Module - 4 Matrices

Matrix : Matrix is a set of n×n elts (real or complex) arranged in a rectangular array of m rows & n columns enclosed by a pair of square brackets. It's written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$A = [a_{ij}]_{m \times n}$$

Inverse of a matrix A:

If a square matrix A with $|A| \neq 0$ then $AA^{-1} = I = A^{-1}A$, where A^{-1} is called of a matrix A.

$$A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj} A$$

Types of Matrices

① Diagonal M: $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ & & c \end{bmatrix}$ Identity M

② Scalar M: All diagonal are equal

eg $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

③ UTM : Below diagonal are zero

$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ ④ LTM: $\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$

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(5) Trace of matrix : Sum of diagonal el.

eg $A = \begin{bmatrix} a & c \\ d & b \end{bmatrix}$ $Tr(A) = a + b$

(6) Transpose of M: If $A = [a_{ij}]_{m \times n}$

Then $A^T = [a_{ji}]_{n \times m}$

(7) Symmetric M: If $A = A^T$

i.e. $[a_{ij}] = [a_{ji}]$

(8) Skew-symmetric: If $A = -A^T$

$[a_{ij}]_{m \times n} = -[a_{ji}]_{n \times m}$

(9) Conjugate of a Matrix: A matrix obtained by taking complex conjugate of a every ells of A

eg $A = \begin{bmatrix} 1+i & 2-3i \\ 5-7i & 3 \end{bmatrix}$

$\bar{A} = \bar{A} = \begin{bmatrix} 1-i & 2+3i \\ 5+7i & 3 \end{bmatrix}$

(10) Conjugate transpose of M.

The transpose of conjugate of matrix A is called conjugat transpose of A is denoted by A^{θ}

eg. $(\bar{A})^T = (\overline{A^T})$

Exercise 2.8

Hermitian & Skew-Hermitian Matrices

1) Hermitian Matrix : A square matrix $A=[a_{ij}]$ is called Hermitian if

$[a_{ij}] = [\overline{a_{ji}}]$ for all i & j

i.e. $A = A^{\theta}$

eg. $A = \begin{bmatrix} 1 & 3+4i & 4-i \\ 3-4i & 0 & 1+3i \\ 4+i & 1-3i & 2 \end{bmatrix}$

2) Skew-Hermitian Matrix :

A square matrix $A=[a_{ij}]$ is called skew Hermitian if

$[a_{ij}] = -[\overline{a_{ji}}]$ for all i, j

i.e. $A = -A^{\theta}$

$A = \begin{bmatrix} 5i & 3-2i \\ -3-2i & 5i \end{bmatrix}$

$\begin{bmatrix} 5i & 3-2i \\ -3-2i & 5i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} -5i & -3+2i \\ 3+2i & -5i \end{bmatrix}$

ex. $\begin{bmatrix} i & 2+3i \\ -2+3i & 0 \end{bmatrix}$ $A^{\theta} = \begin{bmatrix} i & 2+3i \\ 2+3i & 0 \end{bmatrix}$

$A = \begin{bmatrix} 5i & -(3-2i) \\ -(3+2i) & 5i \end{bmatrix}$

$= [5i$

$-A^{\theta} = \begin{bmatrix} -i & -(2-3i) \\ -(2+3i) & 0 \end{bmatrix}$ $-A^{\theta} = [\ -i \quad 2+3i$

3) orthogonal : $AA^T = I$ self inverse

$A^T = A^{-1}$ $A^TA = I$

4) Unitary M.

$AA^{\theta} = I = A^{\theta}A$

$A = \begin{bmatrix} i & 2+3i \\ 2-3i & 0 \end{bmatrix}$. $\bar{A} = \begin{bmatrix} -i & 2-3i \\ 2+3i & 0 \end{bmatrix} = (\bar{A})^T = A^{\theta} = \begin{bmatrix} -i & 2+3i \\ 2-3i & 0 \end{bmatrix}$

$= -\begin{bmatrix} i & -(2+3i) \\ -(2-3i) & 0 \end{bmatrix} = -\begin{bmatrix} i & -2 \\ -2+3i & \end{bmatrix}$

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Ex (2) Express matrix A as sum of symmetric & skew symmetric matrix where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$\rightarrow$

