Lecture notes on flow equation approach to singular stochastic PDEs

Paweł Duch
Faculty of Mathematics and Computer Science
Adam Mickiewicz University in Poznań
ul. Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland
pawel.duch@amu.edu.pl

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Abstract

The flow equation approach is applicable to a large class of singular SPDEs including equations with fractional Laplacian in the whole subcritical regime. The main idea of the approach, borrowed from the Wilsonian renormalization group theory, is to study the so-called coarse-grained process, which captures the behavior of the solution of the original equation at different spatial scales. The dynamics of the coarse-grained process is described by the effective equation. The flow equation governs the evolution of the non-linear term in the effective equation in the coarse-graining scale and plays an analogous role to the Polchinski equation in QFT. The renormalization problem is solved using an inductive argument and amounts to imposing appropriate boundary conditions when solving the flow equation.

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1 Singular subcritical SPDEs

These lecture notes aim to give an overview of an approach to singular SPDEs based on the renormalization group flow equation. Let us recall that in general a singular SPDE is any PDE with random terms or coefficients that cannot be solved using classical PDE tools. The problem is the irregular nature of sample paths of random fields and the insufficient regularity of the solution which does not allow to define products appearing in the non-linear terms of the equation. As a result, it is not even clear what it means to be a solution of a singular SPDE. Some singular SPDEs can be made sense of using certain non-linear properties of the random fields. As we will see, often renormalization of non-linear terms is necessary.

The flow equation approach is applicable to a large class of singular SPDEs in full subcritical regime. In these notes we study only equations driven by the white noise (or its periodization). Recall that the white noise ξ on $\mathbb{R}^n$ is the unique Gaussian random variable valued in $\mathscr{S}'(\mathbb{R}^n)$ with mean zero such that

$$\mathbb{E}(\langle\xi,f\rangle\langle\xi,g\rangle)=\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x$$

for all $f,g\in\mathscr{S}(\mathbb{R}^n)$. Let us list a couple of examples of interesting singular SPDEs:

(1) Dynamical Φ^4_d model, also known as the parabolic stochastic quantization equation of the Φ^4_d model, with $d\in\{2,3\}$ – a parabolic SPDE of the form

$$(\partial_t+1-\Delta)\Phi=\xi-\lambda\,\Phi^3+\infty\,\Phi$$

posed in spacetime $\mathbb{R}_+\times\mathbb{R}^d$ driven by the spacetime white noise ξ.

(2) Dynamical fractional $\Phi^4_{d,\sigma}$ model with $d\in\{2,3,4\}$ and $\sigma\in(d/2,d]$ – a parabolic non-local SPDE of the form

$$(\partial_t+1+(-\Delta)^{\sigma/2})\Phi=\xi-\lambda\,\Phi^3+\infty\,\Phi$$

posed in spacetime $\mathbb{R}_+\times\mathbb{R}^d$ driven by the spacetime white noise ξ and involving the fractional Laplacian $(-\Delta)^{\sigma/2}$ of order σ. Note that for $\sigma=2$ the above SPDE coincides with the standard dynamical Φ^4_d model. Formally, the $\Phi^4_{d,\sigma}$ measure

$$\mu(\mathrm{d}\phi)=\frac{1}{\mathcal{Z}}\exp\left(-\int_{\mathbb{R}^d}\Big(\phi(x)\big(1+(-\Delta)^{\sigma/2}\big)\phi(x)+\lambda\,\phi(x)^4/2-\infty\,\phi(x)^2\Big)\,\mathrm{d}x\right)\mathrm{d}\phi,$$

is invariant for the above SPDE, i.e. if Φ is a solution of the above SPDE with the initial condition $\Phi(0,\bullet)=\phi$ distributed according to the above measure, then $\Phi(t,\bullet)$ is also distributed according to this measure for all $t\in\mathbb{R}_+$.

(3) Elliptic stochastic quantization equation of Φ^4_d model with $d \in \{2,3\}$ – an elliptic SPDE of the form

$$(1-\Delta)\varPhi = \xi - \lambda\,\varPhi^3 + \infty\,\varPhi$$

posed in space $\mathbb{R}^{2+d}$ driven by the white noise ξ. Formally, the solution of the above equation evaluated at the hyperspace of co-dimension two $\varPhi(0,0,\bullet)$ is distributed according to the $\Phi^4_{d,\sigma=2}$ measure.

(4) Dynamical Sine-Gordon model with $\beta \in (0,\sqrt{8\pi})$ – a parabolic SPDE of the form

$$(\partial_t + 1 - \Delta)\varPhi = \xi - \lambda\,\infty\,\beta\,\sin(\beta\varPhi)$$

posed in spacetime $\mathbb{R}_+ \times \mathbb{R}^2$ driven by the spacetime white noise ξ. Formally, the measure

$$\mu(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}}\exp\left(-\int_{\mathbb{R}^2}\Big(\phi(x)(1-\Delta)\phi(x) + 2\lambda\,\infty\,\cos(\beta\phi(x))\Big)\mathrm{d}x\right)\,\mathrm{d}\phi$$

is invariant for the above SPDE.

All of the above SPDEs are singular if $\lambda \neq 0$. For $\lambda = 0$ the equations are linear and can be easily solved but the solution is a Schwartz distribution that is almost surely not a function. This indicates that for $\lambda \neq 0$ we should try to solve the above equations in some space of distributions. However, since multiplication of distributions is in general not a well-defined operation it is not clear how to interpret the non-linear terms. The standard solution is to introduce some UV regularization and subsequently prove that it can be removed provided the nonlinear terms are appropriately renormalized.

In these notes we shall study the following singular elliptic SPDE

$$(1-\Delta)^{\sigma/2}\varPhi = \xi + \lambda\,\varPhi^3 - \infty\,\varPhi \tag{1.1}$$

posed in space $\mathbb{R}^d$ driven by the periodization ξ of the white noise with the period 2π. For concreteness, we assume that $d = 5$ and $\sigma \in (d/3, d/2]$, which corresponds to the so-called full subcritical regime. For pedagogical reason we prefer to study an elliptic problem even though the above equation does not seem to have any practical applications. The method we present can in principle be extended to all of the equations listed above.

In order to make sense of Eq. (1.1) we have to introduce some UV regularization. To this end, let $\vartheta \in C^\infty(\mathbb{R}^d)$ be an even positive function supported in the unit ball that integrates to one. For $\kappa \in [0,1]$ define $\vartheta_\kappa(x) := [\kappa]^{-d}\vartheta(x/[\kappa])$, where $[\kappa] := \kappa^{1/\sigma}$. The function ϑ_κ is supported in a ball of radius $[\kappa]$ and converges to the Dirac delta as $\kappa \searrow 0$. For $\kappa \in (0,1]$ we define the regularized noise by $\xi_\kappa := \vartheta_\kappa * \xi \in C^\infty(\mathbb{R}^d)$, where $*$ denotes the convolution. We have $\xi_\kappa \in C^\infty(\mathbb{R}^d)$ for all $\kappa \in (0,1]$ and $\lim_{\kappa\searrow 0}\xi_\kappa = \xi \in \mathscr{S}'(\mathbb{R}^d)$ almost surely. We rewrite the singular SPDE (1.1) in the following regularized mild form

$$\varPhi = G * F_\kappa[\varPhi], \qquad \kappa \in (0,1], \tag{1.2}$$

where $G \in L^1(\mathbb{R}^d)$ is the fundamental solution for the pseudo-differential operator $(1-\Delta)^{\sigma/2}$. The functional $F_\kappa[\varphi]$, called the force, is defined by

$$F_\kappa[\varphi](x) := \xi_\kappa(x) + \lambda\varphi^3(x) + \sum_{i=1}^{i_\sharp} \lambda^i c_\kappa^{(i)} \varphi(x), \tag{1.3}$$

where $i_\sharp := \lfloor \sigma/(3\sigma - d) \rfloor$ and the parameters $c_\kappa^{(i)} \in \mathbb{R}$ depending on the UV cutoff κ are called the counterterms. Note that the number of counterterms that are needed to renormalize the cubic nonlinearity diverges in the limit $\sigma \searrow d/3$. Thus, we expect that the renormalization problem becomes increasingly more difficult as one approaches the threshold $\sigma = d/3$ at which the equation becomes critical. Let us state the main result of these lecture notes.

Theorem 1.1. *Let $d \in \{1, \ldots, 6\}$ and $\sigma \in (d/3, d/2]$. There exist a choice of counterterms and random variables $\lambda_\star \in [0,1]$ and $\Phi_0 \in \mathscr{S}'(\mathbb{R}^d)$ such that: (0) for every random variable $\lambda \in [-\lambda_\star, \lambda_\star]$ and $\kappa \in (0,1]$ Eq. (1.2) has a solution $\Phi_\kappa \in C^\infty(\mathbb{R}^d)$, (1) it holds $\Phi_0 = \lim_{\kappa\searrow 0} \Phi_\kappa$ almost surely in $\mathscr{S}'(\mathbb{R}^d)$, (2) it holds $\mathbb{E}(\lambda_\star^{-n}) < \infty$ for every $n \in \mathbb{N}_+$.*

Remark 1.2. Note that λ, the prefactor of the non-linear term in the equation, is assumed to be random and sufficiently small. In the case of parabolic equations the method discussed in these notes can be used to construct a solution in a sufficiently small time interval without any assumption about the strength of the non-linearity.

1.1 Da Prato-Debussche regime and beyond

In order to quantify the regularity/irregularity of a function/distribution one can use the Hölder-Besov spaces $\mathscr{C}^\alpha(\mathbb{R}^d) \equiv B^\alpha_{\infty,\infty}(\mathbb{R}^d)$. For positive non-integer α the space $\mathscr{C}^\alpha(\mathbb{R}^d)$ coincides with the Hölder space. For $\alpha \in (-\infty, 0]$ the space $\mathscr{C}^\alpha(\mathbb{R}^d)$ consists of distributions $\phi \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|\phi\|_{\mathscr{C}^\alpha(\mathbb{R}^d)} := \sup_{\mu\in(0,1]} [\mu]^{-\alpha} \|K_\mu * \phi\|_{L^\infty(\mathbb{R}^d)} < \infty, \qquad [\mu] := \mu^{1/\sigma},$$

where $K_\mu(x) := [\mu]^{-d} K(x/[\mu])$, $\mu \in (0,1]$, for some non-negative $K \in C^{\lceil -\alpha \rceil}(\mathbb{M})$ of fast decay that integrates to one. We say that $\phi \in \mathscr{S}'(\mathbb{R}^d)$ has regularity $\alpha \in \mathbb{R}$ if $\phi \in \mathscr{C}^\alpha(\mathbb{R}^d)$. Let us list some well-known facts about the Besov spaces (see e.g. [BCD11, Sec. 2.7, 2.8]):

(1) If $\alpha + \beta > 0$, then the pointwise product of test functions extends continuously as a map $\mathscr{C}^\alpha(\mathbb{R}^d) \times \mathscr{C}^\beta(\mathbb{R}^d) \to \mathscr{C}^{\alpha\wedge\beta}(\mathbb{R}^d)$.

(2) The convolution with the Green function G, the fundamental solution of $(1-\Delta)^{\sigma/2}$, maps continuously $\mathscr{C}^\alpha(\mathbb{R}^d)$ into $\mathscr{C}^{\alpha+\sigma}(\mathbb{R}^d)$.

(3) For $\alpha \geq \beta$ there is a continuous inclusion $\mathscr{C}^\alpha(\mathbb{R}^d) \to \mathscr{C}^\beta(\mathbb{R}^d)$.

Furthermore, we note that sample paths of ξ, the periodization of the white noise on $\mathbb{R}^d$, belong almost surely to the space $\mathscr{C}^{-d/2-\varepsilon}(\mathbb{R}^d)$ for every $\varepsilon > 0$ (see e.g. [MWX16, Thm. 5]).

Consequently, the solution of Eq. (1.2) with $\kappa = 0$ and $\lambda = 0$ given by $G * \xi$ belongs almost surely to the Besov space $\mathscr{C}^\alpha(\mathbb{R}^d)$ with $\alpha = \sigma - d/2 - \varepsilon$. In the subcritical regime Eq. (1.2) with $\lambda \neq 0$ is in some sense a small perturbation of the linear equation with $\lambda = 0$. Consequently, we expect that the solution of Eq. (1.2) also lives in the Besov space $\mathscr{C}^\alpha(\mathbb{R}^d)$.

Remark 1.3. In what follows, $\alpha = \sigma - d/2 - \varepsilon$ denotes the expected regularity of the solution of Eq. (1.2) and $\varepsilon \in (0,\infty)$ is assumed to be sufficiently small. Note that for smaller values of σ the equation is more singular as the regularizing effect of the Green function G is weaker.

Using the above-mentioned facts one easily shows that for $\sigma \in (d/2,\infty)$ the equation we study, $\Phi = G * (\xi - \lambda\,\Phi^3)$, is not singular and can be solved in a Hölder space $\mathscr{C}^\alpha(\mathbb{R}^d)$ with $\alpha > 0$ using the Banach fixed-point theorem. In this regime no regularization is needed. Let us see what happens for smaller values of σ.

Lemma 1.4. *The statement of Theorem 1.1 holds true for $\sigma \in (5d/12, d/2]$.*

Sketch of the proof. Recall that we want to solve the equation

$$\Phi_\kappa = G * (\xi_\kappa + \lambda\,\Phi_\kappa^3 + \lambda\, c_\kappa^{(1)}\,\Phi_\kappa). \tag{1.4}$$

and prove the existence of the limit $\lim_{\kappa\searrow 0}\Phi_\kappa$. We have $G * \xi_0 \in \mathscr{C}^\alpha(\mathbb{R}^d)$ and we expect that $\Phi_0 \in \mathscr{C}^\alpha(\mathbb{R}^d)$. But since $\alpha < 0$ the cube Φ_0^3 is classically ill-defined. We shall employ the so-called Da Prato-Debussche trick [DPD03]. Introducing the notation $\bullet_\kappa := \xi_\kappa$ and $x \longrightarrow y := G(x-y)$ we define $\mathsf{I}_\kappa = G * \bullet_\kappa := G * \xi_\kappa$ and make the following ansatz for the solution of our equation

$$\Phi_\kappa = \mathsf{I}_\kappa + \Psi_\kappa.$$

As we shall see, the remainder Ψ_0 has much better regularity than $\mathsf{I}_0, \Phi_0 \in \mathscr{C}^\alpha(\mathbb{R}^d)$. Using Eq. (1.4) one shows that Ψ_κ satisfies the following equation

$$\Psi_\kappa = \lambda\, G * (\Psi_\kappa^3 + 3\Psi_\kappa^2\,\mathsf{I}_\kappa + 3\Psi_\kappa\,\mathsf{V}_\kappa + \mathsf{W}_\kappa), \tag{1.5}$$

where

$$\mathsf{V}_\kappa := (\mathsf{I}_\kappa)^2 + c_\kappa^{(1)}/3, \qquad \mathsf{W}_\kappa := (\mathsf{I}_\kappa)^3 + c_\kappa^{(1)}\,\mathsf{I}_\kappa, \qquad c_\kappa^{(1)} := -\mathbb{E}\,(\mathsf{I}_\kappa)^2/3$$

for all $\kappa \in (0,1]$.

