Self-Evaluation

Name of student: **Class & Div:** **Roll No:**

1. Do you understand how to evaluate complex line Integral?

(a) Yes (b) No

2. Will you be able to Identify the singularity?

(a) Yes (b) No

3. Are you able to Expand in Taylors and Laurent series of complex valued function?

(a) Yes (b) No

4. Are you able to change to evaluate Residues of function?

(a) Yes (b) No

5. Do you understand how to real integral by Residue theorem?

(a) Yes (b) No

6. Do you understood this module?

(a) Fully understood (b) Partially understood (c) Not at all

sources:

Self-Assessment

Level 1

1. Evaluate by Cauchy's integral formula $\int_c \frac{3z-1}{(z^2-2z-3)}dz$ where C is the circle $|z| = 4$
2. Expand in Taylor's series $f(z) = \frac{1}{(z-2)}$ where $|z| < 2$
3. Evaluate $\int_c \frac{Sin\,\pi z^2 + cos\pi z^2}{(z^2-3z+2)}dz$ where c is the circle $|z| = 3$
4. Expand Sinz in powers of $(z - \frac{\pi}{4})$.
5. Evaluate $\int_c \frac{1}{(z^2+1)(z^2+4)}dz$ where c is the circle $|z| = 3/4$

Level 2

1. Evaluate $\int_{|z-1|=2} \frac{ze^z}{(z-1)^3}dz$
2. Expand $\frac{1}{z^2-3z+2}$ in the region

   (i) $0 < |z| < 1$ (ii) $1 < |z| < 2$ (iii) $2 < |z|$
3. Find all zeros and poles of $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$
4. Which of the poles of $f(z) = \frac{1}{(z^2+1)}$ lies in the upper half of the z-plane
5. Classify the poles of $f(z) = \frac{1}{z^3(z^2+4)}$

Level 3

1. Find all zeros and poles of $\frac{CosZ}{(z^2+1)}$.
2. Evaluate: $\int_c \frac{3z^2+2}{(z-1)(z^2+9)}dz$ by Cauchy's Residue Theorem, where C is

   i) the circle $|z-2| = 2$

   ii) the circle $|z| = 4$
3. Evaluate $\int_0^{2\pi} \frac{d\theta}{(cos\theta+2)^2}$
4. Evaluate $\int_{-\infty}^{+\infty} \frac{x^2-x+2}{x^4+10x^2+9}dx$

Add to Knowledge:

Integrals are extremely important in the study of functions of a complex variable mainly for two reasons. Some properties of analytic functions can be proved by complex integration easily. For instance, the existence of higher derivatives of analytic functions. Secondly in applications real integrals occur which cannot be evaluated by usual methods but can be evaluated by complex integration.

We know that definite integral of a real function is defined on an interval of the real line. But integral of a complex valued function of a complex variable is defined on a curve or arc in the complex plane. A complex definite integral is called a (complex) line integral.

Learning Outcomes

1. **Know:** Student should be able to evaluate integral of complex variable
2. **Comprehend:** Learners should be able to evaluate different types of Complex integrals
3. **Apply, analyze and synthesize:** Student should be able to solve complex line integrals

by the method of residues.

Solution:
The integrand has a simple pole at s = -1 and a double pole at s = 2.

The residue at the simple pole s = -1 is

$$\lim_{s\to -1}(s+1)\left\{\frac{e^{st}}{(s+1)(s-2)^2}\right\} = \frac{1}{9}e^{-t}$$

The residue at the double pole s = 2 is

$$\lim_{s\to 2}\frac{1}{1!}\frac{d}{ds}\left[(s-2)^2\left\{\frac{e^{st}}{(s+1)(s-2)^2}\right\}\right] = \lim_{s\to 2}\frac{d}{ds}\left[\frac{e^{st}}{(s+1)}\right] = \lim_{s\to 2}\frac{(s+1)te^{st}-e^{st}}{(s+1)^2}$$

$$= \frac{1}{3}te^{2t} - \frac{1}{9}e^{2t}$$

Thus

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{e^{st}\,ds}{(s+1)(s-2)^2} = \sum residues = \frac{1}{9}e^{-t} + \frac{1}{3}te^{2t} - \frac{1}{9}e^{2t}$$

Exercise 45

1. Evaluate $\int_{-\infty}^{+\infty}\frac{1}{1+x^6}dx$

2. Evaluate $\int_{-\infty}^{+\infty}\frac{1}{1+x^4}dx$

Let's check take away from the lecture

1. The integration on the circular part of Bromwich contour, when $R\to\infty$ is
   (a) 0 (b) 1 (c) 2 (d) ∞

2. The Residue of $f(z)=\frac{1+z}{z^2-2z^4}$ at the pole of order 2 is
   (a) 1 (b) -1 (c) 2 (d) 0

Homework Problems for the day

1. Evaluate $\int_{-\infty}^{\infty}\frac{1}{1+x^8}dx$

2. Evaluate $\int_{-\infty}^{\infty}\frac{x^2}{1+x^6}dx$

Evaluation of certain Improper Integration Using Bromwich Contour (Self-Study)

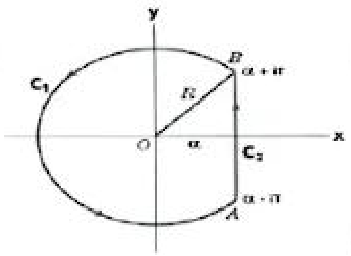
Lecture: 45

1. Learning Objective: Student will be able identify the Bromwich contour and apply it to solve real integral.

2. Introduction: Contour Integral can be applied for solving the definite integral. Bromwich contour can be applied for solving the definite integral .

3. Key Definitions:

1) **The Bromwich contours.** The simple closed curve about which the integration is performed in evaluating formula as shown in Fig. and is called the Bromwich contour. The curve C consists of two parts, C_1 and C_2, as shown in the figure. C_1 is the portion of a circle of radius R, centered at the origin, shown in the figure. C_2 is the vertical line AB located at a distane α to the right of the origin. Integration takes place in the counterclockwise direction on a limiting case of the curve shown in which the radius R can approach infinity. The integration corresponding to formula 1) takes place along the C_2 portion of the curve.



According to the residue theorem

$$\frac{1}{2\pi i}\oint_{C_1} e^{st} f(s)\, ds + \frac{1}{2\pi i}\oint_{C_2} e^{st} f(s)\, ds$$

= sum of residues of all isolated singular points inside C (i)

As R $\rightarrow \infty$, the curve C will encompass all isolated singular points. In Fig, points A and B have the complex coordinates α - iT and α + iT, respectively, where

$$T = \sqrt{R^2 - \alpha^2}$$

Thus (i) becomes

$$\lim_{R\to\infty}\frac{1}{2\pi i}\oint_{C_1} e^{st} f(s)\, ds + \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\alpha - iT}^{\alpha + iT} e^{st} f(s)\, ds$$

= sum of residues of all isolated singular points inside C

4. Sample Problem:

1. Evaluate

$$\frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty}\frac{e^{st}\, ds}{(s+1)(s-2)^2}$$

$$= \lim_{z \to ai} (z-ai) \cdot \frac{z^2}{(z-ai)(z+ai)(z^2+b^2)}$$

$$= \frac{-a^2}{2ai(-a^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

Similarly, Residue (at $z = bi$)

$$= \frac{-b^2}{2bi(a^2-b^2)} = \frac{-b}{2i(a^2-b^2)}$$

(v) Hence,

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = 2\pi i \left[\frac{a}{2i(a^2-b^2)} + \frac{-b}{2i(a^2-b^2)} \right]$$

$$= \frac{\pi}{a+b}$$

Exercise 44

1.Prove that $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$

2.Show that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = \frac{\pi(a+2b)}{2ab^3(a+b)^2}$ $a, b > 0$

Let's check take away from the lecture

1. The requirement for of application of counter Integration $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ is degree of P(x) is greater than Q(x) by

(a) 0 (b) 1 (c) 2 (d) None of these

2. $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$ to solve the contour integral is root of Q(x) are

(a) real (b) complex (c) real or complex (d) None of these

Homework Problems for the day

1. Show that $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3}$

2. Show that $\int_0^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{\pi}{8a^2}$

Learning from the topic: Students will be able to solve problems of definite integral using contour integral.

3. Show that $\int_0^{2\pi} \frac{\sin^2\theta d\theta}{a+b\cos\theta} = \frac{2\pi}{b^2}(a-\sqrt{a^2-b^2})\ (a>b>0)$

Let's check take away from the lecture

1. The residue of f(z)=cot z at each pole is

(a) 0 (b) 1 (c) ½ (d) none

2. Evaluate $\int_c \frac{e^z}{z+1} dz$ where c is the circle $|z|=2$

(a) $\frac{2\pi i}{e}$ (b) $2\pi e$ (c) 0 (d) 1

Homework Problems for the day

1. Evaluate $\int_0^{2\pi} \frac{\sin\theta}{5+4\cos\theta} d\theta$

2. Show that $\int_0^{2\pi} \frac{\sin^2\theta d\theta}{5+4\cos\theta} = \frac{\pi}{4}$

3. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta} d\theta$

Learning from the topic: Students will be able to solve problems by using Residues Theorem.

EVALUATION OF INDEFINITE INTEGRAL

Lecture: 44

1. Learning Objective: Students shall be able to understand definite integral to solve problem.

2. Introduction: This topic is used to solve problems of definite integral contains the algebraic function.

3. Sample Problem:

(1.). Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx, a>0, b>0$

Solution:

(i) Consider as the contour consisting of a semi-circle and diameter on the real axis with center at the origin.

(ii) Now $zf(z) = \frac{z^3}{(z^2+a^2)(z^2+b^2)}$

(iii) Now $\left(z^2+a^2\right)\left(z^2+b^2\right)=0\ i.e.\ z=ai, -ai, bi, -bi$. Of these $z=ai, z=bi$ lie in the upper half of the z-plane.

1. Learning Objective: Students shall be able to understand definite integral to solve problem.

2. Introduction: Cauchy's residue theorem can be utilized to evaluate the real integral. The integral which contains the sine and cosine function and need to be evaluated on the interval $[0,2\pi]$ can be solved with the help of Cauchy's residue theorem.

3. Sample Problem:

1). Using residue theorem evaluate $\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$

Solution: Let $e^{i\theta} = z$,

$$ie^{i\theta} d\theta = dz \quad \therefore d\theta = \frac{dz}{iz} \; and \; \cos\theta = \frac{z+z^{-1}}{2}$$

$$\therefore I = \int_C \frac{1}{\left(2+\frac{z^2+1}{2z}\right)^2} \cdot \frac{dz}{iz} = \frac{4}{i} \cdot \int_C \frac{zdz}{\left(z^2+4z+1\right)^2}$$

Where C is the circle $|z| = 1$.

Now the poles of $f(z)$ are given by $z^2+4z+1=0$.

$$\therefore z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

Let $\alpha = -2-\sqrt{3}$ and $\beta = -2+\sqrt{3}$

Both are poles of order 2. But the pole α lies outside the circle and the pole β lies inside the circle.

$$\because f(z) = \frac{z}{\left(z^2+4z+1\right)^2} = \frac{z}{(z-\alpha)^2 (z-\beta)^2}$$

Residue (at $z = \beta$)

$$= \lim_{z\to\beta} \frac{1}{1!} \frac{d}{dz} \left[(z-\beta)^2 \cdot \frac{z}{(z-\alpha)^2 (z-\beta)^2} \right]$$

$$= \lim_{z\to\beta} \frac{d}{dz} \left[\frac{z}{(z-\alpha)^2} \right] = \lim_{z\to\beta} \frac{-z-\alpha}{(z-\alpha)^3} = -\frac{\beta+\alpha}{(\beta-\alpha)^3}$$

But $\beta+\alpha = -4$ and $\beta-\alpha = 2\sqrt{3}$

Residue (at $z = \beta$) $= \frac{4}{24\sqrt{3}} = \frac{1}{6\sqrt{3}}$

$$\therefore I = 2\pi i \cdot \frac{4}{i} \cdot \frac{1}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}$$

Exercise 43

1. Evaluate the following integral $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \frac{\pi}{2}$

2. Show that $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \frac{\pi}{6}$

$$=\frac{1}{2}\left[30.\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2-6\left(\frac{1}{2}\right)^6\right]=\frac{21}{32}$$

$$\therefore \oint_c f(z)\,dz=2\pi i\left(\frac{21}{32}\right)=\frac{21}{16}\pi i$$

Exercise 42

1. Evaluate the integral using residue theorem

i) $\oint_c \frac{z+4}{z^2+2z+5}dz$ where c is the circle $|z+1+i|=2$ & $|z+1-i|=2$

ii) $\oint_C \frac{12Z-7}{(Z-1)^2(2Z+3)}dz$ where c: $|z+i|=\sqrt{3}$

Let's check take away from the lecture

1. Cauchy's integral theorem is

(a) $f(z_0)=\frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}dz$ (b) $\oint_c f(z)dz=0$ (c) $\int_{c_1} f(z)dz=\int_{c2} f(z)dz$ (d) $\int_c \frac{f(z)}{(z-z_0)^n}dz=\frac{2\pi i}{(n-1)!}f^{n-1}(z_0)$

2. Cauchy's integral formula is

(a) $f(z_0)=\frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}dz$ (b) $\oint_c f(z)dz=0$ (c) $\int_{c_1} f(z)dz=\int_{c2} f(z)dz$ (d) $\int_c \frac{f(z)}{(z-z_0)^n}dz=\frac{2\pi i}{(n-1)!}f^{n-1}(z_0)$

3. Show that $\int_c \frac{e^{2z}}{(z+1)^4}dz=\frac{8\pi i e^{-2}}{3}$ where c is $|z|=3$

Homework Problems for the day

1. Evaluate the following integral using residue theorem

(i) $\oint_c \tan z\,dz$ where c: $|z|=2$ (ii) $\oint_c \frac{z\cos z}{(z-\frac{\pi}{2})^2}dz$ where c: $|z-1|=1$

(iii) $\oint_c \frac{(z-3)}{z^2+2z+5}dz$ where c: $|z+1-i|=2$

2. Evaluate by Cauchy Residue Theorem $\oint_c \frac{(5z-2)}{z(z-1)}dz$ where c is the circle $|z|=2$ taken counter clockwise.

Learning from the topic: Students will be able to solve problems by using Residues Theorem.

EVALUATION OF DEFINITE INTEGRAL

Lecture: 43

Homework Problems for the day

1. Determine the poles of the following functions & residue at each pole

(i) $\dfrac{z^2}{(z^2+a^2)}$ (ii) $\dfrac{\cot \pi z}{(z-a)^2}$ (iii) $\dfrac{z}{\cosh z - \cos z}$

2. Compute residues at double poles of $f(z) = \frac{z^2+2z+3}{(z-i)^2(z+4)}$.

Learning from the topic: Students will be able to find zero's, poles, and residues of given problems.

RESIDUE THEOREM
Lecture: 42

1. Learning Objective: Students shall be able to understand and apply Residue's theorem.

2. Introduction: This topic is used to solve problems by using Residues Theorem and it can be applied for solving the line integral of the function around a singularity.

3. Key Definitions:

Cauchy's Residue Theorem: If $f(z)$ is analytic within and on a simple closed curve C, except at a finite number of isolated singular points $z_1, z_2, ..., z_n$ inside C then

$$\oint_C f(z)dz = 2\pi i(\text{Sum of the residues at } z_1, z_2, ..., z_n)$$

4. Sample Problems

1). Using Residue theorem evaluate $\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where C is $|z| = 1$

Solution: Clearly $z = \pi/6$ is a pole of order 3.

$|z| = 1$ is a circle with center at (0,0) and radius 1.

Then pole $\left(\pi/6, 0\right)$ i.e. $\left(3.14/6, 0\right)$ lies inside C.

Residue at $\left(z = \frac{\pi}{6}\right)$

$$= \frac{1}{2!} \lim_{z \to \frac{\pi}{6}} \frac{d^2}{dz^2}\left[\left(z - \pi/6\right)^3 . \frac{\sin^6 z}{\left(z - \pi/6\right)^3}\right]$$

$$= \frac{1}{2!} \lim_{z \to \frac{\pi}{6}} \frac{d}{dz}\left[6 \sin^5 z \cos z\right]$$

$$= \frac{1}{2!} \lim_{z \to \frac{\pi}{6}} \left[6.5 \sin^4 z \cos^2 z - 6 \sin^5 z \sin z\right]$$

4. Sample Problems:

1). Determine the pole of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and find the residue at each pole.

Solution:

$\because (z-1)^2(z+2) = 0$ *gives* $z = -2, 1$ *and* 1.

Hence $f(z)$ has a simple pole of order 2 at $z = 1$.

(i) Residue of $f(z)$ at $z = -2$ = $\lim_{z\to -2}(z+2)f(z) = \lim_{z\to -2}\frac{z^2}{(z-1)^2} = \frac{4}{9}$.

(ii) Residue of $f(z)$ at $z = 1$ = $\lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)] = \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^2}{z+2}\right] = \frac{5}{9}$.

Exercise 41

1. Determine the poles of the following functions & residue at each pole

(i) $\frac{1}{(z^2-1)}$ (ii) $\frac{z}{(z-1)(z-2)(z-3)}$ (iii) $\frac{\cos z}{z \sin z}$

2. Find poles and residues at poles of $f(z) = \frac{1}{z(z-1)^2}$

3. Find the residue of $f(z) = \frac{1}{(z^2+1)^3}$ about z=i

Let's check take away from the lecture

1. If $f(z) = \frac{Sin z}{z}$, then $z = 0$ is singularity of type

(a) Removable singularity (b) Isolated singularity (c) Essential singularity (d) None of these

2. If $f(z) = \frac{1}{z(z-1)^2}$ then z=1 is the singularity of it as

(a) zero (b) Simple Pole (c) Pole of Order 2 (d) Pole of order 3

3. The singular points of $f(z) = \frac{1}{z(z-1)(z-2)}$ are.

(a) 0,1 (b) 1,2 (c) 0, 2 (d) 0,1,2

4. $f(z) = (z+2)\sin\left(\frac{1}{z-1}\right)$ has an isolated essential singularity at point

$(a)\ 1$ $(b)\ 3$ $(c)\ 0$ $(d) -1$

5. Determine the nature of a pole if $f(z) = \frac{1-e^z}{z}$

(a) *simple* (b) *essential* (c) *removable* (d) *ordinary*

(a) $|z-a| < R$ (b) $|z-a| > R$ (c) $|z-a| = R$ (d) $|z| = R$

2. The region of validity of $\frac{1}{1+z}$ for its Taylor's series expansion about $z = 0$ is
(a) $|z| < 1$ (b) $|z| > 1$ (c) $|z| = 1$ (d) $|z| = 2$

3. The expansion of $\frac{1}{z-2}$ is valid for the region.

(a) $|z| < 1$ (b) $|z| > 2$ (c) $|z| = 1$ (d) $|z| < 2$

4. If F(z) is an analytic function at z = a, then it has a power series expansion about z = a.
(a) Statement is true (b) Statement is false (d) None of these

Homework Problems for the day

1. Obtain Laurent's series expansion of the function $f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)}$

(i) $1<|z|<4$ (ii) $|z|>4$

2. Expand $f(z) = \frac{3z-3}{(2z-1)(z-3)}$ in a Laurent's series about z=1 convergent in
$\frac{1}{2} < |Z-1| < 1$

Learning from the topic: Students will be able to problem by using Laurent's Series and Taylor's Series.

ZERO'S OF ANALYTIC FUNCTION, SINGULARITIES & POLES

Lecture: 41

1. Learning Objective: Students shall be able to understand zeroes of analytic function, singularities & poles and apply it to solve problems.

2. Introduction: The zeros, singularity, and poles of any function of complex variable is of importance as these concepts are used in its application in control theory.

3. Key Definitions:

i) Singular Point or Singularity: If a function $f(z)$ is analytic at every point in the neighborhood of a point z_0 except at z_0 itself then $z = z_0$ is called a singular point or singularity of $f(z)$.

ii) Residue: If $z = z_0$ is an isolated singularity of $f(z)$ then the constant b_1 i. e. the co-efficient of $\frac{1}{z-z_0}$ in the Laurent's expansion of $f(z)$ about $z = z_0$ is called the residue of $f(z)$ at $z = z_0$.

4. Sample Problems:

1) Find Laurent's series which represents the function $f(z)=\dfrac{1}{(z-1)(z-2)}$ when

(i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$

Solution: Let $\dfrac{1}{(z-1)(z-2)}=\dfrac{a}{z-1}+\dfrac{b}{z-2}$

$$\therefore 1=a(z-2)+b(z-1)$$

When $z=1$, $1=-a$ $\therefore a=-1$

When $z=2$, $1=b$

$$\therefore \frac{1}{(z-1)(z-2)}=\frac{-1}{z-1}+\frac{1}{z-2}$$

Case (i) : When $|z|<1$, clearly $|z|<2$

$$\therefore f(z)=\frac{1}{1-z}-\frac{1}{2\left[1-\left(z/2\right)\right]}$$

$$=2[1-z]^{-1}-\left[1-\left(z/2\right)\right]^{-1}$$

$$=2\left[1+z+z^2+z^3+....\right]-\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{3}\right)^3+....\right]$$

Case (ii) : When $1<|z|<2$, we write

$$\frac{1}{(z-1)(z-2)}=-\frac{1}{z-1}+\frac{1}{z-2}\ as$$

$$=-\frac{1}{z[1-(1/z)]}-\frac{1}{2[1-(z/2)]}$$

$$=-\frac{1}{z}[1-(1/z)]^{-1}-\frac{1}{2}[1-(z/2)]^{-1}$$

$$=-\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+....\right]-\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{3}\right)^3+....\right]$$

Exercise 40

1. Obtain Taylor's or Laurent's series expansion of the function

$$f(z)=\frac{1}{z^2-3z+2}$$

when (i) $|z|<1$ (ii) $1<|z|<2$

2. Find all possible Laurent's expansion of the function

$$f(z)=\frac{7z-2}{z(z-2)(z+1)}$$

about z = -1

Let's check take away from the lecture

1. A power series $R=\sum_{n=0}^{\infty}a_n\,(z-a)^n$ converges if

(a) $2\pi i$ (b) $4\pi i$ (c) $6\pi i$ (d) 0

2. The value of the Integral $\int_c \frac{z}{(z-1)^2(z-4)} dz$ where $|z-1| = 1$.

(a) $2\pi i$ (b) $4\pi i$ (c) $\frac{-32\pi i}{9}$ (d) 0

Homework Problems for the day

1. Evaluate $\oint_c \frac{12z-7}{(z-1)^2(2z-3)} dz$ $where\ \ c\ is\ |z+i| = \sqrt{3}$.

2. Evaluate $\int_c \frac{z+2}{z^3-2z^2} dz$, where C is the circle $|Z-2-i| = 2$.

3. If f(k)= $\oint_{c|z|=2} \frac{3z^2+7z+1}{z-k} dz$ find f(1-i) and f″ (1-i).

Learning from the topic: Students will be able to solve the problem with the multiple singularity inside the region of integration.

TAYLOR'S SERIES, LAURENT'S SERIES

Lecture: 40

1.Learning Objective: Students shall be able to understand Taylor's Series and Laurent's series and apply it to solve problems.

2. Introduction: This topic explains about the evaluation of Laurent series and Taylors series of a function of complex variable. It also helps the student to identify the region of convergence for the function.

3. Key Definitions:

(1) Taylor's Series: If $f(z)$ is analytic inside a circle C with centre at z_0 then for all z inside C, $f(z)$ can be expanded as

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + ...$$

The series is convergent at every point inside C and is known as Taylor's series.

(2) Laurent's Series: If there are two concentric circles of radii $r_1\, and\, r_2$ with center at z_0 and if $f(z)$ is analytic within and on annular region R between two circles, then for any point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$$where\ \ a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-z_0)^{n+1}} dw\ ,\ b_n = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-z_0)^{-n+1}} dw$$

Learning from the topic: Students will be able to use the Cauchy's integral theorem and formula

Cauchy's Integral formula for derivatives

Lecture: 39

1.Learning Objective:

Students shall be able to understand Cauchy's integral formula for derivative and apply it to solve the contour integrals.

2. Introduction:

This topic is used to find value of contour integral in complex plane. The Cauchy's integral formula for the derivative can be applied if the singularity of the integrand is appearing in multiples.

3. Key Definitions:

Cauchy's Integral Formula for derivatives:

$$\oint_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{n!} f^{n-1}(z_0)$$

5. Sample Problems:

1. Evaluate $\oint_c \frac{z^2}{(z^2+1)^2} dz$ where c is $|z+1-i| = 2$

Solution: The singularity of this function of Integrand is $\frac{z^2}{(z^2+1)^2}$ is z=I and z=-I since z=I lies inside the curve c

$$\oint_c \frac{z^2/(z+i)^2}{(z-i)^2} dz = \frac{2\pi i}{1}\left[\frac{d}{dz}\frac{z^2}{(z+i)^2}\right]_{z=i}$$

$$= \frac{2\pi i}{1}\left[\frac{(z+i)2z-2z^2}{(z+i)^3}\right]_{z=i}$$

$$= \frac{\pi}{2}$$

Exercise 39

1. Evaluate $\oint_c \frac{z+6}{z^2-4} dz$ where c is (i) $|z| = 1$ (ii) $|z-2| = 1$

2. Evaluate $\int_c \frac{z^2}{(z-1)^2(z-2)} dz$ where c is the circle $|z| = 2.5$.

3. If $f(a) = \oint_c \frac{4z^2+z+5}{z-a} dz$ where c is the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ find the value of

(i) f(3.5) (ii) f(i), f'(-1), f''(-i)

Let's check take away from the lecture

1. The value of the of the integral $\int_c \frac{\sin z}{(z-\frac{\pi}{2})^2} dz$ where C: is $|z| = 2$.

$\therefore (z+1+2i)(z+1-2i)=0$

$\therefore z=-1-2i, z=-1+2i$

(i) Now, $|z|=1$ is a circle with center at the origin and radius 1. Hence, both the points lie outside the circle C and $f(z)$ is analytic in C. By Cauchy's Theorem

$$\int_c \frac{z+3}{z^2+2z+5}dz=0$$

(ii) Now, $|z+1-i|=2$, is a circle with center at A $(-1+i)$ i.e. $(-1,1)$ and radius 2. The point B (-1,2) lies inside and the point C (-1, -2) lies outside the circle. Hence, we put $\frac{z+3}{z+1+2i}$ which is analytic in C.

By Cauchy's formula,

$$\int_c \frac{z+3}{z^2+2z+5}dz=\int_c \frac{z+3/(z+1+2i)}{(z+1-2i)}dz$$

$$=\int_c \frac{f(z)}{z-z_0}dz=2\pi i\, f(z_0)$$

$$=2\pi i.\frac{(2+2i)}{4i}$$

1. *Evaluate* $\int_c \frac{-3z+4}{(z-1)(z-2)}dz$ *where* c *is the circle* $|z|=3/2$

2. *Evaluate* $\int_c \frac{3z^2+7z+1}{(z+1)}dz$ *where* c *is the circle* (i) $|z|=3/2$ (ii) $|z+1|=1$

1. *The value of* $\int_c z^2\,dz$ *is*

   (a) $2\pi i$ (b) 1 (c) 0 (d) $4\pi i$

2. The value of the Integral $\int_c \frac{1}{(z-2)(z-3)}dz$ *where* $|z|=4$

   (a) 0 (b) $2\pi i$ (c) $4\pi i$ (d) πi

1. $\int_c \frac{z^2+z+1}{z-1}dz$ *where* c *is* $|z|=2$ & $|z|=1/2$

2. $\int_c \frac{z^2+4z}{(z-2)(z-2i)}dz$ *where* c *is* (i) $|z|=1$ (ii) $|z-2|=2$ (iii) $|z+i|=1$

D then it is called analytic or regular function of z in the domain D.

(2) Harmonic function:

Any function of x, y which has continuous partial derivatives of the first and second order and satisfies Laplace equation is called a harmonic function.

