There will be two sections in this chapter. One on set theory and one on combinatorics

1 Set theory

Formula for $P(E,F) = P(E) + P(F) - P(EF)$. For a die example coming up to or less than 3, there is two sets

$$E = \{1, 2, 3\} F = \{3\} \tag{1}$$

This yields

$$P(E,F) = P(E) + P(F) - P(EF) = \frac{3}{6} + \frac{1}{6} - \frac{1}{6} = \frac{3}{6} \tag{2}$$

The expansion of the formula is the following:

$$P(A \cup B \cup C) = P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A(B \cup C)) = P(A) + P(B) + P(C) - P(BC) - P(A(B \cup C) \tag{3}$$

It can be shown that

$$P(A(B \cup C)) = P(AB \cup AC) = P(AB) + P(AC) - P(ABC) \tag{4}$$

Therefore

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(BC) - P(AB) - P(AC) + P(ABC) \tag{5}$$

Also it is important to note that

$$P[A \cup C|B] = \frac{P[(A \cup C) \cap B]}{P[B]} = \frac{P[(A \cap B)] \cup (C \cap B)]}{P[B]} = \frac{P[(A \cap B)] \cup P[(C \cap B)]}{P[B]} = P[A|B] + P[C|B] \tag{6}$$

The above equation gives us idea of the total law of probability . If sample space S is defined into N partitions $B_1, \cdots, B_N$ such that $S = \cup_{i=1}^{N} B_i$ and $B_i \cap B_j = \emptyset$, then $P[A]$ can be rewritten in this form:

$$P[A] = P[A \cap S] = P[A \cap (\cup_{i=1}^{N} B_i)] = P[(A \cap B_1) \cup (A \cap B_2) \cdots (A \cap B_N)] \tag{7}$$

This gives us

$$P[A] = \sum_{i=1}^{N} P[A \cap B_i] \tag{8}$$

, which given the formula above yields

$$P[A] = \sum_{i=1}^{N} P[A|B_i] P[B_i] \tag{9}$$

This can be used in problems when probability can be expanded into a tree form (e.g. calculating probability of selecting a red balls from N urns, each having $P[A_i]$ red ball proportion).

2 Combinatorics

There is an urn with k red balls and $N-k$ black balls. We want to define event where red ball is chosen followed by a black one *with replacement.* This is represented by the event $E = \{(z_1, z_2) : z_1 = 1, \cdots, k, z_2 = k+1, \cdots, N\}$. This is defined as

$$P[E] = \frac{N_E}{N_S} = \frac{k(N-k)}{N^2} = \frac{k}{N}(N-k)\frac{1}{N} = \frac{k}{N}(1-\frac{k}{N}) \tag{10}$$

Observe that if we set $p = \frac{k}{N}$ we will get a proportion $p(1-p)$
Now, consider sampling *without replacement.*

$$P[E] = \frac{k(N-k)}{N(N-1)} = \frac{k}{N}\frac{(N-k)N}{N}\frac{1}{N-1} = p(1-p)\frac{N}{1-N} \tag{11}$$

There are two options to consider,

1. Sampling with replacement - we use N^n formula
2. Sampling without replacement - there is two options in here
   (a) ordered sample - for which we will use $\frac{N!}{(N-n)!}$. These are referred to as permutations, and expressed as:

   $$\frac{N!}{(N-n)!} = \frac{N(N-1)(N-2)\cdots 2\cdot 1}{N(N-1)(N-2)\cdots(N-n+1)(N-n)} \tag{12}$$

   This is sometimes referred to as x to the k falling, and can be rewritten as

   $$x^k = (N)_k = x(x+1)(x+2)\cdots(x+k-1) = \prod_{i=0}^{k=1}(x-i) \tag{13}$$

   (b) unordered sample - for which we will use the binomial coefficient $nk = \frac{N!}{n!(N-n)!}$. These are called combinations.

In general, there is fewer results for combinations, than permutations. Also, as the well-known fact states a combination lock is actually a permutation lock. For a permutation lock both sequences $\{1, 4, 6, 9\}$ and $\{1, 4, 6, 9\}$ are different. While for a combination lock both are treated as one sequence.