Indeterminate form

Form $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 1^{\infty}, 0^{0}, \infty^{0}$

Ex. 7.

1) $\lim_{x\to\frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2xe} \quad \left(\frac{0}{0}\right)$

Using L' Hospital Room

$$\lim_{x\to\frac{1}{2}} \frac{2\cos\pi x(-\sin\pi x)\pi}{e^{2x}\cdot 2 - 2e} = \lim_{x\to\frac{1}{2}} \frac{-\pi\sin 2\pi x}{2e^{2x} - 2e} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\frac{1}{2}} \frac{-2\pi^2\cos 2\pi x}{4e^{2x}} = \frac{-2\pi^2\cos 2\pi\times\frac{1}{2}}{4e^{2\times\frac{1}{2}}} = \frac{-2\pi^2\cos\pi}{4e}$$

$$= \frac{-\cancel{2}\pi^2(-1)}{{}_2\cancel{4}e} = \frac{\pi^2}{2e}$$

Show that

2) $\lim\limits_{x \to 0} \log_{\tan x} \tan 2x = 1$

$\underline{\text{Sol}^n}$

$$\lim_{x\to 0} \log_{\tan x} \tan 2x = \lim_{x\to 0} \frac{\log \tan 2x}{\log \tan x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan 2x} \sec^2 2x \cdot 2}{\frac{1}{\tan x} \sec^2 x}$$

$$= \lim_{x\to 0} \frac{\frac{\cancel{\cos 2x}}{\sin 2x} \cdot \frac{1}{\cos^{\cancel{2}} 2x} \cdot 2}{\frac{\cancel{\cos x}}{\sin x} \cdot \frac{1}{\cos^{\cancel{2}} x}} = \lim_{x\to 0} \frac{\frac{2}{\sin 2x \cdot \cos 2x}}{\frac{1}{\sin x \cos x}}$$

$$= \cancel{2\sin} \lim_{x\to 0} \frac{2\sin x \cos x}{\sin 2x \cos 2x} = \lim_{x\to 0} \frac{\cancel{\sin 2x}}{\cancel{\sin 2x} \cdot \cos 2x}$$

$$= \frac{1}{\cos 2\times 0} = \frac{1}{\cos 0^\circ} = 1$$

3) $\lim\limits_{x\to\infty}\left[\dfrac{1^{\frac{1}{x}}+2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}}{4}\right]^{4x}$ $\quad(1^{\infty})$

Using formula for 1^{∞} form $=\exp\lim\limits_{x\to\infty} g(x)\,[f(x)-1]$

$$=\exp\lim_{x\to\infty} 4x\left[\frac{1^{\frac{1}{x}}+2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}}{4}-1\right]$$

$$=\exp\lim_{x\to\infty} \cancel{4}x\left[\frac{1^{\frac{1}{x}}+2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}-4}{\cancel{4}}\right] \qquad [\infty\times 0]$$

$$=\exp\lim_{x\to\infty} \frac{1^{\frac{1}{x}}+2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}-4}{\frac{1}{x}} \qquad \left(\frac{0}{0}\right)$$

Using L' Hospital Rule

$$=\exp\lim_{x\to\infty} \frac{1^{\frac{1}{x}}\log 1\left(-\frac{1}{x^2}\right)+2^{\frac{1}{x}}\log 2\left(-\frac{1}{x^2}\right)+3^{\frac{1}{x}}\log 3\left(-\frac{1}{x^2}\right)+4^{\frac{1}{x}}\log 4\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$=\exp\lim_{x\to\infty} \frac{0\cancel{-\frac{1}{x^2}}\left[2^{\frac{1}{x}}\log 2+3^{\frac{1}{x}}\log 3+4^{\frac{1}{x}}\log 4\right]}{\cancel{-\frac{1}{x^2}}}$$

Now putting limit we get

$\exp\left[2^0 \log 2 + 3^0 \log 3 + 4^0 \log 4\right]$

$= \exp\left[\log 2 + \log 3 + \log 4\right] = \exp \log 2\times 3\times 4$

$= 24$

4) $\lim_{x\to \pi/2} (\cos x)^{\cos^2 x}$ $\quad (0^0)$

Sol^n

Let $y = \lim_{x\to \pi/2} (\cos x)^{\cos^2 x} \Rightarrow \log y = \lim_{x\to\pi/2} \log\left[\cos x\right]^{\cos^2 x}$

$\Rightarrow \log y = \lim_{x\to\pi/2} \cos^2 x \log \cos x \quad (0\times\infty)$

$= \lim_{x\to\pi/2} \dfrac{\log\cos x}{\frac{1}{\cos^2 x}} = \lim_{x\to\frac{\pi}{2}} \dfrac{\log \cos x}{\sec^2 x} \quad \left(\frac{\infty}{\infty}\right)$