(3) Cauchy's Integral Theorem:

If $f(z)$ is an analytic function and its derivative $f'(z)$ is continuous at each point within and on a simple closed curve C then the integral of $f(z)$ along the closed curve C is zero.

$$\text{i. e. } \oint_C f(z)dz = 0.$$

(4) Extension of Cauchy's Integral Theorem:

If $f(z)$ is analytic in R between two simple closed curves, $C_1 and\, C_2$ then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz.$$

(5) Cauchy's Integral Formula:

If $f(z)$ is an analytic function within and on a simple closed curve C of a simply connected region R and if z_0 is any point within C then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)}dz.$$

(6) Extension of Cauchy's Integral Formula:

If $f(z)$ is an analytic in a region R bounded by two closed curves $C_1 and\, C_2$ one within the other, if z_0 is any point in the region R then

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-z_0)}dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{(z-z_0)}dz$$

4. Key Notations:

i: iota (unit of imaginary number)
z: complex number
$\bar{z}$: conjugate of complex number
Re z : real part of complex number z
Im z : imaginary part of z
r : |z| (modulus of z)
θ: arg(z) (argument of z)
sinhx : Sine hyperbolic x
coshx : Cosine hyperbolic x
log z : Principle value of logarithm
Log z: General value of logarithm
$f'(z), f''(z),...$: first order derivative, second order derivative,

5. Sample Problems:

1). Evaluate $\int_C \frac{z+3}{z^2+2z+5}$ where C is the circle (i) $|z|=1$ (ii) $|z+1-i|=2$

Solution: Now,

$z^2+2z+5=0 \text{ gives } (z+1)^2+2^2=0$

MODULE 06
Complex Variable – Integration

1. Motivation:

This topic deals with the integration of function of complex variable. It developed the facility with the methods of complex analysis and applications to problems in mathematical physics. In particular, the student should be able use the residue theorem to evaluate contour integrals as commonly arises in Fourier and Laplace transform theory.

2. Syllabus:

Module	Content	Duration	Self-study duration	Weightage
6	Contour integrals, Cauchy-Goursat theorem (without proof), Cauchy Integral formula (without proof), Taylor's series, zeros of analytic functions, singularities, Laurent's series; Residues, Cauchy Residue theorem (without proof), Evaluation of definite integral involving sine and cosine, Evaluation of certain improper integrals using the Bromwich contour	8 Hrs	16 Hrs	17-19 Marks

3. Prerequisite:

Concept of complex Numbers, formulae of derivative and formulae of integration.

4. Learning Objective: Learners shall be able to

1. Understand counter integral and apply it to solve problems.
2. Understand Cauchy-Goursat theorem, Cauchy Integral formula and apply it to solve problems of complex integration.
3. Understand Taylors Series, Laurent Series of function of complex variable.
4. Understand Zeros of analytic functions, singularities, residues and apply it to solve problems.
5. Understand Cauchy residue theorem and apply it to solve problems.
6. Understand Evaluation of certain improper integrals using the Bromwich contour and apply it to solve problems.

Counter Integration
Lecture: 38

1. Learning Objective:

Students shall be able to understand counter integral and apply it to solve problems.

2. Introduction:

This topic is used to find value of contour integral in complex plane . The Cauchy's integral theorem and Cauchy's integral formula and Cauchy's integral formula for multiply connected region.

3. Key Definitions:

(1) Analytic Function:

A single valued function $w = f(z)$ is defined and differentiable at each point of domain

Self-Evaluation

Name of student: **Course Code: BSC104**
Class &Div: **Roll No:**

1. Do you understand how to evaluate double integration?
   (a) Yes (b) No

2. Will you be able to change the order of integration and evaluate?
   (a) Yes (b) No

3. Are you able to draw the region of integration?
   (a) Yes (b) No

4. Are you able to change to polar form and evaluate the integral?
   (a) Yes (b) No

5. Do you understand how to find the limits of integration in the given region?
   (a) Yes (b) No

6. Do you understood this module?
   (a) Fully understood (b) Partially understood (c) Not at all

Add to Knowledge:

Centre of mass and centre of gravity are the same, so long as gravity is constant. Additionally, if the density is constant (as in our example) then the centre of mass coincides with the centroid.

If the density were not constant, then the centre of mass calculation would be: $= \iiint_{v} \rho(x, y, z) z\, dv$

Moment of inertia is a property which is fundamental to rotational behaviour. It can be thought of as the rotational equivalent of mass, so that, just as we have

Force = mass × acceleration

so also

Torque = moment of inertia × angular acceleration

It is possible to calculate many different moments of inertia for a body, the correct choice depending on which axis it is rotating about. The moment of inertia about an axis is the second moment of the mass about that axis.

i.e. $I = \iiint y^2\, dM = \iiint_{v} \rho y^2\, dv$

Learning Outcomes:

1. **Know:** Students should be able to know the applications of multiple integrals as area, mass and volume
2. **Comprehend:** Students should be able to apply the concept of multiple integrals to find area, mass, volume
3. **Apply, analyze and synthesize:** Student should be able to solve problems based on applications of multiple integrals

Self-Assessment

1. Evaluate $\int_0^2 \int_0^x \int_0^{2x+2y} e^{x+y+z}\, dx\, dy\, dz$

[Hint : Direct Evaluation]] **[Level 1]**

2. *Evaluate* $\iiint \dfrac{dxdydz}{x^2+y^2+z^2}$ *throughout the volume of the sphere* $x^2+y^2+z^2=a^2$

[Hint : Use spherical polar coordinates] **[Level 2]**

3. *Find the area bounded by* $y^2 = x,\ x^2 = -8y$

[Hint : Use Cartesian coordinates] **[Level 3]**

4. Find the mass of the lamina bounded by $x^2+2y-4=0$ and X- axis if density at any point

varies as its distance from X-axis

[Hint : The given curve is a parabola] **[Level 4]**

5. Find the volume in first octant bounded by the circular cylinder $x^2+y^2=2$ and planes $z = x + y,\ y = x,\ z = 0,\ x = 0$.

Hint : Use z as it is given then use cylindrical coordinates] **[Level 5]**

Let's check take away from the lecture

1.According to Greens theorem formula for the area is

(a) $\frac{1}{2}\int_c (ydx - xdy)$ (b) $\frac{1}{2}\int_c (xdx + ydy)$

(c) $\frac{1}{2}\int_c xydxdy$ (d) None

1. According to Gauss divergence theorem

(a) $\iint_s \vec{F}.d\vec{s} = \iiint_v (\nabla.\vec{F})dv$ (b) $\iint_s \vec{F}.ds = \iiint_v (\nabla\times\vec{F})dv$

(c) $\iiint_v (\nabla\times\vec{F})ds = \iint_s (\nabla.\vec{F})dv$ (d) None

2. Evaluate $\iint_s (\nabla r^2)d\vec{s}$

(a) 2v (b) 3v (c) 6v (d) None

Home Work Problems for the day

1.Evaluate $\iint_S \overline{F}.d\overline{s}$ where $\overline{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the region bounded by (i) $y^2 = 4x, x = 1, z = 0, z = 3$ (ii) $x^2 + y^2 = 4, z = 0, z = 3$.

2.Evaluate by Gauss Divergence theorem for $\overline{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$.

3.Evaluate by Gauss Divergence theorem for $\overline{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2 + y^2 = 4, z = 0, z = 3$.

4.Prove that $\int_S (lx + my + nz)ds = 32\pi$ where l, m, n are the direction cosines of the outer normal to the surface whose radius is 2 cm.

Learning from the topic: Students will be able to evaluate triple integration using divergence theorem

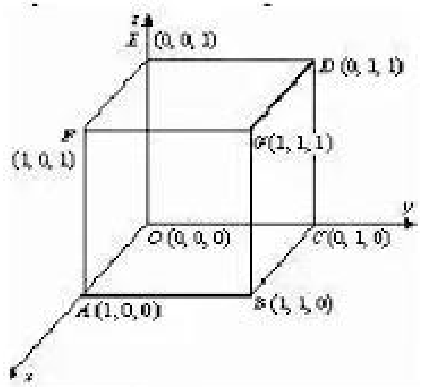
$$\iiint_T div\ \vec{F}\ dv = \iint_S \vec{F}.\hat{n}\ dA$$

3.Sample Problem:

1) Evaluate by Gauss divergence theorem for $\overline{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ over the cube $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0, z = 1$.

Solu : By Gauss divergence theorem,

$$\iiint_v \nabla\cdot\overline{F}dv = \iint_s \overline{F}\cdot\hat{n}ds$$



(i) $\overline{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$$\nabla\cdot\overline{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$= 4z - y$$

(ii) For the cube:

$x$varies from 0 to 1

y varies from 0 to 1

$z$varies from 0 to 1

$$\iiint_v \nabla\cdot\overline{F}dv = \int_0^1\int_0^1\int_0^1(4z - y)dxdydz$$

$$= \int_0^1\int_0^1\left|2z^2 - yz\right|_0^1 dxdy$$

$$= \int_0^1 dx\int_0^1(2 - y)dy$$

$$= |x|_0^1\cdot\left|2y - \frac{y^2}{2}\right|_0^1$$

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2}$$

Exercise 37

1. Evaluate by Gauss Divergence theorem for $\overline{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ over the cube $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.
2. Show that $\iint_s \frac{ds}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = \frac{4\pi}{\sqrt{abc}}$, where S is the surface $ax^2 + by^2 + cz^2 = 1$.
3. Prove that the surface of the sphere $x^2 + y^2 + z^2 = 1$,
$\iint_s (lx\hat{i} + my\hat{j} + nz\hat{k}).\overline{ds} = \frac{4\pi}{3}(l + m + n)$.
4. Prove that $\iint_s \overline{F}.\overline{ds}$ where $\overline{F} = x\hat{i} - y\hat{j} + 2z\hat{k}$ over the surface of the sphere $x^2 + y^2 + (z - a)^2 = a^2$.

3.Find the work done by the force field $\bar{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$ along c_1 where

$c_1 :\ 0 \le x \le 1 \quad y = x, \quad z = 0$

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) None

Home Work Problems for the day

1. Using Stoke's theorem evaluate $\oint_C (y\,dx + z\,dy + x\,dz)$ where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and plane $x + z = a$.

2. Using Stoke's theorem find the work done in moving a particle once around the perimeter of the triangle with vertices at (2,0,0), (0,3,0) and (0,0,6) under the force field $\bar{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$.

3. Using appropriate theorems evaluate the following integral $\iint_S (\nabla \times \vec{F}).d\vec{s}$ where $\bar{F} = (x^2 + y - 4)\hat{i} + 3xy\hat{j} + (x + z^2)\hat{k}$ and S is hemisphere $x^2 + y^2 + z^2 = 4$ lying above XOY - plane.

4.Evaluate $\iint_S A.\hat{n}\,ds$, $A = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the that part of the plane $2x + 3y + 6z = 12$ which is locates in the first octant.

5.Evaluate $\iint_S A.\hat{n}\,ds$, $A = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Learning from the topic: Students will be able to evaluate line integral by converting into it in double integral by using Stoke's theorem

Evaluation of Gauss Divergence Theorem

Lecture: 37

1. **Learning Objective:** Learners shall be able to evaluate surface integral by converting it into triple over the given three dimensional region using divergence theorem.
2. **Divergence Theorem:**

This theorem relates between the surface integral and volume integral.

Statement: Let T be a closed bounded region in space whose boundary is a piecewise smooth oriented surface S. Let F(x, y, z) be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,ds=\iint_R(-\hat{i}+\hat{j}-\hat{k})\cdot\frac{(2\hat{i}+\hat{j}+\hat{k})}{\sqrt{6}}\sqrt{6}\,dx\,dy$$

$$=\int_0^1\int_0^{2-2x}(-2+1-1)\,dx\,dy$$

$$=-2\int_0^1\left|y\right|_0^{2-2x}dx$$

$$=-2\int_0^1(2-2x)\,dx$$

$$=-4\left|x-\frac{x^2}{2}\right|_0^1$$

$$=-4\left(1-\frac{1}{2}\right)$$

$$\iint_S \nabla\times\overline{F}\cdot\hat{n}\,ds=-2 \qquad \ldots(1)$$

$$\text{or}\iint_S \nabla\times\overline{F}\cdot\hat{n}\,ds=-2\iint_R dx\,dy$$

$$=-2(\text{area of}\,\Delta OAB)$$

$$=-2\cdot\frac{1}{2}\cdot 1\cdot 2$$

$$=-2$$

Exercise 36

1. Evaluate by Stoke's theorem for: $\overline{F}=(x+y)\hat{i}+(y+z)\hat{j}-x\hat{k}$ and S is the surface of plane $2x+y+z=2$ which is in the first octant.

2. Evaluate by Stoke's theorem $\oint_C (e^x\,dx+2y\,dy-dz)$ where C is the curve $x^2+y^2=4$ & $z=2$.

3. Evaluate by Stoke's theorem for the function $\overline{F}=z\hat{i}+x\hat{j}+y\hat{k}$ where C is the unit circle in xy – plane bounded by the hemisphere $z=\sqrt{1-x^2-y^2}$.

Let's check take a way from the lecture

1.By stokes theorem

(a) $\int_c \vec{F}.d\vec{r}=\iint_s(\nabla\times\vec{F})d\vec{s}$

(b) $\int_c \vec{F}\times d\vec{r}=\iint_s(\nabla.\vec{F}).d\vec{s}$

(c) $\int_c \vec{F}\times d\vec{s}=\iint_s(\nabla.\vec{F})d\vec{r}$

(d) None

2.Evaluate $\oint_c \vec{F}.d\vec{r}$ where $c: y=2x^2$ & $\vec{F}=3xy\hat{i}-y^2\hat{j}$ from (0,0) to (1,2)

(a) $-\frac{5}{2}$ (b) 6 (c) $-\frac{7}{6}$ (d) None

Module 5: Multivariable Calculus-II (Triple Integration)

Evaluation of Stoke's Theorem

1.Learning Objective:

Learners shall be able to evaluate the line integral by concerting into double integration over the given two dimension al space using Stoke's theorem.

2.Stoke's Theorem:

Its relates between the integral of the curl of the vector field over some surface, to the line integral of the vector field around the boundary of the surface.

Statement: Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple closed curve C. Let be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

Here is a unit nornal vector of S and depending on , the integration around C is taken.

3.Sample Problem:

1) Evaluate by Stoke's theorem for $F \quad (x \quad y)\hat{i} \quad (y \quad z)\hat{j} \quad x\hat{k}$ and s is the surface of the plane $2x \quad y \quad z \quad 2$ which is in the first octant.

Solu: By Stoke's theorem $\iint_S \nabla \times F \cdot \hat{n}\, ds = \int_c$

(i) $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & y+z & -x \end{vmatrix}$

$= \hat{i}\,(0-1) - \hat{j}\,(-1-0) + \hat{k}(0-1)$

$= -\hat{i} + \hat{j} - \hat{k}$

(ii) Let $\phi = 2x + y + z$

$\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\hat{i} + \hat{j} + \hat{k}}{4+1+1} = \dfrac{2\hat{i} + \hat{j} + \hat{k}}{6}$

(iii) Projection of the plane $2x + y + z = 2$ on xy-plane $(z = 0)$ is the triangle OAB bounded by the lines $x = 0$, $y = 0$, $2x + y = 2$

(iv) $ds = \dfrac{dxdy}{\hat{n} \cdot \hat{k}} = \quad 6\, dxdy$

(v) Let R be the region bounded by the triangle OAB in xy-plane.

Along the vertical strip PQ, y varies from 0 to $(2 - 2x)$ and in the region R, x varies from 0 to 1

1. By Greens theorem

(a) $\int_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$ (b) $\int_c Pdx - Qdy = \iint_R \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \right) dxdy$

(c) $\iint_R Pdx - Qdy = \int_c \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \right) dxdy$ (d) None

2. According to Greens theorem formula for the area is

(a) $\frac{1}{2}\int_c (ydx - xdy)$ (b) $\frac{1}{2}\int_c (xdx + ydy)$

(c) $\frac{1}{2}\int_c xydxdy$ (d) None

3. Evaluate $\int_c \left(y^2dx + 2xydy\right)$ where $c : x^2 + y^2 = 1$ from (0,0) to(1,0)

(a) 1 (b) 0 (c) 2π (d) None

Home Work Problems for the day

1.State Green's Theorem , using it evaluate $\int_C (2xydx - y^2dy)$ where C is the curve given by $3x^2 + 4y^2 = 12$.

2. Evaluate $\oint_C (x^2 + 2y)dx + (4x + y^2)dy$ where C is the boundary of the region bounded by $y = 0$ and $y = 2x$ and $y + x = 3$.

3. Evaluate $\oint_C \frac{ydx - xdy}{x^2 + y^2}$ where C is the circle $x^2 + y^2 = a^2$.

4.Evaluate by Green's Theorem in plane $\int_C [(x^2 - \cosh y)dx + (y + \sin x)dy]$

Where C is the rectangle with vertices $(0,0)$, $(\pi ,0)$, $(\pi,1)$, $(0,1)$.

5. Evaluate by Green's Theorem for $\oint_C (x^2 - y^3)dx + (x^3 + y^2)dy$ where C is $x^2 + 4y^2 = 64$.

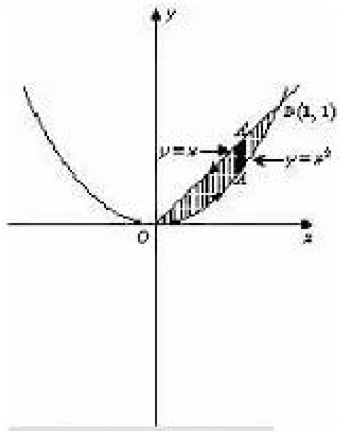
Learning from the topic: Students will be able to evaluate double integration on the given region on the plane where the path of integration along C is anticlockwise.

3. Sample Problems:

1) Evaluate by Greens theorem for $\oint_c \left[(x^2-2xy)dx+(x^2y+3)dy\right]$ where c is the boundary of the region bounded by the parabola $y=x^2$ and the line y = x.

Solution :(i) Point of intersection of the parabola $y=x^2$ and the line y = x is $x=x^2, x=0,1$ and $y=0,1$

Hence, B :(1, 1)



(ii) $M=x^2-2xy, \quad N=x^2y+3$

$\frac{\partial M}{\partial y}=-2x, \qquad \frac{\partial N}{\partial x}=2xy$

(iii) Let R be the region bounded by the line $y=x$ and the parabola $y=x^2$.

Along the vertical strip, y varies from x^2 to x and in the region R, x varies from 0 to 1

$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\int_0^1\int_{x^2}^{x}(2xy+2x)\,dy\,dx$

$=\int_0^1 \left|xy^2+2xy\right|_{x^2}^{x}dx$

$=\int_0^1 (x^3+2x^2-x^5-2x^3)\,dx$

$=\left|\frac{-x^4}{4}+\frac{2x^3}{3}-\frac{x^6}{6}\right|_0^1$

$=-\frac{1}{4}+\frac{2}{3}-\frac{1}{6}$

$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\frac{1}{4} \quad ...(3)$

From equation (2) and (3),

$\oint_c (M\,dx+N\,dy)=\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\frac{1}{4}$

Exercise 35

1. Evaluate by Green's Theorem for $\oint_C 2y^2\,dx+3xdy$ where C is the boundary of the closed region bounded by $y=x^2$ and $y=x$.
2. Evaluate by Green's Theorem in the plane for $\oint_C \frac{1}{y}dx+\frac{1}{x}dy$ where C is the boundary of the region defined by $y=1$, $x=4$ and $y=\sqrt{x}$.
3. Evaluate by Green's Theorem in the plane for $\oint_C (3x^2-8y^2)dx+(4y-6xy)dy$ where C is the boundary of the region defined by $y=x^2$ and $y=\sqrt{x}$.

Let's check take away from the lecture

Let's check take away from the lecture

1. The surface integral $\iint_S \frac{1}{4}\left(\bar{F}\cdot\bar{n}\right) ds$ to be evaluated over a sphere for the given steady velocity vector field F = xi + yj + zk , where S is the sphere, $x^2+y^2+z^2 = 1$ and $\bar{n}$ is the outward unit normal vector to the sphere. The value of the surface integral is

(a) 2π (b) π (c) 4π (d) $\frac{3\pi}{4}$

2. The surface integral $\iint_S \frac{1}{\pi}(9x\,i - 3y\,j).\bar{n}\,ds$ over the sphere $x^2+y^2+z^2 = 9$ is given by

(a) 216 (b) 116 (c) 156 (d) 200

Home Work Problems for the day

1. Evaluate $\iint_S \bar{F}\cdot d\bar{S}$ where $\bar{F}$ = (x – 4) i + z j - y k and S is the portion of $x = 4 - y^2 - z^2$ that lies in front of x = −2 oriented in the negative X-axis direction .

2. Evaluate $\iint_S \bar{F}\cdot d\bar{S}$ where $\bar{F}$ = (x + y) i + x j + x^2z k and S is the portion of $x^2 + y^2 = 36$ between z = -3 and z = 1 oriented outwards (i.e away from Z-axis)

3. Evaluate $\iint_S \bar{F}\cdot d\bar{S}$ where $\bar{F}$ = i + 4z j + (z – y) k and S is the portion of $y = 4z + x^3 + 6$ that lies over the region in the xz-plane with bounded by $z = x^3$, x = 1 and the x-axis oriented in the positive Y-axis direction.

Learning from the topic: Students will be able to evaluate vector surface integral problems over the open surfaces in 3-dimensional space.

Evaluation of Green's Theorem

Lecture: 35

1. Learning Objective:

Learners shall be able to evaluate double integration over the given two dimensional region.

2. Green's Theorem:

Green's theorem relates a line integral around a simple closed curve *C* to a double integral over the plane region *D* bounded by *C*.

Statement: Let C be a piecewise smooth, simple closed curve in a plane, and let *D* be the region bounded by C. If *M (x,y)* and *N (x,y)* are defined on an open region containing *D* and having continuous partial derivatives there, then

$$= \iint_D \left(-P g_x - Q g_y + R\right) dA$$

3. Sample Problems:

1) Evaluate $\iint_S F \cdot dS$ where F = -x i + 2y j − z k and S is the portion of y = 3x^2 + 3z^2 that lies behind y = 6 oriented in the positive y-axis direction.

Solution: Here D will be the circle or disk we get by setting the two equations equal or, 6 = 3x^2 + 3z^2 or x^2 + z^2 = 2. So, D will be the disk x^2 + z^2 ≤ 2

The equation of surface is f(x, y, z) = 3x^2 + 3z^2 − y = 0

The unit normal vector for the surface is $n = \frac{\nabla f}{\nabla f} = \frac{6xi - j + 6zk}{\nabla f}$

We will not compute the magnitude of the gradient as it will just cancel out when we start working out on integral.

The normal vector we computed above does not have the correct orientation.

It is given that the orientation was in the positive y-axis direction

Therefore, we multiply the above normal vector by minus one so as to get the desire orientation. So, the unit normal vector for the surface becomes $n = -\frac{\nabla f}{\nabla f} = \frac{-6xi + j - 6zk}{\nabla f}$

$$\therefore \iint_S F \cdot dS = \iint_S F \cdot n \; ds = \iint_S \left(-xi + 2y\,j - zk\right) \cdot \frac{(-6xi + j - 6zk)}{\nabla f} ds = \iint_S \frac{6x^2 + 2y + 6z^2}{\nabla f} ds$$

$$= \iint_S \frac{6x^2 + 2(3x^2 + 3z^2) + 6z^2}{\nabla f} ds = \iint_S \frac{6x^2 + 2(x^2 + z^2) + 6z^2}{\nabla f} ds$$

$$= \iint_S \frac{12x^2 + 12z^2}{\nabla f} ds = \iint_D \frac{12(x^2 + z^2)}{\nabla f} \nabla f \; dA = \iint_D 12(x^2 + z^2)\, dx dz$$

Since D is the disk x^2 + z^2 ≤ 2

Therefore, to solve the integral further we use polar coordinates

x = r cos θ and z = r sin θ gives dx dz = r dr d θ and x^2 + z^2 = r^2

The polar limits of D are 0 ≤ θ ≤ 2π and 0 ≤ r ≤ √2

$$\therefore \iint_S F \cdot dS = \iint_D 12(x^2 + z^2)\, dxdz = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 12\, r^2 \; r\, dr\, d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 12\, r^3\, dr\, d\theta = \int_{\theta=0}^{2\pi}\left(\frac{12r^4}{4}\right)_0^2 d\theta = \int_{\theta=0}^{2\pi} 12\, d\theta = 24\pi$$

1. Evaluate $\iint_S F \cdot dS$ where F = 3x i + 2z j + (1 − y^2) k and S is the portion of z = 2 − 3y + x^2 that lies over the triangle in the XY-plane with vertices (0,0) , (2,0) , (2,-4) oriented in the negative Z-axis direction.

2. Evaluate $\iint_S F \cdot dS$ where F =x^2 i + 2z j − 3y k and S is the portion of y^2 + z^2 = 4 between x = 0 and x = 3 − z oriented outwards (i.e away from X-axis)

3. Evaluate $\iint_S F \cdot dS$ where F = (z − y) i + x j + 4y k and S is the portion of x + y + z = 2 in

4. Find the work done in moving a particle along the curve
$\bar{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + b\theta\,\hat{k}$ from $\theta=\pi/4$ to $\theta=\pi/2$ under the force field given by $\bar{F} = (-3a\sin^2\theta.\cos\theta)\,\hat{i} + a(2\sin\theta - 3\sin^3\theta)\,\hat{j} + b\sin 2\theta\,\hat{k}$.

5. Prove that $\int_A^B (2xy^3 - y^2\cos x)dx + (1-2y\sin x + 3x^2y^2)dy = \frac{\pi^2}{4}$
along arc $2x = \pi y^2$ from A (0,0) to B ($\pi/2$,1).

Learning from the topic: Students will be able to evaluate vector line integral problems over the open curve in 2 or 3 dimensional spaces.

Vector Surface Integral (Self-Study)

Lecture: 34

1. Learning Objective:
Learners shall be able to evaluate the vector surface integral over the given region in three dimensional space.

2. The vector surface integral:

After discussing how to solve vector line integrals we now need to discuss on how to solve surface integrals of vector fields. In line integrals we discussed that the orientation of the curve we were integrating along could change the answer. The same thing will hold true with surface integrals as well. Therefore, before we start solving surface integrals of vector fields, we first need to introduce the idea of an oriented surface. Every point on the surface has two unit normal vectors $\bar{n}_1$ and $\bar{n}_2 = -\bar{n}_1$ the one which is pointing outward (positive orientation) and other is pointing inward (negative orientation). If S is a closed surface, then S has a positive orientation if we choose the set of unit normal vectors that point outward from the surface while the negative orientation will be the set of unit normal vectors that point in towards the surface.

Suppose z = g(x , y) be any function of x , y then we define new function
f(x , y , z) = z – g(x , y) then f(x , y ,z) = 0 represents equation of the surface. The unit normal to the surface is given by $\bar{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{-g_x i - g_y j + k}{\sqrt{g_x^2 + g_y^2 + 1}}$

Now if $\bar{F}$ = P i + Q j + R k and the surface S is oriented in the positive z-axis direction (i.e. upward direction) then the surface integral of $\bar{F}$ over surface S is given by

$$\iint_S \bar{F}\cdot d\bar{S} = \iint_S \bar{F}\cdot\bar{n}\,ds = \iint_S (Pi + Qy\,j + Rk).\frac{\nabla f}{\|\nabla f\|}\,ds = \iint_D (Pi + Qy\,j + Rk).\frac{\nabla f}{\|\nabla f\|}\,\|\nabla f\|\,dA$$

$$= \iint_D (Pi + Qy\,j + Rk).\frac{(-g_x i - g_y j + k)}{\sqrt{g_x^2 + g_y^2 + 1}}\sqrt{g_x^2 + g_y^2 + 1}\,dA$$

$\overline{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1,\ y = 2t^2,\ z = t^3$ from $t = 1$ to $t = 2$.

4. Find the work done when the force $\overline{F} = (x^2 + x - 2y^3)\hat{i} - (6xy^2 + y)\hat{j}$ moves a particle in xy – plane from A(0,0) to B(1,1) along the curve $y^2 = x$. Is the work done different if the path is $y^2 = x^3$?

Let's check take away from the lecture

1. Evaluate $\int_c (2y\,dx + 3x\,dy)$ where $c: x^2 + y^2 = 4$

(a) 2π (b) 5π (c) 4π (d) None

2. Evaluate $\int_c (y^2 dx + 2xy\,dy)$ where $c: x^2 + y^2 = 1$ from (0,0) to (1,0)

(a) 1 (b) 0 (c) 2π (d) None

3. Evaluate $\int_c (y\,dx + x\,dy)$ where $c: y^2 = 4x$ from (0,0) to (1,2)

(a) 0 (b) 2 (c) 5 (d) None

4. Find the work done by the force field $\bar{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$ along c_1 where

$c_1:\ 0 \le x \le 1 \quad y = x,\ z = 0$

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) None

Home Work Problems for the day

1. Integrate $\overline{F} = x^2 i + xy\,j$
   (a) from O to P along OP, P being (0,1)
   (b) along the x-axis from $x = 0$ to $x = 1$
   (c) along the line $x = 1$ from $y = 0$ to $y = 1$

   Also integrate F from (0,0) to (1,1) along the parabola $y^2 = x$.

