

Logic and sets**Logical formulas**

1. Prove that the following sentences are true:

1. $\neg(\neg p) \Leftrightarrow p$
2. $p \vee \neg p$
3. $\neg(p \wedge q) = (\neg p) \vee (\neg q)$
4. $\neg(p \vee q) = (\neg p) \wedge (\neg q)$
5. $(p \Rightarrow q) \Leftrightarrow (\neg p) \vee q$
6. $0 \Rightarrow 1$

2. Prove that:

1. $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$
2. $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$

Basic set theory**Rough ideas**

3. Prove that $\{1, 1, 2, 2, 2\} = \{1, 2\}$

4. Let X be a set built from all sets such that $A \notin A$. Prove that X does not exist. Hint: what if $X \in X$? What if $X \notin X$?

A few ways of constructing new sets

5. Let A, B, C be sets. Prove that:

1. $A \cup A = A$
2. $A \cup B = B \cup A$
3. $A \cup (B \cup C) = (A \cup B) \cup C$
4. $A \cap A = A$
5. $A \cap B = B \cap A$
6. $A \cap (B \cap C) = (A \cap B) \cap C$
7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
8. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

6. Prove that there is no set of all sets. Hint: assume there is one. Then you can select some sets to form a set that does not exist.

Subsets and complements

7. Prove that $A = B$ iff $A \subseteq B \wedge B \subseteq A$.

8. Prove that for any set A , $\emptyset \subseteq A$.

9. Here you'll prove that there is just one empty set. Let \emptyset and \emptyset' be empty sets. Prove that $\emptyset = \emptyset'$.

10. Prove the following set identities:

1. Let $A \subseteq B$. Prove that $(A^c)^c = A$.
2. Let $A, B \subset U$. Prove that $(A \cup B)^c = A^c \cap B^c$
3. Let $A, B \subset U$. Prove that $(A \cap B)^c = A^c \cup B^c$
4. $\{a \in A : a \in B\} = \{b \in B : b \in A\}$

11.

1. Let $A = \{1, 2, 3\}$. Find 2^A . What is the number of elements in 2^A ? How is it related to the number of elements of A ?
2. Let A be a finite set with n elements. Using the approach in which you choose which elements belong to a subset, prove that 2^A has 2^n elements.

Infinite collections of sets

12. Prove that for finite families of sets, these new definitions agree with the previous.

13. Let $\mathcal{X} \subset 2^U$ be a family of sets and define: $\mathcal{Y} = \{X^c : X \in \mathcal{X}\}$, where $X^c = U \setminus X$. Prove that:

1. $(\bigcup \mathcal{X})^c = \bigcap \mathcal{Y}$
2. $(\bigcap \mathcal{X})^c = \bigcup \mathcal{Y}$

14. Let $A_i \subseteq U$, $i \in I$ and

$$\sigma = \bigcup_{i \in I} A_i, \pi = \bigcap_{i \in I} A_i$$

Prove that:

1. if $k \in I$, then $A_k \cup \sigma = \sigma$
2. $\pi \subseteq \sigma$
3. $\sigma \cap \pi = \pi$

15. Find sum and intersection of family of subsets of \mathbb{R} : $A_r = \{r, -r\}$ for $r \geq 0$.

16. Let $A \subseteq X_i \subseteq U$ for $i \in I$. Prove that

$$A \subseteq \bigcup_{i \in I} X_i$$

17. For every point $a \in A$ there is a set $U_a \subseteq A$ such that $a \in U_a$. Prove that

$$A = \bigcup_{a \in A} U_a.$$

Cartesian product

18. Let $A = \{\{a\}, \{a, b\}\}$, $B = \{\{c\}, \{c, d\}\}$. Prove that $A = B$ iff $a = c \wedge b = d$. Such a set A we call **the ordered pair** (a, b) as it has the property $(a, b) = (c, d)$ iff $a = c$ and $b = d$. Now you can forget how it has been constructed, and just remember this property.

19. Prove that $(a, (b, c)) = (d, (e, f))$ iff $a = d \wedge b = e \wedge c = f$.

- 20.** Check that defining (a, b, c) as $((a, b), c)$ also works (so two ordered tuples are the same if they have the same first element, the same second element, ...)
- 21.** Check that, in terms of sets, $(a, (b, c)) \neq ((a, b), c)$, so formally we do need to stick to one convention. However as we are interested in the property of ordered tuple, we will not distinguish them and denote both of them just as (a, b, c) . Such notational problems appear in various places in mathematics, so we need to try to get used to them.
- 22.** Do you remember the identification of $(a, (b, c))$ and $((a, b), c)$? Prove that $A \times (B \times C) = (A \times B) \times C$. Therefore we'll write it just as $A \times B \times C$ without parenthesess.

Natural numbers and mathematical induction

- 23.** You can prove that $2^n > n$ for every natural number n .
1. Prove that the formula works for $n = 0$ (punch the first domino).
 2. Assume that for some n you proved on some way that $2^n > n$. Using this, prove that $2^{n+1} > (n + 1)$ (if n -th domino falls, then $n + 1$ -th domino also falls)
- 24.**
1. Prove¹ that 6 divides $n^3 - n$ for all natural n .
 2. Prove² that 6 divides $n^3 - n$ for all integers n . You can use a slight modification mathematical induction principle proving the implication „if the theorem works for n , it works also for $n - 1$ ”.
- 25.** (Bernoulli's inequality) Prove that for real $x > -1$ and natural $n \geq 1$, the following inequality holds:

$$(1 + x)^n \geq 1 + nx.$$

- 26.** In Mathsland there are $n \geq 2$ cities. Between each pair of them there is a *one-way* road.
1. Prove that there is a city from which you can drive to all the other cities. Hint: assume that the hypothesis works for some n and any country with n cities. Now consider an arbitrary $n + 1$ -city country. Hide one city and use your assumption.
 2. Prove that there is a city³ to which you can drive from all the others.
- 27.** Let $S \subseteq \mathbb{R}$. We say that S is **well-ordered** iff any non-empty subset $X \subset S$ has the smallest element.
1. Prove that reals and integers with the default ordering are not well-ordered.
 2. Assume that $X \subseteq \mathbb{N}$ doesn't have the smallest element. Define $A = \{n \in \mathbb{N} : \{0, 1, \dots, n\} \cap X = \emptyset\}$ and use mathematical induction to prove that X is empty.
 3. Why are natural numbers well-ordered?

