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Q. Verify Rolle's theorem for $f(x) = e^{-x}[\sin x - \cos x]$, in $[\frac{\pi}{4}, \frac{5\pi}{4}]$

Sol. Given $f(x) = e^{-x}[\sin x - \cos x]$ in $[\frac{\pi}{4}, \frac{5\pi}{4}]$

(i) $f(x)$ is continuous in closed interval $[\frac{\pi}{4}, \frac{5\pi}{4}]$

(ii) $f'(x) = e^{-x}\cdot(\cos x + \sin x) + (\sin x - \cos x)(-e^{-x})$

$= e^{-x}[\cos x + \sin x - \sin x + \cos x]$

$= e^{-x}[2\cos x]$

$= 2\cos x \cdot e^{-x}$

$\therefore f(x)$ is differentiable in open interval $(\frac{\pi}{4}, \frac{5\pi}{4})$

(iii) $a = \pi/4$ $b = 5\pi/4$

$f(a) = f(\pi/4) = \cancel{4}\, e^{-\pi/4}[1/\sqrt{2} - 1/\sqrt{2}]$ — (1)

$= 0$

$f(b) = f(5\pi/4) = e^{-5\pi/4}[-1/\sqrt{2} + 1/\sqrt{2}]$ — (2)

$= 0$

$\therefore f(a) = f(b)$ from (1) & (2)

from (i), (ii) & (iii) Rolle's theorem is applicable

$\therefore \exists$ a point c in open interval where or such that

$f'(c) = 0$

$2e^{-c}\cdot\cos c = 0$

$\cos c = 0$

$c = \pi/2$

$\therefore c = \pi/2 \in [\frac{\pi}{4}, \frac{5\pi}{4}]$

Q. Verify Rolle's theorem for $f(x) = \begin{cases} x^2+1 & , 0 \le x \le 1 \\ 3-x & , 1 < x \le 2 \end{cases}$

Sol.

Closed interval $[0,2]$

① $LHL = \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} (-h)^2+1$

$x = 1-h$ $= \lim_{h \to 0} h^2+1 = 1$

$RHL = \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(+h) = \lim_{h \to 0} 3-(+h) = \lim_{h \to 0} 3-h =$

$= 3$

$LHL \neq RHL$

$\therefore f(x)$ is discontinuous at $x = 1$

Sir's method

① continuity at $x = 1$

when $x < 1$, $f(x) = x^2+1$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (x^2+1) = 2$

when $x > 1$, $f(x) = 3-x$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (3-x) = 2$

when $x = 1$, $f(x) = x^2+1$

$f(1) = 2$

$f(x)$ is continuous in $[0,2]$

(ii) $f'(x) = \begin{cases} 2x & , 0 \le x \le 1 \\ -1 & , 1 < x \le 2 \end{cases}$

$f'(1^-) = 2(1) = 2$

$f'(1^+) = -1$

$f'(1^-) \neq f'(1^+)$ not differentiability.

$$f'(1^-) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1} \frac{(x^2+1)-2}{x-1} = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 2$$

$$f'(1^+) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{(3-x)-2}{x-1} = \lim_{x\to 1} \frac{1-x}{x-1}$$

$$= -1$$

⑤. Use Rolle's theorem to prove that the equation $ax^2+bx = \frac{a}{3} + \frac{b}{2}$ has a root between 0 and 1

sol.

let $f'(x) = ax^2 + bx - \frac{a}{3} - \frac{b}{2} = 0$

$$f(x) = \int f'(x)dx$$

$$= \int ax^2 + bx - \frac{a}{3} - \frac{b}{2}$$

$$= \frac{ax^3}{3} + \frac{bx^2}{2} - \left(\frac{a}{3} + \frac{b}{2}\right)x \quad \text{in } [0,1]$$

∴ This $f(x)$ is continuous in $[0,1]$

(ii) $f'(x) = ax^2+bx - \frac{a}{3} - \frac{b}{3}$ (since polynomial & numerator value is finite ∴ differentiable btw $[a,b]$

$f(x)$ ~~is~~ diff in open interval $(0,1)$

(iii) $a=0$, $b=1$

$f(a) = f(b)$. let us check

$f(a) = f(0) = 0$

$f(b) = f(1) = 0$

∴ $f(a) = f(b)$

from (i), (ii) & (iii) since Rolle's theorem is applicable.

$\therefore \exists$ a point c in $(0,1)$ such that $f'(c)=0$

$ac^2+bc-\frac{a}{3}-\frac{b}{2}=0$

$\therefore$ root of equation lies in $(0,1)$

$\therefore ax^2+bx=\frac{a}{3}+\frac{b}{2}$ has a root between 0 & 1

Q. To prove that eq $2x^3-3x^2-x+1=0$ has atleast one root between 1 and 2.

Sol. Given, $f'(c)=0$

$$f'(x)=2x^3-3x^2-x+1=0$$

$$f(x)=\frac{2x^4}{4}-\frac{3x^3}{3}-\frac{x^2}{2}+x$$

$$f(x)=\frac{x^4}{2}-x^3-\frac{x^2}{2}+x$$

(1) $\therefore$ This $f(x)$ is continuous btw $[1,2]$ closed interval

(2) $f'(x)=2x^3-3x^2-x+1$

$f(x)$ is differentiable in open interval $(1,2)$

(3). $f(a)=f(b)$ let us check

$f(1)=2(1)^3-3(1)^2-(1)+1$

$=2-3-1+1$

$f(1)=-1$

$f(2)=2(2)^3-3(2)^2-2+1$

$=16-12-1$

$=3$

$f(1)\neq f(2)$

$\therefore$ Rolle's theorem is not applicable for $f(x)$ in interval $(1,2)$

$f(a) = f(b)$ let us check

$$f(1) = \frac{1}{2} - 1 - \frac{1}{2} + 1 = 0$$

$$f(2) = \frac{2^4}{2} - 2^3 - \frac{2^2}{2} + 2$$

$$= 2^3 - 2^3 - 2 + 2$$

$$= 0$$

$f(1) = f(2)$

$\therefore$ From 1, 2 & 3 Rolle's theorem is applicable

$\therefore \exists$ a point c in $(1, 2)$ such that $f'(c) = 0$

$$2x^3 - 3x^2 - x + 1 = 0$$

$\therefore 2x^3 - 3x^2 - x + 1 = 0$ has atleast one root in $(1, 2)$

* Lagrange's Mean Value Theorem (LMVT)

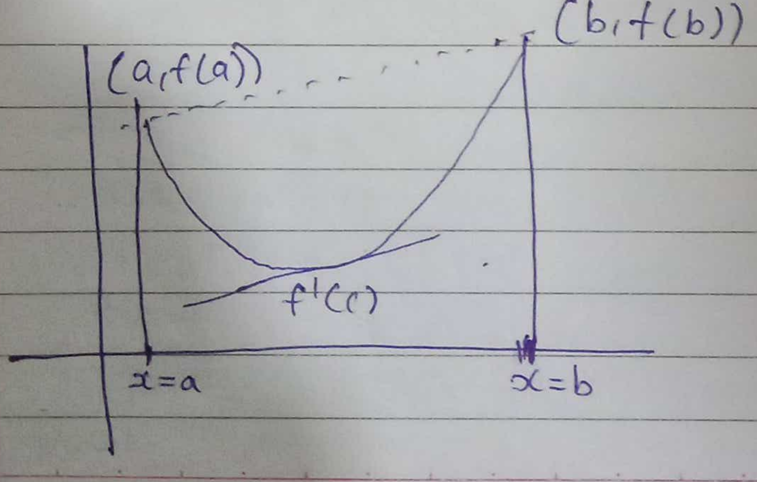
If a function $f(x)$ is

① Continuous in closed interval $[a, b]$

② Differentiable in open interval (a, b)

Then there exist point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



① Verify Lagrange's mean value theorem for $f(x) = x^2+x-1$ in $[0,4]$.

sol Given $f(x) = x^2 + x - 1$ in $[0,4]$

① $f(x)$ is continuous in closed interval $[0,4]$

② $f'(x) = 2x+1$

$\therefore f(x)$ is differentiable in open interval $(0,4)$

$\therefore$ Lagrange mean value theorem is applicable then there exist a point c in $(0,4)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{———} ①$$

$a = 0$

$f(a) = f(0) = -1$

$b = 4$

$f(b) = f(4) = 16+4-1$

$= 19$

from ①

$$\frac{19-(-1)}{4-0} = 2c+1$$

$$\frac{19+1}{4} = 2c+1$$

$$\frac{20}{4} = 2c+1$$

$5 = 2c+1$

$c = 2$

$2 \in (0,4)$

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② Using Lagrange's mean value theorem prove that

$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ and hence deduce

that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

sol. let $f(x) = \tan^{-1}x$ in $[a,b]$ $a>0, b>0$

① $f(x)$ is continuous in closed interval $[a,b]$

② $f'(x) = \frac{1}{1+x^2}$

$f(x)$ is differentiable in open interval (a,b)

$\because$ Lagrange's mean value theorem is applicable.

there exist a point c in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} = \frac{1}{1+c^2}$$

$$\tan^{-1}(b) - \tan^{-1}(a) = \frac{b-a}{1+c^2} \quad \text{--- ①}$$

$\because \quad a < c < b$

$\therefore \quad a^2 < c^2 < b^2$

$\therefore \quad 1+a^2 < 1+c^2 < 1+b^2$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

i.e. $\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$

$$\frac{b-a}{1+b^2} < \frac{b-a}{1+c^2} < \frac{b-a}{1+a^2} \quad \text{—— (2)}$$

from (1) & (2), we get

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

$$\tan^{-1}b - \tan^{-1}a = \frac{b-a}{1+b^2}$$

$b = \frac{4}{3}$, $a = 1$

$$\frac{4/3 - 1}{1 + \frac{16}{7}} < \tan^{-1}\frac{4}{3} - \tan^{-1}(1) < \frac{\frac{4}{3} - 1}{1+1}$$

i.e. $\frac{\frac{1}{3}}{\frac{25}{9}} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1/3}{2}$

i.e. $\frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$

(3) Using lagrange's mean value theorem prove that
→

$$\left(1 - \frac{a}{b}\right) < \log\left(\frac{b}{a}\right) < \left(\frac{b}{a} - 1\right), \quad 0 < a < b$$

and hence deduce that $\frac{1}{4} < \log\frac{4}{3} < \frac{1}{3}$.

sol. let $f(x) = \log x$ in closed interval (a, b)

(1) $f(x)$ is a continuous in $[a, b]$
(2) $f'(x) = \frac{1}{x}$ in (a, b). $f(x)$ is differentiable in open interval (a, b)

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(3) Lagrange's mean value theorem is applicable.

there exist a point c in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\log b - \log a}{b-a} = \frac{1}{c}$$

$$\log\left(\frac{b}{a}\right) = \frac{b-a}{c}$$

$a < c < b$

③ Cauchy's Mean Value theorem:
If $f(x)$ & $g(x)$ is defined in same interval. in $[a,b]$
If function $f(x)$ and $g(x)$ are

① both function is continuous in $[a,b]$
② both function is differentiable in (a,b)
③ $g'(x) \neq 0$

Then there exist a point c in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{g(b)-g(a)}{b-a} = g'(c)$$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Q. Expand $\sqrt{1+\sin x}$ in power of x

Sol. Let $f(x) = \sqrt{1+\sin x}$

$$= \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}$$

$$= \cos\frac{x}{2} + \sin\frac{x}{2}$$

$$= \left(1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4!}\left(\frac{x}{2}\right)^4 \cdots\cdots\right) + \left(\frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 \cdots\right)$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \cdots\cdots$$

Q. Prove that $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\cdots$

Sol.

Let $f(x) = \sinh x$

$$= \frac{1}{2}(e^x - e^{-x})$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots\right)\right]$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \cdots\cdots$$

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$\sinh 0 = 0$
$\cosh 0 = 1$

$\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\ldots$

$f(x) = \sinh x, \; f(0) = 0$
$f'(x) = \cosh x, \; f'(0) = 1$
$f''(x) = \sinh x, \; f''(0) = 0$
$f''''(x) = \cosh x, \; f''''(0) = 1$

Q. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

Sol. Let $f(x) = \log(1+x)$, $f(0) = 0$

$f'(x) = \frac{1}{1+x}$, $f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$, $f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$, $f'''(0) = 2$

$f^{iv}(x) = \frac{-6}{(1+x)^4}$, $f^{iv}(0) = -6$

By Maclaurin's Series

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$

$f(x) = 0 + x - \frac{x^2}{2!} + \frac{x^3}{3!} \times 2 \ldots\ldots$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

Q. Show that $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots\cdots$

Sol. $\log(1+\sin x) = \sin x - \left(\frac{\sin x}{2}\right)^2 + \left(\frac{\sin x}{3}\right)^3 + \cdots\cdots$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \cdots\right)^2 +$$

$$= \frac{1}{3}\left(x - \frac{x^3}{3!} + \cdots\cdots\right)^3$$

$$= \left(x - \frac{x^3}{6} + \cdots\cdots\right) - \frac{1}{2}(x^2) + \frac{1}{3}(x^3 + \cdots$$

$$= x - \frac{x^3}{6} - \frac{1}{2}x^2 + \frac{1}{8}x^3 + \cdots\cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6}$$

② ~~England~~ Expand $\log[1+x+x^2+x^3]$ upto a term in x^8.

Sol. Let $f(x) = \log \cancel{(1+(x^2+x)}$.

$= \log[(1+x) + x^2(x+1)]$

$= \log[1+x(1+x^2)]$

$= \cancel{\log(1+x)}$

$= \log[(1+x)(1+x^2)$

$= \cancel{\log} \log(1+x) + \log(1+x^2)$

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \right)$

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$$\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots\right)$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3}{4}(x^4) + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7}$$

$$- \frac{3}{8}x^8 + \cdots$$

Q. Prove that $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \cdots$

if $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

Sol.

Let $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

$-y = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots$

$-y = \log(1-x)$

$e^{-y} = 1-x$

$x = 1 - e^{-y}$

$$= 1 - \left(1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \frac{y^4}{4!} - \cdots\right)$$

Q4. Prove that $\sin^{-1}x = x + \frac{1}{2}x^3 + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots$

$y = \sin^{-1}x$ or $\cos^{-1}x$ or $\tan^{-1}x$ or $\log(1+x)$ or $\log(1-x)$

Sol.: Let $y = \sin^{-1}x$ ——— (1)

Diff equation (1) w.r.t. x.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$= (1-x^2)^{-1/2}$$

$$= [1 + (-x^2)]^{-1/2}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$

$$= 1 + (-\tfrac{1}{2})(-x^2) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(-x^2)^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}(-x^2)^3$$

$$+ \dots\dots$$

$$= 1 + \frac{1}{2}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}x^4 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-x^6)}{6} + \dots$$

$$= 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \dots\dots$$

By integration

$$y = \int \frac{dy}{dx} \cdot dx + c$$

$$= c + \int \left(1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \dots\right) dx$$

$$y = c + x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \dots\dots$$

Put $x = 0$

$y(0) = c$

$\boxed{c = 0}$

$$\left(\because y = \sin^{-1}(x) \quad y(0) = \sin^{-1}(0) = 0 \right)$$

$$\therefore \sin^{-1}(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4.5}x^5 + \frac{1.3.5}{2.4.6.7}x^7 + \ldots$$

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$$

$$\cos^{-1}(x) = \frac{\pi}{2} - x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4.5}x^5 + \frac{1.3.5}{2.4.6.7}x^7 + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots$$
$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \mp \ldots$$

Similarly, we can prove for $\tan^{-1}x$, $\log(1+x)$

* Indeterminant form :-

① $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$,

1^∞, 0^0, ∞^0

$f(x) = \frac{x^2+4}{x+2}$, $f(2) = 2$

$f(x) = \frac{x^2-4}{x-2}$ $\left(\frac{0}{0}\right)$

$$f(2) = \lim_{x\to 2} \frac{x^2-4}{x-2}$$
$$= 4$$

L-Hospital Rule :-

$$L = \lim_{x\to a} \frac{f(x)}{g(x)} \quad \left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$$

then,

$$= \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$$

again,

$$= \lim_{x\to a} \frac{f''(x)}{g''(x)}$$ till you get $g''(a) \neq 0$

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(1) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

(2) $\lim\limits_{x \to 0} \cos x = 1$

(3) $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$

Q1 Evaluate $\lim\limits_{x \to 1/2} \frac{\cos^2 \pi x}{e^{2x} - 2xe}$

Ans. it is $\frac{0}{0}$ form

$= \lim\limits_{x \to 1/2} \frac{\cos^2 \pi x}{e^{2x} - 2xe}$ applying L-Hospital

$= \lim\limits_{x \to 1/2} \frac{2\cos \pi x . (-\sin \pi x) . \pi}{e^{2x}.2 - 2e}$

$= \lim\limits_{x \to 1/2} \frac{-\cancel{2}\pi \cos \pi x . \sin \pi x}{\cancel{2}(e^{2x} - e)}$

$= \lim\limits_{x \to 1/2} \frac{-\pi (\cos \pi x . \cos \pi x \pi + \sin \pi x (-\sin \pi x)(\pi))}{e^{2x}.2 - 0}$ ×

×

$= \lim\limits_{x \to 1/2} -\pi^2 \frac{\cos^2 \pi x - \sin^2 \pi x}{2e^{2x}}$ ×

$= \lim\limits_{x \to 1/2} \frac{-2\pi \cos \pi x \sin \pi x}{2(e^{2x} - e)}$

$= \lim\limits_{x \to 1/2} \frac{-\pi \sin 2\pi x}{2(e^{2x} - e)}$

$$= \lim_{x\to 1/2} \frac{-\pi \cos 2\pi x \cdot 2\pi}{2(e^{2x}\cdot 2 - 0)}$$

$$= \lim_{x\to 1/2} -\frac{2\pi^2 \cos 2\pi x}{4e^{2x}}$$

$$= \frac{-2\pi^2(-1)}{4e}$$

$$= \frac{\cancel{2}\pi^2}{\cancel{4}2e}$$

$$= \frac{\pi^2}{2e}$$

Date 11.09.23

Note: $L = \lim_{x\to a} (f(x))^{g(x)}$ $(1^\infty, 0^0, \infty^0)$

Taking log on both side

$$\log L = \lim_{x\to a} \log (f(x))^{g(x)}$$

$$= \lim_{x\to a} g(x) \log f(x) \quad (0\times\infty \text{ or } \infty\times 0)$$ (we cannot apply L, Hospital's rule here.)

$$= \lim_{x\to a} \frac{\log f(x)}{1/g(x)} \quad \left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$$

$\log L = L_1$

$L = e^{L_1}$ $\qquad \frac{1}{\infty} = 0 \quad a^0 = 1$

Q. Evaluate $L = \lim_{x\to\infty} \left[\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + 4^{1/x}}{4}\right]^{4x}$ (1^∞)

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Taking log on both side

$$\log L = \lim_{x\to\infty} \log\left(\frac{1^{1/x}+2^{1/x}+3^{1/x}+4^{1/x}}{4}\right)^{4x}$$

$$= \lim_{x\to\infty} 4x \log\left(\frac{1^{1/x}+2^{1/x}+3^{1/x}+4^{1/x}}{4}\right)$$

$$= \lim_{x\to\infty} 4. \frac{\log(1^{1/x}+2^{1/x}+3^{1/x}+4^{1/x}) - \log 4}{1/x} \left(\frac{0}{0}\right)$$

By L-Hospital rule.

$$= \lim_{x\to\infty} 4. \frac{\frac{1}{(1^{1/x}+2^{1/x}+3^{1/x}+4^{1/x})} \cdot \cancel{\left(\frac{-1}{x^2}\right)} (1^{1/x}\log 1 + 2^{1/x}\log 2 + 3^{1/x}\log 3 + 4^{1/x}\log 4)}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} 4. \frac{(1^{1/x}\log 1 + 2^{1/x}\log 2 + 3^{1/x}\log 3 + 4^{1/x}\log 4)}{(1^{1/x}+2^{1/x}+3^{1/x}+4^{1/x})}$$

$$= \cancel{4}! \frac{(0+\log 2+\log 3+\log 4)}{\cancel{4}1}$$

$$= \log 2\times 3\times 4$$

$$\boxed{= \log 24}$$

$$\log L = \log 24$$

$$\boxed{L = 24}$$

4). Evaluate $\lim_{x \to \pi/2} (\cos x)^{\cos^2 x}$

let $L = \lim_{x \to \pi/2} (\cos x)^{\cos^2 x}$ $\quad \left(\frac{0}{0}\right) 0^0$

$\log L = \cancel{\log} \lim_{x \to \pi/2} \log (\cos x)^{\cos^2 x}$

$= \lim_{x \to \pi/2} \cos^2 x \log(\cos x)$ $\quad \log 0 = \infty$ $\quad (0 \times \infty)$

$= \lim_{x \to \pi/2} \frac{\log(\cos x)}{1/\cos^2 x}$ $\quad \left(\frac{\infty}{\infty}\right)$

L Hospital rule.

