Module 1: Calculus-I

Infographics

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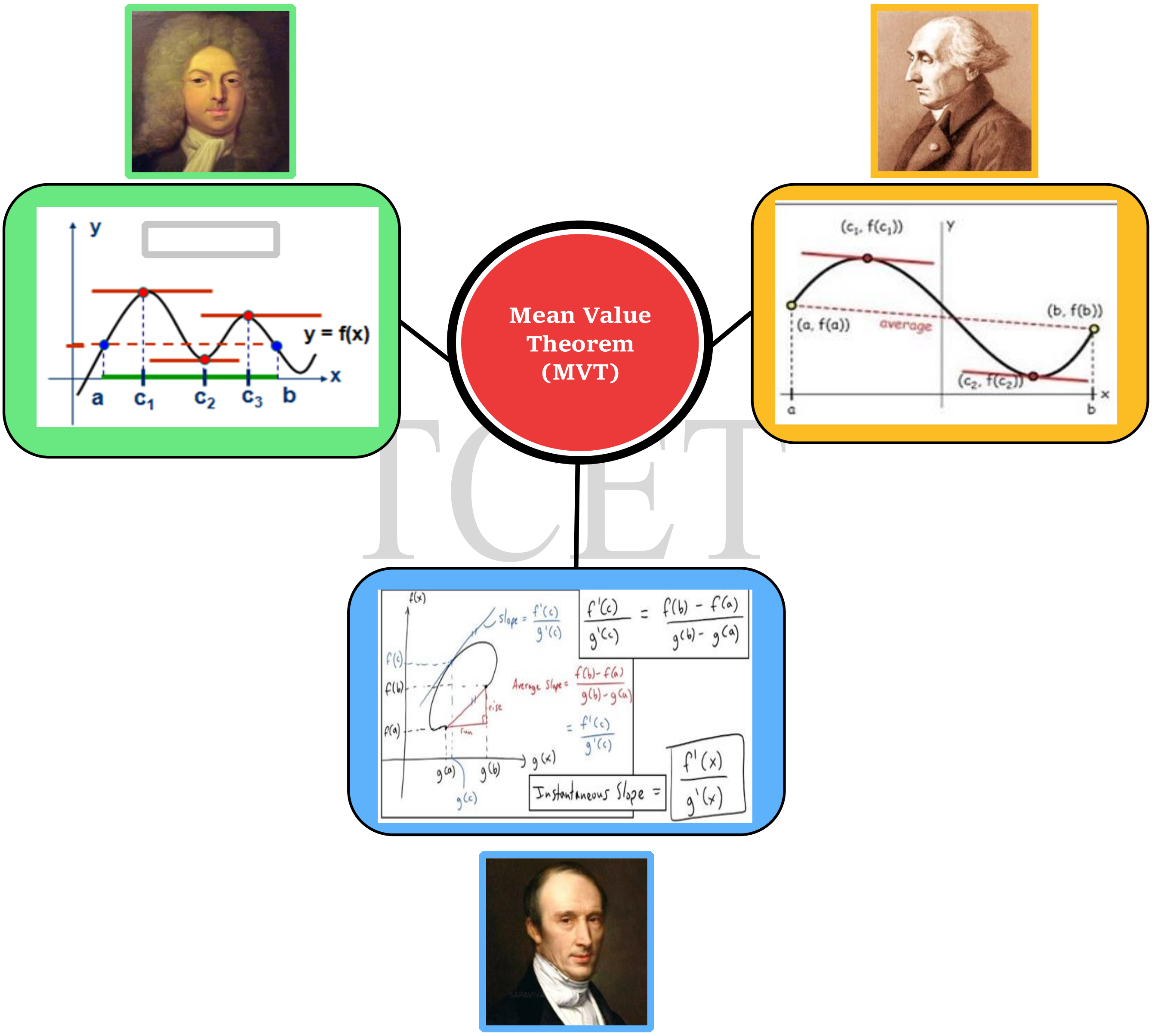


Augustin-Louis Cauchy

Module 1: Calculus-I

1.Motivation:

Mean Value Theorem is one of the important topics of the calculus. This theorem relates the slope of the tangent at one point of the arc of the curve to the slope of its secant. In other word mean value theorem relates the average value of the function with it derivative. Mean value theorem can be applied in the comparison of instantaneous velocity with the average velocity in its engineering applications. The mean value theorem is a very important result in Real Analysis and is very useful for analyzing the behavior of functions in higher mathematics.

The operation of adding infinite term is called infinite series. In mathematics the enumerated no of objects when added they give the structure of infinite series which are required for the study of finite structures with help of generating functions. Infinite series emphasizes on methods for discussing convergence and divergence of series. It also helps in expanding different functions which can be expressed into convergent series.

There are seven indeterminate forms which arise in the algebraic calculation of functions. Indeterminate forms generally contain the 0,1 or∞. We solve this issue of indeterminate form by the concept of limits. The most common indeterminate form arises in the evaluation of limit of ratios. Indeterminate forms can be determined by applying L'Hospital rule.

2.Syllabus:

Lecture No.	Title	Duration (Hrs.)	Self-study (Hrs.)
1	Rolle's Mean value theorem	1	2
2	Lagrange's Mean value theorem	1	2
3	Cauchy's Mean value theorem	1	2
4	Taylor's series	1	2
5	Maclaurin series	1	2
6	Maclaurin theorems	1	2
7	Indeterminate forms	1	2
8	Indeterminate forms continued	1	2
9	De 'Alembert's ratio test and Cauchy's n^{th} root test	1	2
10	Cauchy's Integral test	1	2

3. **Prerequisite:** Concept of limits and its properties, concept of differentiability, continuity, Concept of differentiation, Sequence & Series.

4. **Learning Objective:** Learners shall be able to

1. Understand the Rolle's mean value theorem and apply it to solve the problems
2. Understand the Lagrange's theorem and apply it to solve the problems.
3. Understand the Cauchy's mean value theorem and apply it to solve the problems
4. Expand the function in the Taylor'sand McLaurin series.
5. Find the limits of algebraic function which gives the Indeterminate forms.
6. Test the convergence and divergence of Infinite series.

Rolle'sMean value theorems
Lecture : 01

1. Learning Objective:
Student shall be able to Understand and apply Role's mean value theorem.

2. Introduction: The mean value theorem tells us (roughly) that if we know the slope of the secant line of a function whose derivative is continuous, then there must be a tangent line nearby with that same slope. This lets us draw conclusions about the behavior of a function based on knowledge of its derivative.

3. Key Definitions:

Role's Theorem: If a function f(x) is

(i) f (x) is continuous on the closed interval [a,b]

(ii) f (x) is differentiable on the open interval (a, b)

(iii) f(a)=f(b)

then there exists at least one point c in (a, b) (i.e.a < c < b) such that f ′(c)=0.

Alternative or Another Statement of Rolle's Theorem: If a function $f(x)$ is

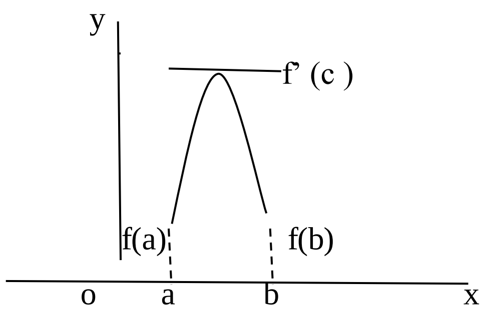
(i) continuous in $[a, a+h]$

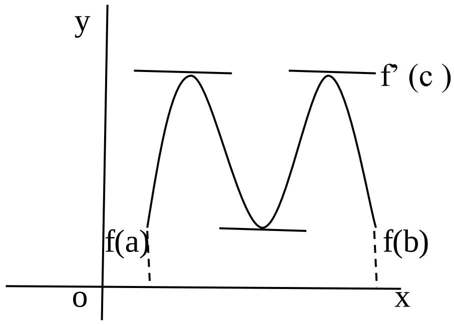
(ii) differentiable in $(a, a+h)$

(iii) $f(a) = f(a+h)$,

then there exists at least one real number θ such that
$f'(a+\theta h)=0$, $for\ \ 0<\theta<1$.

Geometrical Interpretation of Rolle's Theorem:
If graph of the function is a continuous curve between $x=a$ and $x=b$ having a unique tangent at all points in (a,b) and $f(a) = f(b)$ then there exists at least one point P between $x=a$ and $x=b$ on the curve, such that the tangent at this point is parallel to X – axis i.e. $f'(c) = 0$.





Algebraic Interpretation of Rolle's Theorem:

Let f(x) be a polynomial in x and let the two roots of f(x)=0 be x=a and x=b. Then according to Rolle's Theorem, at least one root of $f^{'}(x)=0$ lies between a and b.

3. Sample Problems

(1) Verify the Rolle's theorem for

$f(x)=(x-a)^m\ (x-b)^n\ \ in\ (a,b)$, where m, n are +ve integers

Solution:(i) Since m, n are positive integers,

$f(x)=(x-a)^m(x-b)^n$, being a polynomial, thus

$f(x)$ is continuous in [a,b].

(ii) $f(x)$ is differentiable in (a,b) as it is a polynomial

$$f'(x)=m\ (x-a)^{m-1}(x-b)^n+n\ (x-a)^m\ (x-b)^{n-1}$$

$$f'(x)=\ (x-a)^{m-1}(x-b)^{n-1}[\ m\ (x-b)+n\ (x-a)]$$

$$f'(x)=\ (x-a)^{m-1}(x-b)^{n-1}[(m+n)x-(mb+na)]$$

f ′(x) exists for every value of x in (a,b). Therefore $f(x)$ is differentiable in (a, b).