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix}$$

P is symmetric

$$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$$

Q is skew sym.

$$P + Q = \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

or

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

Ex. 30

(2) Express matrix A as sum of Hermetian & skew Hermetian matrix where

$$A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$$

→ $$A' = \begin{bmatrix} 3i & 1+i & -3-2i \\ -1+i & -i & -1+2i \\ 3-2i & 1+2i & 0 \end{bmatrix}$$

$$A^{\theta} = (\bar{A})' = \begin{bmatrix} -3i & 1-i & -3+2i \\ -1-i & i & -1-2i \\ 3+2i & 1-2i & 0 \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^{\theta}) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Herm

$$Q = \frac{1}{2}(A - A^{\theta}) = \frac{1}{2}\left\{\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} - \begin{bmatrix} 3i & 1-i & -3+2i \\ -1-i & -i & -1-2i \\ 3+2i & 1-2i & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 6i & -2+2i & 6-4i \\ 2+2i & -2i & 2+4i \\ -6-4i & -2+4i & 0 \end{bmatrix} = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$$

Skew Her

$$\therefore A = P + Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + Q = A$$

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Orthogonal Matrix

A square matrix A is called orthogonal if $AA^T = I = A^T A$

Exercise 3.1

1) Show that if A is orthogonal then $|A| = \pm 1$

→ Let A be orthogonal matrix

$\therefore AA^T = I$ ---- (1)

To prove $|A| \neq 0$

from eqⁿ (1) $|AA^T| = |I|$

$\Rightarrow |A||A^T| = 1$ $\because |I| = 1$

$\Rightarrow |A||A| = 1$ $\because |A| = |A^T|$

$\Rightarrow |A|^2 = 1$

$\Rightarrow |A| = \pm 1 \neq 0$

(2) Verify following matrix is orthogonal & hence find its inverse

→

$$A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ -1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

Ans $$A' = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$AA' = \frac{1}{81}\begin{bmatrix} 64+16+1 & -8+16-8 & 32-28-4 \\ 8-16+8 & 1+16+64 & 4+28-32 \\ 32-28+4 & 4+28-32 & 16+49+16 \end{bmatrix}$$

$$AA' = \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal matrix

$\therefore A^{-1} = A'$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

(3) Find a, b, c & A^{-1} if

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \text{ is orthogonal}$$

→ Given A is orthogonal

$\therefore AA^T = I$ ... (1)

$$A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix}$$

$$AA^T = I \Rightarrow \frac{1}{9}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5+a^2 & 4+ab & -2+ac \\ 4+ab & 5+b^2 & 2+bc \\ 2+ac & 2+bc & 8+c^2 \end{bmatrix} = 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ Equating both sides

$\therefore 5+a^2 = 9$ & $4+ab=0$ ... (2)

$5+b^2 = 9$ & $-2+ac=0$ ... (3)

$8+c^2 = 9$ & $2+bc=0$ ... (4)

$\therefore a^2 = 4, \Rightarrow a = \pm 2$

$b^2 = 4 \Rightarrow b = \pm 2$

$c^2 = 1 \Rightarrow c = \pm 1$

case (1) If we choose $a=2, b=2$ & $c=1$ then eq^n (2) & (4) are not satisfied as $4+4 \neq 0$, $2+2 \neq 0$

case (2) If we choose $a=2, b=-2, c=1$ or $a=-2, b=2, c=-1$

then eq^n (2), (3) & (4) are satisfied

$\therefore a=2, b=-2, c=1$ or $a=-2, b=2, c=-1$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

or

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$$

Ex (2) E

Sym

whe

→ A^T

P=

P

Q=

Q

P+Q

Rank of Matrix.

A non negative integer 'r' is said to be rank of a matrix A if

(i) There exist atleast one non-zero minor of order 'r'.

(ii) Every minor of order greater than r is equal to zero.

Eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ A has 9 minors of order 1
A has 9 " " of 2
A has 1 minor " " 3

$|A| = 0$, $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$

$\Rightarrow \rho(A) < 3$, $\rho(A) = 2$.

