Module 5:

Calculus II

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Gamma function :

for $n > 0$, the definite integration $\boxed{\int_0^\infty e^{-x} x^{n-1} dx}$ if exist

Then it is called Gamma function and is denoted by Γn i.e $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

and also

$$\int_0^\infty e^{-x} x^n dx = \Gamma n+1$$

properties of Gamma function :-

1. $\Gamma 1 = 1$
2. $\Gamma n+1 = n \Gamma n$; for n - positive fraction } (+ve)
   $\Gamma n = (n-1) \Gamma (n-1)$
3. $\Gamma n+1 = n!$; for n - positive fraction
   $\Gamma n = (n-1)!$
4. $\Gamma n = \frac{\Gamma n+1}{n}$; for n - Negative fraction } (-ve)
5. $\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin(n\pi)}$
6. $\Gamma \frac{1}{2} = \sqrt{\pi}$

$\Gamma \frac{1}{4} \, \Gamma \frac{3}{4}$

e.g $\Gamma \frac{1}{4} \, \Gamma 1-\frac{1}{4} = \frac{\pi}{\sin(\pi/4)}$

$= \frac{\pi}{\frac{1}{\sqrt{2}}} = \sqrt{2}\pi$

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$\Gamma n+1 = n \underbrace{\Gamma n}$

$= (n-1) \underbrace{\Gamma n-1}$

$= (n-1)(n-2) \underbrace{\Gamma (n-2)}$

$= (n-1)(n-2)(n-3) \Gamma (n-3)$

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eg 5 marks

(1) prove that $\sqrt{\tfrac{3}{2}-x}\,\sqrt{\tfrac{3}{2}+x} = \left(\tfrac{1}{4}-x^2\right)\pi\sec(\pi x)$

provided $-1 < 2x < 1$

$\Rightarrow$ $L.H.S = \sqrt{\tfrac{3}{2}-x}\,\sqrt{\tfrac{3}{2}+x}$

$= \sqrt{\tfrac{1}{2}+1-x}\,\sqrt{\tfrac{1}{2}+1+x}$

$= \sqrt{\left(\tfrac{1}{2}-x\right)+1}\,\sqrt{\left(\tfrac{1}{2}+x\right)+1}$

For $-1<2x<1$

$-\tfrac{1}{2}<x<\tfrac{1}{2}$

$\Rightarrow \left(\tfrac{1}{2}-x\right)>0$ and

$\left(\tfrac{1}{2}+x\right)>0$

are positive fraction

$= \left(\tfrac{1}{2}-x\right)\sqrt{\left(\tfrac{1}{2}-x\right)}\;\left(\tfrac{1}{2}+x\right)\sqrt{\left(\tfrac{1}{2}+x\right)}$

using $\sqrt{n+1} = n\sqrt{n}$

$= \left(\left(\tfrac{1}{2}\right)^2-x^2\right)\sqrt{\left(\tfrac{1}{2}-x\right)}\,\sqrt{\left(\tfrac{1}{2}+x\right)}$

$= \left(\tfrac{1}{4}-x^2\right)\sqrt{\left(\tfrac{1}{2}-x\right)}\,\sqrt{1-\left(\tfrac{1}{2}-x\right)}$

$= \left(\tfrac{1}{4}-x^2\right)\dfrac{\pi}{\sin\left\{\left(\tfrac{1}{2}-x\right)\pi\right\}}$

using $\sqrt{n}\,\sqrt{1-n} = \dfrac{\pi}{\sin(n\pi)}$

$= \left(\tfrac{1}{4}-x^2\right)\dfrac{\pi}{\sin\left(\pi/2-x\pi\right)}$

$= \left(\tfrac{1}{4}-x^2\right)\dfrac{\pi}{\cos\pi x}$ $= \left(\tfrac{1}{4}-x^2\right)\pi\sec(\pi x)$

5 marks

Hint - 1st convert into definition format

(2) prove that $\int_0^\infty \frac{e^{-\sqrt{x}}}{x^{7/4}}\,dx = \frac{8}{3}\sqrt{\pi}$

$\Rightarrow$ put $\sqrt{x} = t$ i.e $x = t^2 \Rightarrow$

$dx = 2t\,dt$

when $x = 0,\ t = 0$

$x = \infty,\ t = \infty$

$= \int_0^\infty \frac{e^{-t}}{t^{7/2}} 2t\,dt$

$= 2\int_0^\infty e^{-t}\, t^1\, t^{-7/2}\,dt = 2\int_0^\infty e^{-t}\, t^{-5/2}\,dt$

$= 2\,\Gamma\left(\frac{-5}{2}+1\right)$ .... By definition $\int_0^\infty e^{-x}x^n\,dx = \Gamma(n+1)$

$= \cancel{2\,\Gamma\left(\frac{-3}{2}\right)} = 2\,\Gamma\left(\frac{-3}{2}\right)$

$= \cancel{2\,\frac{\Gamma\left(\left(\frac{-5}{2}+1\right)+1\right)}{-5/2}} = 2\,\frac{\Gamma\left(-3/2+1\right)}{-3/2}$

$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$

$= \frac{4}{-3}\Gamma\left(-\frac{1}{2}\right)$

$= \frac{4}{-3}\,\frac{\Gamma\left(-1/2+1\right)}{-1/2} = \frac{8}{3}\Gamma\left(\tfrac{1}{2}\right)$

$= \frac{8}{3}\sqrt{\pi}$

$= \frac{4}{\cancel{-3}} \times 2\sqrt{\pi}$

$= \frac{8}{3}\sqrt{\pi}$

$\int_0^\infty e^{-x}\, x^n = \Gamma(n+1)$

$\int_0^\infty e^{-x}\, x^{n-1} = \Gamma(n)$

** 5 marks

(4) Evaluate $\int_0^\infty 7^{-4x^2}\,dx$

$\Longrightarrow$ put $7^{-4x^2} = e^{-t}$

$\Rightarrow \log(7^{-4x^2}) = \log e^{-t}$

$\cancel{-}4x^2(\log 7) = \cancel{-}t(\log e)$

$\cancel{-}4x^2(\log 7) = \cancel{-}t$

i.e $x = \sqrt{\dfrac{t}{4(\log 7)}} = \dfrac{\sqrt{t}}{2\sqrt{\log 7}}$

$\Longrightarrow dx = \underbrace{\left(\dfrac{1}{2\sqrt{\log 7}}\right)}_{\text{constant}} \dfrac{1}{2\sqrt{t}}\,dt$

when $\boxed{x=0} \rightarrow 4(0)^2(\log 7) = t$ $\boxed{t=0}$

$\boxed{x=\infty}$ $4(\infty)^2(\log 7) = t$ $\boxed{t=\infty}$

$\therefore \int_0^\infty 7^{-4x^2}\,dx = \int_0^\infty e^{-t} \dfrac{1}{(2\sqrt{\log 7})} \dfrac{1}{2\sqrt{t}}\,dt$

