Econ 424/Amath 540
Single Index Model

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Sharpe's Single Index Model

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}$$
$$i = 1, \ldots, N;\ t = 1, \ldots, T$$

where

$$\alpha_i,\ \beta_i \text{ are constant over time}$$
$$R_{Mt} = \text{ return on diversified market index portfolio}$$
$$\varepsilon_{it} = \text{ random error term unrelated to } R_{Mt}$$

Assumptions

- cov$(R_{Mt}, \varepsilon_{is}) = 0$ for all t, s

- cov$(\varepsilon_{is}, \varepsilon_{jt}) = 0$ for all $i \neq j$, t and s

- $\varepsilon_{it} \sim$ iid $N(0, \sigma^2_{\varepsilon,i})$

- $R_{M,t} \sim$ iid $N(\mu_M, \sigma^2_M)$

Interpretation of ε_{it}:

$$\varepsilon_{it} = R_{it} - \alpha_i - \beta_i R_{Mt}$$

- Return on market index, R_{Mt}, captures common "market-wide" news.
- β_i measures sensitivity to "market-wide" news
- Random error term ε_{it} captures "firm specific" news unrelated to market-wide news.
- Returns are correlated only through their exposures to common "market-wide" news captured by β_i.

Remark:

The CER model is a special case of Single Index (SI) Model where $\beta_i = 0$ for all $i = 1, \ldots, N$.

$$R_{it} = \alpha_i + \varepsilon_{it}$$

In this case, $\alpha_i = E[R_i] = \mu_i$

Statistical Properties of the SI Model (Unconditional)

- $\mu_i = E[R_{it}] = \alpha_i + \beta_i \mu_M$

- $\sigma_i^2 = \mathrm{var}(R_{it}) = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2$

- $\sigma_{ij} = \mathrm{cov}(R_{it}, R_{jt}) = \sigma_M^2 \beta_i \beta_j$

- $R_{it} \sim N(\mu_i, \sigma_i^2) = N(\alpha_i + \beta_i \mu_M, \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2)$

Derivations:

$$
\begin{aligned}
\mathsf{var}(R_{it}) &= \mathsf{var}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}) \\
&= \beta_i^2 \mathsf{var}(R_{Mt}) + \mathsf{var}(\varepsilon_{it}) \\
&= \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_i^2 \sigma_M^2 &= \text{variance due to market news} \\
\sigma_{\varepsilon,i}^2 &= \text{variance due to non-market news}
\end{aligned}
$$

Next

$$
\begin{aligned}
\sigma_{ij} &= \mathsf{cov}(R_{it}, R_{jt}) \\
&= \mathsf{cov}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}, \alpha_j + \beta_j R_{Mt} + \varepsilon_{jt}) \\
&= \mathsf{cov}(\beta_i R_{Mt}, \beta_j R_{Mt}) + \mathsf{cov}(\beta_i R_{Mt}, \varepsilon_{jt}) \\
&+ \mathsf{cov}(\beta_j R_{Mt}, \varepsilon_{it}) + \mathsf{cov}(\varepsilon_{it}, \varepsilon_{jt}) \\
&= \beta_i \beta_j \mathsf{cov}(R_{Mt}, R_{Mt}) \\
&= \sigma_M^2 \beta_i \beta_j
\end{aligned}
$$

Implications:

- $\sigma_{ij} = 0$ if $\beta_i = 0$ or $\beta_j = 0$ (asset i or asset j do not respond to market news)

- $\sigma_{ij} > 0$ if $\beta_i, \beta_j > 0$ or $\beta_i, \beta_j < 0$ (asset i and j respond to market news in the same direction)

- $\sigma_{ij} < 0$ if $\beta_i > 0$ and $\beta_j < 0$ or if $\beta_i < 0$ and $\beta_j > 0$ (asset i and j respond to market news in opposite direction)

Statistical Properties of the SI Model (Conditional on R_{Mt})

