Module - 4

MATRICES

Previous: Sum of symmetric and skew-symmetric matrices

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Types of matrices

(1) A Square Matrix → equal No. of row and equal No. of column

$\boxed{m = n}$

e.g $\begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}_{\underset{m}{2} \times \underset{n}{2}}$, $\begin{bmatrix} 4 & 1 & 1 \\ 3 & 4 & 3 \\ 2 & 0 & 9 \end{bmatrix}_{3 \times 3}$ $[3]_{1 \times 1}$

(2) diagonal element → square matrix the element lying in diagonal of the matrix

e.g $\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ $\begin{bmatrix} 7 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

diagonal element 2, 4 ; 7, 3, 4

(3) diagonal matrix :- A square matrix whose all non-diagonal element is zero

e.g $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(4) Scalar matrix :- all diagonal element are same

e.g $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(5) Trace of matrix The sum of all the principal diagonal element of square matrix

$A = \begin{bmatrix} 3 & 4 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Trace of $A = 3 + 2 + 5 = \underline{\underline{10}}$

$|A| = 0$

(6) Singular matrix : A square matrix A whose determinant is zero

e.g $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 & 3 \\ 3 & 5 & 3 \\ 1 & -7 & 1 \end{bmatrix}$

$|A| = 6 - 6$
$= 0$

$|B| = 3(5+21) - 4(3-3) + 3(-21-5)$
$= 3(26) - 0 - 3(26)$
$= 0$

$|A| \neq 0$

(7) Non-singular matrix : A square matrix A whose determinant is not zero

e.g $\begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix}$

$|A| = 12 - 3$
$= 9$

(8) Unit matrix or Identity matrix → All diagonal element is one
All non-diagonal element is zero

e.g $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(9) Transpose of a matrix → interchanging row and column

e.g $A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 6 \\ -4 & 1 & 5 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 & -4 \\ -1 & 2 & 1 \\ 3 & 6 & 5 \end{bmatrix}$

(10) upper triangular matrix : A square matrix, in which all element below the diagonal are zero

e.g
$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \quad \begin{bmatrix} 4 & 5 & 4 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{4\times 3}$$

(11) lower triangular matrix : A square matrix, in which all element above the diagonal are zero

e.g
$$\begin{bmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

(12) conjugate of a matrix : replacing each element by its complex conjugate

e.g
$$A = \begin{bmatrix} 2 & 1+i & i \\ -i & 3i+2 & 0 \\ 2-3i & 5i & 4 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 2 & 1-i & -i \\ i & -3i+2 & 0 \\ 2+3i & -5i & 4 \end{bmatrix}$$

• Symmetrix matrix

(13) Transposed conjugated of a matrix : Transpose of the conjugate of matrix A

$$\boxed{A^{\theta} = (\bar{A})^{T} = \overline{(A^{T})}}$$

$$A = \begin{bmatrix} 1+2i & 3-i & 4 \\ 5i & 3+2i & 0 \\ -2i & 1-i & 2-i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1-2i & 3+i & 4 \\ -5i & 3-2i & 0 \\ 2i & 1+i & 2+i \end{bmatrix}$$

$$A^{\theta} = (\bar{A})^{T} = \begin{bmatrix} 1-2i & -5i & 2i \\ 3+i & 3-2i & 1+i \\ 4 & 1+i & 2+i \end{bmatrix}$$

(14) Symmetric matrix : upper all Non-diagonal = lower all non-diagonal

$A = A^T$

e.g
$$\begin{bmatrix} 2 & 4 & 3 \\ 4 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix}$$

(15) Skew - Symmetric matrix : $A = -A^T$ — diagonal part is zero

e.g
$$\begin{bmatrix} 0 & 4 & -3 \\ -4 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$$

upper and lower are equal but conjugate of each other

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$\boxed{A^\theta = (\overline{A})^T = (\overline{A^T})}$

(16) Hermitian matrices : $A = A^\theta$

e.g $\begin{bmatrix} a & c+id \\ c-id & b \end{bmatrix}$, $\begin{bmatrix} 2 & 3+2i & -i \\ 3-2i & 3 & 5 \\ i & 5 & 4 \end{bmatrix}$, $\begin{bmatrix} 2 & 2+3i & 4-i \\ 2-3i & 6 & 6+2i \\ 4+i & 6-2i & 7 \end{bmatrix}$

If $(i,j)^{th}$ element of A is equal to complex conjugate of $(j,i)^{th}$ element

(17) Skew Hermition matrices

upper and lower non-all diagonal elem are equal but negative of conjugate of each other

diagonal i+me (i+-theta hai

$A = -A^\theta$

e.g $\begin{bmatrix} 0 & 2+3i \\ -2+3i & i \end{bmatrix}$, $\begin{bmatrix} 2i & 1+i & -3+2i \\ -1+i & 0 & 2-i \\ 3+2i & -2-i & -3i \end{bmatrix}$

Note :- Purely Imaginory or zero

If $(i,j)^{th}$ element of A is equal to negative complex conjugate of $(j,i)^{th}$ element

$\begin{bmatrix} 3i & 2+i & -1+i \\ -2+i & i & 3-i \\ 1+i & -3-i & 0 \end{bmatrix}$

Note :- every real symmetric matrix is Hermitian and vice-versa

- Every real skew-symmetric matrix is skew-Hermition and vice-versa

Types of matrices

If A is a square matrix $A = [a_{ij}]$ of order n :
then (1) A is said to be Symmetric if

$$\boxed{a_{ij} = a_{ji}} \quad \forall\ i, j$$

(2) A is said to be skew-symmetric if

$$\boxed{a_{ij} = -a_{ji}} \quad \forall\ i, j$$

(3) A is said to be Hermitian

if $\boxed{a_{ij} = \overline{a_{ji}}}$ $\forall\ i, j$

(4) A is said to be skew-Hermitian

if $\boxed{a_{ij} = -\overline{a_{ji}}}$ $\forall\ i, j$

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Notes :

(1) Diagonal elements of ~~skew~~ Skew-Symmetric matrix are alway zero.

(2) Diagonal element of $\boxed{\text{Hermitians}}$ matrices are always real.

(3) Diagonal elements of $\boxed{\text{Skew-Hermitian}}$ matrices are either $\boxed{\text{zero}}$ or purely imaginary

* Symmetric matrix : $A = (\bar{A})$

equal to its transpose

* Skew Symmetric matrix $A = -(\bar{A})$

equal ~~the item~~ to the negative of its transpose.

* Hermition matrix :- $A = A^{\theta}$ $A^{\theta} = (\bar{A})^T$

Square matrix that is equal to the transpose of its conjugate matrix.

