Chapter 1

Logic and sets

1.1 Logical formulas

1. Prove that the following sentences are true:

1. $\neg(\neg p) \Leftrightarrow p$
2. $p \vee \neg p$
3. $\neg(p \wedge q) = (\neg p) \vee (\neg q)$
4. $\neg(p \vee q) = (\neg p) \wedge (\neg q)$
5. $(p \Rightarrow q) \Leftrightarrow (\neg p) \vee q$
6. $0 \Rightarrow 1$

2. Prove that:

1. $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$
2. $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$

1.2 Basic set theory

1.2.1 Rough ideas

3. Prove that $\{1, 1, 2, 2, 2\} = \{1, 2\}$

4. Let X be a set built from all sets such that $A \notin A$. Prove that X does not exist. Hint: what if $X \in X$? What if $X \notin X$?

1.2.2 A few ways of constructing new sets

5. Let A, $B\,C$ be sets. Prove that:

1. $A \cup A = A$
2. $A \cup B = B \cup A$

3. $A \cup (B \cup C) = (A \cup B) \cup C$
4. $A \cap A = A$
5. $A \cap B = B \cap A$
6. $A \cap (B \cap C) = (A \cap B) \cap C$
7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
8. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

6. Prove that there is no set of all sets. Hint: assume there is one. Then you can select some sets to form a set that does not exist.

1.2.3 Subsets and complements

7. Prove that $A = B$ iff $A \subseteq B \wedge B \subseteq A$.

8. Prove that for any set A, $\emptyset \subseteq A$.

9. Here you'll prove that there is just one empty set. Let $\emptyset$ and $\emptyset'$ be empty sets. Prove that $\emptyset = \emptyset'$.

10. Prove the following set identites:

1. Let $A \subseteq B$. Prove that $(A^c)^c = A$.
2. Let A, $B \subset U$. Prove that $(A \cup B)^c = A^c \cap B^c$
3. Let A, $B \subset U$. Prove that $(A \cap B)^c = A^c \cup B^c$
4. $\{a \in A : a \in B\} = \{b \in B : b \in A\}$

11.

1. Let $A = \{1, 2, 3\}$. Find 2^A. What is the number of elements in 2^A? How is it related to the number of elements of A?
2. Let A be a finite set with n elements. Using the approach in which you choose which elements belong to a subset, prove that 2^A has 2^n elements.

1.2.4 Infinite collections of sets

12. Prove that for finite families of sets, these new definitions agree with the previous.

13. Let $\mathcal{X} \subset 2^U$ be a family of sets and define: $\mathcal{Y} = \{X^c : X \in \mathcal{X}\}$, where $X^c = U \setminus X$. Prove that:

1. $(\bigcup \mathcal{X})^c = \bigcap \mathcal{Y}$
2. $(\bigcap \mathcal{X})^c = \bigcup \mathcal{Y}$

14. Let $A_i \subseteq U$, $i \in I$ and
$$\sigma = \bigcup_{i\in I} A_i,\ \pi = \bigcap_{i\in I} A_i$$
Prove that:

1. if $k \in I$, then $A_k \cup \sigma = \sigma$
2. $\pi \subseteq \sigma$
3. $\sigma \cap \pi = \pi$

15. Find sum and intersection of family of subsets of $\mathbb{R}$: $A_r = \{r, -r\}$ for $r \geq 0$.

16. Let $A \subseteq X_i \subseteq U$ for $i \in I$. Prove that
$$A \subseteq \bigcup_{i\in I} X_i$$

17. For every point $a \in A$ there is a set $U_a \subseteq A$ such that $a \in U_a$. Prove that
$$A = \bigcup_{a\in A} U_a.$$

1.2.5 Cartesian product

18. Let $A = \{\{a\}, \{a, b\}\}$, $B = \{\{c\}, \{c, d\}\}$. Prove that $A = B$ iff $a = c \wedge b = d$. Such a set A we call **the ordered pair** (a, b) as it has the property $(a, b) = (c, d)$ iff $a = c$ and $b = d$. Now you can forget how it has been constructed, and just remember this property.

19. Prove that $(a,(b, c)) = (d,(e, f))$ iff $a = d \wedge b = e \wedge c = f$.

20. Check that defining (a, b, c) as $((a, b), c)$ also works (so two ordered tuples are the same if they have the same first element, the same second element, ...)

