Logic and sets

Logical formulas

1. Prove that the following sentences are true:

1. $\neg(\neg p) \Leftrightarrow p$
2. $p \vee \neg p$
3. $\neg(p \wedge q) = (\neg p) \vee (\neg q)$
4. $\neg(p \vee q) = (\neg p) \wedge (\neg q)$
5. $(p \Rightarrow q) \Leftrightarrow (\neg p) \vee q$
6. $0 \Rightarrow 1$

2. Prove that:

1. $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$
2. $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$

Basic set theory

Rough ideas

3. Prove that $\{1, 1, 2, 2, 2\} = \{1, 2\}$

4. Let X be a set built from all sets such that $A \notin A$. Prove that X does not exist. Hint: what if $X \in X$? What if $X \notin X$?

A few ways of constructing new sets

5. Let A, $B\,C$ be sets. Prove that:

1. $A \cup A = A$
2. $A \cup B = B \cup A$
3. $A \cup (B \cup C) = (A \cup B) \cup C$
4. $A \cap A = A$
5. $A \cap B = B \cap A$
6. $A \cap (B \cap C) = (A \cap B) \cap C$
7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
8. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

6. Prove that there is no set of all sets. Hint: assume there is one. Then you can select some sets to form a set that does not exist.

Subsets and complements

7. Prove that $A = B$ iff $A \subseteq B \wedge B \subseteq A$.

8. Prove that for any set A, $\varnothing \subseteq A$.

9. Here you'll prove that there is just one empty set. Let $\varnothing$ and $\varnothing'$ be empty sets. Prove that $\varnothing = \varnothing'$.

10. Prove the following set identites:

1. Let $A \subseteq B$. Prove that $(A^c)^c = A$.
2. Let $A,\, B \subset U$. Prove that $(A \cup B)^c = A^c \cap B^c$
3. Let $A,\, B \subset U$. Prove that $(A \cap B)^c = A^c \cup B^c$
4. $\{a \in A : a \in B\} = \{b \in B : b \in A\}$

11.

1. Let $A = \{1, 2, 3\}$. Find 2^A. What is the number of elements in 2^A? How is it related to the number of elements of A?
2. Let A be a finite set with n elements. Using the approach in which you choose which elements belong to a subset, prove that 2^A has 2^n elements.

Infinite collections of sets

12. Prove that for finite families of sets, these new definitions agree with the previous.

13. Let $\mathcal{X} \subset 2^U$ be a family of sets and define: $\mathcal{Y} = \{X^c : X \in \mathcal{X}\}$, where $X^c = U \setminus X$. Prove that:

1. $(\bigcup \mathcal{X})^c = \bigcap \mathcal{Y}$
2. $(\bigcap \mathcal{X})^c = \bigcup \mathcal{Y}$

14. Let $A_i \subseteq U$, $i \in I$ and

$$\sigma = \bigcup_{i \in I} A_i,\ \pi = \bigcap_{i \in I} A_i$$

Prove that:

1. if $k \in I$, then $A_k \cup \sigma = \sigma$
2. $\pi \subseteq \sigma$
3. $\sigma \cap \pi = \pi$

15. Find sum and intersection of family of subsets of $\mathbb{R}$: $A_r = \{r, -r\}$ for $r \geq 0$.

16. Let $A \subseteq X_i \subseteq U$ for $i \in I$. Prove that

$$A \subseteq \bigcup_{i \in I} X_i$$

17. For every point $a \in A$ there is a set $U_a \subseteq A$ such that $a \in U_a$. Prove that

$$A = \bigcup_{a \in A} U_a.$$

Cartesian product

18. Let $A = \{\{a\}, \{a, b\}\}$, $B = \{\{c\}, \{c, d\}\}$. Prove that $A = B$ iff $a = c \wedge b = d$. Such a set A we call **the ordered pair** (a, b) as it has the property $(a, b) = (c, d)$ iff $a = c$ and $b = d$. Now you can forget how it has been constructed, and just remember this property.