The list of trees $(\mathsf{I}_\kappa, \mathsf{V}_\kappa, \mathsf{W}_\kappa)$ is called the enhanced noise. Of course the products appearing in the definitions of the above trees are ill-defined deterministically for $\kappa = 0$. However, the point is that the tress are simple and quite explicit objects. It turns out that it is possible to control the convergence of the enhanced noise as $\kappa \searrow 0$ by studying its covariance. One shows that with the above choice of the counterterm the limit

$$\lim_{\kappa\searrow 0}(\mathsf{I}_\kappa, \mathsf{V}_\kappa, \mathsf{W}_\kappa) =: (\mathsf{I}_0, \mathsf{V}_0, \mathsf{W}_0) \in \mathscr{C}^\alpha(\mathbb{R}^d) \times \mathscr{C}^{2\alpha}(\mathbb{R}^d) \times \mathscr{C}^{3\alpha}(\mathbb{R}^d) \tag{1.6}$$

exists almost surely and there exists a random variable $R \in [1,\infty]$ such that $\mathbb{E}R^n < \infty$ for all $n \in \mathbb{N}_0$ and

$$\|\tone_\kappa\|_{\mathscr{C}^{\alpha}(\mathbb{R}^d)} \vee \|\ttwo_\kappa\|_{\mathscr{C}^{2\alpha}(\mathbb{R}^d)} \vee \|\tthree_\kappa\|_{\mathscr{C}^{3\alpha}(\mathbb{R}^d)} \leq R$$

for all $\kappa \in [0,1]$. We call the above bounds the stochastic estimates for the enhanced noise.

To proceed we rewrite Eq. (1.5) as the fixed point equation of the map

$$\mathbf{Q}[\psi] \equiv \mathbf{Q}\big[\psi; \tone, \ttwo, \tthree\big] := \lambda\, G * \big(\psi^3 + 3\psi^2 \tone + 3\psi \ttwo + \tthree\big).$$

We claim that the map

$$\mathbf{Q}:\ \mathscr{C}^{3\alpha+\sigma}(\mathbb{R}^d) \to \mathscr{C}^{3\alpha+\sigma}(\mathbb{R}^d)$$

is well defined for all $(\tone, \ttwo, \tthree) \in \mathscr{C}^{\alpha}(\mathbb{R}^d) \times \mathscr{C}^{2\alpha}(\mathbb{R}^d) \times \mathscr{C}^{3\alpha}(\mathbb{R}^d)$. The above choice of the domain of $\mathbf{Q}$ is dictated by the regularity of $G * \tthree \in \mathscr{C}^{3\alpha+\sigma}(\mathbb{R}^d)$. Note that $3\alpha + \sigma > 0$. Hence, ψ in the domain of $\mathbf{Q}$ is a Hölder continuous function and the products

$$\psi^2, \psi^3 \in \mathscr{C}^{3\alpha+\sigma}(\mathbb{R}^d) \subset \mathscr{C}^{3\alpha}(\mathbb{R}^d)$$

are well defined. In order to make sure that the products

$$\psi^2 \tone \in \mathscr{C}^{\alpha}(\mathbb{R}^d) \subset \mathscr{C}^{3\alpha}(\mathbb{R}^d) \qquad \psi \ttwo \in \mathscr{C}^{2\alpha}(\mathbb{R}^d) \subset \mathscr{C}^{3\alpha}(\mathbb{R}^d)$$

are well defined we have to check whether $(3\alpha+\sigma) + l\alpha > 0$ for $l \in \{1,2\}$. Since $\alpha < 0$ it is enough to check this for $l = 2$. We obtain the condition $5\alpha + \sigma > 0$, which is equivalent to $\sigma > 5d/12$. Consequently, all of the products are well defined and belong to $\mathscr{C}^{3\alpha}(\mathbb{R}^d)$. As a result, the map $\mathbf{Q}:\ \mathscr{C}^{3\alpha+\sigma} \to \mathscr{C}^{3\alpha+\sigma}(\mathbb{R}^d)$ is well defined. $\square$

Exercise 1.1. *Complete the deterministic part of the above proof. Assume that the maps in Items (1), (2), (3) above have norms bounded by some constant $c \in [1,\infty)$. For $R \in [1,\infty)$ let $\mathscr{B}_R$ be the closed ball in $\mathscr{C}^{3\alpha+\sigma}(\mathbb{R}^d)$ of radius R and let $\lambda_\star := 1/(100c^3R^2)$. For all $\kappa \in [0,1]$ let $\mathbf{Q}_\kappa[\psi] := \mathbf{Q}[\psi; \tone_\kappa, \ttwo_\kappa, \tthree_\kappa]$. Prove that for $\lambda \in [-\lambda_\star, \lambda_\star]$ and $\kappa \in [0,1]$ the map $\mathbf{Q}_\kappa : \mathscr{B}_R \to \mathscr{B}_R$ is well defined and is a contraction with the Lipschitz constant less than $1/2$. Conclude that for all $\kappa \in [0,1]$ the map $\mathbf{Q}_\kappa$ has a unique fixed point in $\mathscr{B}_R$ denoted by $\varPsi_\kappa$. Next, using (1.6) show that for all $\psi \in \mathscr{B}_R$ it holds $\lim_{\kappa\searrow 0} \mathbf{Q}_\kappa[\psi] = \mathbf{Q}_0[\psi]$. Conclude that $\lim_{\kappa\searrow 0} \varPsi_\kappa = \varPsi_0 \in \mathscr{C}^{3\alpha+\sigma}(\mathbb{R}^d)$ and $\lim_{\kappa\searrow 0} \varPhi_\kappa = \varPhi_0 \in \mathscr{C}^{\alpha}(\mathbb{R}^d)$.*

The method of the above proof does not work for $\sigma \leq 5d/12$ because the product of $\varPsi_0$ and $\ttwo_0$ is no longer well defined classically. The problem is that the Da Prato-Debussche remainder $\varPsi_0$ does not have enough regularity. To proceed we apply the Da Prato-Debussche trick once again to remove the most singular tree $\tthree_\kappa$ on the RHS of Eq. (1.5). To this end, we make the following ansatz

$$\varPhi_\kappa = \tone_\kappa + \varPsi_\kappa = \tone_\kappa + \lambda \tthreeG_\kappa + \tilde{\varPsi}_\kappa, \qquad \tthreeG_\kappa = G * \tthree_\kappa.$$

Eq. (1.4) implies that the remainder $\tilde{\Psi}_\kappa$ satisfies the equation

$$\tilde{\Psi}_\kappa = \lambda\, G * \Big((\tilde{\Psi}_\kappa + \lambda \mathsf{Y}_\kappa)^3 + 3(\tilde{\Psi}_\kappa + \lambda \mathsf{Y}_\kappa)^2\, \mathsf{I}_\kappa + 3\tilde{\Psi}_\kappa \mathsf{V}_\kappa + 3\lambda\, \mathsf{W}_\kappa \Big), \tag{1.7}$$

where the new tree is defined by

$$\mathsf{W}_\kappa := \mathsf{Y}_\kappa\, \mathsf{V}_\kappa.$$

Suppose that $\sigma \in (2d/5, 5d/12]$. Even though the product in the definition of the above tree is ill-defined deterministically no renormalization is necessary. One shows that

$$\lim_{\kappa \searrow 0} \mathsf{W}_\kappa =: \mathsf{W}_0 \in \mathscr{C}^{2\alpha}(\mathbb{R}^d)$$

exists almost surely and there exists a random variable $R \in [1,\infty]$ such that $\mathbb{E}R^n < \infty$ for all $n \in \mathbb{N}_0$ and

$$\|\mathsf{W}_\kappa\|_{\mathscr{C}^{2\alpha}(\mathbb{R}^d)} \leq R$$

for all $\kappa \in [0,1]$.

Exercise 1.2 (♠). *Reformulate Eq.* (1.7) *as a fixed-point problem in a ball in* $\mathscr{C}^{2\alpha+\sigma}(\mathbb{R}^d)$ *and prove Theorem 1.1 for all* $\sigma \in (2d/5, 5d/12]$.

It turns out at $\sigma = 2d/5$ the Da Prato-Debussche method breaks down completely. Since $\mathsf{V}_0 \in \mathscr{C}^{2\alpha}(\mathbb{R}^d)$ with $\alpha = \sigma - d/2 - \varepsilon$ the term $\tilde{\Psi}_0 \mathsf{V}_0$, which cannot be removed using Da Prato-Debussche trick, can at best be defined as an element of $\mathscr{C}^{2\alpha}(\mathbb{R}^d)$. But then one can only hope that $G*(\tilde{\Psi}_0 \mathsf{V}_0) \in \mathscr{C}^{2\alpha+\sigma}(\mathbb{R}^d)$. Thus, we should expect that $\tilde{\Psi}_0 \in \mathscr{C}^{2\alpha+\sigma}(\mathbb{R}^d)$. However, for $\tilde{\Psi}_0 \in \mathscr{C}^{2\alpha+\sigma}(\mathbb{R}^d)$ the product $\tilde{\Psi}_0 \mathsf{V}_0$ is well defined only if $(2\alpha+\sigma) + 2\alpha \geq 0$, which implies that $\sigma > 2d/5$.

For $\sigma = 2d/5$ in order to control the product $\tilde{\Psi}_0 \mathsf{V}_0$ and close the estimates more information about $\tilde{\Psi}_0$ is needed that cannot be quantified in terms of just the regularity. It turns out that writing an equation that is equivalent to the original equation (1.2) and makes sense in the limit $\kappa \searrow 0$ is quite challenging. We refer the interested reader to [Hai15, Eq. (6.3), Prop. 7.5] for the solution of the problem in the framework of regularity structures and to [MW17, Sec. 1.1, 1.2] or [JP23, Thm. 2.1] for the solution involving the decomposition of the products into the resonant term and the paraproducts. For smaller values of σ the problem becomes even more complicated. Let us mention that using the framework of regularity structures or the approach discussed in these notes one can study the full subcritical regime corresponding to $\sigma > d/3$. For $\sigma \leq d/3$ the limit $\lim_{\kappa\searrow 0} \mathsf{V}_\kappa \in \mathscr{S}'(\mathbb{R}^d)$ does not exist and consequently controlling the cubic term in the equation seems impossible.

In Sec. 2 we discuss a reformulation of Eq. (1.2) meaningful in the limit $\kappa \searrow 0$ in the flow equation framework. Let us only mention that the system of equations (2.8) that we will study does not involve the product $\Phi_\kappa \mathsf{V}_\kappa$ at all. Instead, there is a term of the form $K_\mu * (\Phi_{\kappa,\mu} \mathsf{V}_{\kappa,\mu})$, where K_μ is some regularizing kernel of characteristic length scale

$[\mu] \in (0,1]$, the tree $\mathsf{V}_{\kappa,\mu}$ is such that $\|K_\mu * \mathsf{V}_{0,\mu}\|_{L^\infty(\mathbb{R}^d)} \lesssim [\mu]^{2\alpha-\varepsilon}$ and $\Phi_{\kappa,\mu}$ is the coarse-grained process capturing the behavior of Φ_0 at spatial scales larger than $[\mu]$. Because $\Phi_{\kappa,\mu}$ is smooth at scales smaller than $[\mu]$ the product $K_\mu * (\Phi_{0,\mu} \mathsf{V}_{0,\mu})$ is always well defined classically and the only non-trivial task is to control its norm uniformly in $\mu \in (0,1]$. Since we expect that $\Phi_0 \in \mathscr{C}^\alpha(\mathbb{R}^d)$ it should hold $\|\Phi_{0,\mu}\|_{L^\infty(\mathbb{R}^d)} \lesssim [\mu]^\alpha$. Then it is possible to show that

$$\|K_\mu * (\Phi_{0,\mu} \mathsf{V}_{0,\mu})\|_{L^\infty(\mathbb{R}^d)} \lesssim [\mu]^{3\alpha-\varepsilon},$$

which will turn out to be sufficient to close the estimates in full sub-critical regime. The fact that the tree $\mathsf{V}_{0,\mu}$ depends on the scale μ is not important. Note that the above bound would be also true for the standard tree V_0. What is crucial is the fact that the system of equations (2.8) involves the coarse-grained process $\Phi_{\kappa,\mu}$ instead of Φ_κ.

1.2 Literature

A general solution theory for singular SPDEs beyond the Da Prato-Debussche regime was developed for the first time in the work [Hai14], where the framework of regularity structures was introduced. An alternative approach using paracontrolled distributions was given later in [GIP15]. The method based on the flow equation, which was developed in [Duc22, Duc21] and which is discussed in these notes, was inspired by the renormalization group [Wil71] approach to singular SPDEs proposed in [Kup16, KM17]. These notes are primarily based on [Duc22, Duc21]. We also use some ideas from [DGR23]. The main advantage of the flow equation framework is that it is applicable in full subcritical regime. Thus, it provides an alternative to the regularity structures framework developed in [CH16, BHZ19, BCCH21, HS23] or [OSSW21, LOTT21].

1.3 Plan of the lectures

(1) Introduction to singular SPDEs.

(2) Reformulation of original SPDE in terms of effective equation involving effective force, statement of conditions guaranteeing well-posedness of effective equation.

(3) Construction of effective force in terms of effective force coefficients using flow equation, definition of enhanced noise as a finite collection of effective force coefficients, statement of stochastic estimates for enhanced noise.

(4) Definition of joint cumulants of effective force coefficients, flow equation for cumulants.

(5) Uniform bounds for cumulants and conclusion of proof of stochastic estimates with the use of a Kolmogorov-type argument.

The material marked with (♠) is of technical nature and is not essential for understanding the main idea. In particular, problems marked with (♠) are meant to fill gaps in the proofs rather than provide new insights.

2 Effective equation

Definition 2.1. We use the notation $\mathbb{M} := \mathbb{R}^d$, $\mathbb{T} := (\mathbb{R}/2\pi\mathbb{Z})^d$ and $\|\bullet\| := \|\bullet\|_{L^\infty(\mathbb{M})}$.

We would like to construct a solution Φ_κ of the equation

$$\Phi_\kappa = G * F_\kappa[\Phi_\kappa], \qquad F_\kappa[\varphi] := \xi_\kappa + \lambda\,\varphi^3 + \sum_{i=1}^{i_\sharp} \lambda^i\, c_\kappa^{(i)} \varphi, \tag{2.1}$$

and prove that for an appropriate choice of the counterterms $(c_\kappa^{(i)})_{i\in\{1,\ldots,i_\sharp\}}$ the limit

$$\lim_{\kappa\searrow 0} \Phi_\kappa =: \Phi_0 \in \mathscr{S}'(\mathbb{M})$$

exists. Recall that $G \in L^1(\mathbb{M})$ denotes the fundamental solution of the differential operator $(1-\Delta)^{\sigma/2}$. The functional F_κ is called the force. In the regime $\sigma \in (d/3, d/2]$, we are interested in, we expect that $\Phi_0 \in \mathscr{S}'(\mathbb{M})$ is not a function. Since multiplication of distributions is in general an ill-defined operation Eq. (2.1) becomes meaningless in the limit $\kappa \searrow 0$. In this section we will formulate a certain system of equations such that under some assumptions: (1) for every $\kappa \in (0,1]$ there is a correspondence between a solution of this system of equations and a solution of Eq. (2.1) and (2) the system of equations is well-posed in the limit $\kappa \searrow 0$.

To this end, let us first introduce the notion of a scale decomposition of the Green function G and an effective force. A scale decomposition of G is a family of integrable kernels $G_\mu \in L^1(\mathbb{M})$ parameterized by $\mu \in [0,1]$ such that $G_0 = G$, $G_1 = 0$ and the map $[0,1] \ni \mu \mapsto G_\mu \in L^1(\mathbb{M})$ is continuous and piecewise continuously differentiable with the derivative denoted by $\dot G_\mu$. Given the force functional F_κ an effective force is a family of functionals $F_{\kappa,\mu} : C(\mathbb{T}) \to C(\mathbb{T})$ parameterized by $\mu \in [0,1]$ such that $F_{\kappa,0} = F_\kappa$ and $[0,1] \ni \mu \mapsto F_{\kappa,\mu}[\varphi] \in C(\mathbb{T})$ is continuous and piecewise continuously differentiable for all $\varphi \in C(\mathbb{T})$. Moreover, we assume that $F_{\kappa,\mu}$ is of polynomial type.

