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Mathematical Physics for Curious People

Geometrical approach

– Monograph –

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For the people whom I learned mathematics from:
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Preface

So what I told you was true, from a certain point of view.

Star Wars, Episode VI — Obi-Wan Kenobi

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1 Introduction

There are many excellent books on mathematical physics and differential geometry, so a question araises - how does this book differ from any other? I had a few aims working on it:

- Understandable for any person that wants to learn. It does not matter if you are a physicists, mathematician, english literature major or a high-school student. If you have enough self-determination, you can understand the mathematics in this book.
- Self-containing. Mathematics is both broad and deep, so it must be split into many different branches. But I personally found discouraging that if you want to read one book, as prerequsities you need to read two other books, and so on. Here, you can understand that everything contained here with no access to libraries or other mathematical books. Obviously, we don't cover the whole subject, but it is a good start to own research.
- Problem-solving approach. I want you to prove all the theorems in this book, with adjustable amount of hints. This way you can understand what we are actually doing, instead of ommiting proofs that look discouraging at the beginning.
- Abstract concepts first. We start with very abstract concepts and then move to examples and special cases. It is not always possible if we want to provide enough examples, but this is the aim. Starting from abstract, more general terms usually makes the whole situation easier - you have less properties and assumptions to use, so the solutions are more straightforward.
- "So what I told you was true, from a certain point of view." - many mathematical objects look differently for different mathematicians. We will always try to cover many "points of view" to increase the understanding of the subject.
- Objects and maps. We define precisely what are our objects and transformations, that are in some sense natural, that change one object into

another. While we don't use the language of the category theory, you can get some taste.

- Properties, then construction. When we talk about a mathematical object, we usually think about it's *properties*. Explicit construction is useful - as it proves the existence of the object under consideration - but usually hides many important properties of the object. Therefore we define objects by a few properties, then we think about theorems that can be proved using these initial properties (so we end up with many more properties) and then think how to construct the object having the initial properties.
- Notation abusements explained. Mathematics has been evolving for centuries in many different countries, so the notation is rather diverse and sometimes is not the best possible. We will abuse it as it is a standard in mathematical world, but you will always understand what objects are involved in expressions you are manipulating with.
- No jumps. In mathematics we prove theorems and then use these theorems to prove other theorems and so on. In many textbooks I know, these auxilary theorems are referenced as „Check section 3, problem 2.". I don't associate theorems with specific numbers and I don't like going to a specified section. Therefore I reference theorems by their true meaning or commonly used name, rather than an artificial number.

We use the following notation: **bold** will be used for definitions of new objects, and *italics* will be used for additional subtle remarks that should be taken into account. We use footnote[1] to provide additional comments.

Remember that the subject is big and it may be very hard to finish the book in just one day. I strongly advise working on it every day starting from just two minutes a day and increasing the time spend every week. I tried to make the learning curve flat, what leghtens the book. Any mistakes are my own failure and I would be grateful if you pointed them to me. Also any suggestions and comments are welcome. You can create new issues on GitHub: `https://github.com/pawel-czyz/MathematicalPhysics` or write an email to `pczyz@protonmail.com`. Good luck on your road!

[1] Like this one.

2

Logic and sets

If you are already fimilar with operations on logical formulas and sets, you may ommit this chapter.

2.1 Logical formulas

Consider declarative sentences as „Water boils at 100°C" or „2+2=5". We can construct new sentences from them using the following rules:

1. conjuntion (and): $p \wedge q$ is true if and only if p is true and q is true
2. disjuntion (or): $p \vee q$ is true if and only if at least one of sentences p, q is true
3. implication: $p \Rightarrow q$ is false if and only if p is true and q is false. Intuitively, if you know that p implies q and p is true, then q also must be true
4. negation (not): $\neg p$ is true if and only if p is false
5. equivalence (iff, if and only if): $p \Leftrightarrow q$ means exactly $(p \Rightarrow q) \wedge (q \Rightarrow p)$. Intuitively if you know that two sentences are equivalent and one of them is true, the other is also true. Or if one of them is false, the other one is automatically false.

Because mathematics is the art of being smart and lazy, we will assign value 1 to true sentences and 0 to false sentences.

2.1. Prove that the following sentences are true:

1. $\neg(\neg p) \Leftrightarrow p$
2. $p \vee \neg p$
3. $\neg(p \wedge q) = (\neg p) \vee (\neg q)$
4. $\neg(p \vee q) = (\neg p) \wedge (\neg q)$
5. $(p \Rightarrow q) \Leftrightarrow (\neg p) \vee q$
6. $0 \Rightarrow 1$

2.2. Prove that:

1. $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$
2. $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$

Equipped with this powerful machinery we can dive into basic set theory.

2.2 Basic set theory

2.2.1 Rough ideas

In modern mathematics we do not define a set nor set membership, so heuristically you can think that set A is a „collection of objects' and $x \in A$ means that the object x is inside this collection. We will assume that any finite collections of elements $\{x_1, x_2, \dots, x_n\}$ is a set (the empty set is called $\varnothing$ rather than $\{\}$), moreover we will assume the existence of the following sets:

1. real numbers $\mathbb{R}$
2. natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$
3. integers $\mathbb{Z}$
4. rational numbers $\mathbb{Q}$

We say that two sets are equal ($A = B$) iff they have the same elements ($x \in A \Leftrightarrow x \in B$). Note, that we do not check how many times x appears in A. We can just say whether it inside or not.

2.3. Prove that $\{1, 1, 2, 2, 2\} = \{1, 2\}$

We assumed the existence of some sets at the beginning. Why? As you can prove not every „collection of objects" is a set:

2.4. Let X be a set built from all sets such that $A \notin A$. Prove that X does not exist. Hint: what if $X \in X$? What if $X \notin X$?

Therefore at the moment we do not have many sets that we assume to exist. Let's try to define some methods of creating new sets from the know ones:

2.2.2 A few ways of constructing new sets

Assume that A and B are sets:

1. Let's make a formula F that for every element $a \in A$, the value $F(a)$ is true or false. We can then construct a set S with all the elements a from A for which the formula $F(a)$ holds. This set is written explicitly as $S = \{a \in A : F(a)\}$.
2. We can form the sum of two sets: $a \in A \cup B$ iff $a \in A$ or $a \in B$.
3. We can contruct the intersection of two sets: $a \in A \cap B$ iff $a \in A$ and $a \in B$.

4. We can construct the difference of two sets: $A \setminus B = \{a \in A : a \notin B\}$

2.5. Let A, $B\,C$ be sets. Prove that:

1. $A \cup A = A$
2. $A \cup B = B \cup A$
3. $A \cup (B \cup C) = (A \cup B) \cup C$
4. $A \cap A = A$
5. $A \cap B = B \cap A$
6. $A \cap (B \cap C) = (A \cap B) \cap C$
7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
8. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2.6. Prove that there is no set of all sets. Hint: assume there is one. Then you can select some sets to form a set that does not exist.

Moreover, we will introduce two new symbols, called positive and negative infinity: ∞ and $-\infty$. These are *not* real numbers, just symbols that are used to name a few useful sets:

$$
\begin{aligned}
(-\infty, b) &= \{x \in \mathbb{R} : x < b\} \\
(-\infty, b] &= \{x \in \mathbb{R} : x \geq b\} \\
(a, \infty) &= \{x \in \mathbb{R} : a < x\} \\
[a, \infty) &= \{x \in \mathbb{R} : a \leq x\}
\end{aligned}
$$

2.2.3 Subsets and complements

As we have some sets, we can try to compare them. We say that $A \subseteq B$ iff $a \in A \Rightarrow a \in B$ (or intuitively, each element of A is also in B. We say that A is a **subset** of B or that B is a **superset** of A.

2.7. Prove that $A = B$ iff $A \subseteq B \wedge B \subseteq A$.

2.8. Prove that for any set A, $\varnothing \subseteq A$.

2.9. Here you'll prove that there is just one empty set. Let $\varnothing$ and $\varnothing'$ be empty sets. Prove that $\varnothing = \varnothing'$.

If we fix the set B, to each subset A we can assign it's **complement**: $A^c = B \setminus A$. [1]

2.10. Prove the following set identites:

1. Let $A \subseteq B$. Prove that $(A^c)^c = A$.
2. Let A, $B \subset U$. Prove that $(A \cup B)^c = A^c \cap B^c$

[1] It is not the best symbol possible as we need to have B in mind.

3. Let A, $B \subset U$. Prove that $(A \cap B)^c = A^c \cup B^c$
4. $\{a \in A : a \in B\} = \{b \in B : b \in A\}$

Moreover, we assume that for a set A there exists it's **power set**: $2^A = \{X : X \subseteq A\}$.

2.11. 1. Let $A = \{1, 2, 3\}$. Find 2^A. What is the number of elements in 2^A? How is it related to the number of elements of A?
2. Let A be a finite set with n elements. Using the approach in which you choose which elements belong to a subset, prove that 2^A has 2^n elements.