$= \lim_{x \to \pi/2} \frac{\log(\cos x)}{\sec^2 x}$

$= \lim_{x \to \pi/2} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{2\sec x \cdot \sec x \tan x}$

$= \lim_{x \to \pi/2} \frac{\sec x(-\sin x)}{2\sec^{\cancel{2}} x \tan x}$

$= \lim_{x \to \pi/2} \frac{\cos x(-\sin x)}{2\sec x \cdot \sin x}$

$= \lim_{x \to \pi/2} -\frac{1}{2\sec^2 x}$

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$$\log L = \frac{-1}{\infty}$$

$$\log L = 0$$

$$L = e^0$$

$$\boxed{L = 1}$$

2. Evaluate $\lim\limits_{x \to 0} \log_{\tan x} \tan 2x$

<u>Sol.</u>

Let $L = \lim\limits_{x \to 0} \dfrac{\log \tan 2x}{\log \tan x} \quad \left(\dfrac{\infty}{\infty}\right)$

L-Hosptal

$$= \lim_{x \to 0} \frac{\frac{1}{\tan 2x} \cdot \sec^2 2x \cdot 2}{\frac{1}{\tan x} \cdot \sec^2 x}$$

$$= \lim_{x \to 0} \frac{\tan x}{\tan 2x} \cdot \frac{\sec^2 2x}{\sec^2 x} \cdot 2$$

$$= \lim_{x \to 0} \left(\frac{\tan x}{\tan 2x}\right) \cdot \lim_{x \to 0} \left(\frac{2 \sec^2 2x}{\sec^2 x}\right)$$

$$= 2 \lim_{x \to 0} \frac{\tan x}{\tan 2x} \quad \left(\frac{0}{0}\right)$$

$$= 2 \lim_{x \to 0} \frac{\sec^2 x}{\sec^2 2x \cdot 2}$$

$$= \frac{2}{2}$$

$$= 1$$

$$\boxed{L = 1}$$

① $\lim_{x \to 0} \frac{\sin x}{x} = 1$ ② $\lim_{x \to 0} \cos x = 1$ ③ $\lim_{x \to 0} \frac{\tan x}{x} = 1$

⑤ Evaluate $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

Sol. Let $L = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$ $(1)^{\infty}$

Taking log on both side

$$\log L = \lim_{x \to 0} \log\left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$\log L = \lim_{x \to 0} \frac{1}{x^2} \log\left(\frac{\sin x}{x}\right)$$

$$\log L = \lim_{x \to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\log(\sin x) - \log x}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x} \cdot \cos x - \frac{1}{x}}{2x}$$

$$= \lim_{x \to 0} \frac{\cot x - \frac{1}{x}}{2x}$$

$$= \lim_{x \to 0} \frac{-\operatorname{cosec}^2 x - \left(-\frac{1}{x^2}\right)}{2}$$

$$= \lim_{x \to 0} \frac{1}{2} \left(\frac{1}{x^2} - \operatorname{cosec}^2 x\right)$$

$$= \lim_{x \to 0} \frac{1}{2} \frac{1 - x^2 \operatorname{cosec}^2 x}{x^2}$$

$$= \lim_{x \to 0} \frac{1}{2} \frac{0 - (2x\operatorname{cosec}^2 x + x^2\, 2\operatorname{cosec}x \cdot (-\operatorname{cosec}x\cot x))}{2x}$$

$$= \lim_{x \to 0} \frac{1}{2} -\operatorname{cosec}^2 x + \operatorname{cosec}^2 x \cot x$$

$$= \lim_{x \to 0} \frac{1}{2} \operatorname{cosec}^2 x(-1 + \cot x)$$

$$\log L = \lim_{x \to 0} \frac{\frac{1}{\frac{\sin x}{x}}\left(\frac{x\cos x - \sin x}{x^2}\right)}{2x}$$

$$= \lim_{x \to 0} \frac{x\cos x - \sin x}{2x^3}$$

$$= \lim_{x \to 0} \frac{x(-\sin x) + \cancel{\cos x} - \cancel{\cos x}}{6x^2}$$

$$= \lim_{x \to 0} -\frac{\cancel{x}\sin x}{6\cancel{x}\cdot x}$$

$$\log L = -1/6$$

$$\boxed{L = e^{-1/6}}$$

$$= \lim_{x \to 0} \frac{\log\left[\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots}{x}\right]}{x^2}$$

$$= \lim_{x \to 0} \frac{\log\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots\right]}{x^2}$$

$(1 - x)$

$$= \lim_{x\to 0} \log\left[1-\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right)\right]$$

$$= \lim_{x\to 0} \frac{\left(\frac{x^2}{3!} - \frac{x^4}{5!}\right) - \frac{\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right)^2}{2}}{x^2}$$

$$\log L = \lim_{x\to 0} \left(-\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^2}{2(3!)^2}\right)$$

$$\log L = \frac{-1}{6}$$

$$L = e^{-1/6}$$

(5) Evaluate $\lim_{x\to\infty} \left(\frac{ax+1}{ax-1}\right)^x$

Sol. Let $L = \lim_{x\to\infty} \left(\frac{ax+1}{ax-1}\right)^x$

$$\log L = \lim_{x\to\infty} \log\left(\frac{ax+1}{ax-1}\right)^x$$

$$\log L = \lim_{x\to\infty} \log\left(\frac{1+1/ax}{1-1/ax}\right)^x \qquad (1^\infty)$$

$$\log L = \lim_{x\to\infty} x \log\left(\frac{1+1/ax}{1-1/ax}\right) \qquad (\infty \times 0)$$

$$= \lim_{x\to\infty} \frac{\log\left(\frac{1+1/ax}{1-1/ax}\right)}{1/x}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{1}{1+1/ax}\right)\left(\frac{-1}{ax^2}\right) - \left(\frac{1}{1-1/ax}\right)\cdot\left(\frac{1}{ax^2}\right)}{\left(-1/x^2\right)}$$

$$= \lim_{x\to\infty} \left(\frac{1}{1+1/ax}\right)\left(\frac{1}{a}\right) + \left(\frac{1}{1-1/ax}\right)\cdot\frac{1}{a}$$

$$= \frac{1}{a} \lim_{x\to\infty} \left(\frac{1}{1+1/ax}\right) + \left(\frac{1}{1-1/ax}\right)$$

$$= \frac{1}{a}(1+1)$$

$$\log L = \frac{2}{a}$$

$$L = e^{2/a}$$

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Math-1 Tutorial

Convergence of Sequence & Series
↳ It gives some finite value

Divergence ↳ It gives infinite value.

Seq :- a_n

$\{a_n\} = a_1, a_2, a_3, a_4, a_5 \ldots\ldots$

Let there be an seq. $\{a_n\} = \{n\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \ldots$ ↳ increasing sequence.

$\{a_n\} = \{\frac{1}{n}\} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \ldots\ldots\}$ Decreasing Sequence

Series is addition.

$$S_n = \sum_{n=1}^{\infty} a_n$$

$$S_n = \sum_{n=1}^{\infty} n$$

$$= 1 + 2 + 3 + 4 + 5 \ldots\ldots$$

Convergence of Sequence & Series

Series: a series is the sum of the terms of an infinit sequence of numbers.
mathematically,

$$S_n = \sum_{n=1}^{\infty} a_n$$

Ex. $a_n = n$

$$\sum_{n=1}^{\infty} a_n = 1 + 2 + 3 + 4 + \cdots\cdots \infty = \sum_{n=1}^{\infty} n$$

* Cratio's Test.
De Alembert's Ratio Test

formula

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

(i) if $L < 1 \rightarrow$ convergent
(ii) if $L > 1 \rightarrow$ divergent
(iii) if $L = 1 \rightarrow$ Can't Say.

Eg:- $a_n = n$

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)$$

$$= 1$$

Q. For the series Σa_n find the $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

Q1. Test for convergent for the series whose n^{th} term is $\frac{n^2}{2^n}$

Sol. $a_n = \frac{n^2}{2^n}$

Ratio test

$$\lim_{n\to\infty} \Sigma a_n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \div \frac{n^2}{2^n}$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \times \frac{2^n}{(n^2)}$$

a^n in num divergent

a^n in denom than convergent

$$\lim_{n\to\infty} \frac{(n^2+1+2n) \times \cancel{2^n}}{\cancel{2^n} \times 2 \times n^2}$$

$$\lim_{n\to\infty} \frac{1}{2n^2}(n^2+2n+1)$$

$$\lim_{n\to\infty} \frac{1}{2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)$$

$$=\frac{1}{2}$$

$L<1$ $\therefore$ convergent.

a^n in denominator convergent

a^n in numerator divergent.

Q2. Discuss the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}.$$

Sol. Ratio Test

$$L=\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \qquad \text{here } a_n = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$$

$$=\lim_{n\to\infty} \frac{x^{n+1-1}}{n+1 . 3^{n+1}} \div \frac{x^{n-1}}{n . 3^n}$$

$$=\lim_{n\to\infty} \frac{\cancel{x^n}}{(n+1).\cancel{3^n}\times 3} \times \frac{\cancel{3^n}\times n}{\cancel{x^n}\times x^{-1}}$$

$$=\lim_{n\to\infty} \frac{x\times n}{3(n+1)}$$

$$= \lim_{n\to\infty} \frac{x}{3} \frac{\cancel{n}}{\cancel{n}(1+1/n)}$$

$$= \lim_{n\to\infty} \frac{x}{3} \times \frac{1}{(1+1/n)}$$

$$L = \frac{x}{3}$$

for $x < 3$
L is convergent
series

for $x > 3$
L is divergent
series

at $x = 3$
can't be say

Q3 Test for the convergent of the Series whose n^{th} term is $\sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$

Sol. Here series $a_n = \sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$

Using Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{(n+1)!\,2^{n+1}}{(n+1)^{(n+1)}} \div \frac{n!\,2^n}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cancel{(n)!}\,\cancel{2^n}\times 2}{(n+1)^{(n+1)}} \times \frac{n^n}{\cancel{n!}\,\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)} \times 2 \times n^n}{(n+1)^n \cdot \cancel{(n+1)}}$$

$$= \lim_{n\to\infty} \frac{2\,n^n}{(n+1)^n}$$

$$L = \lim_{n\to\infty} \frac{2n^n}{(n+1)^n}$$

$$\log L = \lim_{n\to\infty} \log \frac{2n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \log 2n^n - \log(n+1)^n$$

$$= \lim_{n\to\infty} \log 2 \cancel{*} n\log n - n\log(n+1)$$

$$= \cancel{\log 2 + \lim_{n\to\infty} n\log n - n\log(n+1)}$$

$$= \lim_{n\to\infty} \log 2 + n\log n - n\log(n+1)$$

$$= \lim_{n\to\infty} \log 2 + n\left(\log \frac{\cancel{n}}{\cancel{n}(1+1/n)}\right)$$

$$= \lim_{n\to\infty} \log 2 + n\log\left(\frac{1}{(1+1/n)}\right)$$

$$= \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n$$

$$= 2\lim_{n\to\infty}\left(\frac{\cancel{n}}{\cancel{n}(1+1/n)}\right)^n$$

$$= 2\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^n$$

$$= 2\left(\frac{1}{1+1/\infty}\right)^\infty$$

$$= 2\times 1$$

$$L = 2$$

Q. Test the convergence of the series

$$1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!}$$

sol.

$$a_n = \frac{n^2}{n!}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

Ratio test

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!} \div \frac{n^2}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{\cancel{2}}}{\cancel{(n+1)}(n!)} \times \frac{n!}{n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{n^2}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n^2}\right)$$

$$= 0$$

$L<1$ convergent.

② Evaluate $\displaystyle\lim_{x\to 0} \frac{\log_{\sin x} \cos x}{\log_{\sin x/2} \cos x/2}$

Sol. Let $\displaystyle L = \lim_{x\to 0} \frac{\log_{\sin x} \cos x}{\log_{\sin x/2} \cos x/2}$

$$= \lim_{x\to 0} \frac{\log \cos x}{\log \sin x} \times \frac{\log \sin x/2}{\log \cos x/2}$$

$$= \lim_{x\to 0} \frac{\log \sin x/2}{\log \sin x} \times \frac{\log \cos x}{\log \cos x/2} \quad (\infty \times 0)$$

$$= \left(\lim_{x\to 0} \frac{\log \cos x}{\log \cos x/2}\right)\left(\lim_{x\to 0} \frac{\log \sin x/2}{\log \sin x}\right)$$

$$L = l_1 . l_2 \quad \text{——} ①$$

$$l_1 = \lim_{x\to 0} \frac{\log \cos x}{\log \cos x/2} \quad \left(\frac{0}{0}\right)$$

L-H rule

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x} . (-\sin x)}{\frac{1}{\cos x/2} - \sin x/2 . 1/2}$$

$$= \lim_{x\to 0} \frac{2 \tan x}{\tan x/2} = \lim_{x\to 0} \frac{2\tan x}{\tan x/2}$$

$$= 2\lim_{x\to 0} \frac{\tan x}{\tan x/2}$$

$$= 2\lim_{x\to 0} \frac{\sec^2 x}{\sec^2 x/2 \cdot 1/2}$$

$$= 4\lim_{x\to 0} \frac{\sec^2 x}{\sec^2 x/2}$$

$$= 4\times\frac{1}{1}$$

$$\boxed{L_1 = 4}$$

$$L_2 = \lim_{x\to 0} \frac{\log\sin x/2}{\log\sin x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x/2}\cdot\cos x/2\cdot\frac{1}{2}}{\frac{1}{\sin x}\cdot\cos x}$$

$$= \lim_{x\to 0}\times\frac{1}{2}\,\frac{\cot x/2}{\cot x}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{-\operatorname{cosec}^2 x/2\cdot 1/2}{-\operatorname{cosec}^2 x}$$

$$= \frac{1}{4}\lim_{x\to 0} \frac{\operatorname{cosec}^2 x/2}{\operatorname{cosec}^2 x}$$

$$= \frac{1}{4}\lim_{x\to 0} \frac{-\cancel{2}\operatorname{cosec} x/2\cdot\operatorname{cosec} x/2\cdot\cot x/2\cdot 1/\cancel{2}}{-2\operatorname{cosec} x\cdot\operatorname{cosec} x\cdot\cot x}$$

$$= \frac{1}{8}\lim_{x\to 0} \frac{\operatorname{cosec}^2 x/2\,\cot x/2}{\operatorname{cosec}^2 x\,\cot x}$$

$$= \frac{1}{8}\lim_{x\to 0} \frac{\sin^2 x}{\sin^2 x/2}\,\frac{\tan x}{\tan x/2}\times\frac{x^3}{x^3}$$

$$= \frac{1}{8}\lim_{x\to 0} \frac{\sin^2 x}{x^2}\times\frac{x^2}{\sin^2 x/2}\times\frac{\tan x}{x}\times\frac{x}{\tan x/2}$$

$$= \frac{1}{8} \lim_{x \to 0} \frac{x^2}{\sin^2 x/2} \times \frac{x}{\tan x/2}$$

$$= \frac{1}{4} \times \frac{1}{2} \lim_{x \to 0} \frac{x^2}{\sin^2 x/2} \times \frac{x}{\tan x/2}$$

$$= \lim_{x \to 0} \frac{x^2/4}{\sin^2 x/2} \times \frac{x/2}{\tan x/2}$$

$$= 1 \times 1$$

$$L_2 = 1$$

$$L = L_1 \times L_2$$

$$\boxed{L = 4 \times 1}$$

OR

$$= \lim_{2x \to 0} \frac{\tan x}{\tan x/2}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{\tan x}{x} \times \frac{x}{\tan x/2}$$

$$= \lim_{x \to 0} \frac{\tan x}{x} \times \frac{x/2}{\tan x/2}$$

$$= 1 \times 1$$

$$\boxed{L_2 = 1}$$

② If $\lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} = 1$, find a & b

Sol.

Let $L = \lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3}$ $(0/0)$

$$= \lim_{x \to 0} \frac{x(-a\sin x) + (1 + a\cos x) - b\cos x}{3x^2}$$

When $x \to 0$, $Dr \to 0$

$Nr \to 0$

$1 + a - b = 0$

$a - b = -1$ —— (1)

$= \lim_{x \to 0} \cancel{-ax\sin x + x}$

$$= \lim_{x \to 0} \frac{x(-a\sin x) + (1 + a\cos x) - b\cos x}{3x^2}$$

$$= \lim_{x \to 0} \frac{x(-a\cos x) + (\bar{a}\sin x) + (-a\sin x) + b\sin x}{6x}$$

$$= \lim_{x \to 0} \frac{-ax\cos x - 2a\sin x + b\sin x}{6x}$$

$$= \lim_{x \to 0} \frac{-a(\cos x + x(-\sin x)) - 2a\cos x + b\cos x}{6}$$

$$= \lim_{x \to 0} \frac{-a\cos x + ax\sin x - 2a\cos x + b\cos x}{6}$$

$$= \frac{-a + 0 - 2a + b}{6}$$

$$1 = \frac{-3a + b}{6}$$

$6 = b - 3a$

$a - b = -1$

$b - 3a = 6$

$+ \quad -2a = 5$

$\boxed{a = -\frac{5}{2}}$

$b = a + 1$

$= -\frac{5}{2} + 1$

$\boxed{b = -\frac{3}{2}}$

4) Prove that $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} = \frac{-e}{2}$

sol.

Let $L = \lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$ × $\frac{d}{dx} a^x = a^x \log a$

$L = \lim_{x \to 0} \frac{(1+x)^{1/x} \log(1+x) - 0}{1}$ ×

$L = \lim_{x \to 0} (1+x)^{1/x} \log(1+x)$ ×

Let $L = \lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$

Let $y = (1+x)^{1/x}$

Taking log on both side

$\log y = \frac{1}{x} \log(1+x)$

$= \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} \dots\right)$

$\log y = \left(1 - \frac{x}{2} + \frac{x^2}{3} \dots\right.$

$y = e^{1 - x/2 + x^2/3}$

$y = e^{1+z}$ $\qquad z = -\frac{x}{2} + \frac{x^2}{3} \dots$

$y = e^1 . e^z$

$y = e\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!}\right]$

$y = e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3}\right) + \frac{1}{2!}\left(-\frac{x}{2} + \frac{x^2}{3}\right)^2 + \frac{1}{3!} \dots\right]$

$y = e\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{2}\left(\frac{x^2}{4} \dots + \frac{x^4}{9}\right.\right.$

$$y = e\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{2} \times \frac{x^2}{4}\right]$$

$$y = e\left[1 - \frac{x}{2} + \frac{11x^2}{24} + \cdots\right]$$

from ①

$$L = \lim_{x \to 0} \frac{\cancel{e} - ex/2 + \frac{11ex^2}{24} + \cdots - \cancel{e}}{x}$$

$$L = \lim_{x \to 0} -e/2 + \frac{11ex}{24} + \cdots\cdots$$

$$\boxed{L = -\frac{e}{2}}$$

12.09.23

Module-02 Partial differentiation

① $y = f(x)$

$\swarrow$ dependent, $\searrow$ independent — variable

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

② $z = f(x, y)$

↳ 2 independent variable.

~~dependent variable on 2 independent variable~~

ⓐ $$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$ Keeping y constant

ⓑ $$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$ Keeping x constant

Ex. $z = x^2y + xy^2$

$$\frac{\partial z}{\partial x} = (2x)y + (1)y^2$$

$$= 2xy + y^2$$

$$\frac{\partial z}{\partial y} = x^2 + 2xy$$

Q① If $u = e^{x^y}$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

Sol. Given $u = e^{x^y}$

① $$\frac{\partial u}{\partial x} = \frac{\partial e^{x^y}}{\partial x}$$

$= e^{x^y} \frac{\partial}{\partial x}(x^y)$

$= e^{x^y} \cdot y\, x^{y-1}$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^{x^y})$

$= e^{x^y} \frac{\partial}{\partial y}(x^y)$

$= e^{x^y} \cdot x^y \log x$

(2) If $u = (1-2xy+y^2)^{-1/2}$, then prove that

$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2 \cdot u^3$

sol: let $u = (1-2xy+y^2)^{-1/2}$

$\frac{\partial u}{\partial x} = \frac{-1}{2}(1-2xy+y^2)^{-\frac{1}{2}-1} \cdot (0-2y+0)$

$= \frac{+1}{\cancel{2}}(1-2xy+y^2)^{-3/2}(+\cancel{2}y)$

$\frac{\partial u}{\partial x} = y(1-2xy+y^2)^{-3/2}$

$\frac{\partial u}{\partial x} = y\left[(1-2xy+y^2)^{-1/2}\right]^3$

$\frac{\partial u}{\partial x} = y.(u)^3$

$$\frac{\partial u}{\partial y} = \frac{-1}{2}(1-2xy+y^2)^{-\frac{1}{2}-1}\cdot(0-\cancel{2}x+\cancel{2}y)$$

$$\frac{\partial u}{\partial y} = (x-y)(1-2xy+y^2)^{-3/2}$$

$$\frac{\partial u}{\partial y} = (x-y)\cdot\left[(1-2xy+y^2)^{-1/2}\right]^3$$

$$\partial u/\partial y = (x-y)\cdot u^3$$

$$= x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}$$

$$= x\cdot y(u^3) - y(x-y)\cdot u^3$$

$$= xyu^3 - (xy - y^2)u^3$$

$$= xyu^3 - (xyu^3 - y^2u^3)$$

$$= \cancel{xyu^3} - \cancel{xyu^3} + y^2u^3$$

$$= y^2 u^3$$

$$\boxed{= y^2\cdot u^3}$$

(3) If $u = \log(\tan x + \tan y + \tan z)$, then prove that $\sin 2x\cdot\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2$

Sol.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log(\tan x + \tan y + \tan z)$$

$$= \frac{1}{(\tan x + \tan y + \tan z)}\cdot(\sec^2 x + 0 + 0)$$

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{(\tan x + \tan y + \tan z)}$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{(\tan x + \tan y + \tan z)}$$

$$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{(\tan x + \tan y + \tan z)}$$

$$= \sin 2x \cdot \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= 2\sin x \cdot \cos x \cdot \frac{\sec^2 x}{(\tan x + \tan y + \tan z)} + \frac{2\sin y \cos y . \sec^2 y}{(\tan x + \tan y + \tan z)} + 2\sin z \cos z \frac{\partial u}{\partial z}$$

$$= 2\left[\frac{\tan x}{(\tan x + \tan y + \tan z)} + \frac{\tan y}{(\tan x + \tan y + \tan z)} + \frac{\tan z}{(\tan x + \tan y + \tan z)}\right.$$

$$= 2\left[\frac{\cancel{\tan x + \tan y + \tan z}}{\cancel{\tan x + \tan y + \tan z}}\right]$$

$$= 2 \times (1)$$

$$= 2$$

Higher order Partial derivatives

$z = f(x, y)$

first order derivative

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Second order derivative

(a) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

$$Z_{xx} = \frac{\partial^2 z}{\partial x^2}$$

(b) $$Z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

(c) $$Z_{zz} = \frac{\partial^2 z}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial z}{\partial z}\right)$$

(d) $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = Z_{xy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = Z_{yx}$$

Sometimes, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

Exercise 12

1. If $z = x^y + y^x$, find $\frac{\partial^2 z}{\partial y \partial x}$

first w.r.t x.

$$Z_{yx} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial z}{\partial x} = yx^{y-1} + y^x \log y$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(yx^{y-1} + y^x \log y\right)$$

$$= \frac{\partial}{\partial y}\left(yx^{y-1} + y^x \log y\right)$$

$$= \frac{\partial}{\partial y}\left(\left(\frac{x^y}{x}\right)y + y^x \log y\right)$$

$$= y\left(\frac{x^y \log x}{x}\right) + \frac{x^y}{x} + x y^{x-1} \cdot \log y + y^x \frac{1}{y}$$

$$\boxed{= y x^{y-1} \log x + x^{y-1} + x y^{x-1} \log y + y^{x-1}}$$

$$= \frac{\partial}{\partial y}\left(\frac{x^y}{x} \cdot y\right)$$

$$= y\left(\frac{1}{x} \cdot x^y \log x\right) + \left(\frac{x^y}{x} \cdot 1\right)$$

$$= y x^{y-1} \log x + x^{y-1}$$

(2) If $z = \tan(y+ax) + (y-ax)^{3/2}$, then prove that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$

