Bivariate Probability Distribution

Example - Two discrete rv's X and Y

Bivariate pdf

		Y		
	%	0	1	Pr(X)
	0	1/8	0	1/8
X	1	2/8	1/8	3/8
	2	1/8	2/8	3/8
	3	0	1/8	1/8
	Pr(Y)	4/8	4/8	1

$$\begin{aligned} p(x,y) &= \Pr(X=x, Y=y) = \text{values in table} \\ e.g.,\ p(0,0) &= \Pr(X=0, Y=0) = 1/8 \end{aligned}$$

Properties of joint pdf $p(x,y)$

$$\begin{aligned} S_{XY} &= \{(0,0),\ (0,1),\ (1,0),\ (1,1), \\ &\quad (2,0),\ (2,1),\ (3,0),\ (3,1)\} \\ p(x,y) &\geq 0 \text{ for } x,y \in S_{XY} \\ \sum_{x,y\in S_{XY}} p(x,y) &= 1 \end{aligned}$$

Marginal pdfs

$$
\begin{aligned}
p(x) = \text{Pr}(X = x) &= \sum_{y \in S_Y} p(x, y) \\
&= \text{sum over columns in joint table}
\end{aligned}
$$

$$
\begin{aligned}
p(y) = \text{Pr}(Y = y) &= \sum_{x \in S_X} p(x, y) \\
&= \text{sum over rows in joint table}
\end{aligned}
$$

Conditional Probability

Suppose we know $Y = 0$. How does this knowledge affect the probability that $X = 0, 1, 2,$ or 3? The answer involves conditional probability.

Example

$$
\begin{aligned}
\text{Pr}(X = 0|Y = 0) &= \frac{\text{Pr}(X = 0, Y = 0)}{\text{Pr}(Y = 0)} \\
&= \frac{\text{joint probability}}{\text{marginal probability}} = \frac{1/8}{4/8} = 1/4
\end{aligned}
$$

Remark

$$
\begin{aligned}
\text{Pr}(X = 0|Y = 0) &= 1/4 \neq \text{Pr}(X = 0) = 1/8 \\
&\implies X \text{ depends on } Y
\end{aligned}
$$

The marginal probability, $\text{Pr}(X = 0)$, ignores information about Y.

Definition - Conditional Probability

- The conditional pdf of X given $Y=y$ is, for all $x \in S_X$,

$$p(x|y) = \text{Pr}(X=x|Y=y) = \frac{\text{Pr}(X=x, Y=y)}{\text{Pr}(Y=y)}$$

- The conditional pdf of Y given $X=x$ is, for all values of $y \in S_Y$

$$p(y|x) = \text{Pr}(Y=y|X=x) = \frac{\text{Pr}(X=x, Y=y)}{\text{Pr}(X=x)}$$

Conditional Mean and Variance

$$\begin{aligned}
\mu_{X|Y=y} &= E[X|Y=y] = \sum_{x \in S_X} x \cdot \text{Pr}(X=x|Y=y), \\
\mu_{Y|X=x} &= E[Y|X=x] = \sum_{y \in S_Y} y \cdot \text{Pr}(Y=y|X=x).
\end{aligned}$$

$$\begin{aligned}
\sigma^2_{X|Y=y} &= \text{var}(X|Y=y) = \sum_{x \in S_X} (x - \mu_{X|Y=y})^2 \cdot \text{Pr}(X=x|Y=y), \\
\sigma^2_{Y|X=x} &= \text{var}(Y|X=x) = \sum_{y \in S_Y} (y - \mu_{Y|X=x})^2 \cdot \text{Pr}(Y=y|X=x).
\end{aligned}$$

Example:

$$
\begin{aligned}
E[X] &= 0 \cdot 1/8 + 1 \cdot 3/8 + 2 \cdot 3/8 + 3 \cdot 1/8 = 3/2 \\
E[X|Y=0] &= 0 \cdot 1/4 + 1 \cdot 1/2 + 2 \cdot 1/4 + 3 \cdot 0 = 1, \\
E[X|Y=1] &= 0 \cdot 0 + 1 \cdot 1/4 + 2 \cdot 1/2 + 3 \cdot 1/4 = 2,
\end{aligned}
$$

$$
\begin{gathered}
\mathrm{var}(X) = \mathrm{var}(X|Y=0) = (0-3/2)^2 \cdot 1/8 + (1-3/2)^2 \cdot 3/8 \\
+(2-3/2)^2 \cdot 3/8 + (3-3/2)^2 \cdot 1/8 = 3/4, \\
\mathrm{var}(X|Y=0) = (0-1)^2 \cdot 1/4 + (1-1)^2 \cdot 1/2 \\
+(2-1)^2 \cdot 1/2 + (3-1)^2 \cdot 0 = 1/2, \\
\mathrm{var}(X|Y=1) = (0-2)^2 \cdot 0 + (1-2)^2 \cdot 1/4 \\
+(2-2)^2 \cdot 1/2 + (3-2)^2 \cdot 1/4 = 1/2.
\end{gathered}
$$