2.2.4 Infinite collections of sets

Now we understand how to construct new sets from finite number of sets. But we can also consider „more general" families of sets, that are not necessarily finite: let $A_i \subset U$ for $i \in I$, where I is some indexing set. For example if $I = \{1, 2, \ldots, n\}$ we have a finite family. But you can imagine infinite families as $A_i = \{i\}$, $i \in \mathbb{R}$. How do we define the sum and intersection of them? We cannot sum them iteratively $A_1 \cup A_2 \cup \ldots$ as the process will never end, so we need alternative definitions:

$$\bigcup_{i \in I} A_i = \{a \in U : a \in A_i \text{ for at least one } i \in I\}$$
$$\bigcap_{i \in I} A_i = \{a \in U : a \in A_i \text{ for every } i \in I\}$$

Alternative notation is a **family of sets**, that is just a name for a subset of 2^U:

$$\mathcal{A} = \{A_i \subseteq U : A_i \text{ has some property}\}$$

and write:

$$\bigcup \mathcal{A}, \ \bigcap \mathcal{A}$$

for the infinite sum and intersection of all of family members.

2.12. Prove that for finite families of sets, these new definitions agree with the previous.

2.13. Let $\mathcal{X} \subset 2^U$ be a family of sets and define: $\mathcal{Y} = \{X^c : X \in \mathcal{X}\}$, where $X^c = U \setminus X$. Prove that:

1. $(\bigcup \mathcal{X})^c = \bigcap \mathcal{Y}$
2. $(\bigcap \mathcal{X})^c = \bigcup \mathcal{Y}$

2.14. Let $A_i \subseteq U$, $i \in I$ and

$$\sigma = \bigcup_{i \in I} A_i, \ \pi = \bigcap_{i \in I} A_i$$

Prove that:

1. if $k \in I$, then $A_k \cup \sigma = \sigma$
2. $\pi \subseteq \sigma$
3. $\sigma \cap \pi = \pi$

2.15. Find sum and intersection of family of subsets of $\mathbb{R}$: $A_r = \{r, -r\}$ for $r \geq 0$.

2.16. Let $A \subseteq X_i \subseteq U$ for $i \in I$. Prove that

$$A \subseteq \bigcup_{i \in I} X_i$$

2.17. For every point $a \in A$ there is a set $U_a \subseteq A$ such that $a \in U_a$. Prove that

$$A = \bigcup_{a \in A} U_a.$$

2.2.5 Cartesian product

First of all, we need a useful concept:

2.18. Let $A = \{\{a\}, \{a, b\}\}$, $B = \{\{c\}, \{c, d\}\}$. Prove that $A = B$ iff $a = c \wedge b = d$. Such a set A we call **the ordered pair** (a, b) as it has the property $(a, b) = (c, d)$ iff $a = c$ and $b = d$. Now you can forget how it has been constructed, and just remember this property.

2.19. Prove that $(a, (b, c)) = (d, (e, f))$ iff $a = d \wedge b = e \wedge c = f$.

Therefore it makes sense to write just (a, b, c) for $(a, (b, c))$ and define similarly such **ordered tuple** for four elements, five elements and so on.

2.20. Check that defining (a, b, c) as $((a, b), c)$ also works (so two ordered tuples are the same if they have the same first element, the same second element, ...)

2.21. Check that, in terms of sets, $(a, (b, c)) \neq ((a, b), c)$, so formally we do need to stick to one convention. However as we are interested in the property of ordered tuple, we will not distinguish them and denote both of them just as (a, b, c). Such notational problems appear in various places in mathematics, so we need to try to get used to them.

We can now introduce another way of creating new sets: let A and B be sets. Then we define their **Cartesian product** as

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

2.22. Do you remember the identification of $(a, (b, c))$ and $((a, b), c)$? Prove that $A \times (B \times C) = (A \times B) \times C$. Therefore we'll write it just as $A \times B \times C$ without parentheness.

Commonly used notation is $X^2 = X \times X = \{(x, y) : x, y \in X\}$ and analogously for other powers.

2.3 Natural numbers and mathematical induction

Have you ever seen falling dominoes? To be sure that every domino falls, we need to:

1. punch the first domino
2. for every domino we must be sure the implication: if this particular domino falls, the next one also falls

And that's all, we can be sure that all the dominoes will eventually fall. This style of reasoning[2] is called **mathematical induction** and formally it is written as: if $0 \in S$ and for every[3] $n \in N$ you can prove the implication $n \in S$ then $n+1 \in S$, you know that $N \subseteq S$.

2.23. You can prove that $2^n > n$ for every natural number n.

1. Prove that the formula works for $n = 0$ (punch the first domino).
2. Assume that for some n you proved on some way that $2^n > n$. Using this, prove that $2^{n+1} > (n+1)$ (if n-th domino falls, then $n+1$-th domino also falls)

You can also modify slightly the induction principle - sometimes you should start with number different than 0 or use different induction step (start 0 and step 2 can lead to theorems valid for even numbers, step 0 and steps 1 and -1 can lead to theorems valid for all integers...)

2.24. 1. Prove[4] that 6 divides $n^3 - n$ for all natural n.
2. Prove[5] that 6 divides $n^3 - n$ for all integers n. You can use a slight modification mathematical induction principle proving the implication „if the theorem works for n, it works also for $n - 1$".

2.25. (Bernoulli's inequality) Prove that for real $x > -1$ and natural $n \geq 1$, the following inequality holds:

$$(1+x)^n \geq 1 + nx.$$

2.26. In Mathsland there are $n \geq 2$ cities. Between each pair of them there is a *one-way* road.

[2] We do not show here formally *why* this principle works. For curious, you define natural numbers in such way this principle works.

[3] I repeat: for every n we need to prove the implication „if works for n, then works for $n+1$". The correct way is to write „I assume that there is a given n for which the formula works. I will prove that is works for $n+1$". Common mistake is to write „I assume that the formula works for every n and I will prove that it works for $n+1$.". As professor Wiktor Bartol says - there is no need to prove the statement as you already assumed that it works in every case.

[4] Another method is to notice that $n^3 - n = (n-1) \cdot n \cdot (n+1)$. Why 2 does divide it? Why 3?

[5] How $n^3 - n$ and $(-n)^3 - (-n)$ are related? Does it simplify the proof?

1. Prove that there is a city from which you can drive to all the other cities. Hint: assume that the hypothesis works for some n and any country with n cities. Now consider an arbitrary $n+1$-city country. Hide one city and use your assumption.
2. Prove that there is a city[6] to which you can drive from all the others.

2.27. Let $S \subseteq R$. We say that S is **well-ordered** iff any non-empty subset $X \subset S$ has the smallest element.

1. Prove that reals and integers with the default ordering are not well-ordered.
2. Assume that $X \subseteq \mathbb{N}$ doesn't have the smallest element. Define $A = \{n \in \mathbb{N} : \{0, 1, \ldots, n\} \cap X = \varnothing\}$ and use mathematical induction to prove that X is empty.
3. Why are natural numbers well-ordered?

2.4 Functions

2.4.1 Basics

Consider two sets A and B. We say that a subset $f \subseteq A \times B$ is a **function** iff the following two conditions hold:

- for every element $a \in A$ there is an element $b \in B$ such that $(a, b) \in f$
- if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$

Therefore for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$. Such b will be called **value of f at point** a and given a symbol $f(a)$.

2.28. (Thanks to Antoni Hanke) How many are there functions from the empty set to $\{1, 2, 3, 4\}$?

We need to introduce more terminology: set A is called **the domain of** f, set B is called **the codomain of** f and the function f is written as $f : A \to B$.

2.29. Consider two functions: $f : \{0, 1\} \to \{0, 1\}$ given by $f(x) = 0$ and $g : \{0, 1\} \to \{0\}$. Prove that $f = g$. [7]

2.30. Let $f : A \to B$ and $g : C \to B$, where $A \neq C$. Is it possible that $f = g$?

2.31. Let $f : A \to B$ and $C \subseteq D \subseteq A$. We define: $f[C] = \{b \in B : b = f(c) \text{ for some } c \in C\}$ and analogously $f[D]$. Prove that $f(C) \subseteq f(D)$.

[6] Nice trick: what does happen if you reverse each way? Can you use the former result?

[7] Some mathematicians, as Bourbaki use an alternative definition of function - for them a function is the triple (A, B, f), where f is defined as in the our case. We see that this definition is incompatible with ours. Fortunately, as in the case with different definitions of ordered tuples, this problem will never occur explicitly in the further chapters.

2.4.2 Injectivity, surjectivity and bijectivity

As we have already seen, there may be some elements in codomain that are not values of f. We define **the image of** f as:

$$\operatorname{Im} f = \{b \in B : \text{there is } a \in A \text{ such that } b = f(a)\}.$$

We say that the function $f : A \to B$ is **surjective** (or **onto**) iff $\operatorname{Im} f = B$.