Functions

Basics

- 28.** (Thanks to Antoni Hanke) How many are there functions from the empty set to $\{1, 2, 3, 4\}$?

¹Another method is to notice that $n^3 - n = (n - 1) \cdot n \cdot (n + 1)$. Why 2 does divide it? Why 3?

²How $n^3 - n$ and $(-n)^3 - (-n)$ are related? Does it simplify the proof?

³Nice trick: what does happen if you reverse each way? Can you use the former result?

- 29.** Consider two functions: $f : \{0, 1\} \rightarrow \{0, 1\}$ given by $f(x) = 0$ and $g : \{0, 1\} \rightarrow \{0\}$. Prove that $f = g$.⁴
- 30.** Let $f : A \rightarrow B$ and $g : C \rightarrow B$, where $A \neq C$. Is it possible that $f = g$?
- 31.** Let $f : A \rightarrow B$ and $C \subseteq D \subseteq A$. We define: $f[C] = \{b \in B : b = f(c) \text{ for some } c \in C\}$ and analogously $f[D]$. Prove that $f(C) \subseteq f(D)$.

Injectivity, surjectivity and bijectivity

- 32.** As we remember, \mathbb{R} stands for well-known real numbers. Are the following functions surjective?
1. $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3$
 2. $g : \mathbb{R} \rightarrow \mathbb{R}, g(x) = x^2$
 3. $h : \mathbb{R} \rightarrow \{5\}$
- 33.** As we remember, \mathbb{R} stands for well-known real numbers. Are the following functions injective?
1. $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$
 2. $h : \{0, 1, 2, 3\} \rightarrow \mathbb{R}, h(x) = x$
- 34.** Construct function that is:
1. surjective, but not injective
 2. injective, but not surjective
 3. neither injective nor surjective
 4. bijective
- 35.** Prove that, if exists, g is unique.
- 36.** Assume that f^{-1} exists. Prove that $(f^{-1})^{-1}$ exists and is equal to f .

Function composition

- 37.** Find functions f, g such that:
1. $g \circ f$ exists, but $f \circ g$ is not defined
 2. both $f \circ g$ and $g \circ f$ exist, but $f \circ g \neq g \circ f$
- 38.** Let $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$. Prove that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- 39.**
1. Prove that composition of two surjections is surjective.
 2. Prove that composition of two injections is injective.
 3. Prove that composition of two bijections is bijective.

⁴Some mathematicians, as Bourbaki use an alternative definition of function - for them a function is the triple (A, B, f) , where f is defined as in the our case. We see that this definition is incompatible with ours. Fortunately, as in the case with different definitions of ordered tuples, this problem will never occur explicitly in the further chapters.

40. We will rephrase the definition of the inverse function as follows:

1. If X is a set, we define **the identity function**

$$\text{Id}_X = \{(x, x) \in X^2 : x \in X\}.$$

Prove that it is indeed a function. What is its domain?

2. Let $f : A \rightarrow B$, $g : B \rightarrow A$. Prove that $f = g^{-1}$ iff

$$g \circ f = \text{Id}_A \text{ and } f \circ g = \text{Id}_B$$

41. Let $f : A \rightarrow B$ be an injection. Prove that there is a function $g : \text{Im } f \rightarrow A$ such that $g \circ f = \text{Id}_A$. Such g is called **left inverse of f** .

Countability

Finite sets

42. What is the cardinality of $\{a, a + 1, a + 2, \dots, a + n\}$?

43. Let A , B and C be finite sets. Prove that:

1. $|2^A| = 2^{|A|}$
2. $|A \cup B| = |A| + |B|$ iff A and B are disjoint.
3. $|A \setminus B| = |A| - |B|$ if $B \subseteq A$.
4. $|A| \geq |B|$ if $B \subseteq A$. When does the equality hold?
5. $|A \cup B| = |A| + |B| - |A \cap B|$
6. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

44. Assume that A and B are finite sets. Prove that $|A| = |B|$ iff there is a bijection between A and B .

45. Above we find the way of saying that two cardinalities are equal using existence of a bijection. Let's find a way to compare which is less using another kind of function.

1. Let $O_n = \{1, 2, \dots, n\}$. Prove that there is no injection from O_{n+1} into O_n . Hint: use mathematical induction.
2. Let A and B be finite. Prove that there is an injection from A to B iff $|A| \leq |B|$.

46. Using the above results, prove in one line⁵ that if there is an injection from A onto B and an injection from B into A , then there exists a bijection from A onto B .

Infinite sets

47. Let A , B and C be sets. Prove that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. Hint: find the bijection between A and C .

48. Prove that:

1. $|\mathbb{N}| = |\mathbb{Z}|$.
2. $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
3. $|\mathbb{N}| = |\mathbb{Q}|$.

⁵The main step is $|A| \leq |B|$ and $|B| \leq |A|$, so $|A| = |B|$.

49. Prove that if $A \subseteq B$, then $|A| \leq |B|$.

50. Let A , B and C be sets. Prove that if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

51. Here you can prove that there are more real numbers than naturals or rationals. We define $X = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and choose one convention of writing reals (e.g $0.999\dots = 1.000\dots$, so we can choose to use nines)

1. Assume that you have written all the elements of X in a single column. Can you find a real number that does not occur in the list?
2. Using the above, prove that $|\mathbb{N}| < |X|$
3. Prove that $|\mathbb{Q}| < |\mathbb{R}|$.

