# Econ 424/Amath 540
# Introduction to Portfolio Theory

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## Introduction to Portfolio Theory

Investment in Two Risky Assets

$$
\begin{aligned}
R_A &= \text{ simple return on asset A} \\
R_B &= \text{ simple return on asset B} \\
W_0 &= \text{ initial wealth}
\end{aligned}
$$

Assumptions

- $R_A$ and $R_B$ are described by the CER model

$$
R_i \sim \text{iid } N(\mu_i, \sigma_i^2),\ i = A, B
$$
$$
\text{cov}(R_A, R_B) = \sigma_{AB},\ \text{cor}(R_A, R_B) = \rho_{AB}
$$

- Investors like high $E[R_i] = \mu_i$

- Investors dislike high $\text{var}(R_i) = \sigma_i^2$

- Investment horizon is one period (e.g., one month or one year)

Note: Traditionally in portfolio theory, returns are simple and not continuously compounded

**Portfolios**

$$x_A = \text{share of wealth in asset A} = \frac{\text{\$ in A}}{W_0}$$

$$x_B = \text{share of wealth in asset B} = \frac{\text{\$ in B}}{W_0}$$

Long position

$$x_A,\ x_B > 0$$

Short position

$$x_A < 0 \text{ or } x_B < 0$$

Assumption: Allocate all wealth between assets A and B

$$x_A + x_B = 1$$

Portfolio Return

$$R_p = x_A R_A + x_B R_B$$

Portfolio Distribution

$$\begin{aligned}
\mu_p &= E[R_p] = x_A\mu_A + x_B\mu_B \\
\sigma_p^2 &= \mathrm{var}(R_p) = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB} \\
&= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\rho_{AB}\sigma_A\sigma_B \\
R_p &\sim \text{iid } N(\mu_p, \sigma_p^2)
\end{aligned}$$

End of Period Wealth

$$\begin{aligned}
W_1 &= W_0(1 + R_p) = W_0(1 + x_AR_A + x_BR_B) \\
W_1 &\sim N(W_0(1 + \mu_p), \sigma_p^2 W_0^2)
\end{aligned}$$

Example Data

$$
\begin{aligned}
\mu_A &= 0.175,\ \mu_B = 0.055 \\
\sigma_A^2 &= 0.067,\ \sigma_B^2 = 0.013 \\
\sigma_A &= 0.258,\ \sigma_B = 0.115 \\
\sigma_{AB} &= -0.004875, \\
\rho_{AB} &= \frac{\sigma_{AB}}{\sigma_A \sigma_B} = -0.164
\end{aligned}
$$

Note: Asset A has higher expected return and risk than asset B

**Example**: Long only two asset portfolio

Consider an equally weighted portfolio with $x_A = x_B = 0.5$. The expected return, variance and volatility are

$$\begin{aligned}
\mu_p &= (0.5) \cdot (0.175) + (0.5) \cdot (0.055) = 0.115 \\
\sigma_p^2 &= (0.5)^2 \cdot (0.067) + (0.5)^2 \cdot (0.013) \\
&\quad + 2 \cdot (0.5)(0.5)(-0.004875) = 0.01751 \\
\sigma_p &= \sqrt{0.01751} = 0.1323
\end{aligned}$$

This portfolio has expected return half-way between the expected returns on assets A and B, but the portfolio standard deviation is less than half-way between the asset standard deviations. This reflects risk reduction via diversification.

**Example**: Long-Short two asset portfolio

Next, consider a long-short portfolio with $x_A = 1.5$ and $x_B = -0.5$. In this portfolio, asset B is sold short and the proceeds of the short sale are used to leverage the investment in asset A. The portfolio characteristics are

$$\begin{aligned}
\mu_p &= (1.5) \cdot (0.175) + (-0.5) \cdot (0.055) = 0.235 \\
\sigma_p^2 &= (1.5)^2 \cdot (0.067) + (-0.5)^2 \cdot (0.013) \\
&\quad + 2 \cdot (1.5)(-0.5)(-0.004875) = 0.1604 \\
\sigma_p &= \sqrt{0.01751} = 0.4005
\end{aligned}$$

This portfolio has both a higher expected return and standard deviation than asset A

## Portfolio Value-at-Risk

- Assume an initial investment of $\$W_0$ in the portfolio of assets A and B.

