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Mathematical Physics for Curious People

Geometrical approach

– Monograph –

January 31, 2018

Springer

For the people whom I learned mathematics from:
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Preface

Here come the golden words

place(s),
month year

First name Surname
First name Surname

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Part I

Mathematical introduction

1

Introduction

There are many excellent books on mathematical physics and differential geometry, so a question raises - how does this book differ from any other? I had a few aims working on it:

- target audience is just a normal person that wants to understand advanced mathematics. It does not matter if you are a physicists, mathematician, english literature major or a high-school student. If you have enough self-determination, you can understand the mathematics in this book
- this book should be self-containing. Mathematics is both broad and deep, so it must be split into different branches. But I found it discouraging that if you want to read one book, as prerequsities you need to read two other books, and so on. Here, you can understand everything without any access to libraries or other mathematical books. Obviously, we need to ommitt some Mathematics.
- we define the most fundamental concepts and then we show how they work together in a more specific setting. Many great lecturers show how an abstract concept works in a specific case, so they provide lots of examples. I would like to do an experiment - show abstract concepts, give huge amount of exercises
- I personally enjoy problem solving approach. Therefore I just do not prove theorems - I want you to prove them, with adjustable amount of hints.

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Logic and sets

If you are already fimilar with operations on logical formulas and sets, you may ommit this chapter.

2.1 Logical formulas

Consider declarative sentences as "Water boils at 100°C" or "2+2=5". We can construct new sentences from them using the following rules:

1. conjuntion (and): $p \wedge q$ is true if and only if p is true and q is true
2. disjuntion (or): $p \vee q$ is true if and only if at least one of sentences p, q is true
3. implication: $p \Rightarrow q$ is false if and only if p is true and q is false. Intuitively, if you know that p implies q and p is true, then q also must be true
4. negation (not): $\neg p$ is true if and only if p is false
5. equivalence (iff, if and only if): $p \Leftrightarrow q$ means exactly $(p \Rightarrow q) \wedge (q \Rightarrow p)$. Intuitively if you know that two sentences are equivalent and one of them is true, the other is also true. Or if one of them is false, the other one is automatically false.

Because mathematics is the art of being smart and lazy, we will assign value 1 to true sentences and 0 to false sentences.

2.1. Prove that the following sentences are true:

1. $\neg(\neg p) \Leftrightarrow p$
2. $p \vee \neg p$
3. $\neg(p \wedge q) = (\neg p) \vee (\neg q)$
4. $\neg(p \vee q) = (\neg p) \wedge (\neg q)$
5. $(p \Rightarrow q) \Leftrightarrow (\neg p) \vee q$
6. $0 \Rightarrow 1$

2.2. Prove that:

1. $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$
2. $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$

Equipped with this powerful machinery we can dive into basic set theory.

2.2 Basic set theory

2.2.1 Rough ideas

In modern mathematics we do not define a set nor set membership, so heuristically you can think that set A is a 'collection of objects' and $x \in A$ means that the object x is inside this collection. We will assume that any finite collections of elements $\{x_1, x_2, \dots, x_n\}$ is a set (the empty set is called $\varnothing$ rather than $\{\}$), moreover we will assume the existence of the following sets:

1. real numbers $\mathbb{R}$
2. natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$
3. integers $\mathbb{Z}$
4. rational numbers $\mathbb{Q}$

We say that two sets are equal ($A = B$) iff they have the same elements ($x \in A \Leftrightarrow x \in B$). Note, that we do not check how many times x appears in A. We can just say whether it inside or not.

2.3. Prove that $\{1, 1, 2, 2, 2\} = \{1, 2\}$

We assumed the existence of some sets at the beginning. Why? As you can prove not every 'collection of objects' is a set:

2.4. Let X be a set built from all sets such that $A \notin A$. Prove that X does not exist. Hint: what if $X \in X$? What if $X \notin X$?

Therefore at the moment we do not have many sets that we assume to exist. Let's try to define some methods of creating new sets from the know ones:

2.2.2 A few ways of constructing new sets

Assume that A and B are sets:

1. Let's make a formula F that for every element $a \in A$, the value $F(a)$ is true or false. We can then construct a set S with all the elements a from A for which the formula $F(a)$ holds. This set is written explicitly as $S = \{a \in A : F(a)\}$.
2. We can form the sum of two sets: $a \in A \cup B$ iff $a \in A$ or $a \in B$.
3. We can contruct the intersection of two sets: $a \in A \cap B$ iff $a \in A$ and $a \in B$.