$= \dfrac{1}{4\sqrt{\log 7}} \int_0^\infty \dfrac{e^{-t}}{t^{1/2}} = \dfrac{1}{4\sqrt{\log 7}} \int_0^\infty e^{-t}\, t^{-1/2}$

$= \dfrac{1}{4\sqrt{\log 7}} \Gamma\left(-\tfrac{1}{2}+1\right)$ ... By definition gamma function

$= \dfrac{1}{4\sqrt{\log 7}} \Gamma\left(\tfrac{1}{2}\right)$

$= \dfrac{1}{4\sqrt{\log 7}} \sqrt{\pi}$

$= \sqrt{\dfrac{\pi}{2\log 7}}$ or $= \dfrac{\sqrt{\pi}}{4\sqrt{\log 7}}$

Convert into this format $\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$

or $x^n \, dx = \Gamma n+1$

(5) Evaluate $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} dx$

put $x^3 = t$ $\quad x = (t)^{1/3}$

$dx = \frac{1}{3}(t)^{-2/3} dt$

when $x = 0 \rightarrow x = (t)^{1/3}$

$0 = (t)^{1/3}$

$\boxed{t = 0}$

and

$x = \infty \longrightarrow \infty = (t)^{1/3}$

$\boxed{t = \infty}$

$$\therefore \int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} dx = \int_0^\infty \frac{e^{-t}}{\sqrt{(t)^{1/3}}} \frac{1}{3} (t)^{-2/3} dt$$

$$= \frac{1}{3}\int_0^\infty \frac{e^{-t}}{(t)^{1/6}} (t)^{-2/3} dt = \frac{1}{3}\int_0^\infty e^{-t} t^{-1/6} t^{-2/3} dt$$

$$= \frac{1}{3}\int_0^\infty e^{-t} t^{-\frac{1-4}{6}} dt$$

$$= \frac{1}{3}\int_0^\infty e^{-t} t^{-5/6} dt$$

$$= \frac{1}{3}\Gamma\left(-5/6+1\right) = \frac{1}{3}\Gamma\left(+\frac{1}{6}\right) = \frac{1}{3}\Gamma\frac{1}{6}$$

(6) Evaluate $\int_0^\infty e^{-\sqrt{x}}\, x^{1/4}\, dx$

$\Rightarrow$ put $\sqrt{x} = t$

$x^{1/2} = t$

$x = t^{2} \Rightarrow dx = 2t\, dt$

when $x = 0$ $\boxed{t = 0}$

$x = \infty$ $\boxed{t = \infty}$

$$\therefore \int_0^\infty e^{-\sqrt{x}}\, x^{1/4}\, dx = \int_0^\infty e^{-\sqrt{x}} (x)^{1/4}\, dx$$

$$= \int_0^\infty e^{-t} (t^{2})^{1/4}\, 2t\, dt$$

$$= 2\int_0^\infty e^{-t}\, t^{1/2}\, t\, dt$$

$$\cancel{= \frac{2}{8}\int_0^\infty e^{-t}\, t^{\frac{-4}{8}}\, dt}$$

$$\cancel{= \frac{1}{2}\int_0^\infty e^{-t}\, t^{-3/8}\, dt}$$

$$\cancel{= \frac{1}{2}\Gamma\left(\frac{-3}{8}+1\right) = \frac{1}{2}\Gamma(5/8)}$$

$$= 2\int_0^\infty e^{-t}\, t^{1/2}\, t\, dt$$

$$= 2\int_0^\infty e^{-t}\, t^{3/2}\, dt$$

$$= 2\,\Gamma(3/2+1)$$

$$= 2\,\Gamma(3/2 + 1)$$

$$= \cancel{2} \times \frac{3}{\cancel{2}}\,\Gamma\left(\frac{3}{2}\right) \quad \ldots \{\Gamma(n+1) = n\Gamma(n)\}$$

$$= 3\,\Gamma\left(\frac{1}{2}+1\right) \quad \ldots \{\Gamma(n+1) = n\Gamma(n)\}$$

$$= \frac{3}{2}\,\Gamma(1/2)$$

$$= \frac{3}{2}\sqrt{\pi}$$

$\Gamma 0 = \infty$

(6) Evaluate $\displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx$

$\Rightarrow$ put $x^2 = t$ $\qquad x = \sqrt{t} \Rightarrow dx = \frac{1}{2\sqrt{t}}\,dt$

when $x = 0$ $\boxed{t=0}$

$x = \infty$ $\boxed{t=\infty}$

$$\therefore \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx = \int_0^\infty \frac{e^{-x^2}}{(x)^{1/2}} = \int_0^\infty \frac{e^{-t}}{(\sqrt{t})^{1/2}}\,\frac{1}{2\sqrt{t}}\,dt$$

$$= \int_0^\infty \frac{e^{-t}}{(\sqrt{t})^{1/2}}\,\frac{dt}{2(\sqrt{t})}$$

$$= \frac{1}{2}\int_0^\infty \frac{e^{-t}}{(t)^{1/4}}\,\frac{dt}{2(t^{1/2})}$$

$$= \frac{1}{2}\int_0^\infty \frac{e^{-t}}{(t)^{3/4}}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\,t^{-3/4}$$

$$= \frac{1}{2}\,\Gamma\left(-\tfrac{3}{4}+1\right)$$

$$= \frac{1}{2}\,\Gamma\tfrac{1}{4}$$

$$= \frac{1}{2}\int_0^\infty \frac{e^{-t}}{(\sqrt{t})^{1/2+1/2}}\,dt$$

$$= \frac{1}{2}\int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\,t^{-1/2}\,dt$$

$$= \frac{1}{2}\,\Gamma\left(-\tfrac{1}{2}+1\right)$$

$$= \frac{1}{2}\,\frac{\Gamma(-\frac{1}{2}+1)+1}{-\frac{1}{2}+1}$$

$$= \frac{1}{2}\,\frac{\Gamma \frac{1}{2}+1}{\frac{1}{2}} = \Gamma\tfrac{3}{2}$$

$$= \Gamma\tfrac{1}{2}+1$$

$$= \Gamma\tfrac{1}{2}+1$$

$=$

$(\log(1/x))^{1/3}$

$\log(1/x) = t$

Gamma function Continued

(1) Evaluate $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$

$\Rightarrow$ Put $\sqrt[3]{x} = t$ $\quad (x)^{1/3} = t$

$x = t^3 \quad \Rightarrow \quad dx = 3t^2\, dt$

when
$x = 0 \quad \boxed{t = 0}$

$x = \infty \quad \boxed{t = \infty}$

$$\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx = \int_0^\infty e^{-t}\, (t^3)^{1/2}\, 3t^2\, dt = \int_0^\infty e^{-t}\, (t)^{3/2}\, 3t^2\, dt$$

