Paweł Czyż

Undergraduate Mathematical Physics

Geometrical approach

– Monograph –

12 grudnia 2017

Springer

For the people whom I learned mathematics from:
Wiktor Bartol,
Michał Bączyk,
Jerzy Bednarczuk,
Beata Czyż,
Frederic Grabowski,
Wojciech Guzicki,
Maciej Kiliszek,
Anna Kowalska,
Andre Lukas,
Jakub Perlin,
Krzysztof Reczek,
Arun Shanmuganathan

Przedmowa

Here come the golden words

<table><tr><td>place(s),
month year</td><td align="right">First name Surname
First name Surname</td></tr></table>

Spis treści

Część I

Mathematical introduction

1

Introduction

There are many excellent books on mathematical physics and differential geometry, so a question raises - how does this book differ from any other? I had a few aims working on it:

- target audience is just a normal person that wants to understand advanced mathematics. It does not matter if you are a physicists, mathematician, english literature major or a high-school student. If you have enough self-determination, you can understand the mathematics in this book
- this book should be self-containing. Mathematics is both broad and deep, so it must be split into different branches. But I found it discouraging that if you want to read one book, as prerequsities you need to read two other books, and so on. Here, you can understand everything without any access to libraries or other mathematical books. Obviously, we need to ommitt some Mathematics.
- we define the most fundamental concepts and then we show how they work together in a more specific setting. Many great lecturers show how an abstract concept works in a specific case, so they provide lots of examples. I would like to do an experiment - show abstract concepts, give huge amount of exercises
- I personally enjoy problem solving approach. Therefore I just do not prove theorems - I want you to prove them, with adjustable amount of hints.

2

Logic and sets

If you are already fimilar with operations on logical formulas and sets, you may ommit this chapter.

2.1 Logical formulas

Consider declarative sentences as ”Water boils at 100°C” or ”2+2=5”. We can construct new sentences:

1. conjuntion (and): $p \wedge q$ is true if and only if p is true and q is true
2. disjuntion (or): $p \vee q$ is true if and only if at least one of sentences p, q is true
3. implication: $p \Rightarrow q$ is false if and only if p is true and q is false. Intuitively, if you know that p implies q and p is true, then q also must be true
4. negation (not): $\neg p$ is true if and only if p is false
5. equivalence (iff, if and only if): $p \Leftrightarrow q$ means exactly $(p \Rightarrow q) \wedge (q \Rightarrow p)$. Intuitively: if you know that two sentences are equivalent and one of them is true, the other is also true

Because mathematics is the art of being smart and lazy, we will assign value 1 to true sentences and 0 to false sentences.

2.1. Prove that the following sentences are true:

1. $\neg(\neg p) \Leftrightarrow p$
2. $p \vee \neg p$
3. $\neg(p \wedge q) = (\neg p) \vee (\neg q)$
4. $\neg(p \vee q) = (\neg p) \wedge (\neg q)$
5. $(p \Rightarrow q) \Leftrightarrow (\neg p) \vee q$
6. $0 \Rightarrow 1$

Equipped with this powerful machinery we can dive into basic set theory.

2.2 Basic set theory

2.2.1 Rough ideas

In modern mathematics we do not define a set nor set membership, so heuristically you can think that set A is a 'collection of objects' and $x \in A$ means that the object x is inside this collection. We will assume that any finite collections of elements $\{x_1, x_2, \ldots, x_n\}$ is a set (the empty set is called $\varnothing$ rather than $\{\}$), moreover we will assume that real numbers form a set $\mathbb{R}$. We say that two sets are equal ($A = B$) iff they have the same elements ($x \in A \Leftrightarrow x \in B$). Note, that we do not check how many times x appears in A. We can just say whether it inside or not.

2.2. Prove that $\{1, 1, 2, 2, 2\} = \{1, 2\}$

Not every 'collection of objects' is a set, as you can prove:

2.3. Let X be a set built from all sets such that $A \notin A$. Prove that X does not exist.
Hint: what if $X \in X$? What if $X \notin X$?