$A = (\bar{A})^T$

* Skew - Hermition matrix

is equal to negative of its conjugate transpose

$A = -(\bar{A})^T$

Symmetric and Skew - Symmetric

(1) Express $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ a sum of symmetric and skew-symmetric matrices

let $\boxed{A = \underbrace{\frac{1}{2}(A + A^T)}_{P} + \underbrace{\frac{1}{2}(A - A^T)}_{Q}}$ → solve it give $= A$

where

$$P = \frac{1}{2}(A + A^T)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right.$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$$

As $P_{ij} = P_{ji}$ which is
P is → symmetric

$$Q = \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right.$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 5 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

which is skew-symmetric

Now,

$A = \cancel{BA} Q + P$

$$A = \frac{1}{2}\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

sym skew $A = P + Q$

$= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

(2) Express $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$ as sum of symmetric and skew-symmetric matrices

=> let $A = P + Q$

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

where

$P = \frac{1}{2}(A + A^T)$

$= \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 11/2 \\ 0 & 7 & 3/2 \\ 11/2 & 3/2 & 0 \end{bmatrix}$

which is symmetric

$Q = \frac{1}{2}(A - A^T)$

$= \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -5/2 \\ -1/2 & 5/2 & 0 \end{bmatrix}$

which is skew-symmetric

Now $A = P + Q$

$A = \begin{bmatrix} 3 & 0 & 11/2 \\ 0 & 7 & 3/2 \\ 11/2 & 3/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -5/2 \\ -1/2 & 5/2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

(3) Express the matrix $\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix}$ as a sum of symmetric and skew-symmetric matrix.

$\Rightarrow$ let $A = P + Q$

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

where

$P = \frac{1}{2}(A + A^T)$

$= \frac{1}{2}\left\{\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix} + \begin{bmatrix} 2 & 14 & 3 \\ -4 & 7 & 5 \\ 9 & 13 & 11 \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 4 & 10 & 12 \\ 10 & 14 & 18 \\ 12 & 18 & 22 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix}$

As $P_{ji} = P_{ij}$

which is symmetric

$Q = \frac{1}{2}(A - A^T)$

$= \frac{1}{2}\left\{\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix} - \begin{bmatrix} 2 & 14 & 3 \\ -4 & 7 & 5 \\ 9 & 13 & 11 \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 0 & -18 & 6 \\ 18 & 0 & 8 \\ -6 & -8 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

As $Q_{ij} = -Q_{ji}$

Q is skew-symmetric

Now,

$A = P + Q$

$= \begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix} + \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix}$

Hermition and Skew - Hermitian matrices

e.g

(1) $A = \begin{bmatrix} 1 & -2+3i \\ -2-3i & 6 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -2-3i \\ -2+3i & 6 \end{bmatrix}$

$(\overline{A^T}) = \begin{bmatrix} 1 & -2+3i \\ -2-3i & 6 \end{bmatrix}$

$A^\theta = (\overline{A^T})$

$\boxed{A = A^\theta}$ A is Hermition matrix

(2) $A = \begin{bmatrix} 0 & 2-3i \\ -2-3i & 2i \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -2-3i \\ 2-3i & +2i \end{bmatrix}$

$(\overline{A^T}) = \begin{bmatrix} 0 & -2+3i \\ 2+3i & -2i \end{bmatrix}$

$A^\theta = (\overline{A^T}) = \begin{bmatrix} 0 & -2+3i \\ 2+3i & -2i \end{bmatrix} = -\begin{bmatrix} 0 & 2-3i \\ -2-3i & 2i \end{bmatrix}$

$A^\theta = -A$

$\boxed{A = -A^\theta}$ A is Skew Hermitian matrix

$A^T = A^* = A'$

necessary and sufficient condition for types of matrices :

(1) A is symmetric iff $\boxed{A = A^T}$

(2) A is skew symmetric iff $\boxed{A = -A^T}$

(3) A is Hermition iff $\boxed{A = A^\theta}$

$\left\{ A^\theta = (\bar{A})^T \text{ or } (\overline{A^T}) \right\}$ is called transpose conjugate of matrix

(4) A is skew Hermitian iff $\boxed{A = -A^\theta}$

5 marks

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Result about Types of matrix :-

(I) Every square matrix can be uniquely expressed as sum of $\boxed{\text{symmetric}}$ and $\boxed{\text{skew-symmetric matrices}}$

(II) Every square matrix can be uniquely expressed as sum of $\boxed{\text{Hermition}}$ and $\boxed{\text{skew-Hermition matrices}}$

$$\boxed{A = P + Q}$$

$$P = \boxed{\tfrac{1}{2}(A + A^{\theta})} \qquad Q = \boxed{\tfrac{1}{2}(A - A^{\theta})}$$

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① Express the matrix $A = \begin{bmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{bmatrix}$

as sum of Hermitian and a Skew Hermitian matrix

$\Rightarrow$ $A^{\theta} = \overline{(A^T)}$ $\qquad A = \begin{bmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{bmatrix}$

$$A^T = \begin{bmatrix} 2+3i & -2i & 4 \\ 2 & 0 & 2+5i \\ 3i & 1+2i & -i \end{bmatrix}$$

$$A^{\theta} = \overline{(A^T)} = \begin{bmatrix} 2-3i & 2i & 4 \\ 2 & 0 & 2-5i \\ -3i & 1-2i & i \end{bmatrix}$$

let $P = \frac{1}{2}(A + A^{\theta})$

$$P = \frac{1}{2}\left\{\begin{bmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{bmatrix} + \begin{bmatrix} 2-3i & 2i & 4 \\ 2 & 0 & 2-5i \\ -3i & 1-2i & i \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} (2+3i)+(2-3i) & 2+2i & 3i+4 \\ 2-2i & 0 & (1+2i)+(2-5i) \\ 4-3i & (2+5i)+(1-2i) & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 2+2i & 4+3i \\ 2-2i & 0 & 3-3i \\ 4-3i & 3+3i & 0 \end{bmatrix}$$

$A = P + Q$

$= \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$ $\quad A^{\theta} = (\bar{A})^T = (\overline{A^T})$

let

$Q = \frac{1}{2}(A - A^{\theta})$

$$= \frac{1}{2}\left\{\begin{bmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{bmatrix} - \begin{bmatrix} 2-3i & 2i & 4 \\ 2 & 0 & 2-5i \\ -3i & 1-2i & i \end{bmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} \cancel{2}+3i-\cancel{2}+3i & 2-2i & 3i-4 \\ -2i-2 & 0 & 1+2i-2+5i \\ 4+3i & 2+5i-1+2i & -2i \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 6i & 2-2i & 3i-4 \\ -2-2i & 0 & -1+7i \\ 3i+4 & 1+7i & -2i \end{bmatrix}$$

