Module 2 : Partial Differentiation

dependent Variable → $y = f(x)$ ← Independent Variable

y depend upon x

only one variable → ordinary differentiation

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

→ change in y with respect to x

• If function of limit exist then get derivative of function (not all case)

1 Types of function

① $\boxed{\text{Composite function}}$ (chain rule)

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

② $\boxed{\text{Inverse function}}$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \qquad \frac{dy}{dx} \neq 0$$

③ $\boxed{\text{logarithmic function}}$

$$[f(x)]^{g(x)}$$

④ $\boxed{\text{Parametric function}}$

$y \to t \quad , \quad x \to t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

⑤ $\boxed{\text{Explicit function}}$

$y = f(x)$

→ y in terms of x

⑥ $\boxed{\text{Implicit function}}$

$$x^2 + 2xy + 3y^2 = 0$$

→ y is not in terms of x

depend upon more than one variable

Partial Differentiation :

when you find $\frac{\partial u}{\partial x}$ keep y terms is constant

$$u = f(x, y)$$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \right)$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta x \to 0} \left(\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} \right)$$

(1) $\frac{\partial u}{\partial x}$ or $u_x \longrightarrow$ 1st order partial differentation of u w.r.t $\boxed{x}$ keeping y Constant

and

$\frac{\partial u}{\partial y}$ or $u_y \longrightarrow$ 1st order - - - - - - - - u w.r.t $\boxed{y}$ keeping x Constant

(2) $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$ or $u_{xx} \longrightarrow$ 2nd order - - - of u w.r.t $\boxed{x}$ keeping y Constant

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$ or $u_{yy} \longrightarrow$ 2nd order - - - of u w.r.t $\boxed{y}$ keeping x constant

AND

mixed partial differential

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$ or $u_{yx} \longrightarrow$ 2nd order - - - of u w.r.t Both x and y ~~Constant~~

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$ or $u_{xy} \longrightarrow$ 2nd order . . . u w.r.t both x and y ~~Constant~~

problem:

(1) If $u = \log(\tan x + \tan y)$ then, prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2$$

$\Rightarrow$ $u = \log(\tan x + \tan y)$

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{(\tan x + \tan y)} , \quad \frac{\partial u}{\partial y} = \frac{\sec^2 y}{(\tan x + \tan y)}$$

$$L.H.S = \sin 2x \cdot \frac{\sec^2 x}{\tan x + \tan y} + \frac{\sin 2y \cdot \sec^2 y}{(\tan x + \tan y)}$$

$$= \frac{2\sin x \cos x \times \frac{1}{\cos^2 x} + 2\sin y \cos y \frac{1}{\cos^2 y}}{\tan x + \tan y}$$

$$= \frac{2\tan x + 2\tan y}{\tan x + \tan y} = \frac{2\cancel{(\tan x + \tan y)}}{\cancel{(\tan x + \tan y)}}$$

$$= 2$$

Hence proved.

(2) If $z(x+y) = (x-y)$ Find $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$

$\Rightarrow$ $z = \frac{(x-y)}{(x+y)}$

$$\frac{\partial z}{\partial x} = \frac{(x+y)(1) - (x-y)(1)}{(x+y)^2} \qquad \frac{\partial z}{\partial y} = \frac{(x+y)(-1) - (x-y)(1)}{(x+y)^2}$$

$$= \frac{\cancel{x}+y - \cancel{x}+y}{(x+y)^2} \qquad \frac{\partial u}{\partial y} = \frac{-x-\cancel{y}-x+\cancel{y}}{(x+y)^2}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{2y}{(x+y)^2}} \qquad \frac{\partial z}{\partial y} = \frac{-2x}{(x+y)^2}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left(\frac{2y - (-2x)}{(x+y)^2}\right)^2 = \left(\frac{2\cancel{(y+x)}}{(x+y)^{\cancel{2}}}\right)^2$$

$$= \frac{4}{(x+y)^2}$$

(3) Evaluate $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for $u = e^{x^y}$

$\Rightarrow \frac{\partial u}{\partial x} = e^{x^y} \frac{\partial x^y}{\partial x} = e^{x^y} y x^{(y-1)}$

$\frac{\partial u}{\partial y} = e^{x^y} \frac{\partial x^y}{\partial y} = e^{x^y} x^y \log x$

(4) If $u = (1-2xy+y^2)^{-1/2}$, then show that
$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2 u^3$

$\Rightarrow u = (1-2xy+y^2)^{-1/2}$

Raising (-2) power on both side

$\boxed{u^{-2} = (1-2xy+y^{+2})^{1}}$ --(1)

on differentiating the eq(1) ~~on both si~~ partial w.r.t x

$-\cancel{2}u^{-3}\frac{\partial u}{\partial x} = -\cancel{2}y$

$\boxed{\frac{\partial u}{\partial x} = y u^3}$

on differentiating the eq(1) partial w.r.t y

$-2u^{-3}\frac{\partial u}{\partial y} = -2x + 2y$

$\cancel{-2}u^{-3}\frac{\partial u}{\partial y} = \cancel{-2}(y+x)$

$\frac{\partial u}{\partial y} = (x-y)u^3$

$L.H.S = x y u^3 - y(x-y)u^3$

$= \cancel{xyu^3} - \cancel{xyu^3} + y^2u^3$

$= y^2u^3$

Hence proved.

(5) If $u = \log(\tan x + \tan y + \tan z)$, then show that

$$\sin 2x \left(\frac{\partial u}{\partial x}\right) + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

$u = \log(\tan x + \tan y + \tan z)$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\sec^2 x}{(\tan x + \tan y + \tan z)}$$

similarly,

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{(\tan x + \tan y + \tan z)}, \quad \frac{\partial u}{\partial z} = \frac{\sec^2 z}{(\tan x + \tan y + \tan z)}$$

$$L.H.S = \frac{2\sin x \cos x \sec^2 x}{(\tan x + \tan y + \tan z)} + \frac{2\sin y \cos y \sec^2 y}{(\tan x + \tan y + \tan z)} + \frac{2\sin z \cos z \sec^2 z}{(\tan x + \tan y + \tan z)}$$

$$= 2\left(\frac{\sin x \cancel{\cos x}\frac{1}{\cos^{\cancel{2}} x} + \sin y \cancel{\cos y}\frac{1}{\cos^{\cancel{2}} y} + \sin z \cancel{\cos z}\frac{1}{\cos^{\cancel{2}} z}}{\tan x + \tan y + \tan z}\right)$$

$$= 2\left(\frac{\cancel{\tan x + \tan y + \tan z}}{\cancel{\tan x + \tan y + \tan z}}\right)$$

$$= \underline{2}$$

(6) If $u = x^3 y + e^{xy^2}$. determine $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ (Part ①)

or

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{(Part ②)}$$

differente w.r.t x

part ①

$$\boxed{\frac{\partial u}{\partial x} = 3x^2 y + e^{xy^2}(y^2)}$$

$$\boxed{\frac{\partial^2 u}{\partial y \partial x} = 3x^2 + e^{xy^2}(2y) + y^2 e^{xy^2}(2y)} \quad -①$$

$$\boxed{\frac{\partial u}{\partial y} = x^3 + e^{xy^2} \cdot 2yx}$$

part ②

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = 3x^2 + e^{xy^2}(2y) + 2y\, e^{xy^2} y^2} \quad --②$$

From ① and ②

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

(7) If $u(x+y) = x^2+y^2$, then show that

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$\Rightarrow$ $u = \dfrac{x^2+y^2}{(x+y)}$

--- w.r.t x

$$\frac{\partial u}{\partial x} = \frac{+(x+y)\,2x - (x^2+y^2)}{(x+y)^2} = \frac{+2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{+x^2+2xy-y^2}{(x+y)^2}}$$

--- w.r.t y

$$\frac{\partial u}{\partial y} = \frac{(x+y)\,2y - (x^2+y^2)}{(x+y)^2} = \frac{2xy+2y^2-x^2-y^2}{(x+y)^2}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{-x^2+2xy+y^2}{(x+y)^2}}$$

$$L.H.S = \left(\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{(-x^2+2xy+y^2)}{(x+y)^2}\right)^2$$

$$= \left(\frac{x^2+\cancel{2xy}-y^2+x^2-\cancel{2xy}-y^2}{(x+y)^2}\right)^2$$

$$= \left(\frac{2(x^2-y^2)}{(x+y)^2}\right)^2 = 4\,\frac{(x^2-y^2)^2}{(x+y)^4}$$

$$= 4\,\frac{(x-y)^2\cancel{(x+y)^2}}{(x+y)^{\cancel{4}\,2}} = \frac{4(x-y)^2}{(x+y)^2}$$

$$R.H.S = 4\left(1 - \frac{(x^2+2xy-y^2)}{(x+y)^2} - \frac{(-x^2+2xy+y^2)}{(x+y)^2}\right)$$

$$= 4\left(\frac{(x^2+\cancel{2xy}+y^2) - \cancel{x^2}-2xy+\cancel{y^2}+\cancel{x^2}-\cancel{2xy}-\cancel{y^2}}{(x+y)^2}\right)$$

$$= 4\,\frac{(x-y)^2}{(x+y)^2}$$

L.H.S = R.H.S

∂x

(8) If $u = \log\left(\frac{x}{y}\right)$ then find $u_x + u_y$

$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\cancel{1/y}}{\frac{x}{\cancel{y}}} = \frac{1}{x}, \quad \frac{\partial u}{\partial y} = \frac{-x/y^2}{x/y} = -\frac{1}{y^2}$

$u_x + u_y = \frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy}$

Mixed Partial differentiation

(9) If $u = x^2y + e^{xy^2}$ then, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial u}{\partial y} = x^2 + e^{xy^2}\, 2xy$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 2x + 2\left[xy\, e^{xy^2} \cdot y^2 + e^{xy^2} y\right]$$

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = 2x + 2xy e^{xy^2} y^2 + 2y\, e^{xy^2}} \quad ---(1)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = y\,2x + e^{xy^2} \cdot y^2$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 2x \; \cancel{+ y^2 e^{xy^2} 2xy} + e^{xy^2}\, 2y + y^2 e^{xy^2}\, 2xy$$

$$\boxed{\frac{\partial^2 u}{\partial y \partial x} = 2x + 2xy\, e^{xy^2} y^2 + 2y e^{xy^2}} \quad --(2)$$

~~tone~~ According to eq(1) and eq(2)

$$\boxed{\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}}$$

(10) If $u = \tan^{-1}\left(\frac{y}{x}\right)$ Find the value $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$\Rightarrow$ $\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2} \cdot -\frac{y}{x^2} = -\frac{y}{\cancel{x^2}\left(\frac{x^2+y^2}{\cancel{x^2}}\right)} = \frac{-y}{(x^2+y^2)}$

$\frac{\partial^2 u}{\partial x^2} = -\left[\frac{(x^2+y^2)\times 0 - y(2x)}{(x^2+y^2)^2}\right] = \frac{+2xy}{(x^2+y^2)^2}$

IIIm

$\frac{\partial u}{\partial y} = \frac{\frac{1}{x}}{1+y^2/x^2} = \frac{x}{x^2+y^2}$

$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\times 0 - x\,2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = \underline{0}$

(11) If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ then show that $\frac{\partial^2 u}{\partial x\partial y} = \frac{x^2-y^2}{x^2+y^2}$

$\Rightarrow$ $\frac{\partial u}{\partial y} = x^2\left\{\frac{\frac{1}{x}}{1+y^2/x^2} \cancel{+\tan^{-1}\left(\frac{y}{x}\right)2x}\right\} - y^2\left\{\frac{(-x/y^2)}{1+x^2/y^2} - \tan^{-1}\left(\frac{x}{y}\right)2y\right\}$