52. We know that $|\mathbb{R}| > |\mathbb{N}|$. Using binary system prove that $\mathbb{R} = 2^{\mathbb{N}}$. Do you see similarity between the previous result and $2^n > n$ for natural n ?

53. Cantor's theorem You will prove that $|A| < |2^A|$ for any set A . Let A be a set and $f : A \rightarrow 2^A$.

1. Consider $X = \{a \in A : a \notin f(a)\} \in 2^A$. Is there $x \in A$ for which $f(x) = X$?
2. Is f surjective?
3. Find an injective function $g : A \rightarrow 2^A$.
4. Prove that $|A| < |2^A|$ for any set A .
5. Use Cantor's theorem to prove that there is no set of all sets.

54. Cantor-Schroeder-Bernstein theorem Let's prove that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$ for any sets.

1. (Knaster-Tarski) Now assume that F has *monotonicity* property: $F(X) \subseteq F(Y)$ if $X \subseteq Y$. Prove that F has a fixed point S (that is $F(S) = S$), where:

$$S = \bigcup_{X \in U} X, \text{ where } U = \{Y \in 2^A : Y \subseteq f(Y)\}.$$

2. (Banach) Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be injections. We introduce new symbol: $f[X] = \{b \in B : b = f(x) \text{ for some } x \in X\}$. Prove that function

$$F : 2^A \rightarrow 2^A, F(X) = A \setminus g[B \setminus f[X]]$$

has the monotonicity property.

3. Prove that $A \setminus S \subseteq \text{Im } g$, where F and S are taken from above.
4. Prove that function

$$h(x) = \begin{cases} f(x), & x \in S \\ g^{-1}(x), & x \notin S \end{cases}$$

is a bijection.

Pre-image of a function

55. Let $f : A \rightarrow B$ and $X, Y \subseteq B$. Then:

1. $f(X \cup Y) = f(X) \cup f(Y)$
2. $f(X \cap Y) \subseteq f(X) \cap f(Y)$

You can also generalise this result to an arbitrary collection of sets.

56. Let $f : A \rightarrow B$. Then $f(A) \subseteq B$ and $A = f^{-1}(B)$.

57. Let $f : A \rightarrow B$ and $X, Y \subseteq B$. Then:

1. $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$
2. $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

You can also generalise this result to an arbitrary collection of sets.

Real numbers

58. Check that

1. real numbers understood informally, have the properties listed above
2. rational numbers form a field

59. Prove that there is only one 0 and only one 1. Hint: assume that 0 and 0' have property such that $a = a + 0 = a + 0'$ and try $a = 0$ and $a = 0'$.

60. Prove that if $a + a' = 0$ and $a + a'' = 0$, then $a' = a''$. Therefore we can introduce special symbol for *the* additive inverse: $a + (-a) = 0$ and define subtraction as $a - b := a + (-b)$.

61. Prove that $-a = (-1) \cdot a$.

62. Prove that a set $A \subseteq \mathbb{R}$ can have no upper bounds or infinitely many of them.

63. Prove that supremum is unique, so if x and x' are supremums of A , then $x = x'$.

64. Prove that $x = \sup A$ if and only if $x \geq a$ for every $a \in A$ and for every $\varepsilon > 0$ there is $a \in A$ such that $x < a + \varepsilon$.

65. Prove that natural numbers are *not* bounded from above. Hint: if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$

66. Prove the **Archimedean axiom**⁶ that for every $r \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $n > r$.

67. Prove that for any $r > 0$ there is $n \in \mathbb{N}$ such that $1/n < r$.

68. Prove that rational numbers do *not* have the completeness property:

1. Let $p, q \in \mathbb{Z} \setminus \{0\}$. Prove that $p^2 \neq 2q^2$.
2. Prove that root of two, defined as $x > 0, x^2 = 2$ is not rational.
3. Find a subset of \mathbb{Q} that is bounded above, but has no rational supremum.

69. You should prove that in each nonempty interval there is at least one rational number:

1. Assume that $0 < a < b$. Define

$$A = \left\{ \frac{m}{N} : m \in \mathbb{N} \right\}, \quad \frac{1}{b-a} < N \in \mathbb{N}$$

and prove that $A \cap (a, b)$ is non-empty.

2. Use the above result to prove that in *each* interval there is at least one rational number.
3. Prove that in each interval there are infinitely but countably many, rational numbers.
4. Prove that in each interval there is an irrational number.
5. How many irrational numbers are in each interval?

⁶In fact we do not need to call it axiom, as we are able to prove it.

Absolute value

70. Prove that for every $x, y \in \mathbb{R}$:

1. $|x| = |-x|$
2. if $|x| = |y|$ then $x = y$ or $x = -y$.
3. $|x + y| \leq |x| + |y|$ (this is called **triangle inequality**)
4. $|x - y| \leq |x| + |y|$
5. $||x| - |y|| \leq |x - y|$ (this is sometimes called **reverse triangle inequality**)

General topology**Basic definitions****Topology, open sets and interior**

71. Using mathematical induction prove that the intersection of finitely many open sets is open.

72. Trivial topology Prove that for any X , set $\{\emptyset, X\}$ is a topology.

73. Discrete topology Prove that for any X , its power set 2^X is a topology.

74. Cofinite topology Prove that for any X , the set: $\mathcal{T}_X = \{\emptyset\} \cup \{A \subseteq X : X \setminus A \text{ is finite}\}$ is a topology. Hint: think in terms of complements.

75. Subspace topology Let X be a set and \mathcal{T}_X a topology on it. For $A \subseteq X$ we define $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}_X\}$. Prove that \mathcal{T}_A is a topology on A .