Note: ① The rank of matrix A is denoted by $\rho(A)$

② Rank of any matrix can be calculated

③ If $|A| = 0$ or singular then $\rho(A) <$ order of A

③ If $|A| \neq 0$ or Nonsingular then $\rho(A) =$ order of A

④ If A is of order $m \times n$ Then $\rho(A) \leq \min(m, n)$

Methods to find rank of matrix

1. Normal form / canonical Form
2. PAQ Normal form
3. Row Echelon form.

1. Normal Form of matrix.

Any matrix of order $m \times n$ can be reduced into the form

$$\begin{bmatrix} I_r & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$$ by finite number of elementary row / column transformation which is called Normal form / first canonical form of matrix.

Note: 1. The equivalent types of normal form of a matrix are

$[I_r : 0]$ or $\begin{bmatrix} I_r \\ \vdots \\ 0 \end{bmatrix}$ or $[I_r]$

2. In normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ the order of identity matrix is called rank.

3. Elementary row transformation are

(i) $R_i \leftrightarrow R_j$ (ii) KR_i $K \neq 0$ constant

(iii) $R_i \pm KR_j$

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Exercise 33

(1) Evaluate the rank of following matrices by reducing them to normal form

(I) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

(I) PAQ Normal form

If A is ~~rectangular~~ any matrix of order $m \times n$ In this method we write given matrix A as

$$A = P_m A Q_n \quad \cdots (1)$$

Then apply row as well column transformation to reduce matrix A on left side to normal form and row transformation on P on right side & column transform on Q where P & Q are nonsingular mt

i.e. $A = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$

Note ; (1) In PAQ normal form

$$\rho(A) = \rho(PAQ)$$

(2) If $PAQ = I$ then $AQ = P^{-1}I = P^{-1}$

$$\Rightarrow A = P^{-1}Q^{-1}$$

$$\Rightarrow A^{-1} = (P^{-1}Q^{-1})^{-1} = QP$$

(3) $\rho(A) = \rho(A^{-1})$

① leading element of each row must be unity
② If any zero row is present then it must be at last
③ In each row, numbers of zeros must be of more than previous row

III) Row Echelon form

A matrix is said to be in row Echelon form if

(i) There are some zero rows appearing at bottom of matrix

(ii) (The leading) The number of zero elements before a nonzero elements in a row is less than next such row

Ex. $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Note: The number of nonzero rows in row Echelon form gives rank of matrix

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_2$ $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{pmatrix} * & & & \\ 0 & * & & \\ 0 & 0 & * & \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

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Exercise ~~33~~

(1) Evaluate the rank of following matrices by reducing them to Normal form & Echelon form

(1) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & -1 & 2 & 2 \\ 4 & 1 & 0 & -1 \\ 9 & 1 & 5 & 6 \end{bmatrix}$

→ Normal form

NF (1) (i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \underset{R_3 - 3R_1}{\overset{R_2 - R_1}{\sim}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$

$C_2 - C_1,\ C_3 - C_1 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$

$R_3 - R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$\frac{1}{2} R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{C_3 - C_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

which is ~~in normal form~~

So $\rho(A) = 2$

by

(ii) Echelon form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \underset{\sim}{\substack{R_2 - R_1 \\ R_3 - 3R_1}} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$$

$$R_3 - R_2 \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in row Echelon form so $\rho(A) = 2$

ie. # of nonzero rows in REF are 2.

(iii) Normal form

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Normal form

(2) $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \underset{\sim}{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$\begin{matrix} R_3 - 3R_1 \\ R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -12 & -7 \\ 0 & 1 & -6 & -3 \end{bmatrix} \begin{matrix} R_3 - R_2 \\ R_4 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$

$\begin{matrix} C_3 - 4C_1 \\ C_4 - 3C_1 \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix} \underset{\substack{C_3+3C_2 \\ C_4+C_2}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$

$-\frac{1}{3} C_3, -\frac{1}{2} R_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_4 - 3R_3$

$R_4 - 3R_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\rho(A) = 3$

which is in normal form

(b) Row Echelon for

$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \underset{\sim}{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

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$\begin{matrix} R_3 - 3R_1 \\ R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -12 & -7 \\ 0 & 1 & -6 & -3 \end{bmatrix} \begin{matrix} R_3 - R_2 \\ R_4 - R_3 \\ \sim \\ R_4 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$