19. Prove that $(a, (b, c)) = (d, (e, f))$ iff $a = d \wedge b = e \wedge c = f$.

20. Check that defining (a, b, c) as $((a, b), c)$ also works (so two ordered tuples are the same if they have the same first element, the same second element, ...)

21. Check that, in terms of sets, $(a,(b,c)) \neq ((a,b),c)$, so formally we do need to stick to one convention. However as we are interested in the property of ordered tuple, we will not distinguish them and denote both of them just as (a, b, c). Such notational problems appear in various places in mathematics, so we need to try to get used to them.

22. Do you remember the identification of $(a,(b,c))$ and $((a,b),c)$? Prove that $A \times (B \times C) = (A \times B) \times C$. Therefore we'll write it just as $A \times B \times C$ without parentheness.

Natural numbers and mathematical induction

23. You can prove that $2^n > n$ for every natural number n.

1. Prove that the formula works for $n = 0$ (punch the first domino).
2. Assume that for some n you proved on some way that $2^n > n$. Using this, prove that $2^{n+1} > (n + 1)$ (if n-th domino falls, then $n + 1$-th domino also falls)

24.

1. Prove[1] that 6 divides $n^3 - n$ for all natural n.
2. Prove[2] that 6 divides $n^3 - n$ for all integers n. You can use a slight modification mathematical induction principle proving the implication ,,if the theorem works for n, it works also for $n - 1$".

25. (Bernoulli's inequality) Prove that for real $x > -1$ and natural $n \geq 1$, the following inequality holds:

$$(1 + x)^n \geq 1 + nx.$$

26. In Mathsland there are $n \geq 2$ cities. Between each pair of them there is a *one-way* road.

1. Prove that there is a city from which you can drive to all the other cities. Hint: assume that the hypothesis works for some n and any country with n cities. Now consider an arbitrary $n + 1$-city country. Hide one city and use your assumption.
2. Prove that there is a city[3] to which you can drive from all the others.

27. Let $S \subseteq R$. We say that S is **well-ordered** iff any non-empty subset $X \subset S$ has the smallest element.

1. Prove that reals and integers with the default ordering are not well-ordered.
2. Assume that $X \subseteq \mathbb{N}$ doesn't have the smallest element. Define $A = \{n \in \mathbb{N} : \{0, 1, \ldots, n\} \cap X = \varnothing\}$ and use mathematical induction to prove that X is empty.
3. Why are natural numbers well-ordered?

Functions

Basics

28. (Thanks to Antoni Hanke) How many are there functions from the empty set to $\{1, 2, 3, 4\}$?

[1] Another method is to notice that $n^3 - n = (n - 1) \cdot n \cdot (n + 1)$. Why 2 does divide it? Why 3?
[2] How $n^3 - n$ and $(-n)^3 - (-n)$ are related? Does it simplify the proof?
[3] Nice trick: what does happen if you reverse each way? Can you use the former result?

29. Consider two functions: $f : \{0,1\} \to \{0,1\}$ given by $f(x) = 0$ and $g : \{0,1\} \to \{0\}$. Prove that $f = g$. [4]

30. Let $f : A \to B$ and $g : C \to B$, where $A \neq C$. Is it possible that $f = g$?

31. Let $f : A \to B$ and $C \subseteq D \subseteq A$. We define: $f[C] = \{b \in B : b = f(c) \text{ for some } c \in C\}$ and analogously $f[D]$. Prove that $f(C) \subseteq f(D)$.

Injectivity, surjectivity and bijectivity

32. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions surjective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2$
3. $h : \mathbb{R} \to \{5\}$

33. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions injective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$
2. $h : \{0,1,2,3\} \to \mathbb{R},\ h(x) = x$

34. Construct function that is:

1. surjective, but not injective
2. injective, but not surjective
3. neither injective nor surjective
4. bijective

35. Prove that, if exists, g is unique.

36. Assume that f^{-1} exists. Prove that $(f^{-1})^{-1}$ exists and is equal to f.

Function composition

37. Find functions $f,\ g$ such that:

1. $g \circ f$ exists, but $f \circ g$ is not defined
2. both $f \circ g$ and $g \circ f$ exist, but $f \circ g \neq g \circ f$

38. Left $f : A \to B, g : B \to C, h : C \to D$. Prove that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

39.