4. We can construct the difference of two sets: $A \setminus B = \{a \in A : a \notin B\}$

2.5. Let A, $B\,C$ be sets. Prove that:

1. $A \cup A = A$
2. $A \cup B = B \cup A$
3. $A \cup (B \cup C) = (A \cup B) \cup C$
4. $A \cap A = A$
5. $A \cap B = B \cap A$
6. $A \cap (B \cap C) = (A \cap B) \cap C$
7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
8. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2.6. Prove that there is no set of all sets. Hint: assume there is one. Then you can select some sets to form a set that does not exist.

Moreover, we will introduce two new symbols, called positive and negative infinity: ∞ and $-\infty$. These are *not* real numbers, just symbols that are used to name a few useful sets:

$$\begin{aligned}
(-\infty, b) &= \{x \in \mathbb{R} : x < b\} \\
(-\infty, b] &= \{x \in \mathbb{R} : x \geq b\} \\
(a, \infty) &= \{x \in \mathbb{R} : a < x\} \\
[a, \infty) &= \{x \in \mathbb{R} : a \leq x\}
\end{aligned}$$

2.2.3 Subsets and complements

As we have some sets, we can try to compare them. We say that $A \subseteq B$ iff $a \in A \Rightarrow a \in B$ (or intuitively, each element of A is also in B. We say that A is a **subset** of B or that B is a **superset** of A.

2.7. Prove that $A = B$ iff $A \subseteq B \wedge B \subseteq A$.

2.8. Prove that for any set A, $\emptyset \subseteq A$.

2.9. Here you'll prove that there is just one empty set. Let $\emptyset$ and $\emptyset'$ be empty sets. Prove that $\emptyset = \emptyset'$.

If we fix the set B, to each subset A we can assign it's **complement**: $A^c = B \setminus A$. [1]

2.10. Prove the following set identites:

1. Let $A \subseteq B$. Prove that $(A^c)^c = A$.
2. Let A, $B \subset U$. Prove that $(A \cup B)^c = A^c \cap B^c$

[1] It is not the best symbol possible as we need to have B in mind.

3. Let A, $B \subset U$. Prove that $(A \cap B)^c = A^c \cup B^c$
4. $\{a \in A : a \in B\} = \{b \in B : b \in A\}$

Moreover, we assume that for a set A there exists it's **power set**: $2^A = \{X : X \subseteq A\}$.

2.11. 1. Let $A = \{1, 2, 3\}$. Find 2^A. What is the number of elements in 2^A? How is it related to the number of elements of A?
2. Let A be a finite set with n elements. Using the approach in which you choose which elements belong to a subset, prove that 2^A has 2^n elements.

2.2.4 Infinite collections of sets

Now we understand how to construct new sets from finite number of sets. But we can also consider „more general" families of sets, that are not necessarily finite: let $A_i \subset U$ for $i \in I$, where I is some indexing set. For example if $I = \{1, 2, \ldots, n\}$ we have a finite family. But you can imagine infinite families as $A_i = \{i\}$, $i \in \mathbb{R}$. How do we define the sum and intersection of them? We cannot sum them iteratively $A_1 \cup A_2 \cup \ldots$ as the process will never end, so we need alternative definitions:

$$\bigcup_{i \in I} A_i = \{a \in U : a \in A_i \text{ for at least one } i \in I\}$$
$$\bigcap_{i \in I} A_i = \{a \in U : a \in A_i \text{ for every } i \in I\}$$

2.12. Prove that for finite I these definitions agree with the previous.

2.13. Let $A_i \subseteq U$, $i \in I$ and

$$\sigma = \bigcup_{i \in I} A_i,\ \pi = \bigcap_{i \in I} A_i$$

Prove that:

1. if $k \in I$, then $A_k \cup \sigma = \sigma$
2. $\pi \subseteq \sigma$
3. $\sigma \cap \pi = \pi$

2.14. Find sum and intersection of family of subsets of $\mathbb{R}$: $A_r = \{r, -r\}$ for $r \geq 0$.

2.15. Let $A \subseteq X_i \subseteq U$ for $i \in I$. Prove that

$$A \subseteq \bigcup_{i \in I} X_i$$

2.16. For every point $a \in A$ there is a set $U_a \subseteq A$ such that $a \in U_a$. Prove that

$$A = \bigcup_{a \in A} U_a.$$

2.2.5 Cartesian product

First of all, we need a useful concept:

2.17. Let $A = \{\{a\}, \{a, b\}\}$, $B = \{\{c\}, \{c, d\}\}$. Prove that $A = B$ iff $a = c \wedge b = d$. Such a set A we call **the ordered pair** (a, b) as it has the property $(a, b) = (c, d)$ iff $a = c$ and $b = d$. Now you can forget how it has been constructed, and just remember this property.