2. If $A = (2y+3)\hat{i} + xz\hat{j} + (yz - x)\hat{k}$ evaluate $\int_c A.dr$ along the following paths
   (a) $x = 2t^2,\ y = t,\ z = t^3$ from $t = 0$ to $t = 1$
   (b) the straight lines from (0,0,0) to (0,0,1) then to (0,1,1) and then to (2,1,1)
   (c) the straight line joining (0,0,0) and (2,1,1).

3. Find the total work done in moving a particle in a Force field given by $\overline{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1,\ y = 2t^2,\ z = t^3$ from

(f) If the circulation of $\overline{F}$ along every closed curve in the region is zero then $\overline{F}$ is called irrotational vector field.

(g) If $\overline{F}$ is conservative vector field then $\oint \overline{F}\cdot d\overline{r}$ along every closed curve will be zero i.e. the circulation will be zero. i.e the field is irrotational. Thus, conservative field is irrotational and vice versa.

4. Sample Problems:

1) Find work done in moving a particle along the straight line segments joining the points (0, 0, 0) to (1, 0, 0) then to (1, 1, 0) and finally to (1, 1, 1) under the force field $\overline{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$.

Solution: Work done $\int \overline{F}\cdot d\overline{r}$

(i) Let $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

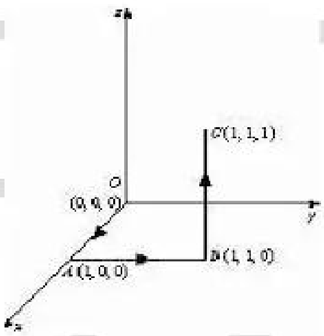
(ii) $\overline{F}\cdot d\overline{r} = \left[(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right]\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$= (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz$

(iii) Path of integration is the line segments joining the points O (0, 0, 0) to A (1, 0, 0), A (1, 0, 0) to B (1, 1, 0) and then B (1, 1, 0) to C (1, 1, 1).

Work done $= \int_c \overline{F}\cdot d\overline{r}$

$= \int_{OA} \overline{F}\cdot d\overline{r} + \int_{AB} \overline{F}\cdot d\overline{r} + \int_{BC} \overline{F}\cdot d\overline{r}$ .....(1)



(a) Along OA : $y = 0, z = 0$

$dy = 0, dz = 0$

x varies from 0 to 1

$\int_{OA} \overline{F}\cdot d\overline{r} = \int_0^1 3x^2 dx = \left|x^3\right|_0^1 = 1$

(b) Along AB : $x = 1, z = 0$

$dx = 0, dz = 0$

y varies from 0 to 1

$\int_{AB} \overline{F}\cdot d\overline{r} = \int_0^1 0\,dy = 0$

(c) Along BC : $x = 1, y = 1$

$dx = 0, dy = 0$

z varies from 0 to 1

$\int_{BC} \overline{F}\cdot d\overline{r} = \int_0^1 20z^2 dz = 20\left|\frac{z^3}{3}\right|_0^1 = \frac{20}{3}$ Substituting in equation (1)

Work done $= 1 + 0 + \frac{20}{3} = \frac{23}{3}$

Exercise 33

1. Evaluate $\int (2xy\hat{i} - x^2\hat{j})d\overline{r}$ over
   (a) the straight line from (0,0) to (2,1)
   (b) parabola $y^2 = 4x$ from (0,0) to (1,2)
   (c) circle $x = \cos t$, $y = \sin t$ $(\pi/2 \le t \le 0)$.

2. Evaluate $\oint_C (x^2 + y)dx + (2x + y^2)dy$ where C is the boundary of the square with vertices (1,1) (1,2) (2,2) (2,1).

Ans: $\frac{1}{6}$

2) Find the volume in first octant bounded by the circular cylinder $x^2+y^2=2$ and planes $z=x+y$, $y=x$, $z=0$, $x=0$.

Ans: $\frac{2\sqrt{2}}{3}$

3) Find the volume within the cylinders $x^2+y^2-4x=0$ cut by the cylinder $z^2=4x$

Ans: $\frac{1024}{25}$

4) Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$

Ans: 8π

5) Find the volume cut off by the paraboloid $x^2+\frac{y^2}{4}+z=1$ cut off by the plane $z=0$

Ans: π

Learning from the topic: Students will be able to find the Volume of the bounded region using the concept of triple integration.

Vector Line Integral (Self-Study)

Lecture: 33

1. Learning Objective:

Learners shall be able to evaluate vector line integration problems over the given two and three dimensional spaces.

2. The vector line integral:

Let $\bar{F}$ be a vector function defined throughout some region of space and let C be any curve in that region. If $\bar{r}$ is the position vector of a point P (x, y, z) on C then the integral $\int_C \bar{F}\cdot d\bar{r}$ is called the line integral of $\bar{F}$ taken over C.

when $\bar{r}$ = xi + yj + zk , we get $d\bar{r}$ = dx i + dy j + dz k and if $\bar{F}$ = F1 i + F2 j + F3 k

then $\int_C \bar{F}\cdot d\bar{r}=\int_C (F_1\,dx+F_2\,dy+F_3\,dz)$

3. Important Properties:

(a) The line integral is depending upon path of integration i.e. If C_1 and C_2 are two different paths joining points A to B then $\int_{C_1} \bar{F}\cdot d\bar{r}\neq\int_{C_2} \bar{F}\cdot d\bar{r}$

(b) If $\bar{F}$ is the gradient of some scalar point function Φ *i.e.* $\bar{F}=\nabla\Phi$ then the line integral is independent of the path from A to B and is depending upon the end points of the curve.

i.e. $\int_C \bar{F}\cdot d\bar{r}=\int_A^B \bar{F}\cdot d\bar{r}=\Phi_B-\Phi_A$

(c) Conversely, if the line integral is independent of the path, then $\bar{F}$ is the gradient of some scalar function Φ known as scalar potential.

(d) If $\bar{F}=\nabla\Phi$ then $\bar{F}$ is called conservative vector field.

(e) If C is the closed curve, then $\oint \bar{F}\cdot d\bar{r}$ is called the circulation of $\bar{F}$ along C

$$Volume = \iint\int_{z=0}^{z=4-y} dxdydz = \iint [z]_0^{4-y} dxdy = \iint [4-y]dxdy.$$

To integrate this integral in the circle $x^2 + y^2 = 4$.

Let us use polar coordinate system.

$$= \int_0^{2\pi}\int_0^{2}(4 - r\sin\theta)\, r dr d\theta = 4\int_0^{\pi/2}\left[4\left(\frac{r^2}{2}\right)_0^2 - \sin\theta\left[\left(\frac{r^3}{3}\right)_0^2\right]\right]d\theta$$

$$= 4\int_0^{\pi/2}\left[4.\frac{4}{2} - \frac{8}{3}\sin\theta\right]d\theta = 32\int_0^{\pi/2} d\theta - \frac{32}{3}\int_0^{\pi/2}\sin\theta d\theta$$

$$= 32(\theta)_0^{\pi/2} - \frac{32}{3}(-\cos\theta)_0^{\pi/2} = 32\frac{\pi}{2} - \frac{32}{3}(0+1) = 16\pi - \frac{32}{3}.$$

Exercise 32

1) Find the volume of the tetrahedron bounded by the coordinate planes $x = 0, y = 0, z = 0$ and

$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1.$

Ans: 4

2) Find the volume bounded by the cylinder $y^2 = x$ and $x^2 = y$ and the planes $z = 0$ and $x + y + z = 2$.

Ans: $\frac{11}{30}$

3) Find the volume bounded by the paraboloid $x^2 + y^2 = az$ and the cylinder $x^2 + y^2 = a^2$

Ans: $\frac{\pi a^3}{2}$

Let's check take awa y from the lecture

1. Using triple integration find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

   (a) πabc (b) $\frac{4}{3}\pi abc$ (c) $\frac{4}{15}\pi a^3 bc$ (d) $\frac{41}{3}\pi abc$

2. Using triple integration find the volume of the sphere $x^2 + y^2 + z^2 = 4$

   (a) $\frac{32}{3}\pi$ (b) $\frac{3}{32}\pi$ (c) 32π (d) none of these

3. Using triple integration find the volume bounded by the paraboloid $x^2 + y^2 = 4z$ and the plane z=5

   (a) 50π (b) 32π (c) 3π (d) 49π

Home Work Problems for the day

1) Find the volume of tetrahedron bounded by x = 0, y = 0 ,z=0 and x +y +z =1

To evaluate volume, Triple integration can be utilized in most effective way.

3. Key Definitions:

Volume: **1.** The triple integration $\iiint_v dxdydz$ on the volume v gives us the volume of the region

2. The volume V of solid of revolution obtain by revolving about the x- axis the plain area bounded by y=f(x), x=a, x=b, x-axis is $V = \int_a^b \pi y^2 dx$.

4 Sample Problems

1. Find the volume bounded by the cylinder $y^2 = x$ and $x^2 = y$ and the planes z = 0 and $x + y + z = 2$.

Solution: The solid is bounded by parabola $y^2 = x$ and $x^2 = y$ and the planes z = 0 which is its base and by the plane $x + y + z = 2$

$$\text{Volume, } V = \int\int\int_0^{z=2-x-y} dxdydz \quad = \int\int [z]_0^{2-x-y} dxdy$$

$$= \int\int [2-x-y]dxdy = \int\int_{y=x^2}^{\sqrt{x}} (2-x-y)dxdy$$

$$= \int \left[2y - xy - \frac{y^2}{2}\right]_{y=x^2}^{\sqrt{x}} dx = \int_{x=0}^{1}\left[2\sqrt{x} - 2x^2 - x\sqrt{x} + x^3 - \frac{x^2}{2} + \frac{x^4}{2}\right]dx$$

$$= \left[2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2\frac{x^3}{3} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^4}{4} - \frac{x^3}{6} + \frac{x^5}{5}\right]_0^1 = \left[\frac{4}{3} - \frac{2}{3} - \frac{2}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5}\right] = \frac{11}{30}$$

2. Find the volume cut off by the paraboloid $x^2 + \frac{y^2}{4} + z = 1$ cut off by the plane z = 0

Solution: The given equation of paraboloid $x^2 + \frac{y^2}{4} + z = 1 \Rightarrow \quad x^2 + \frac{y^2}{4} = -(z-1)$

This has vertex (0,0,1) the intersect of it with z=0 is $x^2 + \frac{y^2}{4} = 1$

$$\text{Volume } V = \int\int\int_{z=0}^{z=1-x^2-\frac{y^2}{4}} dxdydz \quad = \int\int\left[1 - x^2 - \frac{y^2}{4}\right]dxdy$$

Now using the polar coordinate putting $x = r\cos\theta$ and $x = 2r\sin\theta$ and r varies from 0 to 1

$$V = \int\int_0^1 \left[1 - r^2\right]2rdrd\theta \ = 2\int\int_0^1\left[r - r^3\right]drd\theta \ = 2\int\left[\frac{r^2}{2} - \frac{r^3}{3}\right]_0^1 d\theta$$

$$= 2\int\left[\frac{1}{2} - \frac{1}{3}\right]d\theta = \frac{2}{6}\int_0^{2\pi} d\theta = \frac{2}{3}\pi$$

3. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.

Solution:

$$= \int_0^{\pi/2}\int_0^1 \left(\frac{z^2}{2}\right)_0^1 r^3 \sin\theta\cos\theta\, d\theta\, dr \qquad = \frac{1}{2}\int_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^1 \sin\theta\cos\theta\, d\theta$$

$$= \frac{1}{8}\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \qquad = \frac{1}{8}\frac{1}{2}\beta(1,1) = \frac{1}{8}\frac{1}{2}\frac{\Gamma 1\Gamma 1}{\Gamma 2} = \frac{1}{16}.$$

Exercise 31

1) *Evaluate* $\iiint (x^2+y^2)\, dx\, dy\, dz$, *over the region bounded by the surface* $x^2+y^2=2z$ *and the plane* $z=2$

Ans : $\frac{16\pi}{3}$

2) *Evaluate* $\iiint \sqrt{x^2+y^2}\; dx\, dy\, dz$ *over the volume bounded by the right circular cone* $x^2+y^2=z^2$, $z>0$ *and the planes* $z=0$ *and* $z=1$

Ans : $\frac{\pi}{6}$

3) Evaluate $\iiint x^2\, dx\, dy\, dz$ *over the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Ans : $\frac{4\pi}{15}a^3bc$

Let's check take away from the lecture

1) Evaluate $\iiint x^2 dxdydz$ throughout of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

(a) $\frac{4}{15}\pi a^3bc$ (b) $\frac{4}{3}\pi abc$ (c) πabc (d) abc

Home Work Problems for the day

1) *Evaluate* $\iiint z^2\, dx\, dy\, dz$ *over the volume common to the sphere* $x^2+y^2+z^2=a^2$ *and cylinder* $x^2+y^2=ax$

Ans : $\frac{4a^2}{15}\left(\frac{\pi}{2}-\frac{8}{15}\right)$

2) Evaluate $\iiint z^2 dxdydz$ over the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$

Learning from the topic: Students will be able to evaluate triple integration on ellipsoid, cone and cylinder

Application of Multiple Integrations (Volume)

Lecture: 32

1. **Learning Objective:**
   Learners shall be able to calculate Volume of a region using triple integration.
2. **Introduction:**

$$= \frac{abc}{2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \beta\left(\frac{1}{4},\frac{3}{4}\right) = \frac{abc}{2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{\Gamma 1}$$

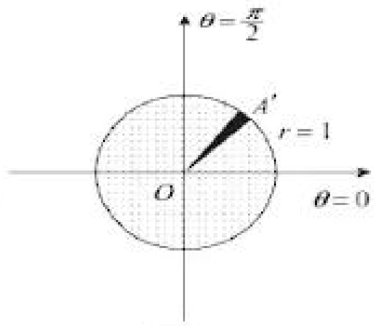
$$= \frac{abc}{2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sqrt{2}} abc \int_0^{2\pi} d\phi \int_0^{\pi} d\theta$$

$$= \frac{\pi}{\sqrt{2}} abc \int_0^{2\pi} d\phi \int_0^{\pi} d\theta = \frac{\pi}{\sqrt{2}} abc \int_0^{2\pi} d\phi\, [\theta]_0^{\pi}$$

$$= \frac{\pi^2}{\sqrt{2}} abc\, [\phi]_0^{2\pi} = \sqrt{2}\pi^3 abc$$

2. Evaluate $\iiint \sqrt{x^2+y^2}\,dx\,dy\,dz$, over the region bounded by the right circular cone $x^2+y^2=z^2, z>0$ and the planes z = 0 and $z=1$

Solution: (1) Putting cylindrical coordinates $x = r\cos\theta,\ y = r\sin\theta,\ z = z$, equation of the cone $x^2+y^2=z^2$ reduces to $r^2=1,\ r=1$.



(2) Draw an elementary volume AB parallel to z-axis in the region which starts from the cone $r = z$ and terminates on the plane $z = 1$

Limits of $z: z = r$ to $z = 1$

(3) Projection of the region in $r\theta$ - plane is the curve of intersection of the cone $r = z$ and the plane $z = 1$ which is obtained as $r = 1$, a circle with center at origin and radius 1.

(4) Draw an elementary radius vector OA' which starts from the origin and terminates on the circle $r = 1$.

Limits of r : $r = 0$ to $r = 1$

Limits of $\theta: \theta = 0$ to $\theta = 2\pi$

$$I = \iiint \sqrt{x^2+y^2}\,dx\,dy\,dz = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{1} r\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^2\,|z|_r^1\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_0^1 r^2(1-r)\,dr$$

$$= |\theta|_0^{2\pi}\cdot\left|\frac{r^3}{3}-\frac{r^4}{4}\right|_0^1 = 2\pi\cdot\frac{1}{12} = \frac{\pi}{6}.$$

3. Evalaute the integral $\iiint_D xyz\,dx\,dy\,dz$ by changing it to cylindrical polar coordinate system where D is bounded by the planes $x=0,\ y=0,\ z=0,\ z=1$ *and the cylinder* $x^2+y^2=1$.

Solution:

$$I = \iiint_D xyz\,dx\,dy\,dz$$

Converting it in cylindrical coordinate system we have, $x = r\cos\theta,\ y = r\sin\theta\ \ z = z$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\int_{z=0}^{1} r\cos\theta\, r\sin\theta\, z r\,dr\,d\theta\,dz$$ ....(for the first octant θ varies from 0 to $\frac{\pi}{2}$

(A) when the region of integration is ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

we put
$$x = a\,r\sin\theta\cos\phi,\ y = b\,r\sin\theta\sin\phi,\ z = c\,r\cos\theta$$
$$dxdydz = abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

In the case of ellipsoid limits of r is always 0 to 1 but limits of θ and ϕ varies as in spherical polar coordinates

(B) If the region of integration is a cylinder of the base radius a we use cylindrical Coordinates we put

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$dxdydz = r\,dr\,d\theta\,dz$

clearly r varies from 0 to a, θ varies from 0 to 2π, z varies from $-\infty$ to ∞

3. Sample Problems

1. Evaluate $\iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\;dx\,dy\,dz$ *where V is the volume of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution : Since the integration is asked on the ellipsoid therefore we use spherical polar

Coordinates we put

$x = a\,r\sin\theta\cos\phi,\ y = b\,r\sin\theta\sin\phi,\ z = c\,r\cos\theta$

$dxdydz = abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$

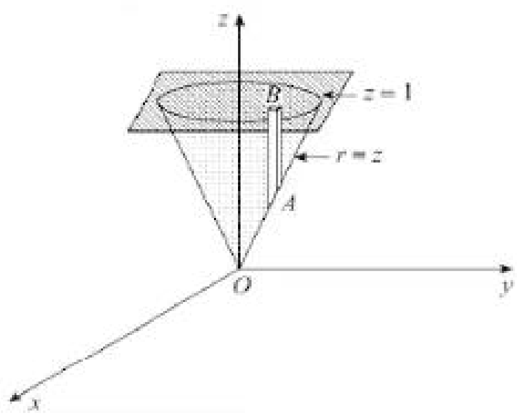
We have

$$I = abc\int_0^{2\pi}\int_0^{\pi}\int_0^1\sqrt{1-r^2}\,dr\,d\theta\,d\phi$$
$$= abc\int_0^{2\pi}\int_0^{\pi}\int_0^1\sqrt{1-r^2}\,dr\,d\theta\,d\phi$$

$$put\ r^2 = t \Rightarrow 2rdr = dt \Rightarrow dr = \frac{dt}{2r^{\frac{1}{2}}}$$

r	0	1
t	0	1

$$= abc\int_0^{2\pi}\int_0^{\pi}\int_0^1(1-t)^{\frac{1}{2}}\frac{dt}{2t^{\frac{1}{2}}}\,d\theta\,d\phi$$
$$= \frac{abc}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt\,d\theta\,d\phi$$



3) *Evaluate* $\iiint x\,y\,z\,(x^2+y^2+z^2)\,dx\,dy\,dz$, *over the first octant of the sphere* $x^2+y^2+z^2=a^2$

Ans : $\frac{a^8}{64}$

4) *Evaluate* $\iiint \frac{dxdydz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ *over the volume bounded by the spheres* $x^2+y^2+z^2=a^2$ *and* $x^2+y^2+z^2=b^2$, $a>b>0$

Ans : $4\pi \log\left(\frac{a}{b}\right)$

1. Using triple integration find $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ the sphere $x^2+y^2+z^2=4$

(a) $\frac{32}{3}\pi$ (b) $\frac{3}{32}\pi$ (c) 32π (d) none of these

1) *Evaluate* $\iiint \frac{z^2\,dx\,dy\,dz}{x^2+y^2+z^2}$ *over the volume of the sphere* $x^2+y^2+z^2=2$

Ans : $\frac{8\pi\ 2}{9}$

2) *Evaluate* $\iiint \frac{dxdydz}{(x^2+y^2+z^2)}$ *over the volume of the sphere* $x^2+y^2+z^2=a^2$

Ans : $-4\pi a$

3) *Evaluate* $\iiint \frac{dxdydz}{a^2-x^2-y^2-z^2}$ *over the sphere* $x^2+y^2+z^2=a^2$

Ans : $a^2\pi^2$

Learning from the topic: Students will be able to evaluate triple integration on tetrahedron and sphere

Evaluation of triple integration over ellipsoid, cone or cylinder

1. **Learning Objective:**
   Learners shall be able to calculate triple integration over the ellipsoid, cone or cylinder.
2. **Introduction:**
   To evaluate triple integration on a cone or cylinder, there is requirement of cylindrical coordinate system.

$$= a^2\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta d\theta \frac{1}{4}\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 2}$$

$$= a^2\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta d\theta \frac{1}{4}\sqrt{\pi}\sqrt{\pi} = \frac{\pi a^2}{4}\int_0^{\frac{\pi}{2}} d\phi\left[-\cos\theta\right]_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4}\int_0^{\frac{\pi}{2}} d\phi\left[-\cos\frac{\pi}{2}+\cos 0\right]$$

$$= \frac{\pi a^2}{4}\int_0^{\frac{\pi}{2}} d\phi\left[0+1\right] = \frac{\pi a^2}{4}\left[\phi\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi a^2}{4}\frac{\pi}{2} = \frac{\pi^2 a^2}{8}$$

$$\text{therefore required integral} = 8\frac{\pi^2 a^2}{8} = \pi^2 a^2$$

3. Evaluate $\iiint x^2 dxdydz$ over the volume of the tetrahedron bounded by $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Solution:

$$I=\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x^2 dxdydz = \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)}\left[z\right]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x^2 dxdy = \int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)x^2 dxdy$$

$$= c\int_0^a x^2\left[\left(1-\frac{x}{a}\right)(y)_0^{b\left(1-\frac{x}{a}\right)} - \frac{1}{b}\left(\frac{y^2}{2}\right)_0^{b\left(1-\frac{x}{a}\right)}\right]dx$$

$$= c\int_0^a x^2\left[\left(1-\frac{x}{a}\right)(b)\left(1-\frac{x}{a}\right) - \frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= c\int_0^a x^2\left[b\left(1-\frac{x}{a}\right)^2 - \frac{b}{2}\left(1-\frac{x}{a}\right)^2\right]dx = c\int_0^a \frac{b}{2}\left(1-\frac{x}{a}\right)^2 x^2 dx = \frac{bc}{2}\int_0^a x^2\left(1-\frac{x}{a}\right)^2 dx$$

Put $\frac{x}{a} = t \Rightarrow dx = adt$

when $x=0, t=0$, & *when* $x=a$, $t=1$.

$$= \frac{bc}{2}\int_0^1 (at)^2(1-t)^2 adt = \frac{a^3bc}{2}\int_0^1 t^{3-1}(1-t)^{3-1}dt = \frac{a^3bc}{2}\beta(3,3) = \frac{a^3bc}{2}\frac{\Gamma 3\Gamma 3}{\Gamma 6} = \frac{a^3bc}{2}\frac{2!2!}{5!} = \frac{a^3bc}{2}\frac{2.2}{120} = \frac{a^3bc}{120}.$$

Exercise 30

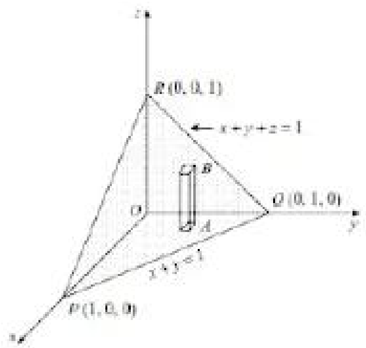
1) *Evaluate* $\iiint \frac{dx\,dy\,dz}{(1+x+y+z)^3}$ *over the volume of the tetrahedron*

$x=0, y=0, z=0, x+y+z=1$

Ans : $\frac{1}{2}\left(\log 2 - \frac{5}{8}\right)$

2) *Evaluate* $\iiint x^2 y z\, dx\,dy\,dz$ *throughout the volume bounded by the planes* $x=0,\ y=0,\ z=0,$ *and* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$x=0,\ y=0, z=0$ *and* $x+y+z=1$



Solution:

$$I=\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}(x+y+z)\,dzdydx$$

$$=\int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{(x+y+z)^2}{2}\right]_0^{1-x-y}dydx$$

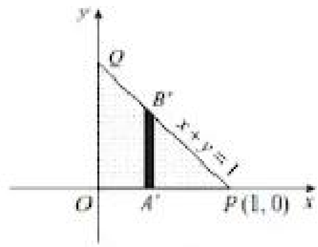
$$=\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{1-x}\left[1-(x+y)^2\right]dydx$$

$$=\frac{1}{2}\int_{x=0}^{1}\left[y-\frac{(x+y)^3}{3}\right]_0^{1-x}dx$$

$$=\frac{1}{2}\int_{x=0}^{1}\left[(1-x)-\frac{1}{3}+\frac{x^3}{3}\right]dx$$

$$=\frac{1}{2}\left[\frac{2}{3}x-\frac{x^2}{2}+\frac{x^4}{12}\right]_0^1$$

$$=\frac{1}{2}\left[\frac{2}{3}-\frac{1}{2}+\frac{1}{12}\right]=\frac{1}{2}.\frac{3}{12}=\frac{1}{8}$$



2. *Evaluate* $\iiint \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}}$ *over the sphere* $x^2+y^2+z^2=a^2$

Solution: since the integration is asked on the sphere therefore, we use spherical polar coordinates by putting

$x=r\sin\theta\cos\phi, y=r\sin\theta\sin\phi, z=r\cos\theta$

$dxdydz=r^2\sin\theta\,dr\,d\theta\,d\phi$

Since the sphere is symmetrical in all four coordinates therefore, we can integrate it in first
coordinate and then take its four times to find the required integral

We have $I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a}\frac{r^2\sin\theta drd\theta d\phi}{\sqrt{a^2-r^2}}$

$$=\int_0^{\frac{\pi}{2}}d\phi\int_0^{\frac{\pi}{2}}\sin\theta d\theta\int_0^{a}\frac{r^2dr}{\sqrt{a^2-r^2}}$$

to solve the last integral put $r=a\sin t \Rightarrow dr=a\cos t\,dt$

r	0	a
t	0	$\frac{\pi}{2}$

$$=\int_0^{\frac{\pi}{2}}d\phi\int_0^{\frac{\pi}{2}}\sin\theta d\theta\int_0^{\frac{\pi}{2}}\frac{a^2\sin^2 t\ a\cos t dt}{a\cos t}\quad=a^2\int_0^{\frac{\pi}{2}}d\phi\int_0^{\frac{\pi}{2}}\sin\theta d\theta\int_0^{\frac{\pi}{2}}\sin^2 t dt$$

$$=a^2\int_0^{\frac{\pi}{2}}d\phi\int_0^{\frac{\pi}{2}}\sin\theta d\theta\frac{1}{2}\beta(\frac{1}{2},\frac{3}{2})\qquad=a^2\int_0^{\frac{\pi}{2}}d\phi\int_0^{\frac{\pi}{2}}\sin\theta d\theta\frac{1}{2}\frac{\Gamma\frac{1}{2}\Gamma\frac{3}{2}}{\Gamma 2}$$

2) $\int_0^4 \int_0^{2\ z} \int_0^{4z-x^2} dz\,dx\,dy$

Ans : 8π

3) $\int_0^1 \int_{y^2}^1 \int_0^{1-x} dx\,dydz$

Ans : $\frac{4}{35}$

4) $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r\,d\theta\,dr\,dz$

Ans : $\frac{5a^3}{64}$

5) $\int_1^e \int_1^{\log y} \int_0^{e^x} \log z\ \ dzdxdy$

Ans : $\frac{1}{4}(e^2-8e+13)$

Learning from the topic: Learners will be able solve direct evaluation on triple integration.

Evaluation of triple integration over tetrahedron and sphere

1. **Learning Objective:**
Learners shall be able to calculate triple integration over the given three-dimensional region.
2. **Introduction:**
To evaluate triple integration on a given volume we use triple integration where we extend the concept of double integration in three dimensions by taking the solid three dentitional beams instead of strip of two dimensions in Cartesian coordinate system.

