Logic and sets

Propositional calculus

New sentences from old

1. Prove that if we know that $p \Leftrightarrow q$ and $q \Leftrightarrow r$, then also $p \Leftrightarrow r$.

2. Let p and q be two sentences. Prove that $p \wedge q$ is true if and only if $q \wedge p$ is true. As we can swap two elements, we say that conjunction is **commutative**.

3. Let p, q, r be three sentences. Prove that $(p \wedge q) \wedge r$ is true if and only if $p \wedge (q \wedge r)$ is true. Such a property is called **associativity** and implies that we do not need to specify the order of calculation. Therefore we can write just $p \wedge q \wedge r$.

4. Prove that disjunction is both associative and commutative.

5. Prove that if $\neg p$ is false if and only if p is true.

6. Assume that you have built a sentence using n sentences: $p_1, p_2, \ldots, p_n$. How many rows does the truth table contain?

7. Prove **distributivity**:

1. $(p \wedge q) \vee r \Leftrightarrow (p \vee r) \wedge (q \vee r)$
2. $(p \vee q) \wedge r \Leftrightarrow (p \wedge r) \vee (q \wedge r)$

8. Prove **De Morgan's laws**:

1. $\neg(p \wedge q) = (\neg p) \vee (\neg q)$
2. $\neg(p \vee q) = (\neg p) \wedge (\neg q)$

9. Prove that $(p \Rightarrow q) \Leftrightarrow (\neg p) \vee q$. Hint: left sentence is false for very specific p and q. Do you need to write down 4 rows in a truth table for right-hand-side sentence?

10. Prove that implication is **transitive**, that is $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$.

11. Let p and q be sentences. Prove that:

1. $\neg(\neg p) \Leftrightarrow p$
2. $p \Rightarrow q$ implies $(\neg q) \Rightarrow (\neg p)$ (Be smart! How many values of p, q do you need to check?)
3. $p \vee (\neg p)$
4. $p \wedge (\neg p)$ is *false* (we could write "Prove $\neg(p \wedge (\neg p))$", but it looks much more terrible!)

12. Prove that $(p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow q) \wedge (q \Rightarrow p))$.

Another point of view

13. Prove transitivity of implication, that is $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$ using transitivity of $\leq$. It simplifies the proof a bit, doesn't it?

Quantifiers

14. Prove that:

1. $\neg\forall_n P(n) \Leftrightarrow \exists_n \neg P(n)$
2. $\neg\exists_n P(n) \Leftrightarrow \forall_n \neg P(n)$

Basic set theory

15. Russel's paradox Let X be a set built from all sets such that $A \notin A$. Prove that X does not exist. Hint: what if $X \in X$? What if $X \notin X$?

16. Prove that $A = B$ iff A is a subset of B and B is a subset of A.

17. Here we will prove that the empty set is a unique set with special property of being a subset of every set:

1. Prove that for every set A, $\varnothing \subseteq A$.
2. Let θ be a set such that $\theta \subseteq A$ for every set A. Prove that $\theta = \varnothing$.

New sets from old

18. Prove that there is *no* set of all sets. Hint: assume there is one and select some elements to create Russel's paradox.

19. Using the axiom of power set and the axiom schema of specification, justify the notation:

$$\{A' \subseteq A : P(A')\},$$

where $P(A')$ assigns true or false to each subset A' of A.

20.

1. Let $A = \{1, 2, 3\}$. Find 2^A. What is the number of elements in $\mathcal{P}(A)$? How is it related to the number of elements of A?
2. Let A be a finite set with n elements. Prove that $\mathcal{P}(A)$ has 2^n elements. Do you see now why $\mathcal{P}(A)$ is also referenced as 2^A? Hint: every subset is specified by elements that are inside it. For every element you have two options - to select it or not.

21. Let A, B and C be sets. Prove that:

1. union defined as $A \cup B = \{x : x \in A \vee x \in B\}$ agrees with $\bigcup\{A, B\}$
2. $A \cup B = B \cup A$ (so union is commutative)
3. $(A \cup B) \cup C = \bigcup\{A, B, C\}$
4. $(A \cup B) \cup C = A \cup (B \cup C)$ (this is called associativity)
5. $A \cup A = A$

22. Let A and B be sets. Prove that $A \subseteq B \cup (A \setminus B)$, where the equality holds iff $B \subseteq A$.

23. Find sum and intersection of family of subsets of $\mathbb{R}$: $A_r = \{r, -r\}$ for $r \geq 0$.

24. Let A, B C be sets. Writing $A \cap B := \bigcap\{A, B\}$, prove that:

1. $A \cap B = B \cap A$ (commutativity)
2. $A \cap (B \cap C) = (A \cap B) \cap C$ (associativity)
3. $A \cap A = A$

25. Prove distributivity:

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Subsets and complements

26. Prove the following set identites:

1. Let $A \subseteq B$. Prove that $(A^c)^c = A$.
2. Let $A,\ B \subset U$. Prove that $(A \cup B)^c = A^c \cap B^c$
3. Let $A,\ B \subset U$. Prove that $(A \cap B)^c = A^c \cup B^c$
4. $\{a \in A : a \in B\} = \{b \in B : b \in A\}$

27. Let $\mathcal{X} \subset \mathcal{P}(U)$ be a family of sets and define: $\mathcal{Y} = \{X^c : X \in \mathcal{X}\}$, where $X^c = U \setminus X$. Prove that:

1. $(\bigcup \mathcal{X})^c = \bigcap \mathcal{Y}$
2. $(\bigcap \mathcal{X})^c = \bigcup \mathcal{Y}$

28. Let $A \subseteq X_i$ for $i \in I$. Prove that

$$A \subseteq \bigcup_{i \in I} X_i$$

29. For every point $a \in A$ there is a set $U_a \subseteq A$ such that $a \in U_a$. Prove that

$$A = \bigcup_{a \in A} U_a.$$

Axiom of choice

Cartesian product

30. Let $A = \{\{a\}, \{a, b\}\}$, $B = \{\{c\}, \{c, d\}\}$. Prove that $A = B$ iff $a = c \wedge b = d$. Such a set A we call **the ordered pair** (a, b) as it has the property $(a, b) = (c, d)$ iff $a = c$ and $b = d$. Now you can forget how it has been constructed, and just remember this property.