2.18. Prove that $(a, (b, c)) = (d, (e, f))$ iff $a = d \wedge b = e \wedge c = f$.

Therefore it makes sense to write just (a, b, c) for $(a, (b, c))$ and define similarily such **ordered tuple** for four elements, five elements and so on.

2.19. Check that defining (a, b, c) as $((a, b), c)$ also works (so two ordered tuples are the same if they have the same first element, the same second element, ...)

2.20. Check that, in terms of sets, $(a, (b, c)) \neq ((a, b), c)$, so formally we do need to stick to one convention. However as we are interested in the property of ordered tuple, we will not distinguish them and denote both of them just as (a, b, c). Such notational problems appear in various places in mathematics, so we need to try to get used to them.

We can now introduce another way of creating new sets: let A and B be sets. Then we define their **Cartesian product** as

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

2.21. Do you remember the identification of $(a, (b, c))$ and $((a, b), c)$? Prove that $A \times (B \times C) = (A \times B) \times C$. Therefore we'll write it just as $A \times B \times C$ without parentheness.

Commonly used notation is $X^2 = X \times X = \{(x, y) : x, y \in X\}$ and analogously for other powers.

2.3 Natural numbers and mathematical induction

Have you ever seen falling dominoes? To be sure that every domino falls, we need to:

1. punch the first domino
2. for every domino we must be sure the implication: if this particular domino falls, the next one also falls

And that's all, we can be sure that all the dominoes will eventually fall. This style of reasoning[2] is called **mathematical induction** and formally it is written as: if $0 \in S$ and for every[3] $n \in N$ you can prove the implication $n \in S$ then $n+1 \in S$, you know that $N \subseteq S$.

2.22. You can prove that $2^n > n$ for every natural number n.

1. Prove that the formula works for $n = 0$ (punch the first domino).
2. Assume that for some n you proved on some way that $2^n > n$. Using this, prove that $2^{n+1} > (n+1)$ (if n-th domino falls, then $n+1$-th domino also falls)

You can also modify slightly the induction principle - sometimes you should start with number different than 0 or use different induction step (start 0 and step 2 can lead to theorems valid for even numbers, step 0 and steps 1 and -1 can lead to theorems valid for all integers...)

2.23. 1. Prove[4] that 6 divides $n^3 - n$ for all natural n.
2. Prove[5] that 6 divides $n^3 - n$ for all integers n. You can use a slight modification mathematical induction principle proving the implication „if the theorem works for n, it works also for $n-1$".

2.24. (Bernoulli's inequality) Prove that for real $x > -1$ and natural $n \geq 1$, the following inequality holds:

$$(1+x)^n \geq 1 + nx.$$

2.25. In Mathsland there are $n \geq 2$ cities. Between each pair of them there is a one-way road.

1. Prove that there is a city from which you can drive to all the other cities. Hint: assume that the hypothesis works for some n and any country with n cities. Now consider an arbitrary $n+1$-city country. Hide one city and use your assumption.
2. Prove that there is a city[6] to which you can drive from all the others.

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[2] We do not show here formally *why* this principle works. For curious, you define natural numbers in such way this principle works.

[3] I repeat: for every n we need to prove the implication „if works for n, then works for $n+1$". The correct way is to write "I assume that there is a given n for which the formula works. I will prove that is works for $n+1$". Common mistake is to write "I assume that the formula works for every n and I will prove that it works for $n+1$.". As professor Wiktor Bartol says - there is no need to prove the statement as you already assumed that it works in every case.

[4] Another method is to notice that $n^3 - n = (n-1) \cdot n \cdot (n+1)$. Why 2 does divide it? Why 3?

[5] How $n^3 - n$ and $(-n)^3 - (-n)$ are related? Does it simplify the proof?

[6] Nice trick: what does happen if you reverse each way? Can you use the former result?

2.26. Let $S \subseteq R$. We say that S is **well-ordered** iff any non-empty subset $X \subset S$ has the smallest element.

1. Prove that reals and integers are not well-ordered.
2. Assume that $X \subseteq \mathbb{N}$ doesn't have the smallest element. Define $A = \{n \in \mathbb{N} : \{0, 1, \ldots, n\} \cap X = \emptyset\}$ and use mathematical induction to prove that X is empty.
3. Why are natural numbers well-ordered?