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\int_0^{2a} f(x)\,dx = 2\int_0^a f(2a-x)\,dx$

$$= 3\int_0^\infty e^{-t}\, t^{7/2}\, dt$$

$$= 3 \,\Gamma\left(\tfrac{7}{2}+1\right)$$

$$= 3\times\frac{7}{2}\Gamma\frac{7}{2} = \frac{3\times 7}{2}\Gamma\left(\frac{6}{2}+1\right) = 3\times\frac{7}{2}\times\frac{6}{2}\Gamma\frac{6}{2}$$

$$= 3\times\frac{7}{2}\times\frac{6}{2}\Gamma\left(\frac{5}{2}+1\right) = 3\times\frac{7}{2}\times\frac{6}{2}\times\frac{5}{2}\Gamma\frac{5}{2} = 3\times\frac{7}{2}\times\frac{6}{2}\times\frac{5}{2}\times\Gamma\left(\frac{4}{2}+1\right)$$

$$= 3\times\frac{7}{2}\times\frac{6}{2}\times\frac{5}{2}\times\Gamma\left(\frac{1}{2}+1\right) = 3\times\frac{7}{2}\times\frac{6}{2}\times\frac{5}{2}\times\frac{1}{2}\Gamma\frac{1}{2}$$

$$= \frac{315}{16}\sqrt{\pi}$$

(3) Evaluate $\int_0^\infty e^{-x^5}\, dx$

$\Rightarrow$ put $x^5 = t$ $\quad x = (t)^{1/5} \quad \Rightarrow \quad dx = \frac{1}{5}(t)^{-4/5}\, dt$

when $\boxed{x=0}$ then $\boxed{t=0}$

$\boxed{x=\infty}$ then $\boxed{t=\infty}$

$$\therefore \int_0^\infty e^{-x^5}\, dx = \int_0^\infty e^{-t}\, \frac{1}{5}\, t^{-4/5}\, dt$$

$$= \frac{1}{5}\int_0^\infty e^{-t}\, t^{-4/5}\, dt$$

(2) Evaluate $\int_0^1 \sqrt[3]{\log(1/x)}\, dx$

$\Rightarrow$ put $\log(1/x) = t$

$\frac{1}{x} = e^{+t}$

$x = e^{-t}$

$\therefore dx = -e^{-t}\, dt$

when $x = 0$ $\boxed{t = \infty}$

$x = 1$ $\boxed{t = 0}$

$\therefore \int_0^1 \sqrt[3]{\log(1/x)}\, dx = \int_\infty^0 \sqrt[3]{t} \quad -e^{-t}\, dt$

$= -\int_\infty^0 3\sqrt{t} + e^{-t}\, dt$

$= -\int_\infty^0 e^{-t} (t)^{1/3}\, dt$

$= \int_0^\infty e^{-t} (t)^{1/3}\, dt \quad \cdots \left\{ \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \right\}$

$= \int_0^\infty e^{-t} (t)^{1/3}\, dt$

$= \overline{\left| \frac{1}{3}+1 \right.} = \frac{1}{3} \overline{\left| \frac{1}{3} \right.}$

$= \frac{1}{5} \overline{\left| \frac{-4}{5}+1 \right.} = \frac{1}{5} \overline{\left| \frac{1}{5} \right.}$

$\Gamma n+1 = n$ $\quad \int_0^\infty e^{-x} x^{n-1} = \Gamma n$

$x^n = \Gamma n+1$

(9) Evaluate $\int_0^\infty \frac{x^4}{4^x} dx.$ $\quad 4^x = e^t$

$\log 4$

$\int_0^\infty x^4 4^{-x} dx$

put $\quad$ or $4^x = e^t$

$4^{-x} = e^{-t}$

$\log(4^{-x}) = \log(e^{-t})$

$\log(4^{-x}) = -t \log e$

$\log(4^{-x}) = -t$

$+x \log(4) = +t$

$x = \frac{t}{(\log 4)}$ $\quad \Rightarrow dx = \frac{1}{(\log 4)} dt$

when $x = 0$ then $\boxed{t=0}$

$x = \infty$ then $\boxed{t=\infty}$

$\therefore \int_0^\infty \frac{x^4}{4^x} dx = \cancel{\int_0^\infty \frac{(t)^4}{(\log 4)^4} e^{-t} dt}$

$= \cancel{\frac{1}{(\log 4)^4} \int_0^\infty e^{-t} t^4 dt}$

$= \cancel{\frac{1}{4(\log 4)} \Gamma 4+1}$

$= \cancel{\frac{4!}{4(\log 4)}}$

$= \cancel{\frac{4 \Gamma 4}{4 \log 4}}$

$= \cancel{\frac{\Gamma 4}{\log(4)^2}}$

$= \cancel{\frac{4 \times 3 \times 2}{4 \times \log 4}}$

$= \cancel{\frac{3 \times 2}{2(\log 2)}}$

$= 3$

$= \int_0^\infty \left(\frac{t}{(\log 4)}\right)^4 \frac{e^{-t}}{(\log 4)} dt$

$= \int_0^\infty \frac{t^4}{(\log 4)^4} \frac{e^{-t}}{(\log 4)} dt$

$= \int_0^\infty \frac{t^4}{(\log 4)^4} \frac{e^{-t}}{(\log 4)} dt$

$= \int_0^\infty \frac{t^4}{(\log 4)^5} e^{-t} dt$

$= \frac{1}{(\log 4)^5} \Gamma 4+1$

$= \frac{\Gamma 5}{(\log 4)^5} =$

(5) Evaluate $\int_0^\infty \frac{x^a}{a^x}\,dx$

$= \int_0^\infty x^a (a^{-x})\,dx$

put $a^{-x} = e^{-t}$

$\log(a^{-x}) = \log(e^{-t})$

$-x\log a = -t\log e$

$\boxed{x = \frac{t}{\log a}} \Rightarrow dx = \frac{1}{(\log a)}dt$

when $x=0$ then $\boxed{t=0}$

$t=0$ then $\boxed{t=\infty}$

$\therefore \int_0^\infty \frac{x^a}{} a^{-x}\,dx = \int_0^\infty x^a a^{-x}\,dx$

$= \int_0^\infty \frac{(t)^a}{(\log a)^a} e^{-t} \frac{dt}{(\log a)}$

$= \int_0^\infty \frac{(t)^a e^{-t}}{(\log a)^{2a}}\,dt$

$= \frac{1}{2a(\log a)} \int_0^\infty (t)^a e^{-t}\,dt$

$= \frac{1}{2a(\log a)} \Gamma a+1$

$= \frac{\cancel{a}\Gamma a}{2\cancel{a}(\log a)}$

$= \frac{\Gamma a}{2\log a}$

(6) prove that $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$

$+\log x = -t$

$\frac{\cancel{\log x}}{x} = e^{t}$

$$L.H.S = \int_0^1 \frac{dx}{\sqrt{-\log x}}$$

put $\log x = -t$

$$x = e^{-t}$$

$$\boxed{dx = -e^{-t}\, dt}$$

when $x = 0$ then $t = \infty$

$x = 1$ then $t = 0$

$$\therefore \int_{\infty} \frac{1}{\sqrt{-\log x}}\, dx = \int_{\infty}^{0} \frac{1}{\sqrt{-(-t)}} - e^{-t}\, dt$$

$$L.H.S = \int_{\infty}^{0} \frac{1}{\sqrt{t}} -e^{-t}\, dt = -\int_{\infty}^{0} \frac{1}{\sqrt{t}} e^{-t}\, dt = -\int_{\infty}^{0} e^{-t}\, t^{-1/2}\, dt$$

$$= \int_{0}^{\infty} e^{-t}\, t^{-1/2}\, dt \cdots \left\{ \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx \right.$$

$$= \int_0^{\infty} \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

L.H.S = R.H.S

Hence proved

or $\int_0 x^m (1-x)^n dx = B(m+1, n+1)$

Beta functions :