21. Check that, in terms of sets, $(a,(b, c)) \neq ((a, b), c)$, so formally we do need to stick to one convention. However as we are interested in the property of ordered tuple, we will not distinguish them and denote both of them just as (a, b, c). Such notational problems appear in various places in mathematics, so we need to try to get used to them.

22. Do you remember the identification of $(a,(b, c))$ and $((a, b), c)$? Prove that $A \times (B \times C) = (A \times B) \times C$. Therefore we'll write it just as $A \times B \times C$ without parentheness.

1.3 Natural numbers and mathematical induction

23. You can prove that $2^n > n$ for every natural number n.

1. Prove that the formula works for $n = 0$ (punch the first domino).
2. Assume that for some n you proved on some way that $2^n > n$. Using this, prove that $2^{n+1} > (n + 1)$ (if n-th domino falls, then $n + 1$-th domino also falls)

24.

1. Prove[1] that 6 divides $n^3 - n$ for all natural n.
2. Prove[2] that 6 divides $n^3 - n$ for all integers n. You can use a slight modification mathematical induction principle proving the implication ,,if the theorem works for n, it works also for $n-1$".

25. (Bernoulli's inequality) Prove that for real $x > -1$ and natural $n \geq 1$, the following inequality holds:

$$(1+x)^n \geq 1 + nx.$$

26. In Mathsland there are $n \geq 2$ cities. Between each pair of them there is a *one-way* road.

1. Prove that there is a city from which you can drive to all the other cities. Hint: assume that the hypothesis works for some n and any country with n cities. Now consider an arbitrary $n+1$-city country. Hide one city and use your assumption.
2. Prove that there is a city[3] to which you can drive from all the others.

27. Let $S \subseteq R$. We say that S is **well-ordered** iff any non-empty subset $X \subset S$ has the smallest element.

1. Prove that reals and integers with the default ordering are not well-ordered.
2. Assume that $X \subseteq \mathbb{N}$ doesn't have the smallest element. Define $A = \{n \in \mathbb{N} : \{0, 1, \dots, n\} \cap X = \emptyset\}$ and use mathematical induction to prove that X is empty.
3. Why are natural numbers well-ordered?

1.4 Functions

1.4.1 Basics

28. (Thanks to Antoni Hanke) How many are there functions from the empty set to $\{1, 2, 3, 4\}$?

29. Consider two functions: $f : \{0, 1\} \to \{0, 1\}$ given by $f(x) = 0$ and $g : \{0, 1\} \to \{0\}$. Prove that $f = g$. [4]

30. Let $f : A \to B$ and $g : C \to B$, where $A \neq C$. Is it possible that $f = g$?

31. Let $f : A \to B$ and $C \subseteq D \subseteq A$. We define: $f[C] = \{b \in B : b = f(c) \text{ for some } c \in C\}$ and analogously $f[D]$. Prove that $f(C) \subseteq f(D)$.

[1] Another method is to notice that $n^3 - n = (n-1) \cdot n \cdot (n+1)$. Why 2 does divide it? Why 3?

[2] How $n^3 - n$ and $(-n)^3 - (-n)$ are related? Does it simplify the proof?

[3] Nice trick: what does happen if you reverse each way? Can you use the former result?

[4] Some mathematicians, as Bourbaki use an alternative definition of function - for them a function is the triple (A, B, f), where f is defined as in the our case. We see that this definition is incompatible with ours. Fortunately, as in the case with different definitions of ordered tuples, this problem will never occur explicitly in the further chapters.

1.4.2 Injectivity, surjectivity and bijectivity

32. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions surjective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2$
3. $h : \mathbb{R} \to \{5\}$

33. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions injective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$
2. $h : \{0, 1, 2, 3\} \to \mathbb{R},\ h(x) = x$

34. Construct function that is:

1. surjective, but not injective
2. injective, but not surjective
3. neither injective nor surjective
4. bijective

35. Prove that, if exists, g is unique.

36. Assume that f^{-1} exists. Prove that $(f^{-1})^{-1}$ exists and is equal to f.

1.4.3 Function composition

37. Find functions $f,\ g$ such that:

1. $g \circ f$ exists, but $f \circ g$ is not defined
2. both $f \circ g$ and $g \circ f$ exist, but $f \circ g \neq g \circ f$

38. Left $f : A \to B, g : B \to C, h : C \to D$. Prove that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

39.

