Econ 424/Amath 540
Time Series Concepts

Eric Zivot

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Time Series Processes

Stochastic (Random) Process

$$\{\ldots, Y_1, Y_2, \ldots, Y_t, Y_{t+1}, \ldots\} = \{Y_t\}_{t=-\infty}^{\infty}$$

sequence of random variables indexed by time

Observed time series of length T

$$\{Y_1 = y_1, Y_2 = y_2, \ldots, Y_T = y_T\} = \{y_t\}_{t=1}^{T}$$

Stationary Processes

- Intuition: $\{Y_t\}$ is stationary if all aspects of its behavior are unchanged by shifts in time

- A stochastic process $\{Y_t\}_{t=1}^{\infty}$ is *strictly stationary* if, for any given finite integer r and for any set of subscripts $t_1, t_2, \ldots, t_r$ the joint distribution of

$$(Y_t, Y_{t_1}, Y_{t_2}, \ldots, Y_{t_r})$$

depends only on $t_1 - t, t_2 - t, \ldots, t_r - t$ but not on t.

Remarks

1. For example, the distribution of (Y_1, Y_5) is the same as the distribution of (Y_{12}, Y_{16}).

2. For a strictly stationary process, Y_t has the same mean, variance (moments) for all t.

3. Any function/transformation $g(\cdot)$ of a strictly stationary process, $\{g(Y_t)\}$ is also strictly stationary. E.g., if $\{Y_t\}$ is strictly then $\{Y_t^2\}$ is strictly stationary.

Covariance (Weakly) Stationary Processes $\{Y_t\}$:

- $E[Y_t] = \mu$ for all t
- $\mathrm{var}(Y_t) = \sigma^2$ for all t
- $\mathrm{cov}(Y_t, Y_{t-j}) = \gamma_j$ depends on j and not on t

Note $\mathrm{cov}(Y_t, Y_{t-j}) = \gamma_j$ is called the j-lag *autocovariance*

Autocorrelations

$$\mathrm{corr}(Y_t, Y_{t-j}) = \rho_j = \frac{\mathrm{cov}(Y_t, Y_{t-j})}{\sqrt{\mathrm{var}(Y_t)\mathrm{var}(Y_{t-j})}} = \frac{\gamma_j}{\sigma^2}$$

Note: By stationarity $\mathrm{var}(Y_t) = \mathrm{var}(Y_{t-j}) = \sigma^2$.

Autocorrelation Function (ACF): Plot of ρ_j against j

Example: Gaussian White Noise Process

$$Y_t \sim \text{iid } N(0, \sigma^2) \text{ or } Y_t \sim GWN(0, \sigma^2)$$
$$E[Y_t] = 0, \ \text{var}(Y_t) = \sigma^2$$
$$Y_t \text{ independent of } Y_s \text{ for } t \neq s$$
$$\Rightarrow \text{cov}(Y_t, Y_{t-s}) = 0 \text{ for } t \neq s$$

Note: "iid" = "**i**ndependent and **i**dentically **d**istributed".

Here, $\{Y_t\}$ represents random draws from the same $N(0, \sigma^2)$ distribution

Example: Independent White Noise Process

$$Y_t \sim \text{iid } (0, \sigma^2) \text{ or } Y_t \sim IWN(0, \sigma^2)$$
$$E[Y_t] = 0, \ \text{var}(Y_t) = \sigma^2$$
$$Y_t \text{ independent of } Y_s \text{ for } t \neq s$$

Here, $\{Y_t\}$ represents random draws from the same distribution. However, we don't specify exactly what the distribution is - only that it has mean zero and variance σ^2.

Example: Weak White Noise Process

$$Y_t \sim WN(0, \sigma^2)$$
$$E[Y_t] = 0,\ \text{var}(Y_t) = \sigma^2$$
$$\text{cov}(Y_t, Y_s) = 0 \text{ for } t \neq s$$

Here, $\{Y_t\}$ represents an uncorrelated stochastic process with mean zero and variance σ^2. Recall, the uncorrelated assumption does not imply independence. Hence, Y_t and Y_s can exhibit non-linear dependence.

Nonstationary Processes

Example: Deterministically trending process

$$Y_t = \beta_0 + \beta_1 t + \varepsilon_t,\ \varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$$
$$E[Y_t] = \beta_0 + \beta_1 t \text{ depends on } t$$

Note: A simple detrending transformation yield a stationary process:

$$X_t = Y_t - \beta_0 - \beta_1 t = \varepsilon_t$$

Example: Random Walk

$$
\begin{aligned}
Y_t &= Y_{t-1} + \varepsilon_t,\ \varepsilon_t \sim WN(0, \sigma_\varepsilon^2),\ Y_0 \text{ is fixed} \\
&= Y_0 + \sum_{j=1}^{t} \varepsilon_j \Rightarrow \text{var}(Y_t) = \sigma_\varepsilon^2 \times t \ \text{ depends on } t
\end{aligned}
$$

Note: A simple detrending transformation yield a stationary process:

$$
\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t
$$

Moving Average (MA) Processes

MA(1) Model

$$
\begin{aligned}
Y_t &= \mu + \varepsilon_t + \theta\varepsilon_{t-1},\ \ -\infty < \theta < \infty \\
\varepsilon_t &\sim iid\ N(0, \sigma_\varepsilon^2)
\end{aligned}
$$