Definition 2.2. A functional of polynomial type is a map $V : C(\mathbb{T}) \to C(\mathbb{T})$ such that the directional derivatives of V at $\varphi \in C(\mathbb{T})$ of order $k \in \mathbb{N}_+$ along $\psi \in C(\mathbb{T})$, i.e.

$$\mathrm{D}^k V[\varphi] \cdot \psi^{\otimes k} := \partial_\tau^k V[\varphi + \tau\psi]\big|_{\tau=0},$$

exist for all $k \in \mathbb{N}_+$ and are non-zero for only finitely many $k \in \mathbb{N}_+$.

In what follows, we will make a specific choice of a scale decomposition of the Green function and an effective force that is suitable for the problem at hand. For the time being, let us continue the informal discussion at a general level. Suppose that Φ_κ is a solution of the original equation $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$. For $\mu \in [0,1]$ we define

$$\Phi_{\kappa,\mu} = G_\mu * F_\kappa[\Phi_\kappa].$$

We call $(\Phi_{\kappa,\mu})_{\mu\in(0,1]}$ the coarse-grained process. By our assumptions about $G_\bullet$ we have $\Phi_{\kappa,0} = \Phi_\kappa$ and $\Phi_{\kappa,1} = 0$. For $\mu \in [0,1]$ the so-called remainder $\zeta_{\kappa,\mu}$ is defined by the equation

$$F_\kappa[\Phi_\kappa] = F_{\kappa,\mu}[\Phi_{\kappa,\mu}] + \zeta_{\kappa,\mu}. \tag{2.2}$$

Because $F_{\kappa,0} = F_\kappa$ and $\Phi_{\kappa,0} = \Phi_\kappa$ we obtain $\zeta_{\kappa,0} = 0$. Since the LHS of Eq. (2.2) does not depend on μ it holds

$$\partial_\mu \Phi_{\kappa,\mu} = \dot{G}_\mu * (F_{\kappa,\mu}[\Phi_{\kappa,\mu}] + \zeta_{\kappa,\mu}) \tag{2.3}$$

and

$$\partial_\mu \zeta_{\kappa,\mu} = -(\partial_\mu F_{\kappa,\mu})[\Phi_{\kappa,\mu}] - \mathrm{D}F_{\kappa,\mu}[\Phi_{\kappa,\mu}] \cdot \partial_\mu \Phi_{\kappa,\mu}. \tag{2.4}$$

Plugging Eq. (2.3) into Eq. (2.4) we obtain

$$\partial_\mu \zeta_{\kappa,\mu} = -(\partial_\mu F_{\kappa,\mu})[\Phi_{\kappa,\mu}] - \mathrm{D}F_{\kappa,\mu}[\Phi_{\kappa,\mu}] \cdot (\dot{G}_\mu * (F_{\kappa,\mu}[\Phi_{\kappa,\mu}] + \zeta_{\kappa,\mu})). \tag{2.5}$$

Using the notation

$$H_{\kappa,\mu}[\varphi] := \partial_\mu F_{\kappa,\mu}[\varphi] + \mathrm{D}F_{\kappa,\mu}[\varphi] \cdot (\dot{G}_\mu * F_{\kappa,\mu}[\varphi]) \tag{2.6}$$

we rewrite Eq. (2.5) in the following way

$$\partial_\mu \zeta_{\kappa,\mu} = -H_{\kappa,\mu}[\Phi_{\kappa,\mu}] - \mathrm{D}F_{\kappa,\mu}[\Phi_{\kappa,\mu}] \cdot (\dot{G}_\mu * \zeta_{\kappa,\mu}). \tag{2.7}$$

Summing up, Eq. (2.3) and Eq. (2.7) together with the boundary conditions $\Phi_{\kappa,1} = 0$ and $\zeta_{\kappa,0} = 0$ imply the following system of equations

$$\begin{cases} \Phi_{\kappa,\mu} = -\int_\mu^1 \dot{G}_\eta * (F_{\kappa,\eta}[\Phi_{\kappa,\eta}] + \zeta_{\kappa,\eta})\,\mathrm{d}\eta \\ \zeta_{\kappa,\mu} = -\int_0^\mu (H_{\kappa,\eta}[\Phi_{\kappa,\eta}] + \mathrm{D}F_{\kappa,\eta}[\Phi_{\kappa,\eta}] \cdot (\dot{G}_\eta * \zeta_{\kappa,\eta}))\,\mathrm{d}\eta\,. \end{cases} \tag{2.8}$$

The above system of equations for $(\Phi_{\kappa,\bullet}, \zeta_{\kappa,\bullet})$ is called the *effective equation.* At a formal level, there is a one to one correspondence between solutions of the original equation $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$ and the above system of equations. Given a solution Φ_κ of the equation $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$ a solution $(\Phi_{\kappa,\bullet}, \zeta_{\kappa,\bullet})$ of the above system of equations with $H_{\kappa,\bullet}$ given by Eq. (2.6) is constructed as outlined above. On the other hand, given a solution $(\Phi_{\kappa,\bullet}, \zeta_{\kappa,\bullet})$ of the above system of equations with $H_{\kappa,\bullet}$ given by Eq. (2.6) one shows that Eq. (2.2) is satisfied and concludes that $\Phi_\kappa = \Phi_{\kappa,0}$ solves the equation $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$. The advantage of the system of equations (2.8) is that, as we will see later, for an appropriate choice of an effective force $F_{\kappa,\bullet}$ it remains meaningful in the limit $\kappa \searrow 0$.

Remark 2.3. In most practical situations the framework discussed above is useful only if the scale decomposition of G is chosen in such a way that for all $\mu \in (0,1]$ the kernel G_μ is smooth (or has sufficiently many derivatives) and G_μ is essentially constant at spatial scales smaller than $[\mu] = \mu^{1/\sigma}$. For such choices of the scale decomposition of G we expect that the coarse grained process $\Phi_{\kappa,\mu}$ captures the behavior of Φ_κ, the solution of $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$, at spatial scales larger than $[\mu] = \mu^{1/\sigma}$ and is essentially constant at smaller scales. Informally, you can imagine that $\Phi_{\kappa,\mu}$ is obtained by averaging Φ_κ over blocks of size $[\mu]$. In particular, $\Phi_{\kappa,\mu}$ is smooth for every $\mu > 0$. Moreover, at least morally, we expect that

$$\|\Phi_\kappa\|_{\mathscr{C}^\alpha(\mathbb{M})} := \sup_{\mu\in(0,1]} [\mu]^{-\alpha} \|K_\mu * \Phi_\kappa\| \simeq \sup_{\mu\in(0,1]} [\mu]^{-\alpha} \|\Phi_{\kappa,\mu}\| \tag{2.9}$$

for all $\alpha < \sigma - d/2$ and we hope to bound the above Besov norm uniformly in $\kappa \in (0,1]$.

Remark 2.4. A possible choice for an effective force $F_{\kappa,\bullet}$, made in [Duc21, Duc22] as well as in [Kup16, KM17], is to define it in such a way that $H_{\kappa,\bullet}$ given by Eq. (2.6) vanishes identically. An effective force satisfies then the so-called flow equation

$$\partial_\mu F_{\kappa,\mu}[\varphi] + \mathrm{D}F_{\kappa,\mu}[\varphi] \cdot (\dot{G}_\mu * F_{\kappa,\mu}[\varphi]) = 0.$$

In a situations in which a small parameter is available solving the above equation is usually unproblematic. However, the solution is not a functional of polynomial type. Since the equation for the remainder $\zeta_{\kappa,\bullet}$ in the system (2.8) is linear and $H_{\kappa,\bullet} = 0$ the unique solution is given by $\zeta_{\kappa,\bullet} = 0$. Consequently, $\Phi_{\kappa,\bullet}$ satisfies the following effective equation

$$\Phi_{\kappa,\mu} = -\int_\mu^1 \dot{G}_\eta * F_{\kappa,\eta}[\Phi_{\kappa,\eta}]\,\mathrm{d}\eta.$$

The advantage of the formulation involving $(\Phi_{\kappa,\bullet}, \zeta_{\kappa,\bullet})$, proposed in [DGR23], is that there is more flexibility in the choice of an effective force $F_{\kappa,\bullet}$ as it has to satisfy the flow equation only up to some error term $H_{\kappa,\bullet}$. In particular, a suitable effective force can usually be constructed without exploiting the presence of a small parameter, even though a small parameter is typically needed anyway to solve (2.8).

In order to prove well-posedness of the system of equations (2.8) an effective force $F_{\kappa,\bullet}$ has to satisfy some additional conditions, which we shall formulate below. To this end, let us first introduce some regularizing kernels and fix a convenient scale decomposition of the Green function.

Definition 2.5. Let $\mathcal{K} \subset \mathscr{S}'(\mathbb{M})$ be the space of signed measures on $\mathbb{M}$ with finite total variation. We set $\|K\|_{\mathcal{K}} = \int_{\mathbb{M}} |K(\mathrm{d}x)|$. For $x \in \mathbb{M}$ we denote by $\delta_x \in \mathcal{K}$ the Dirac delta at x.

Remark 2.6. It holds $\|\delta_x\|_{\mathcal{K}} = 1$ and $\|K\|_{\mathcal{K}} = \|K\|_{L^1(\mathbb{M})}$ for all $K \in L^1(\mathbb{M}) \subset \mathcal{K}$.

Definition 2.7. Let $\mu \in [0,1]$ and $[\mu] = \mu^{1/\sigma}$. The kernel $\tilde{K}_\mu \in \mathcal{K}$ is the unique solution of $\tilde{\mathbf{P}}_\mu \tilde{K}_\mu = \delta_0$, where $\tilde{\mathbf{P}}_\mu := (1 - [\mu]^2\Delta)^{d+2}$. We define $K_\mu := \tilde{K}_\mu * \tilde{K}_\mu * \tilde{K}_\mu \in \mathcal{K}$ and $\mathbf{P}_\mu := \tilde{\mathbf{P}}_\mu^3$.

The kernels $K_\mu, \tilde{K}_\mu$ are of exponential decay with the characteristic length scale $[\mu]$. The kernels $K_\mu, \tilde{K}_\mu$ are not smooth. However, for $\mu \in (0,1]$ the convolution of a distribution with K_μ or $\tilde{K}_\mu$ has some smoothing effect. In what follows, we often use the fact $\mathbf{P}_\mu K_\mu = \delta_0$ to introduce a regularizing kernel where it is needed, e.g. $\psi * \phi = (\mathbf{P}_\mu \psi) * (K_\mu * \phi)$. We will frequently use the properties of the regularizing kernels stated in the exercise below.

Exercise 2.1. *Prove the following statements:*

(1) $\tilde{K}_0 = \delta_0$ *and* $\tilde{K}_\mu \in L^1(\mathbb{M}) \cap C^d_{\mathrm{b}}(\mathbb{M})$ *for* $\mu \in (0,1]$.

(2) For all $\mu \in [0,1]$ *the kernel* $\tilde{K}_\mu$ *is a positive measure and* $\|\tilde{K}_\mu\|_{\mathcal{K}} = 1$.

(3) For all $0 < \eta \le \mu \le 1$ *there exist* $\tilde{K}_{\mu,\eta} \in \mathcal{K}$ *such that* $\|\tilde{K}_{\mu,\eta}\|_{\mathcal{K}} = 1$ *and* $\tilde{K}_\mu = \tilde{K}_{\mu,\eta} * \tilde{K}_\eta$.

Conclude that the kernels K_μ, $\mu \in [0,1]$, *also have the above properties. Hint for Item (3): Let* $\hat{K}_\mu \in \mathcal{K}$ *be the solution of* $\hat{\mathbf{P}}_\mu \hat{K}_\mu = \delta_0$, *where* $\hat{\mathbf{P}}_\mu := (1-[\mu]^2\Delta)$. *It holds* $\hat{K}_\mu = \hat{K}_{\mu,\eta} * \hat{K}_\eta$ *for* $\hat{K}_{\mu,\eta} = [\eta/\mu]^2\,\delta_0 + (1-[\eta/\mu]^2)\,\hat{K}_\mu \in \mathcal{K}$.

Remark 2.8. Recall that $G \in L^1(\mathbb{M})$ is the fundamental solution for the pseudo-differential operator $(1-\Delta)^{\sigma/2}$. Note that G is smooth outside the origin. For every multi-index $a \in \mathbb{N}_0^d$ it holds $|\partial^a G(x)| \lesssim |x|^{\sigma-d-|a|}$ uniformly for $x \in \mathbb{R}^d \setminus \{0\}$. Furthermore, $\partial^a G$ is of fast decay at infinity for every $a \in \mathbb{N}_0^d$.

Definition 2.9 (Scale decomposition of G)**.** Fix $\chi \in C^\infty(\mathbb{R})$ such that $\chi(r) = 0$ for $|r| \le 1/4$ and $\chi(r) = 1$ for $|r| \ge 1/2$ and let $\chi_\mu(r) := \chi(r(1-\mu)/\mu)$ for $\mu \in (0,1]$. For $\mu \in (0,1]$ the smooth kernels $G_\mu, \dot{G}_\mu \in C^\infty_{\mathrm{c}}(\mathbb{M})$ are defined by

$$G_\mu(x) := \chi_\mu(|x|^\sigma)\,G(x), \qquad \dot{G}_\mu := \partial_\mu G_\mu.$$

Remark 2.10. Note that we chose to work with the scale decomposition of G in position space. An alternative would be to introduce the decomposition of G in Fourier space. The advantage of the position space decomposition is that the kernel $\dot{G}_\mu$ is supported in a shell $\{x \in \mathbb{M} \,|\, \mu/4 < (1-\mu)|x|^\sigma < \mu/2\}$. More specifically, it will play an important role that $\operatorname{supp}\dot{G}_\mu \subset \{x \in \mathbb{M} \,|\, |x| \le [\mu]\}$ for $\mu \in (0,1/2]$. The above support property will allow us to completely avoid using weights in the subsequent analysis. The price to pay is that the last equality in Eq. (2.9) is not obvious and a little extra argument, which is part of Exercise B.2, is needed to control the convergence in a Besov norm. The latter fact would be obvious in the case of the Fourier space decomposition of G but then one would have to work with weighted spaces, which makes some estimates look more complicated.

Lemma 2.11. *For all* $l \in \mathbb{N}_0$ *it holds* $\|\tilde{\mathbf{P}}^l_\mu \dot{G}_\mu\|_{\mathcal{K}} \lesssim 1$ *uniformly in* $\mu \in (0,1]$.

