$$

$$R.H.S = \cosh^{-1}\left(\sqrt{1+x^2}\right)$$
$$= \log\left(\sqrt{1+x^2} + \sqrt{\left(\sqrt{1+x^2}\right)^2 - 1}\right)$$
$$= \log\left(\sqrt{1+x^2} + \sqrt{1+x^2-1}\right)$$
$$\log\left(\sqrt{1+x^2} + \sqrt{x^2}\right)$$
$$= \log\left(\sqrt{1+x^2} + x\right)$$
$$R.H.S = \log\left(x + \sqrt{1+x^2}\right)$$
$$L.H.S = R.H.S$$

(iii) $\tanh^{-1}(\cos\theta) = \cosh^{-1}(\operatorname{cosec}\theta)$

$$L.H.S = \tanh^{-1}(\cos\theta)$$
$$= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$
$$= \frac{1}{2}\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$$
$$= \log\left(\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\right)$$
$$= \log\left(\sqrt{\frac{\cancel{2}\cos^2\theta/2}{\cancel{2}\sin^2\theta/2}}\right)$$
$$= \log\left(\sqrt{\cot^2\theta/2}\right)$$
$$= \log(\cot\theta/2)$$

$$R.H.S = \cosh^{-1}(\operatorname{cosec}\theta)$$
$$= \log(\operatorname{cosec}\theta + \cot\theta)$$
$$= \log\left(\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)$$
$$= \log\left(\frac{\cancel{2}\cos^2\theta/2}{\cancel{2}\sin\theta/2\,\cos\theta/2}\right)$$
$$= \log(\cot\theta/2)$$

$$L.H.S = R.H.S$$

(iv) $\tanh^{-1}(\sin\theta) = \cosh^{-1}(\sec\theta)$

$$L.H.S = \tanh^{-1}(\sin\theta)$$
$$= \log\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$$
$$= \log\sqrt{\frac{(\cos\theta/2 + \sin\theta/2)^2}{(\cos\theta/2 - \sin\theta/2)^2}}$$
$$= \log\left(\frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}\right)$$
$$= \log\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)$$
$$L.H.S = \log\left[\tan\left(\frac{\pi}{4} + \theta/2\right)\right]$$

$$R.H.S = \cosh^{-1}(\sec\theta)$$
$$= \log(\sec\theta + \tan\theta)$$
$$= \log\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)$$
$$= \log\left(\frac{1+\sin\theta}{\cos\theta}\right)$$
$$= \log\left(\frac{(\cos\theta/2 + \sin\theta/2)^2}{\cos^2\theta/2 - \sin^2\theta/2}\right)$$
$$= \log\left(\frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}\right)$$
$$= \log\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)$$
$$R.H.S = \log\left[\tan\left(\frac{\pi}{4} + \theta/2\right)\right]$$

$$L.H.S = R.H.S$$