Econ 424/Amath 540
Portfolio Risk Budgeting

Eric Zivot

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Portfolio Risk Budgeting

Idea: Additively decompose a measure of portfolio risk into contributions from the individual assets in the portfolio.

- Show which assets are most responsible for portfolio risk

- Help make decisions about rebalancing the portfolio to alter the risk.

Example: 2 risky asset portfolio

$$\begin{aligned} R_p &= x_1 R_1 + x_2 R_2 \\ \sigma_p^2 &= x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12} \\ \sigma_p &= \left(x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12}\right)^{1/2} \end{aligned}$$

Case 1: $\sigma_{12} = 0$

$$\begin{aligned} \sigma_p^2 &= x_1^2\sigma_1^2 + x_2^2\sigma_2^2 \\ x_1^2\sigma_1^2 &= \text{ portfolio variance contribution of asset A} \\ x_2^2\sigma_2^2 &= \text{ portfolio variance contribution of asset B} \\ \frac{x_1^2\sigma_1^2}{\sigma_p^2} &= \text{percent variance contribution of asset A} \\ \frac{x_2^2\sigma_2^2}{\sigma_p^2} &= \text{percent variance contribution of asset B} \end{aligned}$$

Note

$$\sigma_p = \sqrt{x_1^2\sigma_1^2 + x_2^2\sigma_2^2} \neq x_1\sigma_1 + x_2\sigma_2.$$

To get an additive decomposition we use

$$\begin{aligned}
\frac{x_1^2\sigma_1^2}{\sigma_p} + \frac{x_2^2\sigma_2^2}{\sigma_p} &= \frac{\sigma_p^2}{\sigma_p} = \sigma_p \\
\frac{x_1^2\sigma_1^2}{\sigma_p} &= \text{portfolio sd contribution of asset A} \\
\frac{x_2^2\sigma_2^2}{\sigma_p} &= \text{portfolio sd contribution of asset B}
\end{aligned}$$

Notice that percent sd contributions are the same as percent variance contributions.

Case 2: $\sigma_{12} \neq 0$

$$\begin{aligned}
\sigma_p^2 &= x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12} \\
&= \left(x_1^2\sigma_1^2 + x_1x_2\sigma_{12}\right) + \left(x_2^2\sigma_2^2 + x_1x_2\sigma_{12}\right).
\end{aligned}$$

Here we can split the covariance contribution $2x_Ax_2\sigma_{12}$ to portfolio variance evenly between the two assets and define

$$\begin{aligned}
x_1^2\sigma_1^2 + x_1x_2\sigma_{12} &= \text{variance contribution of asset A} \\
x_2^2\sigma_2^2 + x_1x_2\sigma_{12} &= \text{variance contribution of asset B}
\end{aligned}$$

We can also define an additive decomposition for σ_p

$$
\begin{aligned}
\sigma_p &= \frac{x_1^2\sigma_1^2 + x_1 x_2 \sigma_{12}}{\sigma_p} + \frac{x_2^2\sigma_2^2 + x_1 x_2 \sigma_{12}}{\sigma_p} \\
\frac{x_1^2\sigma_1^2 + x_1 x_2 \sigma_{12}}{\sigma_p} &= \text{sd contribution of asset A} \\
\frac{x_2^2\sigma_2^2 + x_1 x_2 \sigma_{12}}{\sigma_p} &= \text{sd contribution of asset B}
\end{aligned}
$$

Euler's Theorem and Risk Decompositions

- When we used σ_p^2 or σ_p to measure portfolio risk, we were able to easily derive sensible risk decompositions.

- If we measure portfolio risk by value-at-risk or some other risk measure it is not so obvious how to define individual asset risk contributions.

- For portfolio risk measures that are homogenous functions of degree one in the portfolio weights, Euler's theorem provides a general method for additively decomposing risk into asset specific contributions.

Homogenous functions and Euler's theorem

First we define a homogenous function of degree one.

Definition 1 *homogenous function of degree one*

Let $f(x_1, \ldots, x_n)$ be a continuous and differentiable function of the variables $x_1, \ldots, x_n$. f is homogeneous of degree one if for any constant c, $f(c \cdot x_1, \ldots, c \cdot x_n) = c \cdot f(x_1, \ldots, x_n)$.