Let $z = \tan(y+ax) + (y-ax)^{3/2}$ —— (1)

Diff z w.r.t. x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[\tan(y+ax) + (y-ax)^{3/2}\right]$$

$$= \sec^2(y+ax).(a) + \frac{3}{2}(y-ax)^{\frac{3}{2}-1}.(-a)$$

$$= \sec^2(y+ax).(a) + \frac{3}{2}(y-ax)^{1/2}.(-a)$$

Again Diff. w.r.t. x

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{a\partial}{\partial x}\left[\sec^2(y+ax) - \frac{3}{2}(y-ax)^{1/2}\right]$$

$$= a\left[2\sec(y+ax).\sec(y+ax)\tan(y+ax).(a) - \frac{3}{2}\times\frac{1}{2}(y-ax)^{-1/2}.(-a)\right]$$

$$\frac{\partial^2 z}{\partial x^2} = a^2\left[2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}\right] \rightarrow ①$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[\tan(y+ax) + (y-ax)^{3/2}\right]$$

$$= \sec^2(y+ax).(1) + \frac{3}{2}(y-ax)^{1/2}.(1)$$

$$= \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$$

$$\frac{\partial^2 z}{\partial y^2} = 2\sec(y+ax).\sec(y+ax).\tan(y+ax).(1) + \frac{3}{2}\times\frac{1}{2}(y-ax)^{-1/2}.(1)$$

$$= \left[2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}\right] - ③$$

from ① & ③

$$\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$$

③ If $z^3 - \cancel{xy}^{zx} - y = 0$, then show that $\frac{\partial^2 z}{\partial x \partial y} = -\frac{(3z^2+x)}{(3z^2-x)^3}$

<u>sol.</u> Given,

$$z^3 - xy - y = 0$$

$$3z^2\frac{\partial z}{\partial y} - x - 1 = 0$$

$$\frac{\partial}{\partial x}\left(3z^2\frac{\partial z}{\partial y} - x - 1\right) = 0$$

$$3\left(2z\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}+z^2\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\right)-1-0)=0$$

$$3\left(2z\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}+z^2\frac{\partial^2 z}{\partial x\partial y}\right)-1=0$$

$$\cancel{z^3-2x-y=0}$$

or

$z^3-2x-y=0$

Diff w.r.t. y

$$3z^2\frac{\partial z}{\partial y}-x\frac{\partial z}{\partial y}-1=0 \quad \text{—— (1)}$$

Diff w.r.t. x

$$3z^2\frac{\partial z}{\partial x}-z-x\frac{\partial z}{\partial x}-0=0$$

$$3z^2\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial x}-z=0$$

$$\frac{\partial z}{\partial x}(3z^2-x)-z=0$$

$$\frac{\partial z}{\partial x}(3z^2-x)=z$$

$$\frac{\partial z}{\partial x}=\frac{z}{(3z^2-x)} \quad \text{—— (2)}$$

from (1)

$$\frac{\partial z}{\partial y}(3z^2-x)=1$$

$$\frac{\partial z}{\partial y}=\frac{1}{(3z^2-x)} \quad \text{—— (3)}$$

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$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(3z^2-x)^{-1}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -1\,(3z^2-x)^{-1-1} \cdot \left(3.2z.\frac{\partial z}{\partial x} - 1\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{(3z^2-x)^2} \cdot \left(6z\frac{\partial z}{\partial x} - 1\right)$$

from (2)

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{(3z^2-x)^2}\left(6z \times \frac{z}{(3z^2-x)} - 1\right)$$

$$= \frac{-1}{(3z^2-x)^2}\left(\frac{6z^2 - 3z^2 + x}{3z^2-x}\right)$$

$$= \frac{-3z^2-x}{(3z^2-x)^3}$$

$$= -\left(\frac{3z^2+x}{(3z^2-x)3}\right)$$

$$= -\left[\frac{3z^2+x}{(3z^2-x)^3}\right]$$

(4) If $u = \log(x^3+y^3-x^2y-xy^2)$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-4}{(x+y)^2}$$

Sol. Let $u = \log(x^3+y^3-x^2y-xy^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3+y^3-x^2y-xy^2)} \cdot (3x^2+0-2xy-y^2)$$

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$$u = \log(x^3 + y^3 - x^2y - xy^2)$$

$$= \log(x^2(x-y) + y^2(y-x))$$

$$= \log(x^2(x-y) - y^2(x-y))$$

$$= \log((x-y)(x^2-y^2))$$

$$= \log((x-y)(x-y)(x+y))$$

$$u = \log((x-y)^2(x+y))$$

$$u = \log(x-y)^2 + \log(x+y)$$

$$u = 2\log(x-y) + \log(x+y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} 2\log(x-y) + \log(x+y)$$

$$\frac{\partial u}{\partial x} = \frac{2\cdot(1-0)}{(x-y)} + \frac{1}{(x+y)}(1-0)$$

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)} + \frac{1}{(x+y)}$$

$$\frac{\partial^2 u}{\partial x^2} = -2(x-y)^{-2}\cdot(1) + -1(x+y)^{(-1-1)}\cdot(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2}{(x-y)^2} - \frac{1}{(x+y)^{+2}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2\log(x-y) + \log(x+y))$$

$$= \frac{2}{(x-y)}\cdot(0-1) + \frac{1}{(x+y)}\cdot(0+1)$$

$$\frac{\partial u}{\partial y} = \frac{-2}{(x-y)} + \frac{1}{(x+y)}$$

$$\frac{\partial^2 u}{\partial y^2} = -2\cdot -1\cdot (x-y)^{-2}\cdot(0-1) + (-1)\,(x+y)^{-2}\cdot(0+1)$$

$$= -\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -2\frac{\partial (x-y)^{-1}}{\partial x} + \frac{\partial (x+y)^{-1}}{\partial x}$$

$$= -2\cdot -1\cdot (x-y)^{-2}\cdot(1) + -1\cdot(x+y)^{-2}\cdot(1)$$

$$= 2(x-y)^{-2} - (x+y)^{-2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}$$

$$\text{LHS} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\,\partial^2 u}{\partial x \partial y}$$

$$= \frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2} + \frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2} +$$

$$2\left(\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}\right)$$

$$= \cancel{\frac{-4}{(x-y)^2}} + \cancel{\frac{4}{(x-y)^2}} - \frac{2}{(x+y)^2} - \frac{2}{(x+y)^2}$$

$$= \frac{-4}{(x+y)^2}$$

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Date
13.09.23 Partial Derivative of composite function

① if $z = f(u)$ and $u = \phi(x,y)$

$$z \longrightarrow u \longrightarrow (x,y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

② If $z = f(x,y)$ and $x = \phi(t)$, $y = \phi(t)$

$$\because \quad z \longrightarrow (x,y) \longrightarrow t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

③. If $z = f(u,v)$ and $u = \phi(x,y)$ and $v = \phi(x,y)$

$$\because \quad z \longrightarrow (u,v) \longrightarrow (x,y)$$

① $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

② $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

④ If ϕ is a function of (x,y,z) and $x = f(u,v,w)$

$y = f(u,v,w)$

$z = f(u,v,w)$

$\because \phi \longrightarrow (x, y, z) \longrightarrow (u, v, w)$

$$① \ \frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial u}$$

$$② \ \frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial v}$$

$$③ \ \frac{\partial \phi}{\partial w} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial w}$$

Q1. If $z = f(u, v)$, $u = \log(x^2 + y^2)$, $v = \frac{y}{x}$, then show that

$$x \frac{\partial z}{\partial y} - y \frac{\partial z}{\partial x} = (1 + v^2) \frac{\partial z}{\partial v}$$

Sol. $\cancel{\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y}}$

$\cancel{\# \ \frac{\partial}{\partial u}}$

$z \longrightarrow (u, v) \longrightarrow (x, y)$

Given,

$u = \log(x^2 + y^2) \qquad v = \frac{y}{x}$

$$\frac{\partial u}{\partial x} = \frac{1}{(x^2 + y^2)} \cdot (2x + 0)$$

$$\frac{\partial u}{\partial x} = \frac{2x}{(x^2 + y^2)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{(x^2+y^2)} \cdot (2y) \qquad \frac{\partial v}{\partial x} = \frac{-y}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{(x^2+y^2)} \qquad \frac{\partial v}{\partial y} = \frac{1}{x}$$

① $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u} \cdot \frac{2x}{(x^2+y^2)} + \frac{\partial z}{\partial v} \cdot \left(\frac{-y}{x^2}\right)$$

$$= \frac{2x}{(x^2+y^2)} \frac{\partial z}{\partial u} + \left(\frac{-y}{x^2}\right) \frac{\partial z}{\partial v}$$

② $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \frac{2y}{(x^2+y^2)} \cdot \frac{\partial z}{\partial u} + \frac{1}{x} \cdot \frac{\partial z}{\partial v}$$

$$\text{LHS} = x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}$$

$$= x\left(\frac{2y}{x^2+y^2} \cdot \frac{\partial z}{\partial u} + \frac{1}{x} \cdot \frac{\partial z}{\partial v}\right) - y\left(\frac{2x}{(x^2+y^2)} \cdot \frac{\partial z}{\partial u} + \left(\frac{-y}{x^2}\right) \frac{\partial z}{\partial v}\right)$$

$$= \cancel{\frac{2xy}{x^2+y^2} \frac{\partial z}{\partial u}} + \frac{\partial z}{\partial v} - \cancel{\frac{2xy}{(x^2+y^2)} \frac{\partial z}{\partial u}} + \frac{y^2}{x^2} \frac{\partial z}{\partial v}$$

$$= \frac{\partial z}{\partial v} + \frac{y^2}{x^2}\frac{\partial z}{\partial v}$$

$$= \left(1 + \frac{y^2}{x^2}\right)\frac{\partial z}{\partial v}$$

$$= \left(1 + \left(\frac{y}{x}\right)^2\right)\frac{\partial z}{\partial v}$$

$$= \left(1 + v^2\right)\frac{\partial z}{\partial v} \Rightarrow = \text{RHS.}$$

(2) If $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$ and ϕ is a function of x, y, z, then show that

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$$

Sol. Given,

$x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$

$\phi \longrightarrow (x, y, z) \longrightarrow (vw, uw, uv)$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \cdot$$

$x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$

$\frac{\partial x}{\partial u} = 0$, $\frac{\partial y}{\partial u} = \frac{\sqrt{w}}{2\sqrt{u}}$, $\frac{\partial z}{\partial u} = \frac{\sqrt{v}}{2\sqrt{u}}$

$\frac{\partial x}{\partial v} = \frac{\sqrt{w}}{2\sqrt{v}}$, $\frac{\partial y}{\partial v} = 0$, $\frac{\partial z}{\partial v} = \frac{\sqrt{u}}{2\sqrt{v}}$

$\frac{\partial x}{\partial w} = \frac{\sqrt{v}}{2\sqrt{w}}$, $\frac{\partial y}{\partial w} = \frac{\sqrt{u}}{2\sqrt{w}}$, $\frac{\partial z}{\partial w} = 0$

$$\because \phi \longrightarrow (x, y, z) \longrightarrow (u, v, w)$$

$$\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial x}\cdot 0 + \frac{\partial\phi}{\partial y}\cdot\frac{\sqrt{w}}{2\sqrt{u}} + \frac{\partial\phi}{\partial z}\cdot\frac{\sqrt{v}}{2\sqrt{u}}$$

$$u\frac{\partial\phi}{\partial u} = u\left[\frac{\sqrt{w}}{2\sqrt{u}}\frac{\partial\phi}{\partial y} + \frac{\sqrt{v}}{2\sqrt{u}}\cdot\frac{\partial\phi}{\partial z}\right]$$

$$u\frac{\partial\phi}{\partial u} = \frac{\sqrt{uw}}{2}\frac{\partial\phi}{\partial y} + \frac{\sqrt{uv}}{2}\frac{\partial\phi}{\partial z}$$

$$u\frac{\partial\phi}{\partial u} = \frac{y}{2}\frac{\partial\phi}{\partial y} + \frac{z}{2}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial v} = \frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= \frac{\partial\phi}{\partial x}\cdot\frac{\sqrt{w}}{2\sqrt{v}} + \frac{\partial\phi}{\partial y}\cdot(0) + \frac{\partial\phi}{\partial z}\cdot\frac{\sqrt{u}}{2\sqrt{v}}$$

$$\frac{\partial\phi}{\partial v} = \frac{\sqrt{w}}{2\sqrt{v}}\frac{\partial\phi}{\partial x} + \frac{\sqrt{u}}{2\sqrt{v}}\cdot\frac{\partial\phi}{\partial z}$$

$$v\frac{\partial\phi}{\partial v} = v\left[\frac{\sqrt{w}}{2\sqrt{v}}\frac{\partial\phi}{\partial x} + \frac{\sqrt{u}}{2\sqrt{v}}\cdot\frac{\partial\phi}{\partial z}\right]$$

$$= \frac{\sqrt{vw}}{2}\frac{\partial\phi}{\partial x} + \frac{\sqrt{vu}}{2}\frac{\partial\phi}{\partial z}$$

$$v\frac{\partial\phi}{\partial v} = \frac{x}{2}\frac{\partial\phi}{\partial x} + \frac{z}{2}\frac{\partial\phi}{\partial z}$$

Date
25.09.23 Homogeneous function: A function $f(x,y)$ is said to be homogenous in two variables x and y & degree n, if by putting $x = xt$, $y = yt$, the function $f(x,y)$ reduces to $t^n f(x,y)$

$$\text{i.e. } f(x,y) = f(xt, yt) = t^n f(x,y)$$

$$f(x,y) = f(xt, yt) = t^n f(x,y)$$

* Euler's theorem:-

① If z is a homogeneous function of two variables x and y of degree n, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Note: If u is a homogeneous function of three variables x, y, z of degree n.

$$\frac{x\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

② If z is a homogeneous function of two variables x and y of degree n, then

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

③ If u is not a homogeneous function of two variables of degree n and $z = f(u)$, then

ⓐ $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$

ⓑ $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)(g'(u)-1)$

where $g(u)=n\frac{f(u)}{f'(u)}$

Q1) If $u=\tan^{-1}\left(\frac{x^2+y^2}{x-y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-2\sin^3 u\cos u$$

sol: let $u(x,y)=\tan^{-1}\left(\frac{x^2+y^2}{x-y}\right)$

Put $x=xt$, $y=yt$

$$u(xt,yt)=\tan^{-1}\left[\frac{x^2t^2+y^2t^2}{xt-yt}\right]$$

$$=\tan^{-1}\left[\frac{t^2(x^2+y^2)}{t(x-y)}\right]$$

$$=\tan^{-1}\left[\frac{t(x^2+y^2)}{(x-y)}\right]$$

$\therefore$ $u(xt,yt)\neq t^n u(x,y)$

$\therefore$ u is not homogeneous function

$$\tan u=t\left(\frac{x^2+y^2}{x-y}\right)$$

$$z = f(u) = \tan u = t'\left(\frac{x^2+y^2}{x-y}\right)$$

$\therefore$ z is homogeneous function of degree 1 ie $n=1$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)(g'(u)-1)$$

where $g(u) = n\dfrac{f(u)}{f'(u)}$

$$g(u) = \frac{\tan u}{\sec^2 u} = \sin u \cos u$$

$$g'(u) = -\sin^2 u + \cos^2 u$$

$= \sin u \cos u (\cos^2 u - \sin^2 u - 1)$

$= \sin u \cos u (-\sin^2 u - \sin^2 u)$

$= \sin u \cos u (-2\sin^2 u)$

$= -2\sin^3 u \cos u$

Q4 If $u = \sin^{-1}\left\{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right\}^{1/2} = \sin^{-1}\left\{\sqrt{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}}\right\}$

prove that

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{12}\tan u$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{144}\left(\dfrac{\tan^2 u + 13}{}\right)$

Sol. Let $u(X,Y) = \sin^{-1}\left\{\dfrac{X^{1/3}+Y^{1/3}}{X^{1/2}-Y^{1/2}}\right\}^{1/2}$

Put $X = xt$, $Y = yt$

$$u(xt, yt) = \sin^{-1}\left\{ \frac{(xt)^{1/3} + (yt)^{1/3}}{(xt)^{1/2} - (yt)^{1/2}} \right\}^{1/2}$$

$\because$ u is not a homogeneous function

$$z = f(u) = \sin u = \left\{ t^{-1/6} \frac{x^{1/3} + y^{1/3}}{x^{1/2} - y^{1/2}} \right\}^{1/2}$$

$$z = \sin u = t^{-1/12} \left(\frac{x^{1/3} + y^{1/3}}{x^{1/2} - y^{1/2}} \right)^{1/2}$$

$\therefore$ z is homogeneous function of degree $-\frac{1}{12}$

i.e $n = -\frac{1}{12}$

(i) $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

$$= -\frac{1}{12} \frac{\sin u}{\cos u}$$

$$= -\frac{1}{12} \tan u$$

(ii) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\left(g'(u) - 1\right)$$

from (i)

$$= -\frac{1}{12} \tan u \left(-\frac{1}{12} \sec^2 u - 1 \right)$$

$$= -\frac{1}{12} \tan u \left(-\frac{1}{12} - \frac{\tan^2 u}{12} - 1 \right)$$

$$= \frac{-1}{12}\tan u\left(-\frac{13}{12} - \frac{\tan^2 u}{12}\right)$$

$$= \frac{1}{144}\tan u\left(13 + \tan^2 u\right)$$

(Q5) If $u = \sin^{-1}(xy)$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$

Tutorial - 02

① If $u = \log\left(\frac{x}{y}\right)$, then find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$

② If $z = f(x, y)$ and $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$

prove that

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

③ If $u = \log(x^2 + y^2)$, find $\frac{\partial^2 u}{\partial x \partial y}$

④ If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, then prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0.$

Ans:- $u = \log\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial x} = \frac{\partial \log(x/y)}{\partial (x/y)}.$$

$$= \log x - \log y$$

$$= \frac{\partial \log x}{\partial x} - \frac{\partial \log y}{\partial x}$$

$$= \frac{1}{x} - 0$$

$$\frac{\partial u}{\partial x} = \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial (\log x - \log y)}{\partial y}$$

$$= 0 - \frac{1}{y} = \frac{-1}{y}$$

Ans ② ~~If~~ $z = f(x, y)$

$x = u\cos\alpha - v\sin\alpha$

$y = u\sin\alpha + v\cos\alpha$

$\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \quad \text{—— (I)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \quad \text{—— (II)}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \cos\alpha + \frac{\partial z}{\partial y} \cdot \sin\alpha$$

$$\frac{\partial z}{\partial u} = \cos\alpha \frac{\partial z}{\partial x} + \sin\alpha \frac{\partial z}{\partial y} \quad \text{—— (3)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot (-\sin\alpha) + \frac{\partial z}{\partial y} \cdot \cos\alpha$$

$$\frac{\partial z}{\partial v} = \cos\alpha \frac{\partial z}{\partial y} - \sin\alpha \frac{\partial z}{\partial x} \quad \text{—— (4)}$$

Squaring ③

$$\left(\frac{\partial z}{\partial u}\right)^2 = \cos^2\alpha \left(\frac{\partial z}{\partial x}\right)^2 + \sin^2\alpha \left(\frac{\partial z}{\partial y}\right)^2 + 2\cos\alpha\sin\alpha \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}$$

Ans ③

Squaring (4)

$$\left(\frac{\partial z}{\partial v}\right)^2 = \cos^2\alpha\left(\frac{\partial z}{\partial y}\right)^2 + \sin^2\alpha\left(\frac{\partial z}{\partial x}\right)^2 - 2\cos\alpha\sin\alpha\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}$$

adding (3) & (4)

$$\left(\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = \cos^2\alpha\left(\frac{\partial z}{\partial y}\right)^2 + \sin^2\alpha\left(\frac{\partial z}{\partial x}\right)^2$$

$$-\cancel{2\cos\alpha\sin\alpha\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial x}} + \cos^2\alpha\left(\frac{\partial z}{\partial x}\right)^2$$

$$+\sin^2\alpha\left(\frac{\partial z}{\partial y}\right)^2 + \cancel{2\cos\alpha\sin\alpha\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}}$$

$$= \cos^2\alpha\left[\left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2\right] + \sin^2\alpha\left[\left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2\right]$$

$$= \left[\left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2\right]\left[\cos^2\alpha + \sin^2\alpha\right]$$

$$= \left(\frac{\partial z}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 = \left(\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

Ans (3) $u = \log(x^2+y^2)$, find $\frac{\partial^2 u}{\partial x \partial y}$

$$\frac{\partial u}{\partial y} = \frac{\partial \log(x^2+y^2)}{\partial(x^2+y^2)} \cdot \frac{\partial(x^2+y^2)}{\partial y}$$

$$= \frac{1}{x^2+y^2} \cdot (0+2y)$$

$$= \frac{2y}{x^2+y^2}$$

④ If
$u = f(x^2-y^2, y^2-z^2, z^2-x^2)$, then

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

$u = f(t, v, w)$

$u = x^2 - y^2$
$v = y^2 - z^2$
$w = z^2 - x^2$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\cdot\frac{\partial t}{\partial x} + \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\cdot\frac{\partial w}{\partial x}$$
$$= \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x}$$

Q1. If $u = x^3\sin^{-1}\left(\frac{y}{x}\right) + x^4\tan^{-1}\left(\frac{y}{x}\right)$, find value of

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

at $x=1, y=1$

1: Given
$$u = x^3\sin^{-1}\left(\frac{y}{x}\right) + x^4\tan^{-1}\left(\frac{y}{x}\right)$$

Put $v = x^3\sin^{-1}\left(\frac{y}{x}\right)$ and $w = x^4\tan^{-1}\left(\frac{y}{x}\right)$

$u = v + w$
Now $v(X, Y) = X^3\sin^{-1}\left(\frac{Y}{X}\right)$

Put $X = xt$, $Y = yt$

$$v(xt, yt) = (xt)^3 \sin^{-1}\left(\frac{yt}{xt}\right)$$

$$= t^3 x^3 \sin^{-1}\left(\frac{y}{x}\right)$$

$$v(xt, yt) = t^3 v(x, y)$$

$\because$ v is homogeneous function of degree 3

By Euler's theorem.