2.4 Functions

2.4.1 Basics

Consider two sets A and B. We say that a subset $f \subseteq A \times B$ is a **function** iff the following two conditions hold:

- for every element $a \in A$ there is an element $b \in B$ such that $(a, b) \in f$
- if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$

Therefore for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$. Such b will be called **value of f at point** a and given a symbol $f(a)$.

2.27. (Thanks to Antoni Hanke) How many are there functions from the empty set to $\{1, 2, 3, 4\}$?

We need to introduce more terminology: set A is called **the domain of** f, set B is called **the codomain of** f and the function f is written as $f : A \to B$.

2.28. Consider two functions: $f : \{0, 1\} \to \{0, 1\}$ given by $f(x) = 0$ and $g : \{0, 1\} \to \{0\}$. Prove that $f = g$. [7]

2.29. Let $f : A \to B$ and $g : C \to B$, where $A \neq C$. Is it possible that $f = g$?

2.30. Let $f : A \to B$ and $C \subseteq D \subseteq A$. We define: $f[C] = \{b \in B : b = f(c) \text{ for some } c \in C\}$ and analogously $f[D]$. Prove that $f(C) \subseteq f(D)$.

2.4.2 Injectivity, surjectivity and bijectivity

As we have already seen, there may be some elements in codomain that are not values of f. We define **the image of** f as:

$$\operatorname{Im} f = \{b \in B : \text{there is } a \in A \text{ such that } b = f(a)\}.$$

We say that the function $f : A \to B$ is **surjective** (or **onto**) iff $\operatorname{Im} f = B$.

[7] Some mathematicians, as Bourbaki use an alternative definition of function - for them a function is the triple (A, B, f), where f is defined as in the our case. We see that this definition is incompatible with ours. Fortunately, as in the case with different definitions of ordered tuples, this problem will never occur explicitly in the further chapters.

2.31. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions surjective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2$
3. $h : \mathbb{R} \to \{5\}$

If $f(a)$ uniquely specifies a (if $f(a) = f(b)$, then $a = b$) we say that the function is **injective** (or **one-to-one**).

2.32. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions injective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$
2. $h : \{0, 1, 2, 3\} \to \mathbb{R},\ h(x) = x$

If a function f is both surjective and injective, we say that is *bijective*[8].

2.33. Construct function that is:

1. surjective, but not injective
2. injective, but not surjective
3. neither injective nor surjective
4. bijective

Notice that if a function $f : A \to B$ is bijective, then we can construct a function $g : B \to A$ such that $f(g(b)) = b$ and $g(f(a)) = a$.

2.34. Prove that, if exists, g is unique.

We call this function **the inverse function** [9]: $g = f^{-1}$.

2.35. Assume that f^{-1} exists. Prove that $(f^{-1})^{-1}$ exists and is equal to f.

2.4.3 Function composition

If we have two functions: $f : A \to B$ and $g : B \to C$, we can construct the **composition** using formula: $g \circ f : A \to C,\ (g \circ f)(a) = g(f(a))$.

2.36. Find functions $f,\ g$ such that:

1. $g \circ f$ exists, but $f \circ g$ is not defined
2. both $f \circ g$ and $g \circ f$ exist, but $f \circ g \neq g \circ f$

Although function composition is not commutative, it is associative:

[8] If you prefer nouns: surjective function is called surjection, injective - injection and bijective - bijection

[9] It becomes confusing when working on real numbers: $f^{-1}(x)$ is **not** $(f(x))^{-1} = 1/f(x)$

2.37. Left $f : A \to B, g : B \to C, h : C \to D$. Prove that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Therefore we can ommit the brackets and write just $h \circ g \circ f$. We will use function composition very often.

2.38. 1. Prove that composition of two surjections is surjective.
2. Prove that composition of two injections is injective.
3. Prove that composition of two bijections is bijective.