Here P is Hermition and Q is skew Hermition matrix

$A = P + Q$

$$= \begin{bmatrix} 2 & \frac{2+2i}{2} & \frac{4+3i}{2} \\ \frac{2-2i}{2} & 0 & \frac{3-3i}{2} \\ \frac{4-3i}{2} & \frac{3+3i}{2} & 0 \end{bmatrix} + \begin{bmatrix} 3i & \frac{2-2i}{2} & \frac{3i-4}{2} \\ \frac{-2-2i}{2} & 0 & \frac{-1+7i}{2} \\ \frac{3i+4}{2} & \frac{1+7i}{2} & \frac{-i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1+i & \frac{4+3i}{2} \\ 1-(1)i & 0 & \frac{3-3i}{2} \\ \frac{4-3i}{2} & \frac{3+3i}{2} & 0 \end{bmatrix} + \begin{bmatrix} 3i & 1-i & \frac{3i-4}{2} \\ -1-i & 0 & \frac{-1+7i}{2} \\ \frac{3i+4}{2} & \frac{1+7i}{2} & -i/2 \end{bmatrix}$$

(2) express matrix A as the sum of a Hermition and skew - Hermition matrix where

$$A = \begin{bmatrix} 2-i & 3+i & 2i \\ 3 & 0 & 4-i \\ -5 & 2-i & 3+i \end{bmatrix}$$

Sol.

$$A^T = \begin{bmatrix} 2-i & 3 & -5 \\ 3+i & 0 & 2-i \\ 2i & 4-i & 3+i \end{bmatrix}$$

$$(\overline{A^T}) = \begin{bmatrix} 2+i & 3 & -5 \\ 3-i & 0 & 2+i \\ -2i & 4+i & 3-i \end{bmatrix}$$

let

$$P = \frac{1}{2}(A + A^{\theta}) = \frac{1}{2}\left\{\begin{bmatrix} 2-i & 3+i & 2i \\ 3 & 0 & 4-i \\ -5 & 2-i & 3+i \end{bmatrix} + \begin{bmatrix} 2+i & 3 & -5 \\ 3-i & 0 & 2+i \\ -2i & 4+i & 3-i \end{bmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 4 & 6+i & 2i-5 \\ 6-i & 0 & 6 \\ -5-2i & 6 & 6 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 2 & \frac{6+i}{2} & \frac{2i-5}{2} \\ \frac{6-i}{2} & 0 & 3 \\ \frac{-5-2i}{2} & 3 & 3 \end{bmatrix}$$

Here is P is Hermition ~~sgst~~ matrix

let

$$Q = \frac{1}{2}(A - A^{\theta})$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 2-i & 3+i & 2i \\ 3 & 0 & 4-i \\ -5 & 2-i & 3+i \end{bmatrix} - \begin{bmatrix} 2+i & 3 & -5 \\ 3-i & 0 & 2+i \\ -2i & 4+i & 3-i \end{bmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} \cancel{2}-i-\cancel{2}-i & \cancel{3}+i-\cancel{3} & 2i+5 \\ \cancel{3}-\cancel{3}+i & 0 & 4-i-2-i \\ -5+2i & 2-i-4-i & \cancel{3}+i-\cancel{3}+i \end{bmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} -2i & +i & 2i+5 \\ i & 0 & 2-2i \\ -5+2i & -2-2i & 2i \end{bmatrix}\right\}$$

$$= \begin{bmatrix} -i & i/2 & \frac{2i+5}{2} \\ i/2 & 0 & 1-i \\ \frac{-5+2i}{2} & -1-i & i \end{bmatrix}$$

Here is Q is Hermition matrix

$$A = P + Q$$

$$= \begin{bmatrix} 2 & \frac{6+i}{2} & \frac{2i-5}{2} \\ \frac{6-2i}{2} & 0 & 3 \\ \frac{-5-2i}{2} & 3 & 3 \end{bmatrix} + \begin{bmatrix} -i & i/2 & \frac{2i+5}{2} \\ i/2 & 0 & 1-i \\ \frac{-5+2i}{2} & -1-i & i \end{bmatrix}$$

(3) Express matrix A as the sum of Hermition and skew - Hermitian matrix where

$$A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$$

$\Rightarrow$

$$A^T = \begin{bmatrix} 3i & 1+i & -3-2i \\ -1+i & -i & -1+2i \\ 3-2i & 1+2i & 0 \end{bmatrix}$$

$$A^\theta = (\overline{A^T}) = \begin{bmatrix} -3i & +1-i & -3+2i \\ -1-i & i & -1-2i \\ 3+2i & 1-2i & 0 \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^\theta)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} + \begin{bmatrix} -3i & +1-i & -3+2i \\ -1-i & i & -1-2i \\ 3+2i & 1-2i & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

?

$$Q = \frac{1}{2}(A - A^\theta)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} - \begin{bmatrix} -3i & 1-i & -3+2i \\ -1-i & i & -1-2i \\ 3+2i & 1-2i & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3i+3i & -1+i-1+i & 3-2i+3-2i \\ 1+i+1+i & -2i & 1+2i+1+2i \\ -3-2i-3-2i & -1+2i-1+2i & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 6i & -2+2i & 6-4i \\ 2+2i & -2i & 2+4i \\ -6-4i & -2+4i & 0 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$$

$$A = P + Q$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$$

Orthogonal matrix $\rightarrow$ is called (for)

definition A is a square matrix of order m and $AA^T = I$, $A^TA = I$ $\rightarrow$ identity matrix $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$AA^{-1}=I$

comparing with $AA^{-1}=I$

If A is orthogonal then $\boxed{A^T = A^{-1}}$ as $AA^T = I$

or

definition : If $A = [a_{ij}]$ is square matrix of order n then A is said to be orthogonal if

$$\boxed{AA^T = I = A^TA}$$

e.g (1) Determine l, m, n and find A^{-1} if $A = \begin{bmatrix}0 & 2m & n\\ l & m & -n\\ l & -m & n\end{bmatrix}$ is orthogonal.