The Beta function is defined as

$$\boxed{\int_0^1 x^{m-1} (1-x)^{n-1} dx = B(m, n)}$$

where $m, n > 0$

$(m-1)+1$ $(n-1)+1$

e.g $\int_0^1 x^4 (1-x)^5 dx = B(5, 6)$

$4+1$ $5+1$

properitties :

(1) ① $B(m, n) = B(n, m)$

(2) Relation b/w **Beta** and **Gamma** function

most ask question for (2 M)

② $B(m, n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n} = \dfrac{(m-1)! \, (n-1)!}{[(m+n)-1]!}$

(3) $\Gamma \frac{1}{2} = \sqrt{\pi}$

e.g $B(3, 4) = \dfrac{\Gamma 3 \, \Gamma 4}{\Gamma 3+4} = \dfrac{(3-1)! \, (4-1)!}{[(3+4)-1]!}$

$= \dfrac{2! \, 3!}{(7-1)!}$

$= \dfrac{2 \times 1 \times 3 \times 2 \times 1}{6!}$

$= \dfrac{\cancel{2} \times \cancel{3} \times \cancel{2} \times 1}{6 \times 5 \times \cancel{4} \times \cancel{3} \times 2}$

$= \dfrac{1}{60}$

$\boxed{\text{Type (I)}}$ $\int_0^a x^m (a-x)^n \, dx = a^{m+n+1} B(m+1, n+1)$

not in form of $(1-x)^n$ or not in std form

therefore we, put $x = at$

(1) Evaluate

$$\int_0^2 x^2 (2-x)^3 \, dx$$

$\Rightarrow$ put $x = 2t$ $\Rightarrow dx = 2\,dt$

when $x=0$ then $t=0$

$x=2$ then $t=1$

$$\therefore \int_0^2 x^2 (2-x)^3 \, dx = \int_0^1 (2t)^2 (2-2t)^3 \, 2 \times dt$$

$$= \int_0^1 2 \times 4 \times t^2 \, (2)^3 (1-t)^3 \, dt$$

$$= 64 \int_0^1 t^2 (1-t)^3 \, dt$$

$$= 64 \, B(3, 4)$$

$$64 \times \frac{\Gamma 3-1 \, \Gamma 4-1}{\Gamma 3+4-1} = \frac{64 \times 2! \times 3!}{6!}$$

$$= \frac{\cancel{64}^{32} \times \cancel{2} \times 1 \times \cancel{3 \times 2}}{6 \times \cancel{4}_{2} \times 5 \times \cancel{3 \times 2}}$$

$$= \frac{32}{6 \times 5}$$

$$= \frac{16}{3 \times 5}$$

$$= \frac{16}{15}$$

$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n)$

(1) Evaluate $\int_0^2 x^4 (8-x^3)^{-1/3}\,dx$

$\Rightarrow$ put $x^3 = 8t$ $\quad x = 2t^{1/3}$

$dx = \frac{2}{3} t^{-2/3}\,dt$

when $x=0$ then $\boxed{t=0}$

$x = 2$ then $\boxed{t=1}$

$$\int_0^2 x^4 (8-x^3)^{-1/3}\,dx = \int_0^1 (2t^{1/3})^4 (8-8t)^{-1/3} \frac{2}{3} t^{-2/3}\,dt$$

$$= \frac{2}{3}\int_0^1 (2)^4 t^{4/3} t^{-2/3} 8^{-1/3} (1-t)^{-1/3}\,dt$$

$$= \frac{2}{3} \times (2)^4 8^{-1/3} \int_0^1 t^{2/3} (1-t)^{-1/3}\,dt$$

$$= \frac{4\times 8}{3}\, 8^{-1/3} \int_0^1 t^{2/3} (1-t)^{-1/3}\,dt$$

$$= \frac{4\times 8^{2/3}}{3} \int_0^1 t^{2/3} (1-t)^{-1/3}\,dt$$

$$= \frac{4\left((2)^3\right)^{2/3}}{3} \int_0^1 t^{2/3} (1-t)^{-1/3}\,dt$$

$$= \frac{16}{3} \int_0^1 t^{2/3} (1-t)^{-1/3}\,dt$$

$$= \frac{16}{3} B\left(\frac{2}{3}+1, -\frac{1}{3}+1\right)$$

$$= \frac{16}{3} B\left(5/3, 2/3\right)$$

$$= \frac{16}{3} \frac{\Gamma 5/3\ \Gamma 2/3}{\Gamma \frac{5}{3}+2/3}$$

$$= \frac{16}{3} \frac{\Gamma \frac{2}{3}+1\ \Gamma 2/3}{\Gamma \frac{7}{3}} = \frac{16\times 2}{3\quad 3} \times \frac{\Gamma 2/3\ \Gamma 2/3}{\Gamma \frac{4}{3}+1}$$

$$= \frac{16}{3} \times \frac{2}{\cancel{3}} \times \frac{4}{\cancel{3}} \frac{\left(\Gamma\frac{2}{3}\right)^2}{\Gamma\frac{4}{3}}$$

$$= \frac{16 \times 2}{3 \times 4} \frac{\left(\Gamma\frac{2}{3}\right)^2}{\Gamma\left(\frac{1}{3}+1\right)}$$

$$= \frac{\overset{4}{\cancel{16}} \times 2}{\cancel{3} \times \cancel{4} \times \frac{1}{\cancel{3}}} \frac{\left(\Gamma\frac{2}{3}\right)^2}{\Gamma\frac{1}{3}}$$

$$= \frac{8\left(\Gamma\frac{2}{3}\right)^2}{\Gamma\frac{1}{3}}$$

10 marks

(2) prove that $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\,dx \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$

$\Rightarrow$ $L.H.S = \int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\,dx \cdot \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$

let $I_1 = \int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\,dx \qquad I_2 = \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$