2.39. We will rephrase the definition of the inverse function as follows:

1. If X if a set, we define **the identity function**

$$\mathrm{Id}_X = \{(x, x) \in X^2 : x \in X\}.$$

   Prove that it is indeed a function. What is it's domain?
2. Let $f : A \to B,\ g : B \to A$. Prove that $f = g^{-1}$ iff

$$g \circ f = \mathrm{Id}_A \text{ and } f \circ g = \mathrm{Id}_B$$

2.5 Countability

2.5.1 Finite sets

For a finite set X we write the number of elements of X as $|X|$. We can calculate their **cardinalities** (sizes, numbers of elements) with ease,

2.40. What is the cardinality of $\{a, a + 1, a + 2, \ldots, a + n\}$?

2.41. Let A, B and C be finite sets. Prove that:

1. $|2^A| = 2^{|A|}$
2. $|A \cup B| = |A| + |B|$ iff A and B are disjoint.
3. $|A \setminus B| = |A| - |B|$ if $B \subseteq A$.
4. $|A| \geq |B|$ if $B \subseteq A$. When does the equality hold?
5. $|A \cup B| = |A| + |B| - |A \cap B|$
6. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

We can also employ functions to compare cardinalities:

2.42. Assume that A and B are finite sets. Prove that $|A| = |B|$ iff there is a bijection between A and B.

2.43. Above we find the way of saying that two cardinalities are equal using existence of a bijection. Let's find a way to compare which is less using another kind of function.

1. Let $O_n = \{1, 2, \dots, n\}$. Prove that there is no injection from O_{n+1} into O_n. Hint: use mathematical induction.
2. Let A and B be finite. Prove that there is an injection from A to B iff $|A| \leq |B|$.

2.44. Using the above results, prove in one line[10] that if there is an injection from A onto B and an injection from B into A, then there exists a bijection from A onto B.

2.5.2 Infinite sets

But how can we measure the number of elements of an infinite set, as $\mathbb{N}$ or $\mathbb{R}$? As natural numbers are „to small" we need to introduce new numbers, as $|\mathbb{N}|$ and be able to compare them. As we have seen above, the existence of a bijection is a good way of saying that two finite sets have equal cardinalities. It intuitively makes sense to employ this observation even in the infinite case: we say that sets (finite or infinite) A and B have the same caridnalities (or $|A| = |B|$) iff there is a bijection between A and B.

2.45. Let A, B and C be sets. Prove that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. Hint: find the bijection between A and C.

Here you can see the difference between finite and infinite sets - for finite sets a proper subset (a subset that is not the whole set) always has smaller number of elements. In the infinite case it is not true, as a proper subset can have *the same* number of elements.

2.46. Prove that:

1. $|\mathbb{N}| = |\mathbb{Z}|$.
2. $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
3. $|\mathbb{N}| = |\mathbb{Q}|$.

Analogously to the finite case, we define $|A| \leq |B|$ as the existence of an injection from A to B. We say that $|A| < |B|$ iff there is an injection from A to B but there is no bijection.

2.47. Prove that if $A \subseteq B$, then $|A| \leq |B|$.

2.48. Let A, B and C be sets. Prove that if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

2.49. Here you can prove that there are more real numbers than naturals or rationals. We define $X = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and choose one convention of writing reals (e.g 0.999... $=$ 1.000..., so we can choose to use nines)

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[10] The main step is $|A| \leq B$ and $|B| \leq |A|$, so $|A| = |B|$.

1. Assume that you have written all the elements of X in a single column. Can you find a real number that does not occur in the list?
2. Using the above, prove that $|\mathbb{N}| < |X|$
3. Prove that $|\mathbb{Q}| < |\mathbb{R}|$.

2.50. We know that $|\mathbb{R}| > |\mathbb{N}|$. Using binary system prove that $\mathbb{R} = 2^{\mathbb{N}}$. Do you see similarity between the previous result and $2^n > n$ for natural n?

2.51. Cantor's theorem You will prove that $|A| < \left|2^A\right|$ for any set A. Let A be a set and $f : A \to 2^A$.

1. Consider $X = \{a \in A : a \notin f(a)\} \in 2^A$. Is there $x \in A$ for which $f(x) = X$?
2. Is f surjective?
3. Find an injective function $g : A \to 2^A$.
4. Prove that $|A| < |2^A|$ for any set A.
5. Use Cantor's theorem to prove that there is no set of all sets.

2.52. Cantor-Schroeder-Bernstein theorem Let's prove that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$ for any sets.

1. (Knaster-Tarski) Now assume that F has *monotonicity* property: $F(X) \subseteq F(Y)$ if $X \subseteq Y$. Prove that $F(S) = S$, where:
$$S = \bigcup_{X \in U} X, \text{ where } U = \{Y \in 2^A : Y \subseteq f(Y)\}.$$
2. (Banach) Let $f : A \to B$ and $g : B \to A$ be injections. We introduce new symbol: $f[X] = \{b \in B : b = f(x) \text{ for some } x \in X\}$. Prove that function
$$F : 2^A \to 2^A,\ F(X) = A - g[B - f[X]]$$
has the monotonicity property.
3. Using the above statements we know that there is S defined above in the our case. Prove that function
$$h(x) = \begin{cases} f(x), x \in S \\ g^{-1}(x), x \notin S \end{cases}$$
is a bijection. You will need to prove that $X \setminus S \subseteq \operatorname{Im} g$.

