Module-5 Calculus II

Improper Integral

$\int_a^b f(x)\,dx$ is said to be an improper integral if

(i) either $a=\infty$ or $b=\infty$ or both a & b are infinite.

(ii) The value of $f(x)$ at atleast one point x between a & b is unbounded.

eg. $\int_{-1}^{1} \frac{1}{x}\,dx$ is improper integral as $\frac{1}{x}$ is unbounded at $x=0$

Gamma Function :-

For $n>0$, the improper integral

$\int_0^\infty e^{-x} x^{n-1}\,dx$ if exists (finite value)

Then is called Gamma Function & is denoted by Γn

where $n>0$ is parameter.

i.e.
$$\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx}$$

$$\Gamma n+1 = \int_0^\infty e^{-x} x^{n}\,dx$$

Note: (1) $\Gamma 0 = \int_0^\infty e^{-x} x^{0-1} dx$

$= \int_0^\infty e^{-x} \frac{1}{x} dx$ which doesn't exist

i.e. $\Gamma 0$ = Not defined.

(2) $\Gamma -1$, $\Gamma -2$, $\Gamma -3$, ... are not defined.

Properties of Gamma function

1) $\Gamma 1 = 1$

2) $\Gamma n+1 = n \Gamma n$, For 'n' positive fraction

3) $\Gamma n+1 = n!$, For 'n' positive integer

4) $\Gamma n = \frac{\Gamma n+1}{n}$, For 'n' negative fraction

5) $\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin(n\pi)}$

6) $\Gamma \frac{1}{2} = \sqrt{\pi}$

Exercise 4

2) Prove that $\int_0^\infty \frac{e^{-\sqrt{x}}}{x^{7/4}} dx = \frac{8}{3}\sqrt{\pi}$

→

let $I = \int_0^\infty \frac{e^{-\sqrt{x}}}{x^{7/4}} dx$

put $\sqrt{x} = t$ i.e. $x = t^2 \Rightarrow dx = 2t\,dt$

when $x = 0$, $t = 0$
$x = \infty$, $t = \infty$

$\therefore I = \int_0^\infty \frac{e^{-t}}{(t^2)^{7/4}} 2t\,dt$

$= 2\int_0^\infty \frac{t\,e^{-t}}{(t)^{7/2}} dt = 2\int_0^\infty t^{1-\frac{7}{2}} e^{-t} dt$

$= 2\int_0^\infty e^{-t}\, t^{-\frac{5}{2}} dt$

$= 2\int_0^\infty e^{-t} \cdot t^{\left(-\frac{3}{2}\right)-1} dt$

$= 2\,\Gamma\left(-\frac{3}{2}\right)$ $\therefore$ by defⁿ of Gamma Fun.

$= 2\,\frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}}$ $\therefore \Gamma n = \frac{\Gamma(n+1)}{n}$, n - negative fraction

$= -\frac{4}{3}\Gamma\left(-\frac{1}{2}\right) = -\frac{4}{3}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-1/2} = \frac{8}{3}\Gamma\frac{1}{2} = \frac{8}{3}\sqrt{\pi}$

$+_{0}^{\to+\infty}$

3) Prove that $\int_0^1 (x \log x)^4 dx = \frac{4!}{5^5}$

$\rightarrow$

Let : $I = \int_0^1 (x \log x)^4 dx$

put $\log x = -t \Rightarrow x = e^{-t}$

$\Rightarrow dx = -e^{-t} dt$

where

$x = 0, \log 0 = -t \Rightarrow -\infty = -t$

$\Rightarrow \boxed{t = \infty}$

For

$x = 1 \Rightarrow \log 1 = -t \Rightarrow \boxed{t = 0}$

$\therefore I = \int_{\infty}^{0} \left[(e^{-t} (-t)) \right]^4 (-e^{-t} dt)$

$= -\int_0^{\infty} e^{-4t} \cdot (-t)^4 (-e^{-t}) dt$

$\because \int_a^b f\, dx = -\int_b^a f\, dx$

$= \int_0^{\infty} e^{-5t} t^4 dt$

put $5t = u \Rightarrow t = \frac{u}{5} \Rightarrow dt = \frac{du}{5}$

$t = 0 \Rightarrow u = 0$

$t = \infty \Rightarrow u = \infty$

$\therefore I = \int_0^{\infty} e^{-u} \left(\frac{u}{5}\right)^4 \frac{du}{5}$

$= \frac{1}{5^5} \int_0^{\infty} e^{-u} (u)^{5-1} du = \frac{1}{5^5} \Gamma 5$ $\quad \because$ def^n of Γ