$I_1 = \int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\,dx$

$= \int_0^3 x^{3/2}(3-x)^{-1/2}\,dx$

put $x = 3t \qquad dx = 3\,dt$

when $x=0$ then $\boxed{t=0}$

$x=3$ then $\boxed{t=1}$

$= \int_0^1 (3t)^{3/2}(3-3t)^{-1/2}\,3\,dt$

$= (3)^{3/2}(3)^{\frac{1}{2}} \int_0^1 t^{3/2}(1-t)^{-1/2}\,dt$

$= (3)^{3/2+\frac{1}{2}}\, B\left(\frac{3}{2}+1, -\frac{1}{2}+1\right)$

$= (3)^{2}\, B\left(\frac{5}{2}, \frac{1}{2}\right)$

$= (3)^{2}\, \frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{6}{2}3}$

$= \frac{(3)^{2}\sqrt{\frac{5}{2}}\,\sqrt{\pi}}{\sqrt{3}}$

$= (3)^2\, \frac{3}{2}\times\frac{1}{2}\,\frac{\sqrt{\pi}}{2\times 2}$

$I_1 = \frac{27\pi}{8}$

$= \cancel{(3)^{5/2}\,\frac{\Gamma\frac{3}{2}+1\,\sqrt{\pi}}{2\times 1}} = (3)^{5/2}\,\frac{3}{2}\,\Gamma\frac{3}{2}\,\sqrt{\pi}$

$= \frac{(3)^{5/2}\,3\,\Gamma\frac{1}{2}+1\,\sqrt{\pi}}{4}$

$= \frac{(3)^{5/2}(3)\,\Gamma\frac{1}{2}\,\sqrt{\pi}}{8}$

$= \frac{(3)^{5/2}\sqrt{\pi}\sqrt{\pi}}{8}$

$I_1 = \frac{27\pi}{8}$

$I_2 = \int_0^1 \frac{1}{\sqrt{1-x^{1/4}}}\,dx$

$= \int_0^1 (1-x^{1/4})^{-1/2}\,dx$

$x^{1/4} = t$

$x = t^4$

$dx = 4t^3\,dt$

when $x=0$ then $\boxed{t=0}$

$x=1$ then $\boxed{t=1}$

$= \int_0^1 (1-t)^{-1/2}\,4t^3\,dt$

$= 4\int_0^1 t^3(1-t)^{-1/2}\,dt$

$= 4\; B\left(3+1, -\frac{1}{2}+1\right) = 4\,B\left(4, \frac{1}{2}\right)$

$= 4\,\frac{\Gamma 4\;\Gamma\frac{1}{2}}{\Gamma 4+\frac{1}{2}}$

$= 4\,\frac{\Gamma 4\;\sqrt{\pi}}{\Gamma\frac{9}{2}}$

$= \frac{4\times 3!\,\sqrt{\pi}}{\Gamma\frac{7}{2}+1} = \frac{4\times 3\times 2\times\sqrt{\pi}}{\frac{7}{2}\,\Gamma\frac{7}{2}} = \frac{4\times 3\times 2\sqrt{\pi}}{\frac{7}{2}\times\Gamma\frac{5}{2}+1}$

$= \frac{4\times 3\times 2\,\sqrt{\pi}}{\frac{7}{2}\times\frac{5}{2}\,\Gamma\frac{5}{2}} = \frac{4\times 3\times 2\,\sqrt{\pi}}{\frac{7}{2}\times\frac{5}{2}\times\Gamma\frac{3}{2}+1} = \frac{4\times 3\times 2\,\sqrt{\pi}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\,\Gamma\frac{3}{2}} = \frac{4\times 3\times 2\,\sqrt{\pi}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\,\Gamma\frac{1}{2}+1}$

$= \frac{4\times 3\times 2\times\sqrt{\pi}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\,\Gamma\frac{1}{2}} = \frac{4\times\cancel{3}\times 2\times\cancel{\sqrt{\pi}}}{\frac{7}{2}\times\frac{5}{2}\times\frac{\cancel{3}}{2}\times\frac{1}{2}\,\cancel{\sqrt{\pi}}} = \frac{8\times(2)^4}{35}$

$L.H.S = I_1 \cdot I_2 = \frac{27\pi}{\cancel{8}}\times\frac{\cancel{8}\times(2)^4}{35} = \frac{27\times 16\,\pi}{35} = \frac{432\pi}{35}$

10 marks

(3) Show that $\int_0^1 \sqrt{1-\sqrt{x}}\,dx \int_0^{1/2} \sqrt{2y-(2y)^2}\,dy = \frac{\pi}{30}$

$\Rightarrow$ L.H.S $= \int_0^1 \sqrt{1-\sqrt{x}}\,dx \cdot \int_0^{1/2} \sqrt{2y-(2y)^2}\,dy$

let $I_1 = \int_0^1 \sqrt{1-\sqrt{x}}\,dx$ $\quad I_2 = \int_0^{1/2} \sqrt{2y-(2y)^2}\,dy$

L.H.S $= (I_1)\cdot(I_2)$

$I_1 = \int_0^1 \sqrt{1-(x)^{1/2}}\,dx$

$= \int_0^1 (1-(x)^{1/2})^{1/2}\,dx$

let $(x)^{1/2} = t$

$x = t^2$ $\quad dx = 2t\,dt$

when $x = 0$ then $\boxed{t=0}$

$x = 1$ then $\boxed{t=1}$

$= \int_0^1 (1-t)^{1/2}\, 2t\,dt$

$= 2\int_0^1 t\,(1-t)^{1/2}\,dt$

$= 2\,B(1+1, \frac{1}{2}+1)$

$= 2\,B(2, 3/2)$

$= 2\,\frac{\Gamma 2\,\Gamma\frac{3}{2}}{\Gamma 2+\frac{3}{2}}$ $= 2\,\frac{1!\,\Gamma\frac{1}{2}+1}{\Gamma\frac{7}{2}}$

$= \frac{2\times\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{7}{2}}$

$I_1 = \frac{\sqrt{\pi}}{\Gamma\frac{7}{2}}$

$I_2 = \int_0^{1/2} (2y-(2y)^2)^{1/2}\,dy$

$= \int_0^{1/2} (2y)^{1/2}(1-2y)^{1/2}\,dy$

put $2y = t$ $\quad y = 1/2\,t$ $\quad dy = \frac{1}{2}\,dt$

when $y=0$ then $\boxed{t=0}$

$y=\frac{1}{2}$ then $\boxed{t=1}$

$= \int_0^1 t\,(1-t)^{1/2}\,\frac{1}{2}\,dt$

$= \frac{1}{2}\int_0^1 (t)^{1/2}(1-t)^{1/2}\,dt$ $\quad = \frac{1}{2}\,B(\frac{1}{2}+1, \frac{1}{2}+1)$

$= \frac{1}{2}\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma\frac{3}{2}+\frac{3}{2}}$

$= \frac{1}{2}\,\frac{\times\frac{1}{2}\Gamma\frac{1}{2}\;\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{6}{2}3}$

$= \frac{1}{8}\,\frac{\sqrt{\pi}\,\sqrt{\pi}}{\Gamma 3}$

$I_2 = \frac{\pi}{8\times 2!}$

L.H.S $= (I_1)\cdot(I_2)$

L.H.S $= \frac{\sqrt{\pi}}{\Gamma\frac{7}{2}} \cdot \frac{\pi}{16}$

$= \frac{\sqrt{\pi}}{\frac{5}{2}\,\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} \cdot \frac{\pi}{8\times 2}$

$= \frac{\pi}{15\times 2}$ $= \frac{\pi}{30}$

Beta function continued...