Note: In matrix notation we have $f(x_1, \ldots, x_n) = f(\mathbf{x})$ where

$\mathbf{x} = (x_1, \ldots, x_n)'$. Then f is homogeneous of degree one if $f(c \cdot \mathbf{x}) = c \cdot f(\mathbf{x})$

Examples

Let $f(x_1, x_2) = x_1 + x_2$. Then

$$f(c \cdot x_1, c \cdot x_2) = c \cdot x_1 + c \cdot x_2 = c \cdot (x_1 + x_2) = c \cdot f(x_1, x_2)$$

Let $f(x_1, x_2) = x_1^2 + x_2^2$. Then

$$f(c \cdot x_1, c \cdot x_2) = c^2 x_1^2 + x_2^2 c^2 = c^2 (x_1^2 + y_2^2) \neq c \cdot f(x_1, x_2)$$

Let $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ Then

$$f(c \cdot x_1, c \cdot x_2) = \sqrt{c^2 x_1^2 + c^2 x_2^2} = c\sqrt{(x_1^2 + x_2^2)} = c \cdot f(x_1, x_2)$$

Repeat examples using matrix notation

Define $\mathbf{x} = (x_1, x_2)'$ and $\mathbf{1} = (1, 1)'$.

Let $f(x_1, x_2) = x_1 + x_2 = \mathbf{x}'\mathbf{1} = \mathbf{f}(\mathbf{x})$. Then

$$f(c \cdot \mathbf{x}) = (c \cdot \mathbf{x})'\,\mathbf{1} = c \cdot (\mathbf{x}'1) = c \cdot f(\mathbf{x}).$$

Let $f(x_1, x_2) = x_1^2 + x_2^2 = \mathbf{x}'\mathbf{x} = f(\mathbf{x})$. Then

$$f(c \cdot \mathbf{x}) = (c \cdot \mathbf{x})'(c \cdot \mathbf{x}) = c^2 \cdot \mathbf{x}'\mathbf{x} \neq c \cdot f(\mathbf{x}).$$

Let $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} = (\mathbf{x}'\mathbf{x})^{1/2} = f(\mathbf{x})$. Then

$$f(c \cdot \mathbf{x}) = \Big((c \cdot \mathbf{x})'(c \cdot \mathbf{x})\Big)^{1/2} = c \cdot \Big(\mathbf{x}'\mathbf{x}\Big)^{1/2} = c \cdot f(\mathbf{x}).$$

Consider a portfolio of n assets $\mathbf{x} = (x_1, \ldots, x_n)'$

$$\begin{aligned}
\mathbf{R} &= (R_1, \ldots, R_n)' \\
\mathbf{x} &= (x_1, \ldots, x_n)' \\
E[\mathbf{R}] &= \boldsymbol{\mu},\ \mathrm{cov}(\mathbf{R}) = \boldsymbol{\Sigma}
\end{aligned}$$

Define

$$\begin{aligned}
R_p &= R_p(\mathbf{x}) = \mathbf{x}'\mathbf{R}, \\
\mu_p &= \mu_p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\mu} \\
\sigma_p^2 &= \sigma_p^2(\mathbf{x}) = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x},\ \sigma_p = \sigma_p(\mathbf{x}) = (\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})^{1/2}
\end{aligned}$$

Result: Portfolio return $R_p(\mathbf{x})$, expected return $\mu_p(\mathbf{x})$ and standard deviation $\sigma_p(\mathbf{x})$ are homogenous functions of degree one in the portfolio weight vector $\mathbf{x}$.

