# Numerical Analysis 1 – Class 2

Thursday, January 18th 2018

## *Subjects covered*

- Root finding in 1D: Bisection method, secant method.
- Newton's method – 1D. Application to quantum mechanics.
- Newton's method – ND. Application to linkages and robotics.
- Broyden's method – ND.
- Convergence domain of Newton's method.
- Homotopy methods

## *Readings*

- N. Kutz, Chapter 1 (Newton-Raphson method).
- "A quasi-Newton method for solving small nonlinear systems of algebraic equations", J. L. Martin (on blackboard). This is a good overview of several Newton's methods variants, including Broyden's method.
- Optional: C. Moler, Chapter 4 (http://www.mathworks.com/moler/zeros.pdf)

## *Problems*

Most of the following problems require you to write a program. When writing the program, please use your preferred programming language from the list of languages given in class. As always, please make sure you also write a test harness which validates your program's implementation. You will be graded on both your program as well as on your test. E-mail your answers to our TA: Mohamed Elbehiry, elbehiry.m@husky.neu.edu.

1. Mechanical linkages are assemblies of rods, cranks, wheels, gears and other parts which manage motion. An example of a common mechanical linkage is the scissors lift, shown in the photo below. You have undoubtedly seen this kind of machine many times. Analyzing mechanical linkages frequently give rise to non-linear equations which can only be solved numerically.

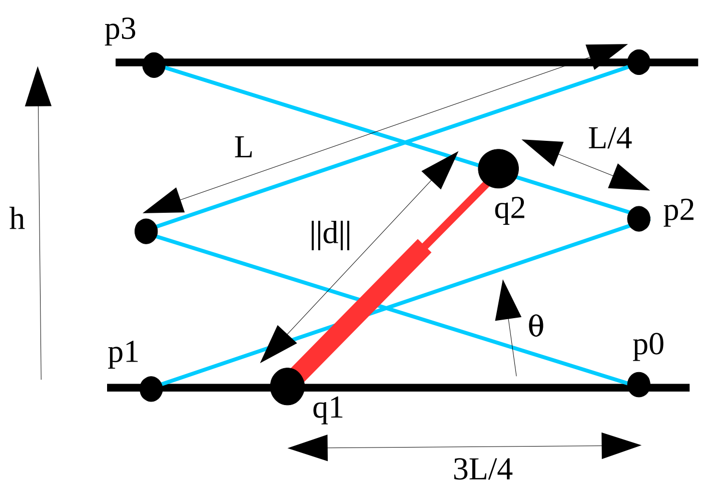
   For this problem, we'll adopt a simple model of the scissors lift, shown below. Each link has length $L$. Take $L = 2.5$. The lift is driven hydraulically using the a piston, shown as the red line segment. The when the lift is fully down, the piston has length $d_{down}$, and when the lift is fully up, the piston's length is $d_{up}$.

Please do the following:


- From the geometry depicted below, use pencil and paper to derive relationships between the piston's length, the height of the lift and the angle θ. The easiest way to do this is:
    - Use point p0 as the origin of your coordinate system. Write down vector expressions for the linkage points p1, p2, and p3 in terms of the angle θ. This gives the height of the lift. (Note that points p1 and p3 slide horizontally as the lift moves up and down.)
    - Then derive vector expressions for the points where the piston is attached to the rods, q1, q2.
    - Knowing q1, q2, you can then get a vector describing the piston, d = q2 – q1. The length of the piston is the Euclidian norm of vector d. For reference, I get the relationship

$$\|d\|_2 = \sqrt{\left(\frac{3L}{4} - \frac{L}{4}\cos\theta\right)^2 + \left(\frac{5L}{4}\sin\theta\right)^2}$$

- Now suppose I ask the simple question: If dup = 0.95$L$, how high is the lift (i.e. what is $h$)? Can you solve this analytically?
- Write a program which solves the question using Newton's method. Feel free to use the program newton1D.m on Blackboard as a starting point. Your test program should run the solver, and then report the height it finds.




2. Consider the 4th order polynomial equation

$$a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \tag{1}$$

recall that the fundamental theorem of algebra states that this equation has 4 (possibly non-unique) solutions over the complex numbers, $x_i \in \mathbb{C}$ . These solutions are called the “roots” of the equation.