$\begin{matrix} R_3 - R_2 \\ R_4 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix} \underset{\sim}{R_4 - \frac{1}{3} R_3} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_4 + 2R_3 \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & \end{bmatrix}$ which is in Echelon form so

$\rho(A) = \#$ non zero rows $= 3$

NF

(3) $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & -1 & 2 & 2 \\ 4 & 1 & 0 & -1 \\ 9 & 1 & 5 & 6 \end{bmatrix}$

(i) Normal form

$C_1 \leftrightarrow C_2 \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 2 & 2 \\ 1 & 4 & 0 & -1 \\ 1 & 9 & 5 & 6 \end{bmatrix} \begin{matrix} R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 5 & 6 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$

$\begin{matrix} C_2 - 2C_1 \\ C_3 - 3C_1 \\ C_4 - 4C_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 5 & 6 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix} R_2 - 2R_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 11 & 16 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$

$R_3 - 2R_1$
$R_4 - 7R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 11 & 16 \\ 0 & 0 & -25 & -37 \\ 0 & 0 & -75 & -110 \end{bmatrix}$$

$C_3 - 11C_2$
$C_4 - 16C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -25 & -37 \\ 0 & 0 & -75 & -110 \end{bmatrix}$$

$$\sim \ \frac{-1}{25} C_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -37 \\ 0 & 0 & 3 & -110 \end{bmatrix}$$

$$R_4 - 3R_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -37 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_4 + 37C_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

$\rho(A) = 4$

(2) Echelon form

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & -1 & 2 & 2 \\ 4 & 1 & 0 & -1 \\ 9 & 1 & 5 & 6 \end{bmatrix} \underset{\sim}{C_1 \leftrightarrow C_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 2 & 2 \\ 1 & 4 & 0 & -1 \\ 1 & 9 & 5 & 6 \end{bmatrix}$$

$R_2 + R_1$
$R_3 - R_1$
$R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 5 & 6 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix} \begin{matrix} 5R_3 - 2R_2 \\ 5R_4 - 7R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 5 & 6 \\ 0 & 0 & -25 & -37 \\ 0 & 0 & -25 & -32 \end{bmatrix}$$

$$R_4 - R_3 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 5 & 6 \\ 0 & 0 & -25 & -37 \\ 0 & 0 & 0 & 5 \end{bmatrix} \rho(A) = 4$$

Exercise 34

1) Find nonsingular matrices P & Q s.t. PAQ is in normal form & hence find $\rho(A)$ & $\rho(PAQ)$ & find inverse of matrix A if exists for following

(i) $A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 1 & 4 & -5 \\ -1 & -5 & -5 & 7 \end{bmatrix}_{3\times 4}$

skip

(ii) $A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

(i) Since A is of order 3×4 so write $A = I_3 A I_4$

$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 1 & 4 & -5 \\ -1 & -5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1$
$R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & -3 & -2 & 3 \\ 0 & -3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$C_2 - 2C_1,\ C_3 - 3C_1,\ C_4 + 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & 3 \\ 0 & -3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by $C_2 \to -\frac{1}{3} C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & -3 & 4 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - R_1$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -3 & 4 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 + 2C_2$
$C_4 - 3C_2$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -5/3 & 2 \\ 0 & -1/3 & -2/3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

$\therefore \rho(A) = \rho(PAQ) \Rightarrow \rho(A) = 2$ so
$\rho(PAQ) = 2$

(5) $A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

$\rightarrow$

$A = PAQ$

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1,\ R_2 \rightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \rightarrow C_2 - 2C_1,\ C_3 \rightarrow C_3 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow 7R_3 - 6R_2$

$-34+1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 5 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 7 & -6 & 4 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -7 & 7 & -7 \\ 7 & -6 & 4 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow -\frac{1}{7}R_2$

$R_3 \rightarrow \frac{1}{5}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 7/5 & -4/5 & 4/5 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 3$

(2) Find A^{-1} & $\rho(A)$ after converting to PAQ form where

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$\rightarrow$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1,\ R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \rightarrow C_2 - 3C_1,\ C_3 \rightarrow C_3 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$I_3 = PAQ$ $\rho(A) = 3$

normal form

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Since $PAQ = I$

$\Rightarrow AQ = P^{-1}I = P^{-1}$

$\Rightarrow A = P^{-1}Q^{-1}$

$\Rightarrow A^{-1} = (P^{-1}Q^{-1})^{-1}$

$A^{-1} = QP \quad \because (AB)^{-1} = B^{-1}A^{-1}$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

LD. L3.