2.32. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions surjective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2$
3. $h : \mathbb{R} \to \{5\}$

If $f(a)$ uniquely specifies a (if $f(a) = f(b)$, then $a = b$) we say that the function is **injective** (or **one-to-one**).

2.33. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions injective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$
2. $h : \{0, 1, 2, 3\} \to \mathbb{R},\ h(x) = x$

If a function f is both surjective and injective, we say that is *bijective*[8].

2.34. Construct function that is:

1. surjective, but not injective
2. injective, but not surjective
3. neither injective nor surjective
4. bijective

Notice that if a function $f : A \to B$ is bijective, then we can construct a function $g : B \to A$ such that $f(g(b)) = b$ and $g(f(a)) = a$.

2.35. Prove that, if exists, g is unique.

We call this function **the inverse function** [9]: $g = f^{-1}$.

2.36. Assume that f^{-1} exists. Prove that $(f^{-1})^{-1}$ exists and is equal to f.

[8] If you prefer nouns: surjective function is called surjection, injective - injection and bijective - bijection

[9] It becomes confusing when working on real numbers: $f^{-1}(x)$ is **not** $(f(x))^{-1} = 1/f(x)$

2.4.3 Function composition

If we have two functions: $f : A \to B$ and $g : B \to C$, we can construct the **composition** using formula: $g \circ f : A \to C,\ (g \circ f)(a) = g(f(a))$.

2.37. Find functions $f,\ g$ such that:

1. $g \circ f$ exists, but $f \circ g$ is not defined
2. both $f \circ g$ and $g \circ f$ exist, but $f \circ g \neq g \circ f$

Although function composition is not commutative, it is associative:

2.38. Left $f : A \to B, g : B \to C, h : C \to D$. Prove that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Therefore we can ommit the brackets and write just $h \circ g \circ f$. We will use function composition very often.

2.39. 1. Prove that composition of two surjections is surjective.
2. Prove that composition of two injections is injective.
3. Prove that composition of two bijections is bijective.

2.40. We will rephrase the definition of the inverse function as follows:

1. If X if a set, we define **the identity function**

$$\mathrm{Id}_X = \{(x, x) \in X^2 : x \in X\}.$$

   Prove that it is indeed a function. What is it's domain?
2. Let $f : A \to B,\ g : B \to A$. Prove that $f = g^{-1}$ iff

$$g \circ f = \mathrm{Id}_A \text{ and } f \circ g = \mathrm{Id}_B$$

2.41. Let $f : A \to B$ be an injection. Prove that there is a function $g : \mathrm{Im}\, f \to A$ such that $g \circ f = \mathrm{Id}_A$. Such g is called **left inverse of** f.

2.5 Countability

2.5.1 Finite sets

For a finite set X we write the number of elements of X as $|X|$. We can calculate their **cardinalities** (sizes, numbers of elements) with ease,

2.42. What is the cardinality of $\{a, a+1, a+2, \ldots, a+n\}$?

2.43. Let A, B and C be finite sets. Prove that:

1. $|2^A| = 2^{|A|}$
2. $|A \cup B| = |A| + |B|$ iff A and B are disjoint.

3. $|A \setminus B| = |A| - |B|$ if $B \subseteq A$.
4. $|A| \geq |B|$ if $B \subseteq A$. When does the equality hold?
5. $|A \cup B| = |A| + |B| - |A \cap B|$
6. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

We can also employ functions to compare cardinalities:

2.44. Assume that A and B are finite sets. Prove that $|A| = |B|$ iff there is a bijection between A and B.

2.45. Above we find the way of saying that two cardinalities are equal using existence of a bijection. Let's find a way to compare which is less using another kind of function.

1. Let $O_n = \{1, 2, \ldots, n\}$. Prove that there is no injection from O_{n+1} into O_n. Hint: use mathematical induction.
2. Let A and B be finite. Prove that there is an injection from A to B iff $|A| \leq |B|$.

2.46. Using the above results, prove in one line[10] that if there is an injection from A onto B and an injection from B into A, then there exists a bijection from A onto B.

2.5.2 Infinite sets

But how can we measure the number of elements of an infinite set, as $\mathbb{N}$ or $\mathbb{R}$? As natural numbers are „to small" we need to introduce new numbers, as $|\mathbb{N}|$ and be able to compare them. As we have seen above, the existence of a bijection is a good way of saying that two finite sets have equal cardinalities. It intuitively makes sense to employ this observation even in the infinite case: we say that sets (finite or infinite) A and B have the same caridnalities (or $|A| = |B|$) iff there is a bijection between A and B.

2.47. Let A, B and C be sets. Prove that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. Hint: find the bijection between A and C.

Here you can see the difference between finite and infinite sets - for finite sets a proper subset (a subset that is not the whole set) always has smaller number of elements. In the infinite case it is not true, as a proper subset can have *the same* number of elements.

2.48. Prove that:

1. $|\mathbb{N}| = |\mathbb{Z}|$.
2. $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
3. $|\mathbb{N}| = |\mathbb{Q}|$.

[10] The main step is $|A| \leq B$ and $|B| \leq |A|$, so $|A| = |B|$.

Analogously to the finite case, we define $|A| \leq |B|$ as the existence of an injection from A to B. We say that $|A| < |B|$ iff there is an injection from A to B but there is no bijection.

2.49. Prove that if $A \subseteq B$, then $|A| \leq |B|$.

2.50. Let A, B and C be sets. Prove that if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

2.51. Here you can prove that there are more real numbers than naturals or rationals. We define $X = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and choose one convention of writing reals (e.g 0.999... = 1.000..., so we can choose to use nines)

1. Assume that you have written all the elements of X in a single column. Can you find a real number that does not occur in the list?
2. Using the above, prove that $|\mathbb{N}| < |X|$
3. Prove that $|\mathbb{Q}| < |\mathbb{R}|$.

2.52. We know that $|\mathbb{R}| > |\mathbb{N}|$. Using binary system prove that $\mathbb{R} = 2^{\mathbb{N}}$. Do you see similarity between the previous result and $2^n > n$ for natural n?

2.53. Cantor's theorem You will prove that $|A| < \left|2^A\right|$ for any set A. Let A be a set and $f : A \to 2^A$.

1. Consider $X = \{a \in A : a \notin f(a)\} \in 2^A$. Is there $x \in A$ for which $f(x) = X$?
2. Is f surjective?
3. Find an injective function $g : A \to 2^A$.
4. Prove that $|A| < |2^A|$ for any set A.
5. Use Cantor's theorem to prove that there is no set of all sets.

2.54. Cantor-Schroeder-Bernstein theorem Let's prove that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$ for any sets.

1. (Knaster-Tarski) Now assume that F has *monotonicity* property: $F(X) \subseteq F(Y)$ if $X \subseteq Y$. Prove that F has a fixed point S (that is $F(S) = S$), where:
$$S = \bigcup_{X \in U} X, \text{ where } U = \{Y \in 2^A : Y \subseteq f(Y)\}.$$
2. (Banach) Let $f : A \to B$ and $g : B \to A$ be injections. We introduce new symbol: $f[X] = \{b \in B : b = f(x) \text{ for some } x \in X\}$. Prove that function
$$F : 2^A \to 2^A, \ F(X) = A \setminus g[B \setminus f[X]]$$
has the monotonicity property.
3. Prove that $A \setminus S \subseteq \operatorname{Im} g$, where F and S are taken from above.
4. Prove that function
$$h(x) = \begin{cases} f(x), x \in S \\ g^{-1}(x), x \notin S \end{cases}$$
is a bijection.

2.5.3 Pre-image of a function

Let $f : A \to B$ and $C \subseteq A$. We used $f[C]$ for a set:

$$f[C] = \{f(c) \in B : c \in C\},$$

but now we will abuse a bit our notation to stick to the common nomenclature. Apparently, many mathematicians write:

$$f(C) = \{f(c) \in B : c \in C\}.$$

This is not correct - as f should take elements $a \in A$ and returns elements $b \in B$, but here f „takes" a subset $C \subseteq A$ and returns a set $f(C) \subseteq B$. We will follow this notation, but you should always check what meaning the object feed to function has (whether it is an element or a subset).

2.55. Let $f : A \to B$ and $X, Y \subseteq B$. Then:

1. $f(X \cup Y) = f(X) \cup f(Y)$
2. $f(X \cap Y) \subseteq f(X) \cap f(Y)$

You can also generalise this result to an arbitrary collection of sets.

To even more abuse the notation, we will also give an additional meaning to f^{-1}. As we know, many functions f *don't* have inverses. But we will write for $D \subseteq B$:

$$f^{-1}(D) = \{a \in A : f(a) \in D\} \subseteq A.$$

We then say that $f^{-1}(D)$ is the **pre-image** of D. To get accustomed with this notation, prove that:

2.56. Let $f : A \to B$. Then $f(A) \subseteq B$ and $A = f^{-1}(B)$.

You should also prove:

2.57. Let $f : A \to B$ and $X, Y \subseteq B$. Then:

1. $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$
2. $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

You can also generalise this result to an arbitrary collection of sets.