1. Prove that composition of two surjections is surjective.
2. Prove that composition of two injections is injective.
3. Prove that composition of two bijections is bijective.

[4] Some mathematicians, as Bourbaki use an alternative definition of function - for them a function is the triple (A, B, f), where f is defined as in the our case. We see that this definition is incompatible with ours. Fortunately, as in the case with different definitions of ordered tuples, this problem will never occur explicitly in the further chapters.

40. We will rephrase the definition of the inverse function as follows:

1. If X if a set, we define **the identity function**
$$\mathrm{Id}_X = \{(x,x) \in X^2 : x \in X\}.$$
Prove that it is indeed a function. What is it's domain?

2. Let $f : A \to B,\ g : B \to A$. Prove that $f = g^{-1}$ iff
$$g \circ f = \mathrm{Id}_A \text{ and } f \circ g = \mathrm{Id}_B$$

41. Let $f : A \to B$ be an injection. Prove that there is a function $g : \mathrm{Im}\, f \to A$ such that $g \circ f = \mathrm{Id}_A$. Such g is called **left inverse of** f.

Countability

Finite sets

42. What is the cardinality of $\{a, a+1, a+2, \ldots, a+n\}$?

43. Let A, B and C be finite sets. Prove that:

1. $|2^A| = 2^{|A|}$
2. $|A \cup B| = |A| + |B|$ iff A and B are disjoint.
3. $|A \setminus B| = |A| - |B|$ if $B \subseteq A$.
4. $|A| \geq |B|$ if $B \subseteq A$. When does the equality hold?
5. $|A \cup B| = |A| + |B| - |A \cap B|$
6. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

44. Assume that A and B are finite sets. Prove that $|A| = |B|$ iff there is a bijection between A and B.

45. Above we find the way of saying that two cardinalities are equal using existence of a bijection. Let's find a way to compare which is less using another kind of function.

1. Let $O_n = \{1, 2, \ldots, n\}$. Prove that there is no injection from O_{n+1} into O_n. Hint: use mathematical induction.
2. Let A and B be finite. Prove that there is an injection from A to B iff $|A| \leq |B|$.

46. Using the above results, prove in one line[5] that if there is an injection from A onto B and an injection from B into A, then there exists a bijection from A onto B.

Infinite sets

47. Let A, B and C be sets. Prove that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. Hint: find the bijection between A and C.

48. Prove that:

1. $|\mathbb{N}| = |\mathbb{Z}|$.
2. $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
3. $|\mathbb{N}| = |\mathbb{Q}|$.

[5] The main step is $|A| \leq B$ and $|B| \leq |A|$, so $|A| = |B|$.

49. Prove that if $A \subseteq B$, then $|A| \leq |B|$.

50. Let A, B and C be sets. Prove that if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

51. Here you can prove that there are more real numbers than naturals or rationals. We define $X = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and choose one convention of writing reals (e.g 0.999... = 1.000..., so we can choose to use nines)

1. Assume that you have written all the elements of X in a single column. Can you find a real number that does not occur in the list?
2. Using the above, prove that $|\mathbb{N}| < |X|$
3. Prove that $|\mathbb{Q}| < |\mathbb{R}|$.

52. We know that $|\mathbb{R}| > |\mathbb{N}|$. Using binary system prove that $\mathbb{R} = 2^{\mathbb{N}}$. Do you see similarity between the previous result and $2^n > n$ for natural n?

53. **Cantor's theorem** You will prove that $|A| < \left|2^A\right|$ for any set A. Let A be a set and $f : A \to 2^A$.

1. Consider $X = \{a \in A : a \notin f(a)\} \in 2^A$. Is there $x \in A$ for which $f(x) = X$?
2. Is f surjective?
3. Find an injective function $g : A \to 2^A$.
4. Prove that $|A| < |2^A|$ for any set A.
5. Use Cantor's theorem to prove that there is no set of all sets.