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m,n)$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

most ask question for 2 Marks ISE / ESE

derival

$$2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} \, d\theta = B(m,n)$$

most imp smoothly *** asked in FA

$(5+1) \to 6$

(2) prove that $\int_0^\infty \frac{x^5}{(2+3x)^{16}}\,dx = \frac{5!\,9!}{2^{10}\,3^6\,15!}$

put $x = \frac{2}{3}t \Rightarrow dx = \frac{2}{3}dt$

when $x=0$ then $\boxed{t=0}$

$x=\infty$ then $\boxed{t=\infty}$

$$L.H.S = \int_0^\infty \frac{x^5}{(2+3x)^{16}}\,dx = \int_0^\infty \frac{(2/3\,t)^5}{(2+\frac{3\times 2}{3}t)^{16}}\,\frac{2}{3}dt$$

$$= \frac{2^5}{3^5}\,\frac{2}{3}\int_0^\infty \frac{t^5}{(2+2t)^{16}}\,dt = \frac{2^6}{3^6}\int_0^\infty \frac{t^5}{2^{16}(1+t)^{16}}\,dt$$

$$= \frac{2^6}{3^6}\times\frac{1}{2^{16}}\int_0^\infty \frac{t^5}{(1+t)^{16}}\,dt = \frac{1}{3^6 2^{10}}\int_0^\infty \frac{t^{6-1}}{(1+t)^{6+10}}$$

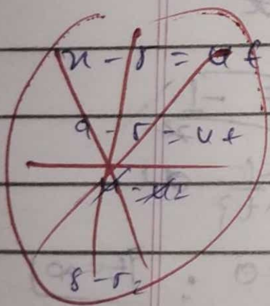
$$= \frac{1}{3^6\,2^{10}}\,B(\overset{5+1}{6}, 10)$$

$\Gamma n = (n-1)!$

$$= \frac{1}{3^6 2^{10}}\,\frac{\Gamma 6\,\Gamma 10}{\Gamma 6+10} = \frac{5!\,9!}{3^6\,2^{10}\,15!}$$

L.H.S = R.H.S

Hence proved



$\int_0^1 x^{m-1}(1-x)^{n-1}$

$\int_0^\infty \frac{x^{m-1}}{x^{m+n}} =$

$\frac{1}{2}-1 \quad \frac{1}{2}+\frac{3}{2} \quad \frac{1}{2} \quad 1\frac{1}{2}$

Hint :- $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m,n)$ → predict by question

$\int_0^\infty \frac{x^{k}}{(1+x)^{k'}} dx = B(m+1, n+1)$

Eg :-

(1) $\int_0^\infty \frac{\sqrt{x}}{1+2x+x^2} dx = \int_0^\infty \frac{(x)^{1/2}}{(1+x)^2} dx = \int_0^\infty \frac{(x)^{3/2-1}}{(1+x)^{\frac{1}{2}+\frac{3}{2}}} dx$

$= B\left(\frac{3}{2}, \frac{1}{2}\right)$

$= \Gamma\frac{3}{2}\,\Gamma\frac{1}{2}$

$(9-x)(x-5)$

$= \Gamma\left(\frac{1}{2}+1\right) \sqrt{\pi}$

$= \frac{1}{2}\Gamma\frac{1}{2}\sqrt{\pi} = \frac{1}{2}\sqrt{\pi}\sqrt{\pi} = \frac{\pi}{2}$

5 marks

(3) Evaluate $\int_5^9 4\sqrt{(9-x)(x-5)}\, dx$ $\qquad \int_0^1 \frac{x^{m-1}}{x^{(m+n)}}$

$x-5 = 4+t$; $9-5 = 4$; $9-4 = t$

put $x = 9t$ ~~→ don't do this~~

$x-5 = 4t$ ✓

Note : General $\left\{ \int_a^b f(x) \Rightarrow \text{put } x-a = (b-a)\times t \right.$

$\therefore \int_5^9 4\sqrt{(9-x)(x-5)}\, dx = \int_0^1 4\sqrt{\quad}$ → $dx = 4\,dt$

when $x=5$; $t=0$

$x=9$; $t=1$

$(9-(4t+5))\,(5+4t-5)$ $\quad 4dt$

$= 4\int_0^1 \sqrt{(4-4t)\;4t}$ $\qquad = 4\int_0^1 (4(1-t)\,4t)^{1/4} dx$

$= \int_0^1$ $\qquad \frac{(\Gamma\frac{1}{4})^2}{8^2 \ldots} \ldots \sqrt{\pi}$ $\qquad = 4\int_0^1 (16)^{1/4} t^{1/4} (1-t)^{1/4} dt$

$= 4\int_0^1 2\, t^{1/4} (1-t)^{1/4} dt$

$= 8 \frac{\Gamma\frac{5}{4}\,\Gamma\frac{5}{4}}{\Gamma\frac{10}{4}\frac{5}{2}} = 8\frac{\Gamma\left(\frac{1}{4}+1\right)\Gamma\left(\frac{1}{4}+1\right)}{\Gamma\left(\frac{3}{2}+1\right)}$ $\qquad = 8\int_0^1 t^{1/4}(1-t)^{1/4} dt$

$= 8\, B\left(\frac{1}{4}+1, \frac{1}{4}+1\right)$

$= 8^2 \frac{\frac{1}{4}\Gamma\frac{1}{4}\,\frac{1}{4}\Gamma\frac{1}{4}}{\frac{3}{2}\Gamma\frac{3}{2}}$ $\qquad = 8\, B\left(\frac{5}{4}, \frac{5}{4}\right)$

$= \frac{1}{3}\frac{(\Gamma\frac{1}{4})^2}{\Gamma\left(\frac{1}{2}+1\right)}$ → $\frac{1}{3\times\frac{1}{2}\sqrt{\pi}}\left(\Gamma\frac{1}{4}\right)^2 = \frac{2}{3}\frac{(\Gamma\frac{1}{4})^2}{\sqrt{\pi}}$

10 marks ★★★

(5) prove that $B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$

Hence find $\int_0^1 \frac{x^2+x^3}{(1+x)^7}\, dx$

$\Rightarrow$ $R.H.S = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$

$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$

$= I_1 + I_2 \quad \cdots (1)$

Consider $I_2 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$

put $x = \frac{1}{t}$ $\quad$ ~~$t = \frac{1}{x}$~~

~~$dt = \frac{-1}{x}$~~

$\Rightarrow dx = \frac{-1}{t^2}\, dt$

when $x = 0$; $\boxed{t=\infty}$

$x = 1$; $\boxed{t=1}$

$I_2 = \int_\infty^1 \frac{\left(\frac{1}{t}\right)^{n-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(-\frac{1}{t^2}\right) dt$

$= -\int_\infty^1 \frac{1}{t^2} \frac{\left(\frac{1}{t}\right)^{n-1}}{\left(\frac{t+1}{t}\right)^{m+n}}\, dt$