Therefore, we see that although f does *not* preserve the set structure, f^{-1} does. This observation underlies the notation of continuity.

2.6 Real numbers

At the beginning we assumed that you had some intuition what real numbers are and how to work with them - to provide examples and make set theory less abstract. But we have not treated them rigorously, as we did not have proper glossary - it's high time we filled this gap and defined them properly. It's high time we defined them properly, as we . A **field** is a tuple $(F, +, \cdot, 1, 0)$. We have many symbols there, let's explain what they mean:

- F is a set
- $+$ and $\cdot$ are functions from F^2 to F. We write $a + b$ for $+(a, b)$ and $a \cdot b$ for $\cdot(a, b)$.
- $1, 0 \in F$ are just distinguished elements of F

We know what objects are in the definition, so we can talk about properties they must have to form a field:

1. $1 \neq 0$ (so F has at least two elements)
2. $a + (b + c) = (a + b) + c$ for all a, b, c (addition is associative)
3. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b, c (multiplication is associative)
4. $a + b = b + a$ for all a, b (addition is commutative)
5. $a \cdot b = b \cdot a$ for all a, b (multiplication is commutative)
6. $a + 0 = a$ for all a (so 0 is neutral element of addition)
7. $a \cdot 1 = a$ for all a (so 1 is neutral element of multiplication)
8. for every a there is a' such that $a + a' = 0$ (existence of an inverse element for addition)
9. $a \cdot (b + c) = a \cdot b + a \cdot c$ for all a, b, c (multiplication distributesover addition)
10. for every $a \neq 0$ there is $\tilde{a}$ such that $a \cdot \tilde{a} = 1$ (multiplication has an inverse element for all non-zero numbers)

2.58. Check that

1. real numbers understood informally, have the properties listed above
2. rational numbers form a field

From the above field axioms, you can derive many facts that may be obvious to you:

2.59. Prove that there is only one 0 and only one 1. Hint: assume that 0 and 0' have property such that $a = a + 0 = a + 0'$ and try $a = 0$ and $a = 0'$.

2.60. Prove that if $a + a' = 0$ and $a + a'' = 0$, then $a' = a''$. Therefore we can introduce special symbol for *the* additiv inverse: $a + (-a) = 0$ and define subtraction as $a - b := a + (-b)$.

2.61. Prove that $-a = (-1) \cdot a$.

As you see, many of the algebraic properties we are used to can be recovered from the axioms, but sometimes it can be complicated. Both real numbers and rational numbers have also an order on them - for example $2 > 1$. It leads to the definition of *total order*. We call a pair $(F, \leq)$ a *totally ordered set* if for every $a, b \in F$ we have:

1. $a \leq b$ or $b \leq a$ (we call this property totality)
2. $a \leq b$ and $b \leq a$ imply $a = b$ (it's called antisymmetry)
3. $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity)

Having relation $\leq$ we can define other: $b \geq a$ means that $a \leq b$ and $a < b$ means that $a \leq b$ and $a \neq b$.

We say that tuple $(F, +, \cdot, 1, 0, \leq)$ is **ordered field** if:

- $(F, +, \cdot, 1, 0)$ is a field
- $(F, \leq)$ is totally ordered
- $a \leq b$ implies $a + c \leq b + c$
- $0 \leq a$ and $0 \leq b$ imply that $0 \leq a \cdot b$

You can check that reals and rationals are ordered fields. These axioms give us much more abilities, for example one is able to prove that $1 > 0$. But we still have no difference in properties that distuingish rationals from reals. This is called the completeness axiom and we will need a few more definitions.

Consider $A \subseteq \mathbb{R}$. We say that x is an **upper bound** of A iff $x \geq a$ for every $a \in A$.

2.62. Prove that a set $A \subseteq \mathbb{R}$ can have no upper bounds or infinitely many of them.

If an upper bound of A exists, we say that A is **bounded from above**. Among them we will distinguish the **supremum** (or **the least upper bound - l.u.b**): $x = \sup A$ iff x is an upper bound of A and for any upper bound y of A we have $x \leq y$.

2.63. Prove that supremum is unique, so if x and x' are supremums of A, then $x = x'$.

2.64. Prove that $x = \sup A$ if and only if $x \geq a$ for every $a \in A$ and for every $\varepsilon > 0$ there is $a \in A$ such that $x < a + \varepsilon$.

Now we can state the **completeness axiom**: each non-empty and bounded from above subset of real numbers has a supremum. This axiom allows us to prove many interesting things:

2.65. Prove that natural numbers are *not* bounded from above. Hint: if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$

2.66. Prove the **Archimedean axiom**[11] that for every $r \in R$, there is $n \in \mathbb{N}$ such that $n > r$.

[11] In fact we do not need to call it axiom, as we are able to prove it.

2.67. Prove that for any $r > 0$ there is $n \in \mathbb{N}$ such that $1/n < r$.

2.68. Prove that rational numbers do *not* have the completeness property:

1. Let $p, q \in \mathbb{Z} \setminus \{0\}$. Prove that $p^2 \neq 2q^2$.
2. Prove that root of two, defined as $x > 0, x^2 = 2$ is not rational.
3. Find a subset of $\mathbb{Q}$ that is bounded above, but has no rational supremum.

2.69. You should prove that in each nonempty interval there is at least one rational number:

1. Assume that $0 < a < b$. Define
$$A = \left\{ \frac{m}{N} : m \in \mathbb{N} \right\}, \quad \frac{1}{b-a} < N \in \mathbb{N}$$
and prove that $A \cap (a, b)$ is non-empty.
2. Use the above result to prove that in *each* interval there is at least one rational number.
3. Prove that in each interval there are infinitely but countably many, rational numbers.
4. Prove that in each interval there is an irrational number.
5. How many irrational numbers are in each interval?

2.6.1 Absolute value

Another concept that will be further useful is the **absolute value** of a real number: if $x \in \mathbb{R}$ we write $|x| \in \mathbb{R}$ for:

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{otherwise} \end{cases}.$$

2.70. Prove that for every $x, y \in \mathbb{R}$:

1. $|x| = |-x|$
2. if $|x| = |y|$ then $x = y$ or $x = -y$.
3. $|x + y| \leq |x| + |y|$ (this is called **triangle inequality**)
4. $|x - y| \leq |x| + |y|$
5. $||x| - |y|| \leq |x - y|$ (this is sometimes calles **reverse triangle inequality**)

3

General topology

We have studied sets and functions between them. Now, we will add a structure - called topology - to a set and look for special functions that will in some sense „preserve the structure". This subject is often called general topology or point-set topology as we will focus on points and special sets called open and closed sets. Eventually you will see, that these new concepts will enable us to derive very quickly and elegantly the majority of the results taught in undergraduate real analysis courses.

3.1 Basic definitions

3.1.1 Topology, open sets and interior

Consider a set X. A topology is a set $\mathcal{T}_X \subseteq 2^X$ such that:

1. $\varnothing, X \in \mathcal{T}_X$
2. if $A, B \in \mathcal{T}_X$, then $A \cap B \in \mathcal{T}_X$
3. if $A_i \in \mathcal{T}_X$ for $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{T}_X$

Members of topology we call **open sets**.

3.1. Using mathematical induction prove that the intersection of finitely many open sets is open.

You may wonder whether, for a given set, topology is unique. As you can prove, there can be many topologies.

3.2. Trivial topology Prove that for any X, set $\{\varnothing, X\}$ is a topology.

3.3. Discrete topology Prove that for any X, it's power set 2^X is a topology.

3.4. Cofinite topology Prove that for any X, the set: $\mathcal{T}_X = \{\varnothing\} \cup \{A \subseteq X : X \setminus A \text{ is finite}\}$ is a topology. Hint: think in terms of complements.

3.5. Subspace topology Let X be a set and $\mathcal{T}_X$ a topology on it. For $A \subseteq X$ we define $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}_X\}$. Prove that $\mathcal{T}_A$ is a topology on A.

3.6. For which sets, there is exactly one topology on them?

3.7. Prove that for an infinite set, there are at least three distinct topologies.

3.8. Real numbers $\mathbb{R}$ are usually equipped with the following topology: $X \subseteq \mathbb{R}$ is open iff for each $x \in X$ there is an open interval (a_x, b_x) such that $x \in (a_x, b_x) \subseteq X$.

1. Prove that it is indeed a topology.
2. Let **a ball** be a set $B(x,r) = \{y \in \mathbb{R} : |x - y| < r\}$ for $r > 0$. Prove that $(a, b) \neq \varnothing$ can be written as $B(x,r)$ for suitable x and r
3. Prove that X is open iff for each $x \in X$ there is a $r > 0$ such that $B(x,r) \subseteq X$. We will see that this results generalises to much broader category of spaces than single $\mathbb{R}$.