Using L'Hospital Rul,

$= \lim_{x\to\pi/2} \dfrac{\frac{1}{\cos x}(-\sin x)}{2\sec^2 x \tan x} = \lim_{x\to\frac{\pi}{2}} \dfrac{-\cancel{\tan x}}{2\sec^2 x\cdot \cancel{\tan x}}$

$= \dfrac{-1}{2\sec^2 \pi/2} = \dfrac{-1}{\infty} = 0$

$\Rightarrow \log y = 0 \Rightarrow \boxed{y = e^0 = 1}$

1) $\lim_{x \to \infty} \left[\frac{ax+1}{ax-1}\right]^x$

$= \lim_{x \to \infty} \left[\frac{a+\frac{1}{x}}{a-\frac{1}{x}}\right]^x \qquad [1^\infty]$

using formula

$\exp \lim_{x \to \infty} g(x)\,[f(x)-1] \qquad$ where $g(x) = x$, $f(x) = \frac{a+\frac{1}{x}}{a-\frac{1}{x}}$

$\cancel{\exp \lim_{x\to\infty} x \left[\frac{a+\frac{1}{x}}{a-\frac{1}{x}}(1)\right]} = \exp \lim_{x \to \infty} x \left[\frac{a+\frac{1}{x}}{a-\frac{1}{x}} - 1\right]$

$= \exp \lim_{x \to \infty} x \left[\frac{\cancel{a}+\frac{1}{x}-\cancel{a}+\frac{1}{x}}{a-\frac{1}{x}}\right]$

ex

$= \exp \lim_{x \to \infty} \frac{\cancel{x} \times \frac{2}{\cancel{x}}}{a-\frac{1}{x}}$

$= \exp \lim_{x \to \infty} \frac{2}{a-\frac{1}{x}}$

$= \exp \frac{2}{a-0} = e^{\frac{2}{a}}$

2) $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$ ——— (1)

let $y = (1+x)^{\frac{1}{x}}$

$\log y = \frac{1}{x} \log(1+x)$

$= \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - - -\right)$

$\log y = \cancel{\frac{x}{x}}\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} - - -\right)$

$y = e^{1 - x/2 + x^2/3 - \frac{x^3}{4} - - - -}$

let $t = -\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + - -$

$y = e^{1+t} = e.e^t$

$Y = e\left[1 + t + \frac{t^2}{2!} + - - - -\right]$

$(1+x)^{\frac{1}{x}} = e\left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} - - - - + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} - -\right)^2 + -\right]$ — (2)

Put the value (2) in eqn (1)

$\lim_{x \to 0} \frac{e\left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} - - - - + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} - -\right)^2 + - - -\right] - e}{x}$

$$= \lim_{x \to 0} e \left[\frac{\cancel{1} - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} - \cdots + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3} \cdots\right)^2 - \cancel{1}}{x} \right]$$

$$= \lim_{x \to 0} \frac{e \cdot \cancel{x} \left[-\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} - \cdots \frac{1}{2!} x \left(-\frac{1}{2} + \frac{x}{3} - \cdots\right)^2 \right]}{\cancel{x}}$$

$$\Rightarrow e\left[-\frac{1}{2} + 0 - 0 - \cdots 0\right]$$

$$= -\frac{e}{2}$$

(4)

$$\lim_{x\to 0} \frac{x(1+a\cos x)-b\sin x}{x^3} = 1$$

$$\Rightarrow 1 = \lim_{x\to 0} \frac{x(1+a\cos x)-b\sin x}{x^3} \quad \left(\frac{0}{0}\right)$$

$$\cancel{\Rightarrow 1 = \lim_{x\to 0} \frac{0-a\sin x-b\cos x}{3x^2}}$$

$$\Rightarrow 1 = \lim_{x\to 0} \frac{x(0-a\sin x)+1\cdot(1+a\cos x)-b\cos x}{3x^2} \quad \text{--- (1)}$$

$$1 = \frac{1+a\cos 0-b\cos 0}{3\times 0} \quad \text{--- (2)}$$

$$\Rightarrow 0 = 1+a-b \quad \Rightarrow \quad a-b=-1 \quad \text{--- (II)}$$

using (2) in (1) we get $\left(\frac{0}{0}\right)$ again apply L'Hospital in (1)

$$\Rightarrow 1 = \lim_{x\to 0} \frac{-a\sin x - ax\cos x - a\sin x + b\sin x}{6x}$$

$$1 = \lim_{x\to 0} \frac{-2a\sin x - ax\cos x + b\sin x}{6x} \quad \left(\frac{0}{0}\right)$$

$$1 = \lim_{x\to 0} \frac{-2a\cos x - a\cos x + ax\sin x + b\cos x}{6}$$

$$1 = \lim_{x\to 0} \frac{-3a\cos x + ax\sin x + b\cos x}{6}$$

$$1 = \frac{-3a+b}{6}$$

$$\Rightarrow -3a+b = 6 \quad \text{———} \quad (3)$$

Solving (2) & (3), we get

$$a = -\frac{5}{2}, \quad b = -\frac{3}{2}$$