- Given that the simple return $R_p \sim N(\mu_p, \sigma_p^2)$, For $\alpha \in (0, 1)$, the $\alpha \times 100\%$ portfolio value-at-risk is

$$\begin{aligned} \mathsf{VaR}_{p,\alpha} &= q_{p,\alpha}^R W_0 \\ &= \left(\mu_p + \sigma_p q_\alpha^z\right) W_0 \end{aligned}$$

where $q_{p,\alpha}^R$ is the $\alpha$ quantile of the distribution of $R_p$ and $q_\alpha^z = \alpha$ quantile of $Z \sim N(0, 1)$.

## Relationship between Portfolio VaR and Individual Asset VaR

**Result**: Portfolio VaR is not a weighted average of asset VaR

$$\mathsf{VaR}_{p,\alpha} \neq x_A \mathsf{VaR}_{A,\alpha} + x_B \mathsf{VaR}_{B,\alpha}$$

unless $\rho_{AB} = 1$.

Asset VaRs for A and B are

$$\begin{aligned}\mathsf{VaR}_{A,\alpha} &= q_{0.05}^{R_A} W_0 = (\mu_A + \sigma_A q_\alpha^z) W_0 \\ \mathsf{VaR}_{B,\alpha} &= q_{0.05}^{R_B} W_0 = (\mu_B + \sigma_B q_\alpha^z) W_0\end{aligned}$$

Portfolio VaR is

$$\begin{aligned}\mathsf{VaR}_{p,\alpha} &= (\mu_p + \sigma_p q_\alpha^z)\, W_0 \\ &= \left[(x_A \mu_A + x_B \mu_B) + \left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}\right)^{1/2} q_\alpha^z\right] W_0\end{aligned}$$

Portfolio weighted asset VaR is

$$\begin{aligned}
x_A \mathsf{VaR}_{A,\alpha} + x_B \mathsf{VaR}_{B,\alpha} &= x_A(\mu_A + \sigma_A q_\alpha^z) W_0 + x_B(\mu_B + \sigma_B q_\alpha^z) W_0 \\
&= \left[(x_A\mu_A + x_B\mu_B) + (x_A\sigma_A + x_B\sigma_B)\, q_\alpha^z\right] W_0 \\
&\neq (\mu_p + \sigma_p q_\alpha^z)\, W_0 = \mathsf{VaR}_{p,\alpha}
\end{aligned}$$

provided $\rho_{AB} \neq 1$.

If $\rho_{AB} = 1$ then $\sigma_{AB} = \rho_{AB}\sigma_A\sigma_B = \sigma_A\sigma_B$ and

$$\begin{aligned}
\sigma_p^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B \sigma_A\sigma_B = (x_A\sigma_A + x_B\sigma_B)^2 \\
&\Rightarrow \sigma_p = x_A\sigma_A + x_B\sigma_B
\end{aligned}$$

and so

$$x_A \mathsf{VaR}_{A,\alpha} + x_B \mathsf{VaR}_{B,\alpha} = \mathsf{VaR}_{p,\alpha}$$

**Example**: Portfolio VaR and Individual Asset VaR

Consider an initial investment of $W_0$ =\$100,000. The 5% VaRs on assets A and B are

$$\text{VaR}_{A,0.05} = q_{0.05}^{R_A} W_0 = (0.175 + 0.258(-1.645)) \cdot 100,000 = -24,937,$$
$$\text{VaR}_{B,0.05} = q_{0.05}^{R_B} W_0 = (0.055 + 0.115(-1.645)) \cdot 100,000 = -13,416.$$