Proof (♠). First note that $\partial_\mu \chi_\mu(|x|^\sigma)$ vanishes unless $\mu/4 < (1-\mu)|x|^\sigma \le \mu/2$. Moreover, for all $a \in \mathbb{N}_0^d$ we have

$$|\partial^a \partial_\mu \chi_\mu(|x|^\sigma)| \lesssim |x|^{\sigma-|a|}/\mu^2$$

uniformly in $\mu \in (0,1]$ and $x \in \mathbb{M}$. Using the properties of the kernel G mentioned in Remark 2.8 we obtain $\|\partial^a \dot{G}_\mu\|_{\mathcal{K}} \lesssim [\mu]^{-|a|}$ for all $a \in \mathbb{N}_0^d$. This implies the lemma since $\tilde{\mathbf{P}}_\mu = (1-[\mu]^2\Delta)^{d+2}$. □

In order to make sense of the system of equations (2.8) in the limit $\kappa \searrow 0$ we shall rewrite it in such a way that it involves only the regularized functionals

$$\tilde{F}_{\kappa,\mu}[\varphi] := K_\mu * F_{\kappa,\mu}[K_\mu * \varphi], \qquad \tilde{H}_{\kappa,\mu}[\varphi] := K_\mu * H_{\kappa,\mu}[K_\mu * \varphi]. \tag{2.10}$$

Note that the above functionals are obtained by convolving the original functionals at scale μ with the regularizing kernel K_μ at the same scale. At a heuristic level, the system of equations (2.8) is equivalent to

$$\begin{cases} \tilde{\Phi}_{\kappa,\mu} = -\int_\mu^1 K_{\eta,\mu} * \tilde{G}_\eta * (\tilde{F}_{\kappa,\eta}[\tilde{\Phi}_{\kappa,\eta}] + \tilde{\zeta}_{\kappa,\eta})\,\mathrm{d}\eta \\ \tilde{\zeta}_{\kappa,\mu} = -\int_0^\mu K_{\mu,\eta} * (\tilde{H}_{\kappa,\eta}[\tilde{\Phi}_{\kappa,\eta}] + \mathrm{D}\tilde{F}_{\kappa,\eta}[\tilde{\Phi}_{\kappa,\eta}] \cdot (\tilde{G}_\eta * \tilde{\zeta}_{\kappa,\eta}))\,\mathrm{d}\eta \end{cases} \tag{2.11}$$

where $\tilde{G}_\mu := \mathbf{P}_\mu^2 \dot{G}_\mu$ and the kernels $(K_{\mu,\eta})_{0\le\eta\le\mu\le1}$ were introduced in Exercise 2.1. Note that $(\tilde{\Phi}_{\kappa,\bullet}, \tilde{\zeta}_{\kappa,\bullet})$ is related to $(\Phi_{\kappa,\bullet}, \zeta_{\kappa,\bullet})$ appearing in the system of equations (2.8) by

$$\tilde{\Phi}_{\kappa,\mu} = \mathbf{P}_\mu \Phi_{\kappa,\mu}, \qquad \tilde{\zeta}_{\kappa,\mu} = K_\mu * \zeta_{\kappa,\mu}.$$

Remark 2.12 (♠). Formally, the equivalence of the systems of equations (2.8) and (2.11) is an immediate consequence of the properties of the regularizing kernel K_μ. More specifically, we use the following identities

$$\Phi_{\kappa,\mu} = K_\mu * \mathbf{P}_\mu \Phi_{\kappa,\mu} = K_\mu * \tilde{\Phi}_{\kappa,\mu}, \qquad \dot{G}_\mu = K_\mu * \mathbf{P}_\mu^2 \dot{G}_\mu * K_\mu = K_\mu * \tilde{G}_\mu * K_\mu$$

as well as $K_\mu = K_{\mu,\eta} * K_\eta$ for $\mu \ge \eta$ and $\mathbf{P}_\mu \dot{G}_\eta = K_{\eta,\mu} * \tilde{G}_\eta * K_\eta$ for $\eta \ge \mu$.

Remark 2.13. Note that $\sup_{\mu\in(0,1]} \|\tilde{G}_\mu\|_{\mathcal{K}} =: C_G < \infty$ by Lemma 2.11 and $\|K_{\eta,\mu}\| = 1$ for all $0 \le \eta \le \mu \le 1$ by Exercise 2.1.

Thanks to the presence of the regularizing kernels in the definition of $\tilde{F}_{\kappa,\mu}$ and $\tilde{H}_{\kappa,\mu}$ we will be able to control the limit of these functionals as $\kappa \searrow 0$. For the choice of an effective force, which will be specified in the next section, $F_{\kappa,\mu}$ is in some sense a small perturbation of the noise ξ_κ. Hence, $\tilde{F}_{\kappa,\mu}$ is in some sense a small perturbation $K_\mu * \xi_\kappa$. Since the bound $\|K_\mu * \xi_\kappa\| \lesssim \mu^{\alpha-\sigma}$ uniform in $\kappa, \mu \in (0,1]$ is satisfied almost surely for all $\alpha < \sigma - d/2 \le 0$ we expect a bound of the form $\|\tilde{F}_{\kappa,\mu}[\varphi]\| \lesssim [\mu]^{\alpha-\sigma}$ uniform in $\kappa, \mu \in (0,1]$ for arbitrary fixed $\varphi \in C(\mathbb{T})$ and all $\alpha < \sigma - d/2$. Then by the first of the equations (2.11) we can hope that

$$\|\tilde{\Phi}_{\kappa,\mu}\| \lesssim \int_\mu^1 [\eta]^{\alpha-\sigma}\,\mathrm{d}\eta \lesssim [\mu]^\alpha$$

uniformly in $\kappa, \mu \in (0,1]$, which, as we argued in Remark 2.3, is consistent with the fact that $\|\Phi_\kappa\|_{\mathscr{C}^\alpha(\mathbb{M})}$ should be uniformly bounded in $\kappa \in (0,1]$. In order to make sense of the second of the equations (2.11) we need a bound of the form $\|\tilde{H}_{\kappa,\mu}[\varphi]\| \lesssim [\mu]^{\beta-\sigma}$ uniform in $\kappa,\mu \in (0,1]$ for arbitrary fixed $\varphi \in C(\mathbb{T})$ and some $\beta > 0$. Then

$$\|\tilde{\zeta}_{\kappa,\mu}\| \lesssim \int_0^\mu [\eta]^{\beta-\sigma}\,\mathrm{d}\eta \lesssim [\mu]^\beta.$$

We stress that it is crucial that $\beta > 0$ for the above bound to be valid.

Definition 2.14. For $\alpha \in (-\infty,0)$, $\beta \in (0,\infty)$ and $R \in [1,\infty)$ we define $\mathscr{B}_R$ to be the set of continuous maps

$$(0,1] \ni \mu \mapsto (\tilde{\Phi}_\mu, \tilde{\zeta}_\mu) \in C(\mathbb{T}) \times C(\mathbb{T})$$

such that

$$\|(\tilde{\Phi}_\bullet, \tilde{\zeta}_\bullet)\|_{\mathscr{B}_R} := \sup_{\mu\in(0,1]} [\mu]^{-\alpha}\,\|\tilde{\Phi}_\mu\| + R \sup_{\mu\in(0,1]} [\mu]^{-\beta}\,\|\tilde{\zeta}_\mu\| \leq R^2.$$

Lemma 2.15. *Fix $\alpha \in (-\infty,0)$, $\beta \in (0,\infty)$ and $R \in [1,\infty)$ such that*

$$R\,(|\alpha| \wedge \beta) > 100\,\sigma\,(C_G \vee 1), \qquad\qquad C_G := \sup_{\mu\in(0,1]} \|\tilde{G}_\mu\|_{\mathcal{K}} < \infty.$$

Suppose that

$$(\tilde{F}_\mu)_{\mu\in(0,1]}, \qquad\qquad (\tilde{H}_\mu)_{\mu\in(0,1]}$$

are families of functionals of polynomial type depending continuously on $\mu \in (0,1]$ such that for some $m_\flat \in \mathbb{N}_0$ it holds

$$[\mu]^{\sigma-\alpha}\,\|\mathrm{D}^k\tilde{F}_\mu[\varphi]\cdot\psi^{\otimes k}\,\| \leq R\,(\lambda^{1/3}[\mu]^{-\alpha}\,\|\psi\|)^k\,(1/2+\lambda^{1/3}[\mu]^{-\alpha}\,\|\varphi\|)^{m_\flat}, \tag{2.12}$$

$$[\mu]^{\sigma-\beta}\,\|\mathrm{D}^k\tilde{H}_\mu[\varphi]\cdot\psi^{\otimes k}\,\| \leq \lambda^{1/3}\,R^2\,(\lambda^{1/3}[\mu]^{-\alpha}\,\|\psi\|)^k\,(1/2+\lambda^{1/3}[\mu]^{-\alpha}\,\|\varphi\|)^{m_\flat} \tag{2.13}$$

for all $k \in \{0,1,2\}$, $\mu \in (0,1]$, $\varphi,\psi \in C(\mathbb{T})$ and $\lambda \in [-1,1]$. Let $\lambda_\star := 1/(2R^2)^3$ and suppose that $\lambda \in [-\lambda_\star,\lambda_\star]$. Under the above assumptions the map $\mathbf{Q} : \mathscr{B}_R \to \mathscr{B}_R$,

$$\mathbf{Q}[\tilde{\Phi}_\bullet,\tilde{\zeta}_\bullet] := \begin{pmatrix} \mu \mapsto -\int_\mu^1 K_{\eta,\mu} * \tilde{G}_\eta * (\tilde{F}_\eta[\tilde{\Phi}_\eta] + \tilde{\zeta}_\eta)\,\mathrm{d}\eta \\ \mu \mapsto -\int_0^\mu K_{\mu,\eta} * (\tilde{H}_\eta[\tilde{\Phi}_\eta] + \mathrm{D}\tilde{F}_\eta[\tilde{\Phi}_\eta]\cdot(\tilde{G}_\eta * \tilde{\zeta}_\eta))\,\mathrm{d}\eta \end{pmatrix}, \tag{2.14}$$

is well defined and is a contraction with the Lipschitz constant less than 1/2.

Remark 2.16. Let us explain the relation between the fixed point equation for the map $\mathbf{Q}$ and the original equation, $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$. Of course, in general, there is no relation at all. However, assume that:

1. $\tilde{F}_\bullet \equiv \tilde{F}_{\kappa,\bullet}$ and $\tilde{H}_\bullet \equiv \tilde{H}_{\kappa,\bullet}$ are defined by Eq. (2.10) in terms of $F_{\kappa,\bullet}$ and $H_{\kappa,\bullet}$,
2. $F_{\kappa,\bullet}$ is an effective force, in particular $F_{\kappa,0} = F_\kappa$,
3. $H_{\kappa,\bullet}$ is defined by Eq. (2.6) in terms of $F_{\kappa,\bullet}$.

Let $\mathbf{Q}_\kappa$ be the map defined by Eq. (2.14) in terms of $\tilde F_\bullet \equiv \tilde F_{\kappa,\bullet}$ and $\tilde H_\bullet \equiv \tilde H_{\kappa,\bullet}$ and suppose that $(\tilde\Phi_{\kappa,\bullet}, \tilde\zeta_{\kappa,\bullet})$ is a fixed point of $\mathbf{Q}_\kappa$. Then one shows that the limit

$$\lim_{\mu\searrow 0} K_\mu * \tilde\Phi_{\kappa,\mu} =: \Phi_\kappa$$

exists in $C(\mathbb{T})$ and satisfies the equation $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$. Moreover, if $\tilde F_{\kappa,\bullet}$ and $\tilde H_{\kappa,\bullet}$ converge in appropriate sense as $\kappa \searrow 0$, then Φ_κ converges in the Besov space $\mathscr{C}^\alpha(\mathbb{M})$. For details see Appendix B.

Remark 2.17. The bounds (2.12) and (2.13) stated in the above lemma say that the functionals $\tilde F_\mu$ and $\tilde H_\mu$ are compatible with the growth of the norm $\|\tilde\Phi_\mu\|$ when μ tends to zero. The bounds also take into account the fact that there is one power of $\lambda^{1/3}$ for each factor of φ in the expression (2.1) for the force $F_\kappa[\varphi]$.

Sketch of the proof. First note that for $(\tilde\Phi_\bullet, \tilde\zeta_\bullet) \in \mathscr{B}_R$ it holds

$$[\mu]^{\sigma-\alpha}\,\|\tilde F_\mu[\tilde\Phi_\mu]\| \leq R, \qquad [\mu]^{\sigma-\beta}\,\|\mathrm{D}\tilde F_\mu[\tilde\Phi_\mu]\cdot(\tilde G_\mu * \tilde\zeta_\mu)\| \leq C_G, \qquad [\mu]^{\sigma-\beta}\,\|\tilde H_\mu[\tilde\Phi_\mu]\| \leq 1.$$

By Remark 2.13 we obtain

$$\begin{aligned}
\|\mathbf{Q}(\tilde\Phi_\bullet, \tilde\zeta_\bullet)\|_{\mathscr{B}_R} \leq C_G \sup_{\mu\in(0,1]} [\mu]^{-\alpha} \int_\mu^1 \|\tilde F_\eta[\tilde\Phi_\eta] + \tilde\zeta_\eta\|\,\mathrm{d}\eta \\
+ R \sup_{\mu\in(0,1]} [\mu]^{-\beta} \int_0^\mu \|\tilde H_\eta[\tilde\Phi_\eta] + \mathrm{D}\tilde F_\eta[\tilde\Phi_\eta]\cdot(\tilde G_\eta * \tilde\zeta_\eta)\|\,\mathrm{d}\eta \\
\leq \sigma/|\alpha|\; C_G\, R + \sigma/|\alpha|\; C_G\, R + \sigma/\beta\; R + \sigma/\beta\; C_G\, R \leq R^2.
\end{aligned}$$

By similar estimates one shows that $\mathbf{Q} : \mathscr{B}_R \to \mathscr{B}_R$ is a contraction with the Lipschitz constant less than 1/2. $\square$

Remark 2.18. Note that the map $\mathbf{Q}$ is only a contraction provided λ is sufficiently small. In the case of parabolic equation an analogous map $\mathbf{Q}$ is a contraction for all values of λ provided the time interval in which we solve the equation is sufficiently small.

3 Construction of effective force

In the previous section we argued that, under certain assumptions, the equation

$$\Phi_\kappa = G * F_\kappa[\Phi_\kappa], \qquad F_\kappa[\varphi] := \xi_\kappa + \lambda\,\varphi^3 + \sum_{i=1}^{i_\sharp} \lambda^i\, c_\kappa^{(i)} \varphi,$$

which want to solve, can be formulated as a fixed point problem for the map $\mathbf{Q}$ defined by Eq. (2.14). Recall that the map $\mathbf{Q}$ involves a scale decomposition $(G_\mu)_{\mu\in[0,1]}$ of the

Green function G and two families of functionals $(\tilde{F}_\mu)_{\mu\in(0,1]}$ and $(\tilde{H}_\mu)_{\mu\in(0,1]}$. The scale decomposition of the Green function was fixed in Def. 2.9. By Remark 2.16 a fixed point of the map $\mathbf{Q}$ corresponds to a solution of the original equation if

$$(\tilde{F}_\mu)_{\mu\in(0,1]} \equiv (\tilde{F}_{\kappa,\mu})_{\mu\in(0,1]}, \qquad (\tilde{H}_\mu)_{\mu\in(0,1]} \equiv (\tilde{H}_{\kappa,\mu})_{\mu\in(0,1]}$$

are defined in terms of an effective force

$$(F_{\kappa,\mu})_{\mu\in[0,1]}$$

by the following equations

$$\tilde{F}_{\kappa,\mu}[\varphi] := K_\mu * F_{\kappa,\mu}[K_\mu * \varphi], \qquad \tilde{H}_{\kappa,\mu}[\varphi] := K_\mu * H_{\kappa,\mu}[K_\mu * \varphi] \tag{3.1}$$

and

$$H_{\kappa,\mu}[\varphi] := \partial_\mu F_{\kappa,\mu}[\varphi] + \mathrm{D}F_{\kappa,\mu}[\varphi] \cdot (\dot{G}_\mu * F_{\kappa,\mu}[\varphi]). \tag{3.2}$$

In view of Lemma 2.15, for all $\kappa \in (0,1]$ we would like to construct an effective force

$$(F_{\kappa,\mu})_{\mu\in[0,1]}$$

such that for some random $R \in [1,\infty]$ with finite moments of all orders it holds

$$[\mu]^{\sigma-\alpha} \,\|\mathrm{D}^k \tilde{F}_{\kappa,\mu}[\varphi] \cdot \psi^{\otimes k}\| \le R\,(\lambda^{1/3}[\mu]^{-\alpha}\,\|\psi\|)^k\,(1/2 + \lambda^{1/3}[\mu]^{-\alpha}\,\|\varphi\|)^{m_\flat}, \tag{3.3}$$

$$[\mu]^{\sigma-\beta} \,\|\mathrm{D}^k \tilde{H}_{\kappa,\mu}[\varphi] \cdot \psi^{\otimes k}\| \le \lambda^{1/3}\,R^2\,(\lambda^{1/3}[\mu]^{-\alpha}\,\|\psi\|)^k\,(1/2 + \lambda^{1/3}[\mu]^{-\alpha}\,\|\varphi\|)^{m_\flat} \tag{3.4}$$

for all $k \in \{0,1,2\}$, $\kappa,\mu \in (0,1]$, $\varphi,\psi \in C(\mathbb{T})$ and $\lambda \in [-1,1]$.