54. **Cantor-Schroeder-Bernstein theorem** Let's prove that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$ for any sets.

1. (Knaster-Tarski) Now assume that F has *monotonicity* property: $F(X) \subseteq F(Y)$ if $X \subseteq Y$. Prove that F has a fixed point S (that is $F(S) = S$), where:
$$S = \bigcup_{X \in U} X, \text{ where } U = \{Y \in 2^A : Y \subseteq f(Y)\}.$$
2. (Banach) Let $f : A \to B$ and $g : B \to A$ be injections. We introduce new symbol: $f[X] = \{b \in B : b = f(x) \text{ for some } x \in X\}$. Prove that function
$$F : 2^A \to 2^A,\ F(X) = A \setminus g[B \setminus f[X]]$$
has the monotonicity property.
3. Prove that $A \setminus S \subseteq \operatorname{Im} g$, where F and S are taken from above.
4. Prove that function
$$h(x) = \begin{cases} f(x), x \in S \\ g^{-1}(x), x \notin S \end{cases}$$
is a bijection.

Pre-image of a function

55. Let $f : A \to B$ and $X, Y \subseteq B$. Then:

1. $f(X \cup Y) = f(X) \cup f(Y)$
2. $f(X \cap Y) \subseteq f(X) \cap f(Y)$

You can also generalise this result to an arbitrary collection of sets.

56. Let $f : A \to B$. Then $f(A) \subseteq B$ and $A = f^{-1}(B)$.

57. Let $f : A \to B$ and $X, Y \subseteq B$. Then:

1. $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$
2. $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

You can also generalise this result to an arbitrary collection of sets.

Real numbers

58. Check that

1. real numbers understood informally, have the properties listed above
2. rational numbers form a field

59. Prove that there is only one 0 and only one 1. Hint: assume that 0 and 0' have property such that $a = a + 0 = a + 0'$ and try $a = 0$ and $a = 0'$.

60. Prove that if $a + a' = 0$ and $a + a'' = 0$, then $a' = a''$. Therefore we can introduce special symbol for *the* additiv inverse: $a + (-a) = 0$ and define subtraction as $a - b := a + (-b)$.

61. Prove that $-a = (-1) \cdot a$.

62. Prove that a set $A \subseteq \mathbb{R}$ can have no upper bounds or infinitely many of them.

63. Prove that supremum is unique, so if x and x' are supremums of A, then $x = x'$.

64. Prove that $x = \sup A$ if and only if $x \geq a$ for every $a \in A$ and for every $\varepsilon > 0$ there is $a \in A$ such that $x < a + \varepsilon$.

65. Prove that natural numbers are *not* bounded from above. Hint: if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$

66. Prove the **Archimedean axiom**[6] that for every $r \in R$, there is $n \in \mathbb{N}$ such that $n > r$.

67. Prove that for any $r > 0$ there is $n \in \mathbb{N}$ such that $1/n < r$.

68. Prove that rational numbers do *not* have the completeness property:

1. Let $p, q \in \mathbb{Z} \setminus \{0\}$. Prove that $p^2 \neq 2q^2$.
2. Prove that root of two, defined as $x > 0, x^2 = 2$ is not rational.
3. Find a subset of $\mathbb{Q}$ that is bounded above, but has no rational supremum.

69. You should prove that in each nonempty interval there is at least one rational number:

1. Assume that $0 < a < b$. Define
$$A = \left\{ \frac{m}{N} : m \in \mathbb{N} \right\}, \ \frac{1}{b-a} < N \in \mathbb{N}$$
and prove that $A \cap (a, b)$ is non-empty.
2. Use the above result to prove that in *each* interval there is at least one rational number.
3. Prove that in each interval there are infinitely but countably many, rational numbers.
4. Prove that in each interval there is an irrational number.
5. How many irrational numbers are in each interval?

[6] In fact we do not need to call it axiom, as we are able to prove it.