$= \frac{x^4}{x^2+y^2}\left(\frac{1}{x}\right) - 2y\tan^{-1}\left(\frac{x}{y}\right) - \frac{y^2}{x^2+y^2}\left(-\frac{x}{y^2}\right)$

$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{y^3 x}{x^2+y^2}$

$= \frac{x^3+y^2x}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$

$= \frac{x\,\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} - 2y\tan^{-1}(x/y) = x - 2y\tan^{-1}(x/y)$

$\frac{\partial^2 u}{\partial x\partial y} = 1 - 2y\,\frac{1}{\frac{x^2+y^2}{y^2}}\cdot\frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)}$

$= \frac{x^2-y^2}{x^2+y^2}$ //

(12) If $z = x^y + y^x$, evalute $\frac{\partial^2 z}{\partial y \partial x}$

$\Rightarrow$ $\frac{\partial z}{\partial x} = y x^{y-1} + y^x \log y$

$$\frac{\partial^2 z}{\partial y \partial x} = y \log x \, x^{(y-1)} + x^{y-1} + \frac{x y^x}{y} + \log y \; x y^{x-1}$$

$$\boxed{\frac{\partial^2 z}{\partial y \partial x} = y x^{(y-1)} \log x + x^{y-1} + y^{x-1} + x y^{x-1} \log y}$$

(13) If $z = \tan(y+ax) + (y-ax)^{3/2}$, then show that

$$\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$$

$\Rightarrow$ L.H.S $= \frac{\partial^2 z}{\partial x^2}$

$$\frac{\partial z}{\partial x} = a \sec^2(y+ax) + \frac{3}{2}(y-ax)^{\frac{1}{2}}(-a)$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = a^2 2\sec(y+ax)\sec(y+ax)\tan(y+ax) - \frac{3a^2}{4}(y-ax)^{-1/2}} \quad (1)$$

R.H.S $= a^2 \frac{\partial^2 z}{\partial y^2}$

$\frac{\partial^2 z}{\partial y^2} \Rightarrow \frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = 2\sec(y+ax)\sec(y+ax)\tan(y+ax) - \frac{3}{4}(y-ax)^{-1/2}} \quad (2)$$

According to (1) and (2)

$$\boxed{\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}}$$

Implicit function wala

$z = x + y$

$\frac{\partial z}{\partial x} = 1$

DATE

Imp (19) If $z^3 - zx - y = 0$, then show that

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(3z^2 + x)}{(3z^2 - x)^3}$$

=> on differentiate w.r.t. y → x = constant

$$3z^2 \frac{\partial z}{\partial y} - x\frac{\partial z}{\partial y} - 1 = 0$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{1}{(3z^2 - x)}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-\left[6z\left(\frac{\partial z}{\partial x}\right) - 1\right]}{(3z^2 - x)^2} \quad \text{---(1)}$$

$$\frac{\partial z}{\partial x} \Rightarrow 3z^2 \frac{\partial z}{\partial x} - \left(z + x\frac{\partial z}{\partial x}\right) = 0$$

$$(3z^2 - x)\frac{\partial z}{\partial x} = z$$

$$\frac{\partial z}{\partial x} = \left(\frac{z}{3z^2 - x}\right) \quad \text{---(2)}$$

$\frac{\partial z}{\partial x}$ put value in eq (1)

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-\left[6z\frac{z}{(3z^2 - x)} - 1\right]}{(3z^2 - x)^2} = \frac{-\left[\frac{6z^2 - 3z^2 + x}{(3z^2 - x)}\right]}{(3z^2 - x)^2}$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = -\frac{(3z^2 + x)}{(3z^2 - x)^3}}$$

(15) If $u = \log(x^3+y^3-x^2y-xy^2)$, then show that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$$

$\Rightarrow$

$$\cancel{\frac{\partial u}{\partial x} = \frac{3x^2-2xy-y^2}{(x^3+y^3-x^2y-xy^2)}}$$

$$u = \log(x^3+y^3-x^2y-xy^2)$$

$$u = \log(x^2(x-y)+y^2(y-x))$$

$$= \log(x^2(x-y)-y^2(x-y)]$$

$$= \log((x-y)\cdot(x^2-y^2))$$

$$= \log[(x-y)(x-y)(x+y)]$$

$$= \log[(x-y)^2(x+y)]$$

$$u = 2\log(x-y) + \log(x+y)$$

diff w.r.t x

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)} + \frac{}{(x+y)}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = -2\frac{1}{(x-y)^2} - \frac{1}{(x+y)^2}} \quad -(1)$$

diff w.r.t y

$$\frac{\partial u}{\partial y} = \frac{2(-1)}{(x-y)} + \frac{}{(x+y)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2(-1)}{(x-y)^2} - \frac{1}{(x+y)^2}$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = \frac{+2}{(x-y)^2} - \frac{1}{(x+y)^2}} \quad --(2)$$

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{+2}{(x-y)^2} - \frac{1}{(x+y)^2}} \quad --(3)$$

$$L.H.S = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$$

$$= \cancel{\frac{-2}{(x-y)^2}} - \frac{1}{(x+y)^2} + \cancel{\frac{4}{(x-y)^2}} - \frac{2}{(x+y)^2} + \cancel{\frac{-2}{(x-y)}} - \frac{1}{(x+y^2)}$$

$$= -\frac{4}{(x+y)^2}$$

Hence proved

★★★

(16) If $x = \cos\theta - r\sin\theta$, $y = \sin\theta + r\cos\theta$, prove that $\frac{\partial r}{\partial x} = \frac{x}{r}$

r [illegible] kalna chai in terms of x

⇒ simplify,

$$x^2 + y^2 = \cos^2\theta + r^2\sin^2\theta - 2r\sin\theta\cos\theta + \sin^2\theta + r^2\cos^2\theta + 2r\sin\theta\cos\theta$$

$$= (\cos^2\theta + \sin^2\theta) + r^2(\sin^2\theta + \cos^2\theta)$$

$$= 1 + r^2(1)$$

$$\boxed{x^2 + y^2 = 1 + r^2}$$

$$\boxed{r^2 = x^2 + y^2 - 1}$$

on differentiating w.r.t x

$$\cancel{2} r \frac{dr}{dx} = \cancel{2} x \qquad \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

(18) Show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$ if $u = x^y$

$\Rightarrow$ $u = x^y$

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \left((y)\, x^{y-1} \log x\right) + \frac{x^y}{x} \quad x^{y-1}$$

$$\cancel{\frac{\partial^3 u}{\partial x^2 \partial y} = (y)\left[(y-1)\, x^{y-2} \log x + \frac{x^{y-1}}{x}\right] + (y-1)\, x^{(y-2)}}$$

$$= x^{y-1} \cdot (y \log x + 1)$$

$$\boxed{\frac{\partial^3 u}{\partial x^2 \partial y} = x^{(y-1)} \left(\frac{y}{x}\right) + (y \log x + 1)(y-1)\, x^{(y-2)}} \quad \text{—(1)}$$

$$\frac{\partial u}{\partial x} = (y)\, x^{y-1}$$

$$\cancel{\frac{\partial^2 u}{\partial x \partial y} = y(y-1)\, x^{(y-2)}}, \quad \frac{\partial^2 u}{\partial y \partial x} = y\, x^{(y-1)} \log x + x^{(y-1)}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = y\left(\frac{x^{(y-1)}}{x} + \log x\,(y-1)\, x^{(y-2)}\right) + (y-1)\, x^{(y-2)}$$

$$\boxed{\frac{\partial^3 u}{\partial x \partial y \partial x} = x^{(y-1)} \frac{y}{x} + (y \log x + 1)(y-1)\, x^{(y-2)}}$$

eq (1) and (2)

$$\boxed{\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}}$$ Hence proved

(19) If $u = \log(x^3+y^3+z^3-3xyz)$, Prove that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

=>

$$L.H.S = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \text{---(1)}$$

$$u = \log(x^3+y^3+z^3-3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{(x^3+y^3+z^3-3xyz)} \text{ - (2)}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{(x^3+y^3+z^3-3xyz)} \text{ - (3)}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3yx}{(x^3+y^3+z^3-3xyz)} \text{ - (4)}$$

from (2), (3) and (4)

-(1)

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3x^2-3yz+3y^2-3yz+3z^2-3xy}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3(x^2+y^2+z^2)-3(xy+yz+zx)}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$= \frac{3}{(x+y+z)}$$

From eq (1)

$$L.H.S = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{(x+y+z)}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2}(1) + \frac{-3}{(x+y+z)^2}(1) + \frac{-3}{(x+y+z)^2}(1)$$

$$= \frac{-9}{(x+y+z)^2}$$

Hence proved

Formula :-

$$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

(20) If $z = x^y + y^x$, verify that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$\Rightarrow$ $\frac{\partial z}{\partial y} = x^y \log x + x\, y^{x-1}$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{x^y}{x} + \log x \; y\, x^{(y-1)} + x\, y^{x-1} \log y + y^{x-1}} \quad -(1)$$

$\frac{\partial z}{\partial x} = y\, x^{y-1} + y^x \log y$

$$\boxed{\frac{\partial^2 z}{\partial y \partial x} = y\, x^{y-1} \log x + x^{y-1} + \frac{y^x}{y} + \log y \cdot x\, y^{x-1}} \quad -(2)$$

eq (1) and eq (2)

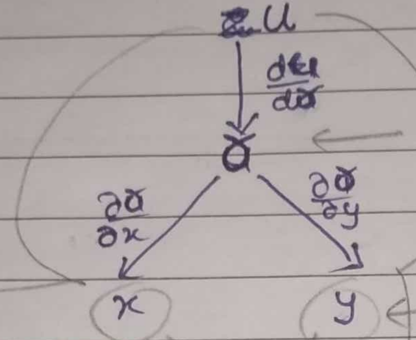
$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

(imp)

Composite function - Total differentiation :

(1) Chain Rule :-

(i) If $u=f(r)$ and $r=g(x, y)$ then u becomes Composite function of x, y



middle one variable part

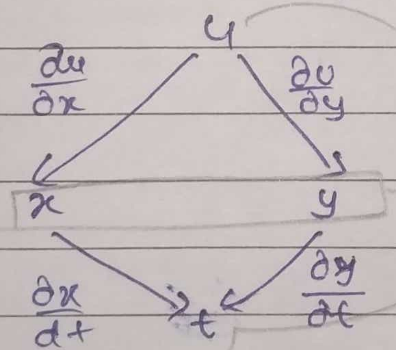
End part two variable

$$\frac{\partial u}{\partial x} = \frac{du}{dr}\left(\frac{\partial r}{\partial x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{du}{dr}\left(\frac{\partial r}{\partial y}\right)$$

→ end part jitna hoga utna separate derivative lena padega

(2) If $u = f(x, y)$ and $x = g_1(t)$, $y = g_2(t)$ then u becomes composite function of 't'



→ middle part jitna hoga utna plus karena padega

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{dt}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$

↳ Called total derivative of u w.r.f t

(3) If $u = f(x, y)$ and $x = g_1(r, \theta)$ and $y = g_2(r, \theta)$ then u becomes composite function of r, θ

$$\begin{tikzcd}
 & u \arrow[dl, "\frac{\partial u}{\partial x}"'] \arrow[dr, "\frac{\partial u}{\partial y}"] & \\
x \arrow[dr, "\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \theta}"'] & & y \arrow[dl, "\frac{\partial y}{\partial r}, \frac{\partial y}{\partial \theta}"] \\
 & r, \theta &
\end{tikzcd}$$