Remark 3.1. After establishing the above-mentioned result we will be able to conclude that for every $\kappa \in (0,1]$ there exists Φ_κ such that $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$ and almost surely $\|\Phi_\kappa\|_{\mathscr{C}^\alpha(\mathbb{M})} \lesssim 1$ uniformly in $\kappa \in (0,1]$. In order to prove almost sure convergence of Φ_κ as $\kappa \searrow 0$ in the Besov space $\mathscr{C}^\alpha(\mathbb{M})$ one has to show in addition that the functionals $\tilde{F}_{\kappa,\bullet}$ and $\tilde{H}_{\kappa,\bullet}$ converge as $\kappa \searrow 0$ in the sense specified in Exercise B.2.

As we will see, $F_{\kappa,\mu}$ is in some sense a small perturbation of the noise ξ_κ and we expect to have the bound (3.3) for all $\alpha < \sigma - d/2 \le 0$ provided the mass counterterm is chosen appropriately. The main problem is to ensure that the bound (3.4) holds for some $\beta > 0$. The idea is to define an effective force $F_{\kappa,\mu}$ so that $H_{\kappa,\mu} = O(\lambda^{i_\flat+1})$ for a sufficiently big $i_\flat \in \mathbb{N}_+$. The hope is that for such $F_{\kappa,\mu}$ the bound (3.4) may be true for some $\beta > 0$.

The starting point of the construction of an effective force is the ansatz

$$\langle F_{\kappa,\mu}[\varphi], \psi\rangle := \sum_{i=0}^{i_\flat} \sum_{m=0}^{3i} \lambda^i \,\langle F^{i,m}_{\kappa,\mu}, \psi \otimes \varphi^{\otimes m}\rangle \tag{3.5}$$

for all $\psi,\varphi \in \mathscr{S}(\mathbb{M})$. The distributions $F^{i,m}_{\kappa,\mu} \in \mathscr{S}'(\mathbb{M}^{1+m})$ that appear on the RHS of the above equality are called the effective force coefficients. By definition $F^{i,m}_{\kappa,\mu} \in \mathscr{S}'(\mathbb{M}^{1+m})$

are such that the expression $\langle F^{i,m}_{\kappa,\mu}, \psi \otimes \varphi_1 \otimes \ldots \otimes \varphi_m \rangle$ is invariant under permutations of the test functions $\varphi_1, \ldots, \varphi_m \in \mathscr{S}(\mathbb{M})$. The coefficients $F^{i,m}_\kappa$ of the force F_κ are defined by an equality analogous to Eq. (3.5). Note that by Eq. (3.2) we have

$$\langle H_{\kappa,\mu}[\varphi], \psi \otimes \varphi^{\otimes m} \rangle := \sum_{i=0}^{2i_\flat} \sum_{m=0}^{3i} \lambda^i \, \langle H^{i,m}_{\kappa,\mu}, \psi \otimes \varphi^{\otimes m} \rangle \tag{3.6}$$

for some $H^{i,m}_{\kappa,\mu} \in \mathscr{S}'(\mathbb{M}^{1+m})$. Recall that we want to construct the effective force such that $H_{\kappa,\mu} = O(\lambda^{i_\flat+1})$, which implies $H^{i,m}_{\kappa,\mu} = 0$ for all $i \in \{0, \ldots, i_\flat + 1\}$.

Remark 3.2. Let us list the non-vanishing force coefficients $F^{i,m}_\kappa$:

$$F^{0,0}_\kappa(x) = \xi_\kappa(x), \qquad F^{1,3}_\kappa(x; \mathrm{d}y_1, \mathrm{d}y_2, \mathrm{d}y_3) = \delta_x(\mathrm{d}y_1)\delta_x(\mathrm{d}y_2)\delta_x(\mathrm{d}y_3),$$
$$F^{i,1}_\kappa(x; \mathrm{d}y_1) = c^{(i)}_\kappa \delta_x(\mathrm{d}y_1), \quad i \in \{0, \ldots, i_\sharp\},$$

where $(c^{(i)}_\kappa)_{i \in \{1,\ldots,i_\sharp\}}$ are the so-called counterterms. It is easy to see that $F^{0,0}_\kappa \in \mathcal{V}^0$, $F^{1,3}_\kappa \in \mathcal{V}^3$, $F^{i,1}_\kappa \in \mathcal{V}^1$, where the spaces $\mathcal{V}^m$ are introduced in the definition below.

Definition 3.3. For $m \in \mathbb{N}_0$ the space $\mathcal{V}^m$ consists of maps $V : \mathbb{M} \times \mathrm{Borel}(\mathbb{M}^m) \to \mathbb{R}$ satisfying the following conditions:

(1) for every $A \in \mathrm{Borel}(\mathbb{M}^m)$ the map $x \mapsto V(x; A + x)$ is continuous and 2π periodic, where $A + x := \{(y_1 + x, \ldots, y_m + x) \in \mathbb{M}^m \,|\, (y_1, \ldots, y_m) \in A\}$ for $A \subset \mathbb{M}^m$,

(2) for every $x \in \mathbb{M}$ the map $A \mapsto V(x; A)$ is a measure with finite total variation,

(3) the following norm
$$\|V\|_{\mathcal{V}^m} := \sup_{x \in \mathbb{M}} \int_{\mathbb{M}^m} |V(x; \mathrm{d}y_1 \ldots \mathrm{d}y_m)|$$
is finite.

Remark 3.4. $(\mathcal{V}^m, \|\bullet\|_{\mathcal{V}^m})$ is a Banach space. For every $V \in \mathcal{V}^m$ and $A \in \mathrm{Borel}(\mathbb{M}^m)$ the map $x \mapsto V(x; A)$ is measurable. We identify $V \in \mathcal{V}^m$ with a distribution $V \in \mathscr{S}'(\mathbb{M}^{1+m})$, denoted by the same symbol, defined by the measure $V(x; \mathrm{d}y_1, \ldots, \mathrm{d}y_m)\, \mathrm{d}x$.

Since we demand that $H_{\kappa,\mu} = O(\lambda^{i_\flat+1})$, by Eq. (3.2) it holds

$$\partial_\mu F_{\kappa,\mu}[\varphi] + \mathrm{D}F_{\kappa,\mu}[\varphi] \cdot (\dot{G}_\mu * F_{\kappa,\mu}[\varphi]) = H_{\kappa,\mu}[\varphi] = O(\lambda^{i_\flat+1}).$$

As a result, the effective force coefficients $(F^{i,m}_{\kappa,\bullet})_{i \in \{0,\ldots,i_\flat\}, m \in \mathbb{N}_0}$ satisfy the following flow equation

$$\partial_\mu F^{i,m}_{\kappa,\mu} = -\sum_{j=0}^{i} \sum_{k=0}^{m} (1+k)\, \mathbf{B}(\dot{G}_\mu, F^{j,1+k}_{\kappa,\mu}, F^{i-j,m-k}_{\kappa,\mu}), \tag{3.7}$$

where the map $\mathbf{B}$ is introduced below.

Definition 3.5. Let $m\in\mathbb{N}_0$, $k\in\{0,\ldots,m\}$. The map $\mathbf{B}:\mathscr{S}(\mathbb{M})\times\mathcal{V}^{1+k}\times\mathcal{V}^{m-k}\to\mathcal{V}^m$ is defined by

$$
\begin{aligned}
&\mathbf{B}(G,W,U)(x;\mathrm{d}y_1,\ldots,\mathrm{d}y_m)\\
&\quad:=\frac{1}{m!}\sum_{\pi\in\mathcal{P}_m}\int_{\mathbb{M}^2}W(x;\mathrm{d}y,\mathrm{d}y_{\pi(1)},\ldots,\mathrm{d}y_{\pi(k)})\,G(y-z)\,U(z;\mathrm{d}y_{\pi(k+1)},\ldots,\mathrm{d}y_{\pi(m)})\,\mathrm{d}z.
\end{aligned}
$$

Exercise 3.1. *Prove that the map $\mathbf{B}:\mathscr{S}(\mathbb{M})\times\mathcal{V}^{1+k}\times\mathcal{V}^{m-k}\to\mathcal{V}^m$ is well defined and*

$$
\|\mathbf{B}(G,W,U)\|_{\mathcal{V}^m}\leq\|G\|_{\mathcal{K}}\,\|W\|_{\mathcal{V}^{1+k}}\|U\|_{\mathcal{V}^{m-k}}.
$$

The basic idea behind the flow equation approach is a recursive construction of the effective force coefficients $(F^{i,m}_{\kappa,\mu})_{i\in\{0,\ldots,i_\flat\},m\in\mathbb{N}_0}$ for all $\kappa\in(0,1]$, $\mu\in[0,1]$. For each $\kappa\in(0,1]$ we define the coefficients in such a way that the map

$$
[0,1]\ni\mu\mapsto F^{i,m}_{\kappa,\mu}\in\mathcal{V}^m
$$

is continuous and continuously differentiable for $\mu\in(0,1]$ using the following recursive algorithm:

(0) We set $F^{0,0}_{\kappa,\mu}=\xi_\kappa$ and $F^{i,m}_{\kappa,\mu}=0$ if $m>3i$,

(I) Assuming that all $F^{i,m}_{\kappa,\mu}$ with $i<i_\circ$, or $i=i_\circ$ and $m>m_\circ$ were constructed we define $\dot F^{i,m}_{\kappa,\mu}$ with $i=i_\circ$ and $m=m_\circ$ to be the RHS of Eq. (3.7).

(II) Subsequently, $F^{i,m}_{\kappa,\mu}$ is defined by $F^{i,m}_{\kappa,\mu}=F^{i,m}_{\kappa}+\int_0^\mu\dot F^{i,m}_{\kappa,\eta}\,\mathrm{d}\eta$.

Definition 3.6. The finite list of the effective force coefficients $(F^{i,m}_{\kappa,\bullet})_{i\in\{0,\ldots,i_\flat\},m\in\{0,\ldots,3i\}}$ is called the *enhanced noise*.

Remark 3.7. The above procedure cannot be used to construct directly $F^{i,m}_{\kappa,\mu}$ with $\kappa=0$. In fact, we expect that $F^{i,m}_{\kappa,\mu}\notin\mathcal{V}^m$. For example, $F^{0,0}_{0,\mu}=\xi\notin C(\mathbb{T})=\mathcal{V}^0$. Instead, the coefficients $F^{i,m}_{0,\mu}$ are defined probabilistically. The stochastic estimates for the enhanced noise are stated in Theorem 3.14 below.

Remark 3.8 (♠). Note that the effective force coefficients depend implicitly on the counterterms $(c^{(i)}_\kappa)_{i\in\{1,\ldots,i_\sharp\}}$. More specifically, the coefficients $F^{i,m}_{\kappa,\mu}$ with $i\in\{1,\ldots,i_\flat\}$ and $m\in\{2,3,\ldots\}$ depend on $(c^{(j)}_\kappa)_{j\in\{1,\ldots,i_\sharp\wedge(i-1)\}}$ and the coefficients $F^{i,m}_{\kappa,\mu}$ with $i\in\{1,\ldots,i_\flat\}$ and $m\in\{0,1\}$ depend on $(c^{(j)}_\kappa)_{j\in\{1,\ldots,i_\sharp\wedge i\}}$.

Exercise 3.2. *Check that the condition $F^{i,m}_{\kappa,\mu}=0$ if $m>3i$ is consistent with the conditions stated in Items (I) and (II). Prove that $F^{i,m}_{\kappa,\mu}=0$ for all $m>(2i+1)\wedge3i$.*

Remark 3.9. Let us list some examples of effective force coefficients. To this end, it will be convenient to use a diagrammatical notation. We view the diagrams as placeholders for certain multi-linear functionals of the noise. Since the diagrams do not play any role in the flow equation approach we refrain from defining precise rules that are used to draw them. Instead, for each diagram we provided an explicit expression it represents. Note that the edges of the diagrams that are introduced below represent the fluctuation propagator $(G-G_\mu)(x-y) =: x \longrightarrow y$ and not the Green function $G(x-y) =: x \longrightarrow y$. First note that

$$F^{0,0}_{\kappa,\mu}(x) = \xi_\kappa(x) =: \bullet_\kappa(x),$$
$$F^{1,3}_{\kappa,\mu}(x;\mathrm{d}y_1,\mathrm{d}y_2,\mathrm{d}y_3) = F^{1,3}_{\kappa}(x;\mathrm{d}y_1,\mathrm{d}y_2,\mathrm{d}y_3) =: \circ\,(x;\mathrm{d}y_1,\mathrm{d}y_2,\mathrm{d}y_3).$$

Let

$$_{\kappa,\mu}(x) := ((G-G_\mu)*\bullet_\kappa)(x)\,, \qquad _{\kappa,\mu}(x) := (\,_{\kappa,\mu}(x))^2 + c^{(1)}_\kappa/3.$$

We have

$$F^{1,2}_{\kappa,\mu}(x;\mathrm{d}y_1,\mathrm{d}y_2) = 3\,_{\kappa,\mu}(x)\,\delta_x(\mathrm{d}y_1)\delta_x(\mathrm{d}y_2) =: 3\,_{\kappa,\mu}(x;\mathrm{d}y_1,\mathrm{d}y_2),$$
$$F^{1,1}_{\kappa,\mu}(x;\mathrm{d}y_1) = 3\,_{\kappa,\mu}(x)\,\delta_x(\mathrm{d}y_1) =: 3\,_{\kappa,\mu}(x;\mathrm{d}y_1),$$
$$F^{1,0}_{\kappa,\mu}(x) = (\,_{\kappa,\mu}(x))^3 + c^{(1)}_\kappa\,_{\kappa,\mu}(x) =: \,_{\kappa,\mu}(x),$$

The coefficient

$$F^{2,5}_{\kappa,\mu}(x;\mathrm{d}y_1,\ldots,\mathrm{d}y_5) =: 3\,_{\mu}(x;\mathrm{d}y_1,\ldots,\mathrm{d}y_5)$$

coincides with the symmetric part of

$$3\,\delta_x(\mathrm{d}y_1)\delta_x(\mathrm{d}y_2)\,(G-G_\mu)(x-y_3)\,\delta_{y_3}(\mathrm{d}y_4)\delta_{y_3}(\mathrm{d}y_5)\,\mathrm{d}y_3.$$

The construction of coefficients $F^{2,4}_{\kappa,\mu}$, $F^{2,3}_{\kappa,\mu}$, $F^{2,2}_{\kappa,\mu}$ is left as an exercise. Let

$$_{\kappa,\mu}(x) := ((G-G_\mu)*\,_{\kappa,\mu})(x).$$

The remaining second order coefficients are given by

$$\begin{aligned} F^{2,1}_{\kappa,\mu}(x;\mathrm{d}y_1) &= 9\,_{\kappa,\mu}(x)\,(G-G_\mu)(x-y_1)\,_{\kappa,\mu}(y_1)\,\mathrm{d}y_1 + c^{(2)}_\kappa\,\delta_x(\mathrm{d}y_1) \\ &\quad + 6\,_{\kappa,\mu}(x)\,_{\kappa,\mu}(x)\,\delta_x(\mathrm{d}y_1) =: 9\,_{\kappa,\mu}(x;\mathrm{d}y_1) + 6\,_{\kappa,\mu}(x;\mathrm{d}y_1) \end{aligned}$$

and

$$F^{2,0}_{\kappa,\mu}(x) = 3\,_{\kappa,\mu}(x)\,_{\kappa,\mu}(x) + c^{(2)}_\kappa\,_{\kappa,\mu}(x) =: 3\,_{\kappa,\mu}(x).$$

Exercise 3.3. *Convince yourself that the expression given in Remark 3.9 satisfy the condition $F^{i,m}_{\kappa,0} = F^{i,m}_{\kappa}$, where $F^{i,m}_{\kappa}$ are the force coefficients listed in Remark 3.2.*

Exercise 3.4. *Using the notation $\dot G_\mu(x-y) := x \longrightarrow y$ draw the diagrams representing the coefficients $\partial_\mu F^{i,m}_{\kappa,\mu}$ for $i\in\{0,1,2\}$. Verify the formulas given in Remark 3.9 and write explicit expressions for the coefficients $F^{2,4}_{\kappa,\mu}$, $F^{2,3}_{\kappa,\mu}$, $F^{2,2}_{\kappa,\mu}$. For simplicity, you can ignore numerical prefactors.*

Definition 3.10. Let $\varepsilon \in [0,\infty)$ and $\alpha \equiv \alpha_\varepsilon := \sigma - d/2 - \varepsilon$, $\gamma \equiv \gamma_\varepsilon := 3\sigma - d - 3\varepsilon$. For $i,m \in \mathbb{N}_0$ we define

$$\varrho_\varepsilon(i,m) := \alpha_\varepsilon - \sigma - m\,\alpha_\varepsilon + i\,\gamma_\varepsilon \in \mathbb{R}.$$

We omit ε if $\varepsilon = 0$. Let $i_\flat, i_\sharp \in \mathbb{N}_+$ be the smallest positive integers such that $\varrho(i_\flat+1,0) > 0$, $\varrho(i_\sharp+1,1) > 0$, respectively.