76. For which sets, there is exactly one topology on them?

77. Prove that for an infinite set, there are at least three distinct topologies.

78. Real numbers \mathbb{R} are usually equipped with the following topology: $X \subseteq \mathbb{R}$ is open iff for each $x \in X$ there is an open interval (a_x, b_x) such that $x \in (a_x, b_x) \subseteq X$.

1. Prove that it is indeed a topology.
2. Let a **ball** be a set $B(x, r) = \{y \in \mathbb{R} : |x - y| < r\}$ for $r > 0$. Prove that $(a, b) \neq \emptyset$ can be written as $B(x, r)$ for suitable x and r
3. Prove that X is open iff for each $x \in X$ there is a $r > 0$ such that $B(x, r) \subseteq X$. We will see that this results generalises to much broader category of spaces than single \mathbb{R} .

79. Prove that each point has an open neighborhood.

80. Prove that A is an open set if and only if each point a has a neighborhood $U_a \in \mathcal{T}$ contained in A (that is $U_a \subseteq A$).

81. Prove that:

1. $\text{int } A$ is an open set.
2. if $A' \subseteq A$ is open, then $A' \subseteq \text{int } A$ (so in some sense, $\text{int } A$ is the biggest open set contained in A)
3. $\text{int } A = A$ iff A is open
4. $\text{int } (\text{int } A) = \text{int } A$ for any A

82. Let $A' \subseteq A$. Prove that:

1. $\text{int } A' \subseteq \text{int } A$
2. $\text{int } A \cup \text{int } B \subseteq \text{int } (A \cup B)$

You can prove also that the union can be arbitrary.

83. We say that a is an **interior point** of A if there is open $U_a \subseteq A$ such that $a \in U_a$. Prove that $\text{int } A$ is the set of all interior points of A .

Closed sets and closure

84. Prove these properties of closed sets in space (X, \mathcal{T}_X) :

1. \emptyset and X are closed
2. If A_1, A_2, \dots, A_n are closed, then their union $A_1 \cup A_2 \cup \dots \cup A_n$ is closed.
3. If \mathcal{A} is any family of closed sets, then the intersection $\bigcap \mathcal{A}$ is closed.

85. We say that p is a limit point of $A \subseteq X$ if for every every open neighborhood U of p there is $q_U \neq p$ such that $q_U \in A \cap U$. Prove that A is closed iff it contains all of its limit points.

86. Prove that:

1. $\text{cl } A$ is a closed set.
2. if C is closed and $A \subseteq C$, then $\text{cl } A \subseteq C$ (so in some sense, $\text{cl } A$ is the smallest closed set containing A)
3. $A \subseteq \text{cl } A$
4. $\text{cl}(A \cup B) = \text{cl } A \cup \text{cl } B$
5. $\text{cl } A = A$ iff A is closed
6. $\text{cl cl } A = \text{cl } A$ for any A

87. We say that p is an **adherent point** of A (or **point of closure**) if for any neighborhood V of p we have $A \cap V \neq \emptyset$. Alternatively, we can say that every neighborhood of p contains a point from A . Prove that $\text{cl } A$ is the set of all adherent points of A .

88. We say that $A \subseteq X$ is **dense** if $\text{cl } A = X$. Prove that A is dense iff for every $U \in \mathcal{T}_X$, $A \cap U \neq \emptyset$

89.

1. Let $r \in \mathbb{R}$. Prove that for every neighborhood V of r there is $q \in \mathbb{Q}$ such that $q \in V$. Hint: each neighborhood must have an interval. And you should have proven that in each interval there is a rational.
2. Conclude that rationals are dense in reals.

Boundary and exterior

90. We say that p is a **frontier** point of A if every open neighborhood of p intersects both A and A^c , so if for every open neighborhood U_p we have $U_p \cap A \neq \emptyset$ and $U_p \cap A^c \neq \emptyset$. Prove that the boundary of A is exactly the set of frontier points of A .

91. Prove that boundary is always closed.

92. Prove that $\partial \partial A \subseteq \partial A$.

93. Prove that $\partial A = \partial A^c$.
94. Prove that $\partial A = \emptyset$ iff A is simultaneously open and closed.
95. Prove that $\partial A = \text{cl } A \cap \text{cl ext } A$

Bases and countability axioms

96. Prove that \mathcal{B} is a basis for (X, \mathcal{T}) iff for every $x \in X$ and every neighborhood U_i of x , there is $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq U_i$.
97. Let \mathcal{B} be a basis of (X, \mathcal{T}) . Prove that:
1. $\bigcup \mathcal{B} = X$
 2. If $U, V \in \mathcal{T}$ and $x \in U \cap V$, then there is a set $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U \cap V$.
98. Consider \mathbb{R} with its standard topology. If $x \in \mathbb{R}$ and U is an open set containing x , we can find a ball $B(x, r)$, $r > 0$ such that $B(x, r) \subseteq U$. Using the fact that rationals are dense in reals, prove that you can find $p, q \in \mathbb{Q}$ such that $x \in (p, q) \subseteq U$.
99. Prove that \mathbb{R} is second countable.

Continuous maps and homeomorphisms

100. Prove that a function is continuous iff preimage of every *closed* set is closed.
101. Let $f : (X, \mathcal{T}) \rightarrow (Y, \tau)$ and \mathcal{B} be a basis of (Y, τ) . Then f is continuous iff $f^{-1}(B) \in \mathcal{T}$ for every $B \in \mathcal{B}$.
102. Assuming that \mathbb{R} is equipped with its standard topology, prove that functions from \mathbb{R} to \mathbb{R} are continuous:
1. $f(x) = ax + b$
 2. $f(x) = x^2$
103. Let $f : (X, \mathcal{T}) \rightarrow (Y, \tau)$. Prove that f is continuous iff $f(\text{cl } A) \subseteq \text{cl } f(A)$ for every $A \subseteq X$.
104. Prove that two *discrete* spaces X and Y are homeomorphic iff $|X| = |Y|$.