The 5% VaR on the equal weighted portfolio with $x_A = x_B = 0.5$ is

$$\text{VaR}_{p,0.05} = q_{0.05}^{R_p} W_0 = (0.115 + 0.1323(-1.645)) \cdot 100,000 = -10,268,$$

and the weighted average of the individual asset VaRs is

$$x_A \text{VaR}_{A,0.05} + x_B \text{VaR}_{B,0.05} = 0.5(-24,937) + 0.5(-13,416) = -19,177.$$

## Portfolio Frontier

Vary investment shares $x_A$ and $x_B$ and compute resulting values of $\mu_p$ and $\sigma_p^2$. Plot $\mu_p$ against $\sigma_p$ as functions of $x_A$ and $x_B$

- Shape of portfolio frontier depends on correlation between assets A and B

- If $\rho_{AB} = -1$ then there exists portfolio shares $x_A$ and $x_B$ such that $\sigma_p^2 = 0$

- If $\rho_{AB} = 1$ then there is no benefit from diversification

- Diversification is beneficial even $0 < \rho_{AB} < 1$

## Efficient Portfolios

**Definition**: Portfolios with the highest expected return for a given level of risk, as measured by portfolio standard deviation, are efficient portfolios

- If investors like portfolios with high expected returns and dislike portfolios with high return standard deviations then they will want to hold efficient portfolios

- Which efficient portfolio an investor will hold depends on their risk preferences
  - Very risk averse investors dislike volatility and will hold portfolios near the global minimum variance portfolio. They sacrifice expected return for the safety of low volatility
  - Risk tolerant investors don't mind volatility and will hold portfolios that have high expected returns. They gain expected return by taking on more volatility.

## Globabl Minimum Variance Portfolio

- The portfolio with the smallest possible variance is called the global minimum variance portfolio.

- This portfolio is chosen by the most risk averse individuals

- To find this portfolio, one has to solve the following *constrained minimization problem*

$$\min_{x_A, x_B} \ \sigma_p^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B \sigma_{AB}$$
$$s.t.\ x_A + x_B = 1$$

**Review of Optimization Techniques: Constrained Optimization**

Example: Finding the minimum of a bivariate function subject to a linear constraint

$$
\begin{aligned}
y &= f(x,z) = x^2 + z^2 \\
\min_{x,z}\ y &= f(x,z) \\
s.t.\ x + z &= 1
\end{aligned}
$$

Solution methods:

- Substitution

- Lagrange multipliers

## Method of Substitution

Substitute $z = x - 1$ in $f(x, z)$ and solve univariate minimization

$$y = f(x, x - 1) = x^2 + (1 - x)^2$$
$$\min_x \; f(x, x - 1)$$

First order conditions

$$\begin{aligned} 0 &= \frac{d}{dx}(x^2 + (1 - x)) = 2x + 2(1 - x)(-1) \\ &= 4x - 2 \\ &\Rightarrow x = 0.5 \end{aligned}$$

Solving for $z$

$$z = 1 - 0.5 = 0.5$$

**Method of Lagrange Multipliers**

Idea: Augment function to be minimized with extra terms to impose constraints

1. Put constraints in homogeneous form

$$x + z = 1 \Rightarrow x + z - 1 = 0$$

2. Form Lagrangian function

$$L(x, z, \lambda) = x^2 + z^2 + \lambda(x + z - 1)$$
$$\lambda = \text{Lagrange multiplier}$$

3. Minimize Lagrangian function

$$\min_{x,z,\lambda} \; L(x,z,\lambda)$$

First order conditions

$$\begin{aligned} 0 &= \frac{\partial L(x,z,\lambda)}{\partial x} = 2 \cdot x + \lambda \\ 0 &= \frac{\partial L(x,z,\lambda)}{\partial z} = 2 \cdot z + \lambda \\ 0 &= \frac{\partial L(x,z,\lambda)}{\partial \lambda} = x + z - 1 \end{aligned}$$