Remark 3.11. Note that $\alpha_\varepsilon \leq 0$ for all $\varepsilon \in [0,\infty)$. Moreover, $\gamma_\varepsilon > 0$ for all $\varepsilon \in [0,\infty)$ in a sufficiently small neighbourhood of $\varepsilon = 0$ by the condition of subcriticality. In particular, $i_\flat, i_\sharp \in \mathbb{N}_+$ are well defined and there are only finitely many $i,m \in \mathbb{N}_0$ such that $m \leq 3i$ and $\varrho(i,m) \leq 0$. Recall that if $m > 3i$, then $F^{i,m}_{\kappa,\mu}$ vanishes identically. For arbitrary $\varepsilon \in (0,\infty)$ and $i,m \in \mathbb{N}_0$ such that $m \leq 3i$ it holds $\varrho_\varepsilon(i,m) < \varrho(i,m)$.

Remark 3.12 (♠). We claim that there exists $\varepsilon_\diamond \in (0,\sigma)$ such that for all $\varepsilon \in (0,\varepsilon_\diamond)$ and all $i,m,l \in \mathbb{N}_0$ it holds $\varrho_\varepsilon(i,m) + l > 0$ if $\varrho(i,m) + l > 0$. In what follows, we assume that $\varepsilon \in (0,\varepsilon_\diamond)$.

Definition 3.13. For $n \in \mathbb{N}_+$ let $\mathcal{K}^n \subset \mathscr{S}'(\mathbb{M}^n)$ be the space of signed measures on $\mathbb{M}^n$ with finite total variation. We set $\|K\|_{\mathcal{K}^n} = \int_{\mathbb{M}^n} |K(\mathrm{d}x_1 \ldots \mathrm{d}x_n)|$. Given $K \in \mathcal{K} = \mathcal{K}^1$ and $n \in \mathbb{N}_+$ we set $K^{\otimes n} := K \otimes \ldots \otimes K \in \mathcal{K}^n$.

In the theorem below we state the *stochastic estimates* for the enhanced noise. Recall that the enhanced noise coincides with the following finite list of the effective force coefficients $(F^{i,m}_{\kappa,\bullet})_{i\in\{0,\ldots,i_\flat\},m\in\{0,\ldots,3i\}}$. By the deterministic results established in Sec. 2 and Corollary 3.17 these estimates imply that for every $\kappa \in (0,1]$ there exists Φ_κ such that $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$ and $\mathbb{E}\big(\sup_{\kappa\in(0,1]} \|\Phi_\kappa\|^n_{\mathscr{C}^\alpha(\mathbb{M})}\big) < \infty$ for all $n \in \mathbb{N}_0$. For the proof of the convergence of Φ_κ as $\kappa \searrow 0$ see Appendix B.

Theorem 3.14. *There exist a choice of the counterterms* $(c^{(i)}_\kappa)_{i\in\{1,\ldots,i_\sharp\}}$ *in the expression* (1.3) *for the force* F_κ *and a random variable* $\tilde{R} \in [1,\infty]$ *such that* $\mathbb{E}\tilde{R}^n < \infty$ *for all* $n \in \mathbb{N}_+$ *and it holds*

$$\|K^{\otimes(1+m)}_\mu * F^{i,m}_{\kappa,\mu}\|_{\mathcal{V}^m} \leq \tilde{R}\,[\mu]^{\varrho_\varepsilon(i,m)}$$

for all $i \in \{0,\ldots,i_\flat\}$, $m \in \{0,\ldots,3i\}$, $\kappa,\mu \in (0,1]$.

Proof. The theorem follows from the bounds for the cumulants of the effective force coefficients established in Theorem 5.3 and Exercise 5.6 together with a Kolmogorov-type argument from Lemma A.1. □

Remark 3.15. Actually, if the bound stated in the above is known for the relevant effective force coefficients, i.e. $F^{i,m}_{\kappa,\mu}$ such that $\varrho(i,m) \leq 0$, it can be easily proved deterministically for the irrelevant coefficients, i.e. $F^{i,m}_{\kappa,\mu}$ such that $\varrho(i,m) > 0$. This is the subject of Exercise 3.5 below. Since the coefficients $F^{i,1}_{\kappa,\mu}$ with $i \in \{i_\sharp+1,\ldots,i_\flat\}$ are irrelevant no renormalization should be necessary to bound them. This is an intuitive reason why only the counterterms $(c^{(i)}_\kappa)_{i\in\{1,\ldots,i_\sharp\}}$ are included in the expression for the force (1.3).

Remark 3.16. The exponent $\varrho_\varepsilon(i,m)$ that appears in the bound in the above theorem follows from the application of the naive rules of power counting. Let us analyse an example. For concreteness, suppose that $d=5$ and $\sigma=2$. Then ${}_{\kappa=0} = G*\xi \in \mathscr{C}^\alpha(\mathbb{M})$ with $\alpha = \sigma - d/2 - \varepsilon = -1/2-\varepsilon$. The above theorem implies in particular that

$$\big\|K_\mu * {}_{\kappa,\mu}\big\| \lesssim [\mu]^{-1/2-2\varepsilon} \tag{3.8}$$

almost surely uniformly in $\kappa,\mu\in(0,1]$. Note that the tree ${}_{\kappa,\mu}$ depends on $\mu\in(0,1]$. Consequently, the above bound says nothing about the regularity of the above tree. Note that it would completely hopeless to prove an analogous bound for the tree

$${}_\kappa(x) := {}_\kappa(x)\,{}_\kappa(x) + 1/3\; c_\kappa^{(2)}\; {}_\kappa(x)$$

with the standard edges $x \longrightarrow y \;=\; G(x-y)$, which do not depend on $\mu\in(0,1]$. The regularity of the above tree cannot possibly be better than the regularity of ${}_\kappa \in \mathscr{C}^{-1-\varepsilon}$. In fact, we have the following almost sure bound

$$\big\|K_\mu * {}_\kappa\big\| \lesssim [\mu]^{-1-\varepsilon}$$

uniform in $\kappa,\mu\in(0,1]$, which is insufficient to close the estimates. For this reason, in the approach to singular SPDEs involving paracontrolled distributions one bounds instead the tree

$${}_\kappa := {}_\kappa \odot {}_\kappa + 1/3\; c_\kappa^{(2)}\; {}_\kappa,$$

where $\odot$ denotes the so-called resonant product. The above tree satisfies the bound

$$\big\|K_\mu * {}_\kappa\big\| \lesssim [\mu]^{-1/2-\varepsilon}$$

uniform in $\kappa,\mu\in(0,1]$. On the other hand, in the regularity structure framework one studies the recentered tree

$${}_\kappa(x,y) := {}_\kappa(y)\Big({}_\kappa(y) - {}_\kappa(x)\Big) + 1/3\; c_\kappa^{(2)}\; {}_\kappa(y)$$

and proves that

$$\sup_{x\in\mathbb{M}}\left|\int K_\mu(x-y)\,{}_\kappa(x,y)\,\mathrm{d}y\right| \lesssim [\mu]^{-1/2-\varepsilon}$$

uniformly in $\kappa,\mu\in(0,1]$. The trees that represent the effective force coefficients does not involve any recentering. Consequently, the structure group does not enter the flow equation framework and there is no positive renormalization. The intuitive reason why the bound (3.8) holds true is the fact that the edges $x \longrightarrow y \;=\; (G-G_\mu)(x-y)$ depend on the scale parameter $\mu\in(0,1]$ and are in some sense small for small μ, for example,

$$\|G-G_\mu\|_{L^1(\mathbb{M})} \le \int_0^\mu \|\dot G_\eta\|_{L^1(\mathbb{M})}\,\mathrm{d}\eta \lesssim [\mu]^\sigma.$$

Corollary 3.17. *There exists deterministic $c \in (0,\infty)$ such that for all $\lambda \in [-1,1]$ the functionals $\tilde F_{\kappa,\bullet}$ and $\tilde H_{\kappa,\bullet}$ defined in terms of the enhanced noise by Eqs. (3.5), (3.2) and (3.1) satisfy the assumptions of Lemma 2.15 with $R = c\,\tilde R$, $\alpha = \sigma - d/2 - \varepsilon$, $\beta = \varrho_\varepsilon(i_\flat + 1, 0)$, $m_\flat = 3i_\flat$ for all $\kappa \in (0,1]$, where $\tilde R \in [1,\infty]$ is the random variable introduced in Theorem 3.14.*

Proof (♠). We have to verify the bounds (3.3), (3.4). Recall that

$$\tilde F_{\kappa,\mu}[\varphi] = K_\mu * F_{\kappa,\mu}[K_\mu * \varphi], \qquad \tilde H_{\kappa,\mu}[\varphi] = K_\mu * H_{\kappa,\mu}[K_\mu * \varphi]$$

and define

$$\tilde F^{i,m}_{\kappa,\mu} := K_\mu^{\otimes(1+m)} * F^{i,m}_{\kappa,\mu}, \qquad \tilde H^{i,m}_{\kappa,\mu} := K_\mu^{\otimes(1+m)} * H^{i,m}_{\kappa,\mu}.$$

Then $\tilde F^{i,m}_{\kappa,\mu}$ and $\tilde H^{i,m}_{\kappa,\mu}$ are related to $\tilde F_{\kappa,\mu}$ and $\tilde H_{\kappa,\mu}$ by formulas analogous to Eqs. (3.5) and (3.6). By Theorem 3.14 we have

$$\|\tilde F^{i,m}_{\kappa,\mu}\|_{\mathcal{V}^m} \leq \tilde R\,[\mu]^{\varrho_\varepsilon(i,m)}.$$

Verification of the bound (3.3) for $\tilde F_{\kappa,\bullet}$ is straightforward. Let us prove the bound (3.4) for $\tilde H_{\kappa,\bullet}$. Using Eqs. (3.2), (3.6) as well as the fact that the effective force coefficients satisfy the flow equation (3.7) we obtain

$$H^{i,m}_{\kappa,\mu} = \sum_{j=i-i_\flat}^{i_\flat} \sum_{k=0}^{m} (1+k)\,\mathbf{B}(\dot G_\mu, F^{j,1+k}_{\kappa,\mu}, F^{i-j,m-k}_{\kappa,\mu})$$

for $i \in \{i_\flat + 1, \ldots, 2i_\flat\}$ and $m \in \mathbb{N}_0$. Since $\mathbf{P}_\mu K_\mu = \delta_0$ this implies that

$$\tilde H^{i,m}_{\kappa,\mu} = \sum_{j=i-i_\flat}^{i_\flat} \sum_{k=0}^{m} (1+k)\,\mathbf{B}(\tilde G_\mu, \tilde F^{j,1+k}_{\kappa,\mu}, \tilde F^{i-j,m-k}_{\kappa,\mu}),$$

where $\tilde G_\mu = \mathbf{P}_\mu^2 \dot G_\mu$. Consequently, by the bounds for $\tilde F^{i,m}_{\kappa,\mu}$, the estimate stated in Exercise 3.1 and the identity

$$\varrho_\varepsilon(j, 1+k) + \varrho_\varepsilon(i-j, m-k) = \varrho_\varepsilon(i,m) - \sigma$$

we have

$$\|\tilde H^{i,m}_{\kappa,\mu}\|_{\mathcal{V}^m} \lesssim \tilde R^2\,[\mu]^{\varrho_\varepsilon(i,m)-\sigma}.$$

Since

$$\langle \tilde H_{\kappa,\mu}[\varphi], \psi \otimes \varphi^{\otimes m}\rangle := \sum_{i=i_\flat+1}^{2i_\flat} \sum_{m=0}^{3i} \lambda^i\,\langle \tilde H^{i,m}_{\kappa,\mu}, \psi \otimes \varphi^{\otimes m}\rangle$$

we obtain the bound (3.4) with $\beta = \varrho_\varepsilon(i_\flat + 1, 0) > 0$. □

Exercise 3.5. *Check that the bound stated in Theorem 3.14 cannot be proved deterministically for all $i, m \in \mathbb{N}_0$ by following the steps (0), (I), (II) of the recursive algorithm presented above. More specifically, assume that the bound holds true for all $F^{i,m}_{\kappa,\mu}$ with $i < i_\circ$, or $i = i_\circ$ and $m > m_\circ$. Prove that*

$$\|K_\mu^{\otimes(1+m)} * \dot{F}^{i,m}_{\kappa,\mu}\|_{\mathcal{V}^m} \lesssim [\mu]^{\varrho_\varepsilon(i,m)-\sigma}$$

for $i = i_\circ$ and $m = m_\circ$, where $\dot{F}^{i,m}_{\kappa,\mu}$ denotes the RHS of Eq. (3.7). By Exercise 2.1 (3)

$$\|K_\mu^{\otimes(1+m)} * F^{i,m}_{\kappa,\mu}\|_{\mathcal{V}^m} \leq \int_0^\mu \|K_\eta^{\otimes(1+m)} * \dot{F}^{i,m}_{\kappa,\eta}\|_{\mathcal{V}^m} \,\mathrm{d}\eta \lesssim \int_0^\mu [\eta]^{\varrho_\varepsilon(i,m)-\sigma} \,\mathrm{d}\eta.$$

Conclude that $\|K_\mu^{\otimes(1+m)} * F^{i,m}_{\kappa,\mu}\|_{\mathcal{V}^m} \lesssim [\mu]^{\varrho_\varepsilon(i,m)}$ provided $\varrho(i,m) > 0$. The upshot is that the so-called irrelevant coefficients $F^{i,m}_{\kappa,\mu}$ such that $\varrho(i,m) > 0$ can be bounded deterministically. For the relevant coefficients $F^{i,m}_{\kappa,\mu}$ such that $\varrho(i,m) \leq 0$ the above procedure fails.

Lemma 3.18. *For all $i \in \{0, \ldots, i_\flat\}$ and $m \in \mathbb{N}_0$ there exists $c \in \mathbb{R}_+$ such that for all $s \in \{0,1\}$, $\kappa \in (0,1]$ and $\mu \in [0,1/2]$ it holds*

$$\operatorname{supp} \partial_\mu^s F^{i,m}_{\kappa,\mu} \subset \{(x, y_1, \ldots, y_m) \in \mathbb{M}^{1+m} \,|\, |x-y_1| \vee \ldots \vee |x-y_m| \leq c\,[\mu]\}.$$

Remark 3.19. To prove the theorem is it enough to use the graphical representation for $F^{i,m}_{\kappa,\mu}$ introduced in Remark 3.9 and observe that for $\mu \in (0,1/2]$ the fluctuation propagator $(G - G_\mu)(x-y)$ represented by edges of the graphs vanishes identically if $|x-y| > [\mu]$ by Remark 2.10. Since we did not introduce precise rules for drawing diagrams in the exercise below we suggest to prove this result using the flow equation. Note that the above support property is not true for μ close to one.

