Econ 424/Amath 540
Portfolio Theory with Matrix Algebra

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Portfolio Math with Matrix Algebra

Three Risky Asset Example

Let R_i $(i = A, B, C)$ denote the return on asset i and assume that R_i follows CER model:

$$R_i \sim iid\ N(\mu_i, \sigma_i^2)$$
$$\mathrm{cov}(R_i, R_j) = \sigma_{ij}$$

Portfolio "$\mathbf{x}$"

$$x_i = \text{share of wealth in asset } i$$
$$x_A + x_B + x_C = 1$$

Portfolio return

$$R_{p,x} = x_A R_A + x_B R_B + x_C R_C.$$

Stock i	μ_i	σ_i	Pair (i,j)	σ_{ij}
A (Microsoft)	0.0427	0.1000	(A,B)	0.0018
B (Nordstrom)	0.0015	0.1044	(A,C)	0.0011
C (Starbucks)	0.0285	0.1411	(B,C)	0.0026

Table 1: Three asset example data.

In matrix algebra, we have

$$\mu = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} = \begin{pmatrix} 0.0427 \\ 0.0015 \\ 0.0285 \end{pmatrix}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} = \begin{pmatrix} (0.1000)^2 & 0.0018 & 0.0011 \\ 0.0018 & (0.1044)^2 & 0.0026 \\ 0.0011 & 0.0026 & (0.1411)^2 \end{pmatrix}$$

Matrix Algebra Representation

$$\mathbf{R} = \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix}, \ \mu = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix}, \ \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \ \mathbf{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix}$$

Portfolio weights sum to 1

$$\begin{aligned} \mathbf{x}'\mathbf{1} &= \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= x_1 + x_2 + x_3 = 1 \end{aligned}$$

Portfolio return

$$R_{p,x} = \mathbf{x}'\mathbf{R} = \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix}$$
$$= x_A R_A + x_B R_B + x_C R_C$$

Portfolio expected return

$$\mu_{p,x} = \mathbf{x}'\mu = \begin{pmatrix} x_A & x_B & x_X \end{pmatrix} \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix}$$
$$= x_A \mu_A + x_B \mu_B + x_C \mu_C$$

R formula

```
t(x.vec)%*%mu.vec
crossprod(x.vec, mu.vec)
```

Excel formula

```
MMULT(transpose(xvec),muvec)
<ctrl>-<shift>-<enter>
```

Portfolio variance

$$
\begin{aligned}
\sigma_{p,x}^2 &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \\
&= \begin{pmatrix} x_A & x_B & x_C \end{pmatrix} \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} \\
&= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 \\
&\quad + 2x_Ax_B\sigma_{AB} + 2x_Ax_C\sigma_{AC} + 2x_Bx_C\sigma_{BC}
\end{aligned}
$$

Portfolio distribution

$$R_{p,x} \sim N(\mu_{p,x}, \sigma_{p,x}^2)$$

R formulas

```
t(x.vec)%*%sigma.mat%*%x.vec
```

Excel formulas

```
MMULT(TRANSPOSE(xvec),MMULT(sigma,xvec))
MMULT(MMULT(TRANSPOSE(xvec),sigma),xvec)
<ctrl>-<shift>-<enter>
```

Covariance Between 2 Portfolio Returns

2 portfolios

$$\mathbf{x} = \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix}$$
$$\mathbf{x}'\mathbf{1} = 1, \ \mathbf{y}'\mathbf{1} = 1$$

Portfolio returns

$$R_{p,x} = \mathbf{x}'\mathbf{R}$$
$$R_{p,y} = \mathbf{y}'\mathbf{R}$$

Covariance

$$\begin{aligned} \mathrm{cov}(R_{p,x}, R_{p,y}) &= \mathbf{x}'\mathbf{\Sigma}\mathbf{y} \\ &= \mathbf{y}'\mathbf{\Sigma}\mathbf{x} \end{aligned}$$

R formula

```
t(x.vec)%*%sigma.mat%*%y.vec
```

Excel formula

```
MMULT(TRANSPOSE(xvec),MMULT(sigma,yvec))
MMULT(TRANSPOSE(yvec),MMULT(sigma,xvec))
<ctrl>-<shift>-<enter>
```