(A) When the region of integration is not bounded by the planes but by sphere then we use spherical polar coordinates as

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

$$dxdydz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

(i) If the region of integration is the whole of the sphere $x^2+y^2+z^2=a^2$ then clearly r varies from 0 to a, θ varies from 0 to π, ϕ varies from 0 to 2π

(ii) If the region of integration is the hemisphere of radius a then clearly r varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$, ϕ varies from 0 to 2π

(iii) If the region of integration is the first octant of the sphere of radius a then clearly r varies from 0 to a, θ varies from 0 to $\frac{\pi}{2}$, ϕ varies from 0 to $\frac{\pi}{2}$

3 Sample Problems

$$= \int_1^e \left[y\log y - 2y + \log y + e - 1 \right] dy$$

$$= \left[\log y \frac{y^2}{2} - \int \frac{1}{y}\frac{y^2}{2}\, dy \right]_1^e - 2\left(\frac{y^2}{2}\right)_1^e + \left[y\log y - y \right]_1^e + (e-1)\left(y \right)_1^e$$

$$= \left[\frac{y^2}{2}\log y - \frac{y^2}{4} \right]_1^e - \left(e^2 - 1\right) + \left[y\log y - y \right]_1^e + (e-1)(e-1)$$

$$\left(\left(\frac{e^2}{2}\log e - \frac{e^2}{4}\right) - \left(0 - \frac{1}{4}\right)\right) - (e^2 - 1) + \left((e-e) - (0-1)\right) + (e-1)^2$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} - (e^2 - 1) - 1 + (e-1)^2 = \frac{e^2}{4} - e^2 + 1 + \frac{1}{4} - 1 + e^2 - 2e + 1 = \frac{e^2}{4} - 2e + \frac{9}{4}.$$

Exercise 29

Evaluate the following:

1) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \; dx\,dy\,dz$

Ans : $\frac{a^6}{48}$

2) $\int_0^{\pi} \int_0^{a(1+\cos\theta)} \int_0^{h} 2\left(1 - \frac{r}{\ldots}\right) r \; dr\, d\theta\, dz$

7. Sample Problems: Direct Evaluation of Integral

1. **Evaluate** $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz$

Solution: $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz$

$$=\int_0^{\log 2}dx\int_0^{x}dy\int_0^{x+y} e^{x+y+z}dz=\int_0^{\log 2}dx\int_0^{x}dy\left(e^{x+y+z}\right)_0^{x+y}$$

$$=\int_0^{\log 2}dx\int_0^{x}dy\left(e^{x+y+x+y}-e^{x+y}\right)=\int_0^{\log 2}dx\int_0^{x}\left(e^{2x+2y}-e^{x+y}\right)dy$$

$$=\int_0^{\log 2}dx\int_0^{x}\left(e^{2x}\,e^{2y}-e^{x}e^{y}\right)dy=\int_0^{\log 2}dx\left[e^{2x}\left(\frac{e^{2y}}{2}\right)_0^{x}-e^{x}\left(e^{y}\right)_0^{x}\right]$$

$$=\int_0^{\log 2}dx\left[e^{2x}\left(\frac{e^{2x}}{2}-\frac{1}{2}\right)-e^{x}\left(e^{x}-1\right)\right]=\int_0^{\log 2}\left[\frac{e^{4x}}{2}-\frac{e^{2x}}{2}-e^{2x}+e^{x}\right]dx$$

$$=\int_0^{\log 2}\left[\frac{e^{4x}}{2}-\frac{e^{2x}}{2}-e^{2x}+e^{x}\right]dx=\int_0^{\log 2}\left[\frac{e^{4x}}{2}-\frac{3e^{2x}}{2}+e^{x}\right]dx=\left[\frac{e^{4x}}{8}-\frac{3e^{2x}}{4}+e^{x}\right]_0^{\log 2}$$

$$=\left[\frac{e^{4\log 2}}{8}-\frac{1}{8}-\frac{3e^{2\log 2}}{4}+\frac{3}{4}+e^{\log 2}-1\right]=\left[\frac{16}{8}-\frac{1}{8}-\frac{12}{4}+\frac{3}{4}+2-1\right]$$

$$=\left[2-\frac{1}{8}-3+\frac{3}{4}+2-1\right]=\frac{5}{8}$$

2. Evaluate $\int_0^{2}\int_1^{z}\int_0^{yz} xyz\,dx\,dy\,dz$

Solution : The inner most limits depend on y and z. Hence integrating first w.r.t. x,

$$I=\int_0^{2}\left\{\int_1^{z}\left(\int_0^{yz}x\,dx\right)y\,dy\right\}z\,dz=\int_0^{2}\left\{\int_1^{z}\left|\frac{x^2}{2}\right|_0^{yz}y\,dy\right\}z\,dz$$

$$=\frac{1}{2}\int_0^{2}\left\{\int_1^{z}(y^2z^2)\,y\,dy\right\}z\,dz=\frac{1}{2}\int_0^{2}\left|\frac{y^4}{4}\right|_1^{z}z^3dz$$

$$=\frac{1}{8}\int_0^{2}(z^4-1)z^3dz=\frac{1}{8}\left|\frac{z^8}{8}-\frac{z^4}{4}\right|_0^{2}$$

$$=\frac{1}{8}(32-4)=\frac{7}{2}.$$

Q.3 Evaluate $\int_1^{e}\int_1^{\log y}\int_0^{e^x}\log z\,dz\,dx\,dy$

$$=\int_1^{e}\int_1^{\log y}\left[z\log z-1\right]_1^{e^x}dxdy$$

$$=\int_1^{e}\int_1^{\log y}\left[\left(e^x\log e^x-e^x\right)-(0-1)\right]dxdy$$

$$=\int_1^{e}\int_1^{\log y}\left[xe^x-e^x+1\right]dxdy$$

$$=\int_1^{e}\left[xe^x-e^x-e^x+x\right]_1^{\log y}dy$$

$$=\int_1^{e}\left[\left(\log y\,e^{\log y}-2e^{\log y}+\log y\right)-\left(e-e-e+1\right)\right]dy$$

Module 5: Triple Integration

Lecture: 29

1. Motivation: This topic deals with vector integration of a function in three dimensional spaces. In this chapter, the vector integral extends the integrals to integrals over curves, surfaces and volumes. These different kinds of integrals can be transformed into one another. Such transformations are done by the powerful formulas of Green (line integrals into double integrals), Gauss (surface integrals into triple integrals), and Stokes (line integrals into surface integrals). These integrals have basic engineering applications in fluid flow, heat problems and in solid mechanics.

2. Syllabus

Module	**Content**	**Class Duration**	**Self-Study duration**	**Weightage**
5	Triple integrals (Cartesian) Orthogonal curvilinear coordinates, Simple applications involving cubes, Sphere Vector line integrals, Vector surface integrals, Theorem of Greens, Gauss and Stokes (only evaluation)	2 hrs lectures 2 hrs lectures 2 hrs lectures 3 hrs lectures	18 hours	**16 Marks**

3. Prerequisite:

Standard formulae of integration, single integration and its applications, evaluation of integration on any region

4. Learning Objective:

1. Students will be able to evaluate triple integration.
2. Students will be able to convert triple integration in spherical polar and cylindrical form.
3. Students will be able find the limits of triple integrals in the given region.
4. Students will be able to apply the concept of multiple integral to find area, mass and volume.

5. Key Notations:

$\iiint_v f(x,y,z)\,dxdydz$ is the integration of f(x,y,z) in the volume v

6. Important Formulae/ Theorems / Properties:

(1) $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

(2) $B(m,n) = \dfrac{m\ n}{m+n}$

Self-Evaluation

Name of student: **Class & Div:** **Roll No:**

1. Do you understand how to evaluate double integration?

(a) Yes (b) No

2. Will you be able to change the order of integration and evaluate?

(a) Yes (b) No

3. Are you able to draw the region of integration?

(a) Yes (b) No

4. Are you able to change to polar form and evaluate the integral?

(a) Yes (b) No

5. Do you understand how to find the limits of integration in the given region?

(a) Yes (b) No

6. Do you understood this module?

(a) Fully understood (b) Partially understood (c) Not at all

Self-Assessment

Level 1

1) *Evaluate* $\int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\, dx\, dy$

2) Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx dy.$

3) *Evaluate* $\iint y^2 dx dy$ *over the area outside* $x^2+y^2-2x=0$ *and inside* $x^2+y^2-4x=0.$

Level 2

1) Express in polar coordinates $\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} f(x,y)\; dx\; dy$

2) Change to polar coordinates and evaluate $\int_0^2 \int_x^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy dx$

3) Change to polar coordinates and evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$

Level 3

1) *Change the order of integration* $\int_0^1 \int_{2y}^{2(1+\sqrt{1-y})} f(x,y)\; dx dy$

2) Change the order of integration and evaluate $\int_0^2 \int_{\sqrt{2y}}^2 \frac{x^2}{\sqrt{x^4-4y^2}}\quad dx\; dy$

3) *Change the order of integration and evaluate* $\int_0^a \int_0^y \frac{x\; dy\; dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$

4) Evaluate $\iint_R \frac{2xy^2}{\sqrt{1+x^2y^2-y^4}}\, dxdy$, where R is a triangle whose vertices are (0, 0), (1, 1), (0, 1)

Add to the Knowledge:
Integral calculus has many applications in our daily life. In Isaac Newton's day, one of the biggest problems was poor navigation at sea. Before calculus was developed, the stars were vital for navigation. Shipwrecks occurred because the ship was not where the captain thought it should be. There was not a good enough understanding of how the Earth, stars and planets moved with respect to each other. Calculus was developed to improve this understanding. Differentiation and integration can help us solve many types of real-world problems.
The Multiple integrals have many interesting applications. Double integrals extend the possibilities of one-dimensional integration. It is basically the concept in one dimension apply to a higher dimension e.g. the line in one dimension becomes the surface in two dimensions. Extending this idea to the integral calculus, the **single integral** get converted to the **double integral**. Although, in real life, objects are three-dimensional. The simplification to two dimensions is instructive, because many of the calculations performed will carry over to the three-dimensional case with little modification. Also, in some physical applications, it is sufficient to approximate a thin three-dimensional object as simply a two-dimensional object. Some of the applications of multiple integrals will be discussed in detail in Module 6.

Learning Outcomes

1. **Know:** Student should be able to evaluate different types of double and triple integrals
2. **Comprehend:** Learners should be able to evaluate different types of double and triple integrals
3. **Apply, analyze and synthesize:** Student should be able to solve the double and triple integrals

$$\bar{x} = \frac{2}{9}\int_{-1}^{2}(x^2+2x-x^3)dx = \frac{2}{9}\left[\frac{x^3}{3}+x^2-\frac{x^4}{4}\right]_{-1}^{2} = \frac{1}{2}$$

$$\bar{y} = \frac{1}{M}\iint_R y\rho(x,y)dxdy = \frac{1}{M}\iint_R y\rho(x,y)dxdy$$

$$= \frac{2}{9}\int_{-1}^{2}\int_{y=x^2}^{x+2} ydydx = \frac{2}{9}\int_{-1}^{2}\left(\frac{y^2}{2}\right)_{x^2}^{x+2}dx$$

$$= \frac{1}{9}\int_{-1}^{2}((x+2)^2-x^4)dx$$

$$= \frac{1}{9}\left[\frac{(x+2)^3}{3}-\frac{x^5}{5}\right]_{-1}^{2} = \frac{1}{9}\left[\frac{1}{3}(64-1)-\frac{1}{5}(32+1)\right]$$

$$= \frac{1}{9}\left(21-\frac{33}{5}\right) = \frac{8}{5}$$

Exercise 28

1) Find the center of gravity of the area bounded by the parabola a $y^2 = x$ and $y + x = 2$.

2) Find the center of gravity of the area between $y = 6x - x^2$ and $y = x$.

Let's check take away from the lecture

1) Find the mass of the lamina in the form of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If the density at any point varies as the product of the distance from the axes of the ellipse.

(a) $\frac{\lambda a^2 b^2}{2}$ (b) $\pi^2 a^2 b^2$ (c) $\frac{\pi a^2 b}{4}$ (d) $\pi a^2 b^2$

Home Work Problems for the day

1) Find the center of gravity of the area in the first quadrant lying between the curves $y^2 = x^3$ and $y = x$

2) Find the center of gravity of the area of the circle $y^2 + x^2 = a^2$ lying in the first quadrant

Mass: Integration $\iint_R f(x,y)dxdy$ on region 'R' gives us mass of the region 'R' where $f(x,y)$ is the density function

Center of Mass: Center of Mass in Cartesian form is given by

$$\bar{x}=\frac{M_y}{M} \quad , \quad \bar{y}=\frac{M_x}{M}$$

$$\bar{x}=\frac{\iint_R y\rho\,dxdy}{\iint_R \rho\,dxdy} \quad ,\bar{y}=\frac{\iint_R x\rho\,dxdy}{\iint_R \rho\,dxdy}$$

Center of Gravity:

(1) In Cartesian form the center of gravity of a plane lamina occupying an area R in xy plane and having density ρ is given by

$$\bar{x}=\frac{\iint_R x\rho\,dxdy}{\iint_R \rho\,dxdy} \quad ,\bar{y}=\frac{\iint_R y\rho\,dxdy}{\iint_R \rho\,dxdy}$$

(2) In Polar form the center of gravity of a plane lamina occupying an area R in xy plane and having density ρ is given by

$$\bar{x}=\frac{\iint_R r^2\cos\theta\rho\,drd\theta}{\iint_R \rho r\,drd\theta} \quad ,\bar{y}=\frac{\iint_R r^2\sin\theta\rho\,drd\theta}{\iint_R \rho r\,drd\theta}$$

4. Sample Problem:

1.Find the center of gravity of a plate whose density $\rho(x,y)$ is constant and is bounded by the curves $y=x^2$ and $y=x+2$.

Solution: density is given by $\rho=k$ Mass of the plate is given by

$$M=\iint_R \rho(x,y)dxdy=k\iint_R dxdy$$

$$=k\int_{x=-1}^{2}\int_{y=x^2}^{x+2} dydx \quad =k\int_{x=-1}^{2}(y)_{x^2}^{x+2}dx$$

$$=k\int_{x=-1}^{2}(x+2-x^2)dx \quad =k\left[-\frac{x^3}{3}+\frac{x^2}{2}+2x\right]_{-1}^{2}$$

$$=k\left[-\frac{9}{3}+\frac{3}{2}+6\right] \quad =\frac{9}{2}k$$

The centre of gravity is given by

$$\bar{x}=\frac{1}{M}\iint_R x\rho(x,y)dxdy \;=\frac{2}{9}\int_{-1}^{2}\int_{x^2}^{x+2} xdydx$$

$$\bar{x}=\frac{2}{9}\int_{-1}^{2}(y)_{x^2}^{x+2}xdx \quad =\frac{2}{9}\int_{-1}^{2}(x+2-x^2)xdx$$

Let's check take away from the lecture

1. To find the area of a region R

(a) $\iiint_R dxdydz$ (b) $\iint_R dxdy$ (c) $\iint_R dxdydz$ (d) $\iiint_R dxdydz$

2. Find the area common to the parabolas $y = x^2$ and $y = 4 - x^2$

(a) $\frac{16\sqrt{2}}{3}$ (b) $16\sqrt{2}$ (c) $\frac{1}{3}$ (d) none of these

1. Using double integration find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(a) πab (b) $\pi^2 ab$ (c) $\pi a^2 b$ (d) $\pi a^2 b^2$

Home Work Problems for the day

1) *Find the area bounded by* $y^2 = x, x^2 = -8y$.

Ans : $\frac{8}{3}$

2) *Find by double integration the smaller of the area bounded by the circle* $x^2 + y^2 = 9$ *and a straight line* $x = 3 - y$

Ans : $\frac{9}{4}(\pi - 2)$

3) Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ans: πab

Learning Outcome: Learners will be able find the area bounded by two curves in cartesian and polar coordinates.

Center of Mass and Gravity

Lecture: 28

1. **Learning Objective:**
Learners shall be able to calculate center of mass and gravity using double integration.
2. **Introduction:**
To evaluate mass of any plane lamina, double integration can be utilized in most effective way, where as to find the mass of a plane lamina the density function is required which is given in every problem.

3. Key Definitions:

$$\text{Area} = \int_0^1 \int_{x^{\frac{3}{2}}}^{x} dy\,dx$$

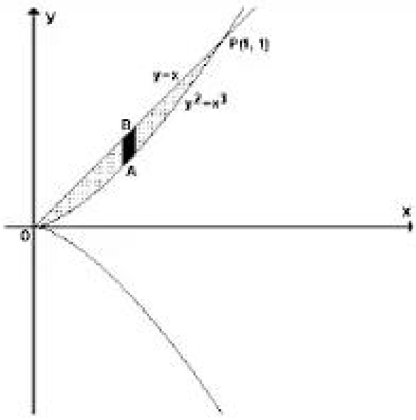
$$= \int_0^1 \left|y\right|_{x^{\frac{3}{2}}}^{x} dx$$

$$= \int_0^1 (x - x^{\frac{3}{2}})dx$$

$$= \left[\frac{x^2}{2} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1$$

$$= \frac{1}{2} - \frac{2}{5}$$

$$= \frac{1}{10}.$$



3. Find the area bounded by the curves $xy = 2, 4y = x^2, y = 4$

Solution: xy=2 (rectangular hyperbola)

$x^2 = 4y$ (Parabola)

y=4 (stright line)

The Point of intersection:

$$x^2 = 4y \Rightarrow y = \frac{x^2}{4}$$

$$\therefore xy = 2 \Rightarrow x.\frac{x^2}{4} = 2 \Rightarrow x^3 = 8.$$

$$\therefore x = 2, y = 1 \Rightarrow (2,1)$$

Considering horizontal strip,

$$\therefore Area = \int_1^4 \int_{x=\frac{2}{y}}^{x=\sqrt{4y}} dxdy = \int_1^4 \left[x\right]_{x=\frac{2}{y}}^{x=\sqrt{4y}} dy = \int_1^4 \left[2y^{\frac{1}{2}} - \frac{2}{y}\right]dy = 2\left(\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right) - 2\left(\log y\right)_1^4 = \frac{4}{3}(8-1) - 2(\log 4 - 0) = \frac{28}{3} - 2\log 4.$$

Exercise 27

1) Find the area between parabola $y = x^2 - 6x + 3$ and the line $y = 2x - 9$

Ans: $\frac{32}{3}$

2) Find the area enclosed by the curve 9xy = 4 and the line 2x+y = 2.

Ans : $-\frac{1}{3} - \frac{4}{9}\log 2$

3) *Find the area bounded by the lines* $y = 2 + x$, $y = 2 - x$ *and* $x = 5$

Ans : 25

Application of Double Integration (Area)

Lecture 27

1. Learning Objective:

Learners shall be able to calculate area of a region using double integration.

2. Introduction:

To evaluate area of any region, double integration can be utilized in most effective way.

3. Key Definitions:

Area: Integration of $\iint_R dxdy$ on any region 'R' gives us area of the region 'R'

4. Sample Problem:

1. Find the area between parabola $y = x^2 - 6x + 3$ and the line $y = 2x - 9$

Solution:

we have $y = x^2 - 6x + 3$ $y = (x-3)^2 - 6$ $y + 6 = (x-3)^2$

Which is a parabola with the vertex at (3,-6) and opening upwards

Required area, $A = \int_{x=2}^{6} \int_{y=x^2-6x+3}^{2x-9} dxdy$

$$= \int_{x=2}^{6} [y]_{y=x^2-6x+3}^{2x-9} dx$$

$$= \int_{x=2}^{6} \left[2x - 9 - x^2 + 6x - 3\right] dx$$

$$= \int_{x=2}^{6} \left[8x - 12 - x^2\right] dx$$

$$= \left[8\frac{x^2}{2} - 12x - \frac{x^3}{3}\right]_2^6$$

$$= \left[4x^2 - 12x - \frac{x^3}{3}\right]_2^6$$

$$= 8 + \frac{8}{3} = \frac{32}{3}$$

2. Find the area enclosed by the curve $y^2 = x^3$ and the line $y = x$.

Solution:

Points of intersection of the curve $y^2 = x^3$ and the line $y = x$ are obtained as

$x^2 = x^3$

$x^3 - x^2 = 0,\ x^2(x-1) = 0,\ x = 0,1$ and $y = 0,1$ Hence, point $O:(0,0)$ and $P:(1,1)$

Draw a vertical strip AB which starts from the parabola $y^2 = x^3$ and terminates on the line $y = x$.

Limits of $y : y = x^{\frac{3}{2}}$ to $y = x$

Limits of $x : x = 0$ to $x = 1$

4) $\iint \frac{dxdy}{\sqrt{xy}}$ *over the region bounded by* $x^2 + y^2 - x = 0,\ y = 0, y > 0.$

Ans : $\frac{\pi}{\sqrt{2}}$

5) $\iint \frac{dx\,dy}{\left(1 + x^2 + y^2\right)^2}$, *over one loop of the lemniscate* $\left(x^2 + y^2\right)^2 = x^2 - y^2$

Ans : $\frac{\pi - 2}{4}$

Let's check take away from lecture

Choose the correct option from the following

1. The value of dx dy in polar form is

   (a) $r^2 dr d\theta$ (b) $r dr d\theta$ (c) $dr d\theta$ (d) none of these

2. The limits of r and θ are identified from the region of integration

   (a) True (b) False

Homework Problems for the day

Evaluate the following

1) $\iint r^4 \cos^3\theta \ dr d\theta$, *over the interior of* $r = 2a\cos\theta$. **Ans :** $\frac{7\pi}{4} a^5$

2) $\int_R \int dx\,dy$, *over the region bounded by* $r = 1 + \cos\theta$. **Ans :** $\frac{3\pi}{2}$

3) $\iint y^2 dx dy$ *over the area outside* $x^2 + y^2 - ax = 0$ *and inside* $x^2 + y^2 - 2ax = 0.$

Ans : $\frac{15\pi a^4}{64}$

4) $\iint \frac{4xy}{x^2 + y^2} e^{-x^2 - y^2} dx\,dy$, *over the +ve quadrant bounded by* $x^2 + y^2 - x\ \theta$ *and* $x\ 4$

Ans : $\frac{1}{e}$

5) $\iint \frac{(x^2 + y^2)^2}{x^2 y^2} dx\,dy$, *over the area common to the circles* $x^2 + y^2 = ax$ *and*

$x^2 + y^2 = by;\ a, b > 0$

6) $\iint r^3\, dr\, d\theta$ *bounded by* $r = 2\sin\theta$ *and* $r = 4\sin\theta.$ **Ans :** $\frac{45\pi}{2}$

Learning Outcome: Learners will be able to convert in to polar form and Evaluate the

Ex. 2 : Evaluate $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$

Solution : Limits of $x : x = \frac{y^2}{4a}$ to $x = y$

Therefore, horizontal strip starts from the parabola $y^2 = 4ax$ and terminates on the line $y = x$ limits of $y : 0$ to $4a$.

The region of integration is bounded by the line $y = x$ and the parabola $y^2 = 4ax$

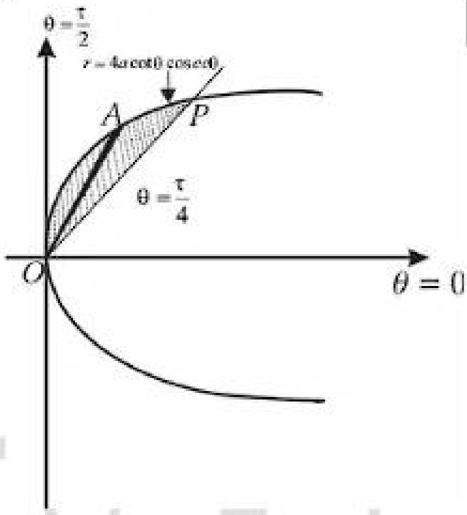
Putting $x = r\cos\theta$ and $y = r\sin\theta$ polar form of the line $y = x$ is $r\sin\theta = r\cos\theta$ gives $\tan\theta = 1$, $\theta = \frac{\pi}{4}$

parabola $y^2 = 4ax$ $r^2\sin^2\theta = 4ar\cos\theta$ $r = 4a\cot\theta\,\text{cosec}\,\theta$

Draw an elementary radius vector OA which starts from the origin and terminates on the parabola $r = 4a\cot\theta\,\text{cosec}\,\theta$.

Therefore limits of $r : r = 0$ to $r = 4a\cot\theta\,\text{cosec}\,\theta$

limits of $\theta : \theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$



$$I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4a\cot\theta\,\text{cosec}\,\theta} \frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(1-2\sin^2\theta\right)\left|\frac{r^2}{2}\right|_0^{4a\cot\theta\,\text{cosec}\,\theta} d\theta = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-2\sin^2\theta)(4a)^2\cot^2\theta\,\text{cosec}^2\theta\,d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\cot^2\theta\,\text{cosec}^2\theta - 2\cot^2\theta\right)d\theta = 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[\left\{-\left(\cot^2\theta\right)\left(-\text{cosec}^2\theta\right)\right\} - 2\,\text{cosec}^2\theta + 2\right]d\theta$$

$$= 8a^2\left|-\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \qquad \left[\because \int\{f(\theta)\}^n f'(\theta)\,d\theta = \frac{\{f(\theta)\}^{n+1}}{n+1}\right]$$

$$= 8a^2\left[-\frac{1}{3}\left(\cot^3\frac{\pi}{2} - \cot^3\frac{\pi}{4}\right) + 2\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right) + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$= 8a^2\left[-\frac{1}{3}(-1) + 2(-1) + 2\cdot\frac{\pi}{4}\right] = 8a^2\left[-\frac{5}{3} + \frac{\pi}{2}\right] = 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]$$

Exercise 26

Change to polar and evaluate the following:

1) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{\left(a^2+x^2+y^2\right)^{\frac{3}{2}}}$ **Ans :** $\frac{2\pi}{a}$

2) $\int_0^1\int_0^x (x+y)\,dy\,dx$ **Ans :** $\frac{1}{2}$

3) $\int_0^1\int_x^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ **Ans :** $\frac{1}{3}\left[4 - \frac{5}{\sqrt{2}}\right]$

Change to polar and evaluate

Lecture: 26

1. Learning Objective: Learners will be able to change the integrand to polar form and evaluate the integral.

2. Introduction:

The double integral can be changed from Cartesian coordinates (x, y) to polar coordinates (r, θ) by putting $x = r\cos\theta \quad y = r\sin\theta$

then $\iint f(x,y)\,dy\,dx = \iint f(r\cos\theta,\ r\sin\theta)|J|\,dr\,d\theta$

where J is the Jacobian (functional determinant)

defined as

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Hence, $\iint f(x,y)dy\,dx = \iint f(r\cos\theta,\ r\sin\theta)|r|\,dr\,d\theta$

$$= \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

3. Sample Problems

Ex. 1: Evaluate $\iint \frac{x^2y^2}{(x^2+y^2)}dx\,dy$, over the region bounded by the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2\ (a>b)$.

Solution: (I) Putting $x = r\cos\theta \quad y = r\sin\theta$, polar form of the

circle $x^2+y^2 = a^2$ is $r^2 = a^2, r = a$

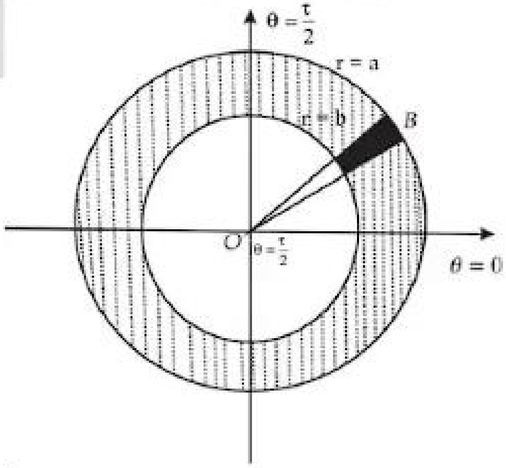
circle $x^2+y^2 = b^2$ is $r^2 = b^2, r = b$

Region of integration is the part bounded between the circles $r = a$ and $r = b$.

Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = b$ and leaves at the circle $r=a$.