Exercise 3.6 (♠). *Prove the above lemma by induction using the flow equation (3.7). Hint: Observe that $F^{i,m}_{\kappa,0} = F^{i,m}_\kappa$ is local, and consequently it satisfies the above support property. Note that for $\mu \in (0,1/2]$ the kernel $\dot{G}_\mu$ is supported in a ball of radius $[\mu]$.*

Remark 3.20 (♠). Since we would like to use Lemma 3.18 in the proof of the stochastic estimates for the enhanced noise we will actually establish the bound stated in Theorem 3.14 only for $\mu \in (0,1/2]$. The bound for $\mu \in [1/2,1]$ can be then easily proved deterministically. Alternatively, one can set

$$\langle F_{\kappa,\mu}[\varphi], \psi\rangle := \sum_{i=0}^{i_\flat} \sum_{m=0}^{3i} \lambda^i \,\langle F^{i,m}_{\kappa,\mu\wedge 1/2}, \psi \otimes \varphi^{\otimes m}\rangle. \tag{3.9}$$

Then $H_{\kappa,\mu} = O(\lambda^{i_\flat+1})$ is true for $\mu \in (0,1/2]$, which is sufficient to show the bound (3.4) for some $\beta > 0$.

4 Cumulants of effective force coefficients

Recall that the finite list of the effective force coefficients $(F^{i,m}_{\kappa,\bullet})_{i\in\{0,\ldots,i_\flat\},m\in\{0,\ldots,3i\}}$ is called the enhanced noise. In order to prove the stochastic estimates for the enhanced noise stated in Theorem 3.14 we will first establish certain bounds for the joint cumulants of the effective force coefficients. Subsequently, we will infer bounds for the moments and apply a Kolmogorov-type argument to conclude. The bounds for cumulants involve the norm $\|\bullet\|_{\mathcal{V}^{\mathfrak{m}}_{\mathrm{t}}}$ introduced in Def. 4.5 and are stated in Theorem 5.3. The proof of the bounds for cumulants is based on a certain flow equation that is derived in Lemma 4.14.

Definition 4.1. Let $p\in\mathbb{N}_+$, $I=\{1,\ldots,p\}$ and ζ_q, $q\in I$, be random variables. The joint cumulant of the multi-set $(\zeta_q)_{q\in I}=(\zeta_1,\ldots,\zeta_p)$ is defined by

$$\mathbb{E}(\zeta_1,\ldots,\zeta_p)\equiv\mathbb{E}(\zeta_q)_{q\in I}=(-\mathrm{i})^p\partial_{t_1}\ldots\partial_{t_p}\log\mathbb{E}\exp(\mathrm{i}t_1\zeta_1+\ldots+\mathrm{i}t_p\zeta_p)\big|_{t_1=\ldots=t_p=0}.$$

In particular, $\mathbb{E}(\zeta_1,\zeta_2)=\mathbb{E}(\zeta_1\zeta_2)-\mathbb{E}\zeta_1\,\mathbb{E}\zeta_2$.

Definition 4.2. Given $n\in\mathbb{N}_+$, $I=\{1,\ldots,n\}$, $m_1,\ldots,m_n\in\mathbb{N}_0$ and random distributions $\zeta_q\in\mathscr{S}'(\mathbb{M}^{1+m_q})$, $q\in I$, the deterministic distribution

$$\mathbb{E}(\zeta_q)_{q\in I}\equiv\mathbb{E}(\zeta_1,\ldots,\zeta_n)\in\mathscr{S}'(\mathbb{M}^n\times\mathbb{M}^{m_1+\ldots+m_n})$$

is defined by the equality

$$\langle\mathbb{E}(\zeta_1,\ldots,\zeta_n),\psi_1\otimes\ldots\otimes\psi_n\otimes\varphi_1\otimes\ldots\otimes\varphi_n\rangle:=\mathbb{E}(\langle\zeta_1,\psi_1\otimes\varphi_1\rangle,\ldots,\langle\zeta_n,\psi_n\otimes\varphi_n\rangle)$$

for all $\psi_q\in\mathscr{S}(\mathbb{M})$, $\varphi_q\in\mathscr{S}(\mathbb{M}^{m_q})$, $q\in I$.

Definition 4.3. A list (i,m,s,r), where $i\in\{0,\ldots,i_\flat\}$, $m\in\mathbb{N}_0$ and $s,r\in\{0,1\}$ is called an index. For $n\in\mathbb{N}_+$ we call

$$\mathtt{I}=((i_1,m_1,s_1,r_1),\ldots,(i_n,m_n,s_n,r_n)) \tag{4.1}$$

a list of indices. We define $n(\mathtt{I}):=n$, $i(\mathtt{I}):=i_1+\ldots+i_n$, $\mathfrak{m}(\mathtt{I}):=(m_1,\ldots,m_n)$, $m(\mathtt{I}):=m_1+\ldots+m_n$, $s(\mathtt{I}):=s_1+\ldots+s_n$ and $r(\mathtt{I}):=r_1+\ldots+r_n$. We use the following notation for the joint cumulants of the effective force coefficients

$$E^{\mathtt{I}}_{\kappa,\mu}:=\mathbb{E}(\partial^{s_1}_\mu\partial^{r_1}_\kappa F^{i_1,m_1}_{\kappa,\mu},\ldots,\partial^{r_n}_\kappa\partial^{s_n}_\mu F^{i_n,m_n}_{\kappa,\mu})\in\mathscr{S}'(\mathbb{M}^{n(\mathtt{I})+m(\mathtt{I})}).$$

Remark 4.4. In order to prove the convergence of the enhanced noise as $\kappa\searrow 0$ one has to study cumulants $E^{\mathtt{I}}_{\kappa,\mu}$ with $r(\mathtt{I})\neq 0$. In what follows, for simplicity, we restrict attention to cumulants $E^{\mathtt{I}}_{\kappa,\mu}$ with $\mathtt{I}=((i_1,m_1,s_1,0),\ldots,(i_n,m_n,s_n,0))$.

Definition 4.5. Let $n\in\mathbb{N}_+$, $\mathfrak{m}=(m_1,\ldots,m_n)\in\mathbb{N}_0^n$ and $m=m_1+\ldots+m_n$. The vector space $\mathcal{V}^{\mathfrak{m}}_{\mathrm{t}}$ consists of maps $V\in C(\mathbb{M}^n\times\mathbb{M}^m)$ such that

(1) the function

$$(x_1,\ldots,x_n) \mapsto V(x_1,\ldots,x_n;\mathrm{y}_1+x_1,\ldots,\mathrm{y}_n+x_n)$$

is 2π periodic in all variables for every

$$(y_1,\ldots,y_m) = (\mathrm{y}_1,\ldots,\mathrm{y}_n) \in \mathbb{M}^{m_1}\times\ldots\times\mathbb{M}^{m_n} = \mathbb{M}^m,$$

where

$$\mathrm{y}+x := (y_1+x,\ldots,y_n+x) \in \mathbb{M}^m$$

for arbitrary $m\in\mathbb{N}_0$, $x\in\mathbb{M}$, $\mathrm{y}=(y_1,\ldots,y_m)\in\mathbb{M}^m$,

(2) it holds

$$V(x_1,\ldots,x_n;y_1,\ldots,y_m) = V(x_1+z,\ldots,x_n+z;y_1+z,\ldots,y_m+z)$$

for all $x_1,\ldots,x_n,y_1,\ldots,y_m,z\in\mathbb{M}$,

(3) the norm

$$\|V\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}} := \sup_{x_1\in\mathbb{M}} \int_{\mathbb{T}^{n-1}\times\mathbb{M}^m} |V(x_1,\ldots,x_n;y_1,\ldots,y_m)|\,\mathrm{d}x_2\ldots\mathrm{d}x_n\,\mathrm{d}y_1\ldots\mathrm{d}y_m$$

is finite.

Remark 4.6. Note that for $n=1$, $\mathfrak{m}=m\in\mathbb{N}^n$ and $V\in\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}$ it holds $\|V\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}} = \|V\|_{\mathcal{V}^m}$.

Remark 4.7. Using the condition of translational invariance stated in Item (2) of the above definition the expression for the norm $\|V\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}}$ can be rewritten in more symmetric form

$$\|V\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^n\times\mathbb{M}^m} |V(x_1,\ldots,x_n;y_1,\ldots,y_m)|\,\mathrm{d}x_1\ldots\mathrm{d}x_n\mathrm{d}y_1\ldots\mathrm{d}y_m.$$

Definition 4.8. For $\varepsilon\in[0,\infty)$ and a list of indices $\mathtt{I}$ of the form (4.1) we define

$$\varrho_\varepsilon(\mathtt{I}) := \varrho_\varepsilon(i_1,m_1)+\ldots+\varrho_\varepsilon(i_n,m_n)\in\mathbb{R}.$$

We also set $\varrho(\mathtt{I}) := \varrho_0(\mathtt{I})$.

Remark 4.9. Using the fact that the law of the noise ξ_κ is invariant under translations in space one proves that the same is true for the effective force coefficients $F^{i,m}_{\kappa,\mu}$. Since $\tilde{K}_\mu\in C(\mathbb{M})\cap L^1(\mathbb{M})$ and $\partial^r_\kappa\partial^s_\mu F^{i,m}_{\kappa,\mu}\in\mathcal{V}^m$ for all $\kappa,\mu\in(0,1]$ one easily shows that

$$\tilde{K}^{\otimes(n+m)}_\mu * E^{\mathtt{I}}_{\kappa,\mu}\in\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}$$

for all $\kappa,\mu\in(0,1]$, where $n=n(\mathtt{I})$, $m=m(\mathtt{I})$ and $\mathfrak{m}=\mathfrak{m}(\mathtt{I})$. In Theorem 5.3 stated in the next section we prove that for an appropriate choice of the counterterms the following bound

$$\|\tilde{K}^{\otimes(n+m)}_\mu * E^{\mathtt{I}}_{\kappa,\mu}\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}} \lesssim [\mu]^{\varrho_\varepsilon(\mathtt{I})-\sigma s(\mathtt{I})+d(n-1)} \tag{4.2}$$

holds uniformly in $\kappa \in (0,1]$ and $\mu \in (0,1/2]$. In order to see that, at least at a heuristic level, the above estimate is compatible with the bound stated in Theorem 3.14 suppose that

$$\mathbb{E}\|\tilde K_\mu^{\otimes(1+m)} * \partial_\mu^s F^{i,m}_{\kappa,\mu}\|^n_{\mathcal{V}^m} \lesssim [\mu]^{n(\varrho_\varepsilon(i,m)-\sigma s)}.$$

The last bound is almost what Theorem 3.14 says. Observe that this bounds implies that

$$\|\tilde K_\mu^{\otimes(n+m)} * E^{\mathtt{I}}_{\kappa,\mu}\|_{\tilde{\mathcal{V}}^{\mathsf{m}}_{\mathsf{t}}} \lesssim [\mu]^{\varrho_\varepsilon(\mathtt{I})-\sigma s(\mathtt{I})}, \tag{4.3}$$

where

$$\|V\|_{\tilde{\mathcal{V}}^{\mathsf{m}}_{\mathsf{t}}} := \sup_{x_1,\ldots,x_n\in\mathbb{M}} \int |V(x_1,\ldots,x_n;y_1,\ldots,y_m)|\,\mathrm{d}y_1\ldots\mathrm{d}y_m.$$

Actually, the bound (4.3) would be also true if $E^{\mathtt{I}}_{\kappa,\mu}$ was defined to be the expected value of a product of the effective force coefficients and not their joint cumulant. The presence of the extra factor $[\mu]^{d(n-1)}$ appearing in the bound (4.2) can be understood at least for $\mu \geq \kappa$ by noting the following support property of the cumulants

$$\begin{aligned}\operatorname{supp} E^{\mathtt{I}}_{\kappa,\mu} \subset \{(x_1,\ldots,x_n;y_1,\ldots,y_m)\in\mathbb{M}^{n+m} \mid \\ |x_2-x_1|\vee\ldots\vee|x_n-x_1|\vee|y_1-x_1|\vee\ldots\vee|y_m-x_1| \leq c\,[\kappa\vee\mu]\}\end{aligned}$$

for all $\kappa,\mu\in(0,1]$ and some $c\in(0,\infty)$ depending only on $\mathtt{I}$. Note that the norm $\|\bullet\|_{\mathcal{V}^{\mathsf{m}}_{\mathsf{t}}}$ is weaker than $\|\bullet\|_{\tilde{\mathcal{V}}^{\mathsf{m}}_{\mathsf{t}}}$. Because of the extra factor $[\mu]^{d(n-1)}$ the bound (4.2) is easier to establish than the bound (4.3).

Remark 4.10 (♠). For $\varepsilon\in(0,\infty)$ and any list of indices $\mathtt{I}$ such that $m(\mathtt{I}) \leq 3i(\mathtt{I})$ it holds $\varrho_\varepsilon(\mathtt{I}) < \varrho(\mathtt{I})$. Moreover, $\varrho_\varepsilon(\mathtt{I}) + (n(\mathtt{I})-1)d > 0$ for $\varepsilon\in(0,\varepsilon_\diamond)$ and lists of indices $\mathtt{I}$ such that $\varrho(\mathtt{I}) + (n(\mathtt{I})-1)d > 0$.

Exercise 4.1. *Draw the graphs representing cumulants $E^{\mathtt{I}}_{\kappa,\mu}$ with $\mathrm{i}(\mathtt{I}) = 2$, $m(\mathtt{I}) = 2$, $s(\mathtt{I}) \in \{0,1\}$, $r(\mathtt{I}) = 0$. For simplicity, ignore numerical prefactors. Use the graphical representation of the effective force coefficients in terms of trees introduced in Sec. 3. Recall that $x \text{———} y = (G-G_\mu)(x-y)$ and $x \text{———} y = \dot G_\mu(x-y)$. Furthermore, use the edge $x \cdots\cdots y := \mathbb{E}\xi_\kappa(x)\xi_\kappa(y)$ to denote the covariance of the regularized noise. Note that, in general, graphs for cumulants are not trees. Hint: For the warm-up, draw the diagrams for the cumulant $\mathbb{E}(\xi_\kappa(x_1)\xi_\kappa(x_2)\xi(x_3),\xi_\kappa(y_1)\xi_\kappa(y_2),\xi_\kappa(z))$.*

Definition 4.11. Fix $n\in\mathbb{N}_+$, $\hat n\in\{1,\ldots,n\}$, $m_1,\ldots,m_{n+1}\in\mathbb{N}_0$. Let

$$\mathsf{m} = (m_1+m_{n+1},m_2,\ldots,m_n)\in\mathbb{N}_0^n, \qquad \tilde{\mathsf{m}} = (1+m_1,m_2,\ldots,m_{n+1})\in\mathbb{N}_0^{n+1},$$
$$\hat{\mathsf{m}} = (1+m_1,m_2,\ldots,m_{\hat n})\in\mathbb{N}_0^{\hat n}, \qquad \check{\mathsf{m}} = (m_{\hat n+1},\ldots,m_{n+1})\in\mathbb{N}_0^{n-\hat n+1}$$

and $\underline{m} = m_1+m_{n+1}$. The bilinear map $\mathbf{A} : \mathscr{S}(\mathbb{M})\times\mathcal{V}^{\tilde{\mathsf{m}}}_{\mathsf{t}} \to \mathcal{V}^{\mathsf{m}}_{\mathsf{t}}$ is defined by

$$\begin{aligned}&\mathbf{A}(G,V)(x_1,\ldots,x_n;\mathsf{y}_1,\mathsf{y}_{n+1},\mathsf{y}_2,\ldots,\mathsf{y}_n)\\ &\qquad := \int_{\mathbb{M}^2} V(x_1,\ldots,x_{n+1};y,\mathsf{y}_1,\ldots,\mathsf{y}_{n+1})\,G(y-x_{n+1})\,\mathrm{d}y\mathrm{d}x_{n+1}.\end{aligned}$$

The trilinear map $\mathbf{B} : \mathscr{S}(\mathbb{M}) \times \mathcal{V}_{\mathsf{t}}^{\hat{\mathsf{m}}} \times \mathcal{V}_{\mathsf{t}}^{\check{\mathsf{m}}} \to \mathcal{V}_{\mathsf{t}}^{\mathsf{m}}$ is defined by

$$\begin{aligned}
&\mathbf{B}(G,W,U)(x_1,\ldots,x_n;\mathrm{y}_1,\mathrm{y}_{n+1},\mathrm{y}_2,\ldots,\mathrm{y}_n)\\
&\quad := \frac{1}{\underline{m}!}\sum_{\pi\in\mathcal{P}_{\underline{m}}}\int_{\mathbb{M}^2} W(x_1,\ldots,x_{\hat{n}};y,y_{\pi(1)},\ldots,y_{\pi(m_1)},\mathrm{y}_2,\ldots,\mathrm{y}_{\hat{n}})\,G(y-x_{n+1})\\
&\qquad\qquad \times\, U(x_{n+1},x_{\hat{n}+1},\ldots,x_n;y_{\pi(m_1+1)},\ldots,y_{\pi(\underline{m})},\mathrm{y}_{\hat{n}+1},\ldots,\mathrm{y}_n)\,\mathrm{d}y\mathrm{d}x_{n+1}.
\end{aligned}$$

In the above equations $\mathrm{y}_j \in \mathbb{M}^{m_j}$, $j \in \{1,\ldots,n+1\}$.