$\Rightarrow$ $AA^T = I$

$$\begin{bmatrix}0 & 2m & n\\ l & m & -n\\ l & -m & n\end{bmatrix} \cdot \begin{bmatrix}0 & l & l\\ 2m & m & -m\\ n & -n & n\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}(0+4m^2+n^2) & (2m^2-n^2) & (-2m^2+n^2)\\ (2m^2-n^2) & (l^2+m^2+n^2) & (l^2-m^2-n^2)\\ (-2mn+n^2) & (l^2-nm-n^2) & (l^2+m^2+n^2)\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$4m^2+n^2 = 1$ ...(1) , $l^2+m^2+n^2 = 1$ $\qquad l^2+mn+n^2=1$

$2m^2-n^2=0$ ...(2)

$4m^2+\cancel{n^2}=1$
$2m^2-\cancel{n^2}=0$
$6m^2=1$

$6m^2=1$ $\quad m^2=\frac{1}{6}$ $\quad \boxed{m=\pm\sqrt{\frac{1}{6}}}$

$2\frac{1}{6}-n^2=0$

$n^2=\frac{1}{3}$ $\quad \boxed{n=\pm\sqrt{\frac{1}{3}}}$

$l^2+m^2+n^2=1$
$l^2+\frac{1}{6}+\frac{1}{3}=1$ $\quad l^2+\frac{3}{6}\frac{1}{2}=1$ $\quad l^2=1-\frac{1}{2}$ $\quad l^2=\frac{1}{2}$ $\quad \boxed{l=\pm\sqrt{\frac{1}{2}}}$

$A^{-1} = A^T$

$A A^{-1} = I$

$A A^T = I$

$$A^{-1} = A^T = \begin{bmatrix} 0 & \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{2}{\sqrt{6}} & \pm\frac{1}{\sqrt{6}} & \pm\frac{1}{\sqrt{6}} \\ \pm\frac{1}{\sqrt{3}} & \mp\frac{1}{\sqrt{3}} & \pm\frac{1}{\sqrt{3}} \end{bmatrix}$$

2) verify the following matrices orthogonal and hence Find its inverse

$$A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

For verify Both condition we check

① $AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

② $AA^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$

$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\boxed{AA^T = I = A^TA}$

③ $A^{-1} = A^T$

$A^{-1} = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$

$$\Rightarrow A^T = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$AA^T = I$

$$\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} (+64+16+1) & (-8+16-8) & (-32+28+4) \\ (-8+16-8) & (1+4+64) & (4+28-32) \\ (-32+28+4) & (4+28-32) & (16+49+16) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A$ is orthogonal matrix

$$A^{-1} = A^T = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \end{bmatrix}$$

unlike $A A^{-1} = I$ determinant

4) If A is orthogonal then show that $|A| = \pm 1$

$\Rightarrow$ A is orthogonal $\Rightarrow$ $AA^T = I$

taking determinant on both side

$|AA^T| = |I|$

$|A||A^T| = 1$ $\quad \ldots \{ |AB| = |A||B| \}$

$|A||A| = 1$ $\quad \ldots \{ |B| = |B^T| \}$

$(|A|)^2 = 1$

$= \sqrt{1}$

$\boxed{|A| = \pm 1}$

$|A|$

$AA^T = I$

(1) Verify the following matrices are orthogonal and hence find its inverse

(i) $A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$

$$AA^T = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} (+4+1+4) & (-4+2+2) & (-2-2+4) \\ (-4+2+2) & (4+4+1) & (2-4+2) \\ (-2-2+4) & (2-4+2) & (1+4+4) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^T A = \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} (+4+4+1) & (-2+4-2) & (-4+2+2) \\ (-2+4-2) & (1+4+4) & (2+2-4) \\ (-4+2+2) & (2+2-4) & (4+1+4) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$AA^T = I = A^T A$

Here is A is orthogonal matrix.

$$A^{-1} = A^T = \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

(ii) $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

$$AA^T = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^TA = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$AA^T = I = A^TA$

where A is orthogonal matrix

$$A^{-1} = A^T = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Afm = fmn1 - fM

fcm = fcm1 - fcm

(iii) $A = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & +\sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^TA = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AA^T = I = A^TA$$

Here A is orthogonal matrix

$$A^{-1} = A^T = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iv) $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -1/\sqrt{2} \\ \frac{1}{\sqrt{3}} & -2/\sqrt{6} & 0 \\ \frac{1}{\sqrt{3}} & 1/\sqrt{6} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

$$AA^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{2}\right) & \left(\frac{1}{3} - \frac{1}{3} + 0\right) & \left(\frac{1}{3} + \frac{1}{6} - \frac{1}{2}\right) \\ \left(\frac{1}{3} - \frac{1}{3} - 0\right) & \left(\frac{1}{3} + \frac{4}{6}\right) & \left(\frac{1}{3} - \frac{2}{6}\right) \\ \left(\frac{1}{3} + \frac{1}{6} - \frac{1}{2}\right) & \left(\frac{1}{3} - \frac{1}{3} \cdot 0\right) & \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{2}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^TA = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) & \left(\frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}}\right) & \left(\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}\right) \\ \left(\frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}}\right) & \left(\frac{1}{6} + \frac{4}{6} + \frac{1}{6}\right) & \left(-\frac{1}{\sqrt{12}} + 0 + \frac{1}{\sqrt{12}}\right) \\ \left(\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}\right) & \left(\frac{1}{\sqrt{12}} \pm 0 - \frac{1}{\sqrt{12}}\right) & \left(\frac{1}{2} + 0 + \frac{1}{2}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AA^T = I = A^TA$$

Here A is orthogonal matrix

$$A^{-1} = A^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -2/\sqrt{6} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} //$$

$\frac{1}{2}(A+A^{\theta}) + \frac{1}{2}(A-A^{\theta})$

$AA^{T} = I$

Unitary matrix

↳ A square matrix A is called unitary if $\boxed{AA^{\theta} = I = A^{\theta}A}$

$(\overline{A^{T}}) = (\overline{A})^{T}$

— A matrix $A = [a_{ij}]$ of order n is said to be unitary matrix if $\boxed{AA^{\theta} = I = A^{\theta}A}$

Note: $A^{-1} = A^{\theta}$

e.g

(1) Show that the matrix $\begin{bmatrix} \frac{1+i}{2} & -\frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is unitary

⟹ let $A = \begin{bmatrix} \frac{1+i}{2} & -\frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$, $A^{T} = \begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ -\frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

$A^{\theta} = (\overline{A^{T}}) = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-1-i}{2} & \frac{1+i}{2} \end{bmatrix}$

$A A^{\theta} = I$

$\frac{1}{2}\begin{bmatrix} (1+i) & (-1+i) \\ (1+i) & (1-i) \end{bmatrix} \frac{1}{2}\begin{bmatrix} (1-i) & (1-i) \\ (-1-i) & (1+i) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\frac{1}{4}\begin{bmatrix} (1-i^2) + (1-i^2) & (1-i^2)+(-1-i+i+i^2) \\ (1-i^2) - (1-i^2) & (1-i^2) + (-i^2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A is unitary matrix