Limits of r : $r = b$ to $r = a$

Limits of $\theta = 0$ to $\theta = 2\pi$



$$I = \iint \frac{x^2y^2}{(x^2+y^2)}dx\,dy = \int_0^{2\pi}\int_b^a \frac{r^4\cos^2\theta\sin^2\theta}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\cos^2\theta\sin^2\theta\left|\frac{r^4}{4}\right|_b^a d\theta = \int_0^{2\pi}\frac{\sin^2 2\theta}{4}\cdot\frac{(a^4-b^4)}{4}d\theta$$

$$= \frac{a^4-b^4}{16}\int_0^{2\pi}\frac{(1-\cos4\theta)}{2}d\theta = \left(\frac{a^4-b^4}{32}\right)\cdot\left|\theta - \frac{\sin4\theta}{4}\right|_0^{2\pi}$$

$$= \left(\frac{a^4-b^4}{32}\right)(2\pi) = \frac{\pi}{16}(a^4-b^4).$$

2. $\int_0^1 \int_x^{2-x} \frac{x}{y}\, dy\, dx$

Ans : $\int_0^1 \int_0^y \frac{x}{y}\, dx\, dy + \int_1^2 \int_0^{2-y} \frac{x}{y}\, dx\, dy = \log\left(\frac{4}{e}\right)$

3. $\int_0^5 \int_{2-x}^{2+x} dy\, dx$

Ans: $\int_{-3}^2 \int_{2-y}^5 dx\, dy + \int_2^7 \int_{y-2}^5 dx\, dy = 25$

4. $\int_0^a \int_0^y \frac{x\, dy\, dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$

Ans : $\int_0^a \int_x^a \frac{x\, dx dy}{\sqrt{(a^2-x^2)(a-y)(y-x)}} = \pi a$

5. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx dy.$

Ans: $\int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy = \frac{2-\sqrt{2}}{2}$

6. $\int_0^2 \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx dy$

Ans : $\int_0^4 \int_0^{\sqrt{4x-x^2}} dx\, dy = 2\pi$

7. $\int_0^{\frac{1}{2}} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\, dx\, dy$

Ans : $\int_0^1 \int_0^{\frac{\sqrt{1-x^2}}{2}} \frac{1+x^2}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}}\, dx\, dy = \frac{\pi^2}{8}$

8. $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\, dx\, dy$ *Ans :* $\frac{\pi^3}{16}$

9. *Show that* $\int_0^3 \int_{\frac{y^2}{9}}^{\sqrt{10-y^2}} dx dy = \frac{1}{2} + 5\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$

10. Change the order of integration and evaluate $\int_0^2 \int_{\sqrt{2y}}^2 \frac{x^2}{\sqrt{x^4-4y^2}}\, dx dy$

Learning outcome: Learners will be able to change the order of integration in the given region and evaluate the integral.

Change the order of integration of the following and evaluate:

1) $$\int_0^{\pi}\int_0^{x} \frac{\sin y\, dy dx}{(\pi - x)(x-y)}$$

$$Ans: \int_0^{\pi}\int_y^{\pi} \frac{\sin y \;\; dxdy}{(\pi - x)\,(x-y)} = 2\pi$$

2) $$\int_0^{1}\int_0^{1-x^2} \frac{e^y}{(e^y+1)\;\; 1-x^2-y^2}\; dxdy$$

$$Ans: \int_0^{1}\int_0^{1-y^2} \frac{e^y}{(e^y+1)\;\; 1-x^2-y^2}\; dy\; dx = \frac{\pi}{2}\log\left(\frac{e+1}{2}\right)$$

3) $$\int_0^{a}\int_{\frac{y^2}{a}}^{y} \frac{y\,dx\,dy}{(a-x)\;\; ax-y^2}$$

$$Ans: \frac{\pi a}{2}$$

4) $$\int_0^{2}\int_{2x}^{2} \frac{y^2}{y^4-4x^2}\; dx\; dy$$

$$Ans: \frac{2\pi}{3}$$

Choose the correct option from the following

1) The identification of order of integration is done with the help of given limits.

(a) True (b) False

2) The change of order of integration is required if integral cannot be evaluated with the given limits

(a) True (b) False

Change the order of integration of the following and evaluate:

1. $$\int_0^{\infty}\int_0^{x} x e^{-\frac{x^2}{y}}\; dy\; dx$$

$$Ans: \int_0^{\infty}\int_y^{\infty} x\, e^{-\frac{x^2}{y}}\; dx\, dy = \frac{1}{2}$$

Ex. 2 : $\int_0^1 \int_x^{1/x} \frac{y}{(1+xy)^2(1+y^2)}\,dx\,dy$

Solution

i. Here the function is integrated first w.r.t. y but evaluation becomes easier by changing the order of integration.
Limits of y : $y = x$ to $y = \frac{1}{x}$

ii. Therefore, vertical strip starts from the line $y = x$ and terminates on the rectangular hyperbola $xy = 1$
Limits of x : $x = 0$ to $x = 1$

iii. The region is bounded by the rectangular hyperbola $xy=1$, the line $y = x$ and y-axis in the +ve quadrant.

iv. Point of intersection of $xy = 1$ and $y = x$ is obtained as $x^2 = 1, x = 1$ and $y = 1$ Hence $P : (1, 1)$

v. To change the order of integration i.e. to integrate first w.r.t. x divide the region into two sub regions OPQ and QPR. Draw a horizontal strip parallel to x-axis in each sub region.

vi. In sub region OPQ strip AB starts from y-axis and terminates on the line $y = x$.
Therefore, limits of x: $x = 0$ to $x = y$
limits of y : $y = 0$ to $y = 1$

vii. In sub region QPR strip CD starts from y-axis and terminates on the rectangular hyperbola $xy=1$
Therefore limits of x : $x = 0$ to $x = \frac{1}{y}$.
limits of y : $y = 1$ to $y \to \infty$
Hence given integral after changing the order can be written as

$$\int_0^1 \int_x^{1/x} \frac{y}{(1+xy)^2(1+y^2)}\,dx\,dy = \int_0^1 \frac{y}{1+y^2}\int_0^y \frac{1}{(1+xy)^2}\,dx\,dy + \int_1^\infty \frac{y}{1+y^2}\int_0^{1/y}\frac{1}{(1+xy)^2}\,dx\,dy$$

$$= \int_0^1 \frac{y}{1+y^2}\left[-\frac{1}{y(1+xy)}\right]_0^y dy + \int_1^\infty \frac{y}{1+y^2}\left[-\frac{1}{y(1+xy)}\right]_0^{1/y} dy$$

$$= -\int_0^1 \frac{1}{1+y^2}\left[\frac{1}{1+y^2}-1\right]dy - \int_1^\infty \frac{1}{1+y^2}\left[\frac{1}{2}-1\right]dy = -\int_0^1\left[\frac{1}{(1+y^2)^2}-\frac{1}{1+y^2}\right]dy + \frac{1}{2}\int_1^\infty \frac{1}{1+y^2}\,dy$$

putting $y = \tan\theta$ in the first from of first integral $dy = \sec^2\theta\, d\theta$,
when $y = 0$, $\theta = 0$, when $y = 1$, $\theta = \frac{\pi}{4}$

$$\int_0^1 \int_x^{1/x} \frac{y}{(1+xy)^2(1+y^2)}\,dx\,dy = -\int_0^{\pi/4}\frac{\sec^2\theta\, d\theta}{\sec^4\theta} + \left[\tan^{-1} y\right]_0^1 + \frac{1}{2}\left[\tan^{-1} y\right]_1^\infty$$

$$= -\int_0^{\pi/4}\frac{(1+\cos 2\theta)}{2}\,d\theta + \left(\tan^{-1} 1 - \tan^{-1} 0\right) + \frac{1}{2}\left(\tan^{-1}\infty - \tan^{-1} 1\right) = -\frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} + \frac{3\pi}{8}$$

$$= -\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} + \frac{3\pi}{8} = \frac{\pi}{4} - \frac{1}{4} = \frac{\pi-1}{4}.$$

Change the order of integration and evaluation

1. **Learning Objective:** Learners will be able to change the order of integration in the given region and evaluate the integral

2. **Introduction:**
In this section method of change the order and then evaluation of the integral will be explained with the help of illustrations.

3. **Sample Problems**

Ex.1: $\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy$

Solution:

Here the function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.

Limits of $x : x = 0$ to $x = a-\sqrt{a^2-y^2}$

Therefore, horizontal strip starts from y-axis and terminates on the circle $(x-a)^2+y^2=a^2$.

limits of $y : y = 0$ to $y = a$

The region is bounded by the circle $(x-a)^2+y^2=a^2$, the line $y = a$ and y-axis.

Point of intersection of $(x-a)^2+y^2=a^2$ and $y=a$ is obtained as $(x-a)^2+a^2=a^2$, $x=a$ Hence $P:(a,a)$

To change the order of integration i.e. to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the circle $(x-a)^2+y^2=a^2$ and terminates on the line $y = a$.

Therefore limits of $y : y=\sqrt{2ax-x^2}$ to $y=a$

limits of $x : x = 0$ to $x = a$

Hence given integral after change of order can be written as

$$\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy = \int_0^a \frac{x\log(x+a)}{(x-a)^2}\int_{\sqrt{2ax-x^2}}^{a} y\,dy\,dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left[\frac{y^2}{2}\right]_{\sqrt{2ax-x^2}}^{a} dx = \int_0^a \frac{x\log(x+a)}{(x-a)^2}\cdot\left(\frac{a^2-2ax+x^2}{2}\right)dx = \frac{1}{2}\int_0^a x\log(x+a)\,dx$$

$$= \frac{1}{2}\left[\left[\frac{x^2}{2}\log(x+a)\right]_0^a - \int_0^a \frac{x^2}{2}\cdot\frac{1}{x+a}\,dx\right] = \frac{1}{2}\left[\frac{a^2}{2}\log 2a - \frac{1}{2}\int_0^a\left\{(x-a)+\frac{a^2}{(x+a)}\right\}dx\right]$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log 2a - \frac{1}{2}\left[\frac{x^2}{2} - ax + a^2\log(x+a)\right]_0^a\right] = \frac{1}{4}\left[a^2\log 2a - \frac{a^2}{2} + a^2 - a^2\log 2a + a^2\log a\right]$$

$$= \frac{1}{4}\left[\frac{a^2}{2} + a^2\log a\right] = \frac{a^2}{8}\left[1+2\log a\right]$$

2) $\int_0^a \int_{\sqrt{\frac{a^2-x^2}{4}}}^{\sqrt{a^2-x^2}} f(x,y)\ dxdy$

Ans : $\int_0^{\frac{a}{2}} \int_{\sqrt{a^2-4y^2}}^{\sqrt{a^2-y^2}} f(x,y)\ dxdy + \int_{\frac{a}{2}}^{a} \int_0^{\sqrt{a^2-y^2}} f(x,y)\ dxdy$

3) $\int_0^a \int_x^{\frac{a^2}{x}} f(x,y) dydx$

Ans: $\int_0^a \int_0^y f(x,y)\ dxdy + \int_a^{\infty} \int_0^{\frac{a^2}{y}} f(x,y)\ dxdy$

Let's check take away from lecture

Choose the correct option from the following

1) Change of order indicates

(a) Change of region (b) change of limits (c) no change (d) none of the above

2) What will be the limit of x after changing the order of $\int_{-a}^{a}\int_0^{\frac{x^2}{a}} f(x,y)\,dx\,dy$

(a) 0 to a (b) -a to a (c) 1 to a (d) none of the above

Homework Problems for the day

Change the order of integration of the following

1) $\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)\ dxdy$

Ans: $\int_0^a \int_{\sqrt{a^2-x^2}}^{a} f(x,y)\ dydx + \int_a^{2a} \int_{x-a}^{a} f(x,y)\ dydx$

2) $\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x,y)\ dxdy$

Ans: $\int_{\sqrt{2}}^{2} \int_{4-y^2}^{2} f(x,y)\ dydx + \int_2^4 \int_0^2 f(x,y)\ dydx + \int_4^{16} \int_0^{4-\sqrt{y}} f(x,y)\ dydx$

3) $\int_0^k \int_{-k+\sqrt{k^2-y^2}}^{k+\sqrt{k^2-y^2}} f(x,y)\ dxdy$

Ans: $\int_{-k}^{0} \int_{\sqrt{k^2-(x+k)^2}}^{k} f(x,y)\ dydx + \int_0^k \int_0^k f(x,y) dydx + \int_k^{2k} \int_0^{\sqrt{2kx-x^2}} f(x,y)\ dydx$

4) $\int_a^b \int_{\frac{k}{x}}^{mx} f(x,y)\ dxdy$

Ans: $\int_{\frac{k}{b}}^{\frac{k}{a}} \int_{\frac{k}{y}}^{b} f(x,y)\ dxdy + \int_{\frac{k}{a}}^{ma} \int_a^b f(x,y)\ dxdy + \int_{ma}^{mb} \int_{\frac{y}{m}}^{b} f(x,y)\ dxdy$

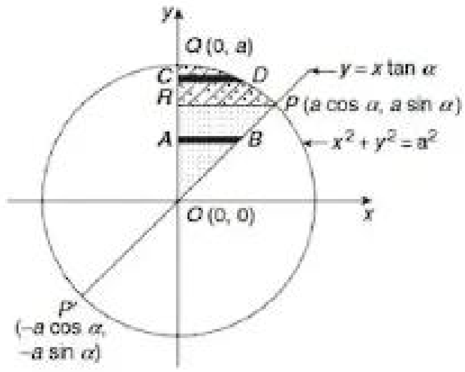
Learning outcome: Learners will be able to draw the region and change the order of integration.

Ex 1. Change the order of integration of $\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$

Solution:

i. Since inner limits depend on x, the function is integrated first w.r.t. y.

ii. Limits of $y: y = x\tan\alpha$ to $y=\sqrt{a^2-x^2}$ along vertical strip $A'B'$. Limits of $x: x = 0$ to $x = a\cos a$



iii. The region is bounded by the line $y = x\tan a$, the circle $x^2+y^2=a^2$ and y-axis.

iv. Points of intersection of $y = x\tan a$ and $x^2+y^2=a^2$ are obtained as

$$x^2+x^2\tan^2\alpha=a^2 \Rightarrow x^2\sec^2\alpha=a^2 \quad \Rightarrow x^2=a^2\cos^2\alpha$$

$$x=\pm a\cos\alpha \text{ and } y=\pm a\sin\alpha$$

Hence, the points of intersection are $P:(a\cos a, a\sin a)$ and $P':(-a\cos a - a\sin a)$.

v. To change the order of integration i.e. to integrate first w.r.t. x, divide the region into two sub regions OPR and PQR as one horizontal strip cannot cover the entire region. Draw a horizontal strip in each subregion.

vi. In sub region OPR strip AB starts from y-axis and terminates on the line $y = x\tan a$.

Limits of x: $x = 0$ *to* $x = y\cot a$ Limits of y: $y = 0$ to $y = a\sin a$

vii. In sub region PQR strip CD starts from y-axis and terminates on the circle $x^2+y^2=a^2$.

Limits of x: $x = 0$ to $x=\sqrt{a^2-y^2}$

Limits of y: $y = a\sin a$ to $y = a$

Hence, the given integral after change of order is

$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx=\int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x,y)\,dx\,dy+\int_{a\sin\alpha}^{a}\int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$$

Exercise 24

Change the order of integration of the following:

1) $$\int_0^4\int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x,y)\,dx\,dy$$

Ans : $$\int_0^2\int_{\frac{y^2}{4}}^{2-\sqrt{4-y^2}} f(x,y)\,dy\,dx+\int_0^2\int_{2+\sqrt{4-y^2}}^{4} f(x,y)\,dy\,dx+\int_2^4\int_{\frac{y^2}{4}}^{4} f(x,y)\,dy\,dx$$

2) Changing the order of integration in the double integral I= $\int_0^8 \int_{\frac{x}{4}}^{2} f(x, y)\, dx\, dy$ leads to

I= $\int_r^s \int_p^q f(x, y)\, dx\, dy$ what is q?

(a) 4y (b) $16y^2$ (c) x (d) 8

Homework Problems for the day

Change the order of integration of the following:

1) $\int_0^4 \int_{\frac{y}{2}}^{9-y} f(x,y)\, dx\, dy$

Ans: $\int_0^2 \int_0^{2x} f(x,y)\, dy\, dx + \int_2^5 \int_0^4 f(x,y)\, dy\, dx + \int_5^9 \int_0^{9-x} f(x,y)\, dy\, dx$

2) $\int_{-a}^{a} \int_0^{\frac{y^2}{a}} f(x,y)\, dx\, dy$

Ans: $\int_0^a \int_{-a}^{-\sqrt{ax}} f(x,y)\, dy\, dx + \int_0^a \int_{\sqrt{ax}}^{a} f(x,y)\, dy\, dx$

3) $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x,y)\, dx\, dy$

Ans: $\int_0^a \int_0^{\sqrt{ay}} f(x,y)\, dx\, dy + \int_a^{2a} \int_0^{2a-y} f(x,y)\, dx\, dy$

4) $\int_0^1 \int_{-\sqrt{2y-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\, dx\, dy$

Ans: $\int_{-1}^{0} \int_{1-\sqrt{1-x^2}}^{1} f(x,y)\, dy\, dx + \int_0^1 \int_{\sqrt{2x-x^2}}^{1} f(x,y)\, dy\, dx$

5) Change the order of integration $\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)\, dx\, dy$

Change of order of integration

Lecture: 24

1. **Learning Objective:**
   Learners will be able to draw the region and change the order of integration.
2. **Introduction**
   Sometimes evaluation of double integral becomes easier by changing the order of integration. To change the order of integration, first, we draw the region of integration with the help of the given limits. Then we draw vertical or horizontal strip as per the required order of integration. This change of order also changes the limits of integration.

3. **Sample Problem**

The region is bounded by the line $y = 4x + 8$,

$y = 4x$, $y = 8$ and x-axis ($y = 0$).

iii. Point of intersection of $y = 4x$ and $y = 8$ is obtained as $8 = 4x$, $x = 2$ Hence $P : (2, 8)$

iv. To change the order of integration i.e., to integrate first w.r.t. y divide the region $OPQR$ into two sub regions OQR and OPQ. Draw a vertical strip parallel to y-axis in each sub region.

v. In sub region OQR strip AB starts from x-axis and terminates on the line $y = 4x + 8$.

Therefore limits of $y : y = 0$ to $y = 4x + 8$ limits of $x : x = -2$ to $x = 0$

Hence the given integral after change of order can be written as

$$\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\, dx\, dy = \int_{-2}^{0} \int_0^{4x+8} f(x, y)\, dy\, dx + \int_0^2 \int_{4x}^{8} f(x, y)\, dy\, dx$$

vi In sub region OPQ strip CD starts from the line $y = 4x$ and terminates on the line $y = 8$.

Therefore limits of $y : y = 4x$ to $y = 8$ limits of $x : x = 0$ to $x = 2$

Change the order of integration of the following:

1) $\int_0^6 \int_{2-x}^{2+x} f(x, y)\, dx\, dy$

Ans: $\int_{-4}^{2} \int_{2-y}^{6} f(x, y)\, dy\, dx + \int_2^8 \int_{y-2}^{6} f(x, y)\, dy\, dx$

2) $\int_0^1 \int_{2x-x^2}^{1+\ 1-x^2} f(x, y)\, dy\, dx$

Ans: $\int_0^1 \int_0^{1-\ 1-y^2} f(x, y)\, dx\, dy + \int_1^2 \int_0^{2y-y^2} f(x, y)\, dx\, dy$

3) $\int_0^a \int_{a^2-x^2}^{x+3a} f(x, y)\, dx\, dy$

Ans: $\int_0^a \int_{a^2-y^2}^{a} f(x, y)\, dx\, dy + \int_0^{3a} \int_0^{a} f(x, y)\, dx\, dy + \int_{3a}^{4a} \int_{y-3a}^{a} f(x, y)\, dx\, dy$

Choose the correct option from the following

1) The curves of the region of integration are identified from the given limits

(a) True (b) False

Change of order of integration

Lecture: 23

1. **Learning Objective:**

   Learners will be able to draw the region and change the order of integration.

2. **Introduction**

   Sometimes evaluation of double integral becomes easier by changing the order of integration. To change the order of integration, first, we draw the region of integration with the help of the given limits. Then we draw vertical or horizontal strip as per the required order of integration. This change of order also changes the limits of integration.

3. **Sample Problem**

Ex. 1 : Change the order of integration $\int_0^1\int_x^{\sqrt{x}} f(x,y)dydx$

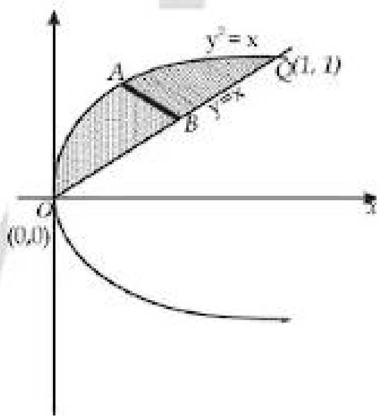
Solution:

i. Here function is integrated first w.r.t. y and then w.r.t. x

ii. Limits of y : $y = x$ to $y=\sqrt{x}$ the parabola

Therefore vertical strip starts from the line $y = x$ and terminates on the parabola $y^2=x$ Limits of x : $x = 0$ to $x = 1$

iii. The region is bounded by the line $y = x$ and the parabola $y^2=x$



iv. Point of intersection of $y^2=x$ and $y = x$ is obtained as $x^2=x$ $x = 0, 1$ and $y = 0, 1$ Hence $Q : (1,1)$

v. To change the order of, integration i.e. to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the parabola $y^2=x$ and terminates on the line $y = x$ Therefore limits of $x : x=y^2$ to $x=y$ Limits of $y : y = 0$ to $y = 1$

Hence the given integral after change of order can be written as $\int_0^1\int_x^{\sqrt{x}} f(x,y)dydx=\int_0^1\int_{y^2}^{y} f(x,y)dxdy$

Ex. 2: Change the order of integration $\int_0^8\int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x,y)dxdy$

Solution:

i. Here the function is integrated first w.r.t. x and then w.r.t. y

ii. Limits of x : $x=\frac{y-8}{4}$ to $x=\frac{y}{4}$

Therefore, horizontal strip starts from the line $y = 4x+8$ and terminates on the line $y = 4x$. Limits of y : $y = 0$ to $y = 8$

(a) $a^4/3$ (b) $a^4/5$ (c) $a^4/7$ (d) $a^4/9$

Evaluate the following

1) $\iint \frac{xy}{1-y^2}\, dx\, dy$ over +ve quadrant of $x^2 + y^2 = 1$

Ans: $\frac{1}{6}$

2) $\iint xy\, dx\, dy$ over the area bounded by the $x-$ axis, the ordinate at $x = 2a$ and the curve $x^2 = 4ay$.

Ans: $\frac{a^4}{3}$

3) $\int\int y\,dxdy$ over the area bounded by $y = x^2$ and $x + y = 2$

Ans: $\frac{36}{5}$

4) $\iint xy \quad 1-x-y \quad dx\, dy$ over the area of the triangle bounded by $x = 0, y = 0$ and $x + y = 1$.

Ans: $\frac{16}{945}$

5) $\iint_R \frac{ye^{2y}}{(1-x)(x-y)}\, dx\, dy$, R is a triangle having vertices $(0,0)$, $(1,0)$ and $(1,1)$.

Ans: $\pi\left[\frac{e^2+1}{4}\right]$

6) $\iint_R y\, dx\, dy$, over the region bounded by $y^2 = 4x$, $x^2 = 4y$.

Ans: $\frac{48}{5}$

7) $\iint_R x^2\, dx\, dy$, where R is the region in the first quadrant bounded by the curve $x = \frac{16}{y}$ and the lines $y = x$, $y = 0$ and $x = 8$.

Ans: $-\frac{2}{3}$

Learning outcome: Learners will be able to find the limits of integration in the given region and evaluate the integral.

Points of intersection of the parabola and the line y=x is obtained a x = 0, 2 and y = 0, 2 Hence 0: (0,0) and Q: (2, 2)

(iii) Here, integration can be done w.r.t. to any variable first. To integrate w.r.t. y first we need to draw vertical strip in the region. But one vertical strip does not cover the entire region, therefore we divide the region OPQR into two sub regions OPR and RPQ and draw one vertical strip in each sub region.

(iv) In the sub region OPR strip starts from the circle and terminates on the parabola Limits of x: x = 0 to 1.

(v) In the sub region RPQ strip starts from the line y = x and terminates on the Parabola Limits of x: x = 1 to x = 2

Evaluate the following

1) $\iint e^{2x-3y}\,dx\,dy$ over the triangle bounded by the lines $x = 0, y = 0$, and $x + y = 1$.

Ans: $\left(\dfrac{e^2}{10}-\dfrac{1}{6}+\dfrac{1}{15e^3}\right)$.

2) $\iint (x^2-y^2)dxdy$ over the triangle with vertices (0, 1), (1, 1), (1, 2).

Ans: $-\dfrac{2}{3}$

3) $\iint_R xy\,(x-1)\,dx\,dy$, R *is the region bounded by* $xy = 4,\ y = 0,\ x = 1$ *and* $x = 4$.

Ans: $8(3-\log 4)$

4) $\iint_R \dfrac{dx\,dy}{x^4+y^2}$, *over* $y \ge x^2, x \ge 1$

Ans: $\dfrac{\pi}{4}$

5) $\iint_R xy\,dx\,dy$, *over the region bounded by* $x^2 = y,\ y^2 = -x$

Ans: $-\dfrac{1}{12}$

Choose the correct option from the following

1. A horizontal strip is taken to evaluate along
   (a) y-axis (b) x-axis (c) line y = x (d) none of these

2. The value of the integral $\iint 2x\,dx\,dy$ over the triangle with vertices A(0,0) ,B(1,0), C(0,1)
   (a) 1 (b) 1/3 (c) 1/8 (d) 1/9

3 The value of the integral $\iint xy\,dx\,dy$ over the region bounded by the x-axis , ordinate at x =2a and the parabola $x^2 = 4ay$ is

4. Sample Problems

Ex. 1 : Evaluate $\iint e^{ax+by}\,dx\,dy$, over the triangle bounded by $x = 0, y = 0, ax + by = 1$.

Solution:

a) The region of integration is the ΔOPQ.

b) Here integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the line $ax + by = 1$.

c) Limits of y : $y = 0$ to $y = \frac{1-ax}{b}$

Limits of x : $x = 0$ to $x = \frac{1}{a}$

$$I = \iint e^{ax+by}\,dx\,dy = \int_0^{\frac{1}{a}} e^{ax} \int_0^{\frac{1-ax}{b}} e^{by}\,dy\,dx$$

$$= \int_0^{\frac{1}{a}} e^{ax} \left[\frac{e^{by}}{b}\right]_0^{\frac{1-ax}{b}} = \frac{1}{b}\int_0^{\frac{1}{a}} e^{ax}\left\{e^{(1-ax)} - 1\right\}dx$$

$$= \frac{1}{b}\int_0^{\frac{1}{a}}(e - e^{ax})\,dx = \frac{1}{b}\left[ex - \frac{e^{ax}}{a}\right]_0^{\frac{1}{a}} = \frac{1}{b}\left[\frac{e}{a} - \frac{e}{a} + \frac{1}{a}\right] = \frac{1}{ab}$$

Ex.2: Evaluate $\iint y\,dx\,dy$ over the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$.

Solution:

a) Region of integration is POQ.

b) Points of intersection of $x^2 = y$ and $y = x + 2$ are obtained as

$$x^2 = x + 2,\ x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2, -1 \text{ and } y = 4, 1$$

Hence points of intersection are $P : (-1, 1)$ and $Q : (2, 4)$

c) The integration can be done w.r.t any variable first. To integrate first w.r.t. y draw a vertical strip AB parallel to y-axis which starts from the parabola $x^2 = y$ and terminates on the line $y = x+2$.

d) Limits of y : $y = x^2$ to $y = x + 2$

Limits of x : $x = -1$ to $x = 2$

$$I = \iint y\,dx\,dy = \int_{-1}^{2}\int_{x^2}^{x+2} y\,dy\,dx$$

$$= \int_{-1}^{2}\left[\frac{y^2}{2}\right]_{x^2}^{x+2} dx = \frac{1}{2}\int_{-1}^{2}\left\{(x+2)^2 - x^4\right\}dx = \frac{1}{2}\left[\frac{(x+2)^3}{3} - \frac{x^5}{5}\right]_{-1}^{2} = \frac{1}{2}\left[\frac{64}{3} - \frac{32}{5} - \frac{1}{3} - \frac{1}{5}\right] = \frac{36}{5}$$

Ex.3: Evaluate over the region enclosed by the circle the parabola and the line y = x.