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n_1 v$$

$$= 3v \quad \text{——— (2)}$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n_1(n_1 - 1)$$

$$w(X, Y) = X^4 \tan^{-1}\left(\frac{y}{x}\right)$$

Put $X = xt$, $Y = yt$

$$w(xt, yt) = (xt)^4 \tan^{-1}\left(\frac{yt}{xt}\right)$$

$$= t^4 . x^4 \tan^{-1}\left(\frac{y}{x}\right)$$

$\therefore$ W is homogeneous of degree $n_2 = 4$

$$LHS = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x^2\frac{\partial^2 (v+w)}{\partial x^2} + 2xy\frac{\partial^2 (v+w)}{\partial x \partial y} + y^2\frac{\partial^2 (v+w)}{\partial y^2} + x\frac{\partial (v+w)}{\partial x} + y\frac{\partial (v+w)}{\partial y}$$

$$= \left(x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2}\right) + \left(x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial w}{\partial y^2}\right) + \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) + \left(x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}\right)$$

= from (2), (3), (4) and (5), we get

$= 6v + 12w + 3v + 4w$
$= 9v + 16w$

$= 9x^3 \sin^{-1}\left(\frac{y}{x}\right) + 16x^4 \tan^{-1}\left(\frac{y}{x}\right) = 9\left(\frac{\pi}{2}\right) + 16\left(\frac{\pi}{4}\right)$

Put $x=1, y=1$

$= \frac{9\pi}{2} + 4\pi = \frac{17\pi}{2}$

(1) If $z = x^n f\left(\frac{y}{x}\right) + y^{-n} f\left(\frac{x}{y}\right)$ to prove that
$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$$

Sol. Put $v = x^n f(y/x)$
$w = y^{-n} f(x/y)$

Put $X = xt \quad Y = yt$

$$V = x^n f(y/x)$$
$$V = (xt)^n f(yt/xt)$$
$$V = x^n t^n f(y/x)$$

V is a homogeneous function with degree n.

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n_1 V$$
$$= nV \quad —① $$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial v}{\partial y^2} = n_1(n_1 - 1)$$
$$= n(n-1)V — ②$$

$$W = Y^{-n} f(X/Y)$$
$$= (yt)^{-n} f(xt/yt)$$
$$= (yt)^{-n} f(x/y)$$
$$= y^{-n} t^{-n} f(x/y)$$

w is a homogeneous function with degree $-n$

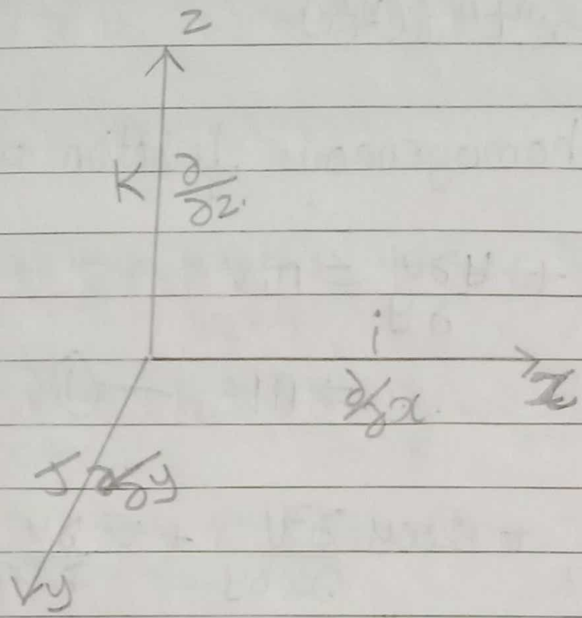
$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = -nw —③$$

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial w}{\partial y^2} = n_2(n_2 - 1)$$
$$= -n(-n-1)$$
$$= (n^2 - n)w — ④$$

$$LHS = x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} +$$
$$y^2\frac{\partial z}{\partial y}.$$

$$= x^2 \frac{\partial^2(v+w)}{\partial x^2} + 2xy \frac{\partial(v+w)}{\partial x \partial y} + y^2 \frac{\partial^2(v+w)}{\partial y^2}$$

$$+ x \frac{\partial(v+w)}{\partial x} + y \frac{\partial(v+w)}{\partial y}$$



Scalar function

1. $\phi = xyz$
2. $\phi = \log(x^2+y^2+z^2)$
3. $\phi = x^4+y^4+z^4$

Vector function

$\bar{F} = F_1 i + F_2 j + F_3 k$

(a) $\bar{F} = xi + yj + zk$

(b) $\bar{r} = xi + yj + zk$

gradient ϕ

1. $\nabla \phi = \text{grad } \phi$
2. $\nabla \cdot \bar{F} = \text{divergence } \bar{F}$
3. $\nabla \times \bar{F} = \text{curl } \bar{F}$

te

1.23 Gradient

① Vector differential operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

② $\phi(x, y, z)$ is a sacalar function

$\nabla\phi$ is a vector quantity which represents an outgoing normal vector to the surface $\phi(x,y,z)$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = c$$

$$= \text{grad } \phi$$

③ Unit Vector $= \frac{\nabla\phi}{|\nabla\phi|}$

④ Maximum directional derivative $= |\nabla\phi|$
magnitude of $|\nabla\phi|$

Ex-18

① if $\phi = 3x^2y - y^3z^2$, find $\nabla\phi$ at point $P(1, -2, -1)$

1. Let $\phi = 3x^2y - y^3z^2$

$$\frac{\partial\phi}{\partial x} = 6xy - 0$$

$$\frac{\partial\phi}{\partial x} = 6xy \quad \text{—①}$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - 3y^2z^2 \quad \text{—②}$$

$$\frac{\partial\phi}{\partial z} = 0 - 2y^3z$$

$$\frac{\partial\phi}{\partial z} = -2y^3z \quad \text{—③}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = 6xy\hat{i} + 3x^2 - 3y^2z^2\hat{j} + \hat{k}\cdot -2y^3z$$

$$\nabla\phi = 6xy\hat{i} + 3(x^2 - y^2z^2)\hat{j} + (-2y^3z)\hat{k}$$

at Point $P(1,-2,-1)$

$$\nabla\phi = -12i + 3(1-4)\hat{j} + (-2\cdot -8\cdot -1)\hat{k}$$

$$\nabla\phi = -12\hat{i} - 9\hat{j} - 16\hat{k}$$

② Find a unit vector normal to surface $xy^3z^2 = 4$ at $(-1,-1,2)$

Sol: ~~Let $\phi = xy^3z^2 = 4$~~

Let $\phi = xy^3z^2 - 4$

$$\frac{\partial\phi}{\partial x} = y^3z^2 \quad — ①$$

$$\frac{\partial\phi}{\partial y} = 3xy^2z^2 \quad — ②$$

$$\frac{\partial\phi}{\partial z} = 2xy^3z \quad — ③$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = (y^3z^2)i + (3xy^2z^2)\hat{j} + (2xy^3z)\hat{k}$$

at point $(-1,-1,2)$

$$\nabla\phi = (-1)^3(4)i + (3(-1)\cdot 4)j + (-2\cdot -1\cdot 2)\hat{k}$$

$$= -4\hat{i} - 12j + 4k$$

$$\nabla\phi = 4(-i - 3j + k)$$

$$\vec{a} = \hat{a}|a|$$
$$\hat{a} = \frac{\vec{a}}{|a|}$$

$$|\nabla\phi| = 4\sqrt{(-1)^2+(-3)^2+(1)^2}$$
$$= 4\sqrt{1+9+1}$$
$$= 4\sqrt{11}$$
$$|\nabla\phi| = 4\sqrt{11}$$

$$\text{unit vector} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{\cancel{4}(-i-3j+k)}{\cancel{4}\sqrt{11}}$$

$$= \frac{-i-3j+k}{\sqrt{11}}$$ let this be $\vec{b}$

Let $\vec{c}$ be a unit vector normal to surface $xy^3z^2=4$ at $(-1,-1,2)$.

$$\vec{c} = ai+bj+ck$$

$$\vec{b}\cdot\vec{c} = 0$$

$$\left[\left(-\frac{1}{\sqrt{11}}i\right)+\left(-\frac{3}{\sqrt{11}}\right)j+\left(\frac{1}{\sqrt{11}}\hat{k}\right)\right]\cdot(ai+bj+ck)=0$$

$$\left[\frac{-a}{\sqrt{11}}+\left(-\frac{b3}{\sqrt{11}}\right)+\frac{c}{\sqrt{11}}\right]=0$$

$$-a-3b+c=0$$
$$\boxed{a+3b-c=0 \quad \text{— (1)}}$$

$$|\vec{c}|=1$$
$$\sqrt{a^2+b^2+c^2}=1$$
$$a^2+b^2+c^2=1$$

③ Show that $\text{grad}\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$

Sol: $\bar{r} = xi + yj + zk$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \ \frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}$

Now, $\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right)$

$\nabla(r^{-1}) = \left(i\frac{\partial r^{-1}}{\partial x} + j\frac{\partial r^{-1}}{\partial y} + k\frac{\partial r^{-1}}{\partial z}\right)$

$= i\left(-1r^{-2}\frac{\partial r}{\partial x}\right) + j\left(-1r^{-2}\frac{\partial r}{\partial y}\right) + k\left(-1r^{-2}\frac{\partial r}{\partial z}\right)$

$= \frac{-i}{r^2}\cdot\frac{x}{r} - \frac{j}{r^2}\cdot\frac{y}{r} - \frac{k}{r^2}\cdot\frac{z}{r} = \frac{-1}{r^3}(xi+yj+zk)$

$= \frac{-\bar{r}}{r^3}$

$\vec{a} = a_1 i + a_2 j + a_3 k$
$\vec{b} = b_1 i + b_2 j + b_3 k$

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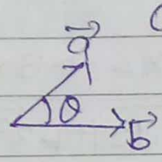
* Directional derivative:

① The directional derivative of ϕ in direction of vector $\bar{a}$ actually vector $\vec{a}$

$$= \frac{\nabla \phi \cdot \bar{a}}{|\bar{a}|}$$



$\vec{a} \cos\theta \, \vec{b} = \vec{a} \cos\theta \frac{\vec{b}}{|b|}$

$= \frac{\vec{a}\vec{b} \cos\phi}{|b|}$

$\vec{a} \cdot \vec{b} = ab \cos\theta$

$a \cos\theta = \frac{a \cdot b}{|b|}$

② The directional derivative of ϕ normal to surface Ψ at (x_1, y_1, z_1)

$$\partial d = \frac{\nabla \phi \cdot \nabla \psi}{|\nabla \psi|}$$

Ex-19

① Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at point $(2, -1, 2)$ in direction of $2i + 3j + 6k$

Sol: Let $\phi = 4xz^3 - 3x^2y^2z$

$$\frac{\partial \phi}{\partial x} = 4z^3 - 6xy^2z \quad \text{—— ①}$$

$$\frac{\partial \phi}{\partial y} = 0 - 3x^2 2yz$$

$$\frac{\partial \phi}{\partial y} = -6x^2yz \quad \text{—— ②}$$

$$\frac{\partial \phi}{\partial z} = 12xz^2 - 3x^2y^2 \quad \text{—— ③}$$

at $(2, -1, 2)$

$$\frac{\partial \phi}{\partial x} = 32 - 24 = 8, \quad \frac{\partial \phi}{\partial y} = 48, \quad \frac{\partial \phi}{\partial z} = 96 - 12 = 84$$

$$\nabla\phi = 8i + 48j + 84k$$

$$\vec{a} = 2i + 3j + 6k$$

$$\text{directional derivative} = \frac{\nabla\phi \cdot \bar{a}}{|a|}$$

$$= \frac{(8\times2 + 3\times48 + 84\times6)}{\sqrt{(2)^2+3^2+6^2}}$$

$$= \frac{16 + 144 + 504}{\sqrt{4+9+36}}$$

$$= \frac{664}{\sqrt{49}}$$

$$= \frac{664}{7}$$

$$= 94.857$$

③ What is the directional derivative of $\phi = xy^2 + yz^3$ at point $(2,-1,1)$ in the direction normal to surface $x\log z - y^2 = -4$ at $(-1,2,1)$

Sol. Solve,

Given, $\phi = xy^2 + yz^3$

$$\frac{\partial\phi}{\partial x} = y^2 + 0$$

$$\frac{\partial\phi}{\partial x} = y^2 \quad \text{--- ①}$$

$$\frac{\partial\phi}{\partial y} = 2xy + z^3 \quad \text{--- ②}$$

$$\frac{\partial\phi}{\partial z} = 3z^2y \quad \text{--- ③}$$

$$\Delta\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\Delta\phi = y^2 i + (2xy + z^3)\hat{j} + 3z^2 y\hat{k}$$

at $(2,-1,1)$

$$\Delta\phi = i + (-3)\hat{j} + (-3)\hat{k}$$

$$\boxed{\nabla\Delta\phi = \hat{i} - 3\hat{j} - 3\hat{k}} \quad — (4)$$

Let $\psi = x\log z - y^2 + 4$

$$\frac{\partial\psi}{\partial x} = \log z \quad — (5)$$

$$\frac{\partial\psi}{\partial y} = -2y \quad — (6)$$

$$\frac{\partial\psi}{\partial z} = \frac{x}{z} \quad — (7)$$

$$\nabla\psi = i\frac{\partial\psi}{\partial x} + j\frac{\partial\psi}{\partial y} + k\frac{\partial\psi}{\partial z}$$

$$\nabla\psi = \log z\, i - 2y\hat{j} + \frac{x}{z}\hat{k}$$

at $(-1, 2, 1)$

$$\nabla\psi = 0 - 4j + (-1k)$$

$$\boxed{\nabla\psi = -4\hat{j} - \hat{k}} \quad — (8)$$

directional derivative of $\nabla\phi = \frac{\nabla\phi \cdot \nabla\psi}{|\nabla\psi|}$

$$= \frac{12 + (3)}{\sqrt{16+1}}$$

$$= \frac{15}{\sqrt{17}}$$

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in its own direction.

(2) (a) In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum?

(b) What is the magnitude of maximum directional derivative?

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(1)

(a)

(b)

* Divergence

$F = x\hat{i} + y\hat{j} + z\hat{k}$ or $a\hat{i} + b\hat{j} + c\hat{k}$ or $F = \nabla\phi$

then Divergence F = div F = $\nabla \cdot F = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot$

$(a\hat{i} + b\hat{j} + c\hat{k})$

$$\nabla \cdot f = \left[\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right]$$

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Module 3 Complex Number

① Power of complex Number:

Given $z = a+ib$ (cartesian form)
where a and b are real number and $i=\sqrt{-1}$

ⓐ Magnitude of $z = |z|$

$$r = \sqrt{a^2+b^2}$$

ⓑ Argument of $z = \tan^{-1}\left(\frac{b}{a}\right)$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

ⓒ Polar form:

$$z = r(\cos\theta + i\sin\theta)$$

where $r = \sqrt{a^2+b^2}$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

ⓓ Exponential form

$z = re^{i\theta}$ where $e^{i\theta} = \cos\theta + i\sin\theta$

ⓔ De-Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

ⓕ Power of z

$$z^n = [r(\cos\theta + i\sin\theta)]^n$$

$= r^n(\cos n\theta + i\sin n\theta)$

or

$z = re^{i\theta}$

$z^n = (re^{i\theta})^n = r^n e^{in\theta}$

Find the value of $(\sqrt{3}-i)^4$ by using De-Moivre's theorem

Let $z = \sqrt{3} - i$

$= \sqrt{3} + (-1)i \quad (a+ib)$

$a = \sqrt{3} \qquad b = -1$

$|z| = \sqrt{(\sqrt{3})^2 + (-1)^2}$

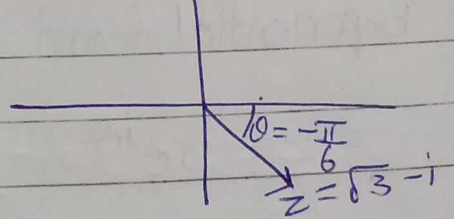
$= \sqrt{3+1}$

$= \sqrt{4}$

$\boxed{\begin{array}{l} |z| = 2 \\ r = 2 \end{array}}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$



$\theta = \frac{11\pi}{6}$ or $-\frac{\pi}{6}$ → clockwise

↳ anti-clockwise

In Polar form,

$z = r(\cos\theta + i\sin\theta)$

$z = 2\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right)$

$z^4 = \left[2\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right]^4$

$$z^4 = 16\left[\cos\left(4\times\frac{11\pi}{6}\right) + i\sin\left(4\times\frac{11\pi}{6}\right)\right]$$

$$z^4 = 16\left[\cos\left(\frac{22\pi}{3}\right) + i\sin\left(\frac{22\pi}{3}\right)\right]$$

ram.

② Prove that by using De-Moivre's theorem $(4n)^{th}$ power of $\frac{1+7i}{(2-i)^2}$ is equal to $(-4)^n$ where n is a positive integer.

Sol. Let

$$z = \frac{1+7i}{(2-i)^2}$$

$$= \frac{1+7i}{4-4i+i^2}$$

$$= \frac{1+7i}{4-4i-1}$$

$$= \frac{1+7i}{3-4i} \times \frac{(3+4i)}{(3+4i)}$$

$$= \frac{3+21i+4i+(+28i^2)}{9-(16(-1))}$$

$$= \frac{3+25i+(28(-1))}{9+16}$$

$$= \frac{25i-25}{25}$$

$$= i-1$$

$$= -1+i$$

$$z = -1+i$$

$$a = -1 \qquad b = 1$$

$$r = \sqrt{a^2+b^2} = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$$

$$\boxed{r = \sqrt{2}}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1)$$

$\cos\pi = -1$

$\cos n\pi = (-1)^n$

$\theta = \frac{3\pi}{4}$ or $\frac{-\pi}{4}$

In polar form

$z = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4}))$

$z^{4n} = \left[\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^{4n}$

$z^{4n} = (4)^n\left[\cos(3n\pi) + i\sin(3n\pi)\right]$

$= 4^n\left[\cos(2n\pi + n\pi) + i\sin(2n\pi + n\pi)\right]$

$= 4^n[\cos(n\pi) + i\sin(n\pi)]$

$= 4^n[(-1)^n + i(0)]$

$= 4^n(-1)^n$

$\boxed{z^{4n} = (-4)^n}$

If α & β are the roots of eq^n $z^2\sin^2\theta - z\sin 2\theta + 1 = 0$ prove that prove that $\alpha^n + \beta^n = 2\cos n\theta \operatorname{cosec}^n\theta$. ($n$ is +ive) by using De-moivre's theorem

The given eq^n is

$z^2\sin^2\theta - z\sin 2\theta + 1 = 0$

$z^2\sin^2\theta - 2\sin\theta\cos\theta\, z + 1 = 0$

$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{2\sin\theta\cos\theta \pm \sqrt{4\sin^2\theta\cos^2\theta - 4\sin^2\theta}}{2\sin^2\theta}$

$$= \frac{2\sin\theta\cos\theta \pm 2\sin\theta\sqrt{\cos^2\theta - 1}}{2\sin^2\theta}$$

$$z = \frac{\cos\theta \pm \sqrt{-\sin^2\theta}}{\sin\theta}$$

$$z = \frac{\cos\theta \pm i\sin\theta}{\sin\theta}$$

$$\alpha = \frac{\cos\theta + i\sin\theta}{\sin\theta}$$

$$\beta = \frac{\cos\theta - i\sin\theta}{\sin\theta}$$

$$\alpha^n + \beta^n = \left(\frac{\cos\theta + i\sin\theta}{\sin\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\sin\theta}\right)^n$$

$$= \left(\frac{\cos n\theta + i\sin n\theta}{\sin^n\theta}\right) + \left(\frac{\cos n\theta - i\sin n\theta}{\sin^n\theta}\right)$$

* $a^3 - b^3 =$
* $a^3 + b^3 =$

Q. Calculate the roots of following equations

(i) $x^4 - x^3 + x^2 - x + 1 = 0$

$$x^2(x^2 - x) + (x^2 - x) + 1 = 0$$

$$(x^2 - x)(x^2 + 1) + 1 = 0$$

$$(x^2 - x)(x^2 + 1) = -1$$

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(3) If $\tan\left(\frac{x}{2}\right) = \tanh\left(\frac{u}{2}\right)$ prove that,

(1) $\sinh u = \tan x$, (2) $\cosh u = \sec x$

(3) $u = \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$

$\sinh 2x = \frac{2\tanh x}{1-\tanh^2 x}$

$\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$

Sol:

Given,

$$\tanh\left(\frac{u}{2}\right) = \tan\left(\frac{x}{2}\right)$$

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \tan\left(\frac{x}{2}\right)$$

multiply LHS by $e^{u/2}$

$$\frac{e^u - 1}{e^u + 1} = \frac{\tan\left(\frac{x}{2}\right)}{1}$$

$$e^u - 1 = \tan\left(\frac{x}{2}\right)(e^u + 1)$$

$$e^u - \tan\left(\frac{x}{2}\right)e^u = 1 + \tan\left(\frac{x}{2}\right)$$

$$e^u\left(1 - \tan\left(\frac{x}{2}\right)\right) = 1 + \tan\left(\frac{x}{2}\right)$$

$$e^u = \frac{1+\tan(x/2)}{1-\tan(x/2)}, \quad e^{-u} = \frac{1-\tan(x/2)}{1+\tan(x/2)}$$

(1) $\cosh u = \frac{e^u + e^{-u}}{2}$

$$= \frac{1}{2}\left[\frac{1+\tan x/2}{1-\tan x/2} + \frac{1-\tan x/2}{1+\tan x/2}\right]$$

$$= \frac{1}{2}\left[\frac{(1+\tan x/2)^2 + (1-\tan x/2)^2}{(1-\tan^2 x/2)}\right]$$

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$$= \frac{1}{2}\left[\frac{1+\tan^2 x/2 + \cancel{2\tan x/2} + \tan^2 x/2 - \cancel{2\tan x/2} + 1}{(1-\tan^2 x/2)}\right]$$

$$= \frac{1+\tan^2 x/2}{1-\tan^2 x/2} = \frac{1}{\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \frac{1}{\cos x} = \sec x$$

(2) $\sinh u = \frac{e^u - e^{-u}}{2}$

$$= \frac{1}{2}\left[\frac{1+\tan x/2}{1-\tan x/2} - \frac{1-\tan x/2}{1+\tan x/2}\right]$$

$$= \frac{1}{2}\left[\frac{(1+\tan x/2)^2 - (1-\tan x/2)^2}{1-\tan^2 x/2}\right]$$

$$= \frac{1}{2}\left[\frac{4\tan x/2}{1-\tan^2 x/2}\right]$$

$$= \frac{2\tan x/2}{1-\tan^2 x/2}$$

$$= \tan x$$

$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$

$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

(3) $e^u = \frac{1+\tan x/2}{1-\tan x/2}$

$$u = \log\left[\frac{1+\tan x/2}{1-\tan x/2}\right]$$

$$u = \log\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$$

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* Inverse of hyperbolic function

$\sinh x = y$

$x = \sinh^{-1}(y)$

① $\sinh^{-1} x = \log\left[x + \sqrt{x^2+1}\right]$ where x is real or complex.