Nonhomogeneous linear Systems of Equations

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Consider a system of 'm' equations in 'n' unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

In matrix the above nonhomogeneous system of eqn can be written as

$$AX = B$$

where
$$A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{\text{coefficient matrix}}, \quad X = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{column of unknown}}$$

$$b = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix}}_{\text{column matrix of constant}}$$

$AX = B$

The system is called Nonhomogeneous if $B \neq 0$

The system $AX = B$ is called Homogeneous if $B = 0$

eg

$2x + 3y + 4z = 5$
$5x + 7y + 3z = 7$
$2x - 7y + 3z = 5$

Nonhomo

$AX = B$

$2x + 3y = 0$
$5x - 7y = 0$

Homo

$AX = 0$

Consistency of an eqⁿ

A system of eqⁿs is said to be consistent if system has solution. If system of eqⁿs has no solution then it is inconsistent.

⇒ **Augmented matrix:** If given system of eqⁿ is $AX = B$ then the matrix $[A|B]$ is an augmented matrix.

Conditions for consistency & Inconsistency

Case

1) If $\rho(A) = \rho[A:B] = n$, n = no. of unknowns the system has unique solⁿ

2) If $\rho(A) = \rho(A:B) < n$, no. of unknowns the system has infinite solⁿs

3) If $\rho(A) \neq \rho(A:B)$, the system is inconsistent i.e. no solution.

Exercise 3.6

① Is the following system of eqⁿ is consistent

$x + y + z = 6$

$x - y + 2z = 5$

$3x + y + z = 8$

$2x - 2y + 3z = 7$

→

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix}$$

$\rho(A) = \rho(A|B) = 3$

∴ Unique solⁿ.

Augmented matrix $[A|B]$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{array}\right]$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{array}\right]$$

$$\begin{array}{l} C_2 - C_1 \\ C_3 - C_1 \\ C_4 - 6C_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{array}\right]$$

$$-\frac{1}{2}C_2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -2 & -10 \\ 0 & 2 & 1 & -5 \end{array}\right]$$

$$\begin{array}{l} R_4 - 2R_2 \\ R_3 - R_2 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -1 & -3 \end{bmatrix} \sim_{3R_4 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim_{\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Discuss consistency of system & if it is

(2) if consistent solve eq

(i) $x+y+z=6,\ x+2y+3z=14,\ 2x+4y+7z=30$

→

$$\begin{bmatrix}1&1&1\\1&2&3\\2&4&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\14\\30\end{bmatrix}$$

The Augmented matrix $[A\|B]$ =

$$\begin{bmatrix}1&1&1&:&6\\1&2&3&:&14\\2&4&7&:&30\end{bmatrix}\ \begin{matrix}R_2-R_1\\R_3-2R_1\end{matrix}\ \begin{bmatrix}1&1&1&:&6\\0&1&2&:&8\\0&2&5&:&18\end{bmatrix}$$

$$R_3-2R_2\ \begin{bmatrix}1&1&1&:&6\\0&1&2&:&8\\0&0&1&:&2\end{bmatrix}$$

which is in Row Echelon Form

$\therefore \rho(A)=3=\rho(A\|B)$

$\therefore$ system is consistent

Further $\rho(A)=3=$ # of unknowns

$\Rightarrow$ system has unique soln

To find soln rewrite eqn from REF

$x+y+z=6$

$y+2z=8$

$z=2 \Rightarrow z=\frac{2}{1}$

$\therefore y+2\left(\frac{2}{1}\right)=8 \Rightarrow y+\frac{4}{1}=8 \Rightarrow y=8-\frac{4}{1}$