Solution:

a) The region of integration is OPQR.

b) Points of intersection of the

(i) circle and the line y = x is obtained as and y = 0, 2

Hence 0: (0, 0) and P: (1, 1)

ii) The circle and the parabola are obtained as and y = 0 Hence 0: (0, 0)

5) $$\int_0^1 \int_0^{1+x^2} \frac{dxdy}{1+x^2+y^2}$$ *Ans:* $\frac{\pi}{4}\log\left(1+\ \ 2\right)$

6) Prove that $$\int_0^{\infty}\int_0^{\infty} \frac{e^{-x^3}}{x}\, y^4\, e^{-y^6}\, dx\, dy = \frac{\pi}{9}$$

7) Prove that $$\int_0^{a\ \ 3}\int_0^{a^2+y^2} \frac{y\, dy dx}{y^2+a^2+x^2} = \frac{\pi a}{4}$$

8) Prove that $$\int_0^{\frac{\pi}{2}}\int_0^{3(1-\cos t)} x^2 \sin t\ dt\, dx = \frac{9}{4}$$

Learning from the topic: Learners will be able to evaluate double integration .

Double Integration over the Region

1. **Learning Objectives:**
   Learners will be able to find the limits of integration in the given region and evaluate the integral.
2. **Introduction:**
   In double integration limits of integration can be obtained if the region of integration is given. In this section the method to find limits of integration is explained by taking an example. We find limit by tracing curve along x axis and along y axis.
3. **Important Formulae/ Theorems/ Properties:**
   Evaluation of Double Integral: Double integral over a given region R may be evaluated by two successive integrations. If R is described as $f_1(x) \le y \le f_2(x)$ *and* $a \le x \le b$, Then $$\iint_R f(x,y)dxdy = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dydx$$

   There are two methods to evaluate double integrals:

a) $$\iint_R f(x,y)dxdy = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)dydx$$

   f(x ,y) is first integrated with respect to 'y' treating 'x' as constant between the limits $f_1(x)$ *and* $f_2(x)$ and then the result is integrated with respect to 'x' between the limits a and b.

b) $$\iint_R f(x,y)dxdy = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy$$

   f(x ,y) is first integrated with respect to 'x' treating 'y' as constant between the limits $g_1(y)$ *and* $g_2(y)$ and then the result is integrated with respect to 'y' between the limits c and d.

Note: For constant limits, we can integrate with respect to any variable first.

$$= -\frac{1}{2}\left|\frac{1}{2}\tan\theta - \theta\right|_0^{\frac{\pi}{4}} = -\frac{1}{2}\left[\frac{1}{2}\tan\frac{\pi}{4} - \frac{\pi}{4} - 0\right] = \frac{1}{8}(\pi - 2)$$

Exercise 21

1) Evaluate $\int_0^1 \int_0^x xy\,dy\,dx$

2) Evaluate $\int_0^1 \int_0^x e^{x+y}\,dy\,dx$ Ans : $\frac{1}{2}(e-1)^2$

3) $\int_0^1 \int_y^{\sqrt{y}} \frac{x}{(1-y)\sqrt{1-x^2}}\,dx\,dy$ Ans : $\frac{\pi}{2}-1$

4) $\int_0^{\frac{\pi}{2}} \int_0^{a(1+\sin\theta)} r^2 \cos\theta\, d\theta\, dr$ Ans : $\frac{5a^3}{4}$

Let's check take away from lecture

Choose the correct option from the following

1) The value of the integral $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x+y)\,dx\,dy$

(a) 0 (b) π (c) -2 (d) 2

2) Find the integration of $\int_0^1 \int_0^x \left(x^2+y^2\right) dx\,dy$

a) 1/3 b) 1/2 c) -1/3 d) -1/2

Homework Problems for the day

Evaluate the following:

1) Evaluate $\int_0^1 \int_0^x xy\,dy\,dx$ Ans : $\frac{1}{8}$

2) Evaluate $\int_0^1 \int_0^x \left(x^2+y^2\right) x\,dy\,dx$ Ans : $\frac{4}{15}$

3) $\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{x} \cos(x+y)\; dx\; dy$ Ans: 0

4) $\int_0^1 \int_0^{\sqrt{\frac{(1-y^2)}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$ Ans : $\frac{\pi}{4}$

Multiple integrals are useful in evaluating plane area, mass of a lamina, mass and volume of solid regions, etc.

1. **Key Notations:**
    a) $dA = dx\,dy$
    b) J : Jacobian

2. **Key Definitions:**
 a. **Double Integration:** Consider the sum $S = \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$

 where $\delta A_r = \delta x_r\,\delta y_r$

If we increase the number of elementary rectangles i.e., n, then the area of each rectangle decreases.

Hence as $n \to \infty$, $\delta A_r \to 0$.

The limit of the sum given by the equation (1), if it exists, is called the double integral of $f(x, y)$ over the region R and is denoted by $\iint_R f(x,y)\,dA$

Hence, $\iint_R f(x,y)\,dA = \lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$ where $dA = dx\,dy$

3. **Sample Problems**

Ex. 1 : Evaluate $\int_0^1\int_0^{1+x^2} \frac{dx\,dy}{1+x^2+y^2}$

Solution :

$$\int_0^1\left\{\int_0^{1+x^2}\frac{dy}{(1+x^2)^2+y^2}\right\}dx = \int_0^1 \left[\frac{1}{1+x^2}\tan^{-1}\frac{y}{1+x^2}\right]_0^{1+x^2} dx$$

$$= \int_0^1 \frac{1}{1+x^2}\left[\tan^{-1}1 - \tan^{-1}0\right]dx = \int_0^1 \frac{1}{1+x^2}\cdot\frac{\pi}{4}\,dx$$

$$= \frac{\pi}{4}\log\left\{x + 1+x^2\right\}_0^1 = \frac{\pi}{4}\log(1+ 2)$$

Ex 2.

$$Evaluate \int_0^{\frac{\pi}{4}}\int_0^{\cos 2\theta}\frac{r}{(1+r^2)^2}\,d\theta\,dr$$

Solution :

$$\int_0^{\frac{\pi}{4}}\int_0^{\cos 2\theta}\frac{1}{2}\cdot\frac{2r}{(1+r^2)^2}\,dr\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}}\left\{\int_0^{\cos 2\theta}(1+r^2)^{-2}\cdot 2r\,dr\right\}d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[-(1+r^2)^{-1}\right]_0^{\cos 2\theta}d\theta \qquad \left[\int [f(r)]^n f'(r)dr = \frac{[f(r)]^{n+1}}{n+1}, \quad n \neq -1\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left\{\frac{1}{1+\cos 2\theta}-1\right\}d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left\{\frac{1}{1+\cos^2\theta}-1\right\}d\theta = -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left\{\frac{1}{2}\sec^2\theta-1\right\}d\theta$$

Module 4: Multivariable Calculus-I (Double Integration)

1. Motivation

Integration of functions of two or more variables is normally called multiple integration. This topic deals with the double integration and its applications. Double integrals are useful in evaluating plane area, mass of a lamina, mass and volume of solid regions etc.

2. Syllabus

Module 1	Content	Class Duration	Self Study duration	Weightage
4	Multiple Integration: Double integrals (Cartesian), change of order of integration in double integrals, Change of variables (Cartesian to polar), Applications: areas, Center of mass and Gravity (constant and variable densities)	3 Hrs. Lectures 2 Hrs. Lectures 2 Hrs. Lectures	14 hours	18-20 Marks

3. Prerequisite:

Concept of determinant, standard integral formulae and curve tracing.

4. Learning Objective: Learners shall be able to

1. Define double integral (DI) and method of evaluation.
2. Evaluate DI over the region.
3. Change the order of integration of DI.
4. Evaluate DI after changing the order of integration.
5. Change to polar coordinate and evaluate.
6. Evaluate after changing to polar coordinates.

Double Integrals (Cartesian)-direct Evaluation

Lecture :21

1. **Learning Objective:**
   Student shall be able to understand and apply Double Integration to Find Area under the Curve. Evaluation of Double Integration.
2. **Introduction:**

Integration of functions of two or more variables is normally called multiple integration. The particular case of integration of functions of two variables is called double integration and that of three variables is called triple integration. Sometimes, we have to change the variables to simplify the integrand while evaluating the multiple integrals. Variables can be changed by substitution or by changing the coordinate system (polar, spherical or cylindrical coordinates).

$$\begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$$

3. Construct an orthonormal basis of $\mathbb{R}$ by applying Gram Schmidt orthogonalization to S={[3,1],[2,2]}

4. Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalizable.

   Also find the Transforming matrix and the diagonal matrix.

Learning Resources:

- Higher Engineering Mathematics, Dr. B. S. Grewal, Khanna Publications.
- A textbook of Applied Mathematics, P.N. and J. N.Wartikar,Volume 1 and 2, Pune Vidyarthi Griha.
- Advanced Engineering Mathematics, Erwin Kreyszing, WileyEastern Limited, 8^{th} Ed.

Self-Assessment

Level 1

1. Prove that characteristic roots of A and A^t are same.

2. Verify that $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ is orthogonal. Also verify that $\frac{1}{\lambda}$ is an eigen value of A^{-1} if λ is an eigen value of A. and verify that eigen values of A are of unit modulus.

3. Find the Eigen values of $A^3 - 3A^2 + A$ where

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

Level 2

1. Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Also find the transforming matrix and the diagonal matrix.

2. Show that the matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ is diagonalizable. Also find the transforming matrix and the diagonal matrix.

3. For a symmetric matrix A the eigen vectors are $[1, 1, 1]^T$, $[1, -2, 1]^T$ corresponding to $\lambda_1 = 6$ $\lambda_2 = 12$ respectively. If $\lambda_3 = 6$ then find the matrix A.

   Hint: Since matrix is symmetric, therefore X_3 will be orthogonal to X_1 and X_2

   Also $D = M^{-1}AM \Rightarrow A = M\ DM^{-1}$

   Ans: $\begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$

Level 3

1. Reduce the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form.

2. Verify that $X = [\,2 \quad 3 \quad -2 \quad -3\,]^T$ is an Eigen vector corresponding to Eigen value $\lambda = 2$ of the matrix

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{\frac{15}{53}(0,2,7)}{\frac{15}{53}\sqrt{53}} = (0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}})$$

The vectors w_1, w_2, w_3 form an orthonormal basis for R^3.

Exercise 20

1. Show that the set V={(1,0,0),(0,1,0),(0,0,1)}is an ortho-normal set in Rwith Euclidean inner product.
2. Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis vectors $u_1=(1,0,0)$, $u_2=(3,7,-2)$, $u_3=(0,4,1)$ in to an orthonormal basis.
3. Find a vector orthogonal to both u=(-6,4,2) and v=(3,1,5) .
4. Find an orthonormal basis of the of the following subspace of R
   S={[1,2,0] , [0,3,1]}

Let's check take away from lecture

1. Let R^3 have the inner product $\langle(x_1, x_2, x_3),(y_1,y_2,y_3)\rangle=2x_1 y_1+x_2 y_2+3x_3 y_3$
   Use the Gram–Schmidt process to trans- form the basis vectors
   $u_1 = (1, 1, 1)$, $u_2 = (1, -1, 1)$, $u_3 = (1, 1, 0)$ into an orthogonal basis.

   a) (2, 1, 1), (2, −5, 1), (3, 0, −2) b) (1, 1, 3), (1, −5, 1), (3, 0, 3)

   c) (3, 1, 1), (1, −5, 1), (3, 0, −2) d) (1, 1, 1), (1,−5, 1), (3, 0, −2)

2. Let u = (4,1,2,3), v=(0,3,8,-2) and w=(3,1,2,2) then ||u||

   a) $\sqrt{30}$ b) 30 c) 20 d) 10

Homework problem for the day

1. Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis vectors u1 = (1, 0, 0), u2 = (3, 7,−2), u3 = (0, 4, 1) into an orthonormal basis.

2. Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis vectors u1 = (1, 0, 3), u2 = (2, 2, 0), u3 = (3, 1, 2) into an orthonormal basis

3. Use Gram–Schmidt process to construct an orthonormal basis for the subspace W of R^4 spanned by the vectors v1 = (1, 1, 0, 0), v2 = (2,−1, 0, 1), v3 = (3, −3, 0, −2), v4 = (1, −2,3)

Learning from the topic: Learners will be able to evaluate Gram Schmidt orthogonalization

V be a non-zero inner product space in R and u=(u_1, u_2 ,u_3) be any base for V. then To find orthogonal basis (v_1,v_2, v_3) for V by following steps

Step 1: Let $v_1 = u_1$ **Step 2** : Let $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Step 3: $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

3. Sample problem:

1) Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis vectors $u_1=(1,0,0)$,$u_2=(3,7,-2)$,$u_3=(0,4,1)$ in to an orthonormal basis.

Solution: Let $v_1=u_1=(1,0,0)$,

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$=(3,7,-2) - \frac{(3,7,-2)\cdot(1,0,0)}{1}(1,0,0)$$

$$= (3,7,-2) - 3(1,0,0)$$
$$= (0,7,-2)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= (0,4,1) - \frac{(0,4,1)\cdot(1,0,0)}{1}(1,0,0) - \frac{(0,4,1)\cdot(0,7,-2)}{53}(0,7,-2)$$

$$=(0,4,1)-(0,182/53,-52/53)$$
$$=(0,30/53,105/53)$$
$$= \frac{15}{53}(0,2,7)$$

The vectors v_1,v_2,v_3 form an orthogonal basis for R.Normalizing v_1,v_2,v_3

$$\|v_1\| = 1, \|v_2\| = \sqrt{53}, \|v_3\| = \frac{15}{53}\sqrt{53}$$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{(1,0,0)}{\sqrt{1}} = (1,0,0)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{(0,7,-2)}{\sqrt{53}} = (0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}})$$

$$\text{Modal matrix } P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Exercise 19

1. If $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ check whether it is orthogonally diagonalization.
2. If $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ check whether it is orthogonally diagonalization
3. Find the symmetric matrix $A_{3\times3}$ having the eigen value $\lambda_1=0$, $\lambda_2=3$, $\lambda_3=15$ with the corresponding eigen vector $X_1=(1,2,2)$, $X_2=(-2,-1,2)$ and X_3.

Homework problem for the day

1. Show that $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ is orthogonally diagonalization.

2. Show that the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalizable.

   Also find transforming matrix and the diagonal matrix.

Gram-Schmidt orthogonalization

Lecture 20

1. **Learning objective:**

   Learner shall be able to use Gram Schmidt orthogonal process

2. **Key Definition**

a) **Gram -Schmidt orthogonalization:**

Gram Schmidt orthogonalization gives a method to find orthonormal vectors. Let

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(iii) For $\lambda = 6$, $\left[A - \lambda_3 I\right]X = 0$ gives

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore -3x_1 - x_2 + x_3 = 0,\ -x_1 - x_2 - x_3 = 0,\ x_1 - x_2 - 3x_3 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -3 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

Hence corresponding to the eigen value $\lambda = 6$,the eigen vector is

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Diagonal matrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

Length of vector $X_1 = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

Length of vector $X_2 = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$

Length of vector $X_3 = \sqrt{(1)^2 + (-2)^2 + (1)^2} = \sqrt{6}$

The normalized eigen vectors are

$$\bar{X}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix},\ \bar{X}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix},\ \bar{X}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\therefore \lambda = 2,3,6$$

(i) For $\lambda = 2$, $\left[A - \lambda_1 I\right]X = 0$ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 - x_2 + x_3 = 0,\ -x_1 + 3x_2 - x_3 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

Hence corresponding to the eigen value $\lambda = 2$,the eigen vector is

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(ii) For $\lambda = 3$, $\left[A - \lambda_2 I\right]X = 0$ gives

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore -x_2 + x_3 = 0,\ -x_1 + 2x_2 - x_3 = 0,\ x_1 - x_2 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

Hence corresponding to the eigen value $\lambda = 3$,the eigen vector is

2. Find orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix having diagonal elements as the characteristic roots of A , where $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

3. Is the arithmetic multiplicity and geometric multiplicity for each eigen value equal for matrix A ? where $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

4. Show that $A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ are similar matrices .

   Hint: Prove that both the matrices have same Eigen values.

Learning from the topic: Learners will be able to diagonalise the given matrix

Orthogonal Transformation

Lecture 19

1. **Learning Objective:**
   Learner shall be able to understand orthogonally diagonalization.
2. **Key Definition:**

a) **Orthogonally diagonalization:**
   A square matrix is said to be orthogonally diagonalization if there exist orthogonal matrix P such that $P^{-1}AP$ is diagonal .
   A square matrix A is orthogonally diagonalization iff A is symmetric matrix.

3. **Sample problem:**

1. Determine the diagonal matrix orthogonally similar to the real symmetric matrix. Also find the modal matrix. $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Solution : $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation is

Exercise 18

1. Examine where the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ is diagonalizable. If so, obtain the matrix P such that $P^{-1}AP$ is a diagonal matrix. **Ans:** non-diagonalizable

2. Is the following matrix diagonalizable? Justify where $A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

   Ans: non-diagonalizable

3. Diagonalize the Hermitian matrix $A = \begin{bmatrix} -3 & 2+2i \\ 2-2i & 4 \end{bmatrix}$

4. Determine whether A is diagonalizable and if so , find the diagonalzing matrix P and diagonal matrix D where $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Let's check take away from lecture

1. Which of the following matrices is not diagonalizable?

   $(a) \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $(b) \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ $(c) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $(d) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

2. If matrix have different Eigen value, then

a) Diagonalizable b) Not diagonalizable

c) Inverse exist d) Cayley Hamilton not applicable

Homework problem for the day

1. Is the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizable? If so find diagonal form of matrix.

$$-8x_1 + 4x_2 + 4x_3 = 0$$

Taking $y = 1$ *and* $z = 1$, $x = 1$

Taking $y = 1$ *and* $z = -1$, $x = 0$

Hence corresponding to the eigen value $\lambda = -1$, the eigen vector is

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(ii) For $\lambda = 3$, algebraic multiplicity = geometric multiplicity = 1.

$[A - \lambda_2 I]X = 0$ gives

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 12x_1 + 4x_2 + 4x_3 = 0,\ -8x_2 + 4x_3 = 0,\ -16x_1 + 8x_2 + 4x_3 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} 4 & 4 \\ 0 & 4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -12 & 4 \\ -8 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -12 & 4 \\ -8 & 0 \end{vmatrix}}$$

$$\therefore \frac{x_1}{16} = \frac{x_2}{16} = \frac{x_3}{32}$$

Hence corresponding to the eigen value $\lambda = 2$, the eigen vector is

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Since algebraic multiplicity for each eigen value is equal to the geometric multiplicity, the matrix A is diagonalizable

Diagonal matrix $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Modal matrix $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

Diagonalization of Matrix

Lecture 18

1. Learning objective: Learner shall be able to understand relation between similar matrix and diagonal matrix.

2. Key Definitions

a) **Similarity of Matrices:** If A and B are two square matrices of order n then is said to be similar to A if there exists a non-singular matrix M such that $B = M^{-1}AM$

b) **Diagonalizable Matrix:** A square matrix A is said to be diagonalizable If it is similar to a diagonal matrix.

3. Sample Problems

1). Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagnosable. Find the diagonal form D and the diagonalizing matrix M.

Solu: $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

The characteristic equation is

$$\begin{vmatrix} 9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

$\therefore \lambda = -1, -1, 3$

(i) For $\lambda = -1, [A - \lambda_1 I]X = 0$ gives

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1, R_3 - 2R_1$

$$\begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 1

Number of unknowns = 3

Number of linearly independent solutions =3-1 = 2

Geometric multiplicity = 2.

4. Compute $A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

Let's check take away from lecture

1. The minimal polynomial associated with the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is

(*a*) $x^3 - x^2 - 2x - 3$ (*b*) $x^3 - x^2 + 2x - 3$ (*c*) $x^3 - x^2 - 3x - 3$ (*d*) $x^3 - x^2 + 3x - 3$

2. If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ by using Cayley-Hamilton

a) $A^8 = 625\ I$ b) $A^6 = 225\ I$ c) $A^6 = I$ d) $A^6 = I$

Home work problem for the day

1. Use Cayley – Hamilton theorem to find A^{-1} where $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and express

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. Ans: $A + 5I$

2. Use Cayley – Hamilton theorem to find A^{-2} where $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

3. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find:

the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ and A^{-1}.

Ans: (i) $-15A^2 + 5A - I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$ (ii) $\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -5 & -3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

> **Learning outcome:** Learners will be able to evaluate different power of given matrix and inverse.

$$= \begin{bmatrix} 132 & -126 & 126 \\ -126 & 132 & -126 \\ 126 & -126 & 132 \end{bmatrix} - \begin{bmatrix} 54 & -45 & 45 \\ -45 & 54 & -45 \\ 45 & -45 & 54 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 4 \\ -4 & 8 & -4 \\ 4 & -4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

2). Apply Cayley-Hamilton theorem to $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and deduce that $A^8 = 625I$.

Solution : $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \quad \text{gives} \quad \lambda^2 - 5 = 0$$

By Cayley-Hamilton theorem,

$$A^2 = 5I$$

$$A^4 = 25I$$

$$A^8 = 625I$$

Exercise 17

1. Verify Cayley – Hamilton theorem for the matrix A and hence find A^{-1} , A^{-2} and A^4 where A is

$$(i)\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad (ii)\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \qquad (iii)\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

2. Obtain characteristic equation of a matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and verify that it is satisfied by matrix A and hence find A^{-1}.

3. Find the characteristic equation of a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & -1 & -1 \end{bmatrix}$ and show that matrix A satisfies the characteristic equation and hence find A^{-1} and A^4.

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

The matrix A satisfies its own characteristic equation. Hence, Cayley-Hamilton theorem is verified. Premultiplying by A^{-1},

$$A^{-1}(A^3 - 6A^2 + 9A - 4I) = 0$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$4A^{-1} = A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Multiplying by A

$$A(A^3 - 6A^2 + 9A - 4I) = 0$$

$$A^4 - 6A^3 + 9A^2 - 4A = 0$$

$$A^4 = 6A^3 - 9A^2 + 4A$$

2. Consider the matrix $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$ Which one of the following statements is true for the Eigen values and Eigen vectors of this matrix?

a) Eigen value 3 has a multiplicity of 2, and only one independent Eigen vector exists.
b) Eigen value 3 has a multiplicity of 2, and two independent Eigen vector exists.
c) Eigen value 3 has a multiplicity of 2, and no independent Eigen vector exists.
d) Eigen value are 3 and -3, and two independent Eigen vectors exist

Homework Problem for the day

1. Find Eigen value and Eigen Vector for following.

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix}$ (iv) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (v) $\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$

Learning outcome: Learners will be able to evaluate Eigen vector.

Cayley-Hamilton Theorem

Lecture 17

1. **Learning objective:** Student shall be able to use Cayley Hamilton to find polynomial of given matrix and different power of matrix
2. **Key Definition:**
   Cayley-Hamilton Theorem: Every square matrix satisfies its own characteristic equation.
3. **Sample problems**

1) Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

Hence find A^{-1} *and* A^4

Solu: $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Now, $X_1^T X_2 = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 0$

$$X_2^T X_3 = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$X_3^T X_1 = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0$$

Hence X_1, X_2 and X_3 are orthogonal to each other

Exercise 16

1. Determine Eigen values and Eigen vectors for the matrix and find the Eigen values of A^3.

(i) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

2. If $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$ then find the Eigen values of $6A^{-1} + A^3 + 2I$.

3. Find Eigen values of A and $4A^{-1}$. Also find Eigen vectors of $A^2 - 4I$ where

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Let's check take away from lecture

1. The number of Linearly independent Eigen vector of matrix $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

a) 0 b) 1 C) 2 d) 3

$$\therefore \frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

Hence corresponding to the Eigen value $\lambda = 0$,the Eigen vector is

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

(ii) For $\lambda = 3, \left[A - \lambda_3 I\right] X = 0$ gives

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\therefore \frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

Hence corresponding to the Eigen value $\lambda = 0$,the Eigen vector is

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

(iii) For $\lambda = 15, \left[A - \lambda_1 I\right] X = 0$ gives

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore -7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\therefore \frac{x_1}{40} = \frac{-x_2}{40} = \frac{x_3}{20}$$

Hence corresponding to the eigen value $\lambda = 15$,the eigen vector is

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(ii) For $\lambda = -1, \left[A - \lambda_2 I\right] X = 0$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 + x_2 + x_3 = 0$$

Taking $x_3 = 0$ *and* $x_2 = 1$, $x_1 = -1$

Taking $x_3 = -2$ *and* $x_2 = 1$, $x_1 = 1$

Hence corresponding to the eigen value $\lambda = -1$, the eigen vector is

$$X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

3) If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, verify whether eigen vectors are mutually orthogonal.

Solution: The characteristic equation is

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\therefore \lambda = 0, 3, 15$$

(i) For $\lambda = 0, \left[A - \lambda_2 I\right] X = 0$ gives

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore 8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

If $x_3 = 1$ *then* $x_1 = 1$.

Hence corresponding to the eigen value $\lambda = 3$,the eigen vector is

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

To verify linear independence of vectors, consider

$$k_1 x_1 + k_2 x_2 + k_3 x_3 = 0$$

$$\therefore k_1 [0,1,1] + k_1 [1,1,1] + k_1 [1,0,1] = 0$$

$$\therefore 0 + k_2 + k_3 = 0 \qquad \text{..........}(i)$$

$$k_1 + k_2 + 0 = 0 \qquad \text{..........}(ii)$$

$$k_1 + k_2 + k_3 = 0 \qquad \text{..........}(iii)$$

From (i) ,(ii) and (iii) , we get $k_1 = 0, k_2 = 0, k_3 = 0$.

Thus, the vectors are linearly independent

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2) Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution: The characteristic equation is

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 3\lambda - 2 = 0$$

$$\therefore \lambda = 2, -1, -1$$

(i) For $\lambda = 2, [A - \lambda_1 I] X = 0$ gives

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore -2x_1 + x_2 + x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_1 + x_2 - 2x_3 = 0$$

$$\therefore \frac{x_1}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

Hence corresponding to the eigen value $\lambda = 2$,the eigen vector is

$R_2 - R_1, R_3 - R_1$

By $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore x_1 - x_2 + x_3 = 0$

$x_2 - x_3 = 0$

Note that the rank of the matrix is 2 and the number of variables is 3.
Hence, there is 3-2=1 linearly independent solutions.
Putting $x_3 = 1$ *we get* $x_2 = 1$ *and* $x_1 = 0$.
Hence corresponding to the eigen value $\lambda = 1$, the eigen vector is

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

II. For $\lambda = 2, [A - \lambda_2 I]X = 0$ gives

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_2 - x_3 = 0,\ x_1 - x_3 = 0,\ x_1 - x_2 = 0$

If $x_3 = 1$, $x_2 = 1$, $x_1 = 1$.

Hence corresponding to the eigen value $\lambda = 2$, the eigen vector is

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

III. For $\lambda = 3, [A - \lambda_3 I]X = 0$ gives

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Use matrix method to obtain the roots of the above equation.

$R_2 + R_1, R_2 + R_3$

By $\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore -x_1 - x_2 + x_3 = 0$

$\therefore 2x_2 = 0$

$\therefore x_2 = 0$

Eigen Vector

Lecture 16

1. **Learning objective:** Learners shall able to find Eigen vector and multiplicity of Eigen vector

2. **Key Definition:**

   I. **Eigen Vectors:** Suppose λ_1 is a root of $|A-\lambda I|=0$ then $|A-\lambda_1 I|=0$. Further we find a non-zero column matrix X such that $[A-\lambda_1 I]X=0$. The vectors X is called the Eigen vector or latent vector corresponding to the root λ_1.

   II. **Algebraic and Geometric Multiplicity of an Eigen value:**
   If λ_1 is an eigen value of the characteristic equation $|A-\lambda I|=0$ repeated t times then t is called the algebraic multiplicity of λ_1. If s is the number of linearly independent eigen vectors corresponding to the eigen value λ_1 then s is called the geometric multiplicity of λ_1. This means the number of linearly independent solutions of $(A-\lambda_1 I)X=0$ is s and the rank of the matrix $A-\lambda_1 I$ will be n-s.

3. Sample Problems

1) Find the eigen values and eigen vectors of the matrix. Also verify that the eigen vectors are

linearly independent. $A=\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution: The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}=0$$

$$\therefore (2-\lambda)\left[(2-\lambda)^2-1\right]+1\left[1(2-\lambda)+1\right]+1\left[-1-(2-\lambda)\right]=0$$

$$\therefore \lambda^3-6\lambda^2+11\lambda-6=0$$

$$\therefore (\lambda-1)(\lambda-2)(\lambda-3)=0$$

$$\therefore \lambda=1,2,3$$

I. For $\lambda=1,[A-\lambda_1 I]X=0$ gives

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Use matrix method to obtain the roots of the above equation.

