1. Prove that the following sentences are true:

1. $\neg(\neg p) \Leftrightarrow p$
2. $p \vee \neg p$
3. $\neg(p \wedge q) = (\neg p) \vee (\neg q)$
4. $\neg(p \vee q) = (\neg p) \wedge (\neg q)$
5. $(p \Rightarrow q) \Leftrightarrow (\neg p) \vee q$
6. $0 \Rightarrow 1$

2. Prove that $\{1, 1, 2, 2, 2\} = \{1, 2\}$

3. Let X be a set built from all sets such that $A \notin A$. Prove that X does not exist. Hint: what if $X \in X$? What if $X \notin X$?

4. Let A, $B\,C$ be sets. Prove that:

1. $A \cup A = A$
2. $A \cup B = B \cup A$
3. $A \cup (B \cup C) = (A \cup B) \cup C$
4. $A \cap A = A$
5. $A \cap B = B \cap A$
6. $A \cap (B \cap C) = (A \cap B) \cap C$
7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
8. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

5. Prove that there is no set of all sets. Hint: assume there is one. Then you can select some sets to form a set that does not exist.

6. Prove that $A = B$ iff $A \subseteq B \wedge B \subseteq A$.

7. Prove the following set identites:

1. Let $A \subseteq B$. Prove that $(A^c)^c = A$.
2. Let A, $B \subset U$. Prove that $(A \cup B)^c = A^c \cap B^c$
3. Let A, $B \subset U$. Prove that $(A \cap B)^c = A^c \cup B^c$
4. $\{a \in A : a \in B\} = \{b \in B : b \in A\}$

8.

1. Let $A = \{1, 2, 3\}$. Find 2^A. What is the number of elements in 2^A? How is it related to the number of elements of A?

2. Let A be a finite set with n elements. Using the approach in which you choose which elements belong to a subset, prove that 2^A has 2^n elements.

9. Let $A_i \subseteq U$ for $i \in I$. Prove that the definitions below agree with the definitions for finite I.

$$\bigcup_{i \in I} A_i = \{a \in U : a \in A_i \text{ for at least one } i \in I\}$$
$$\bigcap_{i \in I} A_i = \{a \in U : a \in A_i \text{ for every } i \in I\}$$

10. Let $A_i \subseteq U$, $i \in I$ and

$$\sigma = \bigcup_{i \in I} A_i,\ \pi = \bigcap_{i \in I} A_i$$

Prove that:

1. if $k \in I$, then $A_k \cup \sigma = \sigma$

2. $\sigma \cap \pi = \pi$

11. Find infinite sum and intersection for the families of subsets of $\mathbb{R}$:

1. $A_i = (0, 1/i)$ for $i = 1, 2, \ldots$

2. $B_i = [0, 1/i)$ for $i = 1, 2, \ldots$

12. Let $A = \{\{a\}, \{a, b\}\}$, $B = \{\{c\}, \{c, d\}\}$. Prove that $A = B$ iff $a = c \wedge b = d$. Such a set A we call **the ordered pair** (a, b) as it has the property $(a, b) = (c, d)$ iff $a = c$ and $b = d$. Now you can forget how it has been constructed, and just remember this property.

13. Prove that $(a, (b, c)) = (d, (e, f))$ iff $a = d \wedge b = e \wedge c = f$.

14. Check that defining (a, b, c) as $((a, b), c)$ also works (so two ordered tuples are the same if they have the same first element, the same second element, ...)

15. Check that, in terms of sets, $(a, (b, c)) \neq ((a, b), c)$, so formally we do need to stick to one convention. However as we are interested in the property of ordered tuple, we will not distinguish them and denote both of them just as (a, b, c).

We define **Cartesian product** as $A \times B = \{(a, b) : a \in A \wedge b \in B\}$.

16. Do you remember the identification of $(a, (b, c))$ and $((a, b), c)$? Prove that $A \times (B \times C) = (A \times B) \times C$. Therefore we'll write it just as $A \times B \times C$ without parentheness.