Derivatives of Simple Matrix Functions

Let $\mathbf{A}$ be an $n \times n$ symmetric matrix, and let $\mathbf{x}$ and $\mathbf{y}$ be an $n \times \mathbf{1}$ vectors. Then

$$\frac{\partial}{\underset{n\times 1}{\partial \mathbf{x}}} \mathbf{x}'\mathbf{y} = \begin{pmatrix} \frac{\partial}{\partial x_1}\mathbf{x}'\mathbf{y} \\ \vdots \\ \frac{\partial}{\partial x_n}\mathbf{x}'\mathbf{y} \end{pmatrix} = \mathbf{y}, \tag{1}$$

$$\frac{\partial}{\underset{n\times 1}{\partial \mathbf{x}}} \mathbf{x}'\mathbf{A}\mathbf{x} = \begin{pmatrix} \frac{\partial}{\partial x_1}\mathbf{x}'\mathbf{A}\mathbf{x} \\ \vdots \\ \frac{\partial}{\partial x_n}\mathbf{x}'\mathbf{A}\mathbf{x} \end{pmatrix} = 2\mathbf{A}\mathbf{x}. \tag{2}$$

Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

First, consider (1). Now

$$\mathbf{x}'\mathbf{y} = x_1 y_1 + x_2 y_2.$$

Then

$$\frac{\partial}{\partial \mathbf{x}}\mathbf{x}'\mathbf{y} = \begin{pmatrix} \frac{\partial}{\partial x_1}\mathbf{x}'\mathbf{y} \\ \frac{\partial}{\partial x_2}\mathbf{x}'\mathbf{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1}(x_1 y_1 + x_2 y_2) \\ \frac{\partial}{\partial x_2}(x_1 y_1 + x_2 y_2) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{y}.$$

Next, consider (2). We have

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2.$$

Then

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}}\mathbf{x}'\mathbf{A}\mathbf{x} &= \begin{pmatrix} \frac{\partial}{\partial x_1}\left(ax_1^2 + 2bx_1x_2 + cx_2^2\right) \\ \frac{\partial}{\partial x_2}\left(ax_1^2 + 2bx_1x_2 + cx_2^2\right) \end{pmatrix} = \begin{pmatrix} 2ax_1 + 2bx_2 \\ 2bx_1 + 2cx_2 \end{pmatrix} \\ &= 2\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= 2\mathbf{A}\mathbf{x}. \end{aligned}$$

Computing Global Minimum Variance Portfolio

Problem: Find the portfolio $\mathbf{m} = (m_A, m_B, m_C)'$ that solves

$$\min_{m_A, m_B, m_C} \sigma_{p,m}^2 = \mathbf{m}'\mathbf{\Sigma}\mathbf{m} \text{ s.t. } \mathbf{m}'\mathbf{1} = 1$$

1. Analytic solution using matrix algebra

2. Numerical Solution in Excel Using the Solver (see 3firmExample.xls)

Analytic solution using matrix algebra

The Lagrangian is

$$L(\mathbf{m},\lambda)=\mathbf{m}'\mathbf{\Sigma}\mathbf{m}+\lambda(\mathbf{m}'\mathbf{1}-1)$$

First order conditions (use matrix derivative results)

$$\underset{(3\times 1)}{\mathbf{0}}=\frac{\partial L(\mathbf{m},\lambda)}{\partial \mathbf{m}}=\frac{\partial \mathbf{m}'\mathbf{\Sigma}\mathbf{m}}{\partial \mathbf{m}}+\frac{\partial}{\partial \mathbf{m}}\lambda(\mathbf{m}'\mathbf{1}-1)=2\cdot\mathbf{\Sigma}\mathbf{m}+\lambda\mathbf{1}$$

$$\underset{(1\times 1)}{0}=\frac{\partial L(\mathbf{m},\lambda)}{\partial \lambda}=\frac{\partial \mathbf{m}'\mathbf{\Sigma}\mathbf{m}}{\partial \lambda}+\frac{\partial}{\partial \lambda}\lambda(\mathbf{m}'\mathbf{1}-1)=\mathbf{m}'\mathbf{1}-1$$