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\begin{bmatrix} \frac{1+i}{2} & -\frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

Given $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$$A^\theta = (\bar{A})^T = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

consider $A A^\theta = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} 1-i^2 + (-1+i)(-1-i) & 1-i^2 + (-1+i)(1+i) \\ 1-i^2 + (1-i)(-1-i) & (1-i^2) + (1-i^2) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1+1 - (1-i)(1+i) & 1-i^2 + i^2 - (1)^2 \\ 1-i^2 - (1-i)(1+i) & (1-i^2 + 1-i^2) \end{bmatrix}$$

$i^2 = -1$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^\theta A = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -(1+i) & (1+i) \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (1-i^2 + 1-i^2) & (1-i)(-1+i) + (1-i)^2 \\ -(1+i)^2 - (1+i)^2 & -(1+i)(-1+i) + 1-i^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2-2i^2 & -1+i+i-i^2+1-2i+i^2 \\ -1-2i-i^2+1+2i+i^2 & -(-1+i-i+i^2)+1+1 \end{bmatrix}$$

$-(-1-1)+(1+1)$

$-(-2)+2$

$=4$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A^{-1} = A^\theta$

$= \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$\therefore \boxed{AA^\theta = I = A^\theta A}$

A is unitary matrix

Skew Hermention matrix

5 marks ★★★

(2) Show that $(I-A)(I+A)^{-1}$ is unitory for $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

Given $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$(I-A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$I+A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$

$(I+A)^{-1} = \frac{1}{|I+A|} \begin{bmatrix} 0 & -1-2i \\ 1-2i & 0 \end{bmatrix}$

where $|I+A| = 1-(1+2i)(-1+2i)$
$= 1-[(2i)^2 - i^2]$
$= 1-[-4-1] = 6$

$(I+A)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$(I-A)(I+A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 1+(-1-2i)(1-2i) & -1-2i-1-2i \\ 1-2i+1-2i & (1+2i)(1-2i)+1 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} 1-1+(2i)^2 & -2-4i \\ 2-4i & -1-4i^2+1 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$

Rank of a matrix :-

If A is a rectangular matrix of order $m \times n$ then a non-negative integer $\boxed{r}$ is said to be rank of a matrix if

(i) There exist atleast one non-zero minor of order r

(ii) every minor of order greater than r is equal to zero

e.g $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ - non-zero minor of order 1 is $|2| \neq 0$
= every minor of order greater than one is zero i.e $\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0$

Note: (1) If A is non singular is $|A| \neq 0$ then rank of A is same as order of A

(2) If A is singular is $|A| = 0$ then rank of A is less than order of A

(3) Rank of null mastrix is always zero $R=0$

(4) Rank of A is denoted by $\rho(A)$.

(5) $\boxed{\rho(A) = \rho(A^{-1})}$

$\rho(A)$

Methods for determining rank:

(1) Raw Echelon form

(2) First canonical form/Normal form

(3) PAQ normal form

(1) Row Echelon form of a matrix:

A matrix said to be in raw echelon form if (i) there are some zero rows appearing at the bottom of the matrix.

and/or

(ii) the no. of zero element before a non-zero element in a row is less than next such r

(iii) The no. of non-zero row in row echel form of a matrix is called rank of a matrix

e.g $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow$ rank is 1

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \Rightarrow$ rank is 3

$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ rank is 2 e.g

~~Reduce~~

(1) Express the following matrix to echelon form and finds its rank $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

=> $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 5R_1$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 8R_2$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix}$$

$R_3 \rightarrow R_3 \times (-\frac{1}{12})$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$$

Rank of A = 3

(2) Evaluate the rank of the following matrices by reducing them to Echelon form :

(i) $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

Rank of (A) $= \rho(A) = 2$

elementary raw transformation

full imp hai

1st $R_i \leftrightarrow R_j$

2nd $R_i \rightarrow KR_i$ for $K \neq 0$ constant

3rd $R_i \rightarrow R_i \pm K R_j$

eg(1) Find $\rho(A)$ by reducing into echelon form where

$$A = \begin{bmatrix} 1 & 3 & -1 \\ -1 & 0 & 3 \\ 3 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} \boxed{1} & 3 & -1 \\ -1 & 0 & 3 \\ 3 & -1 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + R_1$ $\quad R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & \boxed{3} & 2 \\ 0 & -10 & 3 \end{bmatrix}$$

this format is not right to use

Note :- Not use $R_3 \rightarrow 3R_3 + 10R_2$

by $R_3 \rightarrow R_3 - \left(\frac{-10}{3}\right) R_2$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & \frac{29}{3} \end{bmatrix}$$

which is raw echelon form

$\boxed{\rho(A) = 3}$

(2) $A = \begin{bmatrix} \boxed{1} & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ find $\rho(A)$ by reducing it into echelon form

$\Rightarrow$ by $R_2 \rightarrow R_2 + 2R_1$ $\quad R_3 \rightarrow R_3 - R_1$ $\quad$ ~~$R_4 \rightarrow R_4$~~

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

zero osandry order me chaiye

by $R_3 \rightarrow R_3 - \left(-\frac{2}{3} R_2\right)$ $\quad R_4 \rightarrow R_4 - \left(\frac{1}{3} R_2\right)$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is raw echelon form

$\boxed{\rho(A) = 2}$

(2) Evaluate the rank of the following matrices by reducing them to Echelon form:

(i) $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

By $R_2 \rightarrow R_2 + R_1$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

By $R_2 \rightarrow R_2 + 2R_3$ $\quad R_3 \rightarrow R_3 + 2R_2$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is row echelon form

$\rho(A) = 3$

(ii) $B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

By $R_2 \rightarrow R_2 - 2R_1$ $\quad R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

By $R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is row echelon form

$\rho(A) = 2$

(3) The rank of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$\Rightarrow$ By $R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$-1(-1) + 1(1)$

2

By $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

which is A is ~~not~~ row echelon form

$\boxed{\rho(A) = 3}$

(2) Rank of matrix, Reduction to normal form

It can reducing by row and column matrix

Normal form / Canonical Form :-

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Indentity matrix

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$$

- Every $m \times n$ matrix can be reduced to Normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by sequence of elementary $\boxed{\text{row}}$ and $\boxed{\text{column}}$ transfor-mation

used to reduce in form of Normal form

- (Then rank of a matrix is same as order of identity matrix of normal form)