Connected spaces

105. Let (X, \mathcal{T}) be a topological space. Prove that these conditions are equivalent:
1. The space is disconnected.
 2. There are two *open* sets $A, B \subseteq X$ such that $A, B \neq \emptyset$, $A \cap B = \emptyset$, $A \cup B = X$.
 3. There are no two *closed* sets $A, B \subseteq X$ such that $A, B \neq \emptyset$, $A \cap B = \emptyset$, $A \cup B = X$.
 4. There is a set $S \subset X$, $S \neq \emptyset, X$ such that S is open and closed simultaneously (sometimes sets that are both open and closed are called **clopen**).
 5. There is a set $S \subset X$, $S \neq \emptyset, X$ such that $\partial S = \emptyset$.
 6. There are subsets $A, B \subseteq X$, $A, B \neq \emptyset$ such that $A \cap \text{cl } B = B \cap \text{cl } A = \emptyset$ and $A \cup B = X$.

106. Let (X, \mathcal{T}) be a topological space. Prove that these conditions are equivalent:

1. The space is connected.
2. There are no two *open* sets $U, V \subseteq X$ such that $U, V \neq \emptyset$, $U \cap V = \emptyset$, $U \cup V = X$.
3. There are no two *closed* sets $U, V \subseteq X$ such that $U, V \neq \emptyset$, $U \cap V = \emptyset$, $U \cup V = X$.
4. The only sets that are open and closed simultaneously are \emptyset and X .
5. All continuous maps from (X, \mathcal{T}) to $(\{0, 1\}, \text{discrete topology})$ are constant.
6. If $S \subseteq X$ and $\partial S = \emptyset$, then $S = \emptyset$ or $S = X$.

Pseudometric spaces

Pseudometric spaces

107. Prove that for a pseudometric d and every $x, y \in X$, there is $d(x, y) \geq 0$.

108. Prove that $d(x, y) = 0$ for any $x, y \in X$ is a pseudometric on X . This is called **trivial pseudometric**.

109. Prove that $d(x, y) = 1$ for $x \neq y$ is a pseudometric on X . This is called **discrete pseudometric**.

110. Prove that $d(x, y) = |x - y|$ is a pseudometric on \mathbb{R} .

111. We say that $S \subseteq X$ is an open set if for every s in S there is r_s such that $B(s, r_s) \subseteq S$. Prove that this is indeed a topology on X .

112. Prove that

1. Topology obtained from trivial pseudometric is the trivial topology
2. Topology obtained from discrete pseudometric is the discrete topology.

113. Let (X, d) be a pseudometric space. Prove that:

1. Any $B(x, r)$ is open
2. $\{B(x, r) : x \in X \text{ and } r > 0\}$ is a basis
3. This space is first-countable. Hint: consider $r = 1/n$ for $n = 1, 2, \dots$

Topology of \mathbb{R}

Linear algebra

Vector spaces

114. We know that the set of vectors must contain at least one vector (neutral element). Construct a vector space that has *exactly* one vector (so in some sense it is the smallest space).

115. We want to modify our notation in the following way:

1. Prove that o is unique element with the property $v + o = v$ for $v \in V$
2. Prove that $0 \cdot v = o$ for every v . Hint: remember that $1 + 0 = 1$. This suggests to write 0 for o (so 0 since now technically has two different meanings, practically we will never have any problems with that)
3. Let $v \in V$ and $\tilde{v} \in V$ be such an element that $v + \tilde{v} = o$. Prove that \tilde{v} is unique (so if $v' + v = o$, then $\tilde{v} = v'$)
4. Prove that the \tilde{v} is exactly $(-1)v$. It suggests to write $-v$ for additive inverse, and we will do it since now.

116. The **characteristic** of a field F is the smallest natural number n such that $1 + 1 + \dots + 1 = 0$, where we have n ones on the left hand side. If there is no such number we say that characteristic is 0.

1. Prove that \mathbb{R} has characteristic 0.
2. Let (F, V) be a vector space with at least two elements. Prove that $v = -v$ for every $v \in V$ if and only if the scalar field has characteristic 2.
3. Prove that if F has characteristic different from 2, then we have $v = -v$ iff $v = 0$.

117. An important example of a vector space over a field F is F^n , where addition and scalar multiplication are defined pointwise: $f \cdot (a_1, a_2, \dots, a_n) = (fa_1, fa_2, \dots, fa_n)$, $(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$. Prove that it is indeed a vector space.

Bases of vector spaces

118. Consider infinite real sequences with addition and multiplication by a real number defined pointwise: $c = a + b$ iff $c_n = a_n + b_n$ for all n and $b = ra$, $r \in \mathbb{R}$ iff $b_n = ra_n$ for all n .

1. Prove that this is a vector space, let's call it $\mathbb{R}^{\mathbb{N}}$.
2. Prove that set $B = \{e_k : k \in \mathbb{N}\}$, e_k has 1 at k -th place and 0 at all the others, does *not* span $\mathbb{R}^{\mathbb{N}}$.
3. Let $\hat{\mathbb{R}}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ contain all the sequences that have a *finite* number of non-zero elements. Prove that this is a vector space and that it is spanned by B defined above.

119. Assume that v_1, v_2, \dots, v_n are *not* linearly dependent. Prove that there is v_i such that v_i is a linear combination of other vectors: $v_i \in \text{span}\{v_1, \dots, v_{i-1}, v_{i+1}, v_n\}$.

120. You can prove that every finite dimensional vector space has a basis in two steps:

1. Assume that you have a set $\{e_1, e_2, \dots, e_n\}$ that spans V . Prove that if $e_1 \in \text{span}\{e_2, \dots, e_n\}$, then $V = \text{span}\{e_2, \dots, e_n\}$.
2. Using the reduction step given above, show an algorithm finding a basis from a finite spanning set.
3. Prove that a vector space is finite dimensional iff it has a finite basis.