$= \frac{1}{5^5} \Gamma 4+1 = \frac{1}{5^5} \cdot 4!$

4) Evaluate $\int_0^{\infty} 7^{-4x^2} dx$

$\rightarrow$

let $I = \int_0^{\infty} 7^{-4x^2} dx$, put $7^{-4x^2} = e^{-t}$

put $7^{-4x^2} = e^{-t}$

$\Rightarrow -4x^2(\log 7) = -t(\log e)$

$\Rightarrow x^2 = \dfrac{t}{4(\log 7)} \Rightarrow x = \dfrac{\sqrt{t}}{2\sqrt{\log 7}}$

diff^n

$dx = \dfrac{1}{2\sqrt{\log 7}} \cdot \dfrac{1}{2\sqrt{t}} dt$

when $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$\therefore I = \int_0^{\infty} e^{-t} \dfrac{1}{2\sqrt{\log 7}} \cdot \dfrac{1}{2\sqrt{t}} dx$

$= \dfrac{1}{4\sqrt{\log 7}} \cdot \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt$

$= \dfrac{1}{4\sqrt{\log 7}} \Gamma\frac{1}{2}$ $\because$ def^n of Gamma

$= \dfrac{\sqrt{\pi}}{4\sqrt{\log 7}}$

Exercise 4

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① prove that

$$\sqrt{\tfrac{3}{2}-x}\cdot\sqrt{\tfrac{3}{2}+x}=\left(\tfrac{1}{4}-x^2\right)\pi\sec(\pi x)$$

provided $-1<2x<1$

→

$LHS = \sqrt{\tfrac{3}{2}-x}\cdot\sqrt{\tfrac{3}{2}+x}$

$-1<2x<1$

$\Rightarrow -\tfrac{1}{2}<x<\tfrac{1}{2} \Rightarrow x<\tfrac{1}{2}$ & $x>-\tfrac{1}{2}$

i.e. $\left(\tfrac{1}{2}-x\right)>0$ & $\left(\tfrac{1}{2}+x\right)>0$.

$\therefore LHS = \sqrt{\left(1+\tfrac{1}{2}\right)-x}\cdot\sqrt{\left(1+\tfrac{1}{2}\right)+x}$

$= \sqrt{\left(\tfrac{1}{2}-x\right)+1}\;\sqrt{\left(\tfrac{1}{2}+x\right)+1}$

$= \left(\tfrac{1}{2}-x\right)\left(\tfrac{1}{2}+x\right)\sqrt{\tfrac{1}{2}-x}\cdot\sqrt{\tfrac{1}{2}+x}$

$\because \sqrt{n+1}=n\sqrt{n}$

$= \left(\tfrac{1}{4}-x^2\right)\sqrt{\tfrac{1}{2}-x}\cdot\sqrt{1-\left(\tfrac{1}{2}-x\right)}$

$= \left(\tfrac{1}{4}-x^2\right)\dfrac{\pi}{\sin\left(\left(\tfrac{1}{2}-x\right)\pi\right)}$ $\quad \because \sqrt{n}\cdot\sqrt{1-n}=\dfrac{\pi}{\sin(n\pi)}$

$= \left(\tfrac{1}{4}-x^2\right)\dfrac{\pi}{\sin\left(\tfrac{\pi}{2}-x\pi\right)}$

$= \left(\tfrac{1}{4}-x^2\right)\cdot\dfrac{\pi}{\cos\pi x}$ $\quad \left(\sin\left(\tfrac{\pi}{2}-\theta\right)=\cos\theta\right)$

$= \left(\tfrac{1}{4}-x^2\right)\pi\sec\pi x = RHS$

Beta Function

For $m>0, n>0$, the improper integral
$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$ if exists
then it is called Beta Function & is denoted by $B(m,n)$

i.e. $$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$
where m,n - parameter.