Write FOCs in matrix form

$$\begin{pmatrix} 2\mathbf{\Sigma} & \mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix}\begin{pmatrix} \mathbf{m} \\ \lambda \end{pmatrix}=\begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}\begin{matrix} 3\times 1 \\ 1\times 1 \end{matrix}.$$

The FOCs are the linear system

$$\mathbf{A}_m \mathbf{z}_m = \mathbf{b}$$

where

$$\mathbf{A}_m = \begin{pmatrix} 2\mathbf{\Sigma} & \mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix}, \ \mathbf{z}_m = \begin{pmatrix} \mathbf{m} \\ \lambda \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

The solution for $\mathbf{z}_m$ is

$$\mathbf{z}_m = \mathbf{A}_m^{-1}\mathbf{b}.$$

The first three elements of $\mathbf{z}_m$ are the portfolio weights $\mathbf{m} = (m_A, m_B, m_C)'$ for the global minimum variance portfolio with expected return $\mu_{p,m} = \mathbf{m}'\mu$ and variance $\sigma^2_{p,m} = \mathbf{m}'\mathbf{\Sigma}\mathbf{m}$.

Alternative Derivation of Global Minimum Variance Portfolio

The first order conditions from the optimization problem can be expressed in matrix notation as

$$\underset{(3\times1)}{\mathbf{0}} = \frac{\partial L(\mathbf{m},\lambda)}{\partial \mathbf{m}} = 2\cdot\mathbf{\Sigma}\mathbf{m}+\lambda\cdot\mathbf{1},$$

$$\underset{(1\times1)}{0} = \frac{\partial L(\mathbf{m},\lambda)}{\partial \lambda} = \mathbf{m}'\mathbf{1}-1.$$

Using first equation, solve for $\mathbf{m}$:

$$\mathbf{m} = -\frac{1}{2}\cdot\lambda\mathbf{\Sigma}^{-1}\mathbf{1}.$$

Next, multiply both sides by $\mathbf{1}'$ and use second equation to solve for λ:

$$\begin{aligned} 1 = \mathbf{1}'\mathbf{m} &= -\frac{1}{2} \cdot \lambda \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1} \\ \Rightarrow \lambda &= -2 \cdot \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \end{aligned}$$

Finally, substitute the value for λ in the equation for $\mathbf{m}$:

$$\begin{aligned} \mathbf{m} &= -\frac{1}{2}(-2)\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\boldsymbol{\Sigma}^{-1}\mathbf{1} \\ &= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \end{aligned}$$

Efficient Portfolios of Risky Assets: Markowitz Algorithm

Problem 1: find portfolio $\mathbf{x}$ that has the highest expected return for a given level of risk as measured by portfolio variance

$$\max_{x_A,x_B,x_C} \mu_{p,x} = \mathbf{x}'\mu \text{ s.t}$$
$$\sigma^2_{p,x} = \mathbf{x}'\mathbf{\Sigma x} = \sigma^0_p = \text{ target risk}$$
$$\mathbf{x}'\mathbf{1} = 1$$

Problem 2: find portfolio $\mathbf{x}$ that has the smallest risk, measured by portfolio variance, that achieves a target expected return.

$$\min_{x_A,x_B,x_C} \sigma^2_{p,x} = \mathbf{x}'\mathbf{\Sigma x} \text{ s.t.}$$
$$\mu_{p,x} = \mathbf{x}'\mu = \mu^0_p = \text{target return}$$
$$\mathbf{x}'\mathbf{1} = 1$$

Remark: Problem 2 is usually solved in practice by varying the target return between a given range.

Solving for Efficient Portfolios:

1. Analytic solution using matrix algebra

2. Numerical solution in Excel using the solver

Analytic solution using matrix algebra

The Lagrangian function associated with Problem 2 is

$$L(x, \lambda_1, \lambda_2) = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} + \lambda_1(\mathbf{x}'\mu - \mu_{p,0}) + \lambda_2(\mathbf{x}'\mathbf{1} - 1)$$

The FOCs are

$$\begin{aligned}
\underset{(3\times1)}{\mathbf{0}} &= \frac{\partial L(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \mathbf{x}} = 2\mathbf{\Sigma}\mathbf{x} + \lambda_1\mu + \lambda_2\mathbf{1}, \\
\underset{(1\times1)}{0} &= \frac{\partial L(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \lambda_1} = \mathbf{x}'\mu - \mu_{p,0}, \\
\underset{(1\times1)}{0} &= \frac{\partial L(\mathbf{x}, \lambda_1, \lambda_2)}{\partial \lambda_2} = \mathbf{x}'\mathbf{1} - 1.
\end{aligned}$$

These FOCs consist of five linear equations in five unknowns

$(x_A, x_B, x_C, \lambda_1, \lambda_2)$.