(iii) $f(a)=f(b)=0$

Thus, f(x) satisfied all the conditions of Roll's theorem. Hence, Rolle 's Theorem is applicable.

Therefore, there exists at least one point c in (a,b) such that $f'(c)=0$

$$f'(c)=m\ (c-a)^{m-1}(c-b)^n+n\ (c-a)^m\ (c-b)^{n-1}=0$$

$$f'(c)=\ (c-a)^{m-1}(c-b)^{n-1}[\ m\ (c-b)+n\ (c-a)]=0$$

$$\therefore mc-mb+nc-na=0\Rightarrow(m+n)c=mb+na$$

$\therefore c = \frac{m\,b+n\,a}{m+n}$

which represents a point c that divides the interval [a, b] in the ratio of m: n,

Thus, c lies between a and b, i.e. a<c<b. Hence Rolle's theorem is verified.

(2) Prove that the equation $2x^3 - 3x^2 - x + 1 = 0$ has at least one root between 1 and 2.

Solution: Consider the function $f'(x) = 2x^3 - 3x^2 - x + 1 = 0$

$f(x) = \int (2x^3 - 3x^2 - x + 1)dx = \frac{x^4}{2} - x^3 - \frac{x^2}{2} + x$

Deliberately ignoring the integral constant

$f(1) = \frac{1}{2} - 1 - \frac{1}{2} + 1 = 0$ and

$$f(2) = \frac{2^4}{2} - 2^3 - \frac{2^2}{2} + 2 = 8 - 8 - 2 + 2 = 0$$

Hence two roots of f(x) are 1 and 2

Therefore, By Algebraic interpretation of Rolle's Theorem, we can say that f'(x) has at least one root between the roots of f(x).

(3) Verify Rolle's theorem for $f(x) = x\,(x-2)e^{\frac{3x}{4}}$ in (0,2)

Solution: f(x) is continuous in [0,2]

f(x) is differentiable in (0,2)

f(0) = 0, f(2)=0

let f′ (c)=0 $\Rightarrow$ $f'(c) = \left\{\left[(x-2) + x + \frac{3}{4}x(x-2)\right]e^{\frac{3}{4}x}\right\}|_{x=c} = 0$

$$\Longrightarrow (3c)^2 + 2c - 8 = 0 \Longrightarrow c = -2 \; or \frac{8}{6}$$

But c =-2 does not lie in (0,2) thus c=8/6 $\in (0,2)$

Hence Rolle's theorem is verified.

Exercise 1

1. Verify Rolle's Theorem

$(i) f(x) = \frac{\sin x}{e^x} in [0, \pi]$

$(ii) f(x) = 1 - 3(x-1)^{\frac{2}{3}} \; in 0 \le x \le 2$

$(iii)\; f(x) = e^{-x}(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

$(iv) f(x) = \begin{cases} x^2+1, & 0 \le x \le 1 \\ 3-x, & 1 \le x \le 2. \end{cases}$

2. Use Rolle's Theorem to prove that the equation $ax^2 + bx = \frac{a}{3} + \frac{b}{2}$ *has a root between* 0 *and* 1.

Let's Check away from lecture

1. *If* $y=(x-1)^3(x-2)^2$ *in* $[1,2]$ *then c by R.M.V.T. theorem is*

   $(a)\ \frac{8}{5}\quad (b)\ \frac{1}{2}\quad (c)\ \frac{3}{4}\quad (d)$ *None*

2. *If* $y=\cos x$ *in* $[-\pi,\pi]$ *then c by R.M.V.T. theorem is*

   $(a)\ \pi\quad (b)\ 0\quad (c)\ \frac{\pi}{4}\quad (d)$ *None*

3. If $f(x) = (x+2)^3(x-3)^4$ in [-2, 3] then c of R.M.V.T. is

   (a) 1/5 (b) 1/6 (c) 1/7 (d) 1/8

4. If $f(x) = \ln\left\{\frac{x^2+6}{5x}\right\}$ in [2, 3] then c of R.M.V.T. is

   (a) $\sqrt{5}$ (b) $\sqrt{6}$ (c) $\sqrt{7}$ (d) $\sqrt{3}$

5. Is Rolle's theorem is applicable for $f(x) = x^2$ in [1,2]

   (a) Applicable (b) Not Applicable (c) Cannot be said

Homework Problems for the day

1. Verify Rolle's Theorem

$(i) f(x) = \tan x,\quad 0 \le x \le \pi \qquad (ii) f(x) = \log\left[\frac{x^2+ab}{(a+b)x}\right] in[a,b],\quad a>0, b>0.$

$(iii) f(x) = |x|\ in[-1,1]$

2. $If\ \ f(x) = x(x+1)(x+2)(x+3), then\ show\ \ that\ f'(x)\ has\ three\ real\ roots.$

3. Show that between any two roots of $e^x cosx - 1 = 0$ there exist at least one root of $e^x sinx - 1 = 0$.

4. Apply the Rolle's theorem and find the value of c for $f(x) = x^3 - 4x$ in the interval [-2,2].

Learning from the topic: Learner will be able to apply Rolle's mean value theorem.

Lagrange’s mean value theorem

Lecture : 02

1. Learning Objective:
Student shall be able to Understand and apply Lagrange’s mean value theorem.

2. Key Definitions:

Lagrange’s mean value theorem

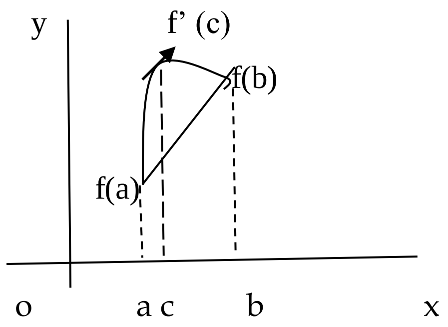
Suppose f(x) is a function that satisfies the following.

(i) f(x) is continuous on the closed interval [a, b]

(ii) f(x) is differentiable on the open interval (a, b)

Then there is a number c such that a < c < b and $f'(c) = \frac{f(b)-f(a)}{b-a}$

Geometrical Interpretation of L.M.V.T.: If curve is continuous from a point A to a point B and has tangent at every point on it then there exists at least one-point c between A and B, such that tangent at this point is parallel to the chord AB.



3. Sample Problems

(1) Verify Lagrange’s mean value theorem for the function $f(x) = x^3 - 2x^2 - 3x - 6$ in $[-1, 4]$.

Solution: We have $f(x) = x^3 - 2x^2 - 3x - 6$ in $[-1, 4]$, f (-1) = -6 and f(4)=14

Since f(x) is polynomial, therefore continuous hence f(x) is continuous in $[-1, 4]$.

$f'(x) = 3x^2 - 4x - 3$ hence f(x) is differentiable in $(-1, 4)$.

All the conditions of Lagrange's mean value theorem are satisfied.

Therefore, there exists $c \in (-1,4)$ such that

$$f^{'}(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(-1)}{4 — 1}$$

$$\Rightarrow 3c^2 - 4c - 3 = \frac{f(4) - f(-1)}{4 - (-1)} = \frac{14 - (-6)}{5} = 4$$

$$\Rightarrow 3c^2 - 4c - 7 = 0$$

$$\Rightarrow \quad (3c - 7)(c + 1) = 0$$

$$\Rightarrow \quad c = \frac{7}{3}, -1$$

The LMVT guarantees that c∈ (-1,4), and since $7/3 \in (-1,4)$

Hence, Lagrange's mean value theorem is verified. Here, x=−1 is excluded since $-1 \notin (-1,4)$.

(2) Using Lagrange's mean value theorem, show that $\sin x \le x$ for $x \ge 0$.

Solution: We have f(x) = x-sin x defined in [0, x].

f (0) = 0 and f(x)=x-sin x

clearly, f(x) is everywhere continuous and differentiable. so

(i) f(x) is continuous in [0, x].

(ii) f′(x)=1-cos x

= f(x) is differentiable in (0, x).

Thus, both the conditions of Lagrange's mean value theorem are satisfied.

$c \in (0, x)$ such that

$f^{'}(c) = \frac{f(x) - f(0)}{x - 0}$, $1 - \cos c = \frac{x - \sin x}{x - 0}$ since $1 - \cos c \ge 0 \ \forall x$

Thus, $\frac{x - \sin x}{x - 0} \ge 0$

$= x \ge \sin x$ Hence $\sin x \le x$ for all $x \ge 0$

(3). Show that for any x≥ 0 , $1 + x < e^x < 1 + x\, e^x$

Solution: Take $f(x) = e^x - (1 + x)$

$$f'(x) = e^x - 1 \ge 0 \ \ for\ x \ge 0$$

so f(x) is increasing function and f(0)=0

$f(x) > 0$ for any x≥ 0

$e^x - (1 + x) > 0\ or\ \ 1 + x < e^x \ldots(1)$

Now consider the function

$g(x) = 1 + xe^x - e^x \Rightarrow g(0) = 0$ *and* $g'(x) = e^x + xe^x - e^x = xe^x \geq 0$ for any $x \geq 0$.
Thus, g(x) is increasing function and therefor g(x) >0
$\Rightarrow 1 + xe^x - e^x > 0 \Rightarrow e^x < 1 + xe^x$ … (2)

From equation (1) and (2) we can say that $1+x<e^x<1+x\,e^x$.

Exercise 2

1. Verify Lagrange's mean value theorem for the function $f(x) = x^2+x-1$ in [0, 4].

2. Using Lagrange's mean value prove that $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2}$,

& *hence deduce that* $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$.