The key result is for volatility $\sigma_p(\mathbf{x}) = (\mathbf{x}'\mathbf{\Sigma}\mathbf{x})^{1/2}$:

$$\begin{aligned}
\sigma_p(c \cdot \mathbf{x}) &= ((c \cdot \mathbf{x})'\mathbf{\Sigma}(c \cdot \mathbf{x}))^{1/2} = 2 \cdot (\mathbf{x}'\mathbf{\Sigma}\mathbf{x})^{1/2} \\
&= c \cdot \sigma_p(\mathbf{x})
\end{aligned}$$

Theorem 2 *Euler's theorem*

Let $f(x_1, \ldots, x_n) = f(\mathbf{x})$ be a continuous, differentiable and homogenous of degree one function of the variables $\mathbf{x} = (x_1, \ldots, x_n)'$. Then

$$\begin{aligned} f(\mathbf{x}) &= x_1 \cdot \frac{\partial f(\mathbf{x})}{\partial x_1} + x_2 \cdot \frac{\partial f(\mathbf{x})}{\partial x_2} + \cdots + x_n \cdot \frac{\partial f(\mathbf{x})}{\partial x_n} \\ &= \mathbf{x}' \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}, \end{aligned}$$

where

$$\underset{(n \times 1)}{\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}} = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{pmatrix}$$

Verifying Euler's theorem

The function $f(x_1, x_2) = x_1 + x_2 = f(\mathbf{x}) = \mathbf{x}'\mathbf{1}$ is homogenous of degree one, and

$$\begin{aligned} \frac{\partial f(\mathbf{x})}{\partial x_1} &= \frac{\partial f(\mathbf{x})}{\partial x_2} = 1 \\ \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} &= \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{1} \end{aligned}$$

By Euler's theorem,

$$\begin{aligned} f(x) &= x_1 \cdot 1 + x_2 \cdot 1 = x_1 + x_2 \\ &= \mathbf{x}'\mathbf{1} \end{aligned}$$

The function $f(x_1, x_2) = (x_1^2 + x_2^2)^{1/2} = f(\mathbf{x}) = (\mathbf{x}'\mathbf{x})^{1/2}$ is homogenous of degree one, and

$$\begin{aligned}
\frac{\partial f(\mathbf{x})}{\partial x_1} &= \frac{1}{2}\left(x_1^2 + x_2^2\right)^{-1/2} 2x_1 = x_1\left(x_1^2 + x_2^2\right)^{-1/2}, \\
\frac{\partial f(\mathbf{x})}{\partial x_2} &= \frac{1}{2}\left(x_1^2 + x_2^2\right)^{-1/2} 2x_2 = x_2\left(x_1^2 + x_2^2\right)^{-1/2}.
\end{aligned}$$

By Euler's theorem

$$\begin{aligned}
f(x) &= x_1 \cdot x_1\left(x_1^2 + x_1^2\right)^{-1/2} + x_2 \cdot x_2\left(x_1^2 + x_2^2\right)^{-1/2} \\
&= \left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2\right)^{-1/2} \\
&= \left(x_1^2 + x_2^2\right)^{1/2}.
\end{aligned}$$

Using matrix algebra we have

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \frac{\partial (\mathbf{x}'\mathbf{x})^{1/2}}{\partial \mathbf{x}} = \frac{1}{2}(\mathbf{x}'\mathbf{x})^{-1/2}\frac{\partial \mathbf{x}'\mathbf{x}}{\partial \mathbf{x}} = \frac{1}{2}(\mathbf{x}'\mathbf{x})^{-1/2} 2\mathbf{x} = (\mathbf{x}'\mathbf{x})^{-1/2} \cdot \mathbf{x}$$

so by Euler's theorem

$$f(\mathbf{x}) = \mathbf{x}'\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{x}'(\mathbf{x}'\mathbf{x})^{-1/2} \cdot \mathbf{x} = (\mathbf{x}'\mathbf{x})^{-1/2}\mathbf{x}'\mathbf{x} = (\mathbf{x}'\mathbf{x})^{1/2}$$

Risk decomposition using Euler's theorem

Let $\text{RM}_p(\mathbf{x})$ denote a portfolio risk measure that is a homogenous function of degree one in the portfolio weight vector $\mathbf{x}$. For example,

$$\text{RM}_p(\mathbf{x}) = \sigma_p(\mathbf{x}) = (\mathbf{x}'\mathbf{\Sigma}\mathbf{x})^{1/2}$$