We can represent the FOCs in matrix notation as

$$\begin{pmatrix} 2\mathbf{\Sigma} & \mu & \mathbf{1} \\ \mu' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mu_{p,0} \\ 1 \end{pmatrix}$$

or

$$\mathbf{A}_x \mathbf{z}_x = \mathbf{b}_0$$

where

$$\mathbf{A}_x = \begin{pmatrix} 2\mathbf{\Sigma} & \mu & \mathbf{1} \\ \mu' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{pmatrix}, \ \mathbf{z}_x = \begin{pmatrix} \mathbf{x} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{b}_0 = \begin{pmatrix} \mathbf{0} \\ \mu_{p,0} \\ 1 \end{pmatrix}$$

The solution for $\mathbf{z}_x$ is then

$$\mathbf{z}_x = \mathbf{A}_x^{-1}\mathbf{b}_0.$$

The first three elements of $\mathbf{z}_x$ are the portfolio weights $\mathbf{x} = (x_A, x_B, x_C)'$ for the efficient portfolio with expected return $\mu_{p,x} = \mu_{p,0}$.

Example: Find efficient portfolios with the same expected return as MSFT and SBUX

For MSFT, we solve

$$\min_{x_A, x_B, x_C} \sigma^2_{p,x} = \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \text{ s.t.}$$

$$\mu_{p,x} = \mathbf{x}'\mu = \mu_{MSFT} = 0.0427$$

$$\mathbf{x}'\mathbf{1} = 1$$

For SBUX, we solve

$$\min_{y_A, y_B, y_C} \sigma^2_{p,x} = \mathbf{y}'\mathbf{\Sigma}\mathbf{y} \text{ s.t.}$$

$$\mu_{p,y} = \mathbf{y}'\mu = \mu_{SBUX} = 0.0285$$

$$\mathbf{y}'\mathbf{1} = 1$$

Example continued

Using the matrix algebra formulas (see R code in Powerpoint slides) we get

$$\mathbf{x} = \begin{pmatrix} x_{msft} \\ x_{nord} \\ x_{sbux} \end{pmatrix} = \begin{pmatrix} 0.8275 \\ -0.0908 \\ 0.2633 \end{pmatrix},\ \mathbf{y} = \begin{pmatrix} y_{msft} \\ y_{nord} \\ y_{sbux} \end{pmatrix} = \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix}$$

Also,

$$\begin{aligned}
\mu_{p,x} &= \mathbf{x}'\mu = 0.0427,\ \mu_{p,y} = \mathbf{y}'\mu = 0.0285 \\
\sigma_{p,x} &= (\mathbf{x}'\mathbf{\Sigma}\mathbf{x})^{1/2} = 0.09166,\ \sigma_{p,y} = (\mathbf{y}'\mathbf{\Sigma}\mathbf{y})^{1/2} = 0.07355 \\
\sigma_{xy} &= \mathbf{x}'\mathbf{\Sigma}\mathbf{y} = 0.005914,\ \rho_{xy} = \sigma_{xy}/(\sigma_{p,x}\sigma_{p,y}) = 0.8772
\end{aligned}$$

Computing the Portfolio Frontier

Result: The portfolio frontier can be represented as convex combinations of any two frontier portfolios. Let $\mathbf{x}$ be a frontier portfolio that solves

$$\begin{aligned}\min_{\mathbf{x}} \sigma^2_{p,x} &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \text{ s.t.} \\ \mu_{p,x} &= \mathbf{x}'\mu = \mu_p^0 \\ \mathbf{x}'\mathbf{1} &= 1\end{aligned}$$