3. If $0 < a < b$, prove that $\left(1 - \frac{a}{b}\right) < \log\frac{b}{a} < \left(\frac{b}{a} - 1\right)$ Hence, prove that $\frac{1}{4} < \log\frac{4}{3} < \frac{1}{3}$

Let's check away from lecture

1. *If* $y = (x-1)^{\frac{3}{2}}$ *in* $[1,2]$ *then c by L.M.V.T. theorem is*

   (a) $\dfrac{1}{2}$ (b) $\dfrac{1}{4}$ (c) $\dfrac{3}{4}$ (d) *None*

2. *If* $y = \cos x$ *in* $[0, 2\pi]$ *then c by L.M.V.T. theorem is*

   (a) π (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{4}$ (d) *None*

3. If $f(x) = x(x-1)(x-2)$ in [0,1/2] then c of LMVT is.

   (a) 0.256 (b) 0.156 (c) 0.236 (d) 0.276

Homework Problems for the day

1. Verify Lagrange's mean value theorem for the function
   (i) $f(x) = x^{\frac{2}{3}}$ *in* $[-8, 8]$.
   (ii) $f(x) = 2x^2 - 7x - 10$ over $[2,5]$ and find c using LMVT
2. Apply Lagrange's mean value theorem for the function log(x) in [a, a+h] & determine θ in terms of a and h and deduce that $0 < \dfrac{1}{\log(1+x)} - \dfrac{1}{x} < 1.$
3. If $f(x) = sin^{-1}x$ [0,1] then find c by using LMVT.
4. If $0 < a < b$, prove that $\left(1 - \frac{a}{b}\right) < \log\frac{b}{a} < \left(\frac{b}{a} - 1\right)$ Hence, prove that $\frac{1}{4} < \log\frac{4}{3} < \frac{1}{3}$

Learning from the topic: Learner will be able to apply Lagrange's mean value theorem.

Cauchy's mean value theorem

Lecture : 03

1. Learning Objective:

Student shall be able to understand and apply Cauchy's mean value theorem.

2. Key Definitions:

Cauchy's mean value theorem

Let functions f(x) and g(x) be

(i) Continuous on [a,b]

(ii) Differentiable on the interior of (a,b).

further that $g'(x) \neq 0, \forall\ x \in (a,b)$.

There exists a point $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

5. **Sample Problems**

(1). If $f(x) = e^x$ and $g(x) = e^{-x}$, prove that **c** of Cauchy's mean value theorem is the Arithmetic mean between a and b, a > 0, b > 0.

Solution: f(x) and g(x), being exponential function, are continuous on [a,b] and $f'(x) = e^x, g'(x) = e^{-x}$ *exists for all x in* (a, b)

therefore $f(x) = e^x$ and $g(x) = e^{-x}$, differentiable on (a,b) and $g'(x) \neq 0$ for any c in (a,b)

Thus, f(x) and g(x) satisfies all the conditions of Cauchy's Mean Value theorem, there exists at least one point c in (a,b).

By Cauchy's Theorem $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, a < c < b$

$\therefore f(x) = e^x, f'(x) = e^x$ and $\quad \therefore g(x) = e^{-x}, f'(x) = -e^{-x}$

$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}, a < c < b \Rightarrow \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -e^{2c},\ a < c < b,$

$\therefore -e^{a+b} = -e^{2c}$.

$\Rightarrow \therefore\ a + b = 2c \Rightarrow \therefore c = \frac{a+b}{2}$

Hence, c is the A.M. between a and b.

(2). Using Cauchy's mean value theorem,

show that $\frac{\sin b - \sin a}{\cos a - \cos b} = \cot c \ w\square ere \ \ a < c < b$,a>0, b>0

Solution: We have to prove that $\frac{\sin b - \sin a}{\cos a - \cos b} = \cot c$

Let f(x)=sin x and g(x)=cos x are defined in the interval (a, b) such that f(x) and g(x) , being trigonometric functions, be continuous on [a,b] and differentiable on (a,b) with g′(x)≠0 for any c in (a,b).

By Cauchy's Theorem$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \quad a < c < b$

$$\frac{\sin b - \sin a}{\cos b - \cos a} = -\frac{\cos c}{\sin c} = -\cot c \quad where\ a < c < b,$$

Hence, $\frac{\sin b - \sin a}{\cos a - \cos b} = \cot c \ \ where \ \ a < c < b,$

Exercise 3

1. $If \ \ f(x) = \sqrt{x} \, and \ \ g(x) = \frac{1}{\sqrt{x}},$

   $then \, prove \, that \, "\ c\ " of \ CMVT \ is \ geometric \ mean \ between \ a \ \& \ b$

2. Using appropriate MVT prove that $\frac{\text{sinb-sina}}{e^b - e^a} = \frac{\text{cosc}}{e^c}$ for a<c<b.

   Hence deduce that $e^c \sin x = (e^x - 1)\cos c$.

3. $If \ U = e^x \ and \ \ V = e^{-x} \ in \ [1,2] \ \ then \ \ c \ \ by \ \ \ C.M.V.T.is$

   $(a)\frac{3}{2}(b)\frac{7}{4}(c)\frac{5}{3}(d)none$

2. $If \ \ U = \log x \, and \, V = \cos x \ \ in \ [1,2] \, then \ \ can \ \ C.M.V.T. theorem \, is \, applicable?$

   $(a) \ applicable \ \ \ (b) not \ applicable \ (c) cannot \ be \ said$

4. If f(x)=$\sqrt{x}$ and g(x)=$\frac{1}{\sqrt{x}}$ in [a,b] then c of C.M. V.T. is

   (a) $\sqrt{ab}$ (b) $\sqrt{\frac{a}{b}}$ (c) ab (d)$\frac{a}{b}$

5. If f(x)=x^2 and g(x)=$\sqrt{x}$ in [1,4] then c of C.M. V.T. is

   (a) $\left(\frac{15}{4}\right)^{\frac{3}{2}}$ (b) $\left(\frac{15}{4}\right)^{\frac{1}{2}}$ (c) $\left(\frac{15}{4}\right)^{\frac{2}{3}}$ (d) $\left(\frac{15}{4}\right)^{\frac{1}{3}}$

6. Lagrange's Mean Value Theorem for $f(x)=\log_e x$ in the interval [1,e] is

(a) False (b) Not always false (c) True (d) Not always true

Homework Problems for the day

1. Verify Cauchy's mean value theorem for

   $(i)\ x^2\ \&\ x^4\ in\ [a,b]\ where\ a>0,\ b>0,$ (ii) $sin\,x\ \&\ cos\,x\ in\ \left[0,\frac{\pi}{2}\right]$

   (iii) $x^2\ \&\ x^3\ in\ [1,2]$

2. *If* $f(x)=\frac{1}{x^2}$ *and* $g(x)=\frac{1}{x}$, *prove that c of CMVT is the harmonic mean between a and b.*

3. Verify Cauchy's MVT for $2x^3$ and x^6 in [a,b] and find c.

Learning from the topic: Learner will be able to apply Cauchy's mean value theorem.

Taylor's and Maclaurin's series

Lecture: 04

1. Learning Objective:

Student shall be able to find the Taylor's series expansion of a function.

2. Introduction:

The Taylor series can be used to calculate the value of an entire function at every point, if the value of the function, and of all its derivatives, are known at a single point. In some engineering problems of integral calculus, it is difficult to find the value of integral by usual means. In such cases by expanding the function into convergent series of infinite terms one can overcome that problem of integration. In similar way expansion of the function plays an important role in finding Laplace Transform of the function.

3. Important Formulae / Theorems / Properties/Definitions:

(1)Taylor's Series: The Taylor series of a real or complex-valued function f (x) that is infinitely differentiable at 'a' real or complex number a is the power series

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots+\frac{(x-a)^n}{n!}f^n(a)+\ldots$$

(2) Alternative form of Taylor's Series: If $x-a=h$ then

$$f(a+h) = f(a)+h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots\ldots\ldots\ldots\ldots + \frac{h^n}{n!} f^n(a) + \ldots\ldots.$$

4. Important Steps to be followed for solving the problems:

- While finding the Taylor's Series expansion of the function one should identify first that expansion is required in powers of which factor i.e. to identify the center of the power series.
- According to the center of power series one should use appropriate form of the Taylor's Series to expand the function.