Let's check take away from lecture

Choose the correct option from the following.

1. The Eigen values of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ are

$(a)\ 5,0 \quad (b)\ 3,0 \quad (c)\ 1,0 \quad (d) -1,0$

2. The minimum and maximum Eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are -2 and 6 respectively. What is the other Eigen value?

$(a)\ 5 \quad (b)\ 3 \quad (c)\ 1 \quad (d)\ -1$

3. The sum of Eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$(a)\ 5 \quad (b)\ 7 \quad (c)\ 9 \quad (d)\ 18$

Homework Problems for the day

1. If λ is Eigen value of non-singular matrix A then prove that $\frac{|A|}{\lambda}$ is Eigen value of (adj A)
2. Prove that characteristic roots of A^n are λ^n
3. If λ is Eigen value of A then show that f(λ) is eigen value of f(A).
4. Find the sum and product of the eigen values of the matrix $A = \begin{bmatrix} -2 & -9 & 5 \\ -5 & -10 & 17 \\ -9 & -21 & 14 \end{bmatrix}$

Learning from the topic: Learners will be able to evaluate Eigen Value

3. Sample Problem

I. Find the eigen values of $A=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$.

Solution: The characteristic equation is

$$\begin{vmatrix}2-\lambda & -1 & 1\\ 1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}=0$$

$$\therefore (2-\lambda)\left[(2-\lambda)^2-1\right]+1\left[1(2-\lambda)+1\right]+1\left[-1-(2-\lambda)\right]=0$$

$$\therefore \lambda^3-6\lambda^2+11\lambda-6=0$$

$$\therefore (\lambda-1)(\lambda-2)(\lambda-3)=0$$

$$\therefore \lambda=1,2,3$$

II. Find the eigen values of the matrix $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$

Solu: The characteristic equation is

$$\begin{vmatrix}-\lambda & 1 & 1\\ 1 & -\lambda & 1\\ 1 & 1 & -\lambda\end{vmatrix}=0$$

$$\therefore \lambda^3-3\lambda-2=0$$

$$\therefore \lambda=2,-1,-1$$

Exercise 15

1. Prove that characteristic roots of real symmetric matrix are real.

2. Prove that characteristic roots of Hermitian matrix are real.

3. Prove that characteristic roots of orthogonal matrix are of unit modulus.

4. Prove that characteristic roots of unitary matrix are of unit modulus.

5. Prove that product of characteristic roots of a square matrix is equal to determinant of a matrix.

6. Key Definitions:

I. **Norm:** If X is a complex n-vector then the positive square root of the inner product (X,X) is called the norm of the vector X and is denoted by $\|X\|$.

Thus, if X= $[x_1, x_2, ..., x_n]$ then

$$\|X\| = \sqrt{(X,X)} = \sqrt{X^\theta X}$$
$$= \sqrt{\left(|x_1|^2 + |x_2|^2 + ... + |x_n|^2\right)}$$

The norm of a vector is also called the length of the vector.

$\|X\| = 0$ iff X=0 i.e. X is a zero vector.

II. **Unit Vector:** If X is a complex n-vector then it is called a unit vector if $\|X\| = 1$. A unit vector is also called a normal vector.

III. **Orthogonal Vectors:** If X and Y are any two complex n-vectors then X is said to be orthogonal to Y if (X,Y)=0 i.e. if $X^\theta Y = 0$

Eigen Value

Lecture: 15

1. **Learning Objective:** Learner shall be able Evaluate Eigen value of given matrix.

2. **Key Definition**

1. **Characteristic Equation:** Let A be a n-rowed square matrix, λ a scalar and I the unit matrix of the same order. The matrix $A - \lambda I$ is called the **characteristic matrix.** The determinant $|A - \lambda I|$ is called the **characteristic polynomial** of A.

   The equation obtained by equating to zero this determinant i.e. the equation $|A - \lambda I| = 0$ is called the **characteristic equation** of the matrix A.

2. **Eigen Values:** The roots of this characteristic equation are called the characteristic roots or latent roots or characteristic values or the **Eigen values** or proper values of the matrix A.

   e.g. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ then, the characteristic equation of A is

   $$\lambda^2 - 4\lambda - 5 = 0$$
   $$\therefore (\lambda - 5)(\lambda + 1) = 0$$
   $$\therefore \lambda = -1, 5.$$

   Hence, -1, 5 are the eigen values of the matrix A.

Module 3: Matrices-II

1. Motivation:

Matrices are a key tool in linear algebra. One use of matrices is to represent linear transformations, which are higher-dimensional analogs of linear functions of the form $f(x) = cx$, where c is a constant; matrix multiplication corresponds to composition of linear transformations. Matrices can also keep track of thecoefficients in a system of linear equations. Matrices find many applications. Physics makes use of matrices in various domains, for example ingeometrical optics and matrix mechanics; the latter led to studying in more detail matrices with an infinite number of rows and columns. Graph theory uses matrices to keep track of distances between pairs of vertices in a graph. Computer graphics uses matrices to project 3-dimensional space onto a 2-dimensional screen.

2. Syllabus:

Module	content	Duration	Self study duration	Weightage:
3	Eigenvalues, eigenvectors, Caley Hamilton theorem, Diagonalization of matrix, Orthogonal transformation, Gram-Schmidt orthogonalization	6	12	14-16 Marks

3. Prerequisite: Matrix, Types of matrices, Inverse of a matrix, Rank of a matrix, Linear dependence and independence.

4. Learning Objective:

1. Learners shall be able evaluate Eigenvalues
2. Learners shall be able evaluate Eigen vectors
3. Learners shall be able evaluate inverse of matrix by Caley Hamilton theorem
4. Learners shall be able find Diagonalization of matrix
5. Learners shall be able know Orthogonal transformation
6. Learners shall be able evaluate Gram-Schmidt Orthogonalization polar coordinate and evaluate.

5. Key Notations:

1. $|A|$: Determinant of A

2. $A = \left[a_{ij}\right]_{m\times n}$: Matrix having i no. of rows and j denotes number of columns.

Self-Evaluation

Name of student: **Class & Div:** **Roll No:**

1. Do you understand how to identify the Second order 4 differential equation?

   (a) Yes (b) No

2. Will you be able to solve differential equations with reducible exact technique?

   (a) Yes (b) No

3. Are you able to identify the reducible linear differential equation?

   (a) Yes (b) No

4. Do you understand how to apply reducible linear technique?

   (a) Yes (b) No

5. Do you understand this module?

   (a) Fully understood (b) Partially understood

1. Solve $(D^2 + a^2)y = \tan ax$

$$\textbf{Ans: } y = c_1 \cos ax + c_2 \sin ax - \left(\frac{1}{a}\right)^2 \cos(ax) \log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

2. Solve $(D^2 + a^2)y = \cot ax$

$$\textbf{Ans: } y = c_1 \cos ax + c_2 \sin ax + \left(\frac{1}{a}\right)^2 \sin ax \log\tan\left(\frac{ax}{2}\right)$$

3. Apply the variation of perimeters to solve $(x^2D^2 + 3xD + 1)y = \frac{1}{(1-x)^2}$

$$\textbf{Ans: } y = c_1 \frac{1}{x} + c_2 \frac{1}{x} \log x + \frac{1}{x} \log\left(\frac{1}{1-x}\right)$$

4. Determine the general solution of the differential equation $x^3\ y''' - 2xy' + 4y = 0$ for x>0

$$\textbf{Ans: } y = (c1 + c2 \log x)\, x^2 + \frac{c_3}{x}$$

5. Solve the initial value problem $\frac{dy}{dx} - y = 0$ where y(0)=1 using power series solution.

$$\textbf{Ans: } y = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Digital references:

1. https://solitaryroad.com/c651.html
2. https://tutorial.math.lamar.edu/
3. https://www.brainkart.com/article/First-order-and-first-degree-differential_38936/

Learning Outcomes

1.**Know**: Student should be able

(i) to solve differential equations with constant and variable coefficient using the appropriate method

(ii) to identify the differential equation appearing in engineering field and to solve it using appropriate method.

2. **Comprehend**: Student should be able

(i) to describe the differential equation with constant and variable coefficient

(ii) Explain the advantages of differential equation appeared in application

3. **Apply**, Analyse, and synthesize: Student should be able to adapt to the class environment for Effective problem solving

Add to the knowledge

In mathematics, **linear differential equations** are differential equations having solutions which can be added together in particular linear combinations to form further solutions. They equate 0 to a polynomial that is linear in the value and various derivatives of a variable; its linearity means that each term in the polynomial has degree either 0 or 1.Linear differential equations can be ordinary (ODEs) or partial (PDEs).The solutions to (homogeneous) linear differential equations form a vector space (unlike non-linear differential equations).

Exercise: 14

1. Derive the Bessel function of first kind of order zero and one in series form.
2. Show that $\left[J_{\frac{-1}{2}}(x)\right]^2 + \left[J_{\frac{1}{2}}(x)\right]^2 = \frac{2}{\pi x}$

Let's check take away from the lecture

1. $x^2y'' + xy' + (x^2 - 25)y = 0$ is a Bessel equation of first kind of order?
   (a) 25 (b) 5 (c) $\sqrt{5}$ (d) None of the above.
2. What is the value of $\int_0^{\frac{\pi}{2}} \sqrt{\pi x} J_{\frac{1}{2}}(2x) = ?$
   (a) 2 (b) 1 (c) 0 (d) -1

Homework Problems for the day

1. Show that $y = x^{\frac{-n}{2}} J_n(2\sqrt{x})$ is the solution of $xy'' + (n+1)y' + y = 0$
2. Prove that $J_{\frac{-3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{-\cos x}{x} - \sin x\right)$

Learning from the topic: Students will be able to solve differential equation of Bessel function of first kind.

Self-assessment

Level 1

1. Solve $(D^3 + 6D^2 + 11D + 6)y = 0$ **Ans:** $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$
2. Solve $(D^4 - 6D^3 + 12D^2 - 8D - 8)y = 0$ **Ans:** $y = c_1 + (c_2 + c_3x + c_4x^2)e^{2x}$
3. Solve $(D^2 - 3D + 2)y = e^{3x}$ **Ans:** $y = c_1e^x + c_2e^{2x} + \frac{1}{2}e^{3x}$
4. Solve $(D^2 + 1)y = \cos 2x$ **Ans:** $y = c_1 \sin x - \frac{1}{3}\cos 2x$
5. Solve $y'' + y = \sin x$ **Ans:** $y = c_1 \cos x + c_2 \sin x - \frac{x}{2}\cos x$

Level 2

1. Solve $(D^2 + 4)y = 0$ given that $y(0) = 2, y'(0) = 0$. **Ans:** $y = 2\cos 2x$.
2. Solve $(D^2 - 3D + 2)y = 0$ given that $y(0) = 0, y'(0) = 0$. **Ans:** $y = 0$
3. Solve $(D^3 - 3D - 2)y = 540x^3e^{-x}$ **Ans:** $c_1e^{2x} + (c_2 + c_3x)e^{-x} - e^{-x}(9x^5 + 15x^4 + 20x^3 + 20x^2)$
4. Solve $\left(D^3 - \frac{4}{x}D^2 + \frac{5}{x^2}D - \frac{2}{x^3}\right)y = 1$ **Ans:** $y = c_1x^2 + c_2x^{\frac{5+\sqrt{21}}{2}} + c_3x^{\frac{5-\sqrt{21}}{2}} - \frac{x^3}{5}$
5. Solve $[(1+2x)^2D^2 - 6(1+2x)D + 16]y = 8(1+2x)^2$ given that $y(0) = 0, y'(0) = 2$.
   Ans: $y = (1+2x)^2 \log(1+2x)[1 + \log(1+2x)]$

Level 3

Bessel's Function of first kind

Lecture: 14

1. Learning Objectives:
Students shall be able to understand the concept of Bessel functions and solve differential equations involving it.

2. Introduction:
The differential equation $x^2y'' + xy' + (x^2 - n^2)y = 0$ occur in advanced studies in applied mathematics, physics, and engineering. They are called Bessel's differential equation of order n. when this differential equation is solved it generate Bessel's functions.

3. Key definition:
Bessel equation of first kind of order n denoted by $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}$ where n is any non-negative constant.

4. Theorems:

a) For a positive integer n, $J_{-n}(x) = (-1)^n J_n(x)$

b) For any integer n, $J_{-n}(x) = (-1)^n J_n(x)$.

5. Sample problems:

1. Prove that $J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Solution: By definition of $J_n(x)$ we have,

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{2 \times 2(n+1)} + \frac{x^4}{2 \times 4 \times 2^2(n+1)(n+2)} - \cdots\right] \text{------}(1)$$

Replacing n by $\frac{-1}{2}$ in (1) and simplifying we get

$$J_{\frac{-1}{2}}(x) = \frac{x^{\frac{-1}{2}}}{2^{\frac{-1}{2}}\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{1 \times 2} + \frac{x^4}{1 \times 2 \times 3 \times 4} - \cdots\right] = \sqrt{\frac{2}{\pi x}} \cos x$$

2. Prove that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

Solution: Replacing n by $\frac{1}{2}$ in (1) and simplifying we get

$$J_{\frac{1}{2}}(x) = \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right)}\left[1 - \frac{x^2}{1 \times 2 \times 3} + \frac{x^4}{1 \times 2 \times 3 \times 4 \times 5} - \cdots\right]$$

$$= \sqrt{\frac{x}{2}} \frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \frac{1}{x}\left[1 - \frac{x^3}{1 \times 2 \times 3} + \frac{x^5}{1 \times 2 \times 3 \times 4 \times 5} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

2. $\frac{d^2y}{dx^2} + xy = 0.$

Ans.: $y = a_0\left(1-\frac{x^3}{3!}+\frac{1.4x^6}{6!}-\frac{1.4.7x^9}{9!}+\ldots\right)+a_1\left(x+\frac{2x^4}{4!}+\frac{2.5x^7}{7!}-\ldots\right)$

3. $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$

Ans. $y = a_0\left(1-x^2-\frac{x^4}{3}-\frac{x^6}{5}\ldots\right)+a_1x$

Let's check take away from the lecture

(1) If $P_0(x) \neq 0$ then x is called

(a) singular point (b) ordinary point (c) can't decide

(2) If $P_0(x) = 0$ then x is called

(a) singular point (b) ordinary point (c) can't decide

Home Work Problems for the day

Solve the following differential equation by series method:

1. $y''+xy'+y=0.$

Ans. : $y=a_0\left(1-\frac{x^2}{2}+\frac{x^4}{2.4}-\frac{x^6}{2.4.6}+\ldots\right)+a_1\left(x+\frac{x^3}{3}+\frac{x^5}{3.5}-\frac{x^7}{3.5.7}+\ldots\right)$

2. $y''-xy'+x^2y=0.$

Ans.: $y=a_0\left(1-\frac{x^4}{12}-\frac{x^6}{90}\ldots\right)+a_1\left(x+\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{144}\ldots\right)$

3. $(2-x^2)y''+2xy'-2y=0.$

Ans. : $y=a_0\left(1+\frac{x^4}{2!}+\frac{x^8}{4!}+\ldots\right)+a_2\left(x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}\ldots\right).$

Learning from the topic: Students will be confident to apply the power series to solve second order linear differential equations.

$a_4 = -\frac{a_0}{3.4}$; $a_5 = -\frac{a_1}{4.5}$; $a_6 = -\frac{a_2}{5.6} = 0$

$a_7 = -\frac{a_3}{6.7} = 0$; $a_8 = -\frac{a_4}{7.8} = \frac{a_0}{3.4.7.8}$; $a_9 = -\frac{a_5}{8.9} = \frac{a_1}{4.5.8.9}$ = and so on.

Substituting the values of a_i's in (2), we get.

$$y = a_0 + a_1x - \frac{a_0}{3.4}x^4 - \frac{a_1}{4.5}x^5 + \frac{a_0}{3.4.7.8}x^8 + \frac{a_1}{4.5.8.9}x^9 + \ldots$$

or $y = a_0\left(1 - \frac{x^4}{3.4} + \frac{x^8}{3.4.7.8} - \ldots\right) + a_1\left(x - \frac{x^5}{4.5} + \frac{x^9}{4.5.8.9} - \ldots\right)$

which is the required general solution of (1) containing two arbitrary constants a_0 and a_1.

(2) Solve the series $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ … (1)

Solution: Since x=0 is an ordinary point of (1), let its series solution be

$y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots\ldots = \sum_{k=0}^{\infty} a_k x^k$ … (2)

Then $\frac{dy}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1}$ and $\frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation (1), we get

$$(1+x^2)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = 0$$

$(1+x)^2\,[2.1a_2 + 3.2a_3x + \ldots + n(n-1)a_n x^{n-2} + (n+1)na_{n+1}x^{n-1} + (n+2)(n+1)a_{n+2}x^n + \ldots]$

$+ x\,[a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots] - [a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots] = 0$

Or $(2a_2 - a_0) + 6a_3x + \ldots[(n+1)(n+2)a_{n+2} + n(n-1)a_n + na_n - a_n]x^n + \ldots = 0$

Or $(2a_2 - a_0) + 6a_3x + \ldots + [(n+1)(n+2)a_{n+2} + (n^2-1)a_n]x^n + \ldots = 0$

Equating to zero the co-efficient of various powers of x, $a_2 = \frac{a_0}{2}$, $a_3 = 0$

Equating to zero the co-efficient of x^n, we have

$(n+1)(n+2)a_{n+2} + (n^2-1)a_n = 0$

Or $a_{n+2} = -\frac{n-1}{(n+2)}a_n$ … (3)

since $n+1 \neq 0$

Putting n= 2,3,4,5, … in (3) successively,

$a_4 = -\frac{1}{4}a_2 = -\frac{a_0}{8}$; $a_5 = -\frac{2}{5}a_3 = 0$; $a_6 = -\frac{1}{2}a_4 = \frac{a_0}{16}$

$a_7 = -\frac{4}{7}a_5 = 0$; $a_8 = -\frac{5}{8}a_6 = -\frac{5a_0}{128}$ and so on.

Substituting the values of a_i's in (2), we get

$$y = a_0 + a_1x + \frac{a_0}{2}x^2 - \frac{a_0}{8}x^4 + \frac{a_0}{16}x^6 + \frac{5a_0}{128}x^8 + \ldots$$

Or $y = a_0\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^5}{128} + \ldots\right) + a_1x$

Which is required general solution of (1) containing two arbitrary constant a_0 and a_1.

Exercise 13

Solve the following equations by series Method:

1. $\frac{d^2y}{dx^2} - y = 0$.

Ans.: $y = a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right) + a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right) = a_0\cosh x + a_1\sinh x$

When x=0 is an irregular singular point of (1), then the differential equation (1) has no series solution of the for $y = \sum_{k=0}^{\infty} a_0 x^{m+k}$

Series Solution when x=0 is an Ordinary Point of the Equation

$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$ .....(1)

Let $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots = \sum_{k=0}^{\infty} a_k x^k$ .....(2)

be the solution of (1).

Then $\frac{dy}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1}$ $\frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$

Substitute the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1)

Equate to zero the co-efficient of various powers of x and find $a_2, a_3, a_4, \ldots$ in terms of a_0 and a_1.

Equate to zero the co-efficient of x^n. The relation so obtained is called the recurrence relation.

Give different values to n in the recurrence relation to determine various a_i's in terms of a_0 and a_1.

Substitute the values of $a_2, a_3, a_4, \ldots$ in (2) to get the series solution of (1) having a_0 and a_1 as arbitrary constant.

5. Sample Problem:

(1) Solve in series the equation $\frac{d^2y}{dx^2} + x^2y = 0$ .....(1)

Solution: Since x=0 is an ordinary point of (1), let its series solution be

Let $Y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots = \sum_{k=0}^{\infty} a_k x^k$ .....(2)

Then $\frac{dy}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1}$ and $\frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} k(k-1) a_k K^{k-2}$

Substituting the values of y and $\frac{d^2y}{dx^2}$ in the given equation (1), we get

$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + x^2 \sum_{k=0}^{\infty} a_k x^k = 0$

OR

$[2.1\, a_2 + 3.2 a_3 x + \ldots + (n+2)(n+1) a_{n+2} x^n + \ldots] + x^2 [a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-2} x^{n-2} + \ldots] = 0$

OR

$2a_2 + 6a_3x + \ldots + [(n+2)(n+1) a_{n+2} + a_{n-2}] x^n + \ldots = 0$

Equating to zero the co-efficient of various powers of x, $a_2 = 0$, $a_3 = 0$ and so on.

Equating to zero the co-efficient of x^n, we have

$(n+1)(n+2) a_{n+2} + a_{n-2} = 0$

or $a_{n+2} = -\frac{a_{n-2}}{(n+1)(n+2)}$ ... (3)

Which is the recurrence relation.

Putting n=2, 3, 4, 5, ...in (3) successively,

1) $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+5y=x^2\sin\log x$ *Ans :* $y=x^2\left(c_1\cos\log x+c_2\sin\log x\right)-\dfrac{x^2}{2}\log x\ \cos\log x$

2) $x^2\dfrac{d^2y}{dx^2}+5x\dfrac{dy}{dx}+3y=\dfrac{\log x}{x^2}$ *Ans :* $y=c_1x^{-1}+c_2x^{-3}+x^{-2}\log x$

3) $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=4\cos\log(x+1)$

Ans : $y=c_1\cos\log(1+x)+c_2\sin\log(1+x)+2\log(1+x)\sin\log(1+x)$

Learning from the topic: Student will be able to find the solution of higher order linear differential equations with variable coefficients.

Power series Solution of Differential Equations (method of Frobenius)

Lecture: 13

1. **Learning Objective:**
   Learner shall be able to understand and solve the differential equations using series method.
2. **Introduction:**
   Solving a differential equation is not always possible by the conventional methods like Bessel's differential equation, Legendre's differential equation etc. To solve these differential equations, we use the series method.
3. **Key notation:**
   Power series $y = a_0 + a_1x + a_2x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$
4. **Important Formulae / Theorem / Properties:**
   Every differential equation of the form (1) does not have series solution. As such, we find the conditions under which the above equation admits of the series solution.

   Dividing (1) by $P_0(x)$, we have $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$ .....(2)

   Where $p(x)=\frac{P_1(x)}{P_0(x)}$ and $q(x)=\frac{P_2(x)}{P_0(x)}$

   Ordinary Point. $x = 0$ is an ordinary point of (1) if $P_0(0)\neq 0$, otherwise it is called a singular point.

   When x=0 is an ordinary point (1), its every solution can be expressed as a series of the form

   $$y = a_0 + a_1x + a_2x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

   Singular Point. $x = 0$ is called a singular point of (1) if $P_0(0)=0$. If $x\,p(x)$ and $x^2q(x)$ possess derivatives of all orders in the neighbourhood of x=0 is called regular singular point of (1)

   When x=0 is a regular singular point of (1), at least one of its solution can be expressed as

   $$y = x^m(a_0 + a_1x + a_2x^2 + \ldots) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

   Where m may be positive or negative integer or a fraction.

(3) Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$

Solution: This is a Legendre's equation.

Putting $x + 1 = e^z$, we get

$(x+1)\ \frac{dy}{dx} = 1 D y$ when $D \equiv \frac{d}{dz}$

$(x+1)^2 \frac{d^2y}{dx^2} = 1^2\ D(D-1)\,y$

On substituting in the given equation,

$D(D-1)\,y + D\,y + y = 4\cos z \Rightarrow (D^2 - D + D + 1)\,y = 4\cos z$

$\Rightarrow (D^2 + 1)\,y = 4\cos z$

The A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i \Rightarrow CF = c_1\cos z + c_2\sin z$

$$PI = \frac{1}{D^2+1}4\cos z = 4\frac{1}{D^2+1}\cos z = 4z\frac{1}{2D}\cos z = 2z\int \cos z\,dz = 2z\sin z$$

The complete solution is $y = CF + PI$

$y = c_1\cos z + c_2\sin z + 2z\sin z$

Replacing z by log (x+1)

$\Rightarrow y = c_1\cos\log(1+x) + c_2\sin\log(1+x) + 2\log(1+x)\sin\log(1+x)$

Exercise 12

Solve the following:

1) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^{-1}$ **Ans :** $y = c_1x + c_2x^2 + \frac{1}{6x}$

2) $3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2\log x$ **Ans :** $y = c_1 + c_2(x)^{\frac{2}{3}} + \frac{x^3}{21}\left(\log x - \frac{16}{21}\right)$

3) $(2x+1)^2\frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$ **Ans :** $y = c_1(2x+1)^3 + c_2(2x+1)^{-1} - \frac{3}{16}(2x+1) - \frac{1}{4}$

Let's check take away from the lecture

1) The equation $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$ is called

(a) Exact (b) Linear

(c) Legendre's equation (d) Cauchy's equation

2) The general solution of the differential equation $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ is

(a) $Ax+Bx^2$ (b) $Ax + B\log x$ (c) $Ax+Bx^2\log x$ (d) $Ax+Bx\log x$
where A& B are constants.

3) The equation $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\{\log(1+x)\}$ is called

(a) Exact (b) Linear

(c) Legendre's equation (d) Cauchy's equation

Home Work Problems for the day

Solve the following:

$$= \frac{1}{3-2D}\cos z + e^z \frac{1}{(D+1)^2 - 2(D+1) + 4}\sin z$$

$$= \frac{1}{9-4D^2}(3+2D)\cos z + e^z \frac{1}{D^2+3}\sin z$$

$$= \frac{1}{13}(3+2D)\cos z + e^z \frac{1}{2}\sin z$$

$$= \frac{1}{13}(3\cos z - 2\sin z) + e^z \frac{1}{2}\sin z$$

The complete solution is $y = CF + PI$

$$y = e^z(c_1\cos\sqrt{3}z + c_2\sin\sqrt{3}z) + \frac{1}{13}(3\cos z - 2\sin z) + e^z\frac{1}{2}\sin z$$

Replacing z by log x

$$y = x\left[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)\right] + \frac{3}{13}\cos(\log x) + \left(\frac{x}{2} - \frac{2}{13}\right)\sin(\log x)$$

2) *Solve* $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

Solution: This is a Legendre's equation.

Putting $3x + 2 = e^z$, we get

$$(3x+2)\frac{dy}{dx} = 3Dy \quad \text{when } D \equiv \frac{d}{dz}$$

$$(3x+2)^2\frac{d^2y}{dx^2} = 3^2 D(D-1)y$$

On substituting in the given equation,

$$\left[9D(D-1) - 3(3D) - 36\right]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1 \Rightarrow (9D^2 - 36)y = \frac{e^{2z}-1}{3}$$

The auxiliary equation is $9D^2 - 36 = 0 \Rightarrow D = 2, -2$

$$CF = c_1 e^{2z} + c_2 e^{-2z}$$

Now, $PI = \frac{1}{9D^2-36}\,\frac{e^{2z}-1}{3}$

$$= \frac{1}{27}\frac{1}{D^2-4}e^{2z} - \frac{1}{27}\frac{1}{D^2-4}e^{0z}$$

$$= \frac{z}{27}\frac{1}{2D}e^{2z} - \frac{1}{27}\frac{1}{0^2-4}e^{0z}$$

$$= \frac{z}{27}\frac{1}{2D}e^{2z} - \frac{1}{27}\frac{1}{0^2-4}e^{0z}$$

$$= \frac{z}{54}\frac{e^{2z}}{2} - \frac{1}{27}\left(\frac{1}{-4}\right) = \frac{1}{108}\left(ze^{2z} + 1\right)$$

The complete solution is $y = CF + PI = c_1 e^{2z} + c_2 e^{-2z} + \frac{1}{108}\left(ze^{2z} + 1\right)$

Replacing z by log (3x+2)

$$y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}\left((3x+2)^2\log(3x+2) + 1\right)$$

The Cauchy's equation will be of the form $a_0 x^2 \dfrac{d^2y}{dx^2} + a_1 x \dfrac{dy}{dx} + a_2 y = X$.

To solve this put $x = e^z \Rightarrow z = \log x$

Then $x\dfrac{dy}{dx} = \dfrac{dy}{dz} = Dy$, *where* $D \equiv \dfrac{d}{dz}$

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D(D-1)y$$

Ily, $x^3 \dfrac{d^3y}{dx^3} = D(D-1)(D-2)y$

Substituting these in Cauchy's equation will reduce it to linear differential equation with constant coefficients.