Properties

$$
\begin{aligned}
E[Y_t] &= \mu + E[\varepsilon_t] + \theta E[\varepsilon_{t-1}] \\
&= \mu + 0 + 0 = \mu
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{var}(Y_t) &= \sigma^2 = E[(Y_t - \mu)^2] \\
&= E[(\varepsilon_t + \theta\varepsilon_{t-1})^2] \\
&= E[\varepsilon_t^2] + 2\theta E[\varepsilon_t\varepsilon_{t-1}] + \theta^2 E[\varepsilon_{t-1}^2] \\
&= \sigma_\varepsilon^2 + 0 + \theta^2\sigma_\varepsilon^2 = \sigma_\varepsilon^2(1+\theta^2) \\
\mathrm{cov}(Y_t, Y_{t-1}) &= \gamma_1 = E[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-1} + \theta\varepsilon_{t-2})] \\
&= E[\varepsilon_t\varepsilon_{t-1}] + \theta E[\varepsilon_t\varepsilon_{t-2}] \\
&\quad + \theta E[\varepsilon_{t-1}^2] + \theta^2 E[\varepsilon_{t-1}\varepsilon_{t-2}] \\
&= 0 + 0 + \theta\sigma_\varepsilon^2 + 0 = \theta\sigma_\varepsilon^2
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\mathrm{cov}(Y_t, Y_{t-2}) &= \gamma_2 = E[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-2} + \theta\varepsilon_{t-3})] \\
&= E[\varepsilon_t\varepsilon_{t-2}] + \theta E[\varepsilon_t\varepsilon_{t-3}] \\
&\quad + \theta E[\varepsilon_{t-1}\varepsilon_{t-2}] + \theta^2 E[\varepsilon_{t-1}\varepsilon_{t-3}] \\
&= 0 + 0 + 0 + 0 = 0
\end{aligned}
$$

Similar calculation show that

$$
\mathrm{cov}(Y_t, Y_{t-j}) = \gamma_j = 0 \text{ for } j > 1
$$

Autocorrelations

$$\rho_1 = \frac{\gamma_1}{\sigma^2} = \frac{\theta\sigma_\varepsilon^2}{\sigma_\varepsilon^2(1+\theta^2)} = \frac{\theta}{(1+\theta^2)}$$
$$\rho_j = \frac{\gamma_j}{\sigma^2} = 0 \text{ for } j > 1$$

Note:

$$\rho_1 = 0 \text{ if } \theta = 0$$
$$\rho_1 > 0 \text{ if } \theta > 0$$
$$\rho_1 < 0 \text{ if } \theta < 0$$

Result: MA(1) is covariance stationary for any value of θ

Example: MA(1) model for overlapping returns

Let r_t denote the 1−month cc return and assume that

$$r_t \sim \text{iid } N(\mu_r, \sigma_r^2)$$

Consider creating a monthly time series of 2−month cc returns using

$$r_t(2) = r_t + r_{t-1}$$

These 2−month returns observed monthly overlap by 1 month

$$r_t(2) = r_t + r_{t-1}$$
$$r_{t-1}(2) = r_{t-1} + r_{t-2}$$
$$r_{t-2}(2) = r_{t-2} + r_{t-3}$$
$$\vdots$$

Claim: The stochastic process $\{r_t(2)\}$ follows a MA(1) process

Autoregressive (AR) Processes

AR(1) Model (mean-adjusted form)

$$\begin{aligned} Y_t - \mu &= \phi(Y_{t-1} - \mu) + \varepsilon_t, \ \ -1 < \phi < 1 \\ \varepsilon_t &\sim \text{iid } N(0, \sigma_\varepsilon^2) \end{aligned}$$

Result: AR(1) model is covariance stationary provided $-1 < \phi < 1$

Properties

$$\begin{aligned} E[Y_t] &= \mu \\ \text{var}(Y_t) &= \sigma^2 = \sigma_\varepsilon^2/(1-\phi^2) \\ \text{cov}(Y_t, Y_{t-1}) &= \gamma_1 = \sigma^2\phi \\ \text{corr}(Y_t, Y_{t-1}) &= \rho_1 = \gamma_1/\sigma^2 = \phi \\ \text{cov}(Y_t, Y_{t-j}) &= \gamma_j = \sigma^2\phi^j \\ \text{corr}(Y_t, Y_{t-j}) &= \rho_j = \gamma_j/\sigma^2 = \phi^j \end{aligned}$$

Note: Since $|\phi| < 1$

$$\lim_{j\to\infty} \rho_j = \phi^j = 0$$

AR(1) Model (regression model form)

$$
\begin{aligned}
Y_t - \mu &= \phi(Y_{t-1} - \mu) + \varepsilon_t \Rightarrow \\
Y_t &= \mu - \phi\mu + \phi Y_{t-1} + \varepsilon_t \\
&= c + \phi Y_{t-1} + \varepsilon_t
\end{aligned}
$$

where

$$
c = (1 - \phi)\mu \Rightarrow \mu = \frac{c}{1 - \phi}
$$

Remarks:

- Regression model form is convenient for estimation by linear regression

The AR(1) model and Economic and Financial Time Series

The AR(1) model is a good description for the following time series

- Interest rates
- Growth rate of macroeconomic variables
  - Real GDP, industrial production
  - Money, velocity
  - Real wages, unemployment