We have three linear equations in three unknowns. Solving gives

$$2x = 2z = -\lambda \Rightarrow x = z$$

$$2z - 1 = 0 \Rightarrow z = 0.5, \; x = 0.5$$

**Example**: Finding the Global Minimum Variance Portfolio

Two methods for solution

- Analytic solution using Calculus
- Numerical solution
  - use the Solver in Excel
  - use R function `solve.QP()` in package quadprog for quadratic optimization problems with equality and inequality constraints

## Calculus Solution

Minimization problem

$$\min_{x_A, x_B} \sigma_p^2 = x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB}$$
$$s.t.\ x_A + x_B = 1$$

Use substitution method with

$$x_B = 1 - x_A$$

to give the univariate minimization

$$\min_{x_A} \sigma_p^2 = x_A^2 \sigma_A^2 + (1 - x_A)^2 \sigma_B^2 + 2 x_A (1 - x_A) \sigma_{AB}$$

First order conditions

$$0 = \frac{d}{dx_A}\sigma_p^2 = \frac{d}{dx_A}\left(x_A^2\sigma_A^2 + (1-x_A)^2\sigma_B^2 + 2x_A(1-x_A)\sigma_{AB}\right)$$

$$= 2x_A\sigma_A^2 - 2(1-x_A)\sigma_B^2 + 2\sigma_{AB}(1-2x_A)$$

$$\Rightarrow x_A^{\mathrm{min}} = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}},\ x_B^{\mathrm{min}} = 1 - x_A^{\mathrm{min}}$$

## Excel Solver Solution

The Solver is an Excel add-in, that can be used to numerically solve general linear and nonlinear optimization problems subject to equality or inequality constraints

- The solver is made by FrontLine Systems and is provided with Excel

- The solver add-in may not be installed in a "default installation" of Excel
  - Tools/Add-Ins and check the Solver Add-In box
  - If Solver Add-In box is not available, the Solver Add-In must be installed from original Excel installation CD

**Portfolios with a Risk Free Asset**

Risk Free Asset

- Asset with fixed and known rate of return over investment horizon

- Usually use U.S. government T-Bill rate (horizons $<$ 1 year) or T-Note rate (horizon $>$ 1 yr)

- T-Bill or T-Note rate is only nominally risk free

**Properties of Risk-Free Asset**

$$
\begin{aligned}
R_f &= \text{return on risk-free asset} \\
E[R_f] &= r_f = \text{ constant} \\
\text{var}(R_f) &= 0 \\
\text{cov}(R_f, R_i) &= 0,\ R_i = \text{ return on any asset}
\end{aligned}
$$

Portfolios of Risky Asset and Risk Free Asset

$$
\begin{aligned}
x_f &= \text{ share of wealth in T-Bills} \\
x_B &= \text{ share of wealth in asset B} \\
x_f + x_B &= 1 \\
x_f &= 1 - x_B
\end{aligned}
$$

Portfolio return

$$\begin{aligned} R_p &= x_f r_f + x_B R_B \\ &= (1 - x_B) r_f + x_B R_B \\ &= r_f + x_B (R_B - r_f) \end{aligned}$$

Portfolio excess return

$$R_p - r_f = x_B (R_B - r_f)$$

Portfolio Distribution

$$\mu_p = E[R_p] = r_f + x_B(\mu_B - r_f)$$

$$\sigma_p^2 = \mathrm{var}(R_p) = x_B^2 \sigma_B^2$$

$$\sigma_p = x_B \sigma_B$$

$$R_p \sim N(\mu_p, \sigma_p^2)$$

**Risk Premium**

$$\begin{aligned}\mu_B - r_f &= \text{excess expected return on asset B} \\ &= \text{expected return on risky asset over return on safe asset}\end{aligned}$$

For the portfolio of T-Bills and asset B

$$\begin{aligned}\mu_p - r_f &= x_B(\mu_B - r_f) \\ &= \text{expected portfolio return over T-Bill}\end{aligned}$$

The risk premia is an increasing function of the amount invested in asset B.