At time $\frac{\partial u}{\partial r}$ ignore θ part

At time $\frac{\partial u}{\partial \theta}$ ignore r part

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial r}\right)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial \theta}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial \theta}\right)$$

called total differential coefficients

(10M)

Note :- when function is Symmetry → much be power Same and also coefficient same.
predict further/next derivative

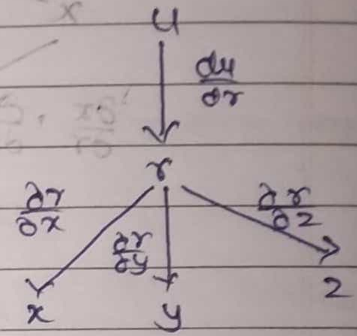
problems :-

① If $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$ then

prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f''(r) + \dfrac{2}{r} f'(r)$

⇒ Here u is composite function of x, y, z

∴ By chain rule



$$\frac{\partial u}{\partial x} = \frac{du}{dr}\left(\frac{\partial r}{\partial x}\right) \text{ or } f'(r)\left(\frac{\partial r}{\partial x}\right)$$

$$r^2 = x^2 + y^2 + z^2$$

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \qquad \cancel{2}r\frac{\partial r}{\partial y} = \cancel{2}y$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}} \qquad \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}} \qquad \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$$

① $\dfrac{\partial u}{\partial x} = \dfrac{du}{dx}\left(\dfrac{\partial r}{\partial x}\right)$ or

$$\frac{\partial u}{\partial x} = f'(r)\left(\frac{\partial r}{\partial x}\right) = f'(r) \times \frac{x}{r}$$

$$\cancel{\frac{\partial^2 u}{\partial x^2} = f'(r) \ldots \frac{1}{r} + \frac{x}{r} f''(r)} \quad \text{---(1)}$$

u v w

② $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left\{ f'(r)\; x\; \frac{1}{r} \right\}$$

$$= f'(r)\, x \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + f'(r)\frac{1}{r}\frac{\partial}{\partial x}(x) + \frac{\partial f'(r)}{\partial x} \cdot \frac{x}{r}$$

$$= x f'(r)\left(-\frac{1}{r^2}\right)\left(\frac{\partial r}{\partial x}\right) + \frac{f'(r)}{r}(1) + f''(r)\frac{x}{r}\left(\frac{\partial r}{\partial x}\right)$$

$$= \frac{-x f'(r)}{r^2} \times \frac{x}{r} + \frac{f'(r)}{r} + f''(r)\frac{x}{r}\cdot\frac{x}{r}$$

$$= \frac{-x^2 f'(r)}{r^3} + \frac{f'(r)}{r} + \frac{f''(r)\, x^2}{r^2}$$

$$= \frac{f'(r)}{r}\left(-\frac{x^2}{r^2}+1\right) + f''(r)\frac{x^2}{r^2}$$

$$= \frac{f'(r)}{r^3}(-x^2+r^2) + f''(r)\frac{x^2}{r^2}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = f'(r)\frac{(y^2+z^2)}{r^3} + f''(r)\frac{x^2}{r^2}} \quad \text{— (1)}$$

similarly, (because $r^2 = x^2+y^2+z^2$, symmetry)

$$\boxed{\frac{\partial^2 u}{\partial y^2} = f'(r)\frac{(x^2+z^2)}{r^3} + f''(r)\frac{y^2}{r^2}} \quad \text{— (2)}$$

$$\boxed{\frac{\partial^2 u}{\partial z^2} = f'(r)\frac{(x^2+y^2)}{r^3} + f''(r)\frac{z^2}{r^2}} \quad \text{--- (3)}$$

Adding (1) and (2) and (3)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = f'(r)\frac{(y^2+z^2)}{r^3} + f''(r)\frac{x^2}{r^2} + f'(r)\frac{(x^2+z^2)}{r^3} + f''(r)\frac{y^2}{r^2}$$

$$+ f'(r)\frac{(x^2+y^2)}{r^3} + f''(r)\frac{z^2}{r^2}$$

$$= \frac{f'(r)2(x^2+y^2+z^2)}{r^3} + \left(\frac{\cancel{x^2+y^2+z^2}}{\cancel{r^2}}\right)f''(r)$$

$$= f'(r)\frac{2}{r}\frac{\cancel{(x^2+y^2+z^2)}}{\cancel{r^2}} + f''(r)(1)$$

$$= f'(r)\frac{2}{r} + f''(r)$$

Hence proved

Hint $\frac{du}{dx} = f'(r^2)\, 2r \frac{dr}{dx}$

** (2) If $u = f(r^2)$ where $r^2 = x^2 + y^2 + z^2$, prove that

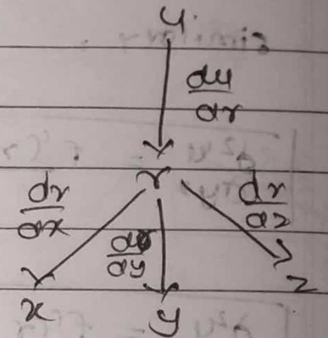
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 4r^2 f''(r^2) + 6 f'(r^2)$$

$\Rightarrow$ Here u is composite function of x, y, z

$\therefore$ By chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial x}\right), \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial y}\right)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial z}\right)$$



$\therefore \quad r^2 = x^2 + y^2 + z^2$

$$\cancel{2} r \frac{dr}{dx} = \cancel{2} x$$

$r^2 = x^2 + y^2 + z^2$

$\cancel{2} r \frac{dr}{dx} = \cancel{2} x \qquad \frac{dr}{dx} = \frac{x}{r}$

$$\boxed{\frac{dr}{dx} = x/r}$$

Similarly $\boxed{\frac{dr}{dy} = y/r}$, $\boxed{\frac{dr}{dz} = z/r}$

$u = f(r^2)$

$$\frac{du}{dr} = f'(r^2)\, 2r \frac{dr}{dx}$$

$$\frac{\partial u}{\partial x} = f'(r^2) \times 2r \,\frac{x}{2}$$

$u = f(r^2)$

$$\frac{du}{dx} = f'(r^2)\, 2r\left(\frac{dr}{dx}\right) = f'(r^2)\, 2\cancel{r}\, \frac{x}{\cancel{r}}$$

$$= 2 f'(r^2) \cdot x$$

Similarly

$$\frac{du}{dy} = f'(r^2)\, 2r\left(\frac{dr}{dy}\right)$$

$$\frac{du}{dz} = f'(r^2)\, 2r\left(\frac{dr}{dz}\right)$$

$$\cancel{\frac{\partial^2 u}{\partial r^2} = 2\left[f'(r^2)\right]} \qquad \frac{\partial u}{\partial x} = 2\left[f'(r^2)\cdot x\right]$$

$$\cancel{\frac{\partial u}{\partial x} = 2\left[f'(r^2)\, r \cdot \frac{\partial r}{\partial x}\right]}$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left[f'(r^2) + x\, f''(r^2)\, 2r\, \frac{\partial r}{\partial x}\right]$$

$$= 2\left[f'(r^2) + 2x^2 f''(r^2)\right]$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = 2\left[f'(r^2) + 2y^2 f''(r^2)\right]$$

$$\frac{\partial^2 u}{\partial z^2} = 2\left[f'(r^2) + 2z^2 f''(r^2)\right]$$

$$L.H.S = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

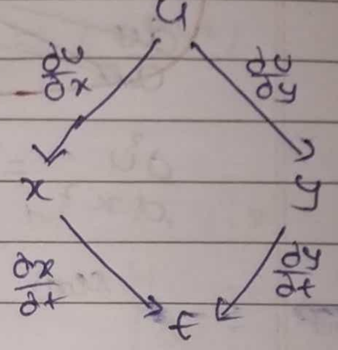
$$= 2\left[3f'(r^2) + 2f''(r^2)\left[x^2+y^2+z^2\right]\right]$$

$$= 6f'(r^2) + 4f''(r^2)\, r^2$$

$$\therefore = R.H.S$$

(3) If $u = x^2 + y^2$ and $x = e^t \cos t$, $y = e^t \sin t$,

Find $\left(\frac{\partial u}{\partial t}\right)$



$\Rightarrow$ $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$

$\boxed{\frac{\partial u}{\partial x} = 2x}$ $\boxed{\frac{\partial u}{\partial y} = 2y}$

$\boxed{\frac{\partial x}{\partial t} = e^t(\sin t) + \cos t\, e^t} = e^t(\cos t - \sin t)$

$\boxed{\frac{\partial y}{\partial t} = e^t(\cos t) + \sin t\, e^t = e^t(\cos t - \sin t)}$

$$\frac{\partial u}{\partial t} = 2x\left(e^t(\cos t - \sin t)\right) + 2y\left(e^t(\cos t + \sin t)\right)$$

$$= 2x\left[e^t \cos t - e^t \sin t\right] + 2y\left[e^t \cos t + e^t \sin t\right]$$

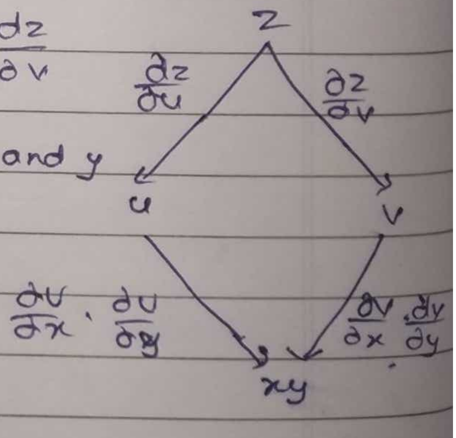
$$= 2x\left[x - y\right] + 2y\left[x + y\right]$$

$$= 2x^2 - \cancel{2xy} + \cancel{2xy} + 2y^2$$

$$= 2(x^2 + y^2) = 2u$$

(4) If $z = f(u, v)$ and $u = \log(x^2 + y^2)$; $v = \frac{y}{x}$

that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1 + v^2)\frac{\partial z}{\partial v}$



$\Rightarrow$ Here z be composite function x and y

∴ By chain rule

$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\right)}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial y}\right)}$$

$$y\frac{\partial z}{\partial x} = y\left[\frac{\partial z}{\partial u}\left(\frac{2x}{x^2+y^2}\right) + \frac{\partial z}{\partial v}\left(\frac{-y}{x^2}\right)\right] = \frac{\partial z}{\partial u}\left(\frac{2xy}{x^2+y^2}\right) + \frac{\partial z}{\partial v}\left(\frac{-y^2}{x^2}\right)$$

$$x\frac{\partial z}{\partial y} = x\left[\frac{\partial z}{\partial u}\left(\frac{2y}{x^2+y^2}\right) + \frac{\partial z}{\partial v}\left(\frac{1}{x}\right)\right] = \frac{\partial z}{\partial u}\left(\frac{2xy}{x^2+y^2}\right) + \frac{\partial z}{\partial v}$$

$$x\frac{dz}{dy} - y\frac{dz}{dx} = \frac{dz}{du}\cancel{\left(\frac{2xy}{x^2+y^2}\right)} + \frac{\partial z}{\partial v} - \frac{dz}{du}\cancel{\left(\frac{2xy}{x^2+y^2}\right)} + \frac{y^2}{x^2}\frac{\partial z}{\partial v}$$

$$= \frac{\partial z}{\partial v}\left(1 + \frac{y^2}{x^2}\right)$$

$$= \frac{\partial z}{\partial v}\left(1+v^2\right) \quad \cdots \left\{\begin{array}{l} v = \frac{y}{x} \\ v^2 = \frac{y^2}{x^2} \end{array}\right.$$