121. Prove that F^n is finite dimensional. Hint: just find a basis.

122. Prove that $\hat{\mathbb{R}}^{\mathbb{N}}$ has a basis.

123. Here you will prove that all the bases of a *finite dimensional* vector space have the same number of elements. Let v_1, v_2, \dots, v_n be a basis of a vector space V and $w_1, w_2, \dots, w_m \in V$, where $m > n$.

1. (**Steinitz exchange lemma**) Prove that if $w_1 \neq 0$, then $v_1 \in \text{span}\{w_1, v_2, v_3, \dots, v_n\}$.
2. Prove that if $w_k \neq 0$ for $k \in \{1, 2, \dots, n\}$, then $w_{n+1} \in V = \text{span}\{w_1, w_2, \dots, w_n\}$.
3. Prove that w_1, w_2, \dots, w_m *cannot* be linearly independent.
4. Prove that each basis of V has the same number of elements. This number is called **the dimension of V** and written as $\dim V$.

124. Let V be a finite dimensional vector space of dimension n . Prove that every linearly independent set of n vectors spans V and every set with n elements spanning V is a basis.

125. Here you will prove that every linearly independent set of vectors can be extended to a basis of a finite dimensional vector space. Let V be a finite dimensional vector space of dimension n .

1. Let $S = \{v_1, v_2, \dots, v_k\} \subseteq V$ be linearly independent. Prove that if $u \in V$, but $u \notin \text{span } S$, then $\{u\} \cup S$ is linearly independent.
2. Prove that there are u_1, u_2, \dots, u_{n-k} such that $v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_{n-k}$ is a basis of V .

126. Assume that you have a basis e_1, e_2, \dots, e_n of a finite dimensional vector space V over a field \mathbb{F} . Therefore every vector v can be written as a sum $v = v_1e_1 + v_2e_2 + \dots + v_ne_n$ for some $v_i \in \mathbb{F}$. Prove that these numbers are unique, that is if $v = v_1e_1 + v_2e_2 + \dots + v_ne_n = v'_1e_1 + v'_2e_2 + \dots + v'_ne_n$, then $v_i = v'_i$ for all i . Hint: $0 = v - v$ and e_i are linearly independent.

Subspaces, direct sum and quotient spaces

127. Prove that $fv + u \in U$ for all $v, u \in U, f \in F$ is equivalent to: for every $v, u \in U$, we have $v + u \in U$ and for every $v \in U, f \in F$ we have $fv \in U$.

128. Let V be a finite dimensional vector space and $U \subseteq V$ be a vector subspace. Prove that U is finite dimensional and $\dim U \leq \dim V$.

129. Let V be a finite dimensional vector space and $U \subseteq V$ be a vector subspace. Prove that $\dim U = \dim V$ iff $U = V$.

130. Prove that each $w \in V \oplus U$ has a *unique* decomposition: $w = v + u, v \in V, u \in U$. That is if $w = v + u = v' + u'$, then $v = v'$ and $u = u'$ for $v, v' \in V, u, u' \in U$.

131. Let U and V be finite dimensional vector spaces. Prove that $\dim U \oplus V = \dim U + \dim V$.

132. Let V_1, V_2, V_n be finite dimensional vector spaces. Prove that

$$\dim V_1 \oplus V_2 \oplus \dots \oplus V_n = \dim V_1 + \dim V_2 + \dots + \dim V_n.$$

133. Prove that the direct product of two vector subspaces is a special case of the general definition if we identify $U \ni u \leftrightarrow (u, 0) \in U \times V, V \ni v \leftrightarrow (0, v) \in U \times V$ employed.

134. Prove directly that direct product of two vector subspaces of V is a vector subspace of V . Hint: check if 0 is inside and use the handy, one-line criterion.

135. Let U be a subspace of V and V be a subspace of W . Prove that U is subspace of W .

136. Let $V = \mathbb{R}^2$ and $U = \{(0, r) : r \in \mathbb{R}\}, W = \{(r, 0) : r \in \mathbb{R}\}$. Prove that:

1. $V = U \oplus W$.
2. Prove that $U \cup V$ is *not* a vector space.

137. Let $U, W \subseteq V$ be two vector subspaces of a finite vector space V . Prove that:

1. $U \cap W$ is a subspace of U, W and V .
2. $U + W := \{u + w : u \in U, w \in W\}$ is a vector subspace⁷ of V
3. Take a basis B_i of $U \cap W$ and extend it using some vectors $B_U \subseteq U$ such that $B_i \cup B_U$ is a basis of U . Repeat this procedure of W defining B_W and prove that $V = \text{span } B_i \cup B_U \cup B_W$.
4. Prove that $B_i \cup B_U \cup B_W$ is linearly independent. Hint: write the condition of linear independence. Express the linear combination of the elements of B_U as a linear combination of B_i and B_W . Why is this linear combination in $U \cap W$? What you can conclude from the fact that $B_i \cup B_U$ is a basis?
5. Prove that $\dim U + \dim W = \dim U + \dim W - \dim U \cap W$.

Quotient spaces

⁷if $U \cap W = \emptyset$ we have just $U + W = U \oplus W$

138. Consider vector space V and its subspace U . We introduce a relation on V : $v \approx u$ iff $v - u \in U$. Prove that \approx is an equivalence relation.

139. Prove that addition and scalar multiplication on V/\approx are well-defined (independent on the class representative), that is:

1. if $v \approx v'$ and $u \approx u'$, then $v + u \approx v' + u'$
2. if α is a scalar and $v \approx v'$ are vectors in V , then $\alpha v \approx \alpha v'$.

140. Prove that under relation \approx , U is identified with 0.

141. Let $U \subseteq V$ be finite-dimensional vector spaces. Prove that $\dim V/U = \dim V - \dim U$. Hint: guess what is the basis of V/U starting with basis of U and completing it to the basis of V .