Let $\mathbf{y} \neq \mathbf{x}$ be another frontier portfolio that solves

$$\begin{aligned}\min_{\mathbf{y}} \sigma^2_{p,y} &= \mathbf{y}'\mathbf{\Sigma}\mathbf{y} \text{ s.t.} \\ \mu_{p,y} &= \mathbf{y}'\mu = \mu_p^1 \neq \mu_p^0 \\ \mathbf{y}'\mathbf{1} &= 1\end{aligned}$$

Let α be any constant. Then the portfolio

$$\mathbf{z} = \alpha \cdot \mathbf{x} + (1 - \alpha) \cdot \mathbf{y}$$

is a frontier portfolio. Furthermore

$$\begin{aligned}
\mu_{p,z} &= \mathbf{z}'\mu = \alpha \cdot \mu_{p,x} + (1-\alpha)\mu_{p,y} \\
\sigma^2_{p,z} &= \mathbf{z}'\mathbf{\Sigma}\mathbf{z} \\
&= \alpha^2\sigma^2_{p,x} + (1-\alpha)^2\sigma^2_{p,y} + 2\alpha(1-\alpha)\sigma_{x,y} \\
\sigma_{x,y} &= \mathrm{cov}(R_{p,x}, R_{p,y}) = \mathbf{x}'\mathbf{\Sigma}\mathbf{y}
\end{aligned}$$

Example: 3 asset case

$$
\begin{aligned}
\mathbf{z} &= \alpha \cdot \mathbf{x} + (1-\alpha) \cdot \mathbf{y} \\
&= \alpha \cdot \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} + (1-\alpha) \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix} \\
&= \begin{pmatrix} \alpha x_A + (1-\alpha) y_A \\ \alpha x_B + (1-\alpha) y_B \\ \alpha x_C + (1-\alpha) y_C \end{pmatrix} = \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix}
\end{aligned}
$$

Example: Compute efficient portfolio as convex combination of efficient portfolio with same mean as MSFT and efficient portfolio with same mean as SBUX.

Let $\mathbf{x}$ denote the efficient portfolio with the same mean as MSFT, $\mathbf{y}$ denote the efficient portfolio with the same mean as SBUX, and let $\alpha = 0.5$. Then

$$
\begin{aligned}
\mathbf{z} &= \alpha \cdot \mathbf{x} + (1-\alpha) \cdot \mathbf{y} \\
&= 0.5 \cdot \begin{pmatrix} 0.82745 \\ -0.09075 \\ 0.26329 \end{pmatrix} + 0.5 \cdot \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix} \\
&= \begin{pmatrix} (0.5)(0.82745) \\ (0.5)(-0.09075) \\ (0.5)(0.26329) \end{pmatrix} + \begin{pmatrix} (0.5)(0.5194) \\ (0.5)(0.2732) \\ (0.5)(0.2075) \end{pmatrix} = \begin{pmatrix} 0.6734 \\ 0.0912 \\ 0.2354 \end{pmatrix} = \begin{pmatrix} z_A \\ z_B \\ z_C \end{pmatrix}.
\end{aligned}
$$

Example continued

The mean of this portfolio can be computed using:

$$\mu_{p,z} = \mathbf{z}'\mu = (0.6734, 0.0912, 0.2354)' \begin{pmatrix} 0.0427 \\ 0.0015 \\ 0.0285 \end{pmatrix} = 0.0356$$

$$\mu_{p,z} = \alpha \cdot \mu_{p,x} + (1-\alpha)\mu_{p,y} = 0.5(0.0427) + (0.5)(0.0285) = 0.0356$$

The variance can be computed using

$$\sigma^2_{p,z} = \mathbf{z}'\mathbf{\Sigma}\mathbf{z} = 0.00641$$

$$\begin{aligned} \sigma^2_{p,z} &= \alpha^2\sigma^2_{p,x} + (1-\alpha)^2\sigma^2_{p,y} + 2\alpha(1-\alpha)\sigma_{xy} \\ &= (0.5)^2(0.09166)^2 + (0.5)^2(0.07355)^2 + 2(0.5)(0.5)(0.005914) = 0.00641 \end{aligned}$$

Example: Find efficient portfolio with expected return 0.05 from two efficient portfolios