5. Sample problems

1. Express f(x) = $2x^3 + 3x^2 - 8x + 7$ in powers of (x-2).

Solution: Let $f(x) = 2x^3 + 3x^2 - 8x + 7$ and $a = 2,\ f(2) = 34$

$$\therefore f'(x) = 6x^2 + 6x - 8, f'(2) = 28$$

$$f''(x) = 12x + 6, f''(2) = 30$$

$$f'''(x) = 12, \quad f'''(2) = 12$$

Hence,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \ldots\ldots\ldots\ldots$$

$$\therefore \mathrm{f(x)} = 2\mathrm{x}^3 + 3\mathrm{x}^2 - 8\mathrm{x} + 7 = 34 + (\mathrm{x}-2) \times 28 + (\mathrm{x}-2)^2 \times 30 + (\mathrm{x}-2)^3 \times 12$$

$$= 34 + 28(\mathrm{x}-2) + 30(\mathrm{x}-2)^2 + 12(\mathrm{x}-2)^3$$

2. Using Taylor's theorem evaluate up to4 places of decimals$\sqrt{25.15}$

Solution: By Taylor's series expansion we have,

$$f(x+h) = f(x)+h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots \qquad (1)$$

Here $f(x) = \sqrt{x} \qquad \therefore f(x+h) = \sqrt{x+h}$

Now $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$; $f''(x) = -\frac{1}{4}x^{-\left(\frac{3}{2}\right)}$; $f'''(x) = \frac{3}{8}x^{-\left(\frac{5}{2}\right)}$

therefore, equation (1) becomes:

$$\sqrt{x+h} = \sqrt{x} + \mathrm{h}\,\frac{1}{2\sqrt{x}} + \frac{h^2}{2!}\left(-\frac{1}{4}x^{-\left(\frac{3}{2}\right)}\right) + \frac{h^3}{3!}\,\frac{3}{8}x^{-\left(\frac{5}{2}\right)} + \ldots$$

Now set x = 25 and h = 0.15 which gives

$$\sqrt{25+0.15} = \sqrt{25} + 0.15\frac{1}{2\sqrt{25}} + \frac{(0.15)^2}{2!}\left(-\frac{1}{4}25^{-\left(\frac{3}{2}\right)}\right) + \frac{(0.15)^3}{3!}\frac{3}{8}25^{-\left(\frac{5}{2}\right)} + \ldots$$

$$\sqrt{25+0.15} = 5.0150$$

(3) Expand sin x in powers of $\left(x-\frac{\pi}{2}\right)$.

Solution: Let $f(x) = \sin x$ and $a = \frac{\pi}{2}, f(a) = \sin\left(\frac{\pi}{2}\right) = 1$

$$\therefore f'(x) = \cos x, f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0, \quad f''(x) = -\sin x, f''\left(\frac{\pi}{2}\right) = -1$$

$$f'''(x) = -\cos x, f'''\left(\frac{\pi}{2}\right) = 0, \quad f^{iv}(x) = \sin x, f^{iv}\left(\frac{\pi}{2}\right) = 1$$

Hence by Taylor's Series we get,

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \ldots\ldots\ldots\ldots$$

$$f(x) = f\left(\frac{\pi}{2}\right) + \left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!}f'''\left(\frac{\pi}{2}\right)$$

$$+ \frac{\left(x-\frac{\pi}{2}\right)^4}{4!}f^{iv}\left(\frac{\pi}{2}\right) + \ldots\ldots\ldots\ldots$$

$$\therefore \sin x = 1 + \left(x-\frac{\pi}{2}\right)(0) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}(-1) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!}(0) + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!}(1) + \ldots\ldots$$

$$\therefore \sin x = 1 + -\frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} + \ldots\ldots$$

Exercise 4

1. Expand $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$ in powers of $(x-1)$ and find f(0.99).

   Ans.: $f(x) = 3(x-1) + 6(x-1)^2 + 7(x-1)^3 + 4(x-1)^4 + (x-1)^5, -0\cdot0294$

2. By using Taylor's series expand $\tan^{-1}(x)$ in positive powers of (x-1) up to first four nonzero terms.

Ans : $\tan^{-1}(x) = \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + .\frac{(x-1)^3}{12}\ldots\ldots\ldots\ldots$

3. Arrange in powers of x, by Taylor's theorem $f(x+2) = 7 + (x+2) + 3(x+2)^3 + (x+2)^4$

   Ans.: $49 + 69x + 42x^2 + 11x^3 + x^4$.

Let's check take away from lecture

Choose the correct option from the following:

1. *By Taylors series* $f(x+h)$

(a) $f(x)+hf'(x)+\frac{h^2}{2!}f''(x)....$ (b) $1+hf(x)+\frac{h^2}{2!}f'(x)+\frac{h^3}{3!}f''(x)....$

(c) $f(h)+hf'(x)+\frac{h^2}{2!}f''(h)....$ (d) *none*

2. In Taylors Series expansion of function in powers of (x-a) centre of power series is:

(a) x=1 (b) x=0 (c) x=a (d) none of these

3. Expand f(x) = x^2+x-1 in powers of $(x-2)$ using Taylors Series.

(a) $(x-2)^2-5(x-2)+5$ (b) $(x-2)^2+5(x-2)-5$ (c) $(x-2)^2+5(x-2)+5$ (d) none of these

4. If $f(x) = x^3 + 3x^2 + 15x - 10$ then the value of $f\left(\frac{11}{10}\right)$ is

(a) 11.461 (b) 10.327 (c) 11.321 (d) 8.724

6. The Taylor's series expansion of $f(x) = e^x$ around x=3 is

(a) $e^3\sum_{n=0}^{\infty}\frac{(x-3)^n}{n!}$ (b) $e^x\sum_{n=0}^{\infty}\frac{(x-3)^n}{n!}$ (c) $e^3\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$ (d) $e^3\sum_{n=0}^{\infty}(x-3)^n$

Homework Problems for the day

1. Expand $\tan^{-1}(x+h)$ in powers of h and hence find the value of $\tan^{-1}(1.003)$ upto 5 places of decimal.

Ans.: 0.78690.

2. Using Taylor's theorem evaluate upto 4 places of decimals.

(i) $\sqrt{1.02}$ (ii) $\sqrt{10}$

Ans.:i) 1.0099 ii) 3.16227

3. Expand log (cosx) about $\frac{\pi}{3}$ using Taylor's expansion.

4. Expand $(2x^3+7x^2+x-1)$ in powers of (x-2)

5. Find the Taylor's series of $f(x) = \log cosx$, around $x = \frac{\pi}{3}$.

6. If $y = \sin log(x^2 + 2x + 1)$,Expand y in ascending powers of x upto x^6.

Learning from the topic: Student will be able to expand a function of single variable into positive ascending integral powers (x-a) using Taylor's Series.

McLaurin's Series

Lecture:05

1. Learning Objective:

Student shall be able to find the Maclaurin's series expansion of any type of function.

2. Introduction:
The Taylor's Series expansion which are done around the x=0 is called Maclaurin's series.

3. Key Definitions:

(1) **Maclaurin's Series**: If f(x) be a given function of x which can be expanded into a convergent series of positive ascending integral powers of x then,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots$$

4. Important Formulae / Theorems / Properties: Standard Expansions:

$(i)\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\ldots\ldots\ldots$

$(ii)\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots\ldots$

$(iii)\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots\ldots$

$(iv)\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots\ldots\ldots$

$(v)\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\ldots\ldots\ldots$

$(vi)\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots\ldots\ldots$

$(vii)\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots\ldots\ldots\ldots$

$(viii)\ \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots$

$(ix)\ \sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \ldots\ldots$

$(x)\ \cos^{-1} x = \frac{\pi}{2} - \left\{x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \ldots\ldots\right\}.$

$(xi)\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots\ldots\ldots$

$(xii)\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\ldots\ldots\ldots$

$(xiii)(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots\ldots\ldots\ldots$ *In particular if* $m = -1$ *then*

$(a)\ (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots$

$(b)\ (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots$

5. Important Steps to be followed for solving the problems:

i. If the required expansion is in powers of x then it is identified as problem of Maclaurin's Series.

ii. While finding the Maclaurin's Series expansion of the function one should identify first that which standard expansion can be used to get the desire series.

iii. Secondly if it is found difficult to use appropriate standard expansion then one can apply Maclaurin's Series to the given function which is to be expanded.

6. Sample problems

(1). Prove that $log(1+e^x) = log\,2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \ldots.$

Solution: Let $f(x) = log(1+e^x)$

$$f'(x) = \frac{e^x}{(1+e^x)} \quad f''(x) = \frac{e^x}{(1+e^x)^2} \quad f'''(x) = \frac{e^x - e^{2x}}{(1+e^x)^3}$$

$$f^{iv}(x) = \frac{(1+e^x)(e^x - 2e^{2x}) - 3e^x(e^x - e^{2x})}{(1+e^x)^4}$$

$$f(0) = \log 2,\ f'(0) = \frac{1}{2},\ f''(0) = \frac{1}{4},\ f'''(0) = 0,\ f^{iv}(0) = \frac{-1}{8}.$$

Using the Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots.. + \frac{x^n}{n!} f^n(0) + \ldots\ldots.$$

$$\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \ldots.$$

(2).Show that $\text{x cosec x} = 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \ldots\ldots\ldots..$

Solution: $\text{x cosec x} = \frac{x}{\sin x} = \frac{x}{x - \frac{x^3}{3!} - \frac{x^5}{5!}\ldots..}$ (using std. expansion of sinx)

$$= \frac{1}{1 - \frac{x^2}{6} + \frac{x^4}{120}\ldots..} = \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120}\right)\ldots.\right]^{-1}$$

$$= 1 + \frac{x^2}{6} - \frac{x^4}{120} + \ldots.\left(\frac{x^2}{6}\ldots..\right)^2 \text{ (Neglecting higher power)}$$

$$= 1 + \frac{x^2}{6} + \left(\frac{1}{36} - \frac{1}{120}\right)x^4\ldots\ldots \quad = 1 + \frac{x^2}{6} + \left(\frac{10-3}{360}\right)x^4\ldots\ldots$$

$$= 1 + \frac{x^2}{6} + \frac{7x^4}{360}\ldots\ldots$$

(3). Show that $e^x\,cosx = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!}\ldots\ldots$

Solution: Given function $f(x) = e^x \cos x \Rightarrow f(0) = 1$

$$f'(x) = e^x \cos x - e^x \sin x \Rightarrow f'(0) = 1$$

$$f''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x \Rightarrow f''(0) = 0$$

$$f'''(x) = -2e^x \cos x - 2e^x sinx \Rightarrow f'''(0) = -2$$

$$f^{iv}(x) = -4e^x \cos x \Rightarrow f^{iv}(0) = -4$$

By Taylor's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) \ldots$$

$e^x cosx$ =1+ x $-\frac{2x^3}{3!} - \frac{2^2x^4}{4!} \ldots$

Exercise 5

1. *Expand* 5^x *upto the first three nonzero terms of the series.*

2. *Expand* $\sqrt{1+\sin x}$. *Ans* : $1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384}$.........