Linear maps

142. Let $L : V \rightarrow W$ be a function between vector spaces over field \mathbb{F} . Prove that the following sentences are equivalent:

1. L is linear
2. for every $u, v \in V$ and $\alpha \in \mathbb{F}$, we have $L(\alpha u + v) = \alpha L(u) + L(v)$
3. for every $v, u \in V$ we have $L(v + u) = L(v) + L(u)$ and for every $v \in V, \alpha \in \mathbb{F}$ we have $L(\alpha v) = \alpha L(v)$

143. Let U be a vector subspace of V . Prove that **the inclusion map** $\iota : U \rightarrow V$ given as $\iota(u) = u$ is linear.

144. Let U be a vector subspace of V . Prove that **the quotient map** $q : V \rightarrow V/U$ given as $q(v) = [v]$ is linear.

Kernel and cokernel

145. Let $L : V \rightarrow W$ be a linear map between vector spaces.

1. Prove that $L(0_V) = 0_W$, where 0_V is the neutral element in V and 0_W is the neutral element in W .
2. Prove that the **kernel of** L defined as: $\ker L = \{v \in V : L(v) = 0_W\}$ is a vector subspace of V .
3. Prove that the image of L is a vector subspace of W . The dimension of $\operatorname{im} L$ is called **the rank** of L : $\operatorname{rk} L = \dim \operatorname{im} L$.
4. Let $V = \operatorname{span} S$. Prove that $\operatorname{im} L = \operatorname{span} L(S)$. Here $L(S)$ has meaning $\{L(s) : s \in S\}$.
5. Prove that if V is finite dimensional, then $\operatorname{im} L$ is also finite dimensional.

146. Prove that $\varphi : V \rightarrow U$ is injective iff $\ker \varphi = \{0\}$.

147. Prove that $\varphi : V \rightarrow U$ is surjective iff $\operatorname{coker} \varphi$ has exactly one element (is a trivial vector space).

Rank-nullity theorem

148. Prove the **rank-nullity theorem** - if V is a finite dimensional vector space and $L : V \rightarrow W$ is linear, then $\dim \ker L + \operatorname{rk} L = \dim V$.

149. You can do a beautiful and simple proof of $\dim U + V = \dim U + \dim V - \dim U \cap V$, where U and V are finite-dimensional subspaces of some greater space S . Consider a map $L : U \times V \rightarrow S$ defined as $L(u, w) = u - w$. Prove that it is a linear map (we give $U \times V$ a linear space structure using entry-wise operations, as in $U \oplus V$). What is its range and kernel?

150. Let $\varphi : V \rightarrow U$ be a map between finite-dimensional vector spaces. Prove that the **index** of φ defined as $\text{index } \varphi = \dim \ker \varphi - \dim \text{coker } \varphi$ is equal to: $\text{index } \varphi = \dim V - \dim U$.⁸

151. Prove that V and $\{0\} \times V$ are isomorphic. (Do you remember the direct sum of two subspaces? In fact we used there this isomorphism).

152. Prove that two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension. Hint: right implication - rank-nullity theorem, left - write down bases.

Exact sequences

153. Prove that if sequence of vector spaces V_i and maps $\varphi_i : V_i \rightarrow V_{i+1}$, is exact, then $\varphi_i \circ \varphi_{i+1} = \mathbf{0}$, where $\mathbf{0}$ is a null ⁹ map $\mathbf{0} : V_i \rightarrow V_{i+2}$ defined as $\mathbf{0}(v) = 0$.

154. Prove the following:

1. sequence $V \xrightarrow{\varphi} U \rightarrow 0$ is exact iff φ is surjective. Hint: what is kernel of the map $U \rightarrow 0$?
2. sequence $0 \rightarrow V \xrightarrow{\varphi} U$ is exact iff φ is injective. Hint: do you remember how injectivity is related to some kernel?
3. sequence $0 \rightarrow V \xrightarrow{\varphi} U \rightarrow 0$ is exact iff V and U are isomorphic.
4. short sequence $0 \rightarrow V_2 \xrightarrow{\varphi_2} V_3 \xrightarrow{\varphi_3} V_4 \rightarrow 0$ is exact iff φ_2 is injective and φ_3 is surjective.

155. Use the rank-nullity theorem and prove that if $0 \rightarrow V_0 \rightarrow V_1 \rightarrow \dots \rightarrow V_{n-1} \rightarrow V_n \rightarrow 0$ is exact, then

$$0 = \sum_{i=0}^n (-1)^i \dim V_i.$$

156. Consider a linear map $L : V \rightarrow U$. Prove that the sequence:

$$0 \rightarrow \ker L \xrightarrow{\kappa} V \xrightarrow{L} U \rightarrow \text{coker } L \rightarrow 0$$

is exact, where $\kappa : \ker L \rightarrow V$ is inclusion map: $\kappa : \ker L \ni l \rightarrow l \in V$.

157. Let $L : V \rightarrow U$ be a map between finite-dimensional vector spaces. Prove once again that the index $L := \dim \ker L - \dim \text{coker } L = \dim V - \dim U$.

Dual spaces

158. Prove that V^* is a vector space by:

1. Finding the neutral element. Hint: function that always yields 0 is linear.
2. Proving that addition and scalar multiplication can be defined, so if $\mu, \omega \in V^*$ and $a \in \mathbb{F}$, then function $a\mu + \omega$, defined as: $(a\mu + \omega)(v) = a \cdot \mu(v) + \omega(v)$ for all $v \in V$, is linear.