31. Prove that $(a, (b, c)) = (d, (e, f))$ iff $a = d \wedge b = e \wedge c = f$.

32. Check that defining (a, b, c) as $((a, b), c)$ also works (so two ordered tuples are the same if they have the same first element, the same second element, ...)

33. Check that, in terms of sets, $(a, (b, c)) \neq ((a, b), c)$, so formally we do need to stick to one convention. However as we are interested in the property of ordered tuple, we will not distinguish them and denote both of them just as (a, b, c). Such notational problems appear in various places in mathematics, so we need to try to get used to them.

34. Do you remember the identification of $(a, (b, c))$ and $((a, b), c)$? Prove that $A \times (B \times C) = (A \times B) \times C$. Therefore we'll write it just as $A \times B \times C$ without parentheness.

Natural numbers and mathematical induction

35. You can prove that $2^n > n$ for every natural number n.

1. Prove that the formula works for $n = 0$ (punch the first domino).
2. Assume that for some n you proved on some way that $2^n > n$. Using this, prove that $2^{n+1} > (n+1)$ (if n-th domino falls, then $n+1$-th domino also falls)

36.

1. Prove[1] that 6 divides $n^3 - n$ for all natural n.
2. Prove[2] that 6 divides $n^3 - n$ for all integers n. You can use a slight modification mathematical induction principle proving the implication ,,if the theorem works for n, it works also for $n-1$".

37. (Bernoulli's inequality) Prove that for real $x > -1$ and natural $n \geq 1$, the following inequality holds:

$$(1+x)^n \geq 1 + nx.$$

38. In Mathsland there are $n \geq 2$ cities. Between each pair of them there is a *one-way* road.

1. Prove that there is a city from which you can drive to all the other cities. Hint: assume that the hypothesis works for some n and any country with n cities. Now consider an arbitrary $n+1$-city country. Hide one city and use your assumption.
2. Prove that there is a city[3] to which you can drive from all the others.

39. Let $S \subseteq R$. We say that S is **well-ordered** iff any non-empty subset $X \subset S$ has the smallest element.

1. Prove that reals and integers with the default ordering are not well-ordered.
2. Assume that $X \subseteq \mathbb{N}$ doesn't have the smallest element. Define $A = \{n \in \mathbb{N} : \{0, 1, \ldots, n\} \cap X = \varnothing\}$ and use mathematical induction to prove that X is empty.
3. Why are natural numbers well-ordered?

Functions

Basics

40. (Thanks to Antoni Hanke) How many are there functions from the empty set to $\{1, 2, 3, 4\}$?

41. Consider two functions: $f : \{0,1\} \to \{0,1\}$ given by $f(x) = 0$ and $g : \{0,1\} \to \{0\}$. Prove that $f = g$. [4]

42. Let $f : A \to B$ and $g : C \to B$, where $A \neq C$. Is it possible that $f = g$?

43. Let $f : A \to B$ and $C \subseteq D \subseteq A$. We define: $f[C] = \{b \in B : b = f(c) \text{ for some } c \in C\}$ and analogously $f[D]$. Prove that $f(C) \subseteq f(D)$.

Injectivity, surjectivity and bijectivity

[1] Another method is to notice that $n^3 - n = (n-1) \cdot n \cdot (n+1)$. Why 2 does divide it? Why 3?
[2] How $n^3 - n$ and $(-n)^3 - (-n)$ are related? Does it simplify the proof?
[3] Nice trick: what does happen if you reverse each way? Can you use the former result?
[4] Some mathematicians, as Bourbaki use an alternative definition of function - for them a function is the triple (A, B, f), where f is defined as in the our case. We see that this definition is incompatible with ours. Fortunately, as in the case with different definitions of ordered tuples, this problem will never occur explicitly in the further chapters.

44. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions surjective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2$
3. $h : \mathbb{R} \to \{5\}$

45. As we remember, $\mathbb{R}$ stands for well-known real numbers. Are the following functions injective?

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$
2. $h : \{0, 1, 2, 3\} \to \mathbb{R},\ h(x) = x$

46. Construct function that is:

1. surjective, but not injective
2. injective, but not surjective
3. neither injective nor surjective
4. bijective

47. Prove that, if exists, g is unique.

48. Assume that f^{-1} exists. Prove that $(f^{-1})^{-1}$ exists and is equal to f.

Function composition

49. Find functions $f,\ g$ such that:

1. $g \circ f$ exists, but $f \circ g$ is not defined
2. both $f \circ g$ and $g \circ f$ exist, but $f \circ g \neq g \circ f$

50. Left $f : A \to B, g : B \to C, h : C \to D$. Prove that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

51.

1. Prove that composition of two surjections is surjective.
2. Prove that composition of two injections is injective.
3. Prove that composition of two bijections is bijective.

52. We will rephrase the definition of the inverse function as follows:

1. If X if a set, we define **the identity function**

$$\mathrm{Id}_X = \{(x, x) \in X^2 : x \in X\}.$$

Prove that it is indeed a function. What is it's domain?

2. Let $f : A \to B,\ g : B \to A$. Prove that $f = g^{-1}$ iff

$$g \circ f = \mathrm{Id}_A \text{ and } f \circ g = \mathrm{Id}_B$$

53. Let $f : A \to B$ be an injection. Prove that there is a function $g : \operatorname{Im} f \to A$ such that $g \circ f = \mathrm{Id}_A$. Such g is called **left inverse of** f.

Countability

Finite sets

54. What is the cardinality of $\{a, a+1, a+2, \ldots, a+n\}$?

55. Let A, B and C be finite sets. Prove that:

1. $|2^A| = 2^{|A|}$
2. $|A \cup B| = |A| + |B|$ iff A and B are disjoint.
3. $|A \setminus B| = |A| - |B|$ if $B \subseteq A$.
4. $|A| \geq |B|$ if $B \subseteq A$. When does the equality hold?
5. $|A \cup B| = |A| + |B| - |A \cap B|$
6. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

56. Assume that A and B are finite sets. Prove that $|A| = |B|$ iff there is a bijection between A and B.

57. Above we find the way of saying that two cardinalities are equal using existence of a bijection. Let's find a way to compare which is less using another kind of function.