17. You can prove that $2^n > n$ for every natural number n.

1. Prove that the formula works for $n = 0$ (punch the first domino).
2. Assume that for some n you proved on some way that $2^n > n$. Using this, prove that $2^{n+1} > (n + 1)$ (if n-th domino falls, then $n + 1$-th domino also falls)

You can also modify slightly the induction principle - sometimes you should start with number different than 0 or use different induction step (start 0 and step 2 can lead to theorems valid for even numbers, step 0 and steps 1 and -1 can lead to theorems valid for all integers...)

18.

1. Prove that 6 divides $n^3 - n$ for all natural n.
2. Prove that 6 divides $n^3 - n$ for all integers n. You can use a slight modification mathematical induction principle proving the implication „if the theorem works for n, it works also for $n - 1$".

19. (Bernoulli's inequality) Prove that for real $x > -1$ and natural $n \geqslant 1$, the following inequality holds:

$$(1 + x)^n \geqslant 1 + nx.$$

20. In Mathsland there are $n \geqslant 2$ cities. Between each pair of them there is a one-way road.

1. Prove that there is a city from which you can drive to all the other cities.
2. Prove that there is a city to which you can drive from all the others.

21. Let $S \subseteq R$. We say that S is **well-ordered** iff any non-empty subset $X \subset S$ has the smallest element.

1. Prove that reals and integers are not well-ordered.
2. Assume that $X \subseteq \mathbb{N}$ doesn't have the smallest element. Define $A = \{n \in \mathbb{N} : \{0, 1, \ldots, n\} \cap X = \emptyset\}$ and use mathematical induction to prove that X is empty.
3. Why are natural numbers well-ordered?

22. (Thanks to Antoni Hanke) How many are there functions from the empty set to $\{1, 2, 3, 4\}$?

23. Consider two functions: $f : \{0, 1\} \to \{0, 1\}$ given by $f(x) = 0$ and $g : \{0, 1\} \to \{0\}$. Prove that $f = g$.

24. Let $f : A \to B$ and $g : C \to B$, where $A \neq C$. Is it possible that $f = g$?

25. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions surjective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2$
3. $h : \mathbb{R} \to \{5\}$

26. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions injective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$
2. $h : \{0, 1, 2, 3\} \to \mathbb{R},\ h(x) = x$

27. Construct function that is:

1. surjective, but not injective
2. injective, but not surjective
3. neither injective nor surjective
4. bijective

Notice that if a function $f : A \to B$ is bijective, then we can construct a function $g : B \to A$ such that $f(g(b)) = b$ and $g(f(a)) = a$.

28. Prove that, if exists, g is unique.
We call this function **the inverse function**: $g = f^{-1}$.

29. Assume that f^{-1} exists. Prove that $(f^{-1})^{-1}$ exists and is equal to f.

If we have two functions: $f : A \to B$ and $g : B \to C$, we can construct the **composition** using formula: $g \circ f : A \to C,\ (g \circ f)(a) = g(f(a))$.

30. Find functions $f,\ g$ such that:

1. $g \circ f$ exists, but $f \circ g$ is not defined
2. both $f \circ g$ and $g \circ f$ exist, but $f \circ g \neq g \circ f$

Although function composition is not commutative, it is associative:

31. Left $f : A \to B, g : B \to C, h : C \to D$. Prove that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Therefore we can ommit the brackets and write just $h \circ g \circ f$. We will use function composition very often.

32.

1. Prove that composition of two surjections is surjective.
2. Prove that composition of two injections is injective.
3. Prove that composition of two bijections is bijective.