$= \int_1^\infty \frac{\frac{1}{t^{n-1}} \frac{1}{t^2}\, t^{m+n}}{(1+t)^{m+n}}\, dt = \int_1^\infty \frac{t^{\cancel{-n}+1-2+m+\cancel{n}}}{(1+t)^{m+n}}$

$= \int_1^\infty \frac{t^{-n+1}\, t^{-2}\, t^{m+n}}{(1+t)^{m+n}}\, dt \qquad I_2 = \int_1^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt \quad \cdots (1)$

=

eq (1) become

$R.H.S = I_1 + I_2$

$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt$

$= \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}} = B(m,n)$

$= L.H.S$

$\therefore \int_0^1 \frac{x^2+x^3}{(1+x)^7}\, dx = \int_0^1 \frac{x^{3-1} + x^{4-1}}{(1+x)^{3+4}}\, dx$

$= B(3,4)$ ... By above result

$= \frac{\Gamma 3\, \Gamma 4}{\Gamma 7} = \frac{2!\, 3!}{6!} = \frac{2! \times \cancel{3!}}{6 \times 5 \times 4 \times \cancel{3!}}$

$= \frac{\cancel{2}}{30 \times \cancel{4}\,2} = \frac{1}{60}$

$x = 1/t$

Duplication formula on Beta function :

For $m>0$, $\boxed{\Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \dfrac{\Gamma 2m \sqrt{\pi}}{2^{(2m-1)}}}$

e.g $m = \frac{1}{4}$ $\Rightarrow$ $m + \frac{1}{2} = \frac{3}{4}$

$$\Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \frac{\Gamma 2m \sqrt{\pi}}{2^{2m-1}}$$

$$\Gamma \frac{1}{4} \, \Gamma \frac{3}{4} = \frac{\Gamma\left(2 \cdot \frac{1}{4}\right) \sqrt{\pi}}{2^{2 \times \frac{1}{4} - 1}} = \frac{\Gamma \frac{1}{2} \sqrt{\pi}}{2^{-1/2}}$$

$$= \sqrt{\pi}\sqrt{\pi} \; 2^{1/2}$$

$$= \sqrt{2}\,\pi$$

Hint $\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\Gamma 2m \sqrt{\pi}}{2^{(2m-1)}}$, $\Gamma m+\frac{1}{2} = \frac{\Gamma 2m \sqrt{\pi}}{2^{(2m-1)} \Gamma m}$

(1) prove that $B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{1}{2^{2n}} \frac{\Gamma n+\frac{1}{2} \sqrt{\pi}}{\Gamma n+1}$

$\Rightarrow \frac{\Gamma n+\frac{1}{2} \, \Gamma n+\frac{1}{2}}{\Gamma 2n+1} = \frac{\Gamma n+\frac{1}{2}}{\Gamma 2n+1} \times \frac{\Gamma 2n \sqrt{\pi}}{2^{(2n-1)} \Gamma n}$

$= \frac{\Gamma n+\frac{1}{2} \, \cancel{\Gamma 2n} \sqrt{\pi}}{2n \, \cancel{\Gamma 2n} \, 2^{(2n-1)} \Gamma n}$ ... $\{\Gamma n+1 = n \Gamma n\}$

$= \frac{\Gamma n+\frac{1}{2} \sqrt{\pi}}{\cancel{2} n \, 2^{(2n)} \, \cancel{2^{1}} \, \Gamma n} = \frac{\Gamma n+\frac{1}{2} \sqrt{\pi}}{(2)^{2n} n \Gamma n}$ ...

$= \frac{\sqrt{\pi} \, \Gamma n+\frac{1}{2}}{(2)^{2n} \Gamma n+1}$ ... $\{\Gamma n+1 = n \Gamma n\}$

Imp

(2) If $B(n, 3) = \frac{1}{105}$ and n is a +ve integer, find n

$\frac{\Gamma n \, \Gamma 3}{\Gamma n+3} = \frac{1}{105}$

$\frac{\Gamma n \; 2!}{\Gamma (n+2)+1} = \frac{1}{105}$ $\Rightarrow \frac{\Gamma n \; 2!}{(n+2) \Gamma n+2} = \frac{1}{105}$

$\Rightarrow \frac{\Gamma n \; 2!}{(n+2) \Gamma (n+1)+1} = \frac{1}{105} = \frac{\Gamma n \; 2!}{(n+2)(n+1) \Gamma (n+1)}$

$\frac{1}{105} = \frac{\cancel{\Gamma n} \; 2!}{(n+2)(n+1) \, n \, \cancel{\Gamma n}}$

$(n+2)(n+1) n = 105 \times 2$

$(n+2)(n+1) n = 210$

$\boxed{n = 5}$

(2) Show that $\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\, dx = \frac{a\sqrt{\pi}\,\Gamma\frac{5}{6}}{\Gamma\frac{1}{3}}$

Put $x^3 = a^3 t$ $\quad x = a t^{1/3}$ $\quad dx = \frac{1}{3} a t^{-2/3} dt$

when $x=0$; $\boxed{t=0}$

$x=a$; $\boxed{t=1}$

$$R.H.S \int_0^a \sqrt{\frac{x^3}{a^3-x^3}} = \frac{a}{3}\int_0^1 t^{-2/3}\sqrt{\frac{a^3 t}{a^3-a^3 t}}\, dt = \frac{a}{3}\int_0^1 \sqrt{\frac{\cancel{a^3} t}{\cancel{a^3}(1-t)}}\, t^{-2/3} dt$$

$$= \frac{a}{3}\int_0^1 \frac{t^{1/2}\, t^{-2/3}}{(1-t)^{1/2}}\, dt = \frac{a}{3}\int_0^1 t^{-\frac{1}{6}} (1-t)^{-1/2} dt$$

$$= \frac{a}{3} B\left(-\frac{1}{6}+1, -\frac{1}{2}+1\right)$$

$\int_0^1 x^{m-1}(1-x)^{n-1} dx = B(m,n)$

$$= \frac{a}{3} B\left(\frac{5}{6}, \frac{1}{2}\right)$$

$$\frac{a}{3} \frac{\Gamma\frac{5}{6}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{5}{6}+\frac{1}{2}\right)} = \frac{a}{3}\frac{\Gamma\frac{5}{6}\sqrt{\pi}}{\Gamma\frac{8}{6}}$$

$$= \frac{a}{3}\frac{\Gamma\frac{5}{6}\sqrt{\pi}}{\Gamma\frac{4}{3}} = \frac{a}{3}\frac{\Gamma\frac{5}{6}\sqrt{\pi}}{\Gamma\left(\frac{1}{3}+1\right)} = \frac{a}{\cancel{3}}\frac{\Gamma\frac{5}{6}\sqrt{\pi}}{\frac{1}{\cancel{3}}\Gamma\frac{1}{3}}$$

$$= \frac{a\sqrt{\pi}\,\Gamma\frac{5}{6}}{\Gamma\frac{1}{3}}$$

(3) Evaluate $\int_0^{\pi/2} \sqrt{\sin x}\, dx \cdot \int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\, dx$

let $I = \int_0^{\pi/2} \sqrt{\sin x}\, dx \cdot \int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\, dx$