Euler's theorem gives the additive risk decomposition

$$\begin{aligned}
\text{RM}_p(\mathbf{x}) &= x_1 \frac{\partial \text{RM}_p(\mathbf{x})}{\partial x_1} + x_2 \frac{\partial \text{RM}_p(\mathbf{x})}{\partial x_2} + \cdots + x_n \frac{\partial \text{RM}_p(\mathbf{x})}{\partial x_n} \\
&= \sum_{i=1}^{n} x_i \frac{\partial \text{RM}_p(\mathbf{x})}{\partial x_i} \\
&= \mathbf{x}' \frac{\partial \text{RM}_p(\mathbf{x})}{\partial \mathbf{x}}
\end{aligned}$$

Here, $\frac{\partial \mathsf{RM}_p(\mathbf{x})}{\partial x_i}$ are called *marginal contributions to risk* (MCRs):

$$\mathsf{MCR}_i^{RM} = \frac{\partial \mathsf{RM}_p(\mathbf{x})}{\partial x_i} = \text{ marginal contribution to risk of asset i,}$$

The *contributions to risk* (CRs) are defined as the weighted marginal contributions:

$$\mathsf{CR}_i^{RM} = x_i \cdot \mathsf{MCR}_i^{RM} = \text{ contribution to risk of asset i,}$$

Then

$$\begin{aligned} \mathsf{RM}_p(\mathbf{x}) &= x_1 \cdot \mathsf{MCR}_1^{RM} + x_2 \cdot \mathsf{MCR}_2^{RM} + \cdots + x_n \cdot \mathsf{MCR}_n^{RM} \\ &= \mathsf{CR}_1^{RM} + \mathsf{CR}_2^{RM} + \cdots + \mathsf{CR}_n^{RM} \end{aligned}$$

If we divide the contributions to risk by $\mathsf{RM}_p(\mathbf{x})$ we get the *percent contributions to risk* (PCRs)

$$1 = \frac{\mathsf{CR}_1^{RM}}{\mathsf{RM}_p(\mathbf{x})} + \cdots + \frac{\mathsf{CR}_n^{RM}}{\mathsf{RM}_p(\mathbf{x})} = \mathsf{PCR}_1^{RM} + \cdots + \mathsf{PCR}_n^{RM},$$

where

$$\mathsf{PCR}_i^{RM} = \frac{\mathsf{CR}_i^{RM}}{\mathsf{RM}_p(\mathbf{x})} = \text{ percent contribution of asset i}$$

Risk Decomposition for Portfolio SD

$$\text{RM}_p(\mathbf{x}) = \sigma_p(\mathbf{x}) = (\mathbf{x}'\mathbf{\Sigma}\mathbf{x})^{1/2}$$

Because $\sigma_p(\mathbf{x})$ is homogenous of degree 1 in $\mathbf{x}$, by Euler's theorem

$$\sigma_p(\mathbf{x}) = x_1 \frac{\partial \sigma_p(\mathbf{x})}{\partial x_1} + x_2 \frac{\partial \sigma_p(\mathbf{x})}{\partial x_2} + \cdots + x_n \frac{\partial \sigma_p(\mathbf{x})}{\partial x_n} = \mathbf{x}' \frac{\partial \sigma_p(\mathbf{x})}{\partial \mathbf{x}}$$

Now

$$\begin{aligned}
\frac{\partial \sigma_p(\mathbf{x})}{\partial \mathbf{x}} &= \frac{\partial (\mathbf{x}'\mathbf{\Sigma}\mathbf{x})^{1/2}}{\partial \mathbf{x}} = \frac{1}{2}(\mathbf{x}'\mathbf{\Sigma}\mathbf{x})^{-1/2} 2\mathbf{\Sigma}\mathbf{x} \\
&= \frac{\mathbf{\Sigma}\mathbf{x}}{(\mathbf{x}'\mathbf{\Sigma}\mathbf{x})^{1/2}} = \frac{\mathbf{\Sigma}\mathbf{x}}{\sigma_p(\mathbf{x})} \\
\Rightarrow & \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} = \text{ith row of } \frac{\mathbf{\Sigma}\mathbf{x}}{\sigma_p(\mathbf{x})}
\end{aligned}$$

Remark: In R, the PerformanceAnalytics function `StdDev()` performs this decomposition