Absolute value

70. Prove that for every $x, y \in \mathbb{R}$:

1. $|x| = |-x|$
2. if $|x| = |y|$ then $x = y$ or $x = -y$.
3. $|x + y| \leq |x| + |y|$ (this is called **triangle inequality**)
4. $|x - y| \leq |x| + |y|$
5. $||x| - |y|| \leq |x - y|$ (this is sometimes calles **reverse triangle inequality**)

General topology

Basic definitions

Topology, open sets and interior

71. Using mathematical induction prove that the intersection of finitely many open sets is open.

72. **Trivial topology** Prove that for any X, set $\{\varnothing, X\}$ is a topology.

73. **Discrete topology** Prove that for any X, it's power set 2^X is a topology.

74. **Cofinite topology** Prove that for any X, the set: $\mathcal{T}_X = \{\varnothing\} \cup \{A \subseteq X : X \setminus A \text{ is finite}\}$ is a topology. Hint: think in terms of complements.

75. **Subspace topology** Let X be a set and $\mathcal{T}_X$ a topology on it. For $A \subseteq X$ we define $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}_X\}$. Prove that $\mathcal{T}_A$ is a topology on A.

76. For which sets, there is exactly one topology on them?

77. Prove that for an infinite set, there are at least three distinct topologies.

78. Real numbers $\mathbb{R}$ are usually equipped with the following topology: $X \subseteq \mathbb{R}$ is open iff for each $x \in X$ there is an open interval (a_x, b_x) such that $x \in (a_x, b_x) \subseteq X$.

1. Prove that it is indeed a topology.
2. Let **a ball** be a set $B(x, r) = \{y \in \mathbb{R} : |x - y| < r\}$ for $r > 0$. Prove that $(a, b) \neq \varnothing$ can be written as $B(x, r)$ for suitable x and r
3. Prove that X is open iff for each $x \in X$ there is a $r > 0$ such that $B(x, r) \subseteq X$. We will see that this results generalises to much broader category of spaces than single $\mathbb{R}$.

79. Prove that each point has an open neighborhood.

80. Prove that A is an open set if and only if each point a has a neighborhood $U_a \in A$ contained in A (that is $U_a \subseteq A$).

81. Prove that:

1. $\operatorname{Int} A$ is an open set.
2. if $A' \subseteq A$ is open, then $A' \subseteq \operatorname{Int} A$ (so in some sense, $\operatorname{Int} A$ is the biggest open set contained in A)
3. $\operatorname{Int} A = A$ iff A is open
4. $\operatorname{Int} \operatorname{Int} A = \operatorname{Int} A$ for any A

82. Let $A' \subseteq A$. Prove that:

1. $\operatorname{Int} A' \subseteq \operatorname{Int} A$
2. $\operatorname{Int} A \cup \operatorname{Int} B \subseteq \operatorname{Int}(A \cup B)$

You can prove also that the union can be arbitrary.

83. We say that a is an **interior point** of A if there is open $U_a \subseteq A$ such that $a \in U_a$. Prove that $\operatorname{Int} A$ is the set of all interior points of A.

Closed sets and closure

84. Prove these properties of closed sets in space $(X, \mathcal{T}_X)$:

1. $\varnothing$ and X are closed
2. If $A_1, A_2, \ldots, A_n$ are closed, then their union $A_1 \cup A_2 \cup \cdots \cup A_n$ is closed.
3. If $\mathcal{A}$ is any family of closed sets, then the intersection $\bigcap \mathcal{A}$ is closed.

85. We say that p is a limit point of $A \subseteq X$ if for every every open neighborhood U of p there is $q_U \neq p$ such that $q_U \in A \cap U$. Prove that A is closed iff it contains all of it's limit points.