3. Pr*ove that* $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\ldots\ldots\ldots.$

Lec't check away from lecture

Choose the correct option from the following:

1. Using Maclaurin's theorem we can expand a^x as:

(a) $1 - x\log a - \frac{(x\log a)^2}{2!} - \ldots$ (b) $1 - x\log a + \frac{(x\log a)^2}{2!} - \ldots$

(c) $1 + x\log a + \frac{(x\log a)^2}{2!} + \ldots$ (d) $1 - x\log a + \frac{(x\log a)^2}{2!} - \ldots$

2. *Expansion of* $x\cos x$ *is* :

(a) $x - \frac{x^3}{2!} + \frac{x^5}{4!} - \ldots$ (b) $x + \frac{x^3}{2!} + \frac{x^5}{4!} + \ldots$ (c) $x - \frac{x^3}{2!} - \frac{x^5}{4!} - \ldots$ (d) *none of these*

Homework Problems for the day

1. Pr*ove that* $\sin\left(e^x - 1\right) = x + \frac{x^2}{2} - \frac{5x^4}{24} + \ldots\ldots.$

2. Pr*ove that* $\log(1+\tan x) = x - \frac{x^2}{2} + \frac{2x^3}{3} + \ldots\ldots\ldots$

3. *Obtain the series for* $\log(1+x)$ *and hence find the series of* $\log_e\left(\frac{1+x}{1-x}\right)$ *and hence find the value of* $\log_e\left(\frac{11}{9}\right)$.

Learning from the topic: Learning from the topic: Student will be able to expand a function of single variable into positive ascending integral powers 'x' using Maclaurin's Series.

MacLaurin's Series continue

Lecture: 06

1. Sample problems

(1). Pr o ve that $sec^2 x = 1 + x^2 + \frac{2x^4}{3} + \ldots\ldots..$

Solution: $\sec^2 x = \left[\frac{1}{\cos x}\right]^2 = \left[\frac{1}{(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\ldots.)}\right]^2$

Using $[1 - x]^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots.$ We get

$\sec^2 x = 1 + 2\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots.\right) + 3\left(\frac{x^2}{2!} - \ldots\ldots\right)^2 + \ldots.$ considering terms up to x^4

$\sec^2 x = 1 + 2\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots.\right) + 3\left(\frac{x^4}{4}\right) + \ldots\ldots.. = 1 + x^2 + \left(\frac{3}{4} - \frac{2}{4!}\right)x^4 + \ldots\ldots..$

$= 1 + x^2 + \frac{2}{3}x^4 + \ldots\ldots..$

(2).Prove that $(1+x)^{\frac{1}{x}} = e - \frac{e}{2}x + \frac{11}{24}x^2 + \ldots.$

Solution: Let $y = (1+x)^{\frac{1}{x}}$

$log\, y = \frac{1}{x} log(1+x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots.\right)$

$= 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots.$

$= 1 + z$ where $z = -\frac{x}{2} + \frac{x^2}{3} - \ldots$

$y = e^{1+z} = e.e^z$

$= e.\left(1 + \left(-\frac{x}{2} + \frac{x}{3} + \ldots\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x}{3} + \ldots\right)^2 + \ldots.\right)$

$= e.\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{x^2}{8} \ldots.\right]$

$$= e - \frac{e}{2}x + \frac{11}{24}x^2 + \ldots.$$

(3).Show that $log(sec\, x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots\ldots..$

Solution: i) $y = \log(\sec x)$ $\quad y(0) = 0$

$y_1 = \tan x$ $\quad y_1(0) = 0$

$y_2 = 1 + y_1^2$ $\quad y_2(0) = 1$

$y_3 = 2y_1y_2$ $\quad y_3(0) = 0$

$y_4 = 2y_2^2 + 2y_1y_3$ $\quad y_4(0) = 2$

$$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots\ldots$$

$$\therefore \log(\sec x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \ldots\ldots\ldots$$

(4) Show that $\log[\sec x] = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \dfrac{x^6}{45}\ldots\ldots\ldots\ldots$

Solution: $\log(\sec x) = \log\left(\dfrac{1}{\cos x}\right) = \log\left(\dfrac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots}\right)$

Let $y = \log(\sec x)$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{1}{(\sec x)}.(\sec x)\tan x \Rightarrow \tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \ldots\ldots$

$\therefore$ an integrating we get

$y = \int\left\{x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \ldots\ldots\right\}dx + C = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \dfrac{x^6}{45} + \ldots\ldots + C$

To find c put x = 0

$\Rightarrow$ $y(0) = \log(\sec 0) = 0 + C$

$\therefore$ $0 = C$

$\therefore$ $\log(\sec x) = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \dfrac{x^6}{45} + \ldots.$

(5).Pr o ve that $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots\ldots\ldots$

Solution: $y = \tan^{-1} x$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{1+x^2} = 1 - x^2 + (x^2)^2 - (x^2)^3 + (x^2)^4\ldots\ldots$

$\therefore \dfrac{dy}{dx} = 1 - x^2 + x^4 - x^6 + \ldots\ldots$

Integrating we get $\quad \int_0^x \dfrac{dy}{dx}dx = \int_0^x (1 - x^2 + x^4 - x^6 + \ldots\ldots)dx$

$$\therefore y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$$

Exercise 6

1. Show that $\log[1+\sin x] = x - \frac{x^2}{2} + \frac{x^3}{6} + ...$

2. Expand $\log\left(1+x+x^2+x^3\right)$ upto a term in x^8.

Ans.: $x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3x^8}{8} + ...$

4. Prove that $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + ...if \;\; y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}.........$

5. Prove that $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + ...$

6. Expand $\log(1+e^x)$ by Maclaurin's theorem as far as in term in x^4.

Let's check away from lecture

Choose the correct option from the following:

1. By Maclaurin's series f(x)

$(a)\;\; f(a) + hf'(a) + \frac{h^2}{2!}f''(a)....$ $\quad (b)\;\; 1 + hf(a) + \frac{h^2}{2!}f'(a) + \frac{h^3}{3!}f''(a)....$

$(c)\;\; f(0) + hf'(0) + \frac{h^2}{2!}f''(0)....$ $\quad (d)\;\; none$

2. By Maclaurin's series function can be expanded into positive ascending integral powers of

$(a)\; h$ $\quad (b)\; x$ $\quad$ (c) a $\quad (d)\; f(x)$

3. In the Maclaurin series expansion of $y = e^x \log(1+x)$
The coefficients of x^5

(a) $\frac{9}{5!}$ $\quad$ (b) $\frac{5}{5!}$ $\quad$ (c) $\frac{4}{5!}$ $\quad$ (d) $\frac{3}{5!}$

Homework Problems for the day

1. *Show that* $(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} + ...$

2. Prove that $e^{e^x} = e\left[1 + x + x^2 + \frac{5x^3}{6} + ...\right]$

3. Prove that $x\,\mathrm{cosec}\,x = 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + ...$

4. Prove that $\log\left(\frac{\sinh x}{x}\right) = \frac{x^2}{6} - \frac{x^4}{180} + ...$

5. Prove that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

6. Find the expansion of $\log\tan\left(\frac{\pi}{4} + x\right)$ up to x^5.

Learning from the topic: Student will be able to expand a function of single variable into positive ascending integral powers 'x' using Maclaurin's Series.

Indeterminate forms

Lecture: 07

1. Learning Objective:
Student shall be able to identify indeterminate forms and apply L'Hospital rules to evaluate the given limit.

2. Introduction: In calculus or in other branch of mathematics it is often occur that the value of an algebraic expression involving the Independent variable cannot be evaluated or its limiting value cannot be evaluated by putting the value of independent variable these are called the Indeterminate forms. To find the limiting value of an indeterminate form L'Hopital has suggested a method.

3. Key Definitions:

(1) L'Hospital Rule: If $f(x)$ and $g(x)$ are two functions of x which can be expanded by Taylor's series in the neighborhood of $x=a$ and if $f(a) = g(a)$ =0, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

4. Important steps to be followed to solve the problem

a) Those limits which cannot be evaluated by using certain rules of limits are known as indeterminate forms.

b) There are seven types of indeterminate forms given as follows:

$(i)\ \frac{0}{0}$ $(ii)\ \frac{\infty}{\infty}$ $(iii)\ 0\times\infty$ $(iv)\ \infty-\infty$

$(v)\ 1^{\infty}$ $(vi)\ 0^0$ $(vii)\ \infty^0$

these limits can be evaluated by using L'Hospital Rule.

c) To apply L'Hospital rule $\lim_{x \to a} \frac{f(x)}{g(x)}$ should be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, if not then first convert it in this form and then apply L'Hospital rule.

5. Sample Problems

(1) *Evaluate* $\lim_{x\to 1} \dfrac{x^x - x}{x-1-\log x}$.