Q. Find derivative of $y = \sinh^{-1} x$

convert $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$

$$y' = \frac{1}{x+\sqrt{x^2+1}}\left[1 + \frac{2x}{2\sqrt{x^2+1}}\right]$$

$$y' = \frac{1}{\sqrt{x^2+1}}$$

② $\cosh^{-1} x = \log\left[x + \sqrt{x^2-1}\right]$

③ $\tanh^{-1} x = \frac{1}{2}\log\left[\frac{1+x}{1-x}\right]$

If $\tanh x = \frac{1}{2}$

$x = \tanh^{-1}\left(\frac{1}{2}\right)$

$x = \frac{1}{2}\log\left[\frac{1+1/2}{1-1/2}\right]$

$$\boxed{x = \frac{1}{2}\log 3}$$

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Ex 2.6

① Prove that $\tanh^{-1}x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

Sol.

let $\tanh^{-1}x = y$

$$x = \tanh y$$

$$RHS. = \frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{1-\tanh^2 y}}$$

$$= \frac{\tanh y}{\sqrt{\operatorname{sech}^2 y}}$$

$$= \frac{\tanh y}{\operatorname{sech} y}$$

$$= \frac{\sinh y}{\cancel{\cosh y}} \cdot \cancel{\cosh y}$$

$$\frac{x}{\sqrt{1-x^2}} = \sinh y$$

$$y = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\tanh^{-1}x = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

② Prove that $\operatorname{sech}^{-1}(\sin\theta) = \log\left[\cot\theta/2\right]$

Sol. let $\operatorname{sech}^{-1}(\sin\theta) = y$

$$\sin\theta = \operatorname{sech} y$$

$$\sin\theta = \frac{1}{\cosh y}$$

$1+\cos x = 2\cos^2 \theta/2$

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$\therefore \cosh y = \frac{1}{\sin\theta} = \operatorname{cosec}\theta$

$y = \cosh^{-1}(\operatorname{cosec}\theta)$

$= \log[\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}]$

$= \log[\operatorname{cosec}\theta + \cot\theta]$

$= \log\left[\frac{1+\cos\theta}{\sin\theta}\right] = \log\left[\frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2}\right]$

$= \log[\cot\theta/2]$

① If $\cosh x = \sec\theta$, prove that ① $x = \log(\sec\theta + \tan\theta)$

Sol. Given,

$\cosh x = \sec\theta$

$x = \cosh^{-1}(\sec\theta)$

$= \log[\sec\theta + \sqrt{\sec^2\theta - 1}]$

$\boxed{x = \log[\sec\theta + \tan\theta]}$

$e^x = \sec\theta + \tan\theta$

$e^{-x} = \frac{1}{\sec\theta + \tan\theta} \times \left(\frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta}\right)$

$e^{-x} = \sec\theta - \tan\theta$

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(iii) $\sinh x = \tan\theta$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\sinh 2x = \frac{2\tanh x}{1-\tanh^2 x}$

$\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$

(iv) $\tanh x = \sin\theta$

$\cosh^2 x - \sinh^2 x = 1$

$1 - \coth^2 x = -\operatorname{cosech}^2 x$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

Q.2. $\dfrac{1}{1 - \dfrac{1}{1 - \dfrac{1}{-\sinh^2 x}}} = \dfrac{1}{1 - \dfrac{1}{1+\operatorname{cosech}^2 x}}$

$= \dfrac{1}{1 - \dfrac{1}{\coth^2 x}} = \dfrac{1}{1-\tanh^2 x} = \dfrac{1}{\operatorname{sech}^2 x}$

$= \cosh^2 x$

Q3. $\tanh x = \frac{1}{2}$ $\qquad \sinh 2x$

$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{2}$

$2e^x - 2e^{-x} = e^x + e^{-x}$

$e^x = 3e^{-x}$

$e^x = \frac{3}{e^x}$

$\boxed{e^{2x} = 3}$

$e^{-2x} = \frac{1}{3}$

$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$

$= \frac{3 - 1/3}{2}$

$= \frac{8}{3} \times \frac{1}{2} = \frac{4}{3}$

$\cosh 2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{3 + 1/3}{2}$

$= \frac{10}{3} \times \frac{1}{2} = \frac{5}{3}$

Q1. $6\sinh x + 2\cosh x + 7 = 0$

$\rightarrow$

$\overset{3}{\cancel{6}}\left(\frac{e^{x} - e^{-x}}{\cancel{2}_1}\right) + \overset{1}{\cancel{2}}\left(\frac{e^{x} + e^{-x}}{\cancel{2}_1}\right) + 7 = 0$

$3e^{x} - 3e^{-x} + e^{x} + e^{-x} + 7 = 0$

$4e^{x} - 2e^{-x} + 7 = 0$

$4e^{2x} + 7e^{x} - 2 = 0$ $\quad e^{x} = t$

$4t^2 + 7t - 2 = 0$ $\quad 8$

$4t^2 + 8t - t - 2 = 0$

$4t(t+2) - 1(t+2) = 0$

$(4t-1)(t+2) = 0$

$t = \frac{1}{4}$, $t = -2$

$e^{x} = \frac{1}{4}$, $e^{x} = -2$

$x = \log(1/4)$ $\quad x = \log(-2) \times$

$\therefore x = \log(1/4)$ or $-2\log(2)$

Math ISE-I

$x^2 + x^{-2} = i$

$x^2 + \frac{1}{x^2} = i$

$(x^2)^2 + 1 = ix^2$

$(x^2)^2 - ix^2 + 1 = 0$

$x^2 = y$

$y^2 - iy + 1 = 0$

$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y = \frac{i \pm i\sqrt{5}}{2}$

$x^2 = \left(\frac{1 \pm \sqrt{5}}{2}\right) i$

* Hyperbolic fuction:- (hyperbola se derived hai) (obtained from hyperbola).

sinh x, cosh x, tanh x, cosech x, coth x, sech x

① $\sinh x = \frac{e^x - e^{-x}}{2}$

② $\cosh x = \frac{e^x + e^{-x}}{2}$

③ $\tanh x = \frac{\sinh x}{\cosh x}$

$= \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$e^{ix} = \cos x + i\sin x$
$e^{-ix} = \cos x - i\sin x$

subtracting,

$$e^{ix} - e^{-ix} = 2i\sin x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

adding, $e^{ix} + e^{-ix} = 2\cos x$

$$\boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}}$$

② Relation between circular and hyperbolic function

① $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

Put x by ix

$$\sin(ix) = \frac{e^{-x} - e^{x}}{2i}$$

$$= \frac{-1}{i}\left[\frac{e^{x} - e^{-x}}{2}\right]$$

$$= \frac{i^2}{i} \sinh x$$

$$\sin(ix) = i\sinh x$$

② $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

Replace x by ix

$$\cos(ix) = \frac{e^{-x} + e^{x}}{2}$$

$$\cos(ix) = \cosh x$$

(3) $\tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i\sinh x}{\cosh x} = i\tanh x$

$\tan(ix) = i\tanh x$

* Identities of hyperbolic function

(1) $\cosh^2 x - \sinh^2 x = 1$

$$\cosh^2 x = (\cosh x)^2$$

(2) $1 - \coth^2 x = -\mathrm{cosech}^2 x$

(3) $1 - \tanh^2 x = \mathrm{sech}^2 x$

(4) (a) $\sinh 2x = 2\sinh x \cosh x$

(b) $\cosh 2x = \cosh^2 x + \sinh^2 x$

$= 1 + 2\sinh^2 x$ or $2\sinh^2 x + 1$

$= 2\cosh^2 x - 1$

(c) $\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$

(5) $\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$

(6) $\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$

Q1. If $\tanh x = 2/3$, find value of x and $\cosh 2x$.

Sol. Given,

$$\tanh x = 2/3$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{2}{3}$$

i.e. $\frac{e^x - 1/e^x}{e^x + 1/e^x} = \frac{2}{3}$

i.e. $3(e^{2x}-1) = 2(e^{2x}+1)$

$3e^{2x} - 3 = 2e^{2x} + 2$

$e^{2x} = 2+3$

$e^{2x} = 5$

$2x = \log 5$

$x = \frac{1}{2}\log 5$

$x = \log \sqrt{5}$

$\cosh 2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{5 + 1/5}{2}$

2. Prove that $(\cosh x - \sinh x)^n = \cosh nx - \sinh nx$

Sol. LHS.

$= (\cosh x - \sinh x)^n$

$= \left[\left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right)\right]^n$

$= \left[\cancel{\frac{e^{x}}{2}} + \frac{e^{-x}}{2} - \cancel{\frac{e^x}{2}} + \frac{e^{-x}}{2}\right]$

$= \left[\frac{e^{-x}}{2} + \frac{e^{-x}}{2}\right]^n$

$= [e^{-x}]^n$

$= e^{-nx}$

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RHS. $= \cosh nx - \sinh nx$ $\qquad \cosh ax = \frac{e^{ax} + e^{-ax}}{2}$

$= \left(\frac{e^{nx} + e^{-nx}}{2}\right) - \left(\frac{e^{nx} - e^{-nx}}{2}\right)$

$= \cancel{\frac{e^{nx}}{2}} + \frac{e^{-nx}}{2} - \cancel{\frac{e^{nx}}{2}} + \frac{e^{-nx}}{2}$

$= \frac{2e^{-nx}}{2}$

$= e^{-nx}$

$\therefore (\cosh x - \sinh x)^n = \cosh nx - \sinh nx.$

② Solve eq. $5\sinh x - \cosh x = 5$, & find $\tanh x$
or
if $5\sinh x - \cosh x = 5$, find $\tanh x$.

sol: given eq is

$5\sinh x - \cosh x = 5$

$5\left(\frac{e^x - e^{-x}}{2}\right) - \left(\frac{e^x + e^{-x}}{2}\right) = 5$

$5e^x - 5e^{-x} - e^x - e^{-x} = 10$

$4e^x - 6e^{-x} = 10$

$4e^{2x} - 6 = 10e^x$

$2e^{2x} - 5e^x - 3 = 0$

$2e^{2x} - 6e^x + e^x - 3 = 0$

$$2e^x(e^x-3)+1(e^x-3)=0$$

$$(e^x-3)(2e^x+1)=0$$

$e^x=3$ $\quad\quad$ $2e^x+1=0$

$\boxed{x=\log 3}$ $\quad\quad$ $x=\log(-1/2)$

when $e^x=3$

$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{3-1/3}{3+1/3}=\frac{8}{10}=\frac{4}{5}$$

Q4) If $\tanh x=\frac{1}{2}$ $\quad\quad$ $p+\cosh 2x=5/3$

$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{1}{2}$$

$$\Rightarrow 2(e^x-e^{-x})=e^x+e^{-x}$$

$$2e^x-2e^{-x}=e^x+e^{-x}$$

$$e^x=3e^{-x}$$

$$\boxed{e^{2x}=3}$$

$$\cosh 2x=\frac{e^{2x}+e^{-2x}}{2}=\frac{3+1/3}{2}=\frac{10/3}{2}$$

$$=5/3$$

$\boxed{p=0}$

1. $\cosh x+\sinh x$

$\cosh x=\frac{e^x+e^{-x}}{2}$, $\sinh x=\frac{e^x-e^{-x}}{2}$

$$=e^x$$

$\sin(ix)=i\sinh x$ $\quad\quad$ $\cos(ix)=\cosh x$

2. $$\frac{d(\cosh 3t)}{dt}=\frac{d\cos(i3t)}{dt}=\frac{d\cos(i3t)}{d(i3t)}\cdot\frac{d(i3t)}{dt}$$

$$=-\sin(i3t)\cdot 3i=-3i\sin(i3t)$$

$$=-3i\,i\sinh 3t=3\sinh 3t$$

Ex 24, Q 5.

3. $\tanh(\log\sqrt{5})$

$= \dfrac{e^{\log\sqrt{5}} - e^{-\log\sqrt{5}}}{e^{\log\sqrt{5}} + e^{-\log\sqrt{5}}} = \dfrac{\sqrt{5} - 1/\sqrt{5}}{\sqrt{5} + 1/\sqrt{5}}$

$= \dfrac{5-1}{\cancel{\sqrt{5}}} \times \dfrac{\cancel{\sqrt{5}}}{5+1} = \dfrac{4}{6} = \dfrac{2}{3}$

4. $2\sinh\left[\dfrac{A+B}{2}\right]\sinh\left[\dfrac{A-B}{2}\right]$

$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$

$\cos(A-B) - \cos(A+B) = 2\sin A\sin B$

$\cos(B) - \cos(A)$

$\cos(hB) - \cosh A$

$\cosh A - \cosh B$

$\cos(ix) = \cosh x$

5. $\cosh^5 x = \left[\dfrac{e^x + e^{-x}}{2}\right]^5$

sol.

$= \dfrac{1}{16\times 2}\left[e^x + e^{-x}\right]^5$

$= \dfrac{1}{32}\left[\dfrac{e^{2x}+1}{e^x}\right]^5$

$= [\cos(ix)]^5$

1.

2. If $\cosh x = \sec\theta$, prove that
(i) $x = \log(\sec\theta + \tan\theta)$ (ii) $\theta = \pi/2 - 2\tan^{-1}(e^{-x})$, (iii) $\sinh x = \tan\theta$
(iv) $\tanh x = \sin\theta$ (v) $\tanh x/2 = \pm \tan\theta/2$

→ Given,

$$\cosh x = \sec\theta$$
$$x = \cosh^{-1}(\sec\theta)$$
$$= \log(\sec\theta + \sqrt{\sec^2\theta - 1})$$
$$x = \log(\sec\theta + \tan\theta)$$

$$\boxed{e^{x} = \sec\theta + \tan\theta}$$

$$\boxed{e^{-x} = \sec\theta - \tan\theta}$$

Now,
$$e^{-x} = \sec\theta - \tan\theta$$
$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$
$$= \frac{1-\sin\theta}{\cos\theta}$$
$$= \frac{1-\cos(\pi/2-\theta)}{\sin(\pi/2-\theta)}$$

$$= \frac{2\sin^2(\pi/4-\theta/2)}{2\sin(\pi/4-\theta/2)\cos(\pi/4-\theta/2)}$$

$$e^{-x} = \frac{\sin(\pi/4-\theta/2)}{\cos(\pi/4-\theta/2)}$$

$$e^{-x} = \tan(\pi/4-\theta/2)$$
$$\tan^{-1}(e^{-x}) = \frac{\pi}{4} - \frac{\theta}{2}$$

$$\frac{\theta}{2} = \frac{\pi}{4} - \tan^{-1}(e^{-x})$$

$$\boxed{\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})}$$

① $\because \cosh x = \sec\theta$

$$\cosh x = \frac{1}{\cos\theta}$$

$$\frac{1+\tanh^2 x/2}{1-\tanh^2 x/2} = \frac{1}{\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}}$$

$$\frac{1+\tanh^2 x/2}{1-\tanh^2 x/2} = \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2}$$

On comparision,

$$\tanh^2 x/2 = \tan^2\theta/2$$

$$\tanh x/2 = \pm\tan\theta/2$$

2. Prove that $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = -\frac{i}{2}\log\left(\frac{a}{x}\right)$

→ let $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = y$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

ie. $i\left(\frac{x-a}{x+a}\right) = \tan y$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$i\left(\frac{x-a}{x+a}\right) = \frac{e^{iy}-e^{-iy}}{i(e^{iy}+e^{-iy})}$$

$$\tan\theta = \frac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$$

$$i^2\left(\frac{x-a}{x+a}\right) = \frac{e^{iy}-e^{-iy}}{e^{iy}+e^{-iy}}$$

$$-1\left(\frac{x-a}{x+a}\right) = \frac{e^{2iy}-1}{e^{2iy}+1}$$

$$\frac{\left(\frac{a-x}{a+x}\right)}{1} = \frac{e^{2iy}-1}{e^{2iy}+1}$$

$$\left(\frac{a-x}{a+x}\right)\left(e^{2iy}+1\right) = e^{2iy}-1$$

$$\left[\left(\frac{a-x}{a+x}\right)e^{2iy} + \left(\frac{a-x}{a+x}\right)\right] = e^{2iy}-1$$

$$\left[\frac{a-x}{a+x} - 1\right]e^{2iy} = -1-\left(\frac{a-x}{a+x}\right)$$

$$\left[\frac{\cancel{-2}x}{\cancel{a+x}}\right]e^{2iy} = \frac{\cancel{-2}a}{\cancel{a+x}}$$

$$e^{2iy} = \frac{a}{x}$$

$$2iy = \log(a/x)$$

$$iy = \frac{1}{2}\log(a/x)$$

$$y = \frac{1}{2i}\log(a/x)$$

$$y = \frac{-i}{2}\log\left(\frac{a}{x}\right)$$

Capital Log is different from log

$$\tan^{-1}\left(i\left(\frac{x-a}{x+a}\right)\right) = \frac{-i}{2}\log\left(\frac{a}{x}\right)$$

another method

$$LHS = \tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right]$$

Put $x = ae^y$

$$= \tan^{-1}\left[i\left(\frac{e^y-1}{e^y+1}\right)\right]$$

multiply by $e^{-y/2}$

$$= \tan^{-1}\left[i\left(\frac{e^{y/2}-e^{-y/2}}{e^{y/2}+e^{-y/2}}\right)\right]$$

$$= \tan^{-1}[i\tanh y/2]$$

$x = ae^y$

$$= \tan^{-1}[\tan(iy/2)]$$

$$= iy/2$$

$$= \frac{i}{2}\log\frac{x}{a}$$

$$= \frac{-i}{2}\log\frac{a}{x}$$

* logarithmic function. for complex numbers

① $\log z = \log_e z$

① $\log z = \log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$

or $= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

(2) $\log(x-iy) = \log\sqrt{x^2+y^2} - i\tan^{-1}\left(\frac{y}{x}\right)$

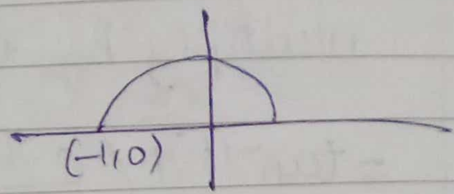
(3) $\operatorname{Log}(x+iy) = 2n\pi i + \log(x+iy)$

$= \log\sqrt{x^2+y^2} + i\left(2n\pi + \tan^{-1}(y/x)\right)$

(4) $\operatorname{Log}(x-iy) = \log\sqrt{x^2+y^2} - i\left(2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right)$

~~5 log(-1) = log(-1) = log log(~~

$\log(-1) = \log(\underset{x}{-1} + i\underset{y}{0})$



$= \log\sqrt{1+0} + i\tan^{-1}\left(\frac{0}{-1}\right)$

$= \log 1 + \tan^{-1}(0/-1)$

$= 0 + i\pi$

$= 0 + i\pi$

$\operatorname{Log}(-1) = \pi i$

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$$\boxed{\log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)}$$

Q(1) Evaluate the value of $\log_2(-3)$

Sol.

$\log_2(-3) = \frac{\log(-3)}{\log 2}$

$= \frac{\log(-3+i0)}{\log 2}$

$$= \frac{\log\sqrt{9+0} + i\tan^{-1}(0/-3)}{\log 2}$$

$$= \frac{\log\sqrt{9} + i\pi}{\log 2}$$

$$= \frac{\log 3 + i\pi}{\log 2}$$

② Prove that i^i is real and hence find value of $\sin(\log i^i)$

Ans Let $z = i^i$

taking log on both side

$$\log z = \log i^i$$

$$= i \log i$$

$$= i \log(0+i)$$

$$= i[\log\sqrt{0+1} + i\tan^{-1}(\tfrac{1}{0})]$$

$$= i(\log 1 + i\,\pi/2)$$

$$= i(i\pi/2)$$

$$\log z = -\pi/2$$

$$z = e^{-\pi/2}$$

$$x + iy = e^{-\pi/2} + 0i$$

$$x = e^{-\pi/2}, \; y = 0$$

$\therefore (i)^i$ is real

$\cos(ix) = \cosh x$
$\sin(iy) = i\sinh y$

① $\sin(x \pm iy) = \sin x \cos(iy) \pm \cos x \sin(iy)$

$= \sin x \cosh y \pm i \cos x \sinh y$

② $\cos(x \pm iy) = \cos x \cos(iy) \mp \sin x \sin(iy)$
$= \cos x \cosh y \mp i \sin x \sinh y$

Q③. Show that $\log\left[\frac{\cos(x - iy)}{\cos(x+iy)}\right] = 2i\tan^{-1}(\tan x \tanh y)$

Ans.

$\because \log\left[\frac{\cos(x-iy)}{\cos(x+iy)}\right] = \log\cos(x-iy) - \log\cos(x+iy)$

$= \log[\cos x \cosh y + i \sin x \sinh y]$
$- \log[\cos x \cosh y - i \sin x \sinh y]$

$= \cancel{\log\sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}} + i\tan^{-1}\left(\frac{\sin x \sinh y}{\cos x \cosh y}\right)$

$- \left(\cancel{\log\sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}} - i\tan^{-1}\left(\frac{\sin x \sinh y}{\cos x \cosh y}\right)\right)$

$= 2i\tan^{-1}(\tan \tanh y)$

④ Show that $\tan\left\{i\log\frac{(a-ib)}{(a+ib)}\right\} = \frac{2ab}{a^2-b^2}$

Ans.