This gives the necessity of the notation of **topological space** that is a pair $(X, \mathcal{T}_X)$, where $\mathcal{T}_X$ is a topology on X. There are spaces, for which just one topology is commonly used in applications. In such situations mathematicians write topological space just as X, assuming that the „preferred" topology is obvious to the reader (as the topology given above on $\mathbb{R}$).

This was „global" point of view - we have a structure of subsets of X. We can also try to express these global properties using local properties - by considering special constructions around a single point and using a set of these points to recover a global property. Consider a topological space $(X, \mathcal{T}_X)$ and a point $x \in X$. If $x \in U \in \mathcal{T}_X$, we say that U is an open neighborhood of x. If $x \in U \subseteq V$, where U is open, we call V a neighborhood of x.

3.9. Prove that each point has an open neighborhood.

3.10. Prove that A is an open set if and only if each point a has a neighborhood $U_a \in A$ contained in A (that is $U_a \subseteq A$).

For a set A in a topological space, we define **the interior of** A as:

$$\operatorname{int} A = \bigcup \mathcal{U}, \text{ where } \mathcal{U} = \{U \in \mathcal{T} : U \subseteq A\}.$$

3.11. Prove that:

1. $\operatorname{int} A$ is an open set.
2. if $A' \subseteq A$ is open, then $A' \subseteq \operatorname{int} A$ (so in some sense, $\operatorname{int} A$ is the biggest open set contained in A)
3. $\operatorname{int} A = A$ iff A is open
4. $\operatorname{int} \operatorname{int} A = \operatorname{int} A$ for any A

3.12. Let $A' \subseteq A$. Prove that:

1. $\operatorname{int} A' \subseteq \operatorname{int} A$
2. $\operatorname{int} A \cup \operatorname{int} B \subseteq \operatorname{int}(A \cup B)$

You can prove also that the union can be arbitrary.

3.13. We say that a is an **interior point** of A if there is open $U_a \subseteq A$ such that $a \in U_a$. Prove that $\operatorname{int} A$ is the set of all interior points of A.

3.1.2 Closed sets and closure

Consider a topological space $(X, \mathcal{T}_X)$. We say that $A \subseteq X$ is closed if and only if $X \setminus A$ is open.

3.14. Prove these properties of closed sets in space $(X, \mathcal{T}_X)$:

1. $\varnothing$ and X are closed
2. If $A_1, A_2, \dots, A_n$ are closed, then their union $A_1 \cup A_2 \cup \dots \cup A_n$ is closed.
3. If $\mathcal{A}$ is any family of closed sets, then the intersection $\bigcap \mathcal{A}$ is closed.

3.15. We say that p is a limit point of $A \subseteq X$ if for every every open neighborhood U of p there is $q_U \neq p$ such that $q_U \in A \cap U$. Prove that A is closed iff it contains all of it's limit points.

We define **the closure** of a set A as:

$$\operatorname{cl} A = \bigcap \mathcal{X},$$

where $\mathcal{X} = \{X \subseteq A : X \text{ is closed}\}$.

3.16. Prove that:

1. $\operatorname{cl} A$ is a closed set.
2. if C is closed and $A \subseteq C$, then $\operatorname{cl} A \subseteq C$ (so in some sense, $\operatorname{cl} A$ is the smallest closed set containing A)
3. $A \subseteq \operatorname{cl} A$
4. $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$
5. $\operatorname{cl} A = A$ iff A is closed
6. $\operatorname{cl} \operatorname{cl} A = \operatorname{cl} A$ for any A

3.17. We say that p is an **adherent point** of A (or **point of closure**) if for any neighborhood V of p we have $A \cap V \neq \varnothing$. Alternatively, we can say that every neighborhood of p contains a point from A. Prove that $\operatorname{cl} A$ is the set of all adherent points of A.

3.18. We say that $A \subseteq X$ is **dense** if $\operatorname{cl} A = X$. Prove that A is dense iff for every $U \in \mathcal{T}_X$, $A \cap U \neq \varnothing$

3.19. 1. Let $r \in \mathbb{R}$. Prove that for every neighborhood V of r there is $q \in \mathbb{Q}$ such that $q \in V$. Hint: each neighborhood must have an interval. And you should have proven that in each interval there is a rational.
2. Conclude that rationals are dense in reals.

3.1.3 Boundary and exterior

We define the **boundary of** A as:

$$\partial A = \operatorname{fr} A = \operatorname{cl} A \setminus \operatorname{int} A$$

3.20. We say that p is a **frontier** point of A if every open neighborhood of p intersects both A and A^c, so if for every open neighborhood U_p we have $U_p \cap A \neq \varnothing$ and $U_p \cap A^c \neq \varnothing$. Prove that the boundary of A is exactly the set of frontier points of A.

3.21. Prove that boundary is always closed.

3.22. Prove that $\partial\partial A \subseteq \partial A$.

3.23. Prove that $\partial A = \partial A^c$.

3.24. Prove that $\partial A = \varnothing$ iff A is simultaneously open and closed.

We define the **exterior of** A as

$$\operatorname{ext} A = X \setminus \operatorname{cl} A$$

3.25. Prove that $\partial A = \operatorname{cl} A \cap \operatorname{cl} \operatorname{ext} A$

3.1.4 Bases and countability axioms

As we have seen, there can be many open sets. Let's try to simplify the situation by considering a smaller family of open sets from which we will be able to recover the whole topology.

Let $(X, \mathcal{T})$ be a topological space. We say that a family of sets $\mathcal{B} \subseteq \mathcal{T}$ is a **basis of topology** iff every open set can be written as a sum of a subfamily of $\mathcal{B}$. Namely for each $U \in \mathcal{T}$ there is $\mathcal{B}_U \subseteq \mathcal{B}$ such that:

$$U = \bigcup \mathcal{B}_U$$

3.26. Prove that $\mathcal{B}$ is a basis for $(X, \mathcal{T})$ iff for every $x \in X$ and every neighborhood U_i of x, there is $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq U_i$.

3.27. Let $\mathcal{B}$ be a basis of $(X, \mathcal{T})$. Prove that:

1. $\bigcup \mathcal{B} = X$
2. If $U, V \in \mathcal{T}$ and $x \in U \cap V$, then there is a set $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U \cap V$.

We say that a space $(X, \mathcal{T})$ is **second countable** iff it has a countable basis.

3.28. Consider $\mathbb{R}$ with it's standard topology. If $x \in \mathbb{R}$ and U is an open set containing x, we can find a ball $B(x, r),\ r > 0$ such that $B(x, r) \subseteq U$. Using the fact that rationals are dense in reals, prove that you can find $p, q \in \mathbb{Q}$ such that $x \in (p, q) \subseteq U$.

3.29. Prove that $\mathbb{R}$ is second countable.

3.2 Continuous maps and homeomorphisms

Consider two topological spaces $(X, \mathcal{T})$ and (Y, τ). We say that function (or **map**) $f : X \to Y$ is **continuous** iff for every open set $U \in \tau$, it's preimage is open: $f^{-1}(U) \in \mathcal{T}$.

3.30. Prove that a function is continuous iff preimage of every *closed* set is closed.

The next problem shows that we can employ the concept of a basis to shorten proofs:

3.31. Let $f : (X, \mathcal{T}) \to (Y, \tau)$ and $\mathcal{B}$ be a basis of (Y, τ). Then f is continuous iff $f^{-1}(B) \in \mathcal{T}$ for every $B \in \mathcal{B}$.

3.32. Assuming that $\mathbb{R}$ is equipped with it's standard topology, prove that functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous:

1. $f(x) = ax + b$
2. $f(x) = x^2$

3.33. Let $f : (X, \mathcal{T}) \to (Y, \tau)$. Prove that f is continuous iff $f(\operatorname{cl} A) \subseteq \operatorname{cl} f(A)$ for every $A \subseteq X$.

We say that a map $f : (X, \mathcal{T}) \to (Y, \tau)$ is a **homeomorphism**[1] iff is bijective and both f and f^{-1} are continuous. We say that two topological spaces are **homeomorphic** iff there is a homeomorphism between them. The aim of topology is to classify all the spaces up to homeomorphisms. This is a hard problem, but we can try to classify a smaller class of spaces:

3.34. Prove that two *discrete* spaces X and Y are homeomorphic iff $|X| = |Y|$.

3.3 Connected spaces

We say that a topological space $(X, \mathcal{T})$ is **disconnected** if there exists two disjoint, non-empty sets such their union is the whole space X. Or using symbols: $(X, \mathcal{T})$ is disconnected if $U, V \in \mathcal{T}$ such that $U, V \neq \varnothing,\ U \cap V = \varnothing,\ U \cup V = X$.