$=4$

$x+\frac{y}{1}+\frac{z}{1}=6 \Rightarrow x+\frac{38}{5}=6$

$\Rightarrow 5x+38=30$

$\Rightarrow 5x=-8 \Rightarrow x=$

now $y=4$, $z=2$, $x+4+2=6$

$\Rightarrow x=0$

(ii) $x+y+z=5$

$x+2y+3z=10$

$x+2y+3z=8$

→

$$\begin{bmatrix}1&1&1\\1&2&3\\1&2&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5\\10\\8\end{bmatrix}$$

Aug $\begin{bmatrix}1&1&1\\1&2&3\\1&2&3\end{bmatrix}\begin{bmatrix}5\\10\\8\end{bmatrix}$

$$\begin{matrix}R_2-R_1\\R_3-R_1\end{matrix}\ \left[\begin{array}{ccc|c}1&1&1&5\\0&1&2&5\\0&1&2&3\end{array}\right]$$

$$R_3-R_2\ \left[\begin{array}{ccc|c}1&1&1&5\\0&1&2&5\\0&0&0&-2\end{array}\right]$$

$\rho(A)=2$, But $\rho(A|B)=3$

$\rho(A)\neq\rho(A|B)$ so inconsistent

(iii) $x-2y+z-w=2$

$x+2y+4w=1$

$4x-z+3w=-1$

→

$$\begin{bmatrix}1&-2&1&-1\\1&2&0&4\\4&0&-1&3\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

Aug

$$\begin{matrix}R_2-R_1\\R_3-4R_1\end{matrix}\ \left[\begin{array}{cccc|c}1&-2&1&-1&2\\0&4&-1&5&-1\\0&8&-5&7&-9\end{array}\right]$$

$$R_3-2R_2\ \left[\begin{array}{cccc|c}1&-2&1&-1&2\\0&4&-1&5&-1\\0&0&-3&-3&-7\end{array}\right]$$

Which is in row Echelon form

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$\therefore \rho(A) = \rho(A|B) = 3 < \text{No. of unknown} = 4$

So system given has infinite sol^n

$3 < 4$

$\therefore$ we introduce $(4-3) = 1$ parameter

put $w = t$, where t is parameter

$\therefore -3z - 3w = -7$

$\Rightarrow -3z - 3t = -7 \Rightarrow z = \frac{7-3t}{3}$

$\therefore 4y - z + 5w = -1$

$\Rightarrow 4y - \left(\frac{7-3t}{3}\right) + 5t = -1$

$\Rightarrow 4y = \left(\frac{7-3t}{3}\right) - 5t - 1$

$4y = \frac{7 - 3t - 15t - 3}{3} = \frac{4 - 18t}{3}$

$\Rightarrow y = \frac{4-18t}{12} = \frac{2(2-9t)}{12} = \frac{2-9t}{6}$

$x - 2y + z - w = 2$

$\Rightarrow x - 2\left(\frac{2-9t}{6}\right) + \frac{7-3t}{3} - t = 2$

$x = \frac{2-9t}{3} + t + \frac{3t-7}{3} = \frac{1-3t}{3}$

$(x, y, z, t) = \left[\frac{1-3t}{3}, \frac{2-9t}{6}, \frac{7-3t}{3}, t\right)$

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Exercise 3.7

6.

(1) Investigate for what values of λ & μ the system of eq^{n}s

$2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$,

$2x + 3y + \lambda z = \mu$

have

(i) No sol^n (ii) Unique sol^n (iii) Many sol^{n}s

$\rightarrow$ In matrix form, $AX = B$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Augmented matrix

$$[A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

$R_2 - \frac{7}{2}R_1$, $R_3 - R_2$

$$[A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15/2 & -39/2 & : & -47/2 \\ 0 & 0 & \lambda-5 & : & \mu-9 \end{bmatrix}$$

which is in Row Echelon form

Case (1) when $\lambda = 5$, $\mu \neq 9$

$\rho(A) = 2$, $\rho(A:B) = 3$

$\therefore \rho(A) \neq \rho(A:B) \Rightarrow$ No sol^n

Inconsistent

Case ②: If $\lambda \neq 5$, $\mu \neq 9$

$\rho(A:B) = 3$, $\rho(A) = 3 \Rightarrow \rho(A:B) = \rho(A)$

∴ Unique solⁿ

Case ③ If $\lambda = 5$, $\mu = 9$

$\rho(A) = 2$, $\rho(A:B) = 2 < 3$

So infinite solⁿ.