Use

$$0.05 = \mu_{p,z} = \alpha \cdot \mu_{p,x} + (1 - \alpha)\mu_{p,y}$$

to solve for α :

$$\alpha = \frac{0.05 - \mu_{p,y}}{\mu_{p,x} - \mu_{p,y}} = \frac{0.05 - 0.0285}{0.0427 - 0.0285} = 1.514$$

Then, solve for portfolio weights using

$$\begin{aligned} \mathbf{z} &= \alpha \cdot \mathbf{x} + (1 - \alpha) \cdot \mathbf{y} \\ &= 1.514 \begin{pmatrix} 0.8275 \\ -0.0908 \\ 0.2633 \end{pmatrix} - 0.514 \begin{pmatrix} 0.5194 \\ 0.2732 \\ 0.2075 \end{pmatrix} = \begin{pmatrix} 0.9858 \\ -0.2778 \\ 0.2920 \end{pmatrix} \end{aligned}$$

Strategy for Plotting Portfolio Frontier

1. Set global minimum variance portfolio = first frontier portfolio

$$\min_{\mathbf{m}} \sigma_{p,m}^2 = \mathbf{m}'\mathbf{\Sigma}\mathbf{m} \text{ s.t. } \mathbf{m}'\mathbf{1} = 1$$

   and compute $\mu_{p,m} = \mathbf{m}'\mu$

2. Find asset i that has highest expected return. Set target return to $\mu^0 = \max(\mu)$ and solve

$$\begin{aligned} \min_{\mathbf{x}} \sigma_{p,x}^2 &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \text{ s.t.} \\ \mu_{p,x} &= \mathbf{x}'\mu = \mu_p^0 = \max(\mu) \\ \mathbf{x}'\mathbf{1} &= 1 \end{aligned}$$

3. Create grid of α values, initially between 1 and -1, and compute

$$\begin{aligned}
\mathbf{z} &= \alpha \cdot \mathbf{m} + (1-\alpha) \cdot \mathbf{x} \\
\mu_{p,z} &= \alpha \cdot \mu_{p,m} + (1-\alpha)\mu_{p,x} \\
\sigma^2_{p,z} &= \alpha^2 \sigma^2_{p,m} + (1-\alpha)^2 \sigma^2_{p,x} + 2\alpha(1-\alpha)\sigma_{m,x} \\
\sigma_{m,x} &= \mathbf{m}' \mathbf{\Sigma} \mathbf{x}
\end{aligned}$$

4. Plot $\mu_{p,z}$ against $\sigma_{p,z}$. Expand or contract the grid of α values if necessary to improve the plot

Finding the Tangency Portfolio

The tangency portfolio $\mathbf{t}$ is the portfolio of risky assets that maximizes Sharpe's slope:

$$\max_{\mathbf{t}} \text{ Sharpe's ratio } = \frac{\mu_{p,t} - r_f}{\sigma_{p,t}}$$

subject to

$$\mathbf{t}'\mathbf{1} = 1$$

In matrix notation,

$$\text{Sharpe's ratio } = \frac{\mathbf{t}'\mu - r_f}{(\mathbf{t}'\mathbf{\Sigma}\mathbf{t})^{1/2}}$$

Solving for Efficient Portfolios:

1. Analytic solution using matrix algebra

2. Numerical solution in Excel using the solver

Analytic solution using matrix algebra

The Lagrangian for this problem is

$$L(\mathbf{t},\lambda) = \left(\mathbf{t}'\mu - r_f\right)(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t})^{-\frac{1}{2}} + \lambda(\mathbf{t}'\mathbf{1} - 1)$$

Using the chain rule, the first order conditions are

$$\underset{(3\times1)}{\mathbf{0}} = \frac{\partial L(\mathbf{t},\lambda)}{\partial \mathbf{t}} = \mu(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t})^{-\frac{1}{2}} - \left(\mathbf{t}'\mu - r_f\right)(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t})^{-3/2}\boldsymbol{\Sigma}\mathbf{t} + \lambda\mathbf{1}$$

$$\underset{(1\times1)}{0} = \frac{\partial L(\mathbf{t},\lambda)}{\partial \lambda} = \mathbf{t}'\mathbf{1} - 1 = 0$$