**Leveraged Investment**

$$x_f < 0,\ x_B > 1$$

Borrow at T-Bill rate to buy more of asset B

Result: Leverage increases portfolio expected return and risk

$$\mu_p = r_f + x_B(\mu_B - r_f)$$
$$\sigma_p = x_B \sigma_B$$
$$x_B \uparrow \ \Rightarrow \mu_p \ \& \ \sigma_p \uparrow$$

## Determining Portfolio Frontier

Goal: Plot $\mu_p$ vs. $\sigma_p$

$$\begin{aligned}
\sigma_p &= x_B \sigma_B \Rightarrow x_B = \frac{\sigma_p}{\sigma_B} \\
\mu_p &= r_f + x_B(\mu_B - r_f) \\
&= r_f + \frac{\sigma_p}{\sigma_B}(\mu_B - r_f) \\
&= r_f + \left(\frac{\mu_B - r_f}{\sigma_B}\right)\sigma_p
\end{aligned}$$

where

$$\begin{aligned}
\left(\frac{\mu_B - r_f}{\sigma_B}\right) &= \mathsf{SR}_B = \text{Asset B Sharpe Ratio} \\
&= \text{excess expected return per unit risk}
\end{aligned}$$

**Remarks**

- The Sharpe Ratio (SR) is commonly used to rank assets.

- Assets with high Sharpe Ratios are preferred to assets with low Sharpe Ratios

## Efficient Portfolios with 2 Risky Assets and a Risk Free Asset

Investment in 2 Risky Assets and T-Bill

$$
\begin{aligned}
R_A &= \text{ simple return on asset A} \\
R_B &= \text{ simple return on asset B} \\
R_f = r_f &= \text{ return on T-Bill}
\end{aligned}
$$

Assumptions

- $R_A$ and $R_B$ are described by the CER model

$$
R_i \sim iid\ N(\mu_i, \sigma_i^2),\ i = A, B
$$
$$
\text{cov}(R_A, R_B) = \sigma_{AB},\ \text{corr}(R_A, R_B) = \rho_{AB}
$$

**Results**:

- The best portfolio of two risky assets and T-Bills is the one with the highest Sharpe Ratio

- Graphically, this portfolio occurs at the tangency point of a line drawn from $R_f$ to the risky asset only frontier.

- The maximum Sharpe Ratio portfolio is called the "tangency portfolio"

**Mutual Fund Separation Theorem**

Efficient portfolios are combinations of two portfolios (mutual funds)

- T-Bill portfolio

- Tangency portfolio - portfolio of assets A and B that has the maximum Shape ratio

Implication: All investors hold assets A and B according to their proportions in the tangency portfolio regardless of their risk preferences.

**Finding the tangency portfolio**

$$\max_{x_A,\, x_B} \mathsf{SR}_p = \frac{\mu_p - r_f}{\sigma_p} \text{ subject to}$$

$$\begin{aligned}
\mu_p &= x_A\mu_A + x_B\mu_B \\
\sigma_p^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_A x_B \sigma_{AB} \\
1 &= x_A + x_B
\end{aligned}$$

Solution can be found analytically or numerically (e.g., using solver in Excel)

Using the substitution method it can be shown that

$$x_A^{\text{tan}} = \frac{(\mu_A - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_{AB}}{(\mu_A - r_f)\sigma_B^2 + (\mu_B - r_f)\sigma_A^2 - (\mu_A - r_f + \mu_B - r_f)\sigma_{AB}}$$

$$x_B^{\text{tan}} = 1 - x_A^{\text{tan}}$$

Portfolio characteristics

$$\mu_p^{\text{tan}} = x_A^{\text{tan}}\mu_A + x_B^{\text{tan}}\mu_B$$

$$\left(\sigma_p^{\text{tan}}\right)^2 = \left(x_A^{\text{tan}}\right)^2 \sigma_A^2 + \left(x_B^{\text{tan}}\right)^2 \sigma_B^2 + 2x_A^{\text{tan}}x_B^{\text{tan}}\sigma_{AB}$$