$\because \log\left(\frac{a-ib}{a+ib}\right) = \log(a-ib) - \log(a+ib)$

$= \left(\cancel{\log\sqrt{a^2+b^2}} - i\tan^{-1}\left(\frac{b}{a}\right)\right) - \left(\cancel{\log\sqrt{a^2+b^2}} + i\tan^{-1}\left(\frac{b}{a}\right)\right)$

$= -2i\tan^{-1}\left(\frac{b}{a}\right)$

$$i\log\left(\frac{a-ib}{a+ib}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$$

$$\tan\left\{i\log\left(\frac{a-ib}{a+ib}\right)\right\} = \tan\left(2\tan^{-1}\frac{b}{a}\right)$$

$$= \frac{2\tan\left(\tan^{-1}\frac{b}{a}\right)}{1-\tan^2\left(\tan^{-1}\left(\frac{b}{a}\right)\right)}$$

$$= \frac{2\, b/a}{1-\left(\tan\left(\tan^{-1} b/a\right)\right)^2}$$

$$= \frac{2\, b/a}{1-\ b^2/a^2}$$

$$= \frac{2ab}{a^2-b^2}$$

(5) If $\tan(\log(x+iy)) = a+ib$ and $a^2+b^2 \neq 1$ then prove that.

$$\tan\left[\log(x^2+y^2)\right] = \frac{2a}{1-(a^2+b^2)}$$

Ans.
Sol.

let, $\tan(\log(x+iy)) = a+ib$

$$\log(x+iy) = \tan^{-1}(a+ib) \quad \text{——— (1)}$$

$$\log(x-iy) = \tan^{-1}[a-ib] \quad \text{——— (2)}$$

Adding (1) & (2)

$$\log(x+iy) + \log(x-iy) = \tan^{-1}(a+ib) + \tan^{-1}(a-ib)$$

$$\log[(x+iy)(x-iy)] = \tan^{-1}\left[\frac{(a+ib)+(a-ib)}{1-(a+ib)(a-ib)}\right]$$

$$\log(x^2+y^2) = \tan^{-1}\left[\frac{2a}{1-(a^2+b^2)}\right]$$

$$\tan(\log(x^2+y^2)) = \frac{2a}{1-(a^2+b^2)} \qquad \boxed{a^2+b^2 \neq 1}$$

(6) Seperate real and imaginary part of $(1+i\sqrt{3})^{(1-i\sqrt{3})}$

Ans.

let $z = (1+i\sqrt{3})^{(1+i\sqrt{3})}$

$$\log z = (1+i\sqrt{3})\log(1+i\sqrt{3})$$
$$= (1+i\sqrt{3})\left[\log\sqrt{1+3} + i\tan^{-1}(\sqrt{3}/1)\right]$$
$$= (1+i\sqrt{3})(\log 2 + i\pi/3)$$

$$= \log 2 + i\pi/3 + i\sqrt{3}\log 2 - \sqrt{3}\pi/3$$

$$\log z = \left(\log 2 - \frac{\pi}{\sqrt{3}}\right) + i\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)$$

$$z = e^{(\log 2 - \pi/\sqrt{3}) + i(\pi/3 + \sqrt{3}\log 2)}$$

$$x+iy = e^{\log 2 - \pi/\sqrt{3}} \cdot e^{i(\pi/3 + \sqrt{3}\log 2)}$$
$$= e^{\log 2 - \frac{\pi}{\sqrt{3}}}\left[\cos(\pi/3 + \sqrt{3}\log 2) + i\sin\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)\right]$$

$$= e^{\log 2 - \frac{\pi}{\sqrt{3}}}\left(\cos\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)\right) + i e^{\log 2 - \frac{\pi}{\sqrt{3}}}\left[\sin\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)\right]$$

$$= \cos\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)e^{\log 2 - \pi/\sqrt{3}} + i\sin\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)e^{\log 2 - \pi/\sqrt{3}}$$

real part

imaginary part

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* Maxima & minima of a function for two independent variable

$f(x,y)$ = given function

Q3) step I : $\frac{\partial f}{\partial x} = ?$, $\frac{\partial f}{\partial y} = ?$

step II : $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

Solving above equation
roots are $(a_1,b_1), (a_2,b_2), (a_3,b_3)$

step III : $r = \frac{\partial^2 f}{\partial x^2} = ?$

$s = \frac{\partial^2 f}{\partial x \partial y} = ?$

$t = \frac{\partial^2 f}{\partial y^2} = ?$

step IV :

Sr.no.	Points	r	s	t	$rt-s^2$	Conclusion
1	(a_1,b_1)				>0	① $rt-s^2>0$ & $r>0, t>0$
2	(a_2,b_2)					function is Minima
					>0	② $rt-s^2>0$ & $r<0, t<0$
						function is maxima
					<0	neither Maxima nor minima (Saddle points)
					$=0$	No conclusion

Ex-16

① Test the following function $xy(3a-x-y)$ for maxima & minima.

Ans. Let $f(x,y) = xy(3a-x-y)$

$$= 3axy - x^2y - xy^2$$

step①: $\frac{\partial f}{\partial x} = 3ay - 2xy - y^2$, $\frac{\partial f}{\partial y} = 3ax - x^2 - 2xy$

step②:

$\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 0$

$3ay - 2xy - y^2 = 0$ $\qquad$ $3ax - x^2 - 2xy = 0$

$y(3a - 2x - y) = 0$ $\qquad$ $x(3a - x - 2y) = 0$

$\boxed{y = 0,}$ $\qquad$ $\boxed{x = 0,}$

$3a - 2x - y = 0$ $\qquad$ $3a - x - 2y = 0$

$\boxed{3a = 2x + y}$ — ① $\qquad$ $\boxed{3a = 2y + x}$ — ②

① When $y=0, x=0$, then point is $(0,0)$

② when $y=0$, then $3a = 2y + x$ is $x = 3a$, $(3a, 0)$

③ When $x=0$, then $3a = 2x + y$ is $\boxed{y = 3a}$, $(0, 3a)$

④ when $3a = 2x + y$ & $3a = 2y + x$

$\boxed{x = a, \quad y = a}$

$\therefore$ (a,a) points.

step (III): $\frac{\partial^2 f}{\partial x^2} = -2y$, $\frac{\partial^2 f}{\partial y^2} = -2x$, $\frac{\partial^2 f}{\partial x \partial y} = 3a - 2x - 2y$

$r = \frac{\partial^2 f}{\partial x^2} = -2y$, $t = \frac{\partial^2 f}{\partial y^2} = -2x$, $\frac{\partial^2 f}{\partial x \partial y} = 3a - 2x - 2y = s$

step (iv)

Sr no	Points	r	t	s	$rt-s^2$	Conclusion
1	$(0,0)$	0	0	$3a$	$-9a^2<0$	Saddle point
2	$(3a,0)$	0	$-6a$	$-3a$	$-9a^2<0$	saddle
3	$(0,3a)$	$-6a$	0	$-3a$	$-9a^2<0$	saddle
4	(a,a)	$-2a$	$-2a$	$-a$	$3a^2>0$	maxima

Maximum value at (a,a)

$$f(x,y) = xy(3a-x-y)$$
$$f(a,a) = a^2(3a-a-a)$$
$$f(a,a) = a^3$$

saddle points: $(0,0), (3a,0), (0,3a)$

② Test the following function $x^3+3xy^2-15x^2-15y^2+72x$ is maxima & minina.

Ans Let $f(x,y) = x^3+3xy^2-15x^2-15y^2+72x$

step ①: $\frac{\partial f}{\partial x} = 3x^2+3y^2-30x+72$

$\frac{\partial f}{\partial y} = 6xy-30y$.

$$\begin{array}{r|l} 2 & 24 \\ 2 & 12 \\ 2 & 6 \\ 3 & 3 \\ & 1 \end{array}$$

step (ii): $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial x} = 0$

$0 = 6xy-30y$

$0 = y(6x-30)$

$\boxed{y=0}$ $x = \cancel{30/6}$ 5

$\boxed{x=5}$

, $0 = 3x^2+3y^2-30x+72$

, $0 = x^2+y^2-10x+24$ — ①

when $y=0$, $3x^2+3y^2-30x+72=0$ in ①

$\cancel{x^2-10x+24=0}$

$\cancel{x^2-12x+2x+24=0}$

$\cancel{x(x-12)+2(x+12)=0}$

$\cancel{(x+2)(x+12)=0}$

x.

$x^2-6x-4x+24=0$

$x(x-6)-4(x-6)=0$

$(x-6)(x-4)=0$

$\boxed{x=6,4}$

Point $(6,0)$, $(4,0)$

Put $x=5$ in — ①

$25+y^2-50+24=0$

$y^2-1=0$

$y^2=1$

$y=\pm 1$

Point $(5,1)$, $(5,-1)$

step $\boxed{IV}$ $r=\dfrac{\partial^2 f}{\partial x^2}=2x+(-10)$ or $6x-30$

$s=\dfrac{\partial^2 f}{\partial x \partial y}=6y$

$t=\dfrac{\partial^2 f}{\partial y^2}=6x-30$

step $\boxed{V}$

Sr.no	Points	r	s	t	$rt-s^2$	conclusion
1	$(6,0)$					
2	$(4,0)$					
3	$(5,1)$					
4	$(5,-1)$					
5	$(5,0)$					
6						

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* Module - 4 Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots\cdots & a_{mn} \end{bmatrix}_{m \times n} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \\ \rightarrow \end{array}$$

$$A = [a_{ij}]_{m \times n}$$

Upper diagonal matrix $\begin{bmatrix} \cdots & 2 & 3 \\ 0 & \cdots & 4 \\ 0 & 0 & \cdots \end{bmatrix}$

Lower diagonal matrix $\begin{bmatrix} \cdots & 0 & 0 \\ 2 & \cdots & 0 \\ 3 & 4 & \cdots \end{bmatrix}$

① Symmetric Matrix: A square matrix A is said to be symmetric if.

$A^T = A$ or $A' = A$ i.e. $a_{ij} = a_{ji}$

② Skew symmetric matrix: A square matrix A is said to be skew symmetric if.

$A^T = -A$ or $A' = -A$ i.e. $a_{ij} = -a_{ji}$

Note = if $i = j$

$a_{ii} = -a_{ii}$

$a_{ii} + a_{ii} = 0$

$2a_{ii} = 0$

$\boxed{a_{ii} = 0.}$

⇒ diagonal element in skew symmetric is 0

Result Every square matrix A can be express as a sum of symmetric and skew symmetric matrix.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \text{ or } A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$\underbrace{P}_{\text{symmetric}} + \underbrace{Q}_{\text{skew-symmetric}} \qquad P + Q$$

Ex - 29

(1) Express matrix A as the sum of symmetric & skew symmetric Matrix, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Now,

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \quad \text{--- (1)}$$

$$P = \frac{1}{2}(A + A^T)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$$

$$Q = \frac{1}{2}(A - A^T)$$

* Hermitian and skew Hermitian matrix
Given,

$$A = [\text{complex element}]_{m\times m}$$

$$A = [a_{ij}]_{m\times m}$$

$$\bar{A} = [\text{Replace } i \text{ by } -i]_{m\times m}$$

$$A^{\theta} = (\bar{A})^T$$

① A matrix A is said to be a Hermitian matrix if $A^{\theta} = A$ i.e, $a_{ij} = \overline{a_{ji}}$

② A matrix A is said to be a skew Hermitian matrix if

$$A^{\theta} = -A$$

i.e. $a_{ij} = -\overline{a_{ji}}$

Result: Every Square matrix A can be expressed as sum of Hermitian and skew Hermitian matrix.

$$A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$$

$A = P + Q \rightarrow$ skew Hermitian matrix

$P \rightarrow$ Hermitian

(1) Express matrix A as the sum of Hermitian and skew Hermitian matrix, where

Given,

$$A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -3i & -1-i & 3+2i \\ 1-i & i & 1-2i \\ -3+2i & -1-2i & 0 \end{bmatrix}$$

$$A^{\theta} = (\bar{A})^T$$

$$= \begin{bmatrix} -3i & 1-i & -3+2i \\ -1-i & i & -1-2i \\ 3+2i & 1-2i & 0 \end{bmatrix}$$

$$\because A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$$

$$A = P + Q \text{ — } (1)$$

$$P = \frac{1}{2}(A + A^{\theta})$$

$$= \frac{1}{2}\left(\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} + \begin{bmatrix} -3i & 1-i & -3+2i \\ -1-i & i & -1-2i \\ 3+2i & 1-2i & 0 \end{bmatrix} \right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\boxed{P = 0}$$

$$Q = \frac{1}{2}(A - A^{\theta})$$

$$= \frac{1}{2}\left(\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} - \begin{bmatrix} -3i & 1-i & -3+2i \\ -1-i & i & -1-2i \\ 3+2i & 1-2i & 0 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 6i & -2+2i & 6-4i \\ 2+2i & -2i & 2+4i \\ -6-4i & -2+4i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$$

(2) Express matrix A as the sum of hermitian and skew hermitian matrix, where.

$$A = \begin{bmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+4i & -i \end{bmatrix}$$

(ii) Reduce the Matrix normal form and find its rank where.

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & -1 & 2 & 2 \\ 4 & 1 & 0 & -1 \\ 9 & 1 & 5 & 6 \end{bmatrix}$$

$C_1 \leftrightarrow C_2$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 2 & 2 \\ 1 & 4 & 0 & -1 \\ 1 & 9 & 5 & 6 \end{bmatrix}$$

$R_2+R_1,\ R_3-R_1,\ R_4-R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 5 & 6 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

$C_2-2C_1,\ C_3-3C_1,\ C_4-4C_4$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 5 & 6 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

R_2-2R_3

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 11 & 16 \\ 0 & 2 & -3 & -5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

$R_3-2R_2,\ R_4-7R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 11 & 16 \\ 0 & 0 & -25 & -37 \\ 0 & 0 & -75 & -110 \end{bmatrix}$$

$C_3 - 11C_2$, $C_4 - 16C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -25 & -37 \\ 0 & 0 & -75 & -110 \end{bmatrix}$$

$R_4 - 3R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -25 & -37 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \times \frac{-1}{25}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -37 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 + 37C_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$A = I_4$

$\rho(A) = 4$

* Reduction of matrix into PAQ normal form

① $A = [a_{ij}]_{m \times n}$

$A = I_m A I_n$

② $A = [a_{ij}]_{3 \times 3}$

i.e $A = I_3 A I_3$

③ i.e, $A = [a_{ij}]_{4 \times 3}$

$A = I_4 A I_3$

④ i.e $A = [a_{ij}]_{3 \times 4}$

$A = I_3 A I_4$

Consider

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times 3}$$

$a_1 = 1$
Then by [illegible] matrix
$a_2 = 0$
$a_3 = 0$

Let $A = I_3 A I_3$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pre matrix   Post matrix

LHS ko normal form me lane ke liye

① When apply row transformation in LHS, then same transformation apply in Pre matrix A and not in Post matrix A

② when applying column transformation in LHS, then same transformation apply in post matrix A & not in Pre matrix A.

① Find non-singular Matrices P and Q such that PAQ is in normal form

Sol. Given, $A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 1 & 4 & -5 \\ -1 & -5 & -5 & 7 \end{bmatrix}_{3\times 4}$

Let $A = I_3 A I_4$

$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 1 & 4 & -5 \\ -1 & -5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & -3 & -2 & 3 \\ 0 & -3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1,\ C_3 - 3C_1,\ C_4 + 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & 3 \\ 0 & -3 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \times -\frac{1}{3}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -3 & 4 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 + 2C_2,\ C_4 - 3C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -5/3 & 2 \\ 0 & -1/3 & -2/3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e. $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$

$\rho(A) = 2$

$$Q = \begin{bmatrix} 1 & 2/3 & -5/3 & 2 \\ 0 & -1/3 & -2/3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

P & Q depends on operation of matrix & it can differ from ~~#~~ Person to other

Q1. Examine the function $f(x,y) = x^4+y^4-x^2-y^2+1$ for extreme values and also find maximum & minimum value.

Date 25/10/23 Orthogonal Matrix.

A square matrix A is said to be orthogonal if

$$AA^T = I = A^T A$$

Inverse of $A = A^T$ $\quad\quad |A^T| = |A|$

i.e. $A^{-1} = A^T$

Ex 3.1

① Show that if A is orthogonal, then $|A| = \pm 1$

Let A be orthogonal matrix, then

$AA^T = I$ $\quad\quad \because |AB| = |A|\,|B|$

$|AA^T| = |I|$ $\quad\quad \because |A^T| = |A|$

$|A|\,|A^T| = 1$

$|A| \cdot |A| = 1$

$|A|^2 = 1$

$|A| = \pm 1$

② Show that matrix is orthogonal and find its inverse where $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

Given $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

$$A^T = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$A.A^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \cdot \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find a, b, c and A^{-1} if $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ is orthogonal.

$$A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix}$$

$$A A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}_{3\times 3} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix}_{3\times 3}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+a^2 & 2+2+ab & 2+(-4)+ac \\ 2+2+ba & 4+1+b^2 & 4-2+bc \\ 2+(-4)+ac & 4-2+bc & 4+4+c^2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 5+a^2 & 4+ab & -2+ac \\ 4+ba & 5+b^2 & 2+bc \\ -2+ac & 2+bc & 8+c^2 \end{bmatrix}$$

$1 = \frac{1}{9}(5+a^2)$, $1 = \frac{1}{9}[5+b^2]$, $1 = \frac{1}{9}(8+c^2)$

$9-5 = a^2$, $9-5 = b^2$, $9-8 = c^2$

$4 = a^2$, $4 = b^2$, $c^2 = 1$

$a = \pm 2$, $b = \pm 2$, $c = \pm 1$

$$A^{-1} = A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ \pm 2 & \pm 2 & \pm 1 \end{bmatrix}$$

* Unitary Matrix

Square matrix A is said to be unitary if

$$AA^\theta = I = A^\theta A$$

Inverse, $A = A^\theta$

i.e $\boxed{A^{-1} = A^\theta}$

Ex 32

① Show that A is unitary and find A^{-1} where A=

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

Ans.

Let

$$A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$\bar{A} = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$$

$A^{\theta} = (\bar{A})^T$

$$A^{\theta} = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

Now $AA^{\theta} = \frac{1}{4}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} (1+i)(1-i)+(-1+i)(-1-i) & (1+i)(1-i)+(-1+i)(1+i) \\ (1+i)(1-i)+(1-i)(-1-i) & (1+i)(1-i)+(1-i)(1+i) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (1-\cancel{i}+\cancel{i}+1)+(1+\cancel{i}-\cancel{i}+1) & 1+1+-[1+1] \\ (1+1)+-(1+1) & (1+1)+(1+1) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ A is unitary matrix.

② Given $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A).(I+A)^{-1}$ is a unitary matrix.

Ans. Given,

$$A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$I-A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$I+A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$|I+A| = \begin{vmatrix} 1 & 1+2i \\ -1+2i & 1 \end{vmatrix} = 1 - (-(1+2i(1-2i)))$$
$$= 1 + (1+4)$$
$$= 6$$

$$\text{cofactor of } |I+A| = \begin{bmatrix} +1 & 1-2i \\ -1-2i & 1 \end{bmatrix}$$

$$\mathrm{adj}(I+A) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Transpose of cofactor.

$$(I+A)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$B = (I-A).(I+A)^{-1}$$
$$= \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1-[1+4) & -1-2i-1-2i \\ 1-2i+1-2i & -(1+4)+1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} -2 & -1-2i \\ 1-2i & -2 \end{bmatrix}$$

$$B = \frac{1}{3}\begin{bmatrix} -2 & -1-2i \\ 1-2i & -2 \end{bmatrix}$$

$$\bar{B} = \frac{1}{3}\begin{bmatrix} -2 & -1+2i \\ 1+2i & -2 \end{bmatrix}$$

$$B^{\theta} = (\bar{B})^{T}$$

$$B^{\theta} = \frac{1}{3}\begin{bmatrix} -2 & 1+2i \\ -1+2i & -2 \end{bmatrix}$$

$$BB^{\theta} = \frac{1}{9}\begin{bmatrix} -2 & -1-2i \\ 1-2i & -2 \end{bmatrix}\begin{bmatrix} -2 & 1+2i \\ -1+2i & -2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 4+(-4-1) & -\cancel{2}-\cancel{4i}+\cancel{2}+\cancel{4i} \\ -\cancel{2}-\cancel{4i}+\cancel{2}+\cancel{4i} & 1+4+4 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Date 10/23 **The rank of the matrix**

The rank of the matrix A is said to be r if possesses the following properties.

① there is atleast one non-zero minor of order r.

② Every minor of order greater than r is zero. The rank of matrix A is denoted by $\rho(A)$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times 3}$$

① $|A| \neq 0$

$\therefore$ rank of $A = 3$

② if $|A|=0$ and atleast one minor of order 2 is non-zero
$\rho(A)=2$ minor $= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}_{2\times 2}$ or $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}_{2\times 2}$

③ If $|A|=0$ and every minor of order $2=0$ $\therefore \rho(A)=1$
To reduce the matrix into Echolon form.

$$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

By R_2-R_1 & R_3-R_1

$$=\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & b_2' & c_2' \\ 0 & b_3' & c_3' \end{bmatrix}$$

$$=\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & b_2' & c_2' \\ 0 & 0 & 0 \text{ or } c_3'' \end{bmatrix}$$

if $a_{33}=0$ $\rho(A)=2$
if $a_{33}=c_3''$ $\rho(A)=3$

$\therefore \rho(A)$ = number of non-zero row

Ex 33

① Evaluate $\rho(A)$ of matrix into Echelon form.

$$A=\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}_{4\times 4}$$

Ans. By $R_2 \to R_2+2R_1$ & $R_3 \to R_3-R_1$

$$A=\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \rightarrow R_2/3$

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_2$ & $R_4 \rightarrow R_4 - R_2$

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)$ = no. of non-zero row

$\rho(A) = 2$

* Reduce the matrix into normal form or canonical form

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times 3}$$

$R_2 - R_1$, $R_3 - R_1$

$$= \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & b_2' & c_2' \\ 0 & b_3' & c_3' \end{bmatrix}$$ apply column transformation using $a_1 = 1$

By $C_2 - C_1$ & $C_3 - C_1$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_2' & c_2' \\ 0 & b_3' & c_3' \end{bmatrix}$$ into 1

$\rightarrow 0$

$R_3 - 2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c_2' \\ 0 & 0 & c_3'' \end{bmatrix}$$

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By $C_3 - C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\text{or}0 \end{bmatrix}$$

no. of non-zero row = 3 if $a_{33} = 1$

$\rho(A) = 3$

if $a_{33} = 0$ then, $\rho(A) = 2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Ex 34

① Reduce the Matrix into Normal form and find its rank where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Ans. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$$

~~$R_1 \to R_2 + 2R_1$~~, ~~$R_3 \to R_3 - R_2$~~

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$$

By $R_3 \to R_3 - R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

By $C_3 \to C_3 - C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By $R_2 \to R_2/-2$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore \rho(A) = 2$

② Reduce the matrix into normal form and find its rank $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}_{4\times4}$

Sol. Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

By $R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$ & $R_4 \to R_4 - R_1$

$$A = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -12 & -7 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

By $C_3 \to C_3 - 4C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -12 & -7 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

$C_4 \to C_4 - 3C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -12 & -7 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

By $R_3 - R_2$, $R_4 - R_2$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

$C_3 + 3C_2$ & $C_4 + C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

By $R_4 - \frac{1}{3}R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4/-9$, $C_4/-6$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - C_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} I_3 \end{bmatrix}$ $\rho(A) = 3$

② Find non-singular matrices P and Q such that PAQ is in normal form and find A^{-1} and $\rho(A)$ where.

Ans: Given, $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}_{3\times3}$

let $A = I_3 A I_3$

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - R_1$ and $R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

if $\rho(A)$ = order of matrix (A) then A^{-1} exists,
else $\rho(A) \neq$ order of matrix (A) then A^{-1} never exists.