Example: 2 asset portfolio

$$
\begin{aligned}
\sigma_p(\mathbf{x}) &= (\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})^{1/2} = \left(x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12}\right)^{1/2} \\
\boldsymbol{\Sigma}\mathbf{x} &= \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1\sigma_1^2 + x_2\sigma_{12} \\ x_2\sigma_2^2 + x_1\sigma_{12} \end{pmatrix} \\
\frac{\boldsymbol{\Sigma}\mathbf{x}}{\sigma_p(\mathbf{x})} &= \begin{pmatrix} \left(x_1\sigma_1^2 + x_2\sigma_{12}\right)/\sigma_p(\mathbf{x}) \\ \left(x_2\sigma_2^2 + x_1\sigma_{12}\right)/\sigma_p(\mathbf{x}) \end{pmatrix}
\end{aligned}
$$

Then

$$\begin{aligned}
\mathsf{MCR}_1^{\sigma} &= \left(x_1\sigma_1^2 + x_2\sigma_{12}\right)/\sigma_p(\mathbf{x}) \\
\mathsf{MCR}_2^{\sigma} &= \left(x_2\sigma_2^2 + x_1\sigma_{12}\right)/\sigma_p(\mathbf{x}) \\
\mathsf{CR}_1^{\sigma} &= x_1 \times \left(x_1\sigma_1^2 + x_2\sigma_{12}\right)/\sigma_p(\mathbf{x}) = \left(x_1^2\sigma_1^2 + x_1x_2\sigma_{12}\right)/\sigma_p(\mathbf{x}) \\
\mathsf{CR}_2^{\sigma} &= x_2 \times \left(x_2\sigma_2^2 + x_2\sigma_2\right)/\sigma_p(\mathbf{x}) = \left(x_2^2\sigma_2^2 + x_1x_2\sigma_{12}\right)/\sigma_p(\mathbf{x})
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{PCR}_1^{\sigma} &= \mathsf{CR}_1^{\sigma}/\sigma_p(\mathbf{x}) = \left(x_1^2\sigma_1^2 + x_1x_2\sigma_{12}\right)/\sigma_p^2(\mathbf{x}) \\
\mathsf{PCR}_2^{\sigma} &= \mathsf{CR}_2^{\sigma}/\sigma_p(\mathbf{x}) = \left(x_2^2\sigma_2^2 + x_1x_2\sigma_{12}\right)/\sigma_p^2(\mathbf{x})
\end{aligned}$$

Note: This is the decomposition we derived at the beginning of lecture.

How to Interpret and Use MCR_i^σ

$$
\begin{aligned}
\mathsf{MCR}_i^\sigma &= \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} \approx \frac{\Delta \sigma_p}{\Delta x_i} \\
&\Rightarrow \Delta \sigma_p \approx \mathsf{MCR}_i^\sigma \cdot \Delta x_i
\end{aligned}
$$

However, in a portfolio of n assets

$$x_1 + x_2 + \cdots + x_n = 1$$

so that increasing or decreasing x_i means that we have to decrease or increase our allocation to one or more other assets. Hence, the formula

$$\Delta \sigma_p \approx \mathsf{MCR}_i^\sigma \cdot \Delta x_i$$

ignores this re-allocation effect.

If the increase in allocation to asset i is offset by a decrease in allocation to asset j, then

$$\Delta x_j = -\Delta x_i$$

and the change in portfolio volatility is approximately

$$\begin{aligned}
\Delta \sigma_p &\approx \mathsf{MCR}_i^\sigma \cdot \Delta x_i + \mathsf{MCR}_j^\sigma \cdot \Delta x_j \\
&= \mathsf{MCR}_i^\sigma \cdot \Delta x_i - \mathsf{MCR}_j^\sigma \cdot \Delta x_i \\
&= \left(\mathsf{MCR}_i^\sigma - \mathsf{MCR}_j^\sigma\right) \cdot \Delta x_i
\end{aligned}$$

μ_1	μ_2	σ_1^2	σ_2^2	σ_1	σ_2	σ_{12}	ρ_{12}
0.175	0.055	0.067	0.013	0.258	0.115	-0.004875	-0.164

Table 1: Example data for two asset portfolio.