Properties of Beta Function

1) $B(m,n)=B(n,m)$

2) $B(m,n)=\dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$, (relation between Gamma & Beta)

3) $B(m,n)=2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta$
(Trignometric Form of Beta)

4) Putting $2m-1=p$, $2n-1=q$ in ③

We get $B\left(\frac{p+1}{2},\frac{q+1}{2}\right)=2\int_0^{\pi/2}(\sin\theta)^{p}(\cos\theta)^{q}d\theta$

5) $B(m,n)=\int_0^{\infty}\dfrac{x^{m-1}}{(1+x)^{m+n}}\,dx$ (2^{nd} def^n of Beta)

ex: ① $B(2,3) = B(3,2)$

$$= \frac{\Gamma 2 \, \Gamma 3}{\Gamma 2+3} = \frac{\Gamma 2 \, \Gamma 3}{\Gamma 5} = \frac{1!\,2!}{4!} = \frac{2}{24}$$

$$= 1/12$$

② $\int_0^{\pi/2} \sin^2\theta \cdot \cos^3\theta \, d\theta$

$$= \frac{1}{2} B\left(\frac{2+1}{2}, \frac{3+1}{2}\right)$$

$$= \frac{1}{2} B\left(\frac{3}{2}, 2\right)$$

$$= \frac{1}{2} \frac{\Gamma 3/2 \, \Gamma 2}{\Gamma \frac{3}{2}+2}$$

$$= \frac{1}{2} \cdot \frac{\frac{1}{2}\Gamma 1/2 \cdot 1!}{\Gamma 7/2}$$

$$= \frac{1}{4} \frac{\sqrt{\pi}}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2}}$$

$$= \frac{1}{4} \frac{\sqrt{\pi}}{\frac{15}{8} \times \sqrt{\pi}} = \frac{1}{4} \times \frac{8}{15}$$

$$= \frac{2}{15}$$

③ $\int_0^{\infty} \frac{x^2}{}$

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Exercise 4.3

① Evaluate $\int_0^2 x^4 (8-x^3)^{-1/3} dx$

$\rightarrow$ $8 - x^3 \rightarrow (1-t)$

$\downarrow$

$8-8t \rightarrow 8(1-t)$

put $x^3 = 8t \Rightarrow x = (8t)^{1/3}$

$x = 2t^{1/3}$

$\Rightarrow dx = \frac{2}{3} \cdot t^{\frac{1}{3}-1} dt$

When $x=0$, $t=0$

$x = 2 \Rightarrow t=1$

$\therefore I = \int_0^1 (2t^{1/3})^4 (8-8t)^{-1/3} \frac{2}{3} t^{\frac{1}{3}-1} dt$

$= \frac{2}{3} \times 2^4 \cdot (8)^{-1/3} \int_0^1 t^{4/3} (1-t)^{-1/3} t^{-2/3} dt$

$= \frac{32}{3} (8)^{-1/3} \int_0^1 t^{\frac{5}{3}-1} (1-t)^{+2/3-1}$

$= \frac{32}{3} \cdot \frac{1}{2} \beta\left(\frac{5}{3}, \frac{2}{3}\right)$

$= \frac{16}{3} \frac{\Gamma 5/3 \cdot \Gamma 2/3}{\Gamma\left(\frac{5}{3}+\frac{2}{3}\right)}$

$= \frac{16}{3} \frac{\Gamma\left(\frac{2}{3}+1\right) \cdot \Gamma 2/3}{\Gamma 7/3} = \frac{\frac{16}{3} \times \frac{2}{3} \Gamma 2/3 \cdot \Gamma 2/3}{\Gamma\left(\frac{4}{3}+1\right)}$

$= 32/9 \frac{(\Gamma 2/3)^2}{\frac{4}{3} \cdot \Gamma \frac{4}{3}} = \frac{\frac{32}{9} (\Gamma 2/3)^2}{\frac{4}{3} \cdot \frac{1}{3} \Gamma 1/3}$

$= \frac{8 (\Gamma 2/3)^2}{\Gamma 1/3}$

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(2) show that $\int_0^1 \sqrt{1-\sqrt{x}}\, dx \cdot \int_0^{1/2} \sqrt{2y-(2y)^2}\, dy = \frac{\pi}{30}$