(5) If $z = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$, prove that

$$\frac{\partial z}{\partial t} = \frac{3}{\sqrt{1-t^2}}$$

$\Rightarrow$

$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{dy}{dt}\right)}$$

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & t & \end{tikzcd}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}}} \quad -(1)$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-1}{\sqrt{1-(x-y)^2}}} \quad -(2)$$

$$\frac{\partial x}{\partial t} = 3 \qquad \frac{\partial y}{\partial t} = 12t^2$$

$$\frac{\partial z}{\partial t} = \frac{(3 - 12t^2)}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(x^2-2xy+y^2)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-x^2+2xy-y^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}}$$

1st $(1-4t^2)^2$ Hint $\sqrt{(\;)^2} = (\;)$

$$= 1 - 8t^2 + 16t^4$$

$$= \frac{3(1-4t^2)}{\sqrt{1-8t^2+16t^4-t^2+8t^4-16t^6}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-8t^2+16t^4) - t^2(1-8t^2+16t^4)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t)^2}}$$

$$= \frac{3}{\sqrt{1-t^2}}\,\frac{\cancel{(1-4t^2)}}{\cancel{(1-4t^2)}}$$

$$= \frac{3}{\sqrt{1-t^2}}$$

Hence proved

let wala

maa maa

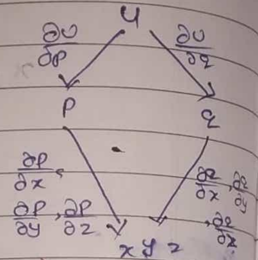
imp for 5 marks

(6). If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ then show that

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$$

$\Rightarrow$ let $p = \frac{y-x}{xy}$, $\boxed{p = \frac{1}{x} - \frac{1}{y}}$, $q = \frac{z-x}{xz}$ $\boxed{q = \frac{1}{x} - \frac{1}{z}}$

$u = f(p, q)$ u becomes composite function of x, y, z



$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial x}\right)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial q}\left(\frac{-1}{x^2}\right) \quad \text{---(1)}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial y}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial y}\right)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial q}(0) \quad \text{---(2)}$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial z}\right)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left(\frac{1}{z^2}\right)$$

eq(1) $\times x^2$ + eq(2) $\times y^2$ + eq(3) $\times z^2$

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = x^2\left[\left(\frac{\partial u}{\partial p}\right)\frac{-1}{x^2} + \frac{\partial u}{\partial q}\frac{-1}{x^2}\right] + y^2\left(\frac{\partial u}{\partial p}\left(\frac{1}{y^2}\right)\right) + z^2\left(\frac{\partial u}{\partial q}\left(\frac{1}{z^2}\right)\right)$$

$$= -\cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial p}}$$

$= 0$

Hence proved.

Born are not because power coefficient

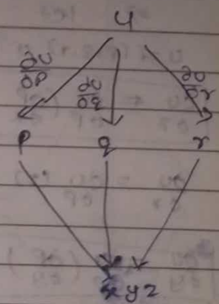
Symmetry and not same

let wala

(7) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ that show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

$\Rightarrow$ let $p = \frac{x}{y}$, $q = \frac{y}{z}$, $r = \frac{z}{x}$



$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial x}\right) + \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial x}\right)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}\left(\frac{-z}{x^2}\right) \quad \text{---(1)}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial y}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial y}\right) + \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial y}\right)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial r}(0) \quad \text{---(2)}$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial z}\right) + \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial z}\right)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{x}\right) \quad \text{---(3)}$$

eq(1) $\times x$ + eq(2) $\times y$ + eq(3) $\times z$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \cancel{\left(\frac{x}{y}\right)\frac{\partial u}{\partial p}} \cancel{-\frac{z}{x}\left(\frac{\partial u}{\partial r}\right)} + \cancel{\left(-\frac{x}{y}\right)\frac{\partial u}{\partial p}} + \cancel{\left(\frac{y}{z}\right)\frac{\partial u}{\partial p}} + \cancel{\left(\frac{-y}{z}\right)\frac{\partial u}{\partial q}} + \cancel{\frac{\partial u}{\partial r}\left(\frac{z}{x}\right)}$$

$= 0$

Hence proved.

DATE PAGE NO.

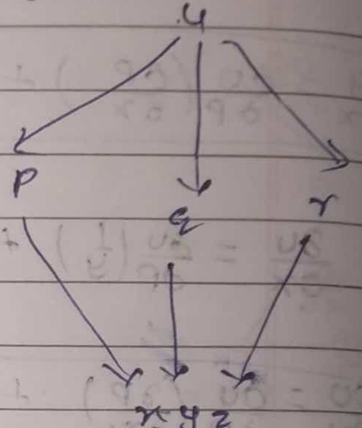
let wala

(8) If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$ then show that

$$6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$$

$\Rightarrow$ let $p = 2x-3y$, $q = 3y-4z$, $r = 4z-2x$

$u = f(p,q,r)$ u becomes composite funcition of x, y, z



$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial x}\right) + \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial x}\right)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(2) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-2) \quad \text{---(1)}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial y}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial y}\right) + \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial y}\right)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-3) + \frac{\partial u}{\partial q}(3) + \frac{\partial u}{\partial r}(0) \quad \text{---(2)}$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial z}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial z}\right) + \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial z}\right)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-4) + \frac{\partial u}{\partial r}(4) \quad \text{---(3)}$$

eq(1) × 6 + eq(2) × 4 + eq(3) × 3

$$6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = \cancel{12\frac{\partial u}{\partial p}} - \cancel{12\frac{\partial u}{\partial r}} - \cancel{12\frac{\partial u}{\partial p}} + \cancel{12\frac{\partial u}{\partial q}}$$

$$- \cancel{12\frac{\partial u}{\partial q}} + \cancel{12\frac{\partial u}{\partial r}}$$

$$= 0$$

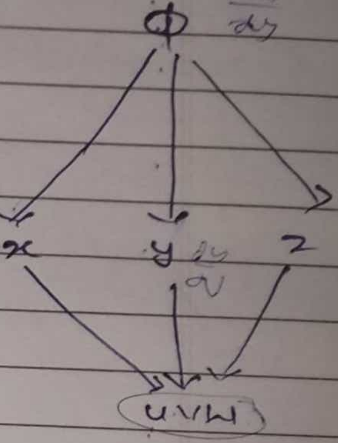
Hence proved.

assumes that $u^\circ = 1$ $v^\circ = 1$ $w^\circ = 1$

(9) If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and ϕ is function of x, y, z then show that

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$$

$\Rightarrow$ ϕ becomes composite function of u, v, w



$$\boxed{\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\left(\frac{\partial x}{\partial u}\right) + \frac{\partial \phi}{\partial y}\left(\frac{\partial y}{\partial u}\right) + \frac{\partial \phi}{\partial z}\left(\frac{\partial z}{\partial u}\right)}$$

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}(0) + \frac{\partial \phi}{\partial y}\left(\frac{w}{2\sqrt{wu}}\right) + \frac{\partial \phi}{\partial z}\left(\frac{v}{2\sqrt{uv}}\right) \quad --(1)$$

$$\boxed{\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\left(\frac{\partial x}{\partial v}\right) + \frac{\partial \phi}{\partial y}\left(\frac{\partial y}{\partial v}\right) + \frac{\partial \phi}{\partial z}\left(\frac{\partial z}{\partial v}\right)}$$

$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\left(\frac{w}{2\sqrt{vw}}\right) + \frac{\partial \phi}{\partial y}(0) + \frac{\partial \phi}{\partial z}\left(\frac{u}{2\sqrt{uv}}\right) \quad ---(2)$$

$$\boxed{\frac{\partial \phi}{\partial w} = \frac{\partial \phi}{\partial x}\left(\frac{\partial x}{\partial w}\right) + \frac{\partial \phi}{\partial y}\left(\frac{\partial y}{\partial w}\right) + \frac{\partial \phi}{\partial z}\left(\frac{\partial z}{\partial w}\right)}$$

$$\frac{\partial \phi}{\partial w} = \frac{\partial \phi}{\partial x}\left(\frac{v}{2\sqrt{vw}}\right) + \frac{\partial \phi}{\partial y}\left(\frac{u}{2\sqrt{wu}}\right) + \frac{\partial \phi}{\partial z}(0) \quad -(3)$$

eq (1) $\times u$

$$u\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial y}\left(\frac{wu}{2\sqrt{uw}}\right) + \frac{\partial \phi}{\partial z}\left(\frac{uv}{2\sqrt{uv}}\right) = \frac{\partial \phi}{\partial y}\left(\frac{\sqrt{wu}}{2}\right) + \frac{\partial \phi}{\partial z}\left(\frac{\sqrt{uv}}{2}\right) \cdots (4)$$

$$= \frac{\partial \phi}{\partial y}\left(\frac{y}{2}\right) + \frac{\partial \phi}{\partial z}\left(\frac{z}{2}\right)$$

eq (2) $\times v$

$$v\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\left(\frac{wv}{2\sqrt{vw}}\right) + \frac{\partial \phi}{\partial z}\left(\frac{uv}{2\sqrt{uv}}\right) = \frac{\partial \phi}{\partial x}\left(\frac{\sqrt{vw}}{2}\right) + \frac{\partial \phi}{\partial z}\left(\frac{\sqrt{uv}}{2}\right) \cdots (5)$$

$$= \frac{\partial \phi}{\partial x}\left(\frac{x}{2}\right) + \frac{\partial \phi}{\partial z}\left(\frac{z}{2}\right)$$

eq (3) $\times w$

$$w\frac{\partial \phi}{\partial w} = \frac{\partial \phi}{\partial x}\left(\frac{vw}{2\sqrt{vw}}\right) + \frac{\partial \phi}{\partial y}\left(\frac{uw}{2\sqrt{wu}}\right) = \frac{\partial \phi}{\partial x}\left(\frac{\sqrt{vw}}{2}\right) + \frac{\partial \phi}{\partial y}\left(\frac{\sqrt{uw}}{2}\right) \cdots (6)$$

$$= \frac{\partial \phi}{\partial x}\left(\frac{x}{2}\right) + \frac{\partial \phi}{\partial y}\left(\frac{y}{2}\right)$$

$$R.H.S = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$$

$$= \left(\frac{y}{2}\right)\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}\left(\frac{z}{2}\right) + \frac{\partial \phi}{\partial x}\left(\frac{x}{2}\right) + \frac{\partial \phi}{\partial z}\left(\frac{z}{2}\right) + \frac{\partial \phi}{\partial x}\left(\frac{x}{2}\right) + \frac{\partial \phi}{\partial y}\left(\frac{y}{2}\right)$$

$$= x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z}$$ Hence proved.

Homogenaus function

Def^n - $u = f(x,y)$ is said to be homogenaus function in x, y of degree $\boxed{n}$, iff $f(xt, yt) = t^n f(x,y)$

e.g ① $f(x,y) = x^2 - y^2$

$\rightarrow f(xt, yt) = x^2t^2 - y^2t^2$

$= t^2 [(x^2 - y^2)]$

$\Rightarrow f(x,y)$ is homogeneaus in ~~function~~ (x,y) of degree ②

② $u = e^{x/y}$

equal to 1

$\rightarrow u(xt, yt) = e^{\cancel{xt}/\cancel{yt}} = e^{x/y} = t^0 e^{x/y}$

$\Rightarrow$ ~~fn~~ u is homogeneaus in (x,y) of degree ⓪

③ $u = \log x - \log y$

$\rightarrow u = (xt, yt) = \log(xt) - \log(yt)$

$= \log\left(\frac{x\cancel{t}}{y\cancel{t}}\right)$

$t^0 = 1$

$= \log(x/y) = t^0 \log(x/y)$

$\rightarrow u$ is homogeneaus in (x,y) of degree 0.