2.5.3 Pre-image of a function

Let $f : A \to B$ and $C \subseteq A$. We used $f[C]$ for a set:
$$f[C] = \{f(c) \in B : c \in C\},$$
but now we will abuse a bit our notation to stick to the common nomenclature. Apparently, many mathematicians write:

$$f(C) = \{f(c) \in B : c \in C\}.$$

This is not correct - as f should take elements $a \in A$ and returns elements $b \in B$, but here f „takes" a subset $C \subseteq A$ and returns a set $f(C) \subseteq B$. We will follow this notation, but you should always check what meaning the object feed to function has (whether it is an element or a subset).

2.53. Let $f : A \to B$ and $X, Y \subseteq B$. Then:

1. $f(X \cup Y) = f(X) \cup f(Y)$
2. $f(X \cap Y) \subseteq f(X) \cap f(Y)$

You can also generalise this result to an arbitrary collection of sets.

To even more abuse the notation, we will also give an additional meaning to f^{-1}. As we know, many functions f *don't* have inverses. But we will write for $D \subseteq B$:

$$f^{-1}(D) = \{a \in A : f(a) \in D\} \subseteq A.$$

We then say that $f^{-1}(D)$ is the **pre-image** of D. To get accustomed with this notation, prove that:

2.54. Let $f : A \to B$. Then $B = f(A)$ and $A = f^{-1}(B)$.

You should also prove:

2.55. Let $f : A \to B$ and $X, Y \subseteq B$. Then:

1. $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$
2. $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

You can also generalise this result to an arbitrary collection of sets.

3

General topology

Now we are prepared enough to dive into general topology (often called point-set topology).

3.1 Basics

3.1.1 Topology and open sets

Consider a set X. A topology is a set $\mathcal{T}_X \subseteq 2^X$ such that:

1. $\varnothing, X \in \mathcal{T}_X$
2. if $A, B \in \mathcal{T}_X$, then $A \cap B \in \mathcal{T}_X$
3. if $A_i \in \mathcal{T}_X$ for $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{T}_X$

Members of topology we call **open sets**.

3.1. Using mathematical induction prove that the intersection of finitely many open sets is open.

You may wonder whether, for a given set, topology is unique. As you can prove, there can be many topologies.

3.2. Trivial topology Prove that for any X, set $\{\varnothing, X\}$ is a topology.

3.3. Discrete topology Prove that for any X, it's power set 2^X is a topology.

3.4. Cofinite topology Prove that for any X, the set: $\mathcal{T}_X = \{\emptyset\} \cup \{A \subseteq X : X \setminus A \text{ is finite}\}$ is a topology.

3.5. For which sets, there is exactly one topology on them? (So at least these listed above must be the same).

3.6. Prove that for an infinite set, there are at least three distinct topologies.

This gives the necessity of the notation of **topological space** that is a pair $(X, \mathcal{T}_X)$, where $\mathcal{T}_X$ is a topology on X. There are spaces, for which just one topology is commonly used in applications. In such situations mathematicians write topological space just as X, assuming that the „preferred" topology is obvious to the reader. Consider a topological space $(X, \mathcal{T}_X)$ and a point $x \in X$. If $x \in U \in \mathcal{T}_X$, we say that U is an open neighborhood of x.

3.7. Prove that each points has an open neighborhood.

3.8. Prove that A is an open set if and only if each point a has a neighborhood $U_a \in A$ contained in A (that is $U_a \subseteq A$).

For a set A in a topological space, we define **the interior of** A as:

$$\operatorname{Int} A = \bigcup_{X \text{ is open and is a subset of A}} X.$$

3.9. Prove that:

1. $\operatorname{Int} A$ is an open set.
2. if $A' \subseteq A$ is open, then $A' \subseteq \operatorname{Int} A$ (so in some sense, $\operatorname{Int} A$ is the biggest open set contained in A)
3. $\operatorname{Int} A = A$ iff A is open
4. $\operatorname{Int}(A \cup B) \subseteq \operatorname{Int} A \cup \operatorname{Int} B$

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