Independence

Let X and Y be discrete rvs with pdfs $p(x)$, $p(y)$, sample spaces S_X, S_Y and joint pdf $p(x,y)$. Then X and Y are independent rv's if and only if

$$
p(x,y) = p(x) \cdot p(y)
$$

for all values of $x \in S_X$ and $y \in S_Y$

Result: If X and Y are independent rv's, then

$$
\begin{aligned}
p(x|y) &= p(x) \quad \text{for all } x \in S_X,\ y \in S_Y \\
p(y|x) &= p(y) \quad \text{for all } x \in S_X,\ y \in S_Y
\end{aligned}
$$

Intuition

Knowledge of X does not influence probabilities associated with Y

Knowledge of Y does not influence probablities associated with X

Bivariate Distributions - Continuous rv's

The joint pdf of X and Y is a non-negative function $f(x, y)$ such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

Let $[x_1, x_2]$ and $[y_1, y_2]$ be intervals on the real line. Then

$$\begin{aligned}
&\Pr(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) \\
&= \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)dxdy \\
&= \text{volume under probability surface} \\
&\text{over the intersection of the intervals} \\
&[x_1, x_2] \text{ and } [y_1, y_2]
\end{aligned}$$

Marginal and Conditional Distributions

The marginal pdf of X is found by integrating y out of the joint pdf $f(x, y)$ and the marginal pdf of Y is found by integrating x out of the joint pdf:

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy,$$
$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx.$$

The conditional pdf of X given that $Y = y$, denoted $f(x|y)$, is computed as

$$f(x|y) = \frac{f(x, y)}{f(y)},$$

and the conditional pdf of Y given that $X = x$ is computed as

$$f(y|x) = \frac{f(x, y)}{f(x)}.$$

The conditional means are computed as

$$\mu_{X|Y=y} = E[X|Y=y] = \int x \cdot p(x|y)dx,$$
$$\mu_{Y|X=x} = E[Y|X=x] = \int y \cdot p(y|x)dy$$

and the conditional variances are computed as

$$\sigma^2_{X|Y=y} = \text{var}(X|Y=y) = \int (x - \mu_{X|Y=y})^2 p(x|y)dx,$$
$$\sigma^2_{Y|X=x} = \text{var}(Y|X=x) = \int (y - \mu_{Y|X=x})^2 p(y|x)dy.$$

Independence.

Let X and Y be continuous random variables. X and Y are independent iff

$$f(x|y) = f(x), \text{ for } -\infty < x, y < \infty,$$
$$f(y|x) = f(y), \text{ for } -\infty < x, y < \infty.$$

Result: Let X and Y be continuous random variables . X and Y are independent iff

$$f(x, y) = f(x)f(y)$$

The result in the above proposition is extremely useful in practice because it gives us an easy way to compute the joint pdf for two independent random variables: we simple compute the product of the marginal distributions.

Example: Bivariate standard normal distribution

Let $X \sim N(0,1)$, $Y \sim N(0,1)$ and let X and Y be independent. Then

$$\begin{aligned} f(x,y) = f(x)f(y) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \\ &= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}. \end{aligned}$$

To find $\Pr(-1 < X < 1, -1 < Y < 1)$ we must solve

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dx dy$$

which, unfortunately, does not have an analytical solution. Numerical approximation methods are required to evaluate the above integral. See R package mvtnorm.

Covariance and Correlation - Measuring linear dependence between two rv's

Covariance: Measures direction but not strength of linear relationship between 2 rv's

$$\begin{aligned} \sigma_{XY} &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= \sum_{x.y \in S_{XY}} (x - \mu_X)(y - \mu_Y) \cdot p(x,y) \ \ \text{(discrete)} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) f(x,y) dx dy \ \ \text{(cts)} \end{aligned}$$

Example: For the data in Table 2, we have

$$\begin{gathered}\sigma_{XY} = \mathsf{Cov}(X,Y) = (0-3/2)(0-1/2)\cdot 1/8 \\ +(0-3/2)(1-1/2)\cdot 0 + \cdots \\ +(3-3/2)(1-1/2)\cdot 1/8 = 1/4\end{gathered}$$