Note: Find the this expression without find single derivative

Euler's theorem : — its applicable for when function is homogeneous function

If $u = f(x, y)$ is homogeneous function in x, y of degree n

$x^2y + x^2 + y^2$

then $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$

$x^3 + xy^2 + y^4$

Deduction on Euler's theorem :

Note:- its applicable for when function is not homogeneous function

If $u = f(x, y)$ is homogeneous function in x, y of degree n

then $\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u}$

let $u = v + w$ and seperately find

(1) If $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + x^4 \tan^{-1}\left(\frac{y}{x}\right)$ find the value of $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ at $x=1$, $y=1$

$\Rightarrow$ Here u is not homogeneous in x, y

let

$\boxed{u = v + w}$ where $v = x^3 \sin^{-1}(y/x)$

$w = x^4 \tan^{-1}(y/x)$

Now, (1)

$$v(xt, yt) = x^3 t^3 \sin^{-1}\left(\frac{yt}{xt}\right)$$
$$= t^3\, x^3 \sin^{-1}(y/x)$$
$$= t^3 v$$

v is homogeneous function in x, y of degree 3

By Euler theorem: $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$

$$\boxed{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3v} \quad \text{--- (1)}$$

By deduction of Euler theorem: $x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} = n(n-1)v$

$$\boxed{x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} = 3(3-1)v = 6v} \quad \text{--- (2)}$$

Now,

(2) $w(xt, yt) = x^4 t^4 \tan^{-1}(yt/xt) = t^4 x^4 \tan^{-1}(y/x)$

$= t^4 u$

w is homogeneous function in x, y of degree 4

By Euler theorem : $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nw$ $\qquad \frac{\partial^2 y}{\partial x \partial y}$

$$\boxed{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 4w} \quad ---(3)$$

By Euler theorem : $x^2\frac{\partial^2 w}{\partial x^2} + y^2\frac{\partial^2 w}{\partial y^2} + \overset{2xy}{\frac{\partial^2 w}{\partial x \partial y}} = n(n-1)w$

$$\boxed{x^2\frac{\partial^2 w}{\partial x^2} + y^2\frac{\partial^2 w}{\partial y^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} = 4(4-1)w = 12w} \quad ---(4)$$

Adding eq ① + eq ③ + eq ② + eq ④

$$\left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}\right] + \left[x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2} + z^2\frac{\partial^2 v}{\partial x \partial y} + x^2\frac{\partial^2 w}{\partial y^2} + y^2\frac{\partial^2 w}{\partial y^2} + 2xy\frac{\partial^2 w}{\partial x \partial y}\right]$$

$$= 3v + 4w + 6v + 12w$$

$\frac{\partial u}{\partial x}$ $\quad \frac{\partial u}{\partial y}$ $\quad \frac{\partial^2 u}{\partial x^2}$ $\quad \frac{\partial^2 u}{\partial y^2}$ $\quad \frac{\partial^2 u}{\partial x \partial y}$

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) + x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$+ 2xy\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y}\right) = 9v + 16w$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 9x^3\sin^{-1}(y/x) + 16x^4\tan^{-1}(y/x)$$

put the value of x and y

$$= 9(1)^3\sin^{-1}(1) + 16(1)^4\tan^{-1}(1)$$

$$= 9\frac{\pi}{2} + 16\frac{\pi}{4} = 17\pi/2$$

(2) If $z = x^n f(y/x) + y^{-n} f(x/y)$.

prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$$

$\Rightarrow$ Here z is not homogeneous function in x, y

let $z = v + w$ where $v = x^n f(y/x)$

$w = y^{-n} f(x/y)$

Now,

$$v(xt, yt) = x^n t^n f\left(\frac{yt}{xt}\right) = t^n x^n f(y/x)$$

$$= t^n v$$

v is homogeneous function in x, y of degree $\boxed{n}$

By Euler's theorem: $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$

$$\boxed{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv} \quad ---(1)$$

By Deduction of Euler's theorem:

$$\boxed{x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} = n(n-1)v} \quad --(2)$$

Now,

$$w(xt, yt) = y^{-n} t^{-n} f\left(\frac{xt}{yt}\right) = t^{-n} y^{-n} f(x/y)$$

$$= t^{-n} w$$

w is homogeneous function in x, y of degree $\boxed{-n}$.

By Euler's theorem: $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nw$

$$\boxed{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = -nw} \quad ---(3)$$

By Deduction of Euler's theorem:

$$x^2\frac{\partial^2 w}{\partial x^2} + y^2\frac{\partial^2 w}{\partial y^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} = n(n-1)w$$

$$\boxed{x^2\frac{\partial^2 w}{\partial x^2} + y^2\frac{\partial^2 w}{\partial y^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} = -n(-n-1)w} \quad --(4)$$

adding eq (1) + eq (3) + eq (2) + eq (4)

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right)=\cancel{nv}-\cancel{nv}$$

$$+y^2\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right)+2xy\left(\cancel{\frac{\partial^2 v}{\partial x\partial y}}+\frac{\partial^2 w}{\partial x\partial y}\right) \qquad +n(n-1)w \quad -n(n-1)w$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial w}{\partial y}+x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 w}{\partial y^2}+2xy\frac{\partial^2 w}{\partial x\partial y}=n^2w-\cancel{nw}+n^2w+\cancel{nw}$$

$$=n^2w$$

Note :- applicable for
when u is not homogeneous
But function of u ($f(u)$) is homogeneous

$\boxed{2^{nd} \text{ Form of Euler's theorem}}$:

Corollary I : If $z = f(u)$ and z is homogeneous function in (x, y) of degree n

then

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}}$$

e.g (1) $u = \sin^{-1}(x+y)$

u is not homogeneous function is x, y

But $\sin u = (x+y) = (z \text{ say})$ become function of u $f(u)$ is homogeneous in (x, y)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = 1\frac{\sin u}{\cos u}$$

Corollary 2 : If $z = f(u)$ and z is homogeneous function in (x, y) of degree n

then

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = g(u)\left[g'(u) - 1\right]}$$

where $g(u) = n\frac{f(u)}{f'(u)}$

e.g

① If $u = \tan^{-1}\left(\frac{x^2+y^2}{x-y}\right)$ prove that

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = -2\sin^3 u \cos u.$$

→ Here u is not homogeneous function in (x, y)

$\tan u = \left(\frac{x^2+y^2}{x-y}\right) = z$ (say) become homogeneous

as

$$z(xt, yt) = \frac{x^2t^2 + y^2t^2}{xt - yt} = \frac{t^{\cancel{2}}(x^2+y^2)}{\cancel{t}(x-y)} = t\,\frac{(x^2+y^2)}{(x-y)}$$

z is homogeneous function ~~of~~ in x, y of degree ①

and $z = \tan u$

$\boxed{n=1}$ and $f(u) = \tan u$

By corollary 2 :

$$x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = g(u)\,[g'(u) - 1]$$

$f(u) = \tan u$

$f'(u) = \sec^2 u$

$$g(u) = n\,\frac{f(u)}{f'(u)}$$

$$= 1\,\frac{\tan u}{\sec^2 u}$$

$$= \frac{\sin u \cos^2 u}{\cos u} = \sin u \cos u$$

$$g'(u) = -\sin u \sin u + \cos u \cos u = \cos^2 u - \sin^2 u$$

$$x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = \sin u \cos u\,[\cos^2 u - \sin^2 u - 1]$$

$$= \sin u \cos u\,[-\sin^2 u + (\cos^2 u - 1)]$$

$$= \text{---}\,[-\sin^2 u + (-\sin^2 u)]$$

$$= (\sin u \cos u)(-2\sin^2 u)$$

$$= -2\sin^3 u \cos u$$

Hence proved

(2) If $u = \sin^{-1}\left(\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}$

prove that (i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{12}\tan u$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + y^2\dfrac{\partial^2 u}{\partial y^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\tan u}{144}(\tan^2 u + 13)$

$\Rightarrow$

Here u is not homogeneous function in (x, y)

$\sin u = \left(\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2} = (z \text{ say})$ Become Homogeneous

$\frac{1}{3}-\frac{1}{2}$

$-\frac{1}{3}$

$z(xt, yt) = \left(\dfrac{x^{1/3}t^{1/3}+y^{1/3}t^{1/3}}{x^{1/2}t^{1/2}-y^{1/2}t^{1/2}}\right)^{1/2} = \left(\dfrac{t^{1/3}}{t^{1/2}}\right)^{1/2}\left(\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}$

$= (t^{-1/6})^{1/2}\, z$

$= t^{-1/12}\, z$

z is homogeneous function in (x, y) of degree $-1/12$

$n = -1/12$ $\quad f(u) = \sin u$

(i) By corollary 1 :

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\dfrac{f(u)}{f'(u)} = -\dfrac{1}{12}\dfrac{\sin u}{\cos u} = -\dfrac{1}{12}\tan u$

(ii) By corollary 2 :

$x^2\dfrac{\partial^2 u}{\partial x^2} + y^2\dfrac{\partial^2 u}{\partial y^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} = g(u)\left[g'(u) - 1\right]$

$g(u) = n\dfrac{f(u)}{f'(u)} = -\dfrac{1}{12}\tan u$

$g'(u) = -\dfrac{1}{12}\sec^2 u$

$x^2\dfrac{\partial^2 u}{\partial x^2} + y^2\dfrac{\partial^2 u}{\partial y^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} = -\dfrac{1}{12}\tan u\left[-\dfrac{1}{12}\sec^2 u - 1\right]$

$= \cancel{\dfrac{-1}{144}\tan u}\ -\dfrac{1}{12}\tan u\left[\dfrac{-\sec^2 u - 12}{12}\right]$

$= \dfrac{1}{12}\tan u\left[\dfrac{\sec^2 u + 12}{12}\right]$

$= \dfrac{1}{144}\tan u\left[\tan^2 u + 1 + 12\right]$

$= \dfrac{1}{144}\tan u\left[\tan^2 u + 13\right]$ // Hence proved.

homogeneous
Euler theorem wala

③ If $u = \frac{x^3y^3}{x^3+y^3} + \log\left(\frac{xy}{x^2+y^2}\right)$, the prove that
$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{6x^3y^3}{x^3+y^3}$

=> Here u is not homogeneous function of x, y

let

$\boxed{u = v + w}$ where $v = \frac{x^3y^3}{x^3+y^3}$

Now,

$v(xt, yt) = \frac{x^3t^3\, y^3t^3}{x^3t^3+y^3t^3} = \frac{t^3x^3y^3 \cancel{t^3}}{\cancel{t^3}(x^3+y^3)} = \frac{t^3}{t^3}v$

$= t^6 v$

v is homogeneous function ~~of~~ in x, y of degree 6

By Euler's theorem

$\boxed{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 3v}$ ...(1)

Now

$w(xt, yt) = \log\left(\frac{xy\cancel{t^2}}{(x^2+y^2)\cancel{t^2}}\right) = \log\left(\frac{xy}{x^2+y^2}\right)$

$= t^0 \log\left(\frac{xy}{x^2+y^2}\right)$

$= t^0 w$

w is homogeneous function in x, y of degree 0

By Euler's theorem

$\boxed{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nw = 0w = 0}$ ...(2)

~~Now~~, adding eq (1) and eq ②

$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = 6v + 0$

$= \frac{6x^3y^3}{x^3+y^3}$

u — w — x

u — w — y

2nd form of Euler's theorem

log wala

(8) If $u=\frac{1}{3}\log\left(\frac{x^3+y^3}{x^2+y^2}\right)$ Find the value (i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$