After much tedious algebra, it can be shown that the solution for $\mathbf{t}$ is

$$\mathbf{t} = \frac{\boldsymbol{\Sigma}^{-1}(\mu - r_f\cdot\mathbf{1})}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}(\mu - r_f\cdot\mathbf{1})}$$

Remarks:

- If the risk free rate, r_f, is less than the expected return on the global minimum variance portfolio, $\mu_{g\,\mathsf{min}}$, then the tangency portfolio has a positive Sharpe slope

- If the risk free rate, r_f, is equal to the expected return on the global minimum variance portfolio, $\mu_{g\,\mathsf{min}}$, then the tangency portfolio is not defined

- If the risk free rate, r_f, is greater than the expected return on the global minimum variance portfolio, $\mu_{g\,\mathsf{min}}$, then the tangency portfolio has a negative Sharpe slope.

Mutual Fund Separation Theorem Again

Efficient Portfolios of T-bills and Risky assets are combinations of two portfolios (mutual funds)

- T-bills
- Tangency portfolio

Efficient Portfolios

$$
\begin{aligned}
x_t &= \text{ share of wealth in tangency portfolio } \mathbf{t} \\
x_f &= \text{share of wealth in T-bills} \\
x_t + x_f &= 1 \Rightarrow \; x_f = 1 - x_t \\
\mu_p^e &= r_f + x_t(\mu_{p,t} - r_f), \; \mu_{p,t} = \mathbf{t}'\mu \\
\sigma_p^e &= x_t\sigma_{p,t}, \; \sigma_{p,t} = \left(\mathbf{t}'\mathbf{\Sigma}\mathbf{t}\right)^{1/2}
\end{aligned}
$$

Remark: The weights x_t and x_f are determined by an investor's risk preferences

- Risk averse investors hold mostly T-Bills ($x_t \approx 0$)

- Risk tolerant investors hold mostly tangency portfolio ($x_t \approx 1$)

- If Sharpe's slope for the tangency portfolio is negative then the efficient portfolio involve shorting the tangency portfolio

Example: Find efficient portfolio with target risk (SD) equal to 0.02

Solve

$$\begin{aligned}
0.02 = \sigma_p^e &= x_t \sigma_{p,t} = x_t(0.1116) \\
&\Rightarrow x_t = \frac{0.02}{0.1116} = 0.1792 \\
x_f &= 1 - x_t = 0.8208
\end{aligned}$$

Also,

$$\begin{aligned}
\mu_p^e &= r_f + x_t(\mu_{p,t} - r_f) = 0.005 + (0.1116)\,(0.05189 - 0.005) = 0.0134 \\
\sigma_p^e &= x_t \sigma_{p,t} = (0.1792)(0.1116) = 0.02
\end{aligned}$$

Example: Find efficient portfolio with target ER equal to 0.07

Solve

$$
\begin{aligned}
0.07 = \mu_p^e &= r_f + x_t(\mu_{p,t} - r_f) \\
\Rightarrow x_t &= \frac{0.07 - r_f}{\mu_{p,t} - r_f} = \frac{0.07 - 0.005}{0.05189 - 0.005} = 1.386
\end{aligned}
$$

Also,

$$
\sigma_p^e = x_t \sigma_{p,t} = (1.386)(0.1116) = 0.1547
$$

Portfolio Value-at-Risk

Let $\mathbf{x} = (x_1, \ldots, x_n)'$ denote a vector of asset share for a portfolio. Portfolio risk is measured by $\text{var}(R_{p,x}) = \mathbf{x}'\mathbf{\Sigma}\mathbf{x}$. Alternatively, portfolio risk can be measured using Value-at-Risk:

$$
\begin{aligned}
\text{VaR}_\alpha &= W_0 q_\alpha^R \\
W_0 &= \text{initial investment} \\
q_\alpha^R &= 100 \cdot \alpha\% \text{ Simple return quantile} \\
\alpha &= \text{ loss probability}
\end{aligned}
$$