Properties of Covariance

$$\begin{aligned}\mathsf{Cov}(X,Y) &= \mathsf{Cov}(Y,X) \\ \mathsf{Cov}(aX,bY) &= a\cdot b\cdot \mathsf{Cov}(X,Y) = a\cdot b\cdot \sigma_{XY} \\ \mathsf{Cov}(X,X) &= \mathsf{Var}(X) \\ X,Y \text{ independent } \implies \mathsf{Cov}(X,Y) &= 0 \\ \mathsf{Cov}(X,Y) &= 0 \not\Rightarrow X \text{ and } Y \text{ are independent} \\ \mathsf{Cov}(X,Y) &= E[XY]-E[X]E[Y]\end{aligned}$$

Correlation: Measures direction and strength of linear relationship between 2 rv's

$$\rho_{XY} = \text{Cor}(X,Y) = \frac{\text{Cov}(X,Y)}{\text{SD}(X)\cdot\text{SD}(Y)}$$
$$= \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y} = \text{ scaled covariance}$$

Example: For the Data in Table 2

$$\rho_{XY} = \text{Cor}(X,Y) = \frac{1/4}{\sqrt{(3/4)\cdot(1/2)}} = 0.577$$

Properties of Correlation

$$
\begin{aligned}
-1 \le \rho_{XY} &\le 1 \\
\rho_{XY} &= 1 \text{ if } Y = aX + b \text{ and } a > 0 \\
\rho_{XY} &= -1 \text{ if } Y = aX + b \text{ and } a < 0 \\
\rho_{XY} &= 0 \text{ if and only if } \sigma_{XY} = 0 \\
\rho_{XY} &= 0 \not\Rightarrow \ X \text{ and } Y \text{ are independent in general} \\
\rho_{XY} &= 0 \implies \text{independence if } X \text{ and } Y \text{ are normal}
\end{aligned}
$$

Bivariate normal distribution

Let X and Y be distributed bivariate normal. The joint pdf is given by

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}$$

where $E[X] = \mu_X$, $E[Y] = \mu_Y$, $\text{SD}(X) = \sigma_X$, $\text{SD}(Y) = \sigma_Y$, and $\rho = \text{cor}(X, Y)$.

Linear Combination of 2 rv's

Let X and Y be rv's. Define a new rv Z that is a linear combination of X and Y :

$$Z = aX + bY$$

where a and b are constants. Then

$$\begin{aligned}\mu_Z = E[Z] &= E[aX + bY] \\ &= aE[X] + bE[Y] \\ &= a \cdot \mu_X + b \cdot \mu_Y\end{aligned}$$

and

$$\begin{aligned}\sigma_Z^2 &= \text{Var}(Z) = \text{Var}(a \cdot X + b \cdot Y) \\ &= a^2\text{Var}(X) + b^2\text{Var}(Y) + 2a \cdot b \cdot \text{Cov}(X, Y) \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2a \cdot b \cdot \sigma_{XY}\end{aligned}$$

If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ then $Z \sim N(\mu_Z, \sigma_Z^2)$

Example: Portfolio returns

$R_A =$ return on asset A with $E[R_A] = \mu_A$ and $\text{Var}(R_A) = \sigma_A^2$

$R_B =$ return on asset B with $E[R_B] = \mu_B$ and $\text{Var}(R_B) = \sigma_B^2$

$\text{Cov}(R_A, R_B) = \sigma_{AB}$ and $\text{Cor}(R_A,\ R_B) = \rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \cdot \sigma_B}$

Portfolio

$x_A =$ share of wealth invested in asset $A, x_B =$ share of wealth invested in asset B

$x_A + x_B = 1$ (exhaust all wealth in 2 assets)

$R_P = x_A \cdot R_A + x_B \cdot R_B =$ portfolio return

Portfolio Problem: How much wealth should be invested in assets A and B?

Portfolio expected return (gain from investing)

$$\begin{aligned} E[R_P] = \mu_P &= E[x_A \cdot R_A + x_B \cdot R_B] \\ &= x_A E[R_A] + x_B E[R_B] \\ &= x_A \mu_A + x_B \mu_B \end{aligned}$$

Portfolio variance (risk from investing)

$$\begin{aligned} \mathsf{Var}(R_P) = \sigma_P^2 &= \mathsf{Var}(x_A R_A + x_B R_B) \\ &= x_A^2 \mathsf{Var}(R_A) + x_B^2 \mathsf{Var}(R_B) + \\ &\quad 2 \cdot x_A \cdot x_B \cdot \mathsf{Cov}(R_A,\ R_B) \\ &= x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \\ \mathsf{SD}(R_P) &= \sqrt{\mathsf{Var}(R_P)} = \sigma_P \\ &= \left(x_A^2 \sigma_A^2 + x_B^2 \sigma_B^2 + 2 x_A x_B \sigma_{AB} \right)^{1/2} \end{aligned}$$

Linear Combination of N rv's.