Solution : $\lim_{x\to 1} \dfrac{x^x - x}{x-1-\log x}$ is of the form $\left[\dfrac{0}{0}\right] = \lim_{x\to 1} \dfrac{x^x(1+\log x)-1}{1-1/x} \quad \left[\dfrac{0}{0}\right]$

$$= \lim_{x\to 1} \frac{x^x\left(1/x\right)+x^x(1+\log x)^2}{1/x^2} = \frac{1+1}{1} = 2$$

(2) Evaluate $\lim_{x\to 0}\left[\dfrac{a}{x} - \cot\dfrac{x}{a}\right]$

Solution: Put $\dfrac{x}{a} = y \;\therefore as\; x\to 0,\; y\to 0$

$$\lim_{x\to 0}\left[\frac{a}{x}-\cot\frac{x}{a}\right] = \lim_{y\to 0}\left[\frac{1}{y}-\cot y\right] \quad [\infty-\infty]$$

$$= \lim_{y\to 0}\left[\frac{1}{y}-\frac{1}{\tan y}\right] = \lim_{y\to 0}\left[\frac{\tan y - y}{y\tan y}\right] \quad \left[\frac{0}{0}\right]$$

$$= \lim_{y\to 0}\frac{\tan y - y}{y^2}\,\frac{y}{\tan y} = \lim_{y\to 0}\frac{\sec^2 y - 1}{2y} = 0$$

(3) Evaluate $\lim_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan\left(\pi x/2a\right)}$

Solution: Let $\dfrac{x}{a} = t$, $\quad \therefore as\; x\to a,\; t\to 1$

$$\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\pi x/2a\right)} = \lim_{t\to 1}(2-t)^{\tan\left(\pi t/2\right)}$$

Let L = $\lim_{t\to 1}(2-t)^{\tan\left(\pi t/2\right)}$

$$\therefore \log L = \lim_{t\to 1}\tan\left(\frac{\pi t}{2}\right)\log(2-t) \quad [\infty\times 0]$$

Exercise 7

1. *Evaluate* $\lim_{x\to \frac{1}{2}} \dfrac{\cos^2 \pi x}{e^{2x} - 2xe}$. **Ans.:** $\dfrac{\pi^2}{2e}$

2. *Show that* $\lim_{x\to 0} \log_{\tan x}\tan 2x = 1$.

3. *Show that* $\lim_{x\to\infty}\left[\dfrac{1^{\frac{1}{x}}+2^{\frac{1}{x}}+3^{\frac{1}{x}}+4^{\frac{1}{x}}}{4}\right]^{4x} = 24$.

4. Evaluate $\lim_{x\to\frac{\pi}{2}}(\cos x)^{\cos^2 x}$ Ans: 1

5. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ Ans: $e^{\frac{-1}{6}}$

Let's check take away from the lecture

1. $\underset{n\to\infty}{Lim}\left(1+\frac{1}{n}\right)^n$ *is*

   $(a)\ e^{-1}$ $(b)\ e$ $(c)\ 1$ (d) *none*

2. $\underset{x\to 0}{Lim}\ x^{\sin x}$ *is*

   $(a)\ 1$ $(b)\ 2$ $(c)\ 3$ (d) *none*

3. $\underset{x\to\frac{\pi}{2}}{Lim}\ (\sec x)^{\cot x}$ *is*

   $(a)\ 0$ $(b)\ 1$ $(c)\ 2$ (d) *none*

Homework Problems for the day

1. Show that $\lim_{x\to 0}\ \log_x \sin x = 1.$

2. *Evaluate* $\lim_{x\to 0}\ \tan x \log x.$ **Ans.:** 0

3. *Evaluate* $\lim_{x\to 0}\left[\frac{1}{x^2}-\cot^2 x\right].$ **Ans.:** $\frac{2}{3}$

4. Prove that $\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}+c^{\frac{1}{x}}+d^{\frac{1}{x}}}{4}\right)^x=(abcd)^{\frac{1}{4}}$

Learning from the topic: Learner will be able to evaluate limiting value of indeterminate forms**.**

Indeterminate forms continued……….

Lecture: 08

(1) Evaluate $\lim_{x\to 0}\frac{e^{2x}-(1+x)^2}{x\log(1+x)}$

Solution: $\lim\limits_{x\to 0} \dfrac{e^{2x} - (1+x)^2}{x\log(1+x)} = \lim\limits_{x\to 0} \dfrac{\left(1+2x+\dfrac{2^2x^2}{2!}+\dfrac{2^3x^3}{3!}+\ldots\right)-\left(1+2x+x^2\right)}{x\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots\right)}$

$$= \lim_{x\to 0} \frac{x^2\left(1+\dfrac{4x}{3}+\dfrac{2x^2}{3}+\ldots\right)}{x^2\left(1-\dfrac{x}{2}+\dfrac{x^2}{3}-\ldots\right)} = 1$$

(2) Evaluate $\lim\limits_{x\to\infty} \dfrac{e^{1/x}+e^{2/x}+\ldots\ldots+e^{x/x}}{x}$ using series expansion.

Solution: $\lim\limits_{x\to\infty} \dfrac{e^{1/x}\left[\left(e^{1/x}\right)^x-1\right]}{(e^{1/x}-1)}\dfrac{1}{x}$ (Numerator is a G.P. having x terms with $r = e^{1/x}$)

$$= \lim_{x\to\infty} \frac{e^{1/x}(e-1)}{x\left(e^{1/x}-1\right)} = \lim_{x\to\infty} \frac{e^{1/x}(e-1)}{x\left(1+\dfrac{1}{x}+\dfrac{1}{2x^2}+\ldots\ldots\ldots-1\right)} = \lim_{x\to\infty} \frac{e^{1/x}(e-1)}{\left(1+\dfrac{1}{x}+\dfrac{1}{2x^2}+\ldots\ldots\ldots-1\right)}$$

$$= \lim_{x\to\infty} \frac{e^{1/x}(e-1)}{\left(1+\dfrac{1}{2x}+\ldots\ldots\ldots\right)} = e-1$$

(3) Expanding in the term of Maclaurin's series Evaluate $\left(\dfrac{\sinh x}{x}\right)^{\frac{1}{x^2}}$.

Solution: Given that $y = \left(\dfrac{\sinh x}{x}\right)^{\frac{1}{x^2}}$

Taking log on both sides log y $=\left(\dfrac{\sinh x}{x}\right)$

$$\Rightarrow \frac{1}{x^2}\log\left(\left(1+x^2\left(\frac{1}{3!}+\frac{x}{4!}+\frac{x^2}{5!}+\ldots\right)\right)\right)$$

$$=\frac{1}{x^2}\left(\left(\frac{x^2}{3!}+\frac{x^3}{4!}\ldots\right)-\frac{\left(\frac{x^2}{3!}+\frac{x^3}{4!}\ldots\right)^2}{2!}+\frac{\left(\frac{x^2}{3!}+\frac{x^3}{4!}\ldots\right)^3}{3!}\ldots\right)$$

$$\Rightarrow \frac{1}{x^2}\left(\frac{x^2}{3!}+\frac{x^4}{4!}+\ldots\ldots\right)=\left(\frac{1}{3!}+\frac{x^2}{4!}+\ldots\ldots\right)=\frac{1}{6}$$

$$\Rightarrow \log y = \frac{1}{6} \Rightarrow y = e^{\frac{1}{6}}$$

Exercise 8

1 Evaluate $\lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^x$. Ans: $e^{\frac{2}{a}}$

2. Prove that $\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}=-\frac{e}{2}$.

3. Prove that $\lim_{x\to 0}\frac{\log_{\sin x}\cos x}{\log_{\sin\frac{x}{2}}\cos\frac{x}{2}}=4$

4. If $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$. Find a and b. **Ans.:** $a=\frac{-5}{2}$, $b=\frac{-3}{2}$

Let's check take away from the lecture

1. $\underset{x\to\frac{\pi}{2}}{Lim}(\sin x)^{\tan x}$ *is*

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) *none*

2. Value of $\frac{tanx}{tan3x}$

(a) 1 (b) 2 (c) 3 (d) 4

3. Value of $\frac{x^2}{e^{x^2}}$

(a) 1 (0) (c) -1 (d) 2

Homework Problems for the day

1. *Evaluate* $\lim_{x\to 0}\frac{\sin^{-1}x-x}{x^3}$. **Ans.:** $\frac{1}{6}$.

2. *Evaluate* $\lim_{x\to 0}\frac{\sin x\sin^{-1}x-x^2}{x^6}$. **Ans.:** 1/18

3 Evaluate $\lim_{x\to 0}\frac{1-x^x}{x\log x}$. Ans: -1

Learning from the topic: Learner will be able to expand the function and evaluate limiting value of indeterminate forms.

Convergence of sequence and Series

Lecture: 09

De 'Alembert's Ratio Test and Cauchy's n^{th} Root Test

1. Learning Objective:

1. Learners shall be able to test the convergence and divergence of Infinite series.
2. Learners shall be able to identify the infinite series and apply the De 'Alembert ratio test

2. Introduction:

A series is the sum of the terms of an infinite sequence of numbers. Given an infinite sequence (a_1, a_2, a_3...), the n^{th} partial sum S_n is the sum of the first n terms of the sequence. That is, $S_n = \sum_{k=1}^{n} a_k$. A series is convergent if the sequence of its partial sums (S_1,S_2,S_3,...) tends to a finite limit, that means that the partial sums become closer and closer to a given number when the number of their terms increases.

3. Important Formulae:

De 'Alembert's ratio test:

$$For the series \sum an find \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

1. if L < 1 the series is absolutely convergent (and hence convergent)
2. if L > 1 the series is divergent.
3. if L=1 the test is inconclusive. The series may be divergent, conditionally convergent or absolutely convergent.