1. Let $O_n = \{1, 2, \ldots, n\}$. Prove that there is no injection from O_{n+1} into O_n. Hint: use mathematical induction.
2. Let A and B be finite. Prove that there is an injection from A to B iff $|A| \leq |B|$.

58. Using the above results, prove in one line[5] that if there is an injection from A onto B and an injection from B into A, then there exists a bijection from A onto B.

Infinite sets

59. Let A, B and C be sets. Prove that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. Hint: find the bijection between A and C.

60. Prove that:

1. $|\mathbb{N}| = |\mathbb{Z}|$.
2. $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
3. $|\mathbb{N}| = |\mathbb{Q}|$.

61. Prove that if $A \subseteq B$, then $|A| \leq |B|$.

62. Let A, B and C be sets. Prove that if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

63. Here you can prove that there are more real numbers than naturals or rationals. We define $X = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and choose one convention of writing reals (e.g 0.999... = 1.000..., so we can choose to use nines)

1. Assume that you have written all the elements of X in a single column. Can you find a real number that does not occur in the list?
2. Using the above, prove that $|\mathbb{N}| < |X|$
3. Prove that $|\mathbb{Q}| < |\mathbb{R}|$.

[5] The main step is $|A| \leq B$ and $|B| \leq |A|$, so $|A| = |B|$.

64. We know that $|\mathbb{R}| > |\mathbb{N}|$. Using binary system prove that $\mathbb{R} = 2^{\mathbb{N}}$. Do you see similarity between the previous result and $2^n > n$ for natural n?

65. **Cantor's theorem** You will prove that $|A| < \left|2^A\right|$ for any set A. Let A be a set and $f : A \to 2^A$.

1. Consider $X = \{a \in A : a \notin f(a)\} \in 2^A$. Is there $x \in A$ for which $f(x) = X$?
2. Is f surjective?
3. Find an injective function $g : A \to 2^A$.
4. Prove that $|A| < |2^A|$ for any set A.
5. Use Cantor's theorem to prove that there is no set of all sets.

66. **Cantor-Schroeder-Bernstein theorem** Let's prove that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$ for any sets.

1. (Knaster-Tarski) Now assume that F has *monotonicity* property: $F(X) \subseteq F(Y)$ if $X \subseteq Y$. Prove that F has a fixed point S (that is $F(S) = S$), where:
$$S = \bigcup_{X \in U} X, \text{ where } U = \{Y \in 2^A : Y \subseteq f(Y)\}.$$
2. (Banach) Let $f : A \to B$ and $g : B \to A$ be injections. We introduce new symbol: $f[X] = \{b \in B : b = f(x) \text{ for some } x \in X\}$. Prove that function
$$F : 2^A \to 2^A,\ F(X) = A \setminus g[B \setminus f[X]]$$
has the monotonicity property.
3. Prove that $A \setminus S \subseteq \operatorname{Im} g$, where F and S are taken from above.
4. Prove that function
$$h(x) = \begin{cases} f(x), x \in S \\ g^{-1}(x), x \notin S \end{cases}$$
is a bijection.

Pre-image of a function

67. Let $f : A \to B$ and $X, Y \subseteq B$. Then:

1. $f(X \cup Y) = f(X) \cup f(Y)$
2. $f(X \cap Y) \subseteq f(X) \cap f(Y)$

You can also generalise this result to an arbitrary collection of sets.

68. Let $f : A \to B$. Then $f(A) \subseteq B$ and $A = f^{-1}(B)$.

69. Let $f : A \to B$ and $X, Y \subseteq B$. Then:

1. $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$
2. $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

You can also generalise this result to an arbitrary collection of sets.

Real numbers

70. Check that

1. real numbers understood informally, have the properties listed above
2. rational numbers form a field

71. Prove that there is only one 0 and only one 1. Hint: assume that 0 and 0' have property such that $a = a + 0 = a + 0'$ and try $a = 0$ and $a = 0'$.

72. Prove that if $a + a' = 0$ and $a + a'' = 0$, then $a' = a''$. Therefore we can introduce special symbol for *the* additiv inverse: $a + (-a) = 0$ and define subtraction as $a - b := a + (-b)$.

73. Prove that $-a = (-1) \cdot a$.

74. Prove that a set $A \subseteq \mathbb{R}$ can have no upper bounds or infinitely many of them.

75. Prove that supremum is unique, so if x and x' are supremums of A, then $x = x'$.

76. Prove that $x = \sup A$ if and only if $x \geq a$ for every $a \in A$ and for every $\varepsilon > 0$ there is $a \in A$ such that $x < a + \varepsilon$.

77. Prove that natural numbers are *not* bounded from above. Hint: if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$

78. Prove the **Archimedean axiom**[6] that for every $r \in R$, there is $n \in \mathbb{N}$ such that $n > r$.

79. Prove that for any $r > 0$ there is $n \in \mathbb{N}$ such that $1/n < r$.

80. Prove that rational numbers do *not* have the completeness property:

1. Let $p, q \in \mathbb{Z} \setminus \{0\}$. Prove that $p^2 \neq 2q^2$.
2. Prove that root of two, defined as $x > 0, x^2 = 2$ is not rational.
3. Find a subset of $\mathbb{Q}$ that is bounded above, but has no rational supremum.

81. You should prove that in each nonempty interval there is at least one rational number:

1. Assume that $0 < a < b$. Define
$$A = \left\{ \frac{m}{N} : m \in \mathbb{N} \right\}, \ \frac{1}{b-a} < N \in \mathbb{N}$$
and prove that $A \cap (a, b)$ is non-empty.
2. Use the above result to prove that in *each* interval there is at least one rational number.
3. Prove that in each interval there are infinitely but countably many, rational numbers.
4. Prove that in each interval there is an irrational number.
5. How many irrational numbers are in each interval?

Absolute value

82. Prove that for every $x, y \in \mathbb{R}$:

1. $|x| = |-x|$
2. if $|x| = |y|$ then $x = y$ or $x = -y$.
3. $|x + y| \leq |x| + |y|$ (this is called **triangle inequality**)
4. $|x - y| \leq |x| + |y|$
5. $||x| - |y|| \leq |x - y|$ (this is sometimes calles **reverse triangle inequality**)

General topology

[6]In fact we do not need to call it axiom, as we are able to prove it.