(ii) $x^2u_{xx}+y^2u_{yy}+2xy\,u_{xy}$

→ Here u is not Homogeneous function in x, y

$$3u=\log\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

$$e^{3u}=\cancel{\log}\left(\frac{x^3+y^3}{x^2+y^2}\right)=(z \text{ say})$$ Become Homogeneous

$$z(xt,yt)=\cancel{\log}\ \frac{x^3t^3+y^3t^3}{x^2t^2+y^2t^2}=\frac{t^{\cancel{3}}(x^3+y^3)}{\cancel{t^2}(x^2+y^2)}=t\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

z is Homogeneous function in x, y of degree 1

$n=1 \qquad f(u)=e^{3u}$

By Collorly I :

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=\frac{1\ \cancel{e^{3u}}}{3\cancel{e^{3u}}}=\frac{1}{3}$$

By Collorly 2 :

$$x^2\frac{\partial^2u}{\partial x^2}+y^2\frac{\partial^2u}{\partial y^2}+2xy\frac{\partial^2u}{\partial x\partial y}=g(u)\left[g'(u)-1\right]$$

$$g(u)=n\frac{f(u)}{f'(u)}=\frac{1}{3}$$

$$g'(u)=0$$

$$x^2\frac{\partial^2u}{\partial x^2}+y^2\frac{\partial^2u}{\partial y^2}+2xy\frac{\partial^2u}{\partial x\partial y}=\frac{1}{3}[0-1]=-\frac{1}{3}$$

do in your practice

(7) If $u = \sin^{-1}(x^2+y^2)^{1/5}$. show that
$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \frac{2}{25} \tan u (2\tan^2 u - 3)$$

=> Here u is not homogeneous function in x, y

$\sin u = (x^2+y^2)^{1/5} = z$ (say) become homogeneous

$u(xt, yt) = (x^2t^2 + y^2t^2)^{1/5} = t^{2/5}(x^2+y^2)^{1/5}$
$= t^{2/5} z$

z is homogeneous function of x, y of degree = 2/5

$\boxed{n = 2/5}$

Collory 2 :-

$f(u) = (x+y^2)^{1/5} = \sin u$

$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = g(u)\,[g'(u) - 1]$

$g(u) = n \frac{f(u)}{f'(u)}$

$= \frac{2}{5}\left(\frac{\sin u}{\cos u}\right)$

$= \frac{2}{5} \tan u$

$g'(u) = \frac{2}{5} \sec^2 u$

$- - - - = \frac{2}{5}\tan u \left[\frac{2}{5}\sec^2 u - 1\right]$

$= \frac{2}{25}\tan u \left[2\sec^2 u - 5\right]$

$= \frac{2}{25}\tan u \left[2(\tan^2 u + 1) - 5\right]$

$= \frac{2}{25}\tan u \left[2\tan^2 u + 2 - 5\right]$

$= \frac{2}{25}\tan u \left[2\tan^2 u - 3\right]$

Hence proved

3rd form of Euler's and Euler diagram

cos → odd = (1) value
sin → odd value = (0) value

(8) If $x = e^u \tan v$, $y = e^u \sec v$, prove that

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 0$$

=> 1st $\frac{\partial u}{\partial x}$? find kaise nikalo

u ko form nikalna hai x and y

$x = e^u \tan v$ — (1)

$y = e^u \sec v$ — (2)

Take square both eq^n and eq(2) − eq(1)

$$y^2 - x^2 = e^{2u}\sec^2 v - e^{2u}\tan^2 v$$

$$= e^{2u}(\sec^2 v - \tan^2 v)$$

$$y^2 - x^2 = e^{2u}(1)$$

Taking log

$$\log e^{2u} = \log(y^2 - x^2)$$

$$2u \log e = \log(y^2 - x^2)$$

$$2u = \log(y^2 - x^2)$$

$$\boxed{u = \frac{1}{2}\log(y^2 - x^2)}$$

Here u is not homogeneous function in x, y

$e^{2u} = (y^2 - x^2) = z$ (say) become Homogeneous

$$z(xt, yt) = y^2t^2 - x^2t^2 = t^2(y^2 - x^2)$$

z is homogeneous function in x, y of degree $\boxed{2}$

$\boxed{n = 2}$ $f(u) = e^{2u}$

Now

Collorally 1 :-

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

$$= \frac{2\,e^{2u}}{2e^{2u}} = (1)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \quad \text{---(1)}$$

2nd wala ise liye

1st simplify in form of v

eq(2) ÷ eq(1)

$$\frac{y}{x} = \frac{\cancel{e^u}}{\cancel{e^u}}\frac{\sec v}{\tan v} = \frac{1}{\cos v}\cdot\frac{\cos v}{\sin v}$$

$$\boxed{\sin v = x/y}$$

$$v = \sin^{-1}(x/y)$$

$$v(xt, yt) = \sin^{-1}\left(\frac{xt}{yt}\right) = t^0 \sin^{-1}(x/y)$$

v is Homogeneous function in x, y of degree 0

By Euler's theorem

$$\boxed{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 0v = 0} \quad \text{---(2)}$$

from eq(1) and eq(2)

$$L.H.S = \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

$$= (1)\times(0)$$

$= 0$ Hence proved

Combination of (5) and (8)

(Q) If $u = \log \frac{x+y}{\sqrt{x^2+y^2}} + \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$. prove

that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin w \cos 2w}{4\cos^3 w}$.

=> Here u is not Homogeneous ~~in~~ x, y

let $v = \log \frac{x+y}{\sqrt{x^2+y^2}}$ $w = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Now.

$v(xt, yt) = \log \frac{xt+yt}{\sqrt{x^2t^2+y^2t^2}} = \log \left(\frac{\cancel{t}\, x+y}{\cancel{t}\sqrt{x^2+y^2}}\right)$

$= t^0 \log\left(\frac{x+y}{\sqrt{x^2+y^2}}\right)$

$= t^0 v$

v is homogeneous function in x, y of degree zero

$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v = 0$

$w = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Now

~~w(xt, yt) = log~~

w is not homogeneous function in x, y

$\sin(w) = \frac{x+y}{\sqrt{x}+\sqrt{y}} = z$ (say) become homogeneous

$w(xt, yt) = \frac{xt + yt}{\sqrt{t}(\sqrt{x}+\sqrt{y})} = t\, t^{-1/2}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$= t^{1/2}(z)$

z is homogeneous function in x, y of degree $\frac{1}{2}$

$n = \frac{1}{2}$ $f(w) = \sin w$

Corollary - 2

$x^2\frac{\partial^2 w}{\partial x^2} + y^2\frac{\partial^2 w}{\partial y^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} = \frac{n f(w)}{f'} g(w)[g'(w) - 1]$

$g(w) = n\frac{f(w)}{f'(w)}$

$= \frac{1}{2}\tan w$ $g'(w) = \frac{1}{2}\sec^2 w$

=> $= \frac{1}{2}\tan w\left[\frac{1}{2}\sec^2 w - 1\right]$

$= \frac{1}{2}\tan w\left[\frac{1}{2\cos^2 w} - 1\right]$

$= \frac{\sin w}{4\cos^2 w \cos w}(\sin^2 w \cos^2 w\ 2\cos^2 w)$

$= \frac{\sin w}{4\cos^3 w}(\sin^2 w - \cos^2 w)$

$= -\frac{\sin w}{4\cos^3 w}(\cos^2 w - \sin^2 w)$

$= -\frac{\sin w \cos 2w}{4\cos^3 w}$ // Hence proved.

(10) If $z = \log(x^2+y^2) + \frac{x^2+y^2}{x+y} - 2\log(x+y)$, prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2+y^2}{x+y}$$

⇒ // 1st simply of your function

$$z = \log(x^2+y^2) - \log(x+y)^2 + \frac{x^2+y^2}{x+y}$$

$$z = \log\left(\frac{x^2+y^2}{(x+y)^2}\right) + \frac{x^2+y^2}{x+y}$$

Here z is not homogeneous function in (x, y)

let

$z = u + v$ where $u = \log\left(\frac{x^2+y^2}{(x+y)^2}\right)$

$$v = \frac{x^2+y^2}{x+y}$$

Now,

$$u(xt, yt) = \log\left(\frac{x^2t^2+y^2t^2}{(xt+yt)^2}\right) = \log\left(\frac{(x^2+y^2)t^2}{[(x+y)t]^2}\right)$$

$$= \log\left(\frac{(x^2+y^2)\cancel{t^2}}{(x+y)^2\cancel{t^2}}\right)$$

$$= t^0 \log\left(\frac{x^2+y^2}{(x+y)^2}\right)$$

u is homogeneous function in (x, y) of degree 0

By Euler's theorem

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 0u = 0} \quad \ldots(1)$$

Now,

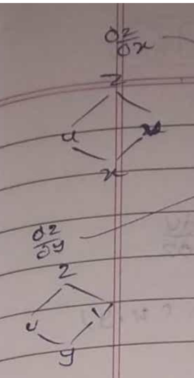
$$v(xt, yt) = \frac{x^2t^2+y^2t^2}{xt+yt} = \frac{t^{\cancel{2}}(x^2+y^2)}{\cancel{t}(x+y)} = t^1\left(\frac{x^2+y^2}{x+y}\right)$$

$$= t^1 v$$

v is homogeneous function in (x, y) of degree 1

By Euler's theorem

$$\boxed{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial x} = nv = (1)v} \quad \ldots(2)$$



Adding eq (1) and eq (2).

$$x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = 0 + 1v$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = v$$

$$= \frac{x^2+y^2}{x+y}$$ // Hence proved

(ii) If $u = \dfrac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

Find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$

$\Rightarrow$ Here u is not homogeneous function in (x, y)

let $u = v + w$ where $v = \dfrac{x^3y^3z^3}{x^3+y^3+z^3}$

$w = \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

Now,

$v(xt, yt) = \dfrac{x^3t^3 \cdot y^3t^3 \cdot z^3t^3}{x^3t^3+y^3t^3+z^3t^3} = \dfrac{t^3t^3\cancel{t^3}(x^3y^3z^3)}{\cancel{t^3}(x^3+y^3+z^3)}$

v is homogeneous function in x, y $= t^6 v$
Euler's theorem on degree (6)

$$\boxed{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 6v} \quad --(1)$$

Now,

$w(xt, yt) = \log\left(\dfrac{xtyt+ytzt+ztxt}{(x^2+y^2+z^2)t^2}\right) = \log\left(\dfrac{t^2(xy+yz+zx)}{t^2(x^2+y^2+z^2)}\right)$

$= \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

$= t^0 \log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

w is homogeneous function in (x, y) in degree 0

Euler's theorem

$$\therefore \boxed{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nw = 0w = 0} \quad --(2)$$

Adding (1) and (2)

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = 6v$$

$$= 6\,\frac{x^3y^3z^3}{(x^3+y^3+z^3)}$$

In ESE For 10 Marks fix question ask By

Application of partial Derivative:

(1) Maximum / minima :-

If $u = f(x, y)$ be function of x and y then to discuss maximum / minimum value

Step 1: Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

Step 2: Solve $\boxed{\frac{\partial u}{\partial x} = 0}$ and $\boxed{\frac{\partial u}{\partial y} = 0}$

to get a pair of x, y value known as **critical / stationary points**

Step 3: Find $r = \frac{\partial^2 u}{\partial x^2}$; $t = \frac{\partial^2 u}{\partial y^2}$; $s = \frac{\partial^2 u}{\partial x \partial y}$

1st check these condition — then → go this condition

Step 4:

(1) If $(rt - s^2) > 0$ and $r > 0$ at point (a, b) then minimum exist.