Special cases of normal form are

$[I_r \; 0]$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, $[I_r]$

(ii) By By By

e.g Evalute the rank of the following matrices by reducing them to Normal form:

(i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

By $R_2 \to R_2 - R_1$ $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$$

$C_2 \to C_2 - C_1$ $C_3 \to C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$$

$C_2 \to C_2 \times (-1/2)$ $C_3 \to C_3 \times (-\frac{1}{2})$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ which is normal form

$\rho(A) = 2$

(ii) $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

By $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

By $R_3 \to R_3 - 3R_1$ $\quad R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -12 & -7 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

By $C_3 \to C_3 - 4C_1$ $\quad C_4 \to C_4 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -12 & -7 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

By $R_3 \to R_3 - R_2$ $\quad R_4 \to R_4 - R_2$

$-12+3$ $\quad -6+3$

-9

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

By $C_3 \to C_3 + 3C_2$ $\quad C_4 \to C_4 + C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

By $C_3 \to C_3 \times (-\frac{1}{3})$ $\quad C_4 \to C_4 \times (-\frac{1}{2})$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

which is Normal form

$\underline{\underline{\rho(A) = 3}}$

(2) Find the rank of the matrix by reducing it to normal form $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

$\Rightarrow$ $R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & -1 & 3 & 6 \\ 0 & 4 & -6 & -10 \\ 5 & 3 & 3 & 11 \end{bmatrix}$$

$R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & -1 & 3 & 6 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix}$$

$C_2 \to C_2 + C_1$

$$\begin{bmatrix} 1 & 0 & 3 & 6 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix}$$

$C_3 \to C_3 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix}$$

$C_4 \to C_4 - 6C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 \div 4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + \frac{3}{2} C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 + 5/2\, C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\cancel{R_3 \to R_3 \times 0}$

$$\cancel{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$C_3 \leftrightarrow C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

rank of (A) $= \rho(A) = 3$

(iii) $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & -1 & 2 & 2 \\ 4 & 1 & 0 & -1 \\ 9 & 1 & 5 & 6 \end{bmatrix}$

$C_1 \leftrightarrow C_2$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 2 & 2 \\ 1 & 4 & 0 & -1 \\ 1 & 9 & 5 & 6 \end{bmatrix}$$

By $R_2 \to R_2 + R_1$ $R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 5 & 6 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

~~By~~ $\cancel{R_2 \to R_2 - 2R_3}$

$$\cancel{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 11 & -17 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}}$$

By $C_2 \to C_2 - 2C_1$ $C_3 \to C_3 - 3C_1$

$C_4 \to C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 5 & 6 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

By $R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 11 & 16 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

By $R_3 \to R_3 - 2R_2$

$R_4 \to R_4 - 7R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 11 & 16 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

By $C_3 \to C_3 - 11C_2$, $C_4 \to C_4 - 16C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

By $R_3 \to R_3 + 2R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

By $R_4 \to R_4 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

By $C_4 \to C_4 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

By $C_4 \to C_4 \times \left(\frac{1}{4}\right)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [I_4]$$

which is normal form

$\boxed{\rho(A) = 4}$

$\to R_3 - 7R_2$

$\to R_4 - 7R_2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 11 & 16 \\ 0 & -3 & -5 \\ 0 & 2 & 2 \end{bmatrix}$$

$C_3 - 11C_2,\ C_4 \to C_4 - 16C_2$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -3 & -5 \\ 2 & 2 \end{bmatrix}$$

$R_3 + 2R_4$

$-5 + 4$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}$$

$R_4 - 2R_3$

$2 + 2$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$$

C_3

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

$R_4 \times \left(\frac{1}{4}\right)$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \left[I_4 \right]$$

(iv) $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

By $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

By $C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - 3C_1$

$C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

By $R_2 \to R_2 \times \left(-\frac{1}{1}\right)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

By $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By $C_3 \to C_3 - C_2$, $C_4 \to C_4 - 3C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

which is normal form

$\boxed{\rho(A) = 2}$

(v) $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

By $R_3 \to R_3 - R_1$
$R_4 \to R_4 + 2R_1$

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & 2 \end{bmatrix}$$

By $C_3 \to C_3 - 2C_1$
$C_4 \to C_4 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & 2 \end{bmatrix}$$

By $C_3 \to C_3 + C_2$
$C_4 \to C_4 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix}$$

By $C_3 \to C_3 + 2C_1$
$C_4 \to C_4 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix}$$

~~ByR~~

By $R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By $C_3' \to C_3 \times \left(\frac{1}{16}\right)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

which is, normal form

$\boxed{\rho(A) = 3}$

~~By $R_4 \to R_4 + R_3$~~

$$\cancel{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

By $R_4 \to R_4 - R$

(vi) $A = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \\ 0 & -4 & -1 & 2 \end{bmatrix}$

By $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \\ 0 & -4 & -1 & 2 \end{bmatrix}$$

By $R_2 \to R_2 - 2R_1$ $\quad R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & -4 & -1 & 2 \end{bmatrix}$$

By $C_3 \to C_3 - C_1$

$C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & -4 & -1 & 2 \end{bmatrix}$$

By $R_2 \to R_2 \times (-1/1)$

$R_3 \to R_3 - 3R_2$

$R_4 \to R_4 - 4R_2$

$R_5 \to R_5 + 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & & & 4 \\ 0 & 0 & 3 & 10 \\ 0 & 0 & 5 & -10 \end{bmatrix}$$

By $C_3 \to C_3 - C_2$

$C_4 \to C_4 - 3C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

~~By $C_3 \to C_3 - C_4$~~

$R_4 \to R_4 \times (\frac{1}{5})$

$R_5 \to R_5 \times (-1/5)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

By $R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$

By $R_5 \to R_5 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ ~~next page~~

pending

3 Rank of Matrix, Reduction to PAQ normal

For a rectangular matrix A of order m×n

we write $A = I_m \, A \, I_n$ → postfactor

I_m → prefactor

- reduce LHS matrix A into Normal form

by using row and column transformation

- On R.H.S we apply only,

row transformation to Prefactor

Column transformation to Postfactor

$$A = I_m \quad A \quad I_n$$

Normal Form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = P \quad A \quad Q$ which is called PAQ normal form

P → non-singular matrix

Note: (1) In PAQ Normal form P and Q are non-singular matrices which are completely determined by transformation used