Let $X_1,\ X_2, \cdots, X_N$ be rvs and let $a_1, a_2, \ldots, a_N$ be constants. Define

$$Z = a_1 X_1 + a_2 X_2 + \cdots + a_N X_N = \sum_{i=1}^{N} a_i X_i$$

Then

$$\begin{aligned} \mu_Z = E[Z] &= a_1 E[X_1] + a_2 E[X_2] + \cdots + a_N E[X_N] \\ &= \sum_{i=1}^{N} a_i E[X_i] = \sum_{i=1}^{N} a_i \mu_i \end{aligned}$$

For the variance,

$$\begin{aligned}\sigma_Z^2 = \mathsf{Var}(Z) &= a_1^2\mathsf{Var}(X_1) + \cdots + a_N^2\mathsf{Var}(X_N) \\ &+ 2a_1a_2\mathsf{Cov}(X_1,\ X_2) + 2a_1a_3\mathsf{Cov}(X_1,\ X_3) + \cdots \\ &+ 2a_2a_3\mathsf{Cov}(X_2,\ X_3) + 2a_2a_4\mathsf{Cov}(X_2,\ X_4) + \cdots \\ &+ 2a_{N-1}a_N\mathsf{Cov}(X_{N-1}, X_N)\end{aligned}$$

Note: N variance terms and $N(N-1) = N^2 - N$ covariance terms. If $N = 100$, there are $100 \times 99 = 9900$ covariance terms!

Result: If $X_1,\ X_2, \cdots, X_N$ are each normally distributed random variables then

$$Z = \sum_{i=1}^{N} a_i X_i \sim N(\mu_Z, \sigma_Z^2)$$

Example: Portfolio variance with three assets

R_A, R_B, R_C are simple returns on assets A, B and C

x_A, x_B, x_C are portfolio shares such that $x_A + x_B + x_C = 1$

$R_p = x_A R_A + x_B R_B + x_C R_C$

Portfolio variance

$$\begin{aligned}\sigma_P^2 &= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 \\ &+ 2x_Ax_B\sigma_{AB} + 2x_Ax_C\sigma_{AC} + 2x_Bx_C\sigma_{BC}\end{aligned}$$

Note: Portfolio variance calculation may be simplified using matrix layout

$$
\begin{array}{cccc}
 & x_A & x_B & x_C \\
x_A & \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\
x_B & \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\
x_C & \sigma_{AC} & \sigma_{BC} & \sigma_C^2
\end{array}
$$

Example: Multi-period continuously compounded returns

$$
\begin{gathered}
r_t = \ln(1 + R_t) = \text{ monthly cc return} \\
r_t \sim N(\mu,\ \sigma^2) \quad \text{for all } t \\
\text{Cov}(r_t, r_s) = 0 \text{ for all } t \neq s
\end{gathered}
$$

Annual return

$$
\begin{aligned}
r_t(12) &= \sum_{j=0}^{11} r_{t-j} \\
&= r_t + r_{t-1} + \cdots + r_{t-11}
\end{aligned}
$$

Then

$$
\begin{aligned}
E[r_t(12)] &= \sum_{j=0}^{11} E[r_{t-j}] \\
&= \sum_{j=0}^{11} \mu \qquad (E[r_t] = \mu \text{ for all } t) \\
&= 12\mu \quad (\mu = \text{mean of monthly return})
\end{aligned}
$$

$$
\begin{aligned}
\text{Var}(r_t(12)) &= \text{Var}\left(\sum_{j=0}^{11} r_{t-j}\right) \\
&= \sum_{j=0}^{11} \text{Var}(r_{t-j}) = \sum_{j=0}^{11} \sigma^2 \\
&= 12 \cdot \sigma^2 \quad (\sigma^2 = \text{monthly variance}) \\
\text{SD}(r_t(12)) &= \sqrt{12} \cdot \sigma \text{ (square root of time rule)}
\end{aligned}
$$

Then

$$
r_t(12) \sim N(12\mu, 12\sigma^2)
$$

For example, suppose

$$r_t \sim N(0.01, (0.10)^2)$$

Then

$$\begin{aligned} E[r_t(12)] &= 12 \times (0.01) = 0.12 \\ \mathsf{Var}(r_t(12)) &= 12 \times (0.10)^2 = 0.12 \\ \mathsf{SD}(r_t(12)) &= \sqrt{0.12} = 0.346 \\ r_t(12) &\sim N(0.12, (0.346)^2) \end{aligned}$$