(2) If $(rt - s^2) > 0$ and $r < 0$ at point (a, b) the maximum exist

(3) If $(rt - s^2) < 0$ at point (a, b) then neither maxima nor minima and such point is called as **Saddle point**

(4) If $\boxed{(rt - s^2) = 0}$ at point (a, b) then NO conclusion can be drawn obtain maxima / minima

① Test the following function for maxima / minima

(i) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

step 1: let $u = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$\boxed{\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 30x + 72}$$

and

$$\boxed{\frac{\partial u}{\partial y} = 6xy - 30y}$$

step 2: solve $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$

$\boxed{3x^2 + 3y^2 - 30x + 72 = 0}$ and $6xy - 30y = 0$

$\cancel{6xy = 30y}$

$\cancel{x = 30}$

$6y(x-5) = 0$

$6y = 0$ $\boxed{x=5}$

$\boxed{y=0}$

when $\boxed{y=0}$ eq (1) ~~becomes~~

$3x^2 - 30x + 72 = 0$

$x^2 - 10x + 24 = 0$ 24: $-6, -4$

$x^2 - 4x - 6x + 24 = 0$

$x(x-4) - 6(x-4) = 0$

$\boxed{x=4}$ $\boxed{x=6}$

$y = 0$ and $x = 6 \Rightarrow$ pt. $(6, 0)$

$y = 0$ and $x = 4 \Rightarrow$ pt. $(4, 0)$

when $\boxed{x=5}$ eq (1)

$\cancel{3x^2 + 75 - 30x + 72 = 0}$

$\cancel{3x^2 - 30x - 3 = 0}$

$\cancel{x^2 - 10x - 1 = 0}$

$75 + 3y^2 - 150 + 72 = 0$

$3y^2 - 150 + 147 = 0$

$3y^2 = 3$

$y^2 = 1$

$y = \pm 1$

$x = 5$ and $y = 1 \Rightarrow$ pt. $(5, 1)$

$x = +5$ and $y = -1 \Rightarrow$ pt. $(5, -1)$

24
3)72
6
12

25

-30×5

Step 3: $r = \frac{\partial^2 u}{\partial x^2}$; $t = \frac{\partial^2 u}{\partial y^2}$; $s = \frac{\partial^2 u}{\partial x \partial y}$

$\boxed{r = 6x - 30}$ $\boxed{t = 6x - 30}$ $\boxed{s = 6y}$

Step 4:

points	r	t	s	$(rt - s^2)$	Conclusion
(6,0)	$6 > 0$	6	0	$36 > 0$	minima
(4,0)	$-6 < 0$	-6	0	$36 > 0$	maxima
(5,1)	0	0	6	$-36 < 0$	Neither maxima nor minima
(5,-1)	0	0	-6	$-36 < 0$	neither maxima nor minima

$u = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$u_{max}(4,0) = (4)^3 + 3(4)\times(0)^2 - 15(4)^2 - 15\times(0)^2 + 72(4)$

$= 64 - 240 + 288$

$= 64 + 48 = 112$

$$\begin{array}{r} \overset{3}{15} \\ \times 16 \\ \hline 90 \\ 15\times \\ \hline 240 \end{array}$$

$u_{min}(6,0) = (6)^3 - 15(6)^2 + 72(6)$

$= 36\times 6 - 540 + 432$

$= 216 - 540 + 432$

$648 - 540$

$= 108$

$$\begin{array}{r} \overset{3}{36} \\ 15 \\ \hline 180 \\ 36\times \\ \hline 540 \end{array} \qquad \begin{array}{r} \overset{3}{36} \; 18 \\ \times 6 \\ \hline 216 \end{array}$$

(2) $xy(3a - x - y)$

let $u = xy(3a - x - y) = 3axy - x^2y - xy^2$

Step 1 :- $\boxed{\frac{\partial u}{\partial x} = 3ay - 2xy - y^2}$

and $\boxed{\frac{\partial u}{\partial y} = 3ax - x^2 - 2xy}$

Step 2 : solve $\frac{\partial u}{\partial x} = 0$ $\qquad$ $\frac{\partial u}{\partial y} = 0$

$3ay - 2xy - y^2 = 0$ $\qquad$ $3ax - x^2 - 2xy = 0$

$y(3a - 2x - y) = 0$ $\qquad$ $x(3a - x - 2y) = 0$

$\boxed{y=0}$ or $\boxed{3a-2x-y=0}$ $\qquad$ $\boxed{x=0}$ or $\boxed{3a-x-2y=0}$

① ② ③ ④

when $\boxed{y=0}$ and $x=0$ $\Rightarrow$ pt. $(0,0)$

$\boxed{y=0}$ and $3a-x-2y=0$ $\Rightarrow$ pt. $(3a,0)$

$\boxed{3a=x}$

$\boxed{x=0}$ and $3a-2x-y=0$ $\Rightarrow$ pt. $(0,3a)$

$\boxed{3a=y}$

$\boxed{3a-2x-y=0}$ ...(1) and $\boxed{3a-x-2y=0}$ ...(2) $\Rightarrow$ pt. (a,a)

2 × eq(1)

$6a - 4x - \cancel{2y} = 0$

$3a - x - \cancel{2y} = 0$

$3a - 3x = 0$

$\boxed{x=a}$

put eq(1)

$3a - 2(a) - y = 0$

$\boxed{y=a}$

Step 3 : $r = \frac{\partial^2 u}{\partial x^2}$; $t = \frac{\partial^2 u}{\partial y^2}$; $s = \frac{\partial^2 u}{\partial x \partial y}$

$\boxed{r = -2y}$ $\qquad$ $\boxed{t = -2x}$ $\qquad$ $\boxed{s = 3a - 2x - 2y}$

Step 4 :

point	r	t	s	$(rt - s^2)$	conclusion
(0,0)	0	0	3a	$-9a^2 < 0$	Neither maxima nor minima
(3a,0)	0	$-6a$	$3a-6a$ $-3a$	$-9a^2 < 0$	— " —
(0,3a)	$-6a<0$	0	$-3a$	$-9a^2 < 0$	— " —
(a,a)	$-2a<0$	$-2a$	$-a$	$4a^2 - a^2$ $= 3a^2 > 0$	maxima

$u_{max}(a,a) = 3axy - x^2y - xy^2$

$= 3(a)(a)(a) - (a)^2(a) - (a)(a)^2$

$= 3a^3 - a^3 - a^3$

$= a^3$

(3) $x^3+y^3-63x-63y+12xy$

let $u = x^3+y^3-63x-63y+12xy$

Step 1: $\boxed{\frac{\partial u}{\partial x} = 3x^2-63+12y}$ and $\boxed{\frac{\partial u}{\partial y} = 3y^2-63+12x}$

Step 2: Solve $\frac{\partial u}{\partial x}=0$ and $\frac{\partial u}{\partial y}=0$

$3x^2-63+12y=0$ $\qquad$ $3y^2-63+12x=0$

$y = \frac{63-3x^2}{12}$ $\qquad$ $3\left(\frac{21-x^2}{4}\right)^2-63+12x=0$

$\boxed{y = \frac{21-x^2}{4}}$ $\qquad$ $\frac{3(441-42x^2+x^4)}{16} - 63 + 12x = 0$

$\frac{3(441-42x^2+x^4)}{16} - \cancel{1008} + 64x = 0$ $\quad$ 336

$441-42x^2+x^4-336+64x=0$

$\boxed{x^4-42x^2+64x+105=0}$

Here $x=-1$ is rt. of above equation by inspection

	x^4	x^3	x^2	x^1	c
-1	1	0	-42	64	105
		-1	1	41	-105
	1	-1	-41	105	0

$\boxed{x-(-1)}$ $(x+1)$ $\qquad$ $\boxed{x^3-x^2-41x+105}$

$x^4-42x^2+64x+105=0$

$(x+1)(x^3-x^2-41x+105)=0$

$x+1=0$ $\qquad$ $x^3-x^2-41x+105=0$

$\boxed{x=-1}$ $\qquad$ $(x+7)(x-5)(x-3)=0$

$\boxed{x=-7}$ $\boxed{x=5}$ $\boxed{x=3}$

16, 21, 336, 63, 378, 1008, 441, 336, 105

$y = \frac{21-x^2}{4}$ $\qquad$ $\frac{21-49}{4}$ $\quad$ 28 $\quad$ 21-25

when

$x=-1$ and $\boxed{y=5}$ $\Rightarrow$ pt. $(-1,5)$

$x=-7$ and $\boxed{y=-7}$ $\Rightarrow$ pt. $(-7,-7)$

$x=5$ and $\boxed{y=-1}$ $\Rightarrow$ pt. $(5,-1)$

$x=3$ and $\boxed{y=3}$ $\Rightarrow$ pt. $(3,3)$

$\therefore$ stationary point is $(-1,5), (-7,-7), (5,-1), (3,3)$

Step 3: $r = \frac{\partial^2 u}{\partial x^2}$; $t = \frac{\partial^2 u}{\partial y^2}$; $s = \frac{\partial^2 u}{\partial x \partial y}$

$\boxed{r = 6x}$ $\qquad$ $\boxed{t = 6y}$ $\qquad$ $\boxed{s = 12}$

Step 4:

Points	r	t	s	$(rt-s^2)$	Conclusion
$(-1,5)$	-6	30	12	$=-180-144$ $=-324<0$	Neither maxima nor minima
$(-7,-7)$	$-42<0$	-42	12	$1764-144$ $=1620>0$	maxima
$(5,-1)$	$30>0$	-6	12	$-180-144$ $=-324<0$	Neither maxima nor minima
$(3,3)$	$18>0$	18	12	$324-144$ $=180>0$	minima

$u = x^3+y^3-63x-63y+12xy$

$u_{max}(-7,-7) = -343-343+441+441+12\times-7\times-7$

$= -686+882+588$

$= 784$

$u_{min}(3,3) = 27+27-189-189+108$

$= -216$

? most Imp For(2M) in ISE

unit vector $\longrightarrow \hat{n} = \frac{\bar{n}}{|\bar{n}|}$

Application of partial Differential :-

Gradient :-

If $\phi(x, y, z)$ be any scalor point function then gradient of ϕ is denoted by

~~grad(ϕ)~~ grad ϕ or $\nabla\phi$

unit vector along x-axis

$$\boxed{\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}}$$

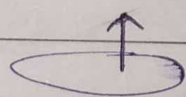
unit vector along z-axis

unit vector along y-axis

$$\boxed{\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}}$$

is called vector different operators

Note : Geometrically $\nabla\phi$ respresentd outword normal to the surface whose equation is $\phi(x, y, z) = c$ or $\boxed{\phi = c}$



$\phi = c$

Formulae :-

To Find the gradient of any scaler valued function at a point (x_1, y_1, z_1) of the function $\phi(x, y, z)$ evalute $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\Big|_{(x_1, y_1, z_1)}$

(1) If $\phi = 3x^2y - y^3z^2$ find $\nabla\phi$ at the point $P(1,-2,-1)$

$\Rightarrow$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = \frac{\partial}{\partial x}(3x^2y - y^2z^2)\hat{i} + \frac{\partial}{\partial y}(3x^2y - y^2z^2)\hat{j} + \frac{\partial}{\partial z}(3x^2y - y^2z^2)\hat{k}$$

$$= (6xy - 0)\hat{i} + (3x^2 - 2yz^2)\hat{j} + (0 - 2y^2z)\hat{k}$$

$$\nabla\phi\Big|_{P(1,-2,-1)} = 6(1)(-2)\hat{i} + (3(1)^2 - 3(-2)^2(-1)^2)\hat{j} + (-2(-2)^3(-1))\hat{k}$$

$$= -12\hat{i} + (3 - 12)\hat{j} - 16\hat{k}$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