86. Prove that:

1. $\operatorname{Cl} A$ is a closed set.
2. if C is closed and $A \subseteq C$, then $\operatorname{Cl} A \subseteq C$ (so in some sense, $\operatorname{Cl} A$ is the smallest closed set containing A)
3. $A \subseteq \operatorname{Cl} A$
4. $\operatorname{Cl}(A \cup B) = \operatorname{Cl} A \cup \operatorname{Cl} B$
5. $\operatorname{Cl} A = A$ iff A is closed
6. $\operatorname{Cl} \operatorname{Cl} A = \operatorname{Cl} A$ for any A

87. We say that p is an **adherent point** of A (or **point of closure**) if for any neighborhood V of p we have $A \cap V \neq \varnothing$. Alternatively, we can say that every neighborhood of p contains a point from A. Prove that $\operatorname{Cl} A$ is the set of all adherent points of A.

88. We say that $A \subseteq X$ is **dense** if $\operatorname{Cl} A = X$. Prove that A is dense iff for every $U \in \mathcal{T}_X$, $A \cap U \neq \varnothing$

89.

1. Let $r \in \mathbb{R}$. Prove that for every neighborhood V of r there is $q \in \mathbb{Q}$ such that $q \in V$. Hint: each neighborhood must have an interval. And you should have proven that in each interval there is a rational.
2. Conclude that rationals are dense in reals.

Boundary and exterior

90. We say that p is a **frontier** point of A if every open neighborhood of p intersects both A and A^c, so if for every open neighborhood U_p we have $U_p \cap A \neq \varnothing$ and $U_p \cap A^c \neq \varnothing$. Prove that the boundary of A is exactly the set of frontier points of A.

91. Prove that boundary is always closed.

92. Prove that $\partial\partial A \subseteq \partial A$.

93. Prove that $\partial A = \partial A^c$.

94. Prove that $\partial A = \varnothing$ iff A is simultaneously open and closed.

95. Prove that $\partial A = \operatorname{Cl} A \cap \operatorname{Cl} \operatorname{Ext} A$

Bases and countability axioms

96. Prove that $\mathcal{B}$ is a basis for $(X, \mathcal{T})$ iff for every $x \in X$ and every neighborhood U_i of x, there is $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq U_i$.

97. Let $\mathcal{B}$ be a basis of $(X, \mathcal{T})$. Prove that:

1. $\bigcup \mathcal{B} = X$
2. If $U, V \in \mathcal{T}$ and $x \in U \cap V$, then there is a set $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U \cap V$.

98. Consider $\mathbb{R}$ with it's standard topology. If $x \in \mathbb{R}$ and U is an open set containing x, we can find a ball $B(x, r),\ r > 0$ such that $B(x, r) \subseteq U$. Using the fact that rationals are dense in reals, prove that you can find $p, q \in \mathbb{Q}$ such that $x \in (p, q) \subseteq U$.

99. Prove that $\mathbb{R}$ is second countable.

Continuous maps and homeomorphisms

100. Prove that a function is continuous iff preimage of every *closed* set is closed.

101. Let $f : (X, \mathcal{T}) \to (Y, \tau)$ and $\mathcal{B}$ be a basis of (Y, τ). Then f is continuous iff $f^{-1}(B) \in \mathcal{T}$ for every $B \in \mathcal{B}$.

102. Assuming that $\mathbb{R}$ is equipped with it's standard topology, prove that functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous:

1. $f(x) = ax + b$
2. $f(x) = x^2$

103. Let $f : (X, \mathcal{T}) \to (Y, \tau)$. Prove that f is continuous iff $f(\operatorname{Cl} A) \subseteq \operatorname{Cl} f(A)$ for every $A \subseteq X$.

104. Prove that two *discrete* spaces X and Y are homeomorphic iff $|X| = |Y|$.

Linear algebra

Vector spaces

105. We know that the set of vectors must contain at least one vector (neutral element). Construct a vector space that has *exactly* one vector (so in some sense it is the smallest space).