(2) In PAQ Normal form $\rho(A)$ = order of identity matrix on L.H.S

(3) If order of identity matrix is same as order of matrix

then $PAQ = I$

$$A = P^{-1} I Q^{-1}$$

$$A = P^{-1} Q^{-1}$$

$$A^{-1} = (P^{-1} Q^{-1})^{-1}$$

$$\boxed{A^{-1} = QP}$$

10 marks

① Find non-singular matrices P and Q such that PAQ is in normal form and hence find $\rho(A)$ and $\rho(PAQ)$ and Find inverse of matrix A if exists for the fellowing.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

we known

prefactor, post factor

$$A = I_3 \, A \, I_3$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 \to R_3 - R_1$
$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & +1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_2 \to C_2 - 3C_1$
$C_3 \to C_3 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & +1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A. \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 3$ $\quad \rho(PAQ) = 3$

$[I_3] = PAQ$ which is a normal form

Now $PAQ = I_r$

$PAQ = I_3$

where,

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = QP$$

$$= \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} (1+3+3) & (-3) & (-3) \\ (-1) & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$ //

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

$\Rightarrow$ we known

$A = I_m A I_n$

$A = I_3 A I_3$

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_2 \to C_2 - 2C_1$

$C_3 \to C_3 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 7 \\ 0 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \to R_2 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 \to R_3 + 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & -16 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 4 & 10 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_3 \to C_3 + 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -16 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 4 & 10 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -9 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 \to R_3 \times (-1/16)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ -1/16 & -4 & -5/8 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -9 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cancel{S(P) = 3}$ $S(PAQ) = 3$

$[I_3] = PAQ$ which is a normal form

Now $PAQ = I_r$

$PAQ = I_3$

where $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ -1/16 & -4 & -5/8 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & -2 & -9 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

$A^{-1} = QP$

$$= \begin{bmatrix} 1 & -2 & -9 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ -\frac{1}{16} & -4 & -5/8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{16} & 34 & -3/8 \\ -5/16 & -19 & -1/8 \\ -1/16 & -4 & -5/8 \end{bmatrix}$$

(iii) $A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 1 & 4 & -5 \\ -1 & -5 & -5 & 7 \end{bmatrix}_{\substack{m \times n \\ 3\times 4}}$

$\Rightarrow$ we known

$A = I_m \times A \times I_n$

$A = I_3 A \times I_4$

$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 1 & 4 & -5 \\ -1 & -5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \to R_2 - 2R_1$ $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 3 \\ 0 & -3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & 3 \\ 0 & -3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_2 \to C_2 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 4 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 4 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_3 \to C_3 + 2C_2$

$C_4 \to C_4 - 3C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -4 & 5 & -16 \\ 0 & 1 & 2 & -3 \\ 0 & -2 & -3 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

$$\boxed{\rho(A) = 2 = \rho(PAQ)}$$

(i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

we known

$$A = I_3 A I_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By ~~$R_2 \times (-1/2)$~~ ~~$R_3 \times (-1/2)$~~ $C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \to R_2 \times (-1/2)$ $R_3 \to R_3 \times (-1/2)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ +1/2 & -1/2 & 0 \\ 3/2 & 0 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 3/2 & 1/2 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 3/2 & 1/2 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

which is PAQ normal form

$\rho(A) = 2$

(i) $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}_{3\times4}$

we known

$$A = I_3 A I_4$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \to R_2 \times (-1/1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_3 \to C_3 - C_2$, $C_4 \to C_4 - 3C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$ which is normal form

$$\boxed{\rho(A) = 2 = \rho(PAQ)}$$

(3) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

we known

$A = I_m \times A \times I_n$

$A = I_3 \, A \, I_4$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - 3C_1$

$C_4 \to C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_2 \to C_2 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -5 \\ 0 & -2 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \to R_2 \times (-1/1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ +2 & -1 & 0 \\ 3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 7 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - 2C_2$

$C_4 \to C_4 - 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 7 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -3 & -6 \\ 0 & 1 & -2 & -5 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_4 \to C_4 \times (-1/12)$

$C_3 \leftrightarrow C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 7 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1/2 & -5 \\ 0 & 1 & 5/2 & -2 \\ 0 & -1 & -5/2 & 3 \\ 0 & 0 & -1/12 & 0 \end{bmatrix}$$

$[I_3 \; 0] = PAQ$ which is PAQ Normal form

$\rho(A) = 3$

Matrix

(II) PAQ normal form :-

For a rectangular matrix A of order $m \times n$ we write $\boxed{A = I_m A I_n}$ → postfactor

Prefactor

(1) reduce LHS matrix A into normal form by using row and column transformation

(2) On R.H.S we apply only row transformation to Prefactor and only column transformation to Postfactor

$$A = I_m A I_n$$

Normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \underbrace{P A Q}$ which is called PAQ normal form

significance → non-singular matrix

Note : (1) In PAQ Normal form P and Q are non-singular matrices which are completely Determined by transformation used.

(2) In PAQ normal form $\rho(A)$ = order of identity matrix on LHS

(3) If order of identity matrix is same as order of matrix.

then $PAQ = I$ $\qquad A = P^{-1} I Q^{-1}$ $\qquad A = P^{-1} Q^{-1}$

$$A^{-1} = (P^{-1} Q^{-1})^{-1}$$

$$A^{-1} = (Q^{-1})^{-1} (P)^{-1}$$

$$\boxed{A^{-1} = QP}$$

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(e.g.) 10 marks

① Find non-singular matrices P and Q such that PAQ is in normal form and hence find $\rho(A)$ and $\rho(PAQ)$ also find A^{-1} if exist

$\Rightarrow$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

we know

$$A = I_3 \, A \, I_3$$

pivot element

$$\begin{bmatrix} \boxed{1} & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \to R_2 - R_1$ ~~$C_2 \to$~~

$R_3 \to R_3 - R_1$ ~~$C_3 \to$~~

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

by $C_2 \to C_2 - 3C_1$

$C_3 \to C_3 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 3$

$\rho(PAQ) = 3$ $\quad [I_3] = PAQ$ → spectrical

In Triangular matrix

$|A| =$ product of all diagonal

$[I_3] = PAQ$ which is a normal form

Note-3

Note-3

according to > Now $PAQ = I_3$ where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = P^{-1} I_3 Q^{-1}$ as P and Q are non-singular matrices

$A = P^{-1} Q^{-1}$

$A^{-1} = (P^{-1} Q^{-1})^{-1}$

$\boxed{A^{-1} = QP}$

$$A^{-1} = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3+3 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Non-homogeneous system of linear algebraic equations

consider

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\}$$ system of m-equations in n-variable/unknown

In matrix form $\boxed{AX = B}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ ... coefficient matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$ ... column matrix of unknown

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$ → column matrix of RHS constant

$[A \mid B]$ ..... Augmented matrix

System $AX = B$ is said to be non-homogeneous if $B \neq 0$ i.e atleast one RHS constant is non-zero

To test consistency of system (i.e to test solution exist or not)

~~Reduce matrix A~~

** Reduce matrix A into row - echelon form along with augmented matrix [A ⋮ B]

<u>case i :</u>

If $\boxed{\rho(A) = \rho(A \vdots B)}$ then System is "Consistent" (i.e it has a solution)

Further (a) if $\rho(A)$ = no. of variable $= n$

=> system has <u>Unique (finite) solution</u>

(b) if $\rho(A)$ < no. of variables

=> system has <u>infinitely many solution</u>

<u>case i :</u>

If $\boxed{\text{If } \rho(A) \neq \rho(A \vdots B)}$ then System is "inconsistent" (i.e it <u>has no solution</u>)

e.g.