The 16th century French mathematician Francois Vieta discovered a set of equations relating the roots of a polynomial of degree n to the coefficients ai. They are:

$$x_1 + x_2 + x_3 + \cdots + x_n = \frac{a_{n-1}}{a_n}$$

$$x_1 ( x_2 + x_3 + \cdots + x_n ) + x_2 ( x_3 + \cdots ) + \cdots + x_{n-1} x_n = \frac{a_{n-2}}{a_n}$$

$$\vdots$$

$$x_1 x_2 x_3 \cdots x_n = \frac{a_0}{a_n}$$

Wikipedia has an excellent article about this theorem if you want more details. Notice that Vieta's equations provide a system of *n* nonlinear equations in *n* unknowns. This suggests that one method to find the solutions of equation (1) is to use a rootfinder like Newton's Method.

Please do the following:

- Consider the 4th degree polynomial equation

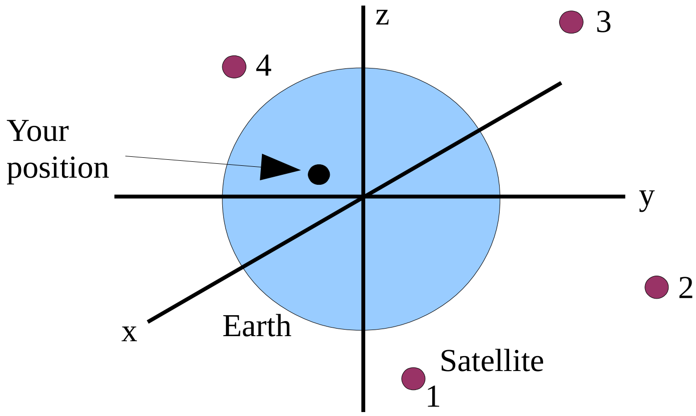
$$x^4 - 2.5\,x^3 + 2.76\,x^2 - 0.52\,x - 0.48 = 0 \qquad (2)$$

- Using Vieta's formulas, write down the system of equations obeyed by the roots $x_i$ .
- From the system, derive and write down the Jacobian matrix.
- Now use Newton's Method to compute the roots of equation (2). Feel free to adapt and modify the 2D Newton's Method code on Blackboard to solve this problem. Your test program should set the coefficients of the polynomial, run Newton's method, and then return the values of the roots it finds. Note that some roots may be complex. Finally, the test should verify that when you put each of the roots into equation (2), the result is zero.

This method of finding the roots of a polynomial is called the Durand-Kerner algorithm.

3. Here is an example of using GPS (global positioning system) to determine your position on earth. This example is explained in more detail in the paper "An underdetermined linear system for GPS", by Dan Kalman (on Blackboard).

Consider a GPS system consisting of four satellites orbiting earth.




Each satellite broadcasts the time on its internal clock, and its position. Some time after broadcast you receive the signal from each of the four satellites. The delay time is determined by the distance between the satellite and you, as well as the speed of radio waves (i.e. the speed of light). You don't have your own clock, so you have only the information sent by the satellites to find your position.

The distance from your position to satellite *i* is

$$d_i = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}$$

where $(x, y, z)$ represents your position, and $(x_i, y_i, z_i)$ the position of satellite $i$. Each satellite broadcasts its position, so you know the triplet $(x_i, y_i, z_i)$ for each satellite. This distance is also

$$d_i = c(t - t_i(0))$$

where $c$ is the velocity of light (c = 0.047 in our units), $t$ is the current time, and $t_i(0)$ is the time at which satellite $i$ broadcast the message you are receiving at time $t$. (This parameter is contained in the position message broadcast by the satellite.)

Set up a system of 4 simultaneous, nonlinear equations, and solve it to get your position $(x, y, z)$ using Newton's method. Feel free to adapt and modify the 2D Newton's Method code on Blackboard to solve this problem. Use the data below, which are the positions and times you receive from each satellite.

| **Satellite** | **x** | **y** | **z** | **t(0)** |
|---|---|---|---|---|
| **1** | 1.2000 | 2.3000 | 0.2000 | -12.3486 |
| **2** | -0.5000 | 1.5000 | 1.8000 | -2.9544 |

| | | | | |
|---|---|---|---|---|
| **3** | -1.7000 | 0.8000 | 1.3000 | -5.8142 |
| **4** | 1.7000 | 1.4000 | -0.5000 | -4.2745 |

Note that the units are normalized so that the earth's radius is 1. (That's why c = 0.047.) Check your result – make sure the position you find is on the earth's surface. As an additional check, you may use the method described in Kalman's paper to create a set of linear equations which are easier to solve than those above. The result you get from Newton's method should match the result you get from the linear equations.