106. We want to modify our notation in the following way:

1. Prove that o is unique element with the property $v + o = v$ for $v \in V$
2. Prove that $0 \cdot v = o$ for every v. Hint: remember that $1 + 0 = 1$. This suggests to write 0 for o (so 0 since now technically has two different meanings, practically we will never have any problems with that)
3. Let $v \in V$ and $\tilde{v} \in V$ be such an element that $v + \tilde{v} = 0$. Prove that $\tilde{v}$ is unique (so if $v' + v = 0$, then $\tilde{v} = v'$)
4. Prove that the $\tilde{v}$ is exactly $(-1)v$. It suggests to write $-v$ for additive inverse, and we will do it since now.

107. The **characteristic** of a field F is the smallest natual number n such that $1 + 1 + \cdots + 1 = 0$, where we have n ones on the left hand side. If there is no such number we say that characteristic is 0.

1. Prove that $\mathbb{R}$ has characteristic 0.
2. Let (F, V) be a vector space with at least two elements. Prove that $v = -v$ for every $v \in V$ if and only if the scalar field has characteristic 2.
3. Prove that if F has characteristic different from 2, then we have $v = -v$ iff $v = 0$.

108. An important example of a vector space over a field F is F^n, where addition and scalar multiplication are defined pointwise: $f \cdot (a_1, a_2, \dots, a_n) = (fa_1, fa_2, \dots, fa_n)$, $(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$. Prove that it is indeed a vector space.

Bases of vectos spaces

109. Consider infinite real sequences with addition and multiplication by a real number defined pointwise: $c = a + b$ iff $c_n = a_n + b_n$ for all n and $b = ra, r \in \mathbb{R}$ iff $b_n = ra_n$ for all n.

1. Prove that this is a vector space, let's call it $\mathbb{R}^{\mathbb{N}}$.
2. Prove that set $B = \{e_k : k \in \mathbb{N}\}$, e_k has 1 at k-th place and 0 at all the others, does *not* span $\mathbb{R}^{\mathbb{N}}$.
3. Let $\hat{\mathbb{R}}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ contain all the sequences that have a *finite* number of non-zero elements. Prove that this is a vector space and that it is spanned by B defined above.

110. Assume that $v_1, v_2, \dots, v_n$ are *not* linearly dependent. Prove that there is v_i such that v_i is a linear combination of other vectors: $v_i \in \operatorname{Span}\{v_1, \dots, v_{i-1}, v_{i+1}, v_n\}$.

111. You can prove that every finite dimensional vector space has a basis in two steps:

1. Assume that you have a set $\{e_1, e_2, \dots, e_n\}$ that spans V. Prove that if $e_1 \in \operatorname{Span}\{e_2, \dots, e_n\}$, then $V = \operatorname{Span}\{e_2, \dots, e_n\}$.
2. Using the reduction step given above, show an algorithm finding a basis from a finite spanning set.
3. Prove that a vector space is finite dimensional iff it has a finite basis.

112. Prove that F^n is finite dimensional. Hint: just find a basis.

113. Prove that $\hat{\mathbb{R}}^{\mathbb{N}}$ has a basis.

Subspaces, direct sum and quotient spaces

114. Prove that $fv + u \in U$ for all v, $u \in U, f \in F$ is equivalent to: for every v, $u \in U$, we have $v + u \in U$ and for every $v \in U, f \in F$ we have $fv \in U$.

115. Prove that each $w \in V \oplus U$ has a *unique* decomposition: $w = v + u$, $v \in V$, $u \in U$. That is if $w = v + u = v' + u'$, then $v = v'$ and $u = u'$ for $v, v' \in V$, $u, u' \in U$.

116. Prove that the direct product of two subspaces is a special case of the general definition, with the identification $v \leftrightarrow (0, v)$ employed.

117. Prove that direct product of two vector subspaces of V is a vector subspace of V. Hint: check if 0 is inside and use the handy, one-line criterion.

118. Let $V = \mathbb{R}^2$ and $U = \{(0, r) : r \in \mathbb{R}\}$, $W = \{(r, 0) : r \in \mathbb{R}\}$. Prove that:

1. $V = U \oplus W$
2. Let $U \hat{\oplus} W = \{u + v : u \in U, v \in V\}$. Prove that this set is *not* a vector space.