→ let $I_1 = \int_0^1 \sqrt{1-\sqrt{x}}\, dx$

put $\sqrt{x} = t \Rightarrow x = t^2$

$\cancel{\frac{1}{2\sqrt{x}} dx = dt}$ $\quad dx = 2t\,dt$

$x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$\therefore I_1 = \int_0^1 \sqrt{1-t}\, dt\,(2t)$

$= \int_0^1 2t \cdot (1-t)^{1/2}\, dt$

$= 2\int_0^1 t^{2-1} (1-t)^{\frac{3}{2}-1}\, dt$

$I_1 = 2\,\beta\left(2, \frac{3}{2}\right)$ ... (1)

$I_2 = \int_0^{1/2} \sqrt{2y-(2y)^2}\, dy$

put $2y = t \Rightarrow y = \frac{t}{2} \Rightarrow dy = \frac{dt}{2}$

when $y = 0,\ t = 0$

$y = 1/2 \Rightarrow t = 1$

$\therefore I_2 = \int_0^1 \sqrt{t-t^2}\, \frac{dt}{2}$

$= \frac{1}{2}\int_0^1 \sqrt{t(1-t)}\, dt$ $= \frac{1}{2}\beta\left(\frac{3}{2}, \frac{3}{2}\right)$

$= \frac{1}{2}\int_0^1 t^{3/2-1} (1-t)^{\frac{3}{2}-1}\, dt$ (2)

$\Gamma\left(\frac{n}{2}+\frac{n}{2}\right) =$

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$\Gamma(n+1) = n\Gamma(n)$ for +ve fraction

∴ Given Integral

$= I_1 I_2$

$= 2\beta\left(2, \frac{3}{2}\right) \cdot \frac{1}{2}\beta\left(\frac{3}{2}, \frac{3}{2}\right)$ ... ② & ①

$= \frac{\Gamma 2 \cdot \Gamma \frac{3}{2}}{\Gamma\left(2+\frac{3}{2}\right)} \cdot \frac{\Gamma\frac{3}{2} \cdot \Gamma\frac{3}{2}}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)}$

$= \frac{\Gamma 2 \cdot \Gamma 3/2}{\Gamma\left(2+\frac{3}{2}\right)}$

$= \frac{\Gamma 2 \frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 7/2} \cdot \frac{\frac{1}{2}\Gamma\frac{1}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 2}$

$= \frac{\Gamma 3/2 \cdot \Gamma 3/2}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)}$

$= \frac{\frac{1}{2}\sqrt{\pi}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}} \cdot \frac{\frac{1}{4}\sqrt{\pi}\cdot\sqrt{\pi}}{2!}$

$= \frac{1!\,\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 7/2}$

$= \frac{\pi}{30}$

$= \frac{\frac{1}{2}\Gamma\frac{1}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 3}$

$= \frac{\frac{1}{4}\sqrt{\pi}\cdot\sqrt{\pi}}{\Gamma 3}$

$= \pi/4$

$\Gamma(n+1)$

$\Gamma 7/2 = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

$\Gamma\frac{5}{2} = \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

ex (2) Take

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Exercise 4 :-

Prove that

1) $$\int_0^\infty \frac{x^5}{(2+3x)^{16}} dx = \frac{5!\,9!}{2^{10}\cdot 3^6 \cdot 15!}$$

→ let $I = \int_0^\infty \frac{x^5}{(2+3x)^{16}} dx$ $\quad 2+\frac{3}{2}\cdot\frac{2}{3}t$

$(2+3x) \to (1+t)$

$(2+3x) \quad 2(1+t) = 2+2t)$

put $3x = 2t \Rightarrow x = \frac{2}{3}t \Rightarrow dx = \frac{2}{3}dt$

$\Rightarrow x$

when

$x = 0 \Rightarrow t = 0$

$x \Rightarrow \infty \Rightarrow t = \infty$

$$\therefore I = \int_0^\infty \frac{(\frac{2}{3}t)^5}{(2+2t)^{16}} \cdot \frac{2}{3} dt$$

$$= \left(\frac{2}{3}\right)^6 \int_0^\infty \frac{t^5}{\{2^{16}(1+t)^{16}\}} dt$$

$$= \frac{2^6}{3^6 \cdot 2^{16}} \int_0^\infty \frac{t^{6-1}}{(1+t)^{6+10}} dt$$

$$= \frac{2^6}{3^6\cdot 2^{16}} B(6,10) = \frac{2^6}{3^6\cdot 2^{16}} \frac{\Gamma 6 \cdot \Gamma 10}{\Gamma 6+10}$$

$$= \frac{1}{3^6 \cdot 2^{10}} \frac{5!\,9!}{15!} \qquad \Gamma n+1 = n!$$

Mode = 3(21.5) − 2(21.5)
= 64.5 − 43.6
=

(3) P.T. $B\left(n+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{1}{2^{2n}} \frac{\Gamma n+1/2}{\Gamma n+1} \sqrt{\pi}$