Basic definitions

Topology, open sets and interior

83. Using mathematical induction prove that the intersection of finitely many open sets is open.

84. **Trivial topology** Prove that for any X, set $\{\varnothing, X\}$ is a topology.

85. **Discrete topology** Prove that for any X, it's power set 2^X is a topology.

86. **Cofinite topology** Prove that for any X, the set: $\mathcal{T}_X = \{\varnothing\} \cup \{A \subseteq X : X \setminus A \text{ is finite}\}$ is a topology. Hint: think in terms of complements.

87. **Subspace topology** Let X be a set and $\mathcal{T}_X$ a topology on it. For $A \subseteq X$ we define $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}_X\}$. Prove that $\mathcal{T}_A$ is a topology on A.

88. For which sets, there is exactly one topology on them?

89. Prove that for an infinite set, there are at least three distinct topologies.

90. Real numbers $\mathbb{R}$ are usually equipped with the following topology: $X \subseteq \mathbb{R}$ is open iff for each $x \in X$ there is an open interval (a_x, b_x) such that $x \in (a_x, b_x) \subseteq X$.

1. Prove that it is indeed a topology.
2. Let **a ball** be a set $B(x, r) = \{y \in \mathbb{R} : |x - y| < r\}$ for $r > 0$. Prove that $(a, b) \neq \varnothing$ can be written as $B(x, r)$ for suitable x and r
3. Prove that X is open iff for each $x \in X$ there is a $r > 0$ such that $B(x, r) \subseteq X$. We will see that this results generalises to much broader category of spaces than single $\mathbb{R}$.

91. Prove that each point has an open neighborhood.

92. Prove that A is an open set if and only if each point a has a neighborhood $U_a \in A$ contained in A (that is $U_a \subseteq A$).

93. Prove that:

1. $\operatorname{int} A$ is an open set.
2. if $A' \subseteq A$ is open, then $A' \subseteq \operatorname{int} A$ (so in some sense, $\operatorname{int} A$ is the biggest open set contained in A)
3. $\operatorname{int} A = A$ iff A is open
4. $\operatorname{int} \operatorname{int} A = \operatorname{int} A$ for any A

94. Let $A' \subseteq A$. Prove that:

1. $\operatorname{int} A' \subseteq \operatorname{int} A$
2. $\operatorname{int} A \cup \operatorname{int} B \subseteq \operatorname{int} (A \cup B)$

You can prove also that the union can be arbitrary.

95. We say that a is an **interior point** of A if there is open $U_a \subseteq A$ such that $a \in U_a$. Prove that $\operatorname{int} A$ is the set of all interior points of A.

Closed sets and closure

96. Prove these properties of closed sets in space $(X, \mathcal{T}_X)$:

1. $\varnothing$ and X are closed
2. If $A_1, A_2, \dots, A_n$ are closed, then their union $A_1 \cup A_2 \cup \dots \cup A_n$ is closed.
3. If $\mathcal{A}$ is any family of closed sets, then the intersection $\bigcap \mathcal{A}$ is closed.

97. We say that p is a limit point of $A \subseteq X$ if for every every open neighborhood U of p there is $q_U \neq p$ such that $q_U \in A \cap U$. Prove that A is closed iff it contains all of it's limit points.

98. Prove that:

1. $\operatorname{cl} A$ is a closed set.
2. if C is closed and $A \subseteq C$, then $\operatorname{cl} A \subseteq C$ (so in some sense, $\operatorname{cl} A$ is the smallest closed set containing A)
3. $A \subseteq \operatorname{cl} A$
4. $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$
5. $\operatorname{cl} A = A$ iff A is closed
6. $\operatorname{cl} \operatorname{cl} A = \operatorname{cl} A$ for any A

99. We say that p is an **adherent point** of A (or **point of closure**) if for any neighborhood V of p we have $A \cap V \neq \varnothing$. Alternatively, we can say that every neighborhood of p contains a point from A. Prove that $\operatorname{cl} A$ is the set of all adherent points of A.

100. We say that $A \subseteq X$ is **dense** if $\operatorname{cl} A = X$. Prove that A is dense iff for every $U \in \mathcal{T}_X$, $A \cap U \neq \varnothing$

101.

1. Let $r \in \mathbb{R}$. Prove that for every neighborhood V of r there is $q \in \mathbb{Q}$ such that $q \in V$. Hint: each neighborhood must have an interval. And you should have proven that in each interval there is a rational.
2. Conclude that rationals are dense in reals.

Boundary and exterior

102. We say that p is a **frontier** point of A if every open neighborhood of p intersects both A and A^c, so if for every open neighborhood U_p we have $U_p \cap A \neq \varnothing$ and $U_p \cap A^c \neq \varnothing$. Prove that the boundary of A is exactly the set of frontier points of A.

103. Prove that boundary is always closed.

104. Prove that $\partial\partial A \subseteq \partial A$.

105. Prove that $\partial A = \partial A^c$.

106. Prove that $\partial A = \varnothing$ iff A is simultaneously open and closed.

107. Prove that $\partial A = \operatorname{cl} A \cap \operatorname{cl} \operatorname{ext} A$

Bases and countability axioms

108. Prove that $\mathcal{B}$ is a basis for $(X, \mathcal{T})$ iff for every $x \in X$ and every neighborhood U_i of x, there is $B_i \in \mathcal{B}$ such that $x \in B_i \subseteq U_i$.

109. Let $\mathcal{B}$ be a basis of $(X, \mathcal{T})$. Prove that:

1. $\bigcup \mathcal{B} = X$
2. If $U, V \in \mathcal{T}$ and $x \in U \cap V$, then there is a set $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U \cap V$.

110. Consider $\mathbb{R}$ with it's standard topology. If $x \in \mathbb{R}$ and U is an open set containing x, we can find a ball $B(x, r)$, $r > 0$ such that $B(x, r) \subseteq U$. Using the fact that rationals are dense in reals, prove that you can find $p, q \in \mathbb{Q}$ such that $x \in (p, q) \subseteq U$.

111. Prove that $\mathbb{R}$ is second countable.

Continuous maps and homeomorphisms

112. Prove that a function is continuous iff preimage of every *closed* set is closed.

113. Let $f : (X, \mathcal{T}) \to (Y, \tau)$ and $\mathcal{B}$ be a basis of (Y, τ). Then f is continuous iff $f^{-1}(B) \in \mathcal{T}$ for every $B \in \mathcal{B}$.