1. Prove that composition of two surjections is surjective.
2. Prove that composition of two injections is injective.
3. Prove that composition of two bijections is bijective.

40. We will rephrase the definition of the inverse function as follows:

1. If X if a set, we define **the identity function**

$$\mathrm{Id}_X = \{(x, x) \in X^2 : x \in X\}.$$

   Prove that it is indeed a function. What is it's domain?

2. Let $f : A \to B,\ g : B \to A$. Prove that $f = g^{-1}$ iff

$$g \circ f = \mathrm{Id}_A \text{ and } f \circ g = \mathrm{Id}_B$$

41. Let $f : A \to B$ be an injection. Prove that there is a function $g : \operatorname{Im} f \to A$ such that $g \circ f = \mathrm{Id}_A$. Such g is called **left inverse of f**.

1.5 Countability

1.5.1 Finite sets

42. What is the cardinality of $\{a, a+1, a+2, \ldots, a+n\}$?

43. Let A, B and C be finite sets. Prove that:

1. $|2^A| = 2^{|A|}$
2. $|A \cup B| = |A| + |B|$ iff A and B are disjoint.
3. $|A \setminus B| = |A| - |B|$ if $B \subseteq A$.
4. $|A| \geq |B|$ if $B \subseteq A$. When does the equality hold?
5. $|A \cup B| = |A| + |B| - |A \cap B|$
6. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

44. Assume that A and B are finite sets. Prove that $|A| = |B|$ iff there is a bijection between A and B.

45. Above we find the way of saying that two cardinalities are equal using existence of a bijection. Let's find a way to compare which is less using another kind of function.

1. Let $O_n = \{1, 2, \ldots, n\}$. Prove that there is no injection from O_{n+1} into O_n. Hint: use mathematical induction.
2. Let A and B be finite. Prove that there is an injection from A to B iff $|A| \leq |B|$.

46. Using the above results, prove in one line[5] that if there is an injection from A onto B and an injection from B into A, then there exists a bijection from A onto B.

[5]The main step is $|A| \leq B$ and $|B| \leq |A|$, so $|A| = |B|$.

1.5.2 Infinite sets

47. Let A, B and C be sets. Prove that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. Hint: find the bijection between A and C.

48. Prove that:

1. $|\mathbb{N}| = |\mathbb{Z}|$.
2. $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
3. $|\mathbb{N}| = |\mathbb{Q}|$.

49. Prove that if $A \subseteq B$, then $|A| \leq |B|$.

50. Let A, B and C be sets. Prove that if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

51. Here you can prove that there are more real numbers than naturals or rationals. We define $X = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and choose one convention of writing reals (e.g 0.999... = 1.000..., so we can choose to use nines)

1. Assume that you have written all the elements of X in a single column. Can you find a real number that does not occur in the list?
2. Using the above, prove that $|\mathbb{N}| < |X|$
3. Prove that $|\mathbb{Q}| < |\mathbb{R}|$.

52. We know that $|\mathbb{R}| > |\mathbb{N}|$. Using binary system prove that $\mathbb{R} = 2^{\mathbb{N}}$. Do you see similarity between the previous result and $2^n > n$ for natural n?

53. **Cantor's theorem** You will prove that $|A| < \left|2^A\right|$ for any set A. Let A be a set and $f : A \to 2^A$.

1. Consider $X = \{a \in A : a \notin f(a)\} \in 2^A$. Is there $x \in A$ for which $f(x) = X$?
2. Is f surjective?
3. Find an injective function $g : A \to 2^A$.
4. Prove that $|A| < |2^A|$ for any set A.
5. Use Cantor's theorem to prove that there is no set of all sets.

54. **Cantor-Schroeder-Bernstein theorem** Let's prove that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$ for any sets.

1. (Knaster-Tarski) Now assume that F has *monotonicity* property: $F(X) \subseteq F(Y)$ if $X \subseteq Y$. Prove that F has a fixed point S (that is $F(S) = S$), where:

$$S = \bigcup_{X \in U} X, \text{ where } U = \{Y \in 2^A : Y \subseteq f(Y)\}.$$

2. (Banach) Let $f : A \to B$ and $g : B \to A$ be injections. We introduce new symbol: $f[X] = \{b \in B : b = f(x) \text{ for some } x \in X\}$. Prove that function

$$F : 2^A \to 2^A,\ F(X) = A \setminus g[B \setminus f[X]]$$

has the monotonicity property.