4.Sample Problems:

1). Determine if the following series is convergent or divergent $\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)}$ **Solution:**

$$L=\lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}\frac{4^{2n+1}(n+1)}{(-10)^{n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(-10)(n+1)}{4^{2}(n+2)}\right|=\lim_{n\to\infty}\left|\frac{(-10)(1+1/n)}{16(1+2/n)}\right|$$

$$=\frac{10}{16}=\frac{5}{8}<1$$

So, by ratio test this series is convergent

2.Determine whether the series is convergent or divergent $\sum_{n=1}^{\infty}\frac{n!}{5^{n}}$.

Solution: $L=\lim_{n\to\infty}\left|\frac{(n+1)!5^{n}}{5^{n+1}\ n!}\right|=\lim_{n\to\infty}\left|\frac{(n+1)!}{5\ n!}\right|=\lim_{n\to\infty}\left|\frac{(n+1)}{5}\right|=\infty>1$

So by the ratio test series is divergent.

3). Determine whether the series is convergent or divergent $\sum_{n=0}^{\infty}\left(\frac{5n-3n^{3}}{7n^{3}+2}\right)^{n}$.

Solution:

$$L=\lim_{n\to\infty}\left|\left(\frac{5n-3n^{3}}{7n^{3}+2}\right)^{n}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\left(\frac{5n-3n^{3}}{7n^{3}+2}\right)\right|=\lim_{n\to\infty}\left|\left(\frac{5/n^{2}-3}{7+2/n^{3}}\right)\right|=\frac{3}{7}<1$$

Hence by root test series is convergent

4. Determine if the following series is convergent or divergent $\sum_{n=1}^{\infty}\frac{n^{n}}{3^{1+2n}}$

Solution:

$$L=\lim_{n\to\infty}\left|\frac{n^{n}}{3^{1+2n}}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{3^{\frac{1}{n}+2}}=\frac{\infty}{3^{2}}=\infty>1$$ so by root test this series is divergent

Exercise 9

1) Test for convergence of the series whose nth term is

i) $\frac{n^{2}}{2^{n}}$ ii) $\frac{3^n}{\log n}$. iii) $\frac{2^2}{2!}\sum_{n=1}^{\infty}[\frac{(3n)!}{n!}]^{1/n}$ iv) $\sum_{n=1}^{\infty}\frac{1}{n^{n}}$

2) Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n.3^{n}}$

Ans: convergent if x<3, divergent if x>3.

3) Test for convergence $\frac{1}{2}x+x^2+\frac{9}{8}x^3+x^4+\frac{25}{32}x^5+\ldots\infty$

4) Test the convergence of the series $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots.\infty$.

5) 2.Discuss the convergence of the series $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots\infty$

Let's check take away from the lecture

1. In the ratio test if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ then series is

(a) convergent (b) divergent (c)Oscillatory (d)cannot be said

2. Investigate the series whose nth term is $\sum_{n=1}^{\infty}\frac{3^n}{n^2}$

(a) convergent (b) divergent (c)Oscillatory (d)cannot be said

3.Investigate the series whose nth term is $\sum_{n=1}^{\infty}\frac{n^3}{(\log 3)^n}$

(a) convergent (b) divergent (c)Oscillatory (d)cannot be said

4. Investigate the series whose nth term is $\sum_{n=1}^{\infty}\frac{n^n}{2^{3n-1}}$

(a) convergent (b) divergent (c)Oscillatory (d)cannot be said

5. Investigate the series whose nth term is $\sum_{n=1}^{\infty}(\frac{n^2}{\log 2)^n}$

(a) convergent (b) divergent (c)Oscillatory (d)cannot be said

Homework Problems for the day

1) Discuss the convergence and divergence of the series $\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\ldots\infty$

2) Test for convergence of the series whose nth term is

i) $\sum_{n=1}^{\infty}\frac{n!2^n}{n^n}$ ii) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n \quad (x>0)$ iii) $\sum_{n=1}^{\infty}\frac{1}{(\log n)^n}$ iv) $\sum_{n=1}^{\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$

3. Test the convergence of the series $1 + (\frac{1}{2})^2 + (\frac{1}{3})^3 + \ldots.$

Learning from the topic: Learner will be able to test the convergence and divergence of an Infinite series using Ratio test and nth root test.

Self-Study

Lecture: 10

Cauchy's Integral Test

1. Learning Objective:

1. Learners shall be able to check whether the given series is convergent or divergent of Infinite series.
2. Learners shall be able to identify the infinite series and apply the Cauchy's integral test.

Introduction:

The **integral test for convergence** is a method used to test infinite series of non-negative terms for convergence. An early form of the test of convergence was developed in India by Madhavain the 14th century, and by his followers at the Kerala School. In Europe, it was later developed by Maclaurin and Cauchy and is sometimes known as the Maclaurin–
Cauchy test.

Definition:

If for $x \geq 1$, f(x) is non-negative, monotonic decreasing function of x such that $f(n) = u_n$ for all positive integral values of n, then the series $\sum u_n$ is convergent if improper integral $\int_1^\infty f(x)dx$ is convergent otherwise $\sum u_n$ divergent.

OR

A positive term series f(1) + f(2) +……. + f(n) +…., where f(n) decrease as n increase, converges or diverges according as the integral $\int_1^\infty f(x)dx$ is finite or infinite.

Note: 1. The integral $\int_1^\infty f(x)dx$ has finite value then the series is convergent.

2. The integral $\int_1^\infty f(x)dx$ has infinite value then the series is divergent.

Sample Problems:

Que 1: Apply the Cauchy's integral test to discuss the behavior of the infinite series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

Sol.: Here $u_n = \frac{1}{n\log n} = f(n)$. Changing n to x, we get $f(x) = \frac{1}{x\log x}$ .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \geq 2$.

$\therefore$ Cauchy's integral test is applicable.

Then $I_n = \int_2^n f(x)dx = \int_2^n \frac{1}{x\log x}dx = \int_2^n \frac{\frac{1}{x}}{\log x}dx = [\log(\log x)]_2^n = \log(\log n) - \log(\log 2)$

$$\therefore \lim_{n\to\infty} I_n = \lim_{n\to\infty}[\log(\log n) - \log(\log 2)] \to \infty - \log(\log 2)$$

$\therefore \lim_{n\to\infty} I_n \to \infty$. $\left[\because \int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$

$\Rightarrow \int_1^\infty f(x)dx$ diverges to ∞. Hence by Cauchy's integral test, the given series is divergent.

Que 2: Apply the Cauchy's integral test to discuss the behavior of the infinite series $\sum_{n=2}^{\infty} \frac{n}{n\sqrt{n^2-1}}$

Sol.: Here $u_n = \frac{n}{n\sqrt{n^2-1}} = f(n)$. Changing n to x, we get $f(x) = \frac{x}{x\sqrt{x^2-1}}$ .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \geq 1$. Then

$$I = \int_1^\infty f(x)dx = \int_1^\infty \frac{x}{(x^2+1)^{1/2}}dx$$

Let $x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta d\theta$.

As $x \to 1$ we have $\theta \to 0$ and as $x \to \infty$ we have $\theta \to \frac{\pi}{2}$.

$$\therefore I = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

$\Rightarrow$ I converges to $\frac{\pi}{2}$, hence by Cauchy's integral test the given series is convergent.

Que 3: Apply the Cauchy's integral test to discuss the behavior of the infinite series $\sum \frac{n^3}{n^4+3}$.

Sol.: Here $u_n = \frac{2n^3}{n^4+3} = f(n)$. Changing n to x, we get $f(x) = \frac{2x^3}{x^4+3}$ .... (1).

Here f(x) is positive and monotonically decreasing $\forall x \geq 1$. Then

$$I_n = \int_1^n f(x)dx = \int_1^n \frac{2x^3}{x^4+3}dx = \left[\frac{1}{2}\log(x^4+3)\right]_1^n = \frac{1}{2}\log(n^4+3) - \frac{1}{2}\log 4$$

$$= \frac{1}{2}\log\left(\frac{n^4+3}{4}\right).$$

$\therefore \lim_{n\to\infty} I_n \to \infty$.

$\Rightarrow \{I_n\}$ diverges to ∞, hence by Cauchy's integral test the given series is divergent.

Exercise 10

1. Use Cauchy's integral test to show that given $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ is convergent or divergent
2. Apply the Cauchy's integral test to discuss the behavior of the series $\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+..$
3. Show that the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent and divergent of $0<p\leq 1$.
4. Examine the convergence of the series $\sum_{n=1}^{\infty}\frac{\log n}{n^2}$.

Let's check take away from the lecture

1. Investigate the series whose nth term is $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.

   a) Convergent b) divergent c) oscillatory d) cannot be said
2. Investigate the series whose nth term is $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$.

   a) Convergent b) divergent c) oscillatory d) cannot be said
3. Examine the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$.

   a) 1 b) 0 c) $\frac{\pi}{2}$ d) 1/2

Homework Problems for the day

Que 1: Apply the Cauchy's integral test to discuss the behavior of the infinite series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

Que 2: Discuss the behavior of the infinite series $\sum_{n=1}^{\infty}\frac{1}{2n+3}$.