Consider two portfolios:

- equal weighted portfolio $x_1 = x_2 = 0.5$
- long-short portfolio $x_1 = 1.5$ and $x_2 = -0.5$.

	σ_i	x_i	MCR_i^{σ}	CR_i^{σ}	PCR_i^{σ}
			$\sigma_p = 0.1323$		
Asset 1	0.258	0.5	0.23310	0.11655	0.8807
Asset 2	0.115	0.5	0.03158	0.01579	0.1193
			$\sigma_p = 0.4005$		
Asset 1	0.258	1.5	0.25540	0.38310	0.95663
Asset 2	0.115	-0.5	-0.03474	0.01737	0.04337

Table 2: Risk decomposition using portfolio standard deviation.

Interpretation: For equally weighted portfolio, increasing x_1 from 0.5 to 0.6 decreases x_2 from 0.5 to 0.4. Then

$$\begin{aligned}
\Delta\sigma_p &\approx (\mathsf{MCR}_1^{\sigma} - \mathsf{MCR}_2^{\sigma}) \cdot \Delta x_i \\
&= (0.23310 - 0.03158)(0.1) \\
&= 0.02015
\end{aligned}$$

For the long-short portfolio, increasing x_1 from 1.5 to 1.6 decreases x_2 from -0.5 to -0.6. Then

$$\begin{aligned}
\Delta\sigma_p &\approx (\mathsf{MCR}_1^{\sigma} - \mathsf{MCR}_2^{\sigma}) \cdot \Delta x_i \\
&= [0.25540 - (-0.03474)]\,(0.1) \\
&= 0.02901
\end{aligned}$$

Beta as a Measure of Asset Contribution to Portfolio Volatility

For a portfolio of n assets with return

$$R_p(\mathbf{x}) = x_1 R_1 + \cdots + x_n R_n = \mathbf{x}'\mathbf{R}$$

we derived the portfolio volatility decomposition

$$\begin{aligned}
\sigma_p(\mathbf{x}) &= x_1 \frac{\partial \sigma_p(\mathbf{x})}{\partial x_1} + x_2 \frac{\partial \sigma_p(\mathbf{x})}{\partial x_2} + \cdots + x_n \frac{\partial \sigma_p(\mathbf{x})}{\partial x_n} = \mathbf{x}' \frac{\partial \sigma_p(\mathbf{x})}{\partial \mathbf{x}} \\
\frac{\partial \sigma_p(\mathbf{x})}{\partial \mathbf{x}} &= \frac{\mathbf{\Sigma x}}{\sigma_p(\mathbf{x})}, \ \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} = \text{ith row of } \frac{\mathbf{\Sigma x}}{\sigma_p(\mathbf{x})}
\end{aligned}$$

With a little bit of algebra we can derive an alternative expression for

$$\mathsf{MCR}_i^{\sigma} = \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} = \text{ith row of } \frac{\mathbf{\Sigma x}}{\sigma_p(\mathbf{x})}$$

Definition: The beta of asset i with respect to the portfolio is defined as

$$\beta_i = \frac{cov(R_i, R_p(\mathbf{x}))}{var(R_p(\mathbf{x}))} = \frac{cov(R_i, R_p(\mathbf{x}))}{\sigma_p^2(\mathbf{x})}$$

Result: β_i measures asset contribution to $\sigma_p(\mathbf{x})$:

$$\begin{aligned} \mathsf{MCR}_i^{\sigma} &= \frac{\partial \sigma_p(\mathbf{x})}{\partial x_i} = \beta_i \sigma_p(\mathbf{x}) \\ \mathsf{CR}_i^{\sigma} &= x_i \beta_i \sigma_p(\mathbf{x}) \\ \mathsf{PCR}_i^{\sigma} &= x_i \beta_i \end{aligned}$$

Remarks

- By construction, the beta of the portfolio is 1

$$\beta_p = \frac{cov(R_p(\mathbf{x}), R_p(\mathbf{x}))}{var(R_p(\mathbf{x}))} = \frac{var(R_p(\mathbf{x}))}{var(R_p(\mathbf{x}))} = 1$$