3.35. Let $(X, \mathcal{T})$ be a topological space. Prove that these conditions are equivalent:

1. The space is disconnected.
2. There are two *open* sets $A, B \subseteq X$ such that $A, B \neq \varnothing,\ A \cap B = \varnothing,\ A \cup B = X$.

[1] from the Ancient Greek *homois* - similar and *morphe* - shape

3. There are no two *closed* sets $A, B \subseteq X$ such that $A, B \neq \varnothing$, $A \cap B = \varnothing$, $A \cup B = X$.
4. There is a set $S \subset X$, $S \neq \varnothing, X$ such that and S is open and closed simultaneously (sometimes sets that are both open and closed are called **clopen**).
5. There is a set $S \subset X$, $S \neq \varnothing, X$ such that $\partial S = \varnothing$.
6. There are subsets $A, B \subseteq X$, $A, B \neq \varnothing$ such that $A \cap \operatorname{cl} B = B \cap \operatorname{cl} A = \varnothing$ and $A \cup B = X$.

If a space is not disconnected, it is called **connected**.

3.36. Let $(X, \mathcal{T})$ be a topological space. Prove that these conditions are equivalent:

1. The space is connected.
2. There are no two *open* sets $U, V \subseteq X$ such that $U, V \neq \varnothing$, $U \cap V = \varnothing$, $U \cup V = X$.
3. There are no two *closed* sets $U, V \subseteq X$ such that $U, V \neq \varnothing$, $U \cap V = \varnothing$, $U \cup V = X$.
4. The only sets that are open and closed simultaneously are $\varnothing$ and X.
5. All continous maps from $(X, \mathcal{T})$ to $(\{0, 1\}, \text{discrete topology})$ are constant.
6. If $S \subseteq X$ and $\partial S = \varnothing$, then $S = \varnothing$ or $S = X$.

4

Pseudometric spaces

We have already seen how general topology works. Now we will focus on another collection of spaces, with richer structure, called pseudometric spaces. We will follow Cain's approach (which is one of my favourite book on this subject).

4.1 Pseudometric spaces

Consider a set X. We say that a function $d : X \times X \to \mathbb{R}$ is a **pseudometric** if:

1. for all $x \in X$, $d(x, x) = 0$
2. for all $x, y \in X$, $d(x, y) = d(y, x)$
3. for all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

4.1. Prove that for a pseudometric d and every $x, y \in X$, there is $d(x, y) \geq 0$.

4.2. Prove that $d(x, y) = 0$ for any $x, y \in X$ is a pseudometric on X. This is called **trivial pseudometric**.

4.3. Prove that $d(x, y) = 1$ for $x \neq y$ is a pseudometric on X. This is called **discrete pseudometric**.

4.4. Prove that $d(x, y) = |x - y|$ is a pseudometric on $\mathbb{R}$.

As we can see above, pseudometric is not determined by the underlying set (as in the case with topology!). Therefore we introduce the concept of pseudometric space (X, d). As we said, these spaces have richer structure - each pseudometric space is a topological space, as you can prove in a minute. Essential concept is the concept of a ball of radius $r > 0$ centered at $x \in X$:

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

4.5. We say that $S \subseteq X$ is an open set if for every s in S there is r_s such that $B(s, r_s) \subseteq S$. Prove that this is indeed a topology on X.

4.6. Prove that

1. Topology obtained from trivial pseudometric is the trivial topology
2. Topology obtained from discrete pseudometric is the discrete topology.

As any pseudometric space is a topological space, we have many results and concepts that may work for them! In this chapter we will try to derive stronger results (we have more assumptions, so we can obtain more results). As we remember basis of the topology can simplify many results. Let's find it.

4.7. Let (X, d) be a pseudometric space. Prove that:

1. Any $B(x, r)$ is open
2. $\{B(x, r) : x \in X \text{ and } r > 0\}$ is a basis
3. This space is first-countable. Hint: consider $r = 1/n$ for $n = 1, 2, \ldots$.

4.2 Topology of $\mathbb{R}$

Now, we will focus on the „natural" topology of the real line. As we remember real numbers is a complete ordered field.

5

Linear algebra

5.1 Vector spaces

Exactly as we did with fields or topological spaces, we will define vector spaces as a tuple with some properties. **Vector space**[1] over a field F is a tuple: $(F, V, +, \cdot)$, where the objects involved are:

- F is a field. It's elements are called **scalars**.
- V is a set. It's elements are called **vectors**.
- $+$ is a function $V \times V \to V$. We always write $v + u$ instead of $+(v, u)$.
- $\cdot$ is a function $F \times V \to V$. We always write $f \cdot v$ instead of $\cdot(f, v)$.

They should have the following properties:

1. $(u + v) + w = u + (v + w)$ for every $u, v, w \in V$ (associativity of addition)
2. $u + v = v + u$ for every $u, v \in V$ (commutativity)
3. There is $o \in V$ such that $v + o = v$ for all v (neutral element of addition)
4. For every $v \in V$ there is a $\tilde{v} \in V$ such that $v + \tilde{v} = o$ (additive inverse)
5. For every $f, g \in F$ and $v \in V$ we have $(fg) \cdot V = f \cdot (g \cdot V)$ (so the multiplication agrees with that for scalars)
6. As F is a field, we have $1 \in F$. For every $v \in V$ we want $1 \cdot v = v$.
7. For every $f \in F$ and $u, v \in V$ we have $f \cdot (u + v) = f \cdot u + f \cdot v$ (distributivity)
8. For every $f, g \in F$ and $u \in V$ we have $(f + g) \cdot u = f \cdot u + g \cdot u$

5.1. We know that the set of vectors must contain at least one vector (neutral element). Construct a vector space that has *exactly* one vector (so in some sense it is the smallest space).

It's high time we started to abuse our notation making it less explicit, but more convenient. First of all we usually ommit (as in the case of field) $\cdot$ for multiplication: $fv = f \cdot v$, for $f \in F, v \in V$. But that's not all!

5.2. We want to modify our notation in the following way:

[1] Sometimes vector spaces are also called**linear spaces**

1. Prove that o is unique element with the property $v + o = v$ for $v \in V$
2. Prove that $0 \cdot v = o$ for every v. Hint: remember that $1 + 0 = 1$. This suggests to write 0 for o (so 0 since now technically has two different meanings, practically we will never have any problems with that)
3. Let $v \in V$ and $\tilde{v} \in V$ be such an element that $v + \tilde{v} = 0$. Prove that $\tilde{v}$ is unique (so if $v' + v = 0$, then $\tilde{v} = v'$)
4. Prove that the $\tilde{v}$ is exactly $(-1)v$. It suggests to write $-v$ for additive inverse, and we will do it since now.

Moreover, if we specify the field od scalars and operations, we will say V for the vector space, without invoking the all tuple elements (as in the case with topological spaces or fields- we write a single $\mathbb{R}$ and everyone knows what field operations we allow and what topology is assumed).

In most cases we will be interested in non-patological vector spaces, namely „big enough" in some sense. Why?

5.3. The **characteristic** of a field F is the smallest natual number n such that $1 + 1 + \cdots + 1 = 0$, where we have n ones on the left hand side. If there is no such number we say that characteristic is 0.

1. Prove that $\mathbb{R}$ has characteristic 0.
2. Let (F, V) be a vector space with at least two elements. Prove that $v = -v$ for every $v \in V$ if and only if the scalar field has characteristic 2.
3. Prove that if F has characteristic different from 2, then we have $v = -v$ iff $v = 0$.

5.4. An important example of a vector space over a field F is F^n, where addition and scalar multiplication are defined pointwise: $f \cdot (a_1, a_2, \dots, a_n) = (fa_1, fa_2, \dots, fa_n)$, $(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$. Prove that it is indeed a vector space.

5.1.1 Bases of vectos spaces

In topology we introduced a basis as a family of open sets from each every open set could be constructed in some natural way. This useful concept occurs also in vector spaces - as you can see, F^n has an interesting property: each element $(a_1, a_2, \dots, a_n) \in F^n$ can be written in a form $a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$, where e_k has 1 at k-th place and 0s in the other. Moreover, if $\lambda_1, \lambda_2, \dots, \lambda_n \in F$, then the only possibility for the equation $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n = 0$ to hold is $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. This suggests the following definitions: let $U \subseteq V$. A set $\operatorname{span} U$ is defined as:

$$\operatorname{span} U = \{\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n : \lambda_1, \dots, \lambda_n \in F, u_1, \dots, u_n \in U\}.$$

Alternatively, we can say that $\operatorname{span} U$ is a subset of V such that each $v \in \operatorname{span} U$ can be wrtitten as a finite **linear combination** of elements from U. It is important to *disallow* infinite combinations - the concept of an infinite sum is essentialy topological and we have *not* assumed any topology on our space yet! Therefore we cannot define convergence and infinite sums.