114. Assuming that $\mathbb{R}$ is equipped with it's standard topology, prove that functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous:

1. $f(x) = ax + b$
2. $f(x) = x^2$

115. Let $f : (X, \mathcal{T}) \to (Y, \tau)$. Prove that f is continuous iff $f(\operatorname{cl} A) \subseteq \operatorname{cl} f(A)$ for every $A \subseteq X$.

116. Prove that two *discrete* spaces X and Y are homeomorphic iff $|X| = |Y|$.

Connected spaces

117. Let $(X, \mathcal{T})$ be a topological space. Prove that these conditions are equivalent:

1. The space is disconnected.
2. There are two *open* sets $A, B \subseteq X$ such that $A, B \neq \varnothing$, $A \cap B = \varnothing$, $A \cup B = X$.
3. There are no two *closed* sets $A, B \subseteq X$ such that $A, B \neq \varnothing$, $A \cap B = \varnothing$, $A \cup B = X$.
4. There is a set $S \subset X$, $S \neq \varnothing, X$ such that and S is open and closed simultaneously (sometimes sets that are both open and closed are called **clopen**).
5. There is a set $S \subset X$, $S \neq \varnothing, X$ such that $\partial S = \varnothing$.
6. There are subsets $A, B \subseteq X$, $A, B \neq \varnothing$ such that $A \cap \operatorname{cl} B = B \cap \operatorname{cl} A = \varnothing$ and $A \cup B = X$.

118. Let $(X, \mathcal{T})$ be a topological space. Prove that these conditions are equivalent:

1. The space is connected.
2. There are no two *open* sets $U, V \subseteq X$ such that $U, V \neq \varnothing$, $U \cap V = \varnothing$, $U \cup V = X$.
3. There are no two *closed* sets $U, V \subseteq X$ such that $U, V \neq \varnothing$, $U \cap V = \varnothing$, $U \cup V = X$.
4. The only sets that are open and closed simultaneously are $\varnothing$ and X.
5. All continous maps from $(X, \mathcal{T})$ to $(\{0, 1\}, \text{discrete topology})$ are constant.
6. If $S \subseteq X$ and $\partial S = \varnothing$, then $S = \varnothing$ or $S = X$.

Pseudometric spaces

Pseudometric spaces

119. Prove that for a pseudometric d and every $x, y \in X$, there is $d(x, y) \geq 0$.

120. Prove that $d(x, y) = 0$ for any $x, y \in X$ is a pseudometric on X. This is called **trivial pseudometric**.

121. Prove that $d(x, y) = 1$ for $x \neq y$ is a pseudometric on X. This is called **discrete pseudometric**.

122. Prove that $d(x, y) = |x - y|$ is a pseudometric on $\mathbb{R}$.

123. We say that $S \subseteq X$ is an open set if for every s in S there is r_s such that $B(s, r_s) \subseteq S$. Prove that this is indeed a topology on X.

124. Prove that

1. Topology obtained from trivial pseudometric is the trivial topology
2. Topology obtained from discrete pseudometric is the discrete topology.

125. Let (X, d) be a pseudometric space. Prove that:

1. Any $B(x, r)$ is open
2. $\{B(x, r) : x \in X \text{ and } r > 0\}$ is a basis
3. This space is first-countable. Hint: consider $r = 1/n$ for $n = 1, 2, \ldots$.

Topology of $\mathbb{R}$

Linear algebra

Vector spaces

126. We know that the set of vectors must contain at least one vector (neutral element). Construct a vector space that has *exactly* one vector (so in some sense it is the smallest space).

127. We want to modify our notation in the following way:

1. Prove that o is unique element with the property $v + o = v$ for $v \in V$
2. Prove that $0 \cdot v = o$ for every v. Hint: remember that $1 + 0 = 1$. This suggests to write 0 for o (so 0 since now technically has two different meanings, practically we will never have any problems with that)
3. Let $v \in V$ and $\tilde{v} \in V$ be such an element that $v + \tilde{v} = 0$. Prove that $\tilde{v}$ is unique (so if $v' + v = 0$, then $\tilde{v} = v'$)
4. Prove that the $\tilde{v}$ is exactly $(-1)v$. It suggests to write $-v$ for additive inverse, and we will do it since now.

128. The **characteristic** of a field F is the smallest natual number n such that $1 + 1 + \cdots + 1 = 0$, where we have n ones on the left hand side. If there is no such number we say that characteristic is 0.

1. Prove that $\mathbb{R}$ has characteristic 0.
2. Let (F, V) be a vector space with at least two elements. Prove that $v = -v$ for every $v \in V$ if and only if the scalar field has characteristic 2.
3. Prove that if F has characteristic different from 2, then we have $v = -v$ iff $v = 0$.

129. An important example of a vector space over a field F is F^n, where addition and scalar multiplication are defined pointwise: $f \cdot (a_1, a_2, \ldots, a_n) = (fa_1, fa_2, \ldots, fa_n)$, $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$. Prove that it is indeed a vector space.

Bases of vectos spaces

130. Consider infinite real sequences with addition and multiplication by a real number defined pointwise: $c = a + b$ iff $c_n = a_n + b_n$ for all n and $b = ra, r \in \mathbb{R}$ iff $b_n = ra_n$ for all n.

1. Prove that this is a vector space, let's call it $\mathbb{R}^{\mathbb{N}}$.
2. Prove that set $B = \{e_k : k \in \mathbb{N}\}$, e_k has 1 at k-th place and 0 at all the others, does *not* span $\mathbb{R}^{\mathbb{N}}$.
3. Let $\hat{\mathbb{R}}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ contain all the sequences that have a *finite* number of non-zero elements. Prove that this is a vector space and that it is spanned by B defined above.

131. Let V be a vector space and consider a finite set $\{v_1, v_2, \ldots, v_n\} \subseteq V$. Prove that it is linearly independent iff the only solution to the equation:

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$$

is $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

132. Consider a finite set $U = \{v_1, v_2, \ldots, v_n\}$ with at least two vectors. Prove that the following two statements are equivalent:

- U is linearly dependent
- there is $v_i \in U$ such can be written as linear combination of other vectors: $v_i \in \operatorname{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, v_n\}$

133. You can prove that every finite dimensional vector space has a basis in two steps:

1. Assume that you have a set $\{e_1, e_2, \ldots, e_n\}$ that spans V. Prove that if $e_1 \in \operatorname{span}\{e_2, \ldots, e_n\}$, then $V = \operatorname{span}\{e_2, \ldots, e_n\}$.
2. Using the reduction step given above, show an algorithm finding a basis from a finite spanning set.
3. Prove that a vector space is finite dimensional iff it has a finite basis.