159. Let $V^{**} = (V^*)^*$ be the dual space of the dual space to V . We will try to prove that, in some sense, V is a subset of it. Prove that:

1. For each $v \in V$ there is a $\tilde{v} \in V^{**}$ such that for every $\omega \in V^*$ such that $\langle \tilde{v}, \omega \rangle = \langle \omega, v \rangle$
2. Prove that the map $v \mapsto \tilde{v}$ is a monomorphism (an injective linear map).

$$\begin{array}{ccc} V & \xrightarrow{f} & U \\ & \searrow f^*(\omega) & \swarrow \omega \\ & \mathbb{R} & \end{array}$$

160. Let $f : V \rightarrow U$ be a linear map between finite dimensional spaces. We define $f^* : U^* \rightarrow V^*$ as follows: Prove that such map is linear and that $f^* \circ \omega = \omega \circ f$ for every $\omega \in U^*$. Here we see how a function sending vectors in one direction can naturally induce a linear map that „pulls-back” dual vectors.

161. Let V be a finite dimensional space with basis e_1, e_2, \dots, e_n . Prove that $\mu^1, \mu^2, \dots, \mu^n \in V^*$ defined as

$$\mu^i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} ,$$

is a basis of V^* .

Tensors

162. Consider a multilinear function f from $V_1 \times V_2 \times \dots \times V_n$ to \mathbb{F} . Prove that f is a linear mapping from $V_1 \oplus V_2 \oplus \dots \oplus V_n$, to \mathbb{F} .

163. Find such f that f is a linear mapping from $V_1 \oplus V_2 \oplus \dots \oplus V_n$, to \mathbb{F} , but f *cannot* be treated as a multilinear function from $V_1 \times V_2 \times \dots \times V_n$ to \mathbb{F} .

Universal property

164. Let V and W be vector spaces. Assume that there is a space T and bilinear map φ have the universal property and space U an bilinear map θ also have the universal property. Prove that T and U are naturally isomorphic.

165. Define $f : V \times W \rightarrow W \otimes V$ as $f(v, w) = w \otimes v$. Using the universal property, prove that there is an isomorphism $v \otimes w \mapsto w \otimes v$.

166. Prove that there is a canonical isomorphism between $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ by:

1. Proving that map $\lambda_u : V \times W \rightarrow (U \otimes V) \otimes W$ given by $\lambda_u(v, w) = (u \otimes v) \otimes w$ is bilinear.
2. As λ_u is bilinear, you can find linear map $\tilde{\lambda}_u : V \otimes W \rightarrow (U \otimes V) \otimes W$. Prove that map $\Lambda : U \times (V \otimes W) \rightarrow (U \otimes V) \otimes W$ given by $\Lambda(u, \omega) = \tilde{\lambda}_u(\omega)$ is bilinear. Find therefore a map $U \otimes (V \otimes W)$ to $(U \otimes V) \otimes W$.

Construction

167. Let V be a set of all functions from set S to field \mathbb{F} such that $f(s) = 0$ for all but finitely many $s \in S$. Prove that V is a vector space.

168. Consider a set S and a free vector space generated by it V_S . We define inclusion mapping $\iota : S \rightarrow V_S$ as $S \ni s \mapsto s \in V_S$. Prove that for every function $f : S \rightarrow U$, where U is a vector space, there is a unique linear map $\tilde{f} : V_S \rightarrow U$ such that $f = \tilde{f} \circ \iota$. This can be written as a commutative diagram:

169. Let V and W be vector spaces over a field \mathbb{F} . Let A be a free vector space over $V \times W$. Consider set S containing elements of the forms

$$(v + v', w + w') - (v', w'), \quad (\alpha v, \alpha w) - \alpha(v, w)$$

for $(v, w) \in V \times W$ and it's free vector space B . Prove that $V \oplus W$ is naturally isomorphic to A/B .

⁸This is a very important result - index, defined with the help of a chosen function is doesn't in fact depend on this function! A beautiful generalisation of this result, is called Atiyah-Singer index theorem.

⁹Later we will refer to this map just as 0 - now this symbol has at least three meanings! It can be an additive neutral element of a field, a neutral element in a vector space or a linear map!

$$\begin{array}{ccc} S & \xrightarrow{\iota} & V_S \\ & \searrow f & \downarrow \tilde{f} \\ & & U \end{array}$$

170. Let V and W be vector spaces over a field \mathbb{F} . Let A be a free vector space over $V \times W$. Consider a set S containing all the elements of the forms:

$$(v + v', w) - (v, w) - (v', w), \quad (\alpha v, w) - \alpha(v, w) \quad (1)$$

$$(v, w + w') - (v, w) - (v, w'), \quad (v, \alpha w) - \alpha(v, w) \quad (2)$$

and it's free vector space B . Prove that A/B has the universal property of tensor product, with $v \otimes w = [(v, w)]$.

171. Let V and W be finite dimensional over \mathbb{F} . Consider a set S of all bilinear functions from $V^* \times W^* \rightarrow \mathbb{F}$. As we remember, for finite dimensional V we have a natural isomorphism $V \approx V^{**}$, so we can write $v(\nu) \in \mathbb{F}$ for $v \in V, \nu \in V^*$. Let's define: $v \otimes w(\nu, \omega) = v(\nu) \times w(\omega)$. Prove that:

1. $v \otimes w \in S$ (so it must be a bilinear function)
2. $\{v_i \otimes w_j\}$ form a basis of S if $\{v_i\}$ and $\{w_j\}$ are bases of V and W .
3. $V \otimes W$ defined as above is naturally isomorphic to S .
4. $\dim V \otimes W = \dim V \cdot \dim W$

172. Let V, W, U be finite dimensional vector spaces over the same field. Prove that there is a natural isomorphism:

$$V \otimes (W \oplus U) \approx (V \otimes W) \oplus (V \otimes U).$$

173. Prove that for finite dimensional V and W , there is a natural isomorphism between $\text{Hom}(V, W)$ and $V^* \otimes W$.

174. Prove that for finite dimensional V , $\text{End } V$ is naturally isomorphic to $\text{End } V^*$.