Given that we observe, $R_{Mt} = r_{Mt}$

- $E[R_{it}|R_{Mt} = r_{Mt}] = \alpha_i + \beta_i r_{Mt}$
- $\sigma_i^2 = \text{var}(R_{it}|R_{Mt} = r_{Mt}) = \sigma^2_{\varepsilon,i}$
- $\sigma_{ij} = \text{cov}(R_{it}, R_{jt}|R_{Mt} = r_{Mt}) = 0$
- $R_{it}|R_{Mt} = r_{Mt} \sim N(\alpha_i + \beta_i r_{Mt}, \sigma^2_{\varepsilon,i})$

Interpretation of β_i

$$\beta_i = \frac{\text{cov}(R_{it}, R_{Mt})}{\text{var}(R_{Mt})} = \frac{\sigma_{iM}}{\sigma_M^2}$$

β_i captures the contribution of asset i to the variance/risk of the market index.

Derivation:

$$\begin{aligned}
\text{cov}(R_{it}, R_{Mt}) &= \text{cov}(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}, R_{Mt}) \\
&= \text{cov}(\beta_i R_{Mt}, R_{Mt}) + \text{cov}(\varepsilon_{it}, R_{Mt}) \\
&= \beta_i \text{var}(R_{Mt}) \\
&\Rightarrow \beta_i = \frac{\text{cov}(R_{it}, R_{Mt})}{\text{var}(R_{Mt})}
\end{aligned}$$

Decomposition of Total Variance

$$\sigma_i^2 = \mathsf{var}(R_{it}) = \beta_i^2\sigma_M^2 + \sigma_{\varepsilon,i}^2$$
$$\text{total variance} = \text{market variance} + \text{non-market variance}$$

Divide both sides by σ_i^2

$$\begin{aligned} 1 &= \frac{\beta_i^2\sigma_M^2}{\sigma_i^2} + \frac{\sigma_{\varepsilon,i}^2}{\sigma_i^2} \\ &= R_i^2 + 1 - R_i^2 \end{aligned}$$

where

$$\begin{aligned} R_i^2 &= \frac{\beta_i^2\sigma_M^2}{\sigma_i^2} = \text{proportion of market variance} \\ 1 - R_i^2 &= \text{proportion of non-market variance} \end{aligned}$$

Sharpe's Rule of Thumb: A typical stock has $R_i^2 = 30\%$; i.e., proportion of market variance in typical stock is 30% of total variance.

Return Covariance Matrix

3 asset example

$$
\begin{aligned}
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it},\ i = 1, 2, 3 \\
\sigma_i^2 &= \mathrm{var}(R_{it}) = \beta_i^2 \sigma_M^2 + \sigma_{\varepsilon,i}^2 \\
\sigma_{ij} &= \mathrm{cov}(R_{it}, R_{jt}) = \sigma_M^2 \beta_i \beta_j
\end{aligned}
$$

Covariance matrix

$$
\begin{aligned}
\boldsymbol{\Sigma} &= \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} \\
&= \begin{pmatrix} \beta_1^2\sigma_M^2 + \sigma_{\varepsilon,1}^2 & \sigma_M^2\beta_1\beta_2 & \sigma_M^2\beta_1\beta_3 \\ \sigma_M^2\beta_1\beta_2 & \beta_2^2\sigma_M^2 + \sigma_{\varepsilon,2}^2 & \sigma_M^2\beta_2\beta_3 \\ \sigma_M^2\beta_1\beta_3 & \sigma_M^2\beta_2\beta_3 & \beta_3^2\sigma_M^2 + \sigma_{\varepsilon,3}^2 \end{pmatrix} \\
&= \sigma_M^2 \begin{pmatrix} \beta_1^2 & \beta_1\beta_2 & \beta_1\beta_3 \\ \beta_1\beta_2 & \beta_2^2 & \beta_2\beta_3 \\ \beta_1\beta_3 & \beta_2\beta_3 & \beta_3^2 \end{pmatrix} + \begin{pmatrix} \sigma_{\varepsilon,1}^2 & 0 & 0 \\ 0 & \sigma_{\varepsilon,2}^2 & 0 \\ 0 & 0 & \sigma_{\varepsilon,3}^2 \end{pmatrix}
\end{aligned}
$$

Simplification using matrix algebra

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} \sigma^2_{\varepsilon,1} & 0 & 0 \\ 0 & \sigma^2_{\varepsilon,2} & 0 \\ 0 & 0 & \sigma^2_{\varepsilon,3} \end{pmatrix}$$

