Econ 424/Amath 540
Statistical Analysis of Efficient Portfolios

Eric Zivot

August 7, 2012

The CER Model and Efficient Portfolios

Let R_{it} denote the return on asset i in month t and assume that R_{it} follows CER model:

$$
\begin{aligned}
R_{it} &\sim iid\ N(\mu_i, \sigma_i^2), \\
& i = 1, \ldots, N \text{ (assets)} \\
& t = 1, \ldots, T \text{ (months)} \\
cov(R_{it}, R_{jt}) &= \sigma_{ij}
\end{aligned}
$$

We estimate the CER model parameters using sample statistics giving

$$
\hat{\mu}_i, \hat{\sigma}_i^2, \hat{\sigma}_{ij}
$$

Remember, the estimates $\hat{\mu}_i, \hat{\sigma}_i^2$ are $\hat{\sigma}_{ij}$ are random variables and are subject to error

Key result: Sharpe ratios and efficient portfolios are functions of $\hat{\mu}_i, \hat{\sigma}_i^2, \hat{\sigma}_{ij}$; they are random variables and are subject to error

Statistical Properties of Efficient portfolios

- Inputs to portfolio theory are estimates from CER model $\hat{\mu}$ and $\hat{\Sigma}$
- Sharpe ratios and efficient portfolios are functions of $\hat{\mu}$ and $\hat{\Sigma}$.
- The estimated Sharpe ratio is

$$\widehat{SR}_i = \frac{\hat{\mu}_i - r_f}{\hat{\sigma}_i}$$

- No easy formula for $SE(\widehat{SR}_i)$

- The estimated global minimum variance portfolio is

$$\hat{\mathbf{m}} = \frac{\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}}$$

  $\hat{\mathbf{m}}$ is estimated with error because we estimate $\mathbf{\Sigma}$ using $\hat{\mathbf{\Sigma}}$.

- No easy analytic formulas for the standard errors of the elements of $\hat{\mathbf{m}} = (\hat{m}_1, \ldots, \hat{m}_n)'$; i.e. no easy formula for $SE(\hat{m}_i)$

- In addition, the expected return and standard deviation of $R_{p,\hat{m}} = \hat{\mathbf{m}}'\mathbf{R}$ have additional sources of error due to the error in $\hat{\mathbf{m}}$. That is,

$$\begin{aligned}\hat{\mu}_{p,\hat{m}} &= \hat{\mathbf{m}}'\hat{\mu} \\ \hat{\sigma}_{p,\hat{m}} &= (\hat{\mathbf{m}}'\hat{\mathbf{\Sigma}}\hat{\mathbf{m}})^{1/2}\end{aligned}$$

  No easy analytic formulas for SE($\hat{\mu}_{p,\hat{m}}$) and SE($\hat{\sigma}_{p,\hat{m}}$)

Bootstrapping Efficient Portfolios

The bootstrap can be used to evaluate the sampling uncertainty of Sharpe ratios and efficient portfolios.

Portfolio statistics to boostrap:

- Portfolio weights
- Portfolio expected returns and standard deviations

The CER Model and Efficient Portfolios

Result: We have seen evidence that the parameters of the CER model for various assets are not constant over time:

- Rolling estimates of μ, σ, and σ_{ij} show variation over time

Implication: Since estimates of μ, σ, and σ_{ij} are inputs to efficient portfolio calculations, then time variation in $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\sigma}_{ij}$ imply time variation in efficient portfolios

Rolling Efficient Portfolios

Idea: Using rolling estimates of μ and $\boldsymbol{\Sigma}$ compute rolling efficient portfolios

- global minimum variance portfolio
- efficient portfolio for target return
- tangency portfolio
- efficient frontier

Look at time variation in resulting portfolio weights

Rolling Global Minimum Variance Portfolio

Idea: compute estimates of portfolio weights $\mathbf{m}$ over rolling windows of length $n < T$:

$$\min_{\mathbf{m}(n)} \; \mathbf{m}_t(n)'\hat{\mathbf{\Sigma}}_t(n)\mathbf{m}_t(n) \;\text{ s.t. }\; \mathbf{m}_t(n)'\mathbf{1} = 1$$

$$t = n, \ldots, T$$

$$\hat{\mathbf{\Sigma}}_t(n) = \text{rolling estimate of } \mathbf{\Sigma} \text{ in month } t$$

If

$$\hat{\mathbf{\Sigma}}_n(n) \approx \hat{\mathbf{\Sigma}}_{n+1}(n) \approx \cdots \approx \hat{\mathbf{\Sigma}}_T(n)$$

then

$$\mathbf{m}_n(n) \approx \mathbf{m}_{n+1}(n) \approx \cdots \approx \mathbf{m}_T(n)$$

Rolling Efficient Portfolios

Idea: compute estimates of portfolio weights $\mathbf{x}$ over rolling windows of length $n < T$ for $t = n, \ldots, T$:

$$\begin{gathered}\min_{\mathbf{x}(n)} \ \mathbf{x}_t(n)'\hat{\mathbf{\Sigma}}_t(n)\mathbf{x}_t(n) \\ \text{s.t. } \mathbf{x}_t(n)'\mathbf{1} = \mathbf{1},\ \mathbf{x}_t(n)'\hat{\mu}_t(n) = \mu_p^{\text{target}} \\ \hat{\mu}_t(n) = \text{ rolling estimate of } \mu \text{ in month } t \\ \hat{\mathbf{\Sigma}}_t(n) = \text{rolling estimate of } \mathbf{\Sigma} \text{ in month } t\end{gathered}$$

If

$$\begin{gathered}\hat{\mu}_n(n) \approx \hat{\mu}_{n+1}(n) \approx \cdots \approx \hat{\mu}_T(n) \\ \hat{\mathbf{\Sigma}}_n(n) \approx \hat{\mathbf{\Sigma}}_{n+1}(n) \approx \cdots \approx \hat{\mathbf{\Sigma}}_T(n)\end{gathered}$$

then

$$\mathbf{x}_n(n) \approx \mathbf{x}_{n+1}(n) \approx \cdots \approx \mathbf{x}_T(n)$$