② Determine values of λ so that eqⁿs

$x+y+z=1$, $x+2y+4z=\lambda$

$x+4y+10z=\lambda^2$

have a solⁿ & solve them completely in each case

→

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

The Augmented matrix $[A|B]$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{array}\right]$$

$R_2 - R_1$, $R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{array}\right]$$

$R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{array}\right] \quad \text{--- } (*)$$

Which is in Echelon form

$\rho(A) = 2$

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2$

To have system solⁿ $\rho(A|B) = 2$

& $\rho(A|B) = 2$ when $\lambda^2 - 3\lambda + 1 = 0$

Case (i) $\lambda = 1$ $(*) \Rightarrow x+y+z=1$

$y+3z=0$

$\rho(A) = \rho(A|B) = 2 < 3 =$ no. of unknowns

∴ Infinite solⁿ

To find solⁿ we use $(3-2) = 1$ parameter (rank)

set $\boxed{z = t_1}$, ∴ $y+3z=0 \Rightarrow y+3t_1=0$

$\Rightarrow \boxed{y = -3t_1}$

& $x+y+z=1 \Rightarrow x-3t_1+t_1=1$

$\Rightarrow x - 2t_1 = 1$

$\Rightarrow \boxed{x = 1+2t_1}$

Case (ii) $\lambda = 2$

$(*) \Rightarrow x+y+z=1$ set $z = t_2$

$y+3z=1$ $y+3z=1 \Rightarrow y = 1-3t_2$

& $x+y+z=1$

$\Rightarrow x+1-3t_2+t_2=1$

$\Rightarrow x = 2t_2$

Exercise 39

(1) Examine whether following vectors are linearly ind. or dependent

(i) $[1,1,1,3]$, $[1,2,3,4]$, $[2,3,4,7]$

$\rightarrow$ let $X_1 = [1,1,1,3]$, $X_2 = [1,2,3,4]$
$X_3 = [2,3,4,7]$

let $K_1X_1 + K_2X_2 + K_3X_3 = 0$

$K_1[1,1,1,3] + K_2[1,2,3,4] + K_3[2,3,4,7] = [0,0,0,0]$

$\Rightarrow K_1 + K_2 + 2K_3 = 0$
$K_1 + 2K_2 + 3K_3 = 0$
$K_1 + 3K_2 + 4K_3 = 0$
$3K_1 + 4K_2 + 7K_3 = 0$

In matrix form

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - R_1$, $R_4 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_4 - R_2,\ R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

ρ(coefficient matrix) $= 2$

and $2 < 3$ (No of unknowns)

$\therefore$ Vectors are linearly dependent

(iii) $(1, 2, -1, 0), (1, 3, 1, 2), (4, 2, 1, 0)$
$(6, 1, 0, 1)$

$\rightarrow$ Let $X_1 = (1, 2, -1, 0)$, $X_2 = (1, 3, 1, 2)$
$X_3 = (4, 2, 1, 0)$, $X_4 = (6, 1, 0, 1)$

Let $K_1 X_1 + K_2 X_2 + K_3 X_3 + K_4 X_4 = 0$

$\Rightarrow K_1(1, 2, -1, 0) + K_2(1, 3, 1, 2)$
$+ K_3(4, 2, 1, 0) + K_4(6, 1, 0, 1) = 0$

$\Rightarrow$ $K_1 + K_2 + 4K_3 + 6K_4 = 0$
$2K_1 + 3K_2 + 2K_3 + K_4 = 0$
$-K_1 + K_2 + K_3 + 0 \cdot K_4 = 0$
$0 \cdot K_1 + 2K_2 + 0K_3 + K_4 = 0$

In matrix form as

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1, \quad R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 2 & 5 & 6 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_4 - R_3$

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

ρ(coefficient matrix) = 4

$\Rightarrow$ System has trivial sol^n

as ρ(coefficient matrix) = No. of variab

$4 = 4$

$\Rightarrow k_1 = 0, k_2 = 0, k_3 = 0, k_4 = 0$

$\Rightarrow$ vectors are line Indepe