2.2.2 A few ways of constructing new sets

Therefore we assume that some sets (as finite sets or real numbers) exist and we will construct new sets from the given ones using a few rules. Assume that A and B are sets:

1. Let's make a formula F such that for every element $a \in A$, the value $F(a)$ is true or false. We can then construct a set S with all the elements from A for wchich the formula: $a \in S$ iff $F(a)$ and $a \in A$. This set is written explicitly as $S = \{a \in A : F(a)\}$.
2. We can form the sum of two sets: $a \in A \cup B$ iff $a \in A$ or $a \in B$.
3. We can contruct the intersection of two sets: $a \in A \cap B$ iff $a \in A$ and $a \in B$.
4. We can construct the difference of two sets: $A \setminus B = \{a \in A : a \notin B\}$

2.4. Prove that there is no set of all sets.
Hint: assume there is one. Then you can select some sets to form a set that does not exist.

2.2.3 Subsets and complements

As we have some sets, we can try to compare them. We say that $A \subseteq B$ iff $a \in A \Rightarrow a \in B$ (or intuitively, each element of A is also in B. We say that A is a **subset** of B or that B is a **superset** of A.

2.5. Prove that $A = B$ iff $A \subseteq B \wedge B \subseteq A$.

If we fix the set B, to each subset A we can assign it's **complement**: $A^c = B \setminus A$. [1] Moreover, we will assume that for a set A there exists it's *power set*: $2^A = \{X : X \subseteq A\}$.

2.6. Prove the following set identites:

1. Let $A \subseteq B$. Prove that $(A^c)^c = A$.
2. Let A, $B \subset U$. Prove that $(A \cup B)^c = A^c \cap B^c$
3. Let A, $B \subset U$. Prove that $(A \cap B)^c = A^c \cup B^c$
4. $\{a \in A : a \in B\} = \{b \in B : b \in A\}$

2.7. Let $A = \{1, 2, 3\}$. Find 2^A. What is the number of elements in 2^A? How is it connected with the number of elements of A?

2.2.4 Cartesian product

First of all, we need a useful concept:

2.8. Let $A = \{a, \{a, b\}\}$ $B = \{c, \{c, d\}\}$. Prove that $A = B$ iff $a = c \wedge b = d$. Such a set A we call **the ordered pair** (a, b) as it has the property $(a, b) = (c, d)$ iff $a = c$ and $b = d$. Now you can forget how it has been constructed, and just remember this property.

2.9. Prove that $(a, (b, c)) = (d, (e, f))$ iff $a = d \wedge b = e \wedge c = f$.

Therefore it makes sense to write just (a, b, c) for $(a, (b, c))$ and define similarily such **ordered tuple** for four elements, five elements and so on.

2.10. Check that defining (a, b, c) as $((a, b), c)$ also works (so two ordered tuples are the same if they have the same first element, the same second element, ...)

2.11. Check that, in terms of sets, $(a, (b, c)) \neq ((a, b), c)$, so formally we do need to stick to one convention. However as we are interested in the property of ordered tuple, we will not distinguish them and denote both of them just as (a, b, c). Such notational problems appear in various places in mathematics, so we need to try to get used to them.

We can now introduce anot her way of creating new sets: let A and B be sets. Then we define their **Cartesian product** as

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

2.12. Do you remember the identification of $(a, (b, c))$ and $((a, b), c)$? Prove that $A \times (B \times C) = (A \times B) \times C$. Therefore we'll write it just as $A \times B \times C$ without parentheness.

[1] It is not the best symbol possible as we need to have B in mind.

2.3 Functions

2.3.1 Basics

Consider two sets A and B. We say that a subset $f \subseteq A \times B$ is a **function** iff the following two conditions hold:

- for every element $a \in A$ there is an element $b \in B$ such that $(a, b) \in f$
- if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$

Therefore for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$. Such b will be called **value of f at point** a and given a symbol $f(a)$.

2.13. (Thanks to Antoni Hanke) How many are there functions from the empty set to $\{1, 2, 3, 4\}$?

We need to introduce more terminology: set A is called **the domain of** f, set B is called **the codomain of** f and the function f is written as $f : A \to B$.