5.5. Consider infinite real sequences with addition and multiplication by a real number defined pointwise: $c = a + b$ iff $c_n = a_n + b_n$ for all n and $b = ra, r \in \mathbb{R}$ iff $b_n = ra_n$ for all n.

1. Prove that this is a vector space, let's call it $\mathbb{R}^{\mathbb{N}}$.
2. Prove that set $B = \{e_k : k \in \mathbb{N}\}$, e_k has 1 at k-th place and 0 at all the others, does *not* span $\mathbb{R}^{\mathbb{N}}$.
3. Let $\hat{\mathbb{R}}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ contain all the sequences that have a *finite* number of non-zero elements. Prove that this is a vector space and that it is spanned by B defined above.

Moreover we will say that a *finite* set of vectors $\{v_1, v_2, \ldots, v_n\}$ is **linearly independent** if the only solution for

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$$

is $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. If vectors are *not* linearly independent, we say that they are **linearly dependent**.

5.6. Assume that $v_1, v_2, \ldots, v_n$ are *not* linearly dependent. Prove that there is v_i such that v_i is a linear combination of other vectors: $v_i \in \operatorname{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, v_n\}$.

We say that a vector space V has **finite dimension** (or is *finite dimensional*) if there is a *finite* $U \subseteq V$ such that $V = \operatorname{span} U$.

5.7. You can prove that every finite dimensional vector space has a basis in two steps:

1. Assume that you have a set $\{e_1, e_2, \ldots, e_n\}$ that spans V. Prove that if $e_1 \in \operatorname{span}\{e_2, \ldots, e_n\}$, then $V = \operatorname{span}\{e_2, \ldots, e_n\}$.
2. Using the reduction step given above, show an algorithm finding a basis from a finite spanning set.
3. Prove that a vector space is finite dimensional iff it has a finite basis.

5.8. Prove that F^n is finite dimensional. Hint: just find a basis.

5.9. Prove that $\hat{\mathbb{R}}^{\mathbb{N}}$ has a basis.

Apparently the proof that *every* vector space is equivalent to the Axiom of Choice!

5.10. Here you will prove that all the bases of a *finite dimensional* vector space have the same number of elements. Let $v_1, v_2, \ldots, v_n$ be a basis of a vector space V and $w_1, w_2, \ldots, w_m \in V$, where $m > n$.

1. **(Steinitz exchange lemma)** Prove that if $w_1 \neq 0$, then $v_1 \in \operatorname{span}\{w_1, v_2, v_3, \ldots, v_n\}$.
2. Prove that if $w_k \neq 0$ for $k \in \{1, 2, \ldots, n\}$, then $w_{n+1} \in V = \operatorname{span}\{w_1, w_2, \ldots, w_n\}$

3. Prove that $w_1, w_2, \dots, w_m$ *cannot* be linearly independent.
4. Prove that each basis of V has the same number of elements. This number is called **the dimension of** V and written as $\dim V$.

5.11. Let V be a finite dimensional vector space of dimension n. Prove that every linearly independent set of n vectors spans V and every set with n elements spanning V is a basis.

5.12. Here you will prove that every linearly independent set of vectors can be extended to a basis of a finite dimensional vector space. Let V be a finite dimensional vector space of dimension n.

1. Let $S = \{v_1, v_2, \dots, v_k\} \subseteq V$ be linearly independent. Prove that if $u \in V$, but $u \notin \operatorname{span} S$, then $\{u\} \cup S$ is linearly independent.
2. Prove that there are $u_1, u_2, \dots, u_{n-k}$ such that $v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_{n-k}$ is a basis of V.

5.13. Assume that you have a basis $e_1, e_2, \dots, e_n$ of a finite dimensional vector space V over a field $\mathbb{F}$. Therefore every vector v can be written as a sum $v = v_1e_1 + v_2e_2 + \cdots + v_ne_n$ for some $v_i \in \mathbb{F}$. Prove that these numbers are unique, that is if $v = v_1e_1 + v_2e_2 + \cdots + v_ne_n = v_1'e_1 + v_2'e_2 + \cdots + v_n'e_n$, then $v_i = v_i'$ for all i. Hint: $0 = v - v$ and e_i are linearly independent.

5.1.2 Subspaces, direct sum and quotient spaces

As in the case of topological spaces, there are many ways of constructing *new* vector spaces from old. In topology we could construct new spaces taking a subset of a known topological space, take disjoint unions (sums) of topological spaces, divide topological spaces by some relations and take product of them. In this subsection we cover first three constructions - fourth one gives a raise to the concept of tensors and multilinear algebra and we will cover it in great detail later. Consider a vector space V over field F and a subset $U \subset V$ such that $0 \in U$ and for all $v, u \in U, f \in F$, we have $fv + u \in U$. You can check that it is indeed a vector space:

5.14. Prove that $fv + u \in U$ for all $v,\, u \in U, f \in F$ is equivalent to: for every $v,\, u \in U$, we have $v + u \in U$ and for every $v \in U, f \in F$ we have $fv \in U$.

Such a U we call a **vector subspace** of V.

5.15. Let V be a finite dimensional vector space and $U \subseteq V$ be a vector subspace. Prove that U is finite dimensional and $\dim U \leq \dim V$.

5.16. Let V be a finite dimensional vector space and $U \subseteq V$ be a vector subspace. Prove that $\dim U = \dim V$ iff $U = V$.

There is also a method of constructing direct sums: assume that you have two vector spaces V, W over *the same field* F. We define their direct sum as:

$$V \oplus W = V \times W = \{(v, w) : v \in V,\, w \in W\}$$

with addition and multiplication defined entrywise:

$$a(v, w) + (v', w') = (av + v', aw + w').$$

We often identify $v \in V$ with $(v, 0) \in V \oplus W$ and $w \in W$ with $(0, w) \in V \oplus W$. Then $av + bu, a, b \in F, v \in V, u \in U$ should be understood as $(av, bu) \in V \oplus U$.

5.17. Prove that each $w \in V \oplus U$ has a *unique* decomposition: $w = v + u$, $v \in V$, $u \in U$. That is if $w = v + u = v' + u'$, then $v = v'$ and $u = u'$ for $v, v' \in V$, $u, u' \in U$.

5.18. Let U and V be finite dimensional vector spaces. Prove that $\dim U \oplus V = \dim U + \dim V$.

5.19. Let V_1, V_2, V_n be finite dimensional vector spaces. Prove that

$$\dim V_1 \oplus V_2 \oplus \cdots \oplus V_n = \dim V_1 + \dim V_2 + \cdots + \dim V_n.$$

Our general definition has a very nice interpretation when we go to subspaces - now assume that you have two subspaces of V: $U, W \subseteq V$ such that $U \cap W = 0$. Their **direct product** is a set:

$$U \oplus W = \{u + v : u \in U,\, v \in V\}$$

5.20. Prove that the direct product of two vector subspaces is a special case of the general definition if we identify $U \ni u \leftrightarrow (u, 0) \in U \times V$, $V \ni v \leftrightarrow (0, v) \in U \times V$ employed.

5.21. Prove directly that direct product of two vector subspaces of V is a vector subspace of V. Hint: check if 0 is inside and use the handy, one-line criterion.

5.22. Let U be a subspace of V and V be a subspace of W. Prove that U is subspace of W.

5.23. Let $V = \mathbb{R}^2$ and $U = \{(0, r) : r \in \mathbb{R}\}$, $W = \{(r, 0) : r \in \mathbb{R}\}$. Prove that:

1. $V = U \oplus W$.
2. Prove that $U \cup V$ is *not* a vector space.

5.24. Let $U, W \subseteq V$ be two vector subspaces of a finite vector space V. Prove that:

1. $U \cap W$ is a subspace of U, W and V.

2. $U + W := \{u + w : u \in U, w \in W\}$ is a vector subspace[2] of V
3. Take a basis B_i of $U \cap W$ and extend it using some vectors $B_U \subseteq U$ such that $B_i \cup B_U$ is a basis of U. Repeat this procedure of W defining B_W and prove that $V = \operatorname{span} B_i \cup B_U \cup B_W$.
4. Prove that $B_i \cup B_U \cup B_V$ is linearly independent. Hint: write the condition of linear independence. Express the linear combination of the elements of B_U as a linear combination of B_i and B_V. Why is this linear combination in $U \cap V$? What you can conclude from the fact that $B_i \cup B_U$ is a basis?
5. Prove that $\dim U + V = \dim U + \dim V - \dim U \cap V$.

5.1.3 Quotient spaces

Let us introduce a relation:

5.25. Consider vector space V and it's subspace U. We introduce a relation on V: $v \approx u$ iff $v - u \in V$. Prove that $\approx$ is an equivalence relation.

We have an equivalence relation, so it splits V into equivalence classes $V/\approx$.