33. We will rephrase the definition of the inverse function as follows:

1. If X if a set, we define **the identity function**
$$\mathrm{Id}_X = \{(x,x) \in X^2 : x \in X\}.$$
Prove that it is indeed a function. What is it's domain?
2. Let $f : A \to B,\ g : B \to A$. Prove that $f = g^{-1}$ iff
$$g \circ f = \mathrm{Id}_A \text{ and } f \circ g = \mathrm{Id}_B$$

34. What is the cardinality of $\{a, a+1, a+2, \ldots, a+n\}$?

35. Let A, B and C be finite sets. Prove that:

1. $|2^A| = 2^{|A|}$
2. $|A \cup B| = |A| + |B|$ iff A and B are disjoint.
3. $|A \setminus B| = |A| - |B|$ if $B \subseteq A$.
4. $|A| \geqslant |B|$ if $B \subseteq A$. When does the equality hold?
5. $|A \cup B| = |A| + |B| - |A \cap B|$
6. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

36. Assume that A and B are finite sets. Prove that $|A| = |B|$ iff there is a bijection between A and B.

37. Above we find the way of saying that two cardinalities are equal using existence of a bijection. Let's find a way to compare which is less using another kind of function.

1. Let $O_n = \{1, 2, \ldots, n\}$. Prove that there is no injection from O_{n+1} into O_n. Hint: use mathematical induction.
2. Let A and B be finite. Prove that there is an injection from A to B iff $|A| \leqslant |B|$.

38. Using the above results, prove in one line that if there is an injection from A onto B and an injection from B into A, then there exists a bijection from A onto B.

We say that sets A and B have the same caridnalities (or $|A| = |B|$) iff there is a bijection between A and B.

39. Let A, B and C be sets. Prove that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. Hint: find the bijection between A and C.

40. Prove that:

1. $|\mathbb{N}| = |\mathbb{Z}|$.
2. $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
3. $|\mathbb{N}| = |\mathbb{Q}|$.

41. Prove that if $A \subseteq B$, then $|A| \leqslant |B|$.

42. Let A, B and C be sets. Prove that if $|A| \leqslant |B|$ and $|B| \leqslant |C|$, then $|A| \leqslant |C|$.

43. Here you can prove that there are more real numbers than naturals or rationals. We define $X = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\}$ and choose one convention of writing reals (e.g $0.999... = 1.000...$, so we can choose to use nines)

1. Assume that you have written all the elements of X in a single column. Can you find a real number that does not occur in the list?
2. Using the above, prove that $|\mathbb{N}| < |X|$
3. Prove that $|\mathbb{Q}| < |\mathbb{R}|$.

44. We know that $|\mathbb{R}| > |\mathbb{N}|$. Using binary system prove that $\mathbb{R} = 2^{\mathbb{N}}$. Do you see similarity between the previous result and $2^n > n$ for natural n?

45. Cantor's theorem You will prove that $|A| < \left|2^A\right|$ for any set A. Let A be a set and $f : A \to 2^A$.

1. Consider $X = \{a \in A : a \notin f(a)\} \in 2^A$. Is there $x \in A$ for which $f(x) = X$?
2. Is f surjective?
3. Find an injective function $g : A \to 2^A$.
4. Prove that $|A| < |2^A|$ for any set A.
5. Use Cantor's theorem to prove that there is no set of all sets.

46. Cantor-Schroeder-Bernstein theorem Let's prove that if $|A| \leqslant |B|$ and $|B| \leqslant |A|$, then $|A| = |B|$ for any sets.

1. (Knaster-Tarski) Now assume that F has *monotonicity* property: $F(X) \subseteq F(Y)$ if $X \subseteq Y$. Prove that $F(S) = S$, where:

$$S = \bigcup_{X \in U} X, \text{ where } U = \{Y \in 2^A : Y \subseteq f(Y)\}.$$

2. (Banach) Let $f : A \to B$ and $g : B \to A$ be injections. We introduce new symbol: $f[X] = \{b \in B : b = f(x) \text{ for some } x \in X\}$. Prove that function

$$F : 2^A \to 2^A,\ F(X) = A - g[B - f[X]]$$

has the monotonicity property.

3. Using the above statements we know that there is S defined above in the our case. Prove that function

$$h(x) = \begin{cases} f(x), x \in S \\ g^{-1}(x), x \notin S \end{cases}$$

is a bijection. You will need to prove that $X \setminus S \subseteq \operatorname{Im} g$.