If returns are normally distributed then

$$
\begin{aligned}
q_\alpha &= \mu_{p,x} + \sigma_{p,x} q_\alpha^Z \\
\mu_{p,x} &= \mathbf{x}'\mu \\
\sigma_{p,x} &= \left(\mathbf{x}'\mathbf{\Sigma}\mathbf{x}\right)^{1/2} \\
q_\alpha^Z &= 100 \cdot \alpha\% \text{ quantile from } N(0,1)
\end{aligned}
$$

Example: Using VaR to evaluate an efficient portfolio

Invest in 3 risky assets (Microsoft, Starbucks, Nordstrom) and T-bills. Assume $r_f = 0.005$

1. Determine efficient portfolio that has same expected return as Starbucks

2. Compare $\text{VaR}_{.05}$ for Starbucks and efficient portfolio based on \$100,000 investment

Solution for 1.

$$
\begin{aligned}
\mu_{\mathsf{SBUX}} &= 0.0285 \\
\mu_p^e &= r_f + x_t(\mu_{p,t} - r_f) \\
r_f &= 0.005 \\
\mu_{p,t} &= \mathbf{t}'\mu = .05186, \sigma_{p,t} = 0.111
\end{aligned}
$$

Solve

$$
\begin{aligned}
0.0285 &= 0.005 + x_t(0.05186 - 0.005) \\
x_t &= \frac{0.0285 - .005}{0.05186 - .005} = 0.501 \\
x_f &= 1 - 0.501 = 0.499
\end{aligned}
$$

Note:

$$
\begin{aligned}
\mu_p^e &= 0.005 + 0.501 \cdot (0.05186 - 0.005) = 0.0285 \\
\sigma_p^e &= x_t \sigma_{p,t} = (0.501)(0.111) = 0.057
\end{aligned}
$$

Solution for 2.

$$
\begin{aligned}
q_{.05}^{\mathsf{SBUX}} &= \mu_{\mathsf{SBUX}} + \sigma_{\mathsf{SBUX}} \cdot (-1.645) \\
&= 0.0285 + (0.141) \cdot (-1.645) \\
&= -0.203 \\
q_{.05}^{e} &= \mu_p^e + \sigma_p^e \cdot (-1.645) \\
&= .0285 + (.057) \cdot (-1.645) \\
&- 0.063
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathsf{VaR}_{.05}^{SBUX} &= \$100,000 \cdot q_{.05}^{\mathsf{SBUX}} \\
&= \$100,000 \cdot (-0.203) = -\$20,300 \\
\mathsf{VaR}_{.05}^{e} &= \$100,000 \cdot q_{.05}^{e} \\
&= \$100,000 \cdot (-0.063) = -\$6,300
\end{aligned}
$$

Efficient Portfolios without Short-Sales

Short Sale

- Borrow asset from broker and sell now
- To close short position, buy back asset and return to broker
- Profit if asset price drops after short sale
- If asset i is sold short then
$$x_i < 0$$
where $x_i =$ share of wealth in asset i

No Short Sale Restrictions

- Exchanges (e.g. NYSE, NASDAQ) may prevent short sales in some assets
- Some institutions (e.g. pension funds) are prevented from short-selling assets
- Certain accounts do not allow short sales (e.g. retirement accounts)
- Short selling often requires substantial credit qualifications

Markowitz Algorithm with No Short Sales Restrictions

$$
\begin{aligned}
\min_{\mathbf{x}} \sigma^2_{p,x} &= \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \quad \text{s.t.} \\
\mu_{p,x} &= \mathbf{x}'\mu = \mu_p^0 \\
\mathbf{x}'\mathbf{1} &= 1 \\
x_i &\geq 0
\end{aligned}
$$

Remarks:

- Problem must be solved numerically (e.g. using the Solver) :(
- Portfolio Frontier can no longer be constructed from any two efficient portfolios :(
- No short sale portfolio frontier must lie "inside" the portfolio frontier that allows short sales

R functions for Minimum Variance Portfolios with No Short Sales Restrictions

- R package tseries function `portfolio.optim()`

- R package quadprog function `solve.QP()`

- Guy Yollin's R in Finance presentation on R tools for portfolio optimization

  http://www.rinfinance.com/RinFinance2009/presentations/yollin_slides.pdf

- Rmetrics (`www.Rmetrics.org`) package fPortfolio