5.26. Prove that addition and scalar multiplication on $V/\approx$ are well-defined (independent on the class representative), that is:

1. if $v \approx v'$ and $u \approx u'$, then $v + u \approx v' + u'$
2. if α is a scalar and $v \approx v'$ are vectors in V, then $\alpha v \approx \alpha v'$.

5.27. Prove that under relation $\approx$, U is identified with 0.

We have vector addition and scalar multiplication, we have a neutral element - we have a new vector space! This vector space is called **quotient space** and usually written as V/U.

5.28. Let $U \subseteq V$ be finite-dimensional vector spaces. Prove that $\dim V/U = \dim V - \dim U$. Hint: guess what is the basis of V/U starting with basis of U and completing it to the basis ov V.

5.2 Linear maps

As continous functions are in some kind, natural mappings between topological spaces, we can define such natural mappings between vector spaces. Let V and W be vector spaces over the same field $\mathbb{F}$. We say that a function $L : V \to W$ is **linear** iff $L(\alpha v + \beta u) = \alpha L(v) + \beta L(u)$ for every $v, u \in V$, $\alpha, \beta \in \mathbb{F}$.

5.29. Let $L : V \to W$ be a function between vector spaces over field $\mathbb{F}$. Prove that the following sentences are equivalent:

[2] if $U \cap W = \varnothing$ we have just $U + W = U \oplus W$

1. L is linear
2. for every $u, v \in V$ and $\alpha \in \mathbb{F}$, we have $L(\alpha u + v) = \alpha L(u) + L(v)$
3. for every $v, u \in V$ we have $L(v + u) = L(v) + L(u)$ and for every $v \in V, \alpha \in \mathbb{F}$ we have $L(\alpha v) = \alpha L(v)$

5.30. Let U be a vector subspace of V. Prove that **the inclusion map** $\iota : U \to V$ given as $\iota(u) = u$ is linear.

5.31. Let U be a vector subspace of V. Prove that **the quotient map** $q : V \to V/U$ given as $q(v) = [v]$ is linear.

5.2.1 Kernel and cokernel

5.32. Let $L : V \to W$ be a linear map between vector spaces.

1. Prove that $L(0_V) = 0_W$, where 0_V is the neutral element in V and 0_W is the neutral element in W.
2. Prove that the **kernel of** L defined as: $\ker L = \{v \in V : L(v) = 0_W\}$ is a vector subspace of V.
3. Prove that the image of L is a vector subspace of W. The dimension of $\operatorname{im} L$ is called **the rank** of L: $\operatorname{rk} L = \dim \operatorname{im} L$.
4. Let $V = \operatorname{span} S$. Prove that $\operatorname{im} L = \operatorname{span} L(S)$. Here $L(S)$ has meaning $\{L(s) : s \in S\}$.
5. Prove that if V is finite dimensional, then $\operatorname{im} L$ is also finite dimensional.

Using kernel we can say whether a linear function is injective:

5.33. Prove that $\varphi : V \to U$ is injective iff $\ker \varphi = \{0\}$.

There is a similar concept, checking the surjectivity. We say that the cokernel of a linear map $L : V \to U$ is a *quotient* vector space: $U/\operatorname{im} L$.

5.34. Prove that $\varphi : V \to U$ is surjective iff $\operatorname{coker} \varphi$ has exactly one element (is a trivial vector space).

5.2.2 Rank-nullity theorem

You can prove the rank-nullity theorem:

5.35. Prove the **rank-nullity theorem** - if V is a finite dimensional vector space and $L : V \to W$ is linear, then $\dim \ker L + \operatorname{rk} L = \dim V$.

5.36. You can do a beuatiful and simple proof of $\dim U+V = \dim U + \dim V - \dim U \cap V$, where U and V are finite-dimensional subspaces of some greater space S. Consider a map $L : U \times V \to S$ defined as $L(u, w) = u - w$. Prove that it is a linear map (we give $U \times V$ a linear space structure using entry-wise operations, as in $U \oplus V$). What is it's range and kernel?

5.37. Let $\varphi : V \to U$ be a map between finite-dimensional vector spaces. Prove that the **index** of φ defined as $\operatorname{index}\varphi = \dim\ker\varphi - \dim\operatorname{coker}\varphi$ is equal to: $\operatorname{index}\varphi = \dim V - \dim U$.[3]

In the category of topological spaces, homeomorphic spaces have the same topological properties (as connectedness or Haussdorf property). We say that a map $L : W \to W$ is a **isomorphism**[4] **of vector spaces** iff is bijective and both L and L^{-1} are linear and we can say that abstract[5] vector spaces are identical if they are isomorphic.

5.38. Prove that V and $\{0\} \times V$ are isomorphic. (Do you remember the direct sum of two subspaces? In fact we used there this isomorphism).

You can also classify all finite-dimensional vector spaces over the same field, up to isomorphism:

5.39. Prove that two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension. Hint: right implication - rank-nullity theorem, left - write down bases.

As we remember $\mathbb{F}^n$ is an n-dimensional vector space over $\mathbb{F}$. Therefore many people learn how to work only on them, as all the vector space properties can be just transferred from $\mathbb{F}^n$. We will not do it, as the isomorphism is usually not-unique and does not preserve additional structures, very important in more advanced geometry. Such chosen isomorphisms will later give a raise to objects called metric tensors and symplectic forms.

5.2.3 Exact sequences

In algebraic topology topological spaces are investigated by assigning to them algebraic objects, like vector spaces or groups. One of frequently-occuring concept are exact sequences. Consider a sequence of vector spaces[6] $V_i,\ i \in \{1, 2, \dots\}$ and linear maps between them $\varphi_i : V_i \to V_{i+1}$. It is often written as:

$$\dots \xrightarrow{\varphi_{i-1}} V_i \xrightarrow{\varphi_i} V_{i+1} \xrightarrow{\varphi_{i+1}} \dots,$$

and map φ_i is referenced as $V_i \to V_{i+1}$. We say that the sequence is **exact** if $\operatorname{im}\varphi_i = \ker\varphi_{i+1}$ for all i.

[3] This is a very important result - index, defined with the help of a chosen function is doesn't in fact depend on this function! A beautiful generalisation of this result, is called Atiyah-Singer index theorem.

[4] from the Ancient Greek *isos* - equal and *morphe* - shape

[5] Later we will define additional structures on vector spaces for which just arbitrary isomorphisms are not sufficient

[6] Or, more generally, Abelian groups as we will need only to use such properties as kernel, image and quotient spaces.

5.40. Prove that if sequence of vector spaces V_i and maps $\varphi_i : V_i \to V_{i+1}$, is exact, then $\varphi_i \circ \varphi_{i+1} = \mathbf{0}$, where $\mathbf{0}$ is a null [7] map $\mathbf{0} : V_i \to V_{i+2}$ defined as $\mathbf{0}(v) = 0$.

Consider a 0-dimensional vector space $\{0\}$, which is usually abbreviated to just 0[8]. An exact sequence (Remember! Here 0 means $\{0\}$):

$$0 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \xrightarrow{\varphi_3} V_4 \xrightarrow{\varphi_4} 0$$

is called a **short exact sequence**. Longer exact sequences are called, obviously, **long exact sequences**.

5.41. Prove the following:

1. sequence $V \xrightarrow{\varphi} U \longrightarrow 0$ is exact iff φ is surjective. Hint: what is kernel of the map $U \to 0$?
2. sequence $0 \longrightarrow V \xrightarrow{\varphi} U$ is exact iff φ is injective. Hint: do you remember how injectivity is related to some kernel?
3. sequence $0 \longrightarrow V \xrightarrow{\varphi} U \longrightarrow 0$ is exact iff V and U are isomorphic.
4. short sequence $0 \longrightarrow V_2 \xrightarrow{\varphi_2} V_3 \xrightarrow{\varphi_3} V_4 \longrightarrow 0$ is exact iff φ_2 is injective and φ_3 is surjective.

5.42. Use the rank-nullity theorem and prove that if $0 \to V_0 \to V_1 \to \cdots \to V_{n-1} \to V_n \to 0$ is exact, then

$$0 = \sum_{i=0}^{n} (-1)^i \dim V_i.$$

5.43. Consider a linear map $L : V \to U$. Prove that the sequence:

$$0 \longrightarrow \ker L \xrightarrow{\kappa} V \xrightarrow{L} U \longrightarrow \operatorname{coker} L \longrightarrow 0$$

is exact, where $\kappa : \ker L \to V$ is inclusion map: $\kappa : \ker L \ni l \to l \in V$.

5.44. Let $L : V \to U$ be a map between finite-dimensional vector spaces. Prove once again that the index $L := \dim \ker L - \dim \operatorname{coker} L = \dim V - \dim U$.

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[7] Later we will refer to this map just as 0 - now this symbol has at least three meanings! It can be an additive neutral element of a field, a neutral element in a vector space or a linear map!

[8] Similar notational discrepancy was in the set theory - we wanted to write $f^{-1}(a)$ for $f^{-1}(\{a\})$.

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