Then

$$\underset{(3\times3)}{\mathbf{\Sigma}} = \sigma^2_M \cdot \underset{(3\times1)}{\beta}\underset{(1\times3)}{\beta}' + \underset{(3\times3)}{\mathbf{D}}$$

where

$$\sigma^2_M \cdot \beta\beta' = \text{ covariance due to market}$$
$$\mathbf{D} = \text{asset specific variances}$$

SI Model and Portfolios

2 asset example

$$
\begin{aligned}
R_{1t} &= \alpha_1 + \beta_1 R_{Mt} + \varepsilon_{1t} \\
R_{2t} &= \alpha_2 + \beta_2 R_{Mt} + \varepsilon_{2t} \\
x_1 &= \text{share invested in asset 1} \\
x_2 &= \text{share invested in asset 2} \\
x_1 + x_2 &= 1
\end{aligned}
$$

Portfolio return

$$\begin{aligned}
R_{p,t} &= x_1 R_{1t} + x_2 R_{2t} \\
&= x_1(\alpha_1 + \beta_1 R_{Mt} + \varepsilon_{1t}) \\
&+ x_2(\alpha_2 + \beta_2 R_{Mt} + \varepsilon_{2t}) \\
&= (x_1\alpha_1 + x_2\alpha_2) + (x_1\beta_1 + x_2\beta_2)\, R_{Mt} \\
&+ (x_1\varepsilon_{1t} + x_2\varepsilon_{2t}) \\
&= \alpha_p + \beta_p R_{Mt} + \varepsilon_{p,t}
\end{aligned}$$

where

$$\begin{aligned}
\alpha_p &= x_1\alpha_1 + x_2\alpha_2 \\
\beta_p &= x_1\beta_1 + x_2\beta_2 \\
\varepsilon_{p,t} &= x_1\varepsilon_{1t} + x_2\varepsilon_{2t}
\end{aligned}$$

SI Model with Large Portfolios

$$
\begin{aligned}
i &= 1, \ldots, N \text{ assets (e.g. } N = 500) \\
x_i &= \frac{1}{N} = \text{equal investment shares} \\
R_{it} &= \alpha_i + \beta_i R_{Mt} + \varepsilon_{it}
\end{aligned}
$$

Portfolio return

$$
\begin{aligned}
R_{p,t} &= \sum_{i=1}^{N} x_i R_{it} \\
&= \sum_{i=1}^{N} x_i \left(\alpha_i + \beta_i R_{Mt} + \varepsilon_{it}\right) \\
&= \sum_{i=1}^{N} x_i \alpha_i + \left(\sum_{i=1}^{N} x_i \beta_i\right) R_{Mt} + \sum_{i=1}^{N} x_i \varepsilon_{it} \\
&= \frac{1}{N} \sum_{i=1}^{N} \alpha_i + \left(\frac{1}{N} \sum_{i=1}^{N} \beta_i\right) R_{Mt} + \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{it} \\
&= \bar{\alpha} + \bar{\beta} R_{Mt} + \bar{\varepsilon}_t
\end{aligned}
$$

where

$$\bar{\alpha} = \frac{1}{N}\sum_{i=1}^{N}\alpha_i$$

$$\bar{\beta} = \frac{1}{N}\sum_{i=1}^{N}\beta_i$$

$$\bar{\varepsilon}_t = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it}$$

Result: For large N,

$$\bar{\varepsilon}_t = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{it} \approx E[\varepsilon_{it}] = 0$$

because $\varepsilon_{it} \sim$ iid $N(0, \sigma^2_{\varepsilon,i})$.

Implications

In a large well diversified portfolio, the following results hold:

- $R_{p,t} \approx \bar{\alpha} + \bar{\beta} R_{Mt}$: all non-market variance is diversified away

- $\text{var}(R_{p,t}) = \bar{\beta}^2 \text{var}(R_{Mt})$: Magnitude of portfolio variance is proportional to market variance. Magnitude of portfolio variance is determined by portfolio beta $\bar{\beta}$

- $R_p^2 \approx 1$: Approximately 100% of portfolio variance is due to market variance