134. Prove that $\mathbb{F}^n$ is finite dimensional. Hint: just find a basis.

135. Prove that $\hat{\mathbb{R}}^{\mathbb{N}}$ has a basis.

136. You can see how to prove the basis existence with the help of Zorn's lemma. Let V be a vector space.

1. Let $\mathcal{A} = \{U \subseteq V : U \text{ is linearly independent}\}$. Prove that $\mathcal{A}$ is not empty.
2. Prove that relation on A given by $A \preceq B \Leftrightarrow A \subseteq B$ is a partial order.
3. Consider any chain $\mathcal{C} \subseteq \mathcal{A}$. Define $C = \bigcup \mathcal{C}$. We want to prove that C is linearly independent.
4. Assume that C is linearly dependent, so $0 = \lambda_1 v_1 + \ldots \lambda_n v_n$ for some $v_i \in C$. If $v_i \in C_i \in \mathcal{C}$, what can you conclude about $C_1 \cup C_2 \cup \cdots \cup C_n$?
5. From Zorn's lemma we know that there is a *maximal* element A in $\mathcal{A}$. What if A does not span V? Hint: add an element that is not in the span and think about linear independence of new set. A is maximal, isn't it?

137. Here you will prove that all the bases of a *finite dimensional* vector space have the same number of elements. Let $v_1, v_2, \ldots, v_n$ be a basis of a vector space V and $w_1, w_2, \ldots, w_m \in V$, where $m > n$.

1. **(Steinitz exchange lemma)** Prove that if $w_1 \neq 0$, then $v_1 \in \operatorname{span}\{w_1, v_2, v_3, \ldots, v_n\}$.
2. Prove that if $w_k \neq 0$ for $k \in \{1, 2, \ldots, n\}$, then $w_{n+1} \in V = \operatorname{span}\{w_1, w_2, \ldots, w_n\}$
3. Prove that $w_1, w_2, \ldots, w_m$ *cannot* be linearly independent.
4. Prove that each basis of V has the same number of elements. This number is called **the dimension of** V and written as $\dim V$.

138. Let V be a finite dimensional vector space of dimension n. Prove that:

1. every linearly independent set of n vectors spans V (so must span V)
2. every set with n elements spanning V is a basis (so must be linearly independent)

139. Here you will prove that every linearly independent set of vectors can be extended to a basis of a finite dimensional vector space. Let V be a finite dimensional vector space of dimension n.

1. Let $S = \{v_1, v_2, \dots, v_k\} \subseteq V$ be linearly independent. Prove that if $u \in V$, but $u \notin \operatorname{span} S$, then $\{u\} \cup S$ is linearly independent.
2. Prove that there are $u_1, u_2, \dots, u_{n-k}$ such that $v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_{n-k}$ is a basis of V.

140. Assume that you have a basis $e_1, e_2, \dots, e_n$ of a finite dimensional vector space V over a field $\mathbb{F}$. Therefore every vector v can be written as a sum $v = v_1e_1 + v_2e_2 + \cdots + v_ne_n$ for some $v_i \in \mathbb{F}$. Prove that these numbers are unique, that is if $v = v_1e_1 + v_2e_2 + \cdots + v_ne_n = v_1'e_1 + v_2'e_2 + \cdots + v_n'e_n$, then $v_i = v_i'$ for all i. Hint: $0 = v - v$ and e_i are linearly independent.

Subspaces, direct sum and quotient spaces

141. Prove that $fv + u \in U$ for all $v, u \in U, f \in F$ is equivalent to: for every $v, u \in U$, we have $v + u \in U$ and for every $v \in U, f \in F$ we have $fv \in U$.

142. Let V be a finite dimensional vector space and $U \subseteq V$ be a vector subspace. Prove that U is finite dimensional and $\dim U \leq \dim V$.

143. Let V be a finite dimensional vector space and $U \subseteq V$ be a vector subspace. Prove that $\dim U = \dim V$ iff $U = V$.

144. Prove that each $w \in V \oplus U$ has a *unique* decomposition: $w = v + u$, $v \in V$, $u \in U$. That is if $w = v + u = v' + u'$, then $v = v'$ and $u = u'$ for $v, v' \in V$, $u, u' \in U$.

145. Let U and V be finite dimensional vector spaces. Prove that $\dim U \oplus V = \dim U + \dim V$.

146. Let V_1, V_2, V_n be finite dimensional vector spaces. Prove that

$$\dim V_1 \oplus V_2 \oplus \cdots \oplus V_n = \dim V_1 + \dim V_2 + \cdots + \dim V_n.$$

147. Prove that the direct product of two vector subspaces is a special case of the general definition if we identify $U \ni u \leftrightarrow (u, 0) \in U \times V$, $V \ni v \leftrightarrow (0, v) \in U \times V$ employed.

148. Prove directly that direct product of two vector subspaces of V is a vector subspace of V. Hint: check if 0 is inside and use the handy, one-line criterion.

149. Let U be a subspace of V and V be a subspace of W. Prove that U is subspace of W.

150. Let $V = \mathbb{R}^2$ and $U = \{(0, r) : r \in \mathbb{R}\}$, $W = \{(r, 0) : r \in \mathbb{R}\}$. Prove that:

1. $V = U \oplus W$.
2. Prove that $U \cup V$ is *not* a vector space.

151. Let $U, W \subseteq V$ be two vector subspaces of a finite vector space V. Prove that:

1. $U \cap W$ is a subspace of U, W and V.
2. $U + W := \{u + w : u \in U, w \in W\}$ is a vector subspace[7] of V
3. Take a basis B_i of $U \cap W$ and extend it using some vectors $B_U \subseteq U$ such that $B_i \cup B_U$ is a basis of U. Repeat this procedure of W defining B_W and prove that $V = \operatorname{span} B_i \cup B_U \cup B_W$.
4. Prove that $B_i \cup B_U \cup B_V$ is linearly independent. Hint: write the condition of linear independence. Express the linear combination of the elements of B_U as a linear combination of B_i and B_V. Why is this linear combination in $U \cap V$? What you can conclude from the fact that $B_i \cup B_U$ is a basis?
5. Prove that $\dim U + V = \dim U + \dim V - \dim U \cap V$.