(2) Legendre's Linear Differential Equation:

The Legendre's equ. will be of the form $P_0 (ax+b)^2 \dfrac{d^2y}{dx^2} + P_1 (ax+b)\dfrac{dy}{dx} + P_2 y = X$ to solve this put

$ax + b = e^z \Rightarrow z = \log(ax+b)$

Then $(ax+b)\dfrac{dy}{dx} = a\dfrac{dy}{dz} = a\,D\,y$ when $D \equiv \dfrac{d}{dz}$

$$(ax+b)^2 \frac{d^2y}{dx^2} = a\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right] = a^2\, D(D-1)\,y \text{ and so on.}$$

Substituting these in Legendre's equation will reduce it to linear differential equation with constant coefficients.

4. Sample Problem:

1) *Solve* $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x)$

Solution: This is a Cauchy's equation.

Putting $x = e^z$, we get

$x\dfrac{dy}{dx} = Dy$, *where* $D \equiv \dfrac{d}{dz}$

& $x^2 \dfrac{d^2y}{dx^2} = D(D-1)y$

On substituting in the given equation,

$$\left[D(D-1) - D + 4\right]y = \cos z + e^z \sin z$$

$$\therefore (D^2 - 2D + 4)y = \cos z + e^z \sin z$$

The AE is $D^2 - 2D + 4 = 0$

$$\therefore \quad D = 1 \pm \sqrt{3}\, i$$

$$\text{CF} = e^z\,(c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z)$$

$$\text{Now, PI} = \frac{1}{D^2 - 2D + 4}\cos z + \frac{1}{D^2 - 2D + 4} e^z \sin z$$

2) $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$ **Ans :** $y = c_1e^x + c_2e^{-x} - 1 - xe^x + \left(e^x - e^{-x}\right)\log(1+e^x)$

3) $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$ **Ans :** $y = c_1\cos ax + c_2\sin ax + \dfrac{1}{a^2}\cos ax\,\log\cos ax + \dfrac{x}{a}\sin ax$

Let's check take away from the lecture

(1) The particular integral when $X = e^{x^2}$ can be found by applying the VPM ?
(a) yes (b) no (c) can't decide

(2) The particular integral when $X = \sin x^2$ can be found by applying the VPM?
(a) yes (b) no (c) can't decide

Home Work Problems for the day

Solve the following:

1) $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$ **Ans :** $y = c_1\cos 2x + c_2\sin 2x - \dfrac{1}{4}\cos 2x\,\log(\sec 2x + \tan 2x)$

2) $(D^2 + a^2)y = \operatorname{cosec} ax$ **Ans :** $y = c_1\cos ax + c_2\sin ax - \dfrac{x}{a}\cos ax + \dfrac{\sin ax}{a^2}\log(\sin ax)$

3) $(D^2 - 1)y = 2(1 - e^{-2x})^{\frac{-1}{2}}$ **Ans :** $y = c_1e^x + c_2e^{-x} - e^x\sin e^{-x} + \sqrt{e^{2x}+1}\;e^{-x}$

4) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^{-2x}\sec^2 x\,(1 + 2\tan x)$

Ans : $y = c_1e^{-2x} + c_2e^{-3x} + \dfrac{e^{-2x}}{4}(1 + 2\tan x)^2 - e^{-2x}\sec^2 x$

Learning from the topic: Students will be confident to apply the VPM to calculate the PI - particular integral of higher order linear differential equations with constant coefficients.

Cauchy's Euler's differential equation

Lecture: 12

1. Learning Objective:

Learner shall be able to develop the skill to identify and solve the second order linear differential equations with variable coefficients and solve it.

2. Introduction:

Here we will learn how second order linear differential equations with variable coefficients will be reduced to linear differential equations with constant coefficients by suitable substitutions (assumption). Then applying Type 1 to Type 5 along with variation parameter method, further this equation will be solved for complete solution.

3. Important Formulae / Theorem / Properties:

(1) Cauchy's Euler's Linear Differential Equation:

Solution: Converting into symbolic form $D^2y + y = \frac{1}{1+\sin x}$, where $D = \frac{d}{dx}$

The auxiliary equation is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$$CF = e^{0x}\left(c_1\cos x + c_2\sin x\right) = c_1y_1 + c_2y_2$$

Here $y_1 = \cos x,\ y_2 = \sin x,\ X = \frac{1}{1+\sin x}$

Let $\quad P.I. = uy_1 + vy_2$

$$\text{Now}\quad W = \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u = -\int \frac{y_2 X}{w}dx = -\int \frac{\sin x}{1}\left(\frac{1}{1+\sin x}\right)dx = \int \frac{\sin^2 x - \sin x}{\cos^2 x}dx$$

$$= \int \tan^2 x\,dx = \int \sec x.\tan x\,dx = \int\left(\sec^2 x - 1\right)dx - \sec x = \tan x - x - \sec x$$

$$v = \int \frac{y_1 X}{w}dx = \int \frac{\cos x}{1}\ \frac{1}{1+\sin x}dx = \log\left(1+\sin x\right)$$

$$PI = uy_1 + vy_2 = (\tan x - x - \sec x)\cos x + \log(1+\sin x)\sin x$$

$$PI = \sin x - 1 - x\cos x + \sin x\log\left(1+\sin x\right)$$

The complete solution is $y = CF + PI$

$$y = c_1\cos x + c_2\sin x + \sin x - 1 - x\cos x + \sin x\log\left(1+\sin x\right)$$

(3) Solve by the method of variation parameters $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

Solution: The auxiliary equation is $D^2 - 6D + 9 = 0 \Rightarrow \left(D-3\right)^2 = 0 \Rightarrow D = 3, 3$

$$CF = c_1e^{3x} + c_2\,x\,e^{3x}$$

Here $y_1 = e^{3x},\ y_2 = x\,e^{3x},\ X = \frac{e^{3x}}{x^2}$

Let $PI = uy_1 + vy_2$

$$\text{Now}\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$$

$$u = -\int \frac{y_2 X}{w}dx = -\int \frac{x\,e^{3x}}{e^{6x}}\ \frac{e^{3x}}{x^2}dx = -\int \frac{1}{x}dx = -\log x$$

$$v = \int \frac{y_1 X}{W}dx = \int \frac{e^{3x}}{e^{6x}}\ \frac{e^{3x}}{x^2}dx = \int \frac{1}{x^2}dx = -\frac{1}{x}$$

$$PI = uy_1 + vy_2 = -\log x\ e^{3x} - \frac{1}{x}\,x\,e^{3x} = -\left(\log x + 1\right)e^{3x}$$

The complete solution is $y = CF + PI = c_1e^{3x} + c_2xe^{3x} - e^{3x}\left(\log x + 1\right)$

Exercise 11

Solve the following:

1) $\frac{d^2y}{dx^2} - y = e^{-x}\sin\left(e^{-x}\right) + \cos\left(e^{-x}\right)$ **Ans:** $y = c_1e^{x} + c_2e^{-x} - e^{x}\sin\left(e^{-x}\right)$

Learners shall be able to learn alternate method to calculate the PI - particular integral of higher order linear differential equations with constant coefficients.

2. Introduction:

In this part, we adapt the method of variation of parameters to a linear second-order differential equation to find its particular solution in general case.

3. Key notation:

The Wronskian W of y_1 and y_2 is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

4. Important Formulae / Theorem / Properties: Variation of Parameter Method (VPM)

This method is particularly used for solving a L.D. E. of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$ where P and Q are constants.

5. Working Rule:

(i) On solving AE say $CF = c_1 y_1 + c_2 y_2$

(ii) Let $PI = u\,y_1 + v\,y_2$, where u and v are functions of x to be determined.

(iii) Find $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

(iv) Evaluate $u = -\int\left(\frac{y_2 X}{W}\right)dx$ & $v = \int\left(\frac{y_1 X}{W}\right)dx$

(v) Simplify $PI = u\,y_1 + v\,y_2$

(vi) Write $CS = y = CF + PI$

6. Sample Problem:

1) Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$ by variation of parameter method.

Solution: The auxiliary equation is $D^2 + 3D + 2 = 0$

$$\therefore \quad (D+1)(D+2) = 0 \quad \therefore D = -1, 2$$

$$\therefore \quad CF = c_1e^{-x} + c_2e^{-2x}$$

Here $y_1 = e^{-x}$, $y_2 = e^{-2x}$, $X = e^{e^x}$

Let $PI = u\,y_1 + v\,y_2$

$$Now, \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

$$u = -\int \frac{y_2 X}{W}dx = -\int \frac{e^{-2x}e^{e^x}}{-e^{-3x}}dx = \int e^{e^x} e^x dx = e^{e^x} \quad [by\ putting\ e^x = t]$$

$$v = \int \frac{y_1 X}{W}dx = \int \frac{e^{-x}e^{e^x}}{e^{-3x}}dx = \int e^{2x}e^{e^x}dx = e^x e^{e^x} - e^{e^x}$$

$$(\ by\ putting\ e^x = t,\ v = \int e^t\, t\, dt = t e^t - e^t\ )$$

$$PI = u\,y_1 + v\,y_2 = e^{e^x}e^{-x} - (e^x e^{e^x} - e^{e^x})e^{-2x} = e^{-2x}e^{e^x}$$

The complete solution is $y = CF + PI = c_1e^{x} + c_2e^{-2x} + e^{-2x}e^{e^x}$

(2) Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$

2) $\left(D^2-4D+4\right)y=x^3$ **Ans:** $y_p=PI=\frac{1}{4}\left(3+\frac{9x}{2}+3x^2+x^3\right)$

3) $\left(D^2+2D+1\right)y=e^{-x}\log x$ **Ans:** $y_p=PI=\frac{x^2}{2}e^{-x}(\log x-\frac{3}{2})$

4) $(D^2+6D+9)y=e^{-3x}\frac{1}{x^2}$ **Ans:** $y_p=PI=e^{3x}\left(\log x+1\right)$

Let's check take away from the lecture

(1) The particular integral can be found for $X = x^m$

(a) if m is positive (b) if m is negative (c) if m is real (d) if m is fractional

(2) For the polynomial $7x^2+5x+2$ its derivative of order 3 and higher is

(a) Constant (b) zero (c) negative (d) can't decide

(3) If $X = e^{ax}V$, where V is a function of x, then

(a) $PI = e^{ax}\frac{1}{f(D+a)}V$ (b) $PI = \frac{1}{f(D+a)}e^{ax}V$

(c) $PI = V\frac{1}{f(D+a)}e^{ax}$ (d) $PI = e^{ax}\frac{1}{f(a)}V$

(4) If $X = e^{5x}x^2$, then

(a) $PI = e^{5x}\frac{1}{f(D+5)}x^2$ (b) $PI = \frac{1}{f(D+5)}e^{5x}x^2$

(c) $PI = x^2\frac{1}{f(D+5)}e^{5x}$ (d) $PI = e^{5x}\frac{1}{f(5)}x^2$

Home Work Problems for the day

Solve the following:

1) $(D^2+3D+2)y=e^x\cos 2x+e^{-2x}$ **Ans:** $y_p=PI=\frac{e^x}{52}(\cos 2x+5\sin 2x)+xe^{-2x}$

2) $\frac{d^2y}{dx^2}+2y=x^2e^{3x}+e^x\cos x$ **Ans:** $y_p=PI=\frac{e^x}{4}(\sin x+\cos x)+\frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$

3) $\frac{d^2y}{dx^2}+2y=x^2e^{3x}+e^x-\cos 2x$ **Ans:** $y_p=PI=\frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)+\frac{e^x}{3}+\frac{\cos 2x}{2}$

Learning from the topic: Student will be able to calculate the particular integral when $X = x^m,\ e^{ax}V$.

Method of Variation of Parameter

Lecture: 11

1. Learning Objective:

$$=\frac{32}{5}e^{x}\left(D-\frac{1}{4}\right)\sin\left(\frac{x}{2}\right)=\frac{32}{5}e^{x}\left[\frac{1}{2}\cos\left(\frac{x}{2}\right)-\frac{1}{4}\sin\left(\frac{x}{2}\right)\right]=\frac{8}{5}e^{x}\left[2\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]$$

(4) Find particular Integral for $\left(D^2-7D-6\right)y=\left(1+x^2\right)e^{2x}$.

Solution: $PI=\frac{1}{f(D)}\ x=\frac{1}{D^2-7D+6}\left(1+x^2\right)e^{2x}=e^{2x}\frac{1}{(D+2)^2-7(D+2)+6}\left(1+x^2\right)$

$$=e^{2x}\frac{1}{D^2-3D-4}\left(1+x^2\right)=\frac{e^{2x}}{-4}\left\{\frac{1}{1-\left(\frac{D^2-3D}{4}\right)}\right\}\left(1+x^2\right)$$

$$=\frac{e^{2x}}{-4}\left\{1+\left(\frac{D^2-3D}{4}\right)+\left(\frac{D^2-3D}{4}\right)^2+\ldots\right\}\left(1+x^2\right)\quad=\frac{e^{2}}{-4}\left\{1+\frac{D^2}{4}-\frac{3D}{4}+\frac{9D^2}{16}\right\}\left(1+x^2\right)$$

$$=\frac{e^{2x}}{-4}\left\{1-\frac{3D}{4}+\frac{13D^2}{16}\right\}\left(1+x^2\right)\ \text{.... considering terms upto } D^2$$

$$=\frac{e^{2x}}{4}\left\{\left(1+x^2\right)-\frac{3}{4}(2x)+\frac{13}{16}(2)\right\}=\frac{e^{2x}}{-4}\left\{1+x^2-\frac{6x}{4}+\frac{26}{16}\right\}$$

(4) Find particular Integral for $\left(D^2+2\right)y=e^{x}\cos x+x^2e^{3x}$

Solution: $PI=\frac{1}{D^2+2}\left(e^{x}\cos x+x^2e^{3x}\right)=PI\ 1\ +PI\ 2$ ..... (1)

$$PI\ 1=\frac{1}{D^2+2}\ e^{x}\cos 3x\ =\ e^{x}\ \frac{1}{(D+1)^2+2}\cos x=e^{x}\frac{1}{D^2+2D+3}\cos x$$

$$=e^{x}\ \frac{1}{2D+2}\cos x\qquad=\frac{e^{x}}{2}\ \frac{D-1}{D^2-1}\cos x=\frac{-e^{x}}{2(2)}\left(-\sin x-\cos x\right)$$

$=\frac{e^{x}}{4}\left(\sin x+\cos x\right)$.... (2)

$$PI\ 2=\frac{1}{D^2+2}e^{3x}\ x^2=e^{3x}\ \frac{1}{(D+3)^3+2}\ x^2=e^{3x}\ \frac{1}{D^2+6D+11}\ x^2$$

$$=\frac{e^{3x}}{11}\left[1+\frac{6D+D^2}{11}\right]^{-1}x^2$$

$$=\frac{e^{3x}}{11}\left[1-\frac{\left(6D+D^2\right)}{11}+\frac{36D^2}{121}+\ldots\right]x^2=\frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]\quad\ldots\ldots\quad(3)$$

From (1), (2) and (3) $\quad PI=\frac{e^{x}}{4}\left(\sin x+\cos x\right)+\frac{e^{3x}}{11}\left[x^2-\frac{121x}{11}+\frac{50}{121}\right]$

Exercise 10

Solve the following:

1) $\left(D^2-4D+4\right)y=8x^2$ **Ans:** $y_p=PI=2x^2+4x+3.$

$$=\frac{1}{4}\left[1-\left(\frac{4D-D^2}{4}\right)\right]^{-1}x^2=\frac{1}{4}\left[1-\left(\frac{4D-D^2}{4}\right)+D^2\right]x^2$$

$$=\frac{1}{4}\left[x^2+\frac{1}{4}(8x-2)+2\right]=\frac{1}{4}\left[x^2+2x+\frac{3}{2}\right]$$

$$y_p=PI=\frac{1}{4}\left[x^2+2x+\frac{3}{2}\right].$$

(2) Find particular Integral for $\frac{d^2y}{dx^2}+2y=x^2e^{3x}$

Solution:

$$PI=\frac{1}{D^2+2}x^2e^{3x}=\frac{1}{D^2+2}e^{3x}\ x^2=e^{3x}\frac{1}{(D+3)^2+2}\ x^2$$

$$=e^{3x}\frac{1}{D^2+6D+11}x^2=\frac{e^{3x}}{11}\left[1-\frac{(6D+D^2)}{11}\right]^{-1}x^2$$

$$=\frac{e^{3x}}{11}\left[1-\frac{(6D+D^2)}{11}+\frac{36D^2}{121}+\ldots\right]x^2$$

$$=\frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}-\frac{2}{11}+\frac{72}{121}\right]$$

$$=\frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]$$

$$y_p=PI=\frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]$$

(3) Find particular Integral for $\left(D^2-3D+2\right)y=2e^x\sin\left(\frac{x}{2}\right)$

Solution: $PI=\frac{1}{(D^2-3D+2)}2e^x\sin\left(\frac{x}{2}\right)=2e^x\frac{1}{\left((D+1)^2-3(D+1)+2\right)}\sin\left(\frac{x}{2}\right)$

$$=2e^x\frac{1}{\left(D^2+2D+1-3D-3+2\right)}\sin\left(\frac{x}{2}\right)=2e^x\frac{1}{\left(D^2-D\right)}\sin\left(\frac{x}{2}\right)$$

$$=2e^x\frac{1}{\left(-\left(\frac{1}{2}\right)^2-D\right)}\sin\left(\frac{x}{2}\right)=2e^x\frac{\left(\frac{-1}{4}+D\right)}{\left(\frac{-1}{4}-D\right)\left(\frac{-1}{4}+D\right)}\sin\left(\frac{x}{2}\right)$$

$$=2e^x\frac{\left(D-\frac{1}{4}\right)}{-\left(D^2-\frac{1}{16}\right)}\sin\left(\frac{x}{2}\right)=2e^x\frac{\left(D-\frac{1}{4}\right)}{\left(\frac{1}{16}--\left(\frac{1}{2}\right)^2\right)}\sin\left(\frac{x}{2}\right)=2e^x\frac{\left(D-\frac{1}{4}\right)}{\left(\frac{5}{16}\right)}\sin\left(\frac{x}{2}\right)$$

(a) yes (b) no (c) can't decide

Home Work Problems for the day

Find particular Integral for the following:

(1) $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{-x}$ **Ans:** $y=c_1e^{-x}+c_2e^{-2x}+xe^{-x}$

(2) $(D^2+D+1)y=(1+\sin x)^2$ **Ans:** $y=c_1e^{x}+(c_2+c_3x)e^{2x}+\frac{x^2}{2}e^{2x}+2xe^{x}-\frac{1}{6}e^{-x}-\frac{1}{2}$

(3) $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=\cosh 2x\ \sinh 3x$ **Ans:** $y_p=PI=\frac{1}{4}\left(\frac{e^{5x}}{42}-\frac{xe^{-x}}{5}+\frac{e^{x}}{6}-\frac{e^{-5x}}{12}\right)$

Learning from the topic: Student will be able to find the particular integral when $X=e^{ax}$, $\sin(ax+b)$ *or* $\cos(ax+b)$

Lecture: 10

1. Learning Objective:

Learners shall be able to calculate the particular integral when $X = x^m,\ e^{ax}V$.

2. Introduction:

In this case we learn to find the particular integral if the nonhomogeneous part of differential equation is either algebraic or exponential multiplied by a trigonometric or algebraic function

3. working Rule:

Type 4: If $X = x^m$, where m is a positive integer, then

$$PI=\frac{1}{f(D)}x^m=\frac{1}{1+\phi(D)}x^m=\{1+\phi(D)\}^{-1}x^m.$$

Now expand the bracket by the formula $(1+x)^{-1}=1-x+x^2-x^3+x^4-\dots\dots$

$(1-x)^{-1}=1+x+x^2+x^3+x^4+\dots\dots$

and operate each term of the expansion on x^m.

Type 5: If $X=e^{ax}V$, where V is a function of x, then

$$PI=\frac{1}{f(D)}e^{ax}V=e^{ax}\frac{1}{f(D+a)}V$$

4. Sample Problem:

(1) Find particular Integral for $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=x^2$.

Solution: $PI=\frac{1}{(D-2)^2}x^2$

$$=\frac{1}{4[1-\frac{4D-D^2}{4}]}x^2$$

(1) Find particular Integral for $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$.

Solution:
$$PI = \frac{1}{(D-2)^2} e^x$$
$$= \frac{1}{D^2-4D+4} e^x = \frac{1}{1-4+4} e^x = e^x$$

$$y_p = PI = e^x$$

(2) Find PI for $\left(D^2 - 5D + 6\right) y = \sin 3x$.

Solution:
$$PI = \frac{1}{D^2 - 5D + 6} \sin 3x = \frac{1}{-3^2 - 5D + 6} \sin 3x = -\frac{1}{5D+3} \sin 3x$$
$$= -\frac{5D-3}{25D^2-9} \sin 3x = -\frac{5D \sin 3x - 3\sin 3x}{25(-3^2)-9} = \frac{5\cos 3x - \sin 3x}{78}$$
$$y_p = PI = \frac{5\cos 3x - \sin 3x}{78}.$$

Exercise 09

Solve the following:

1) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$ **Ans :** $y_p = PI = \frac{e^x}{6}$

2) $\left(D^2 + 6D + 9\right) y = 3^x$ **Ans :** $y_p = \frac{3^x}{(\log_e 3 + 3)^2}$

3) $D^2 y - 4y = 2\cos h(2x)$ **Ans :** $y_p = \frac{x}{2} \sin h\ 2x$

4) $\left(D^2 + 4\right) y = \cos 2x$ **Ans :** $y_p = \frac{x}{4} \sin 2x$

5) $\frac{d^2y}{dx^2} + y = \sin x \sin 2x$ **Ans :** $y_p = \frac{1}{16}\cos 3x + \frac{x}{4}\sin x$

Let's check take away from the lecture

(1) For differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$, the particular integral is

(a) $\frac{1}{15}e^{2x}$ (b) $\frac{1}{5}e^{2x}$ (c) $3e^{2x}$ (d) $c_1e^{-x} + c_2e^{-3x}$

(2) The particular integral when $X = e^{3x^2}$ can be found by the above type

(a) yes (b) no (c) can't decide

(3) For the differential equation $\frac{d^2y}{dx^2} + 3y = \cos 2x$, the particular integral is

(a) $-\cos 2x$ (b) $\cos 2x \sin 2x$ (c) $-\sin 2x$ (d) $c_1e^{-x} + c_2e^{-3x}$

(4) The particular integral when $X = \sin x^2$ can be found by applying the type: 2

Home Work Problems for the day

Solve the following:

1) $\left(D^2-5D+6\right)y=0$ — *Ans :* $y=c_1e^{2x}+c_2e^{3x}$.

2) $D^2y-(a+b)Dy+aby=0$ — *Ans :* $y=c_1e^{ax}+c_2e^{bx}$

3) $(D^2+D+1)y=0.$ — *Ans :* $y=e^{\frac{-x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)$.

4) $\frac{d^2y}{dx^2}-y=0.$ — *Ans :* $y=c_1e^{x}+c_2e^{-x}$

5) $\frac{d^2y}{dx^2}+y=0.$ — *Ans :* $y=c_1\cos x+c_2\sin x$

Learning from the topic: Student will be able to find the complementary function of second order linear differential equations with constant coefficients.

Lecture: 09

1. Learning Objective:
Learners shall be able to calculate the particular integral if the nonhomogeneous part of the differential equation as $X=e^{ax}$, $\sin(ax+b)$ *or* $\cos(ax+b)$.

2. Introduction:
The solution of Nonhomogeneous linear differential equation with constant coefficients has two parts in its solution as complementary function and particular integral. The evaluation of particular integral depends on the function which are present in the function in the Nonhomogeneous part in this section we will learn how to find PI for exponential and sine/cosine function.

3. Working rule: The working rule is explained as the differential equation is categorised as per the type of function on right hand side of differential equation

Type 2: If $X=e^{ax}$, *then* $PI=\frac{1}{f(D)}e^{ax}=\begin{cases}\frac{e^{ax}}{f(a)} & \text{if } f(a)\neq 0\\ \frac{xe^{ax}}{f'(a)} & \text{if } f'(a)\neq 0\\ \text{and proceed in this way}\end{cases}$

Note: Exponential functions whose argument is linear factor can be evaluated by this method.

Type 3: If $X=\sin(ax+b)$ *or* $\cos(ax+b)$, *then* $PI=\frac{1}{f(D)}\sin(ax+b)$

Replace D^2 by $-a^2$ & remaining linear factor in D converts to D^2 (quadratic) by rationalizing.

Note: Only sine and cosine function whose argument is linear factor can be evaluated by this method.

4. Sample Problem:

$\therefore (D+2)(D+1)=0$

$\therefore D=-1,-2$

$\therefore\ CF=c_1e^{-x}+c_2e^{-2x}$

In this case $PI = 0$

The complete solution is $CS = y = CF + PI = c_1e^{-x}+c_2e^{-2x}$.

(2) Solve $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=0.$

Solution: The auxiliary equation is

A.E. $D^2-4D+4=0$

$\therefore (D-2)^2=0 \qquad \therefore D=2,2$

$CF=(c_1+c_2x)e^{2x}$

In this case $PI = 0$

The complete solution is $CS = y = CF + PI = (c_1+c_2x)e^{2x}$.

(3) Solve $\dfrac{d^2y}{dx^2}+2y=0.$

Solution: The auxiliary equation is

A.E. $(D^2+2)=0$

$\therefore \quad D=\sqrt{2}\,i,\ -\sqrt{2}\,i$

$\therefore\ CF=c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x$

In this case $PI = 0$

The complete solution is $CS = y = CF + PI = c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x$.

Exercise 08

Solve the following:

1) $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+3y=0$ — ***Ans :*** $y=c_1e^{-x}+c_2e^{-3x}$

2) $\left(D^2+2D+1\right)y=0.$ — ***Ans :*** $y=\left(c_1+c_2x\right)e^{-x}$

3) $\left(D^2-2D+2\right)y=0.$ — ***Ans :*** $y=e^{x}\left(c_1\cos x+c_2\sin x\right).$

Let's check take away from the lecture

(1) The order of the differential equation $\dfrac{d^2y}{dt^2}+\left(\dfrac{dy}{dt}\right)^3+y^4=e^{-t}$ *is*

(a) 1 (b) 2 (c) 3 (d) 4

(2) The coefficients of the differential terms in the equation $\dfrac{d^2y}{dt^2}+\left(\dfrac{dy}{dt}\right)^3+y^4=e^{-t}$ are

(a) constant (b) variable (c) none

3. Key notation:

Differential Operator: $D = \frac{d}{dx}$

4. Important Formulae / Theorem / Properties:

(i) Second Order Differential equation (SODE):

An ordinary differential equation of order 2 is of the form

$$P_0 \frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = X \qquad \text{--------} \quad (1)$$

Where the coefficients P_0, P_1, P_2 and X are functions of x or constants.

Using $\frac{d}{dx} \equiv D$, (1) can be written as $f(D)\ y = X$ -------- (2)

The general solution of (2) is given as

y = Complete Solution = Complementory Function + Particular Integral

or $y = y_c + y_p$ or CS = CF + PI

(ii) Complementary Function (CF):

Auxiliary equation of the D.E. (1) isf (D) = 0 *or* f(m) = 0.

The roots of the auxiliary equation may occur in three different ways:

(1) Roots are real and distinct say m_1, m_2 then

$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

(2) Roots are real and repeated say $m_1 = m_2$

$CF = (c_1 + c_2 x)\, e^{m_1 x}$

(3) Roots are complex and not repeated say $m_1, m_2 = \alpha \pm i\beta$, then

$CF = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$

Note :(1) The number of arbitrary constants and order of D.E. must be same.

(2) Complex roots always occur in pair.

(iii) Particular Integral (PI):

Particular integral of the D.E. (1) is given as $PI = \frac{1}{f(D)} X$

(iv) Type 1: If $X = 0$ *then* $PI = \frac{1}{f(D)} 0 = 0$

5. Working rule:

The working rule for finding the solution of second order differential equation with constant Coefficients is as follows:

(1) Identify the equation as second order differential equation with constant coefficients.

(2) Using $\frac{d}{dx} \equiv D$ convert the given equation into symbolic form $f(D)\ y = X$.

(3) Construct the auxiliary equation (AE) by considering $f(D) = 0$. Find roots of auxiliary equation and write Complementary function (CF) as per the nature of roots.

(4) Consider $PI = \frac{1}{f(D)} X$ and calculate it as per Type 1 to Type 7.

(5) Write the complete solution CS = CF + PI or $y = y_c + y_p$.

6. Sample Problems:

(1) Solve $(D^2 + 3D + 2)y = 0$.

Solution: The auxiliary equation is $D^2 + 3D + 2 = 0$