Duplication formula : of Beta fun

for $m>0$ $\Gamma m \cdot \Gamma m+\frac{1}{2} = \frac{\Gamma 2m \cdot \sqrt{\pi}}{2^{2m-1}}$

→ LHS $= B\left(n+\frac{1}{2}, n+\frac{1}{2}\right)$

$= \frac{\Gamma n+\frac{1}{2} \cdot \Gamma n+\frac{1}{2}}{\Gamma n+\frac{1}{2}+\frac{1}{2}+n} = \frac{\Gamma n+\frac{1}{2} \, \Gamma n+\frac{1}{2}}{\Gamma 2n+1}$

$= \frac{\left(\frac{\Gamma 2n \cdot \sqrt{\pi}}{\Gamma n \cdot 2^{2n-1}}\right) \Gamma n+\frac{1}{2}}{\Gamma 2n+1}$ by duplicat formul

$= \frac{\Gamma 2n \cdot \sqrt{\pi} \cdot \Gamma n+1/2}{\Gamma n \cdot \Gamma 2n+1 \cdot 2^{2n-1}}$

$= \frac{\Gamma 2n \cdot \Gamma n+1/2 \, \sqrt{\pi}}{\Gamma n \cdot 2n \, \Gamma 2n \cdot 2^{2n-1}}$ ∵ $\Gamma n+1 = n\Gamma n$

$= \frac{\Gamma n+1/2 \cdot \sqrt{\pi}}{n \Gamma n \cdot 2^{2n}} = \frac{\Gamma n+1/2 \, \sqrt{\pi}}{\Gamma n+1 \cdot 2^{2n}}$

= RHS

Exercise 45

① $\int_5^9 \sqrt[4]{(9-x)(x-5)}\, dx = \frac{2\left(\Gamma \frac{1}{4}\right)^2}{3\sqrt{\pi}}$

→ let $I = \int_5^9 \sqrt[4]{(9-x)(x-5)}\, dx$

put $(x-5) = (9-5)t$

$\Rightarrow x = 5 + 4t \Rightarrow dx = 4\, dt$

when $x = 5$, $t = 0$

$x = 9$, $t = 1$

$I = \int_0^1 \sqrt[4]{[9-(5+4t)]\, 4t} \cdot 4\, dt$

$= 4\int_0^1 \sqrt[4]{(4-4t)(4t)}\, dt$

$= 4\int_0^1 [4-4t]^{1/4} [4t]^{1/4}\, dt$

$= 4\int_0^1 [4(1-t)]^{1/4}\, 4^{1/4} \cdot t^{1/4}\, dt$

$= 4\; 4^{1/4} \cdot 4^{1/4} \int_0^1 (1-t)^{1/4} t^{1/4}\, dt$

$\because B(m+1, n+1) = \int_0^1 x^m (1-x)^n dx$

$= 8\, B\left(\frac{1}{4}+1, \frac{1}{4}+1\right)$

$= 8\, B\left(\frac{5}{4}, \frac{5}{4}\right)$

$= 8 \frac{\Gamma 5/4 \cdot \Gamma 5/4}{\Gamma\left(\frac{5}{4}+\frac{5}{4}\right)} = \frac{8 \cdot \frac{1}{4}\Gamma 1/4 \cdot \frac{1}{4}\Gamma 1/4}{\Gamma \frac{5}{2}}$

$4^{1/4 + 1/4} = 4^{1/2}$

$(2^2)^{1/4}$

$2^{1/2}$

$2^{1/2} \cdot 2^{1/2}$

2

$\Gamma n+1 = n\Gamma n$ for +ve fra

Med = 21.5

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$3(\cancel{\ }21.5) = 2($

$$= \frac{\frac{8}{16}\left(\Gamma\frac{1}{4}\right)}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$$