(1) Discuss the consistency of the system and solve them if consistent.

(i)
$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$2x + 4y + 7z = 30$$

→ ∴ In matrix form $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

∴ Augmented matrix : $[A \vdots B]$

pivot element

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 4 & 7 & 30 \end{array}\right]$$

by $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 18 \end{array}\right] \cancel{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \cancel{\begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}}$$

by $R_3 \to R_3 - 2R_2$

pivot element

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \cancel{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

which is in row echelon form

$$\rho(A) = \rho(A \vdots B) = 3$$

$\rho(A) = \rho(A \mid B) = 3$

$\therefore$ system is consistent

further $\rho(A) = 3$ = no. of variable

$\Rightarrow$ System has finite (unique) solution

To find solution rewrite equation from row echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x + y + z = 6$ $\qquad$ $x + 4 + 2 = 6$ $\qquad$ $\boxed{x = 0}$

$y + 2z = 8$ $\qquad$ $\therefore$ $y + 4 = 8$ $\qquad$ $\boxed{y = 4}$

$\boxed{z = 2}$

$\Rightarrow$ sol^n $\boxed{x = 0 \; ; \; y = 4 \; ; \; z = 2}$

(ii) $x+y+z=5$; $x+2y+3z=10$; $x+2y+3z=8$

→ In matrix form $AX=B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 8 \end{bmatrix}$$

∴ Augmented matrix : $[A \vdots B]$

Pivot element

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 5 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & 3 & 8 \end{array}\right]$$

By $R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$

Pivot element

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & \boxed{1} & 2 & 10 \\ 0 & 1 & 2 & 8 \end{array}\right]$$

By $R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 0 & 8 \end{array}\right]$$ which is in row-echelon form

$\rho(A) = 2$ and $\rho(A \vdots B) = 3$

$\rho(A) \neq \rho(A \vdots B)$

∴ System is not consistent

Note :- x, y, z coefficient is same therefore we can say that there is non solution

(iii) $x - 2y + z - w = 2$, $x + 2y + 4w = 1$, $4x - z + 3w = -1$

$\rightarrow$ In matrix form $AX = B$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 2 & 0 & 4 \\ 4 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore$ Augmented matrix : $[A \,\vdots\, B] = \left[\begin{array}{cccc|c} \boxed{1} & -2 & 1 & -1 & 2 \\ 1 & 2 & 0 & 4 & 1 \\ 4 & 0 & -1 & 3 & -1 \end{array}\right]$

pivot element

By $R_2 \rightarrow R_2 - R_1$ $R_3 \rightarrow R_3 - 4R_1$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 2 \\ 0 & 4 & -1 & 5 & -1 \\ 0 & 8 & -5 & 7 & -9 \end{array}\right]$$

$R_3 \rightarrow R_3 - 2R_2$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 2 \\ 0 & 4 & -1 & 5 & -1 \\ 0 & 0 & -3 & -3 & -7 \end{array}\right]$$ which is in row echelon form

$\rho(A) = 3 = \rho(A \,\vdots\, B)$

System is consistent

Further $\rho(A) = 3 < 4 =$ no. of variables

infinite solution

To find infinite solution

$x - 2y + z - w = 2$

$4y - z + 5w = -1$

$-3z - 3w = -7$

Note: { we solve the equation we need a parameter }

it's difference b/w rank and no. variable

we can have any value

put $\boxed{w=t}$ t ... parameter

$$-3z - 3w = -7$$

$$-3z - 3t = -7$$

$$\boxed{z = \frac{7-3t}{3}}$$

$$4y - z + 5w = -1$$

$$4y - \frac{7-3t}{3} + 5w = -1$$

$$12y - 7 - 3t + 15w = -3$$

$$12y - 3t + 15w = 4$$

$$12y - 3t + 15t = 4$$

$$12y + 12t = 4$$

$$\boxed{y = \left(\frac{4-12t}{12}\right)}$$

$$y = \left(\frac{1-3t}{3}\right) \quad ?$$

(i) when $\boxed{\cancel{\lambda \neq 5 \quad \mu = 5}}$ $\boxed{\lambda = 5 \quad \mu \neq 9}$

$\lambda = 5 \Rightarrow (\lambda - 5) = 0 \Rightarrow \rho(A) = 2$

$\mu = 9 \Rightarrow (\mu - 9) \neq 0 \Rightarrow \rho(A:B) = 3$

$\therefore \rho(A) \neq \rho(A:B) \Rightarrow$ No solution

(ii) when $\boxed{\lambda \neq 5}$ and you can have any value

$\lambda \neq 5 \Rightarrow (\lambda - 5) \neq 0 \Rightarrow \rho(A) = 3$ and $\rho(A:B) = 3$

$\therefore \rho(A) = 3 = \rho(A:B) =$ no. of variable

$\Rightarrow$ system has unique solution

(iii) when $\lambda = 5 \quad \mu = 9$

$(\lambda - 5) = 0 \Rightarrow \rho(A) = 2$

$(\mu - 9) = 0 \Rightarrow \rho(A:B) = 2$

$\rho(A) = \rho(A:B) = 2$ $\rho(A) <$ no. of variable

$\Rightarrow$ System has infinite solution.

(4) Investigate for what value of λ and μ the system of equations $2x + 3y + 5z = 9$;
$7x + 3y - 2z = 8$; $2x + 3y + \lambda z = \mu$

have (1) No solution (2) Unique solution (3) many solⁿ.

$\Rightarrow$ In matrix form $AX = B$

$$\begin{bmatrix} 2 & 3 & +5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Augmented matrix $[A \mid B]$

$$\left[\begin{array}{ccc|c} 2 & 3 & +5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - \frac{7}{2} R_1 \qquad R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15/2 & \frac{-39}{2} & \frac{-47}{2} \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$ which is row echelon form