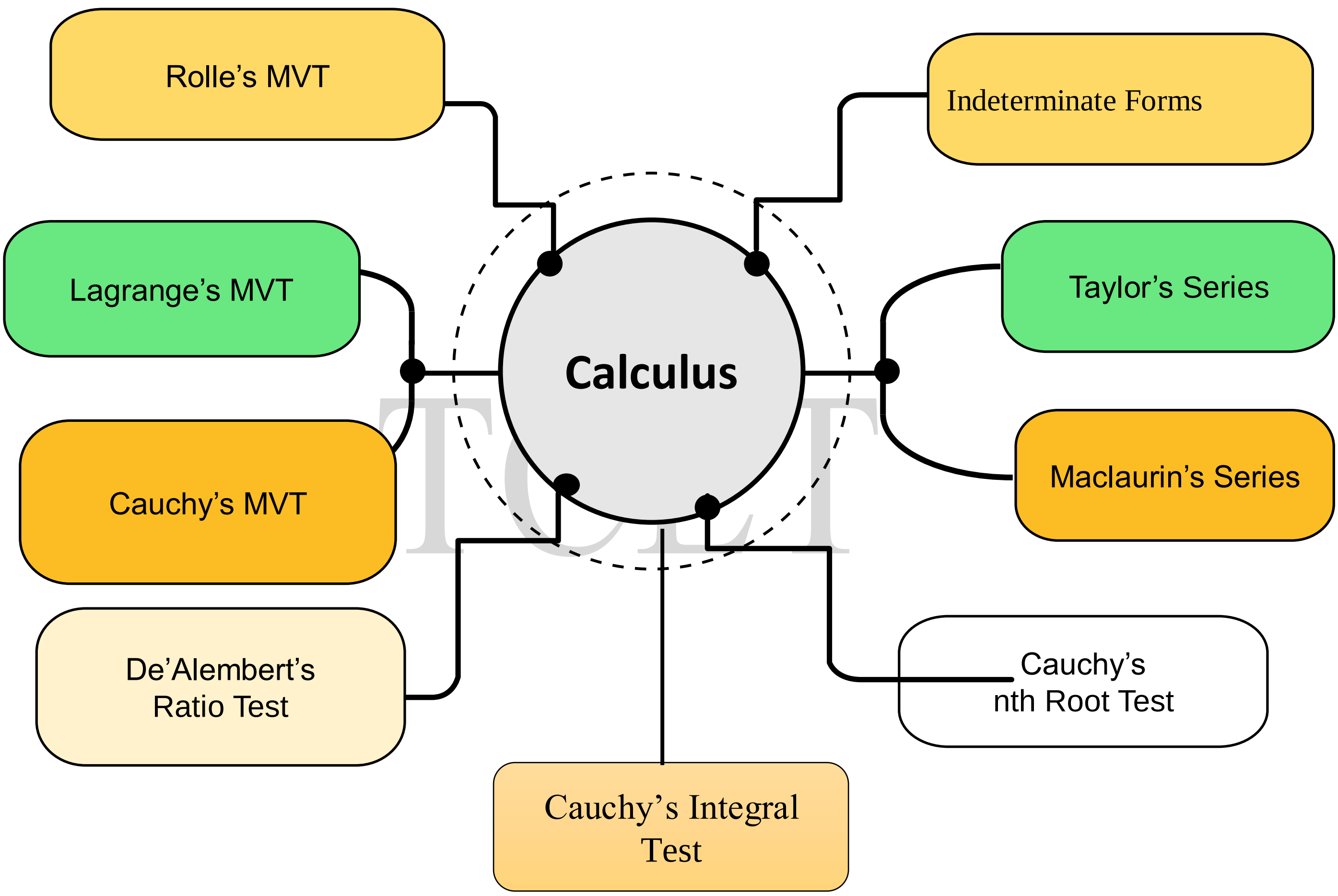
Que 3: Apply the Cauchy's integral test to discuss the behavior of the infinite series $\sum\frac{n^3}{n^4+3}$.

Learning from the topic: Learner will be able to test the convergence and divergence of an Infinite series using n^{th} root test.

Tutorial Questions

1) Verify Roll's Theorem for the function f(x) = x (x+3) $e^{-x/2}$ in $-3 \le x \le 0$.
2) Verify Lagrange's mean value theorem for the function $f(x) = \log x \ in \ [1. e]$.
3) Verify Cauchy's mean value theorem for the function $f(x) = x^2 + 2 \ \ and g(x) = x^3 - 1$ in [1,2].
4) Expand $\sqrt{1 + \sin x}$ by standard expansion.
5) $Evaluate \lim_{x \to 0} \frac{\log(1+x^3)}{sin^3 x}$
6) $Prove \ that \ \lim_{n \to 0} \sin x \log x = 0$
7) Prove that $\lim_{x \to \infty} [\frac{1}{x}]^{\frac{1}{x}} = 1$
8) Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$
9) Test for convergence of the series $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$.
10) Test for convergence of the series $\frac{1}{2} + \frac{2}{3} x + (\frac{3}{4})^2 x^2 + (\frac{4}{5})^3 x^3 + \cdots$
11) Apply the Cauchy's integral test to discuss the behavior of the infinite series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Concept Map


Rolle's MVT
Indeterminate Forms
Lagrange's MVT
Calculus
Taylor's Series
Cauchy's MVT
Maclaurin's Series
De'Alembert's Ratio Test
Cauchy's nth Root Test
Cauchy's Integral Test

Problems for Self-assessment:

Level 1

1) Consider the function $f(x)=\sqrt{x-2}$. On what intervals are the hypotheses of the Mean Value Theorem satisfied?
(A) [0, 2] (B) [1, 5] (C) [2, 7] (D) None of these

2) Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all the numbers c that satisfy the conditions of Rolle's Theorem
i) $f(x) = 5 - 12x + 3x^2$ in [1, 3], ii) $f(x) = x^3 - x^2 - 6x + 2$ in [0, 3]

3) Expand log (cosx) about $\frac{\pi}{3}$ using Taylor's expansion

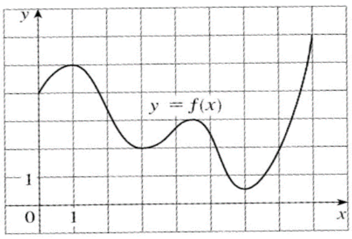
4) Prove that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots\ldots\ldots if \quad y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots\ldots$

5) Evaluate $\lim_{x\to 0} \left(\frac{1}{x}\right)^{2\sin x}$.

Level 2

1) Verify that the function $f(x) = \sin x$ satisfies the hypotheses of the Mean Value Theorem on the interval [2, 11]. Then approximate to 3 decimal places all the values of c in (2, 11) that satisfy the Mean Value Theorem equation.

2) Use the graph of **f(x)** to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval [0, 8].

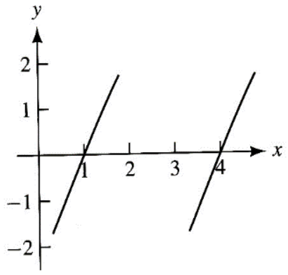


3) Show that $\log[1+\sin x] = x - \frac{x^2}{2} + \frac{x^3}{6} + \ldots\ldots\ldots\ldots$

4) Using Taylor's theorem evaluate upto 4 places of decimals $\sqrt{9\cdot 12}$

5) Prove that $\lim_{x\to 1} \left[\frac{1}{\log x} - \frac{x}{x-1}\right] = -\frac{1}{2}$

Level 3



1) f is a continuous function. A portion of the graph of f is shown to the right. Explain why f must have a root in the interval (1, 4).

2) Prove that $\sec^{-1}\left(\frac{1}{1-2x^2}\right) = 2\left\{x + \frac{x^3}{6} + \frac{3x^5}{40} + ...\right\}..$

3) If $\lim_{x\to 0} \frac{\sin 2x + p\sin x}{x^3}$ is finite, find the value of p and the limit

Ans: p = -2 and limit -1

4) Evaluate $\lim_{x\to a}\left[\frac{1}{2}\left(\sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}}\right)\right]^{\frac{1}{x-a}}$

5) Discuss the convergence of the series $\frac{x^2}{2\log 2} + \frac{x^3}{3\log 3} + \frac{x^4}{4\log 4} + ...$

6) Prove that $e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 - \frac{x^5}{5} + ...$

Learning Outcomes:

1.Know: Student should be able to

a) All three mean value theorem

b) Taylor's and McLaurin's series of various function

c) Define limits of various types of standard function, indeterminate forms, L'Hospital rules.

2. Comprehend: Student should be able to comprehend all three mean value theorem and expansion of function in the form of Taylor's and McLaurin's series and L'Hsopitals rule.

3. Apply, analyze and synthesize: Student should be able to

Apply mean value theorem to find the roots of an equation in an interval and apply Taylor's series to find the approximate value of function and Solve the problems of indeterminate forms with help of L' Hospital rule.

Digital references:

1. https://math.libretexts.org/Bookshelves/Calculus/Map%3A_Calculus__Early_Transcendentals_(Stewart)/04%3A_Applications_of_Differentiation/4.02%3A_The_Mean_Value_Theorem
2. https://openstax.org/books/calculus-volume-1/pages/4-4-the-mean-value-theorem
3. https://en.wikipedia.org/wiki/Mean_value_theorem

Add to Knowledge: The application of mean value theorem is in the calculation of average value of any function or the root calculation of any polynomial. Whereas the expansion of any function gives the quantize factorization of any function and Indeterminate forms arise in many areas of Mathematics and Engineering where to calculate the value of the function we use the technique of Indeterminate forms.

Self-Evaluation

Name of student: **Class &Div.:** **Roll No:**

1. Are you able to evaluate the different forms of limit using L′ Hospital Rule?
   (a) Yes (b) No

2. Do you understand how to expand function of one variable in positive ascending integral powers of (x-a)?
   (a) Yes (b) No

3. Will you able to identify how to fit a curve for a given bivariate data?
   (a) Yes (b) No

4. Are you able to find the two regression lines for a given bivariate data?
   (a) Yes (b) No

5. Do you understand this module?
   (a) Fully understood (b) Partially understood (c) Not at all