- When $\beta_i = 1$

$$\begin{aligned} \mathsf{MCR}_i^{\sigma} &= \sigma_p(\mathbf{x}) \\ \mathsf{CR}_i^{\sigma} &= x_i \sigma_p(\mathbf{x}) \\ \mathsf{PCR}_i^{\sigma} &= x_i \end{aligned}$$

- When $\beta_i > 1$

$$
\begin{aligned}
\mathsf{MCR}_i^\sigma &> \sigma_p(\mathbf{x}) \\
\mathsf{CR}_i^\sigma &> x_i \sigma_p(\mathbf{x}) \\
\mathsf{PCR}_i^\sigma &> x_i
\end{aligned}
$$

- When $\beta_i < 1$

$$
\begin{aligned}
\mathsf{MCR}_i^\sigma &< \sigma_p(\mathbf{x}) \\
\mathsf{CR}_i^\sigma &< x_i \sigma_p(\mathbf{x}) \\
\mathsf{PCR}_i^\sigma &< x_i
\end{aligned}
$$

Derivation of Result:

Recall,

$$\frac{\partial \sigma_p(\mathbf{x})}{\partial \mathbf{x}} = \frac{\mathbf{\Sigma x}}{\sigma_p(\mathbf{x})}$$

Now,

$$\mathbf{\Sigma x} = \begin{pmatrix} \sigma_1^1 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The first row of $\mathbf{\Sigma x}$ is

$$x_1\sigma_1^2 + x_2\sigma_{12} + \cdots + x_n\sigma_{1n}$$

Now consider

$$\begin{aligned} cov(R_1, R_p) &= cov(R_1, x_1R_1 + \cdots + x_nR_n) \\ &= cov(R_1, x_1R_1) + \cdots + cov(R_1, x_nR_n) \\ &= x_1\sigma_1^1 + x_2\sigma_{12} + \cdots + x_n\sigma_{1n} \end{aligned}$$

Next, note that

$$\beta_1 = \frac{cov(R_1, R_p)}{\sigma_p^2} \Rightarrow cov(R_1, R_p) = \beta_1\sigma_p^2(\mathbf{x})$$

Hence, the first row of $\mathbf{\Sigma x}$ is

$$x_1\sigma_1^2 + x_2\sigma_{12} + \cdots + x_n\sigma_{1n} = \beta_1\sigma_p^2(\mathbf{x})$$

and so

$$\begin{aligned}
\mathsf{MCR}_1^\sigma &= \frac{\partial\sigma_p(\mathbf{x})}{\partial x_1} = \text{first row of } \frac{\mathbf{\Sigma x}}{\sigma_p(\mathbf{x})} \\
&= \frac{\beta_1\sigma_p^2(\mathbf{x})}{\sigma_p(\mathbf{x})} = \beta_1\sigma_p(\mathbf{x})
\end{aligned}$$

In a similar fashion, we have

$$\begin{aligned}
\mathsf{MCR}_i^\sigma &= \frac{\partial\sigma_p(\mathbf{x})}{\partial x_i} = \text{first row of } \frac{\mathbf{\Sigma x}}{\sigma_p(\mathbf{x})} \\
&= \frac{\beta_i\sigma_p^2(\mathbf{x})}{\sigma_p(\mathbf{x})} = \beta_i\sigma_p(\mathbf{x})
\end{aligned}$$

Beta as a Measure of Portfolio Risk

Key points:

- Asset specific risk can be diversified away by forming portfolios. What remains is “portfolio risk”.

- Riskiness of an asset should be judged in a portfolio context

- Beta measures the portfolio risk of an asset

Beta and Risk Return Tradeoff

$$R_p = \text{ return on any portfolio}$$
$$R_i = \text{ return on any asset } i$$
$$\beta_{i,p} = \frac{\mathsf{cov}(R_i, R_p)}{\mathsf{var}(R_p)} = \frac{\sigma_{i,p}}{\sigma_p^2}$$

Conjecture: If $\beta_{i,p}$ is the appropriate measure of the risk of an asset, then the asset's expected return, μ_i, should depend on $\beta_{i,p}$. That is

$$E[R_i] = \mu_i = f(\beta_{i,p})$$

The *Capital Asset Pricing Model* (CAPM) formalizes this conjecture.