3. Prove that $A \setminus S \subseteq \operatorname{Im} g$, where F and S are taken from above.

4. Prove that function

$$h(x) = \begin{cases} f(x), x \in S \\ g^{-1}(x), x \notin S \end{cases}$$

is a bijection.

1.5.3 Pre-image of a function

55. Let $f : A \to B$ and $X, Y \subseteq B$. Then:

1. $f(X \cup Y) = f(X) \cup f(Y)$
2. $f(X \cap Y) \subseteq f(X) \cap f(Y)$

You can also generalise this result to an arbitrary collection of sets.

56. Let $f : A \to B$. Then $f(A) \subseteq B$ and $A = f^{-1}(B)$.

57. Let $f : A \to B$ and $X, Y \subseteq B$. Then:

1. $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$
2. $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

You can also generalise this result to an arbitrary collection of sets.

Chapter 2

General topology

2.1 Basic definitions

2.1.1 Topology and open sets

58. Using mathematical induction prove that the intersection of finitely many open sets is open.

59. Trivial topology Prove that for any X, set $\{\varnothing, X\}$ is a topology.

60. Discrete topology Prove that for any X, it's power set 2^X is a topology.

61. Cofinite topology Prove that for any X, the set: $\mathcal{T}_X = \{\emptyset\} \cup \{A \subseteq X : X \setminus A \text{ is finite}\}$ is a topology. Hint: think in terms of complements.

62. For which sets, there is exactly one topology on them? (So at least these listed above must be the same).

63. Prove that for an infinite set, there are at least three distinct topologies.

64. Prove that each points has an open neighborhood.

65. Prove that A is an open set if and only if each point a has a neighborhood $U_a \in A$ contained in A (that is $U_a \subseteq A$).

66. Prove that:

1. $\operatorname{Int} A$ is an open set.
2. if $A' \subseteq A$ is open, then $A' \subseteq \operatorname{Int} A$ (so in some sense, $\operatorname{Int} A$ is the biggest open set contained in A)
3. $\operatorname{Int} A = A$ iff A is open
4. $\operatorname{Int} \operatorname{Int} A = \operatorname{Int} A$ for any A

67. Let $A' \subseteq A$. Prove that:

1. $\operatorname{Int} A' \subseteq \operatorname{Int} A$
2. $\operatorname{Int} A \cup \operatorname{Int} B \subseteq \operatorname{Int}(A \cup B)$

You can prove also that the union can be arbitrary.

68. We say that a is an **interior point** of A if there is open $U_a \subseteq A$ such that $a \in U_a$. Prove that $\operatorname{Int} A$ is the set of all interior points of A.

2.1.2 Closed sets

69. Prove these properties of closed sets in space $(X, \mathcal{T}_X)$:

1. $\varnothing$ and X are closed
2. If $A_1, A_2, \ldots, A_n$ are closed, then their union $A_1 \cup A_2 \cup \cdots \cup A_n$ is closed.
3. If $\mathcal{A}$ is any family of closed sets, then the intersection $\bigcap \mathcal{A}$ is closed.

70. Prove that:

1. $\operatorname{Cl} A$ is a closed set.
2. if C is closed and $A \subseteq C$, then $\operatorname{Cl} A \subseteq C$ (so in some sense, $\operatorname{Cl} A$ is the smallest closed set containing A)
3. $\operatorname{Cl} A = A$ iff A is closed
4. $\operatorname{Cl} \operatorname{Cl} A = \operatorname{Cl} A$ for any A

71. We say that p is an **adherent point** of A (or **point of closure**) if there is a set S_p such that $p \in S_p$ (we do *not* require S to be open) and $A \cap S_p \neq \emptyset$. Alternatively, we can say that every neighborhood of p, whether open or not, contains a point from A. Prove that $\operatorname{Cl} A$ is the set of all adherent points of A.

72. We say that p is a **frontier** point of A if every open neighborhood of p contains intersects both A and A^c, so if for every open neighborhood U_p we have $U_p \cap A \neq \emptyset$ and $U_p \cap A^c \neq \emptyset$. Prove that the boundary of A is exactly the set of frontier points of A.

73. Prove that boundary is always closed.

74. Prove that $\partial\partial A \subseteq \partial A$.

75. Prove that $\partial A = \partial A^c$.

76. Prove that $\partial A = \partial A^c$.

77. Prove that $\partial A = \emptyset$ iff A is simultaneously open and closed.