(2) Find a unit vector normal to the surface $x^3 + y^3 + 3xy = 3$ at $(1, 2, -1)$

$\nabla\phi = c \rightarrow$

$\phi = x^3 + y^3 + 3xy$

The normal to the surface $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$$= [(3x^2) + (3y)]\hat{i} + (3y^2 + 3x)\hat{j} + 0\hat{k}$$

$15 \times 15 = 225$, $225 + 81 = 306$

$$\nabla\phi\Big|_{(1,2,-1)} = (3(1)^2 + 3(2))\hat{i} + (3(2)^2 + 3(1))\hat{j} + 0\hat{k}$$

$$= 9\hat{i} + 15\hat{j} + 0\hat{k}$$

unit vector normal to the surface is : $\dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{9\hat{i} + 15\hat{j} + 0\hat{k}}{\sqrt{306}}$

$\Delta\phi = c$

(3) Find a unit vector normal to the surface $xy^3z^2 = 4$ at $(-1,-1,2)$

$\Rightarrow \quad \phi = xy^3z^2$

The normal to the surface is $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$= (y^3z^2)\hat{i} + (3xy^2z^2)\hat{j} + (2xy^3z)\hat{k}$

$\nabla\phi\Big|_{(-1,-1,2)} = (-1)^3(2)^2\hat{i} + (3(-1)(-1)^2(2)^2)\hat{j} + (2(-1)(-1)^3(2))\hat{k}$

$= -4\hat{i} - 12\hat{j} + 4\hat{k}$

$\frac{144}{32}$

a unit vector normal to the surface is $\frac{\nabla\phi}{|\nabla\phi|} = \frac{-4 - 12\hat{j} + 4\hat{k}}{\sqrt{176}}$

$= \frac{+4(-\hat{i} - 3\hat{j} + \hat{k})}{\sqrt{16 \times 11}}$

$= \frac{4(-\hat{i} - 3\hat{j} + \hat{k})}{4\sqrt{11}}$

$= \frac{(-\hat{i} - 3\hat{j} + \hat{k})}{\sqrt{11}}$

(4) If $\phi = \log(x^2+y^2+z^2)$, Find $\nabla\phi$ at $(2,1,1)$

$\log(4+1+1)$
$\log 6$

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$= \frac{2x}{\cancel{\log}(x^2+y^2+z^2)}\hat{i} + \frac{2y}{\cancel{\log}(x^2+y^2+z^2)}\hat{j} + \frac{2z}{\cancel{\log}(x^2+y^2+z^2)}\hat{k}$

$\boxed{\nabla\phi = \frac{4}{\cancel{\log} 6}\,\frac{2}{3}\hat{i} + \frac{2}{\cancel{\log} 6}\,\frac{1}{3}\hat{j} + \frac{2}{\cancel{\log} 6}\,\frac{1}{3}\hat{k}}$

(5) Show that $\text{grad}(1/r) = \frac{-\bar{r}}{r^3}$

$\rightarrow$ Here, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is radius vector / position vector of a point (x,y,z)

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$

Similarly

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$ $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

$\Delta f(x) = f(x+h) - f(x)$

$\nabla f(x) = f(x) - f(x-h)$

$grad(1/r) = \nabla(1/r) = \frac{\partial}{\partial x}(1/r)\hat{i} + \frac{\partial}{\partial y}(1/r)\hat{j} + \frac{\partial}{\partial z}(1/r)\hat{k}$

$= -\frac{1}{r^2}\left(\frac{\partial r}{\partial x}\right)\hat{i} - \frac{1}{r^2}\left(\frac{\partial r}{\partial y}\right)\hat{j} - \frac{1}{r^2}\left(\frac{\partial r}{\partial y}\right)\hat{k}$

$= -\frac{1}{r^2}\frac{x}{r}\hat{i} - \frac{1}{r^2}\frac{y}{r}\hat{j} - \frac{1}{r^2}\frac{z}{r}\hat{k}$

$= -\frac{1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k})$

$= -\frac{\bar{r}}{r^3}$ // Hence proved.

(5) Find $\nabla\phi$, when $\phi = xyz$ at $(1, 2, 3)$

$\nabla\phi = (yz)\hat{i} + (xz)\hat{j} + (xy)\hat{k}$

$= (2\times3)\hat{i} + 1\times3\hat{j} + 1\times2\hat{k}$

$= 6\hat{i} + 3\hat{j} + 2\hat{k}$

(6) Find a unit vector to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 2)$

$\Rightarrow$ $\phi = x^2y + 2xz$

~~The normal to the surface~~ : $\nabla\phi = (2xy + 2z)\hat{i} + (x^2)\hat{j} + (2x)\hat{k}$

$\nabla\phi|_{(2,-2,2)} = (-8+4)\hat{i} + 4\hat{j} + 4\hat{k}$

$= -4\hat{i} + 4\hat{j} + 4\hat{k}$

16
16
16
48

$r^2 = x^2 + y^2$

$r = \sqrt{x^2 + y^2 + z^2}$

$\frac{dr}{dx} = \frac{x}{r}$ $\quad \frac{dr}{dy} = \frac{y}{r}$ $\quad \frac{dz}{dy} = \frac{z}{r}$

$grad(\frac{1}{r}) = \nabla(\frac{1}{r}) = \frac{\partial}{\partial x}(\frac{1}{r})\hat{i} +$

$= -\frac{1}{r^2}\frac{\partial r}{\partial x}$

~~To~~ The Normal to the surface : $\nabla\phi = (2xy+2z)\hat{i} + (x^2)\hat{j} + (2x)\hat{k}$

$$\nabla\phi\Big|_{(2,-2,2)} = -4\hat{i} + 4\hat{j} + 4\hat{k}$$

unit vector normal to the surface $= \dfrac{-4\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{16+16+16}}$

$$= \frac{\cancel{4}(-\hat{i} + \hat{j} + \hat{k})}{\cancel{4}\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

(7) Find a unit vector normal to the surface $x^3+y^3+3xy = 3$ at $(1, 2, -1)$

$$\phi = x^3+y^3+3xy$$

$\Rightarrow$ The normal to the surface : $\nabla\phi = (3x^2+3y)\hat{i} + (3y^2+3x)\hat{j} + 0\hat{k}$

$$\nabla\phi\Big|_{(1,2,-1)} = (3(1)+6)\hat{i} + (3(2)^2+3(1))\hat{j} + 0\hat{k}$$

$$= 9\hat{i} + 15\hat{j} + 0\hat{k}$$

The unit vector of normal to the surface $= \dfrac{9\hat{i} + 15\hat{j} + 0\hat{k}}{\sqrt{306}}$

225
81

How to check question of directional derivative

most ask 2M for ISE

$\boxed{\text{Directional derivative}}$:

If $\phi(x, y, z)$ be any scalar point function then the directional derivative of ϕ in the direction of $\bar{a}$ is given by $\frac{\nabla\phi \cdot \bar{a}}{|\bar{a}|}$

dot product

$\nabla\phi \cdot \hat{n}$

Note: ① D.D is always maximum in the direction of gradient vector.

② Maximum value of D.D is $|\nabla\phi|$

① Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction of $2\hat{i} + 3\hat{j} + 6\hat{k}$

$\Rightarrow$ Given direction $\bar{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

Given surface $\phi = 4xz^3 - 3x^2y^2z$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}(\hat{k})$$

$$= (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (-3x^2y^2)\hat{k}$$

$[12xz^2 -$

$$\nabla\phi\Big|_{(2,-1,+2)} = (4\times 8 - 6\times 2(-1)^2 2)\hat{i} + (-6\times(2)^2 - 1\times 2)\hat{j} + (-3(2)^2(-1)^2]\hat{k}$$

$$= (32-24)\hat{i} + 48\hat{j} + (96-12)\hat{k}$$

$$= 8\hat{i} + 48\hat{j} + 84\hat{k}$$

Directional derivative of ϕ in the direction of $2\hat{i}+3\hat{j}+6\hat{k}$ $= \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

$$= \frac{(8\hat{i}+48\hat{j}+84\hat{k})\cdot(2\hat{i}+3\hat{j}+6\hat{k})}{\sqrt{4+9+36}}$$

$$= \frac{16+144+504}{\sqrt{49}} = \frac{664}{7}$$

② Find the D.D of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the vector $(\hat{i}+2\hat{j}+2\hat{k})$

$\Rightarrow$ $\nabla\phi = (y^2)\hat{i} + (2xy + z^3)\hat{j} + (3yz^2)\hat{k}$

$$\nabla\phi\Big|_{(2,-1,1)} = \hat{i} + (-4+1)\hat{j} + (-3\hat{k}) = \hat{i} - 3\hat{j} - 3\hat{k}$$

direction derivative of ϕ in the direction of $\bar{a}$

$$= \frac{(\hat{i}-3\hat{j}-3\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{1+4+4}}$$

$$= \frac{1-6-6}{\sqrt{9}} = \frac{-11}{3}$$

~~③ Find the d.d of $\phi = 4x$~~

③ (a) In what direction from the point $(2,1,-1)$ in the directional derivative of $\phi = x^2yz^3$ a maximum?

(b) what is the magnitude of this maximum?

=> (a) $\nabla\phi = (2xyz^3)\hat{i} + (x^2z^3)\hat{j} + (3x^2yz^2)\hat{k}$

$\nabla\phi|_{(2,1,-1)} = (2\times2\times1(-1)^3)\hat{i} + (4\times(-1)^3)\hat{j} + (3(2)^2(1)(-1)^2)\hat{k}$

$= -4\hat{i} - 4\hat{j} + 12\hat{k}$

D.D is alway maximum in the direction of gradient vector

$\nabla\phi = -4\hat{i} - 4\hat{j} + 12\hat{k}$

(b) $|\nabla\phi| = \sqrt{16 + 16 + 144} = \sqrt{176}$

(4) What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at $(-1, 2, 1)$

→ Given direction is normal to the surface.

$x \log z - y^2 = -4$

$$\psi = x \log z - y^2$$

$$\nabla \psi = \frac{\partial \psi}{\partial x}\hat{i} + \frac{\partial \psi}{\partial y}\hat{j} + \frac{\partial \psi}{\partial z}\hat{k}$$

$$\nabla \psi = (\log z)\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k}$$

$$\nabla \psi \Big|_{(-1,2,1)} = (\log 1)\hat{i} - 2\times 2\hat{j} + \frac{-1}{1}\hat{k}$$

$$= 0\hat{i} - 4\hat{j} - \hat{k}$$

$$\bar{a} = 0\hat{i} - 4\hat{j} - \hat{k}$$

∴ Given direction $\boxed{\bar{a} = 0\hat{i} - 4\hat{j} - \hat{k}}$

Now,

$$\phi = xy^2 + yz^3$$

$$\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= (y^2)\hat{i} + (2xy + z^3)\hat{j} + 3yz^2\hat{k}$$

$$\nabla \phi \Big|_{(2,-1,1)} = (-1)^2\hat{i} + (2\times 2 - 1 + 1)\hat{j} + 3(-1)(1)^2\hat{k}$$

$$= \hat{i} + (-4 + 1)\hat{j} - 3\hat{k}$$

$$= \hat{i} - 3\hat{j} - 3\hat{k}$$

directional derivative of ϕ in the direction at $(0\hat{i} - 4\hat{j} - \hat{k})$

$$= \frac{\nabla \phi \cdot \bar{a}}{|\bar{a}|}$$

$$= \frac{(\hat{i} - 3\hat{j} - 3\hat{k}) \cdot (0\hat{i} - 4\hat{j} - \hat{k})}{\sqrt{16 + 1}}$$

$$= \frac{+12 + 3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$