$\because \Gamma\frac{5}{2} = \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

$\frac{43}{2} = 21.5$

$$= \frac{2}{3}\frac{\left(\Gamma\frac{1}{4}\right)^2}{\sqrt{\pi}}$$

(2) show that

$$\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\,dx = \frac{a\sqrt{\pi}\,\Gamma 5/6}{\Gamma 1/3}$$

$\rightarrow$ $(a^3-x^3) \rightarrow (1-t)$

$\downarrow$

$a^3 t$

put $x^3 = a^3 t \Rightarrow x = (a^3 t)^{1/3}$

$\Rightarrow dx = \frac{a}{3} t^{1/3-1}\,dt$

when $x = 0 \,.\, t = 0$

$x = a \,,\, t = 1$

$$I = \int_0^1 \sqrt{\frac{a^3 t}{a^3(1-t)}}\,\frac{a}{3} t^{\frac{1}{3}-1}\,dt$$

$$= \frac{a}{3}\int_0^1 \frac{t^{1/2}\cdot t^{1/3-1}}{(1-t)^{1/2}}\,dt$$

$\frac{2}{3}+\frac{1}{2}$

$$= \frac{a}{3}\int_0^1 t^{\frac{5}{6}-1}(1-t)^{\frac{1}{2}-1}\,dt$$

$$= \frac{a}{3} B\left(\frac{5}{6},\frac{1}{2}\right)$$

$$= \frac{a}{3} \frac{\Gamma 5/6 \; \Gamma 1/2}{\Gamma \left(\frac{5}{6} + 1/2\right)}$$

$$= \frac{a}{3} \frac{\Gamma 5/6 \cdot \sqrt{\pi}}{\Gamma 4/3}$$

$$= \frac{a}{3} \frac{\Gamma 5/6 \cdot \sqrt{\pi}}{\Gamma \left(1 + \frac{1}{3}\right)} = \frac{a}{3} \frac{\Gamma 5/6 \; \sqrt{\pi}}{\frac{1}{3} \Gamma 1/3}$$

$$= \frac{a \, \Gamma \frac{5}{6} \cdot \sqrt{\pi}}{\Gamma 1/3}$$

(1) $$\int_0^a \frac{\sqrt{x^3}}{\sqrt{a^3 - x^3}} \, dx = \int_0^a (x^3)^{1/2} (a^3 - x^3)^{-1/2} \, dx$$

$x = a t$

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Exercise 44

ex. (2) Prove that

$$B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$$

Hence find $\int_0^1 \frac{x^2 + x^3}{(1+x)^7}\,dx$

→ $$RHS = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= I_1 + I_2 \quad \text{---} (1)$$

consider $$I_2 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\,dt$

When $x = 0 \Rightarrow t = \infty$

$x = 1 \Rightarrow t = 1$

$$\therefore I_2 = \int_\infty^1 \frac{(1/t)^{n-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(\frac{-1}{t^2}\right)dt$$

$$= -\int_{\infty}^{1} \frac{(1/t)^{n-1}}{\left(\frac{t+1}{t}\right)^{m+n}} \cdot \frac{1}{t^2}\, dt$$

$$= \int_{1}^{\infty} \frac{t^{m+n} \cdot \frac{1}{t^{n-1}} \cdot \frac{1}{t^2}\, dt}{(1+t)^{m+n}}$$

$$= \int_{1}^{\infty} \frac{t^{m+n+1-n-2}}{(1+t)^{m+n}}\, dt$$

$$= \int_{1}^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}}\, dt$$

$\therefore$ eq^n ① becomes

$$RHS = \int_{0}^{1} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_{1}^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}}\, dt$$

$$= \int_{0}^{\infty} \frac{u^{m-1}}{(1+u)^{m+n}}\, du$$

$$= \beta(m,n) = LHS.$$

$$\therefore \int_{0}^{1} \frac{x^2+x^3}{(1+x)^7}\, dx = \int_{0}^{1} \frac{x^{3-1} + x^{4-1}}{(1+x)^{3+4}}$$

$$= \beta(3,4)$$

$$= \frac{\Gamma 3\, \Gamma 4}{\Gamma 3+4} = \frac{\Gamma 3 \cdot \Gamma 4}{\Gamma 7} = \frac{2!\,3!}{6!} = \frac{1}{60}$$