By $C_2 - 3C_1$, & $C_3 - 3C_1$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. $I_3 = PAQ$ ——— ①

where.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Echlon form $\left[\begin{matrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1000 \end{matrix}\right]$ go like this

rank of A = 3 , i.e, $\rho(A) = 3$

$\because$ rank of A = order of matrix A

$3 = 3$

$\therefore A^{-1}$ exist.

from equation — (1)

$$P^{-1} I_3 = P^{-1} P A Q$$

$$P^{-1} = I_3 A Q$$

$$P^{-1} = AQ$$

$$A^{-1} P^{-1} = A^{-1} A Q$$

$$A^{-1} P^{-1} = I_3 Q$$

$$A^{-1} P^{-1} = Q$$

$$A^{-1} P^{-1} P = QP$$

$$A^{-1} I_3 = QP$$

$$\boxed{A^{-1} = QP}$$

$2x + 3y = 5 \longrightarrow$ non-homogeneous equation

$2x + 3y = 0 \longrightarrow$ homogeneous equation.

* Non-homogeneous linear equation P & Q such that PAQ is in normal form.

Given, $a_1 x + b_1 y + c_1 z = d_1$ — (1)

$a_2 x + b_2 y + c_2 z = d_2$ — (2)

$a_3 x + b_3 y + c_3 z = d_3$ — (3)

In matrix form,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B$$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ $[A:B] = \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 1 \end{array}\right]$

$\rho(A:B) = 2$

Reduce matrix A into Echelon form

$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & c_2' \\ 0 & 0 & 1000 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2' \\ d_3'' \end{bmatrix} \qquad \begin{aligned} d_2' &= d_2 - d_1 \\ d_3' &= d_3 - d_2 \\ d_3'' &= d_3' - d_2' \end{aligned}$$

Rank of A = ?
Rank of [A : B] = ?

① If Rank of A ≠ Rank of [A : B]
⇒ System is inconsistent
⇒ No solution is obtained.

② If Rank of A = Rank of [A : B]
⇒ System is consistent
⇒ Solution is obtained.

ⓐ ∵ rank of A = number of unknown
$\boxed{r = n}$
∴ Unique solution is obtained.

ⓑ ∵ rank of A < number of unknown
$r < n$
infinite solution or many solution.

① Discuss the consistency of the system and if consistent solve the equation.

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$2x + 4y + 7z = 30$$

Ans: In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$AX = B$

By $R_2 - R_1$ & $R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 18 \end{bmatrix}$$

By $R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix}$$

rank of $A = 3$

rank of $[A:B] = 3$

$\because$ rank of A = rank of $[A:B]$

$3 = 3$

$\therefore$ system is consistent

$\because$ rank of A = number of unknown

$r = n$

$3 = 3$

$\therefore$ Unique solution is obtained

$x + y + z = 6$ — ①

$y + 2z = 8$ — ②

$z = 2$ — ③

② Discuss the consistency of the system and if consistent solve the equation

$x + y + z = 5$

$x + 2y + 3z = 10$

$x + 2y + 3z = 8$

Ans In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 8 \end{bmatrix}$$ i.e. $AX = B$

By $R_2 - R_1$ & $R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}$$

By $R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

rank of $A = 2$

rank of $[A : B] = 3$

rank of $A \neq$ rank of $[A : B]$

$\therefore$ system is inconsistent.

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③ Discuss the consistency of the system and if consistent solve the equation.

$x-2y+z-w=2$, $x+2y+4w=1$, $4x-z+3w=-1$

Ans. In matrix form

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 2 & 0 & 4 \\ 4 & \cancel{0} & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

i.e. $AX=B$

By R_2-R_1 , R_3-4R_1

$$\begin{bmatrix} 1 & -2 & +1 & -1 \\ 0 & 4 & -1 & 5 \\ 0 & 8 & -5 & 7 \end{bmatrix}_{3\times4}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}_{4\times1} = \begin{bmatrix} 2 \\ -1 \\ -9 \end{bmatrix}_{3\times1}$$

By R_3-2R_2

$$\begin{bmatrix} 1 & -2 & +1 & -1 \\ 0 & 4 & -1 & 5 \\ 0 & 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -7 \end{bmatrix}$$

By

∴ rank of $A=3$

∴ rank of $[A:B]=3$

rank of $A=$ rank of $[A:B]$

$3=3$

∴ System is consistent.

∵ $r<n$

i.e $3<4$

∴ Infinite solution is obtained.

$\because$ $n-r=4-3=1$ Linear independent solution is obtained.

Inequation form

$$x-2y+z-w=2 \quad \text{—— (1)}$$
$$4y-z+5w=-1 \quad \text{—— (2)}$$
$$-3z-3w=-7 \quad \text{—— (3)}$$

Put $w=t$

from (3)

$$-3z-3t=7$$
$$z=\frac{7-3t}{3}$$

from (2)

$$4y=-1+(12-54)$$
$$4y=-1+\left(\frac{7.3t}{30}\right)-5t$$

$$y=\frac{4-18t}{12}$$

$$z=\frac{2-9t}{6}$$

from (1)

$$x=2+2y-z+w$$
$$x=2+2\left(\frac{2-9t}{6}\right)-\left(\frac{7.3t}{3}\right)+t$$

$$x=\frac{1-3t}{3}$$ $\therefore$ $\boxed{x=}$

Ex 37

① Investigate for what value of λ and μ the system of equation.
$2x+3y+5z=9$, $7x+3y-2z=3$, $2x+3y+\lambda z=\mu$ have ① no solution ② Unique solution (iii) infite solution.

Ans.

Let the matrix form be

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ \mu \end{bmatrix}$$

$AX = B$

By $R_2 - 3R_1$

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & -6 & -17 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -24 \\ \mu \end{bmatrix}$$

By $R_2 \longleftrightarrow R_1$

$$\begin{bmatrix} 1 & -6 & -17 \\ 2 & 3 & 5 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -24 \\ 9 \\ \mu \end{bmatrix}$$

By $R_2 - 2R_1$, $R_3 - 2R_1$

$$\begin{bmatrix} 1 & -6 & -17 \\ 0 & 15 & 39 \\ 0 & 15 & \lambda+34 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -24 \\ 57 \\ \mu+48 \end{bmatrix}$$

By $R_3 - R_2$

$$\begin{bmatrix} 1 & -6 & -17 \\ 0 & 15 & 39 \\ 0 & 0 & \lambda-5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -24 \\ 5y \\ \mu-9 \end{bmatrix}$$

① No solution

$\therefore$ rank of $A \neq$ rank of $[A:B]$

$\lambda = 5$

$\therefore$ ie ~~$\lambda \neq 5$~~ and $\mu \neq 9$

② Unique Solution:-

① rank of A = rank of $[A:B]$

② $r = n$

$= 3$

$\lambda \neq 5$, μ have any value

③ Infinite solution

① rank of A = rank of $[A:B]$

② $r < n$

$r < 3$

$\lambda = 5$, $\mu = 9$

(2) Determine the value of λ so that equation

$x+y+z=1$, $x+2y+4z=\lambda$, $x+4y+10z=\lambda^2$ have a solution and solve them for each value of λ

Ans. In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

$$AX = B$$

By $R_2 - R_1$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ \ldots \end{bmatrix}\begin{bmatrix} x \\ y \\ \ldots \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \ldots \end{bmatrix}$$

$10z = \lambda$ have a solution and

them for each value of λ

Ans. In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

$$AX = B$$

By $R_2 - R_1$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 \end{bmatrix}$$

By $R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 1 \end{bmatrix}$$

By $R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 1 - 3\lambda + 3 \end{bmatrix} \quad \text{— (3)}$$

$\because$ system have a solution

rank of A = rank of $[A : B]$

$$2 = 2$$

$$\lambda^2 - 3\lambda + 2 = 0 \quad \text{— (1)}$$

$$\lambda^2 - 2\lambda - \lambda + 2 = 0$$

$$\lambda(\lambda - 2) - 1(\lambda - 2) = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\boxed{\lambda = 2, 1}$$

Put $\lambda = 1$ in (3)

$$\cancel{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & \end{bmatrix}} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$x+y+z=1$ — (1)
$y+3z=0$ — (2)

Put $z=t$

$y=-3t$

$\therefore x-3t+t=1$

$\boxed{x=1+2t}$

Put $\lambda=2$ in (3)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$x+y+z=1$ — (4)
$y+3z=1$ — (5)

Put $z=t$

$y=1-3t$

$\therefore x+\cancel{1}-3t+t=\cancel{1}$

$\boxed{x=2t}$

* Linear Dependent and Independent of vectors.

(1) Linear Dependent vectors: A set of r vectors $x_1, x_2, x_3, \ldots x_r$ is said to be linear dependent if there exist a scalar $k_1, k_2, k_3, \ldots k_r$ (not all zero) such that $k_1x_1+k_2x_2+\cdots+k_rx_r=0$ at least one not equal to zero.

(2) Linear Independent vector: A set of r vectors $x_1, x_2, x_3, \ldots x_r$ is said to linearly independent if $\exists$ a scalar $k_1, k_2, k_3, \ldots k_r$ (all zero) such that $k_1x_1+k_2x_2+\cdots+k_rx_r=0$.

Ex. 39

(1) Examine whether the following vectors are linearly dependent or independent.

$(1,1,1,3), (1,2,3,4), (2,3,4,7)$

Sol: let $X_1 = (1,1,1,3)$

$X_2 = (1,2,3,4)$

$X_3 = (2,3,4,7)$

Consider Matrix Equation.

$$K_1X_1 + K_2X_2 + K_3X_3 = 0$$

i.e $K_1(1,1,1,3) + K_2(1,2,3,4) + K_3(2,3,4,7) = (0,0,0,0)$

$$K_1 + K_2 + 2K_3 = 0 \quad \text{--- (1)}$$
$$K_1 + 2K_2 + 3K_3 = 0 \quad \text{--- (2)}$$
$$K_1 + 3K_2 + 4K_3 = 0 \quad \text{--- (3)}$$
$$3K_1 + 4K_2 + 7K_3 = 0 \quad \text{--- (4)}$$

In matrix form,

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}}_{4\times3} \underbrace{\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}}_{3\times1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{4\times1}$$

By $R_2 - R_1$, $R_3 - R_1$, $R_4 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 2R_2$ & $R_4 - R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank of $A = 2 = r$

number of unknown $= 3 = n$

Result:

① if $r = n$

$\Rightarrow$ zero solution then linearly independent, $k_1, k_2, k_3 = 0$

② if $r < n$

$\Rightarrow$ non-zero solution

$\because \quad r < n$

$2 < 3$

$\therefore$ non-zero solution is obtained.

$\because n - r = 3 - 2 = 1$ linear independent solution is obtained.

$k_1 + k_2 + 2k_3 = 0$ —— (5)

$k_2 + k_3 = 0$ —— (6)

Put $k_3 = t$

$k_2 = -t$

$k_1 + (-t) + 2t = 0$

$k_1 = -t$

$\therefore k_1, k_2, k_3$ not all zero.

$\therefore$ Vector X_1, X_2, X_3 are linearly dependent.

$k_1X_1 + k_2X_2 + k_3X_3 = 0$

$-tX_1 - tX_2 + tX_3 = 0$

$-X_1 - X_2 + X_3 = 0$

$\boxed{X_3 = X_1 + X_2}$

② Examine the following set of vectors are linearly dependent or independent.

$(1,2,-1,0), (1,3,1,2), (4,2,1,0), (6,1,0,1)$

Ans. let

$X_1 = (1,2,-1,0)$ $\quad X_3 = (4,2,1,0)$

$X_2 = (1,3,1,2)$ $\quad X_4 = (6,1,0,1)$

Matrix equation is

$$K_1X_1 + K_2X_2 + K_3X_3 + K_4X_4 = 0$$

i.e. $K_1(1,2,-1,0) + K_2(1,3,1,2) + K_3(4,2,1,0) + K_4(6,1,0,1) = (0,0,0,0)$

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1, R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 2 & 5 & 6 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_2, R_4 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & 12 & 23 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_4 - \frac{12}{17}R_3$

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & 0 & 55/17 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$r = 4, n = 4$

$r = n$

$\therefore$ zero-solution is obtained.

$K_1 = K_2 = K_3 = K_4 = 0$

Vectors are linealy independent.

Module-06 Numerical Integration

$y = f(x)$ $[a, b]$ $h = \dfrac{b-a}{n}$

$x: a, a+h, a+2h, a+3h, \ldots b$

$f(x): y_0, y_1, y_2, y_3, \ldots y_n$

$\Delta f(x):$

① Forward difference operator (Δ)

$y_1 - y_0, y_2 - y_1, y_3 - y_2$

del $\Delta f(x) = f(x+h) - f(x)$

② Backward difference operator (∇)

debloy $\nabla f(x) = f(x) - f(x-h)$

③ $f(x+h) = Ef(x)$ $\to \Delta$

④ $f(x-h) = E^{-1} f(x)$

$\hookrightarrow$ operator $\Delta^{-1} = \nabla$.

⑤ $\Delta = E - 1$ or $E = 1 + \Delta$

⑥ $\nabla = 1 - E^{-1}$ or $E^{-1} = 1 - \nabla$

$E^r f(x) = f(x+2h)$

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Ex5.2

① Find value of (i) Δe^{ax} (ii) $\Delta^2 e^x$, (iii) $\nabla \log x$

Ans Given,

$$f(x) = e^{ax},\ f(x+h) = e^{a(x+h)}$$

$$\Delta = E - 1$$

$$\Delta f(x) = (E-1) f(x)$$

$$\Delta f(x) = E f(x) - f(x)$$

$$\Delta f(x) = f(x+h) - f(x)$$

You can also use direct formula.

$$\Delta e^{ax} = e^{a(x+h)} - e^{ax}$$

$$= e^{ax} \cdot e^{ah} - e^{ax}$$

$$= e^{ax}(e^{ah} - 1)$$

$$\Delta e^{ax} = e^{ax}(e^{ah} - 1)$$

(ii) given, $f(x) = e^x$

$$\Delta = E - 1$$

$$\Delta^2 = (E-1)^2$$

$$\Delta^2 = E^2 - 2E + 1$$

$$\Delta^2 f(x) = E^2 f(x) - 2E f(x) + f(x)$$

$$\Delta^2 e^x = f(x+2h) - 2f(x+h) + f(x)$$

2nd order derivative

$$\Delta^2 e^x = e^{x+2h} - 2e^{x+h} + e^x$$

$$= e^x \cdot e^{2h} - 2e^x \cdot e^h + e^x$$

$$= e^x(e^{2h} - 2e^h + 1)$$

$$= e^x(e^h - 1)^2$$

(iii) $f(x) = \log x$, $f(x+h) = \log(x+h)$

$$\Delta = E - 1$$

$$\Delta fx = E f(x) - f(x)$$

$$= f(x+h) - f(x)$$

$$= \log(x+h) - \log x$$

$$= \log\left(\frac{x+h}{x}\right)$$

(2) Show that $(1+\Delta)(1-\nabla)=1$

Ans. $\because 1+\Delta = E$

$1-\nabla = E^{-1}$

$LHS = (1+\Delta)(1-\nabla)$

$= E \cdot E^{-1}$

$= E \cdot \frac{1}{E}$

$= 1$

RHS

$\Delta f(x) = f(x+h) - f(x)$

$\Delta^2 f(x) = \Delta(f(x+h) - f(x))$

$= \Delta f(x+h) - \Delta f(x)$

$= f(x+h+h) - f(x+h)$

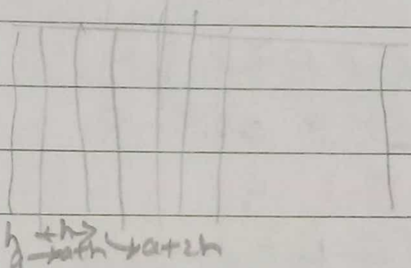
$- f(x+h) + f(x)$

$= f(x+2h) - 2f(x+h)$

$+ f(x)$

* $\int_a^b f(x)\,dx$ into n subinterval.

$h = \frac{b-a}{n}$



(1) Rectangular method or Mid point Method.

$$\int_a^b f(x)\,dx = h\left[f(x_1^*) + f(x_2^*) + f(x_3^*) + \ldots\ldots + f(x_n^*)\right]$$

where x_i^* is the mid point of subinterval.

$x:$ a a+h a+2h a+3h

x_1^* x_2^* x_3^*

10

(1) Evaluate: $\int_0^{10} \frac{1}{1+x^2}\,dx$ by using rectangular Method.

Ans. $a=0,\ b=10$

$h = \frac{b-a}{n} = \frac{10-0}{10} = 1$

x: sub-interval: (0,1), (1,2), (2,3), (3,4), (4,5), (5,6), (6,7),
(7,8), (8,9), (9,10).

x_i: midpoint: 0.5 1.5 2.5 3.5 4.5 5.5 6.5 7.5

$\downarrow$ x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*

8.5 $\to x_9^*$, 9.5 $\to x_{10}^*$

$f(x_i^*) = 0.8$ 0.307 0.137 0.075 0.047 0.032 0.023
0.017 0.0136 0.0109.

$$= \frac{1}{10}[0.8 + 0.307 + 0.137 + 0.075 + 0.047 + 0.032 + 0.023 + 0.01 + 0.0736 + 0.0109]$$

$$\int_0 \frac{1}{1+x^2}\,dx \cong 1.46$$
$\approx$

Evaluate: $\int_0^2 x^2 dx$ by using rectangular Method with (4) subintervals.

$a = 0$, $b = 2$, $n = 4$

$$h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

4 subinterval: [0, 0.5) [0.5, 1.0], [1.0, 1.5), [1.5, 2.0].

x_i: midpoint : 0.25 0.75 1.25 1.75

$f(x_i^*)$: 0.0625 0.5625 1.5625 3.0625

$$\approx 0.5[0.0625 + 0.5625 + 1.5625 + 3.0625]$$
$$\approx 2.625$$

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Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\dots+y_{n-1})\right]$$

$$= \frac{h}{2}[X+2R]$$

where $X = y_0 + y_n$

$R = y_1 + y_2 + \dots + y_{n-1}$

Q. $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)dx$ using Trapezoidal Rule.

$h = \frac{b-a}{n} = \frac{1.4-0.2}{6} = \frac{1.2}{6} = 0.2$

x:	0.2 (x_0)	0.4 (x_1)	0.6 (x_2)	0.8 (x_3)	1.0 (x_4)	1.2 (x_5)	1.4 (x_6)
$y=f(x_0)$ =	2.83	2.41	2.34	2.46	2.73	3.15	3.74

By Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{h}{2}[X+2R]$$

$$= \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

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(2) Evaluate: $\int_0^{\pi/2} \frac{\sin x}{x} dx$ by using Trapezoidal Rule

Ans. Here, $a=0$, $b=\pi/2$, $f(x) = \frac{\sin x}{x}$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{n} = \frac{\pi/2}{6} = \frac{\pi}{12}$$

x: 0 $\frac{\pi}{12}$ $\frac{\pi}{6}$ $\frac{\pi}{4}$ $\frac{\pi}{3}$ $\frac{5\pi}{12}$ $\frac{\pi}{2}$

~~y midpoint: $\frac{\pi}{24}$ $\frac{\pi}{8}$ $\frac{5\pi}{24}$ $\frac{7\pi}{24}$ $\frac{9\pi}{24}$ $\frac{11\pi}{24}$~~

y :	1	0.98	0.954	0.900	0.826	0.737	0.636
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{h}{2}[X + 2R]$$

$$\int_0^{\pi/2} \frac{\sin x}{x} dx = \frac{\pi}{12\times 2}[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{\pi}{24}[(1+0.636) + 2(0.98+0.954+0.900+0.826+0.737)]$$

$$= 1.365$$

(3) The velocity of train start from rest is given by following table the time being reckoned in minutes from the start and velocity in km/hr.

Time :	3	6	9	12	15	18
Velocity :	22	29	31	20	4	0

Estimate approximate distance covered in 18 minutes by Trapezoidal Rule

Ans. velocity $= \frac{ds}{dt}$ $v = \frac{ds}{dt}$

$$S = \int_0^{18} v\,dt$$

velocity of train start from rest i.e. $v=0$

$$S = \int_0^{18} 0\,dt = 0$$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
Time :	0	3	6	9	12	15	18
Velocity :	0	22	29	31	20	4	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$h = 3$ minutes

$= \frac{3}{60}$ hrs. $= \frac{1}{20} = 0.05$ hrs

By trapezoidal rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{0.05}{2}\left[0+2(106)\right]$$

$$= \frac{0.05}{2} \times 212$$

$$= 5.3$$

Simpson $\frac{1}{3}$rd rule :

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\dots+y_{n-3}) + 2(y_2+y_4+\dots+y_{n-2})\right]$$

$$= \frac{h}{3}\left[x+4O+2E\right)$$

even subintervals.

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Simpson's $\frac{3}{8}$ th Rule:

$$\int_a^b f(x)dx = \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots+y_{n-1})$$

→ y_6 nahi hoga

$$+2(y_3+y_6+\dots+y_{n-3})\Big]$$

↳ even intervals & multiple of 3. $n=6,12$

Ex 56

① Evaluate $\int_{0.2}^{1.4}(\sin x-\log_e x+e^x)dx$ by using

① Simpson 1/3rd rule
② Simpson 3/8th rule

Ans.

$a=0.2,\ b=1.4,\ f(x)=\sin x-\log x+e^x$

$$h=\frac{b-a}{n}=\frac{1.4-0.2}{n}=\frac{1.2}{6}=0.2$$

x:	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y:	2.834	2.41	2.34	2.46	2.73	3.15	3.74

$$\int_a^b f(x)dx=\frac{h}{3}\Big[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\Big]$$

$$\int_{0.2}^{1.4} f(x)dx=\frac{3\times0.2}{3}\Big[(6.57)+4(2.41+2.46+3.15)+2(2.34+2.74)\Big]$$

$$=3.2529$$

Simpson's 3/8th rule

$$\int_a^b f(x)dx=\frac{3h}{8}\Big[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\Big]$$

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$$= \frac{3(0.2)}{8}\left[(2.834 + 3.74) + 3(\qquad\qquad) + 2(2.46)\right]$$

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② Evaluate $\int_{4}^{5.2} \log x \, dx$ by simpson's $\frac{1}{3}$rd rule.

Ans: Here $a = 4$, $b = 5.2$, $f(x) = \log x$

$h = \frac{b-a}{n} = \frac{5.2-4}{6} = \frac{1.2}{6} = 0.2$

x:	4	4.2	4.4	4.6	4.8	5.0	5.2
	x_0	x_1	x_2	x_3	x_4	x_5	x_6
	y_0	y_1					
$y f(x)$:	1.386	1.435	1.481	1.526	1.568	1.609	1.648

By simpson's $\frac{1}{3}$rd rule.