$I = I_1 \cdot I_2$

$I_1 = \int_0^{\pi/2} (\sin x)^{1/2}\, dx = \int_0^{\pi/2} (\sin x)^{1/2} (\cos x)^0\, dx$

$= \frac{1}{2} B\left(\frac{\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)$

$= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{2}\right)$

$I_2 = \int_0^{\pi/2} \sin^{-1/2} x\, dx = \int_0^{\pi/2} (\sin x)^{-1/2} \cos^0 x\, dx = \frac{1}{2} B\left(\frac{-\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)$

$= \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right)$

$I = I_1 \cdot I_2 = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{2}\right) \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{2}\right)$

$= \frac{1}{4} \frac{\cancel{\Gamma\frac{3}{4}}\, \Gamma\frac{1}{2}}{\Gamma\frac{5}{4}} \cdot \frac{\Gamma\frac{1}{4}\, \Gamma\frac{1}{2}}{\cancel{\Gamma\frac{3}{4}}}$

$= \frac{1}{4} \frac{\Gamma\frac{1}{4} \left(\Gamma\frac{1}{2}\right)^2}{\Gamma\frac{5}{4}} = \frac{1}{4} \frac{\Gamma\frac{1}{4} (\sqrt{\pi})^2}{\Gamma\left(\frac{1}{4}+1\right)} = \frac{\cancel{\Gamma\frac{1}{4}}\, \pi}{\frac{1}{4}\, \cancel{\Gamma\frac{1}{4}}} = \frac{1}{\cancel{4}} \frac{\pi}{\cancel{4}}$

$= \cancel{\frac{\pi}{16}}$

$= \pi$

$1/2$

(4) Evaluate $\int_0^{\pi/6} \cos^6 3\theta \sin^2 6\theta \, d\theta$

put $3\theta = t$ $\qquad d\theta = \frac{1}{3} dt$

when $\theta = 0$; $\boxed{t=0}$

$\theta = \pi/6$; $\boxed{t = \pi/2}$

$$\therefore \int_0^{\pi/6} \cos^6 3\theta \sin^2 6\theta \, d\theta = \int_0^{\pi/2} \cos^6 t \sin^2 2t \, \frac{1}{3} dt$$

$$= \int_0^{\pi/2} \cos^6 t (\sin 2t)^2 \frac{1}{3} dt = \frac{1}{3} \int_0^{\pi/2} \cos^6 t (2\sin t \cos t)^2 dt$$

$$= \frac{1}{3} \int_0^{\pi/2} (2)^2 \cos^6 t \cdot \sin^2 t \cos^2 t \, dt = \frac{4}{3} \int_0^{\pi/2} \sin^2 t \cos^8 t \, dt$$

$$= \frac{4}{3}\left[\frac{1}{2} B\left(\frac{2+1}{2}, \frac{8+1}{2}\right)\right] = \frac{4}{3} \times \frac{1}{2} B(3/2, 9/2)$$

$$= \frac{2}{3} \frac{\Gamma 3/2 \, \Gamma \frac{9}{2}}{\Gamma \frac{12}{2}}$$

or $$= \frac{2}{3} \frac{\Gamma \frac{1}{2}+1 \; \Gamma \frac{7}{2}+1}{\Gamma 6} = \frac{2}{3} \frac{\frac{1}{2}\Gamma\frac{1}{2} \; \frac{7}{2}\Gamma \frac{7}{2}}{5!}$$

$$= \frac{2}{3}\left(\frac{\frac{1}{2}\sqrt{\pi} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{8 \times 4 \times 3 \times 2 \times 1}\right) = \frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}} \frac{\sqrt{\pi}}{5!} \; \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{\cancel{3}}{2} \cdot \frac{1}{2}\sqrt{\pi}$$

$$\left(= \frac{7 \cdot \pi}{3 (2)^7}\right)$$

$$= \frac{\sqrt{\pi}\sqrt{\pi} \times 7 \times 5}{2 \times 2 \times 2 \times 2 \times 5!} = \frac{\pi \times 7 \times \cancel{5}}{2 \times 2 \times 2 \times 2 \times \cancel{5} \times 4 \times 3 \times 2}$$

$$= \frac{7 \times \pi}{3 (2)^7}$$

5marks ★★★ most Imp

$$\int_0^{2a} f(x) = 2\int_0^{a} f(2a-x)\,dx$$

(5) Calculate $\int_0^{\pi} \sin^2\theta\,(1-\cos^3\theta)\,d\theta$

$$\int_0^{\pi} \sin^2\theta\,(1-\cos^3\theta) = 2\int_0^{\pi/2} \sin^2(\pi-\theta)\,[1-\cos^3(\pi-\theta)]\,d\theta$$

Hint ....using $\int_0^{2a} f(x)dx = 2\int_0^{a} f(2a-x)dx$

$$= 2\int_0^{\pi/2} \sin^2\theta\,[1-(-\cos\theta)^3]\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^2\theta\,[1-(-\cos^3\theta)]\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^2\theta\,(1+\cos^3\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} [\sin^2\theta + \sin^2\theta\cos^3\theta]\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^2\theta\,\cancel{d\theta} + 2\int_0^{\pi/2} \sin^2\theta\cos^3\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^2\theta\cos^0\theta\,d\theta + 2\int_0^{\pi/2} \sin^2\theta\cos^3\theta\,d\theta$$

$$= 2\frac{1}{2}B\left(\frac{2+1}{2}, \frac{0+1}{2}\right) + 2\frac{1}{2}B\left(\frac{2+1}{2}, \frac{3+1}{2}\right)$$

$$= 2\frac{1}{2}B(3/2, 1/2) + 2\frac{1}{2}B(3/2, \cancel{4}/\cancel{2})$$

$$= 2\frac{1}{2}B(3/2, 1/2) + 2\frac{1}{2}B(3/2, 2)$$

$$= 2\frac{1}{2}\frac{\Gamma\frac{3}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)} + 2\frac{1}{2}\frac{\Gamma 3/2\,\Gamma 2}{\Gamma\left(\frac{3}{2}+2\right)}$$

$$= \cancel{2}\frac{1}{\cancel{2}}\frac{\cancel{\tfrac{1}{2}}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 2} + \cancel{2}\frac{1}{\cancel{2}}\frac{\cancel{\tfrac{1}{2}}\Gamma\frac{1}{2}\Gamma 2}{\Gamma\frac{7}{2}}$$

$$= \frac{\pi}{2} + \frac{\cancel{\Gamma\tfrac{1}{2}}}{2\times 5/2\; 3/2\; \frac{1}{2}\;\cancel{\Gamma\tfrac{1}{2}}}$$

$$= \frac{\pi}{2} + \frac{8}{15}\times\frac{1}{\cancel{2}}$$

$$= \frac{\pi}{2} + \frac{4}{15}$$

$$= \frac{15\pi+8}{30}$$