## Efficient Portfolios: tangency portfolio plus T-Bills

$$
\begin{aligned}
x_{\mathsf{tan}} &= \text{ share of wealth in tangency portfolio} \\
x_f &= \text{share of wealth in T-bills} \\
x_{\mathsf{tan}} + x_f &= 1 \\
\mu_p^e &= r_f + x_{\mathsf{tan}}(\mu_p^{\mathsf{tan}} - r_f) \\
\sigma_p^e &= x_{\mathsf{tan}}\sigma_p^{\mathsf{tan}}
\end{aligned}
$$

Result: The weights $x_{\mathsf{tan}}$ and $x_f$ are determined by an investor's risk preferences

- Risk averse investors hold mostly T-Bills

- Risk tolerant investors hold mostly tangency portfolio

**Example**

For the two asset example, the tangency portfolio is

$$
\begin{aligned}
x_A^{\mathsf{tan}} &= .46,\ x_B^{\mathsf{tan}} = 0.54 \\
\mu_p^{\mathsf{tan}} &= (.46)(.175) + (.54)(.055) = 0.11 \\
\left(\sigma_p^{\mathsf{tan}}\right)^2 &= (.46)^2(.067) + (.54)^2(.013) \\
&\quad + 2(.46)(.54)(-.005) \\
&= 0.015 \\
\sigma_p^{\mathsf{tan}} &= \sqrt{.015} = 0.124
\end{aligned}
$$

Efficient portfolios have the following characteristics

$$
\begin{aligned}
\mu_p^e &= r_f + x_{\mathsf{tan}}(\mu_p^{\mathsf{tan}} - r_f) \\
&= 0.03 + x_{\mathsf{tan}}(0.11 - 0.03) \\
\sigma_p^e &= x_{\mathsf{tan}}\sigma_p^{\mathsf{tan}} \\
&= x_{\mathsf{tan}}(0.124)
\end{aligned}
$$

**Problem**: Find the efficient portfolio that has the same risk (SD) as asset B? That is, determine $x_{\text{tan}}$ and $x_f$ such that

$$\sigma_p^e = \sigma_B = 0.114 = \text{target risk}$$

Note: The efficient portfolio will have a higher expected return than asset B

**Solution**:

$$
\begin{aligned}
.114 = \sigma_p^e &= x_{\mathsf{tan}}\sigma_p^{\mathsf{tan}} \\
&= x_{\mathsf{tan}}(.124) \\
&\Rightarrow x_{\mathsf{tan}} = \frac{.114}{.124} = .92 \\
x_f &= 1 - x_{\mathsf{tan}} = .08
\end{aligned}
$$

Efficient portfolio with same risk as asset B has

$$
\begin{aligned}
(.92)(.46) &= .42 \text{ in asset A} \\
(.92)(.54) &= .50 \text{ in asset B} \\
&\phantom{=}\ .08 \text{ in T-Bills}
\end{aligned}
$$

If $r_f = 0.03$, then expected Return on efficient portfolio is

$$
\mu_p^e = .03 + (.92)(.11 - 0.03) = .104
$$

**Problem**: Assume that $r_f = 0.03$. Find the efficient portfolio that has the same expected return as asset B. That is, determine $x_{\textsf{tan}}$ and $x_f$ such that

$$\mu_p^e = \mu_B = 0.055 = \text{target expected return}$$

Note: The efficient portfolio will have a lower SD than asset B

**Solution**:

$$\begin{aligned} 0.055 = \mu_p^e &= 0.03 + x_{\text{tan}}(.11 - .03) \\ x_{\text{tan}} &= \frac{0.055 - 0.03}{.11 - .03} = .31 \\ x_f &= 1 - x_{\text{tan}} = .69 \end{aligned}$$

Efficient portfolio with same expected return as asset B has

$$\begin{aligned} (.31)(.46) &= .14 \text{ in asset A} \\ (.31)(.54) &= .17 \text{ in asset B} \\ &\phantom{=}\ .69 \text{ in T-Bills} \end{aligned}$$

The SD of the efficient portfolio is

$$\sigma_p^e = .31(.124) = .038$$