2.14. Consider two functions: $f : \{0, 1\} \to \{0, 1\}$ given by $f(x) = 0$ and $g : \{0, 1\} \to \{0\}$. Prove that $f = g$. [2]

2.15. Let $f : A \to B$ and $g : C \to B$, where $A \neq C$. Is it possible that $f = g$?

2.3.2 Injectivity, surjectivity and bijectivity

As we have already seen, there may be some elements in codomain that are not values of f. We define **the image of** f as:

$$\operatorname{Im} f = \{b \in B : \text{there is } a \in A \text{ such that } b = f(a)\}.$$

We say that the function $f : A \to B$ is **surjective** (or **onto**) iff $\operatorname{Im} f = B$.

2.16. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions surjective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2$
3. $h : \mathbb{R} \to \{5\}$

If $f(a)$ uniquely specifies a (if $f(a) = f(b)$, then $a = b$) we say that the function is **injective** (or **one-to-one**). Are the following functions injective?

2.17. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions surjective?

[2] Some mathematicians, as Bourbaki use an alternative definition of function - for them a function is the triple (A, B, f), where f is defined as in the our case. We see that this definition is incompatible with ours. Fortunately, as in the case with different definitions of ordered tuples, this problem will never occur explicitly in the further chapters.

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$
2. $h : \{0, 1, 2, 3\} \to \mathbb{R},\ h(x) = x$

If a function f is both surjective and injective, we say that is *bijective*[3].

2.18. Construct function that is:

1. surjective, but not injective
2. injective, but not surjective
3. neither injective nor surjective
4. bijective

Notice that if a function $f : A \to B$ is bijective, then we can construct a function $g : B \to A$ such that $f(g(b)) = b$ and $g(f(a)) = a$.

2.19. Prove that, if exists, g is unique.

We call this function **the inverse function** [4]: $g = f^{-1}$.

2.20. Assume that f^{-1} exists. Prove that $(f^{-1})^{-1}$ exists and is equal to f.

2.3.3 Function composition

[3] If you prefer nouns: surjective function is called surjection, injective - injection and bijective - bijection

[4] It becomes confusing when working on real numbers: $f^{-1}(x)$ is **not** $(f(x))^{-1} = 1/f(x)$

3

General topology

Your text goes here. Separate text sections with the standard LaTeX sectioning commands.

4

Linear algebra

5

$\mathbb{R}^{\boldsymbol{n}}$

Your text goes here. Separate text sections with the standard LaTeX sectioning commands.

5.1 Section Heading

Your text goes here. Use the LaTeX automatism for your citations [1].

5.1.1 Subsection Heading

Your text goes here.

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \tag{5.1}$$

Subsubsection Heading

Your text goes here. Use the LaTeX automatism for cross-references as well as for your citations, see Sect. 5.1.

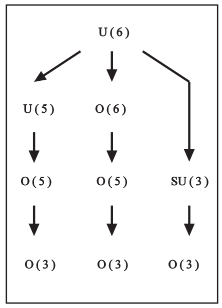
Paragraph Heading

Your text goes here.

Subparagraph Heading. Your text goes here.

Tabela 5.1. Please write your table caption here

first	second	third
number	number	number
number	number	number



Rysunek 5.1. Please write your figure caption here

Theorem 5.1. *Theorem text goes here.*

Lemma 5.2. *Lemma text goes here.*

Problems

5.1. The problem[1] is described here. The problem is described here. The problem is described here.

5.2. Problem Heading
(a) The first part of the problem is described here.
(b) The second part of the problem is described here.

[1] Footnote

Solutions

Problems of Chapter 1

5.1 The solution is revealed here.

5.2 Problem Heading
(a) The solution of first part is revealed here.
(b) The solution of second part is revealed here.

Literatura

1. Kajan E (2002) Information technology encyclopedia and acronyms. Springer, Berlin Heidelberg New York
2. Broy M (2002) Software engineering – From auxiliary to key technologies. In: Broy M, Denert E (eds) Software Pioneers. Springer, Berlin Heidelberg New York
3. Che M, Grellmann W, Seidler S (1997) Appl Polym Sci 64:1079–1090
4. Ross DW (1977) Lysosomes and storage diseases. MA Thesis, Columbia University, New York