Quotient spaces

152. Consider vector space V and it's subspace U. We introduce a relation on V: $v \approx u$ iff $v - u \in V$. Prove that $\approx$ is an equivalence relation.

153. Prove that addition and scalar multiplication on $V/\approx$ are well-defined (independent on the class representative), that is:

1. if $v \approx v'$ and $u \approx u'$, then $v + u \approx v' + u'$
2. if α is a scalar and $v \approx v'$ are vectors in V, then $\alpha v \approx \alpha v'$.

154. Prove that under relation $\approx$, U is identified with 0.

155. Let $U \subseteq V$ be finite-dimensional vector spaces. Prove that $\dim V/U = \dim V - \dim U$. Hint: guess what is the basis of V/U starting with basis of U and completing it to the basis ov V.

Linear maps

156. Let $L : V \to W$ be a function between vector spaces over field $\mathbb{F}$. Prove that the following sentences are equivalent:

1. L is linear
2. for every $u, v \in V$ and $\alpha \in \mathbb{F}$, we have $L(\alpha u + v) = \alpha L(u) + L(v)$
3. for every $v, u \in V$ we have $L(v + u) = L(v) + L(u)$ and for every $v \in V, \alpha \in \mathbb{F}$ we have $L(\alpha v) = \alpha L(v)$

157. Let U be a vector subspace of V. Prove that **the inclusion map** $\iota : U \to V$ given as $\iota(u) = u$ is linear.

158. Let U be a vector subspace of V. Prove that **the quotient map** $q : V \to V/U$ given as $q(v) = [v]$ is linear.

Kernel and cokernel

159. Let $L : V \to W$ be a linear map between vector spaces.

1. Prove that $L(0_V) = 0_W$, where 0_V is the neutral element in V and 0_W is the neutral element in W.
2. Prove that the **kernel of** L defined as: $\ker L = \{v \in V : L(v) = 0_W\}$ is a vector subspace of V.
3. Prove that the image of L is a vector subspace of W. The dimension of $\operatorname{im} L$ is called **the rank** of L: $\operatorname{rk} L = \dim \operatorname{im} L$.
4. Let $V = \operatorname{span} S$. Prove that $\operatorname{im} L = \operatorname{span} L(S)$. Here $L(S)$ has meaning $\{L(s) : s \in S\}$.
5. Prove that if V is finite dimensional, then $\operatorname{im} L$ is also finite dimensional.

[7]if $U \cap W = \varnothing$ we have just $U + W = U \oplus W$

160. Prove that $\varphi : V \to U$ is injective iff $\ker \varphi = \{0\}$.

161. Prove that $\varphi : V \to U$ is surjective iff $\operatorname{coker} \varphi$ has exactly one element (is a trivial vector space).

Rank-nullity theorem

162. Prove the **rank-nullity theorem** - if V is a finite dimensional vector space and $L : V \to W$ is linear, then $\dim \ker L + \operatorname{rk} L = \dim V$.

163. You can do a beuatiful and simple proof of $\dim U + V = \dim U + \dim V - \dim U \cap V$, where U and V are finite-dimensional subspaces of some greater space S. Consider a map $L : U \times V \to S$ defined as $L(u, w) = u - w$. Prove that it is a linear map (we give $U \times V$ a linear space structure using entry-wise operations, as in $U \oplus V$). What is it's range and kernel?

164. Let $\varphi : V \to U$ be a map between finite-dimensional vector spaces. Prove that the **index** of φ defined as $\operatorname{index} \varphi = \dim \ker \varphi - \dim \operatorname{coker} \varphi$ is equal to: $\operatorname{index} \varphi = \dim V - \dim U$.[8]

165. Prove that V and $\{0\} \times V$ are isomorphic. (Do you remember the direct sum of two subspaces? In fact we used there this isomorphism).

166. Prove that two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension. Hint: right implication - rank-nullity theorem, left - write down bases.

Exact sequences

167. Prove that if sequence of vector spaces V_i and maps $\varphi_i : V_i \to V_{i+1}$, is exact, then $\varphi_i \circ \varphi_{i+1} = \mathbf{0}$, where $\mathbf{0}$ is a null [9] map $\mathbf{0} : V_i \to V_{i+2}$ defined as $\mathbf{0}(v) = 0$.

168. Prove the following:

1. sequence $V \xrightarrow{\varphi} U \longrightarrow 0$ is exact iff φ is surjective. Hint: what is kernel of the map $U \to 0$?
2. sequence $0 \longrightarrow V \xrightarrow{\varphi} U$ is exact iff φ is injective. Hint: do you remember how injectivity is related to some kernel?
3. sequence $0 \longrightarrow V \xrightarrow{\varphi} U \longrightarrow 0$ is exact iff V and U are isomorphic.
4. short sequence $0 \longrightarrow V_2 \xrightarrow{\varphi_2} V_3 \xrightarrow{\varphi_3} V_4 \longrightarrow 0$ is exact iff φ_2 is injective and φ_3 is surjective.

169. Use the rank-nullity theorem and prove that if $0 \to V_0 \to V_1 \to \cdots \to V_{n-1} \to V_n \to 0$ is exact, then

$$0 = \sum_{i=0}^{n} (-1)^i \dim V_i.$$

170. Consider a linear map $L : V \to U$. Prove that the sequence:

$$0 \longrightarrow \ker L \xrightarrow{\kappa} V \xrightarrow{L} U \longrightarrow \operatorname{coker} L \longrightarrow 0$$

is exact, where $\kappa : \ker L \to V$ is inclusion map: $\kappa : \ker L \ni l \to l \in V$.

171. Let $L : V \to U$ be a map between finite-dimensional vector spaces. Prove once again that the $\operatorname{index} L := \dim \ker L - \dim \operatorname{coker} L = \dim V - \dim U$.

[8] This is a very important result - index, defined with the help of a chosen function is doesn't in fact depend on this function! A beautiful generalisation of this result, is called Atiyah-Singer index theorem.