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + (y_4+y_2)\cdot 2\right]$$

$= 1.82746$

Ex 57.

① The velocity of a train starts from rest is given by the following table the time being reckoned in minutes from start and speed in km/hour

Time:	0	3	6	9	12	15	18
Velocity:	0	22	29	31	20	4	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Estimate approximately the distance covered in 18 minutes by simpson $\frac{1}{3}$rd rule.

Ans.

$V = \frac{ds}{dt}$

$\int_0^{18/60} ds = \int V dt$

$S = V \int_0^{18/60} dt = \int_0^{3/10} V dt$

$\therefore x_i$:	0	1/20	1/10	3/20	1/5	5/20	3/10
y_i:	0	22	29	31	20	4	0

$h = \text{diff} = 3\text{ min} = \frac{3}{60} = \frac{1}{20}\text{ hr.}$

$$\int_0^{3/10} v\,dt = \frac{1/20}{3}\left[(0+0)+4(22+31+4)+2(29+20)\right]$$

$$= \frac{1}{60}\left[228+98\right] = \frac{\overset{54.3}{\cancel{326}}}{\cancel{60}} = 5.43$$

Ex 58

① Evaluate $\int_{0.2}^{1.4}(\sin x - \log x + e^x)\,dx$ by using simpson's $\frac{3}{8}$th rule

② Evaluate $\int_0^{\pi/2} \frac{\sin x}{x}\,dx$ using simpson's $\frac{3}{8}$th rule.

done completed.

E 59

① Find the volume ~~of~~ of solid of revolution formed by rotating about the x-axis bounded by the lines $x=0$, $x=1.5$, $y=0$ and the curve passing through

X:	0	0.25	0.50	0.75	1.0	1.25	1.50
y:	1	0.9655	0.9195	0.8215	0.7081	0.5812	0.5799
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By using Simpson 3/8th Rule.

∴ volume $= \pi\,(\text{Area})$

$= 3.14\,(\text{Area})$

$$\int dV = \int \pi\,\text{Area}\,dx$$

$$V = \int_0^{1.50} \pi\,\text{Area}\,dx$$

$$\text{Area} = \int_a^b y^2\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3\times 0.25}{8}\left[(0.5799+1)+3(0.9655+0.9195+0.7081+0.5812)+2(0.8215)\right]$$

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Module-05
Gamma function

① $\Gamma n = \int_0^\infty e^{-x} . x^{n-1} . dx$

② $\Gamma n+1 = \int_0^\infty e^{-x} . x^{n} . dx$

③ $\Gamma n+1 = n \Gamma n$ (subtract +1 or -1)

$= n.(n-1)\Gamma n-1$

$= n(n-1)(n-2) \ldots 3.2.1$

$\Gamma n+1 = n!$

or $\Gamma n = (n-1)!$

④ $\Gamma 5 = 4!$

$= 4.3.2.1 = 24$

⑤ $\Gamma 7 = 6!$

$= 6.5.4.3.2.1$

⑥ $\Gamma -3$ value does not excist

⑦ $\Gamma 3/2 = 1/2 \Gamma 1/2$

$= \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$ $\Gamma 1/2 = \sqrt{\pi}$

⑧ $\Gamma \frac{5}{2} = \frac{3}{2} \Gamma 3/2$

$= \frac{3}{2} . \frac{1}{2} \Gamma 1/2 = \frac{3}{4}\sqrt{\pi}$

(9) $\Gamma 7/2 = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma 1/2$

$= \frac{15}{8}\sqrt{\pi}$

$\Gamma n = \frac{\Gamma n+1}{n}$ → use for Γ -ive negative

(10) $\Gamma -3/2 = \frac{\Gamma -1/2}{-3/2}$

$= \frac{\Gamma 1/2}{-3/2 \cdot -1/2}$

$= \frac{4}{3}\sqrt{\pi}$

te

23 * Result: $\Gamma p \, \Gamma 1-p = \frac{\pi}{\sin p\pi}$

Ex $\Gamma 1/4 \, \Gamma \frac{3}{4} = \Gamma \frac{1}{4} \, \Gamma 1-\frac{1}{4}$

$= \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{1/\sqrt{2}} = \sqrt{2}\pi$

Ex 4.1

Prove that $\Gamma \frac{3}{2}-x \, \Gamma \frac{3}{2}+x = \left(\frac{1}{4}-x^2\right)\pi \sec \pi x$, provided $-1<2x<1$

$= \Gamma 1+\frac{1}{2}-x \, \Gamma 1+\frac{1}{2}+x$

$= \Gamma \left(\frac{1}{2}-x\right)+1 \, \Gamma \left(\frac{1}{2}+x\right)+1$

$= \left(\frac{1}{2}-x\right)\Gamma \frac{1}{2}-x \cdot \left(\frac{1}{2}+x\right)\Gamma \frac{1}{2}+x$

$$=\left(\frac{1}{4}-x^2\right)\sqrt{\frac{1}{2}-x}\;\sqrt{\frac{1}{2}+x}$$

$$=\left(\frac{1}{4}-x^2\right)\sqrt{\frac{1}{2}-x}\;\sqrt{1-\left(\frac{1}{2}-x\right)}$$

$$p=\frac{1}{2}-x$$

$$=\left(\frac{1}{4}-x^2\right)\frac{\pi}{\sin\left(\frac{1}{2}-x\right)\pi}$$

$$=\left(\frac{1}{4}-x^2\right)\frac{\pi}{\sin\left(\frac{\pi}{2}-x\pi\right)}$$

$$=\left(\frac{1}{4}-x^2\right)\frac{\pi}{\cos\pi x}$$

$$=\left(\frac{1}{4}-x^2\right)\pi\sec\pi x$$

To find the integration of $\int_0^\infty e^{-x^2}dx$, $\int_0^\infty e^{-x^m}dx$ or $\int_0^\infty e^{-\sqrt{x}}dx$

② Prove that $\int_0^\infty \frac{e^{-\sqrt{x}}}{x^{7/4}}\,dx=\frac{8}{3}\sqrt{\pi}$

ns.

let $I=\int_0^\infty \frac{e^{-\sqrt{x}}}{x^{7/4}}\,dx$

$$=\int_0^\infty e^{-\sqrt{x}}\cdot x^{-7/4}\,dx$$

Put $\sqrt{x}=t$

$x = t^2$

$dx = 2t\,dt$

when $x = 0$, $t = 0$

$x = \infty$, $t = \infty$

$$= \int_0^\infty e^{-t} (t^2)^{-7/4} 2t\,dt$$

$$= 2\int_0^\infty e^{-t} \cdot t^{-7/2} \cdot t\,dt$$

$$= 2\int_0^\infty e^{-t} \cdot t^{-5/2} dt$$

$$= 2\,\Gamma\left(\frac{-5}{2}+1\right)$$

$$= 2\,\Gamma\left(\frac{-3}{2}\right)$$

$$= 2\,\frac{\Gamma(-1/2)}{(-3/2)}$$

$$= 2\,\frac{\Gamma(1/2)}{(-3/2)(-1/2)}$$

$$= \frac{2\sqrt{\pi}}{3/4} = \frac{8\sqrt{\pi}}{3}$$

③ Evaluate $\int_0^\infty 7^{-4x^2} dx$

Ans. Let $I = \int_0^\infty 7^{-4x^2} dx$

Put $7^{-4x^2} = e^{-t}$

Taking log on both side

$$-4x^2 \log 7 = -t \log e$$

$$x^2 = \frac{t}{4\log 7}$$

$$x = \frac{t^{1/2}}{2\sqrt{\log 7}}$$

$$dx = \frac{1}{4} \frac{t^{-1/2}}{\sqrt{\log 7}} dt$$

$x=0, t=0$
$x=\infty, t=\infty$

$$= \int_0^\infty e^{-t} \cdot \frac{1}{4\sqrt{\log 7}} t^{-1/2} dt$$

$$= \frac{1}{4\sqrt{\log 7}} \int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \frac{1}{4\sqrt{\log 7}} \sqrt{-\frac{1}{2}+1}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\log 7}}$$

Q $\int_0^1 (\log x)^m dx$

Put $\log x = -t$

$x = e^{-t}$

(4) Prove that $\int_0^1 (x\log x)^4 dx = \frac{4!}{5^5}$

Ans.

Let $I = \int_0^1 (x \log x)^4 dx$

$= \int_0^1 x^4 (\log x)^4 dx$

Put $\log x = -t$

$x = e^{-t}$

$dx = -e^{-t} dt$

when $x = 0$, $t = \infty$

$x = 1$, $t = 0$

$= \int_\infty^0 (e^{-t})^4 \cdot (-t)^4 \cdot (-e^{-t} dt)$

$= -\int_\infty^0 e^{-4t} \cdot t^4 \cdot e^{t} dt$

$= \int_0^\infty e^{-5t} \cdot t^4 dt$

Put $5t = u$

$t = u/5$

$dt = \frac{du}{5}$

When $t=0,\ u=0$

$t=\infty,\ u=\infty$

$$=\int_0^\infty e^{-u}\left(\frac{u}{s}\right)^4 \frac{du}{s}$$

$$=\frac{1}{s^5}\int_0^\infty e^{-u}u^4\,du$$

$$=\frac{1}{s^5}\,\Gamma(4+1)$$

$$=\frac{\Gamma(5)}{s^5}$$

$$=\frac{4!}{s^5}$$

* Beta-function

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

(1) $B(m,n)=B(n,m)$

(2) $B(m,n)=\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

(3) $B(m,n)=2\displaystyle\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta$

$x=\sin^2\theta \quad dx=2\sin\theta\cos\theta$

$= (\sin^2\theta)^{m-1} \cdot \sin\theta$

$= \sin^{2m-2+1}\theta$

$= \sin^{2m-1}\theta$

(4) $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

(1) Evaluate $\int_0^2 x^4(8-x^3)^{-1/3} dx$

Ans. Let $I = \int_0^2 x^4(8-x^3)^{-1/3} dx$

Put $x^3 = 8t$

$x = 2t^{1/3}$

$dx = \frac{2}{3} t^{-2/3} dt$

$x=0, \; t=0$

$x=2, \; t=1$

$= \int_0^1 (2t^{1/3})^4 \cdot (8-8t)^{-1/3} \cdot \frac{2}{3} t^{-2/3} dt$

$= \int_0^1 2^4 t^{4/3} \cdot 8^{-1/3} (1-t)^{-1/3} \cdot \frac{2}{3} t^{-2/3} dt$

$= 2^4 \cdot (2^3)^{-1/3} \frac{2}{3} \int_0^1 t^{2/3} (1-t)^{-1/3} dt$

$= 2^4 \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \int_0^1 t^{5/3-1} (1-t)^{5/3-1} dt$

$$=\frac{2^4}{3} B(5/3, 5/3)$$

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$$= \frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right)$$

$$= \frac{16}{3} \cdot \frac{\Gamma(5/3)\,\Gamma(2/3)}{\Gamma(5/3+2/3)}$$

$$= \frac{16}{3} \frac{\Gamma(5/3)\,\Gamma(2/3)}{\Gamma(7/3)}$$

$$= \frac{16}{3} \cdot \frac{2/3\,\Gamma(2/3)\,\Gamma(2/3)}{\frac{4}{3} \times \frac{1}{3}\,\Gamma(1/3)}$$

$$= \frac{32}{4} \frac{(\Gamma(2/3))^2}{\Gamma(1/3)}$$

$$= \cancel{16}\, 8 \frac{(\Gamma(2/3))^2}{\Gamma(1/3)}$$

② Show that $\int_0^1 \sqrt{1-\sqrt{x}}\,dx \cdot \int_0^{1/2} \sqrt{2y-(2y)^2}\,dy = \frac{\pi}{30}$

Ans. Let $I = \int_0^1 \sqrt{1-\sqrt{x}}\,dx \cdot \int_0^{1/2} \sqrt{2y-(2y)^2}\,dy.$

$$I = I_1 . I_2$$

Now, $I_1 = \int_0^1 \sqrt{1-\sqrt{x}}\,dx$

Put $\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

$x = 0,\ t = 0$

$x = 1,\ t = 1$

$$= \int_0^1 \sqrt{1-t}\; 2t\,dt$$

$$= 2\int_0^1 t(1-t)^{1/2}\,dt$$

$$= 2\int_0^1 t^{2-1}(1-t)^{3/2-1}\,dt$$

$$= B(2, 3/2)$$

$$I_2 = \int_0^{1/2} \sqrt{2y-(2y)^2}\,dy$$

Put $2y = t$

$y = t/2$

$dy = \frac{dt}{2}$

$y = 0,\ t = 0$

$y = 1/2,\ t = 1$

$$= \int_0^1 \sqrt{t}\cdot\sqrt{1-t}\;\frac{1}{2}dt$$

$$= \frac{1}{2}\int_0^1 t^{1/2}(1-t)^{1/2}\,dt$$

$$= \frac{1}{2}\int t^{3/2-1}(1-t)^{3/2-1}dt$$

$$I_2 = \frac{1}{2}B(3/2, 3/2)$$

from ①

$$I = 2B\left(2, \frac{3}{2}\right) \cdot \frac{1}{2}B\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= \frac{\Gamma 2\,\Gamma 3/2}{\Gamma 7/2} \cdot \frac{\Gamma 3/2 \cdot \Gamma 3/2}{\Gamma 3}$$

$$= \frac{\Gamma 3/2 \cdot \Gamma 3/2 \cdot \cancel{\Gamma 3/2} \cdot \cancel{3/2}}{5/2 \cdot 3/2 \cdot \cancel{\Gamma 3/2} \cdot 2 \cdot 1}$$

$$= \frac{1 \cdot 1/2\sqrt{\pi} \cdot 1/2\sqrt{\pi}}{5/2 \cdot 3/2 \cdot 2 \cdot 1}$$

$$= \frac{\pi}{5 \cdot 3 \cdot 2}$$

$$= \frac{\pi}{30}$$

Result:

① Another form of Beta function

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m, n)$$

② Duplication formula

$$\Gamma m\,\Gamma m+\tfrac{1}{2} = \frac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}} = \frac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}$$

Ex. 44

① Prove that $\int_0^\infty \frac{x^5}{(2+3x)^{16}}\,dx = \frac{5!\,9!}{2^{10}.3^6.15!}$

Let $I = \int_0^\infty \frac{x^5}{(2+3x)^{16}}\,dx$

Put $3x = 2t$

$x = \frac{2}{3}t$

$dx = \frac{2}{3}dt$

When $x=0$, $t=0$

$x=\infty$, $t=\infty$

$= \int_0^\infty \frac{(\frac{2}{3}t)^5}{(2+2t)^{16}} \cdot \frac{2}{3}dt$

$= \left(\frac{2}{3}\right)^5 \cdot \frac{1}{2^{16}} \cdot \frac{2}{3} \cdot \int_0^\infty \frac{t^5}{(1+t)^{16}}\,dt$

$= \frac{2^5}{3^5} \cdot \frac{1}{2^{16}} \cdot \frac{2}{3} \int_0^\infty \frac{t^{6-1}}{(1+t)^{6+10}}\,dt$

$= \frac{1}{2^{10}} \cdot \frac{1}{3^6} \cdot B(6,10)$

$= \frac{1}{2^{10}} \cdot \frac{1}{3^6} \frac{\Gamma 6\,.\,\Gamma 10}{\Gamma 16}$

$= \frac{1}{2^{10}} \cdot \frac{1}{3^6} \cdot \frac{5!\,9!}{15!}$

$$= \frac{5 \cdot 9!}{2^{10} \cdot 3^6 \cdot 15!}$$

(2) Prove that $B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$ and hence $\int_0^1 \frac{x^2 + x^3}{(1+x)^7}\, dx$

Ans. $\because B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + I_1 \quad \text{—— (1)}$$

where $I_1 = \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

Put $x = 1/t$

$$dx = -\frac{1}{t^2}\, dt$$

when $x = 1,\ t = 1$
$x = \infty,\ t = 0$

$$\because I_1 = \int_1^0 \frac{(1/t)^{m-1}}{(1 + 1/t)^{m+n}} \cdot \left(\frac{-1}{t^2}\right) dt$$

$$= \int_0^1 \frac{t^{1-m}\, t^{m+n}\, t^{-2}}{(t+1)^{m+n}}\, dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\, dt$$

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

$$I_1 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

from (1)

$$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\therefore \int_0^1 \frac{x^2 + x^3}{(1+x)^7}\, dx = \int_0^1 \frac{x^{3-1} + x^{4-1}}{(1+x)^{3+4}}\, dx$$

$$= B(3,4)$$

$$= \frac{\Gamma 3\, \Gamma 4}{\Gamma 7} = \frac{2!\, 3!}{6!}$$

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Simpson $1/3^{rd}$ → n = even n = 4, 6, 8, 10, ...

Simpson $3/8^{th}$ → n = even but multiple of 3 i.e n = 6, 12, ...

Trapezoidal → n = even or odd

③ Prove that $B(n+\frac{1}{2}, n+\frac{1}{2}) = \frac{\Gamma n+1/2}{2^{2n}\Gamma n+1}\sqrt{\pi}$

Ans.

$$\because B(n+1/2, n+\tfrac{1}{2}) = \frac{\cancel{\Gamma n+\frac{1}{2}}\cdot \cancel{\Gamma n+\frac{1}{2}}}{\Gamma 2n+1}$$

$$= \frac{\Gamma n+1/2\ \Gamma n+1/2}{\Gamma n+\frac{1}{2}+n+\frac{1}{2}}$$

$$= \frac{\Gamma n+1/2 . \Gamma n+1/2}{\Gamma 2n+1}$$

$$= \frac{\Gamma n+1/2}{2n.}\left(\frac{\Gamma n+1/2}{\Gamma 2n}\right) \quad —①$$

By Duplication formula,

$$\Gamma m\ \Gamma m+1/2 = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m$$

$$\frac{\Gamma m+1/2}{\Gamma 2m} = \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \frac{1}{\Gamma m}$$

$\because$ from ①

$$= \frac{\Gamma n+1/2}{2n} \cdot \frac{\sqrt{\pi}}{2^{2n-1}\Gamma n}$$

$$= \frac{\Gamma n+1/2\ \ \sqrt{\pi}}{(2.2^{2n-1}).(n\Gamma n)}$$

$$= \frac{\Gamma n+1/2 \cdot \sqrt{\pi}}{2^{2n}\,\Gamma n+1}$$

Ex 45

(1) Show that $\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\,dx = \frac{a\sqrt{\pi}\,\Gamma 5/6}{\Gamma 1/3}$

Ans. Let $I = \int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\,dx$

$$= \int_0^a \frac{x^{3/2}}{(a^3-x^3)^{1/2}}\,dx$$

$$= \int_0^a x^{3/2}(a^3-x^3)^{-1/2}\,dx$$

Put $x^3 = a^3 t$

$x = a\,t^{1/3}$

$$dx = \frac{a}{3}\,t^{-2/3}\,dt$$

$x=0$, $t=0$

$x=a$, $t=1$

$$= \int_0^1 (at^{1/3})^{3/2}\cdot(a^3-a^3t)^{-1/2}\,\frac{a}{3}\,t^{-2/3}\,dt$$

$$= a^{3/2}\cdot a^{-3/2}\cdot \frac{a}{3}\int_0^1 t^{1/2}(1-t)^{-1/2}\cdot t^{-2/3}\,dt$$

$$= \frac{a}{3}\int_0^1 t^{-1/6}(1-t)^{-1/2}\,dt$$

$$= \frac{a}{3} \int_0^1 t^{5/6-1} (1-t)^{1/2-1}\, dt$$

$$= \frac{a}{3} B\left(5/6, 1/2\right)$$

$$= \frac{a}{3} \frac{\Gamma 5/6 \cdot \Gamma 1/2}{\Gamma\left(\frac{5}{6}+\frac{1}{2}\right)}$$

$$= \frac{a}{3} \cdot \frac{\Gamma 5/6 \cdot \sqrt{\pi}}{\Gamma 4/3}$$

$$= \frac{a}{\cancel{3}} \cdot \frac{\Gamma 5/6 \cdot \sqrt{\pi}}{\frac{1}{\cancel{3}} \cdot \Gamma 1/3}$$

$$= \frac{a\sqrt{\pi}\, \Gamma 5/6}{\Gamma 1/3}$$

② Prove that $\int_5^9 \sqrt[4]{(9-x)(x-5)}\, dx = \frac{2\left(\Gamma 1/4\right)^2}{3\sqrt{\pi}}$

Ans. Let $I = \int_5^9 \sqrt[4]{(9-x)(x-5)}\, dx$

rule:

$9-x+x-5$

= Put $9-x$ or $x-5 = (9-5)t$

(upper limit − lower limit)

$\therefore$

$$x-5 = 4t$$

$$x = 4t+5$$

$$dx = 4dt$$

$x = 9 \quad ; \quad t = 1$
$x = 5 \quad ; \quad t = 0$

$$= \int_0^1 \sqrt[4]{(9-4t-5).4t} \,.\, 4dt$$

$$= \int_0^1 \sqrt[4]{(4-4t).4t} \,.\, 4dt$$

$$= \int_0^1 \sqrt[4]{4(1-t)4t} \,.\, 4dt$$

$$= 4.(16)^{1/4} \int_0^1 t^{1/4}.(1-t)^{1/4} dt$$

$$= 4.2 \int_0^1 t^{5/4-1}.(1-t)^{5/4-1}.dt$$

$$= 8\, B\left(\frac{5}{4}, \frac{5}{4}\right)$$

$$= 8 . \frac{\Gamma 5/4 . \Gamma 5/4}{\Gamma 5/4 + 5/4}$$

$$= 8 . \frac{\Gamma 5/4 . \Gamma 5/4}{\Gamma 5/2}$$

$$= \frac{8 . \frac{1}{4}\Gamma 1/4 . \frac{1}{4}\Gamma 1/4}{\frac{3}{2}.\frac{1}{2}\sqrt{\pi}} = \frac{2(\Gamma 1/4)^2}{3\sqrt{\pi}}$$

Tutorial – 06

③ $\int_0^1 (\log x)^4 dx$

④ Prove that,

$$\int_0^1 \frac{x^{3/2}}{\sqrt{3-x}} dx \cdot \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}} = \frac{432\pi}{35}$$

Tutorial – 07

Q1. Evaluate $\Delta^2\left(\frac{1}{x}\right)$

Q2. Evaluate $\int_0^4 x^2 dx$ by Rectangular method.

Q3. Evaluate $\int_{-1}^1 \frac{1}{1+x^2} dx$ by using Trapezoidal rule

Q4. Evaluate $\int_{-3}^3 x^4 dx$ by ① simpson $1/3^{rd}$ Rule

② simpson $3/8^{th}$ rule.