[9] Later we will refer to this map just as 0 - now this symbol has at least three meanings! It can be an additive neutral element of a field, a neutral element in a vector space or a linear map!

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "f^*(\omega)"'] & & U \arrow[dl, "\omega"] \\ & \mathbb{R} & \end{tikzcd}$$

Dual spaces

172. Prove that V^* is a vector space by:

1. Finding the neutral element. Hint: function that always yields 0 is linear.
2. Proving that addition and scalar multiplication can be defined, so if μ, $\omega \in V^*$ and $a \in \mathbb{F}$, then function $a\mu+\omega$, defined as: $(a\mu+\omega)(v) = a \cdot \mu(v) + \omega(v)$ for all $v \in V$, is linear.

173. Let $V^{**} = (V^*)^*$ be the dual space of the dual space to V. We will try to prove that, in some sense, V is a subset of it. Prove that:

1. For each $v \in V$ there is a $\tilde{v} \in V^{**}$ such that for every $\omega \in V^*$ such that $\langle \tilde{v}, \omega \rangle = \langle \omega, v \rangle$
2. Prove that the map $v \mapsto \tilde{v}$ is a monomorphism (an injective linear map).

174. Let $f : V \to U$ be a linear map between finite dimensional spaces. We define $f^* : U^* \to V^*$ as follows: Prove that such map is linear and that $f^* \circ \omega = \omega \circ f$ for every $\omega \in U^*$. Here we see how a function sending vectors in one direction can naturally induce a linear map that ,,pulls-back" dual vectors.

175. Let V be a finite dimensional space with basis $e_1, e_2, \ldots, e_n$. Prove that $\mu^1, \mu^2, \ldots, \mu^n \in V^*$ defined as

$$\mu^i(e_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases},$$

is a basis of V^*.

Tensors

176. Consider a multilinear function f from $V_1 \times V_2 \times \cdots \times V_n$ to $\mathbb{F}$. Prove that f is a linear mapping from $V_1 \oplus V_2 \oplus \cdots \oplus V_n$, to $\mathbb{F}$.

177. Find such f that f is a linear mapping from $V_1 \oplus V_2 \oplus \cdots \oplus V_n$,to $\mathbb{F}$, but f *cannot* be treated as a mutlilinear function from $V_1 \times V_2 \times \cdots \times V_n$ to $\mathbb{F}$.

Universal property

178. Let V and W be vector spaces. Assume that there is a space T and bilinear map φ have the universal property and space U an bilinear map θ also have the universal property. Prove that T and U are naturally isomorphic.

179. Define $f : V \times W \to W \otimes V$ as $f(v, w) = w \otimes v$. Using the universal property, prove that there is an isomorphism $v \otimes w \mapsto w \otimes v$.

180. Prove that there is a canonical isomorphism between $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ by:

1. Proving that map $\lambda_u : V \times W \to (U \otimes V) \otimes W$ given by $\lambda_u(v, w) = (u \otimes v) \otimes w$ is bilinear.
2. As λ_u is bilinear, you can find linear map $\tilde{\lambda}_u : V \otimes W \to (U \otimes V) \otimes W$. Prove that map $\Lambda : U \times (V \otimes W) \to (U \otimes V) \otimes W$ given by $\Lambda(u, \omega) = \tilde{\lambda}_u(\omega)$ is bilinear. Find therefore a map $U \otimes (V \otimes W)$ to $(U \otimes V) \otimes W$.

Construction

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "f"'] & V_S \arrow[d, "\tilde{f}"] \\ & U \end{tikzcd}$$

181. Let V be a set of all functions from set S to field $\mathbb{F}$ such that $f(s) = 0$ for all but finitely many $s \in S$. Prove that V is a vector space.

182. Consider a set S and a free vector space generated by it V_S. We define inclusion mapping $\iota : S \to V_S$ as $S \ni s \mapsto s \in V_s$. Prove that for every function $f : S \to U$, where U is a vector space, there is a unique linear map $\tilde{f} : V_S \to U$ such that $f = f \circ \iota$. This can be written as a commutative diagram:

183. Let V and W be vector spaces over a field $\mathbb{F}$. Let A be a free vector space over $V \times W$. Consider set S containing elements of the forms

$$(v + v', w + w') - (v', w'), \quad (\alpha v, \alpha w) - \alpha(v, w)$$

for $(v, w) \in V \times W$ and it's free vector space B. Prove that $V \oplus U$ is naturally isomorphic to A/B.

184. Let V and W be vector spaces over a field $\mathbb{F}$. Let A be a free vector space over $V \times W$. Consider a set S containing all the elements of the forms:

$$(v + v', w) - (v, w) - (v', w), \ (\alpha v, w) - \alpha(v, w) \tag{1}$$
$$(v, w + w') - (v, w) - (v, w'), \ (v, \alpha w) - \alpha(v, w) \tag{2}$$

and it's free vector space B. Prove that A/B has the universal property of tensor product, with $v \otimes w = [(v, w)]$.

185. Let V and W be finite dimensional over $\mathbb{F}$. Consider a set S of all bilinear functions from $V^* \times W^* \to \mathbb{F}$. As we remember, for finite dimensional V we have a natural isomorphism $V \approx V^{**}$, so we can write $v(\nu) \in \mathbb{F}$ for $v \in V$, $\nu \in V^*$. Let's define: $v \otimes w(\nu, \omega) = v(\nu) \times w(\omega)$. Prove that:

1. $v \otimes w \in S$ (so it must be a bilinear function)
2. $\{v_i \otimes w_j\}$ form a basis of S if $\{v_i\}$ and $\{w_j\}$ are bases of V and W.
3. $V \otimes W$ defined as above is naturally isomorphic to S.
4. $\dim V \otimes W = \dim V \cdot \dim W$

186. Let V, W, U be finite dimensional vector spaces over the same field. Prove that there is a natural isomorphism:

$$V \otimes (W \oplus U) \approx (V \otimes W) \oplus (V \otimes U).$$

187. Prove that for finite dimensional V and W, there is a natural isomorphism between $\mathrm{Hom}(V, W)$ and $V^* \otimes W$.

188. Prove that for finite dimensional V, $\operatorname{End} V$ is naturally isomorphic to to $\operatorname{End} V^*$.