Lemma 4.12. *Let $p \in \mathbb{N}_+$, $I = \{1,\ldots,p\}$ and $\zeta_1,\ldots,\zeta_p,\Phi,\Psi$ be random variables. It holds*

$$\mathbb{E}(\zeta_1\ldots\zeta_p) = \sum_{r=1}^{p}\sum_{\substack{I_1,\ldots,I_r\subset I,\\ I_1\cup\ldots\cup I_r=I\\ I_1,\ldots,I_r\neq\emptyset}} \mathbb{E}(\zeta_q)_{q\in I_1}\ldots\mathbb{E}(\zeta_q)_{q\in I_r}, \tag{4.4}$$

$$\mathbb{E}((\zeta_q)_{q\in I},\Phi\Psi) = \mathbb{E}((\zeta_q)_{q\in I},\Phi,\Psi) + \sum_{\substack{I_1,I_2\subset I\\ I_1\cup I_2=I}} \mathbb{E}((\zeta_q)_{q\in I_1},\Phi)\ \mathbb{E}((\zeta_q)_{q\in I_2},\Psi). \tag{4.5}$$

Proof. See e.g. Proposition 3.2.1 in [PT11]. □

Lemma 4.13. *Let $n \in \mathbb{N}_+$, $i_1 \in \mathbb{N}_0$, $m_1,\ldots,m_n \in \mathbb{N}_0$ and $I \equiv \{2,\ldots,n\}$. For any random distributions $\zeta_q \in \mathscr{S}'(\mathbb{M}^{1+m_q})$, $q \in I$, the cumulant*

$$\mathbb{E}(\partial_\mu F^{i_1,m_1}_{\kappa,\mu},(\zeta_q)_{q\in I}) \in \mathscr{S}'(\mathbb{M}^{n+m_1+\ldots+m_n})$$

is a linear combination of the expressions

$$\mathbf{A}\big(\dot{G}_\mu,\mathbb{E}(F^{j,1+k}_{\kappa,\mu},(\zeta_q)_{q\in I},F^{i_1-j,m_1-k}_{\kappa,\mu})\big) \tag{4.6}$$

or

$$\mathbf{B}\big(\dot{G}_\mu,\mathbb{E}(F^{j,1+k}_{\kappa,\mu},(\zeta_q)_{q\in I_1}),\mathbb{E}(F^{i_1-j,m_1-k}_{\kappa,\mu},(\zeta_q)_{q\in I_2})\big), \tag{4.7}$$

where $j \in \{1,\ldots,i_1\}$, $k \in \{0,\ldots,m_1\}$, $r \in \{0,r_1\}$ and the subsets $I_1,I_2 \subset I$ are such that $I_1 \cup I_2 = I$ and $I_1 \cap I_2 = \emptyset$. The coefficients of the above linear combination do not depend on $\kappa,\mu \in (0,1]$. We used the notation introduced in Def. 4.2.

Proof. The statement follows immediately from the flow equation (3.7) and Eq. (4.5). □

Lemma 4.14. *Let $\mathtt{J} \equiv (\mathtt{J}_1,\ldots,\mathtt{J}_n) = ((i_1,m_1,s_1,0),\ldots,(i_n,m_n,s_n,0))$ be a list of indices such that $s_1 = 1$.*

(A) The distribution $E^{\mathtt{J}}_{\kappa,\mu}$ can be expressed as a linear combination of distributions of the form

$$\mathbf{A}\big(\dot{G}_\mu,E^{\mathtt{K}}_{\kappa,\mu}\big) \qquad \text{or} \qquad \mathbf{B}\big(\dot{G}_\mu,E^{\mathtt{L}}_{\kappa,\mu},E^{\mathtt{M}}_{\kappa,\mu}\big),$$

where the lists of indices $\mathtt{K}$, $\mathtt{L}$, $\mathtt{M}$ *satisfy the following conditions*

$$\begin{aligned} &n(\mathtt{K})=n(\mathtt{J})+1, && && n(\mathtt{L})+n(\mathtt{M})=n(\mathtt{J})+1,\\ &i(\mathtt{K})=i(\mathtt{J}), && && i(\mathtt{L})+i(\mathtt{M})=i(\mathtt{J}),\\ &m(\mathtt{K})=m(\mathtt{J})+1, && \qquad or \qquad && m(\mathtt{L})+m(\mathtt{M})=m(\mathtt{J})+1,\\ &s(\mathtt{K})=s(\mathtt{J})-1, && && s(\mathtt{L})+s(\mathtt{M})=s(\mathtt{J})-1,\\ &\varrho_\varepsilon(\mathtt{J})-\sigma=\varrho_\varepsilon(\mathtt{K})-d, && && \varrho_\varepsilon(\mathtt{J})-\sigma=\varrho_\varepsilon(\mathtt{L})+\varrho_\varepsilon(\mathtt{M}). \end{aligned}$$

(B) *Suppose that the bound*

$$\|\tilde{K}_\mu^{\otimes(n(\mathtt{I})+m(\mathtt{I}))}\ast E^{\mathtt{I}}_{\kappa,\mu}\|_{\mathcal{V}^{\mathrm{m}(\mathtt{I})}_{\mathrm{t}}}\lesssim[\mu]^{\varrho_\varepsilon(\mathtt{I})-\sigma s(\mathtt{I})+(n(\mathtt{I})-1)d}$$

holds uniformly in $\kappa\in(0,1]$, $\mu\in(0,1/2]$ *for all lists of indices* $\mathtt{I}$ *such that* $i(\mathtt{I})<i(\mathtt{J})$, *or* $i(\mathtt{I})=i(\mathtt{J})$ *and* $m(\mathtt{I})>m(\mathtt{J})$. *Then the above bound holds uniformly in* $\kappa\in(0,1]$, $\mu\in(0,1/2]$ *for* $\mathtt{I}=\mathtt{J}$.

Remark 4.15. Let $\mathtt{J}\equiv(\mathtt{J}_1,\ldots,\mathtt{J}_n)$ be a list of indices. For a permutation $\pi\in\mathcal{P}_n$ we set $\pi(\mathtt{J}):=(\mathtt{J}_{\pi(1)},\ldots,\mathtt{J}_{\pi(n)})$. By Remark 4.7 it holds

$$\|E^{\mathtt{J}}_{\kappa,\mu}\|_{\mathcal{V}^{\mathrm{m}(\mathtt{J})}_{\mathrm{t}}}=\|E^{\pi(\mathtt{J})}_{\kappa,\mu}\|_{\mathcal{V}^{\mathrm{m}(\pi(\mathtt{J}))}_{\mathrm{t}}}.$$

Hence, the above lemma is true for all list of indices $\mathtt{J}$ such that $s(\mathtt{I})\neq 0$.

Proof (♠). Part (A) of the lemma follows immediately from Lemma 4.13 applied with

$$\zeta_q\equiv\partial_\mu^{s_q}F^{i_q,m_q,a_q}_{\kappa,\mu},\qquad q\in\{2,\ldots,n\}.$$

It holds

$$\mathtt{K}=((j,k+1,0,0),\,\mathtt{J}_2,\ldots,\mathtt{J}_n,\,(i_1-j,m_1-k,0,0)),$$
$$\mathtt{L}=(j,k+1,0,0)\sqcup(\mathtt{J}_q)_{q\in I_1},\qquad \mathtt{M}=(i_1-j,m_1-k,0,0)\sqcup(\mathtt{J}_q)_{q\in I_2},$$

where $\sqcup$ denotes the concatenation of lists, $I_1\cup I_2=I=\{2,\ldots,n\}$, $I_1\cap I_2=\emptyset$ and $j\in\{1,\ldots,i_1\}$, $k\in\{0,\ldots,m_1\}$ coincide with the respective objects in Eqs. (4.6) and (4.7). This together with Def. (4.8) implies that the lists $\mathtt{K}$, $\mathtt{L}$, $\mathtt{M}$ satisfy the conditions stated in Part (A). To prove Part (B) we use Part (A) and Remark 4.18. □

In the remaining part of this section, we collect some technical results that were used in the proof of the above lemma.

Definition 4.16 (♠). Let $\mathbb{T}:=\mathbb{M}/(2\pi\mathbb{Z})^d$. For $K\in L^1(\mathbb{M})$ we define $\mathbf{T}K\in L^1(\mathbb{T})$ by

$$\mathbf{T}K(x):=\sum_{y\in(2\pi\mathbb{Z})^d}K(x+y).$$

Lemma 4.17 (♠)**.** *The maps* $\mathbf{A} : \mathscr{S}(\mathbb{M}) \times \mathcal{V}^{\tilde{\mathfrak{m}}}_{\mathfrak{t}} \to \mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}$, $\mathbf{B} : \mathscr{S}(\mathbb{M}) \times \mathcal{V}^{\hat{\mathfrak{m}}}_{\mathfrak{t}} \times \mathcal{V}^{\check{\mathfrak{m}}}_{\mathfrak{t}} \to \mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}$ *are well defined. It holds*

$$\|\mathbf{A}(G,V)\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}} \leq \|\mathbf{T}|G|\|_{L^\infty(\mathbb{T})}\, \|V\|_{\mathcal{V}^{\tilde{\mathfrak{m}}}_{\mathfrak{t}}},$$
$$\|\mathbf{B}(G,W,U)\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}} \leq \|G\|_{L^1(\mathbb{M})}\, \|W\|_{\mathcal{V}^{\hat{\mathfrak{m}}}_{\mathfrak{t}}}\, \|U\|_{\mathcal{V}^{\check{\mathfrak{m}}}_{\mathfrak{t}}}.$$

Proof. See [Duc22, Lemma 14.10]. □

Remark 4.18 (♠). Note that Lemmas 2.11 and 4.19 (D) imply that $\|\tilde{\mathbf{P}}^2_\mu \dot{G}_\mu\|_{L^1(\mathbb{M})} \lesssim 1$ and

$$\|\mathbf{T}|\tilde{\mathbf{P}}^2_\mu \dot{G}_\mu|\|_{L^\infty(\mathbb{T})} \leq \|\mathbf{T}\tilde{K}_\mu\|_{L^\infty(\mathbb{T})}\, \|\tilde{\mathbf{P}}^3_\mu \dot{G}_\mu\|_{L^1(\mathbb{M})} \lesssim [\mu]^{-d}\, \|\tilde{\mathbf{P}}^3_\mu \dot{G}_\mu\|_{L^1(\mathbb{T})},$$

uniformly in $\mu \in (0,1]$. Moreover, using the fact that $\tilde{\mathbf{P}}_\mu \tilde{K}_\mu = \delta_0$ one shows that for all $\mu \in (0,1]$ it holds

$$\tilde{K}^{\otimes(n+m)}_\mu * \mathbf{A}(G,V) = \mathbf{A}\big(\tilde{\mathbf{P}}^2_\mu G, \tilde{K}^{\otimes(n+m+2)}_\mu * V\big)$$

and

$$\tilde{K}^{\otimes(n+m)}_\mu * \mathbf{B}(G,W,U) = \mathbf{B}\big(\tilde{\mathbf{P}}^2_\mu G, \tilde{K}^{\otimes(\hat{n}+\hat{m}+1)}_\mu * W, \tilde{K}^{\otimes(n-\hat{n}+m-\hat{m}+1)}_\mu * U\big),$$

where $m = m_1 + \ldots + m_{n+1}$ and $\hat{m} = m_1 + \ldots + m_{\hat{n}}$. Consequently, by the above lemma we have

$$\|\tilde{K}^{\otimes(n+m)}_\mu * \mathbf{A}(\dot{G}_\mu, V)\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}} \lesssim [\mu]^{-d}\, \|\tilde{K}^{\otimes(n+m)}_\mu * V\|_{\mathcal{V}^{\tilde{\mathfrak{m}}}_{\mathfrak{t}}}$$

and

$$\|\tilde{K}^{\otimes(n+m)}_\mu * \mathbf{B}(\dot{G}_\mu, W, U)\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}} \lesssim \|\tilde{K}^{\otimes(\hat{n}+\hat{m}+1)}_\mu * W\|_{\mathcal{V}^{\hat{\mathfrak{m}}}_{\mathfrak{t}}}\, \|\tilde{K}^{\otimes(n-\hat{n}+m-\hat{m}+1)}_\mu * U\|_{\mathcal{V}^{\check{\mathfrak{m}}}_{\mathfrak{t}}}$$

uniformly in $\mu \in (0,1]$ and $V \in \mathcal{V}^{\tilde{\mathfrak{m}}}_{\mathfrak{t}}$, $W \in \mathcal{V}^{\hat{\mathfrak{m}}}_{\mathfrak{t}}$, $U \in \mathcal{V}^{\check{\mathfrak{m}}}_{\mathfrak{t}}$.

Lemma 4.19 (♠)**.** *Let* $a \in \mathbb{N}^d_0$ *and* $p \in [1,\infty]$. *The following is true:*

(A) If $|a| \leq d$, *then* $\|\partial^a \tilde{K}_\mu\|_{\mathcal{K}} \lesssim [\mu]^{-|a|}$ *uniformly in* $\mu \in (0,1]$.

(B) It holds $\|\tilde{\mathbf{P}}_\mu \partial_\mu \tilde{K}_\mu\|_{\mathcal{K}} \lesssim [\mu]^{-\sigma}$ *uniformly in* $\mu \in (0,1]$.

(C) $\|\tilde{K}_\mu\|_{L^p(\mathbb{M})} \lesssim [\mu]^{-d(p-1)/p}$ *uniformly in* $\mu \in (0,1]$.

(D) $\|\mathbf{T}\tilde{K}_\mu\|_{L^p(\mathbb{T})} \lesssim [\mu]^{-d(p-1)/p}$ *uniformly in* $\mu \in (0,1]$.

Exercise 4.2 (♠)**.** *Prove the above lemma. Hint for Item (D): Identify* $\mathbb{T}$ *with* $[-\pi,\pi)^d \subset \mathbb{R}^d$. *Write* $\mathbf{T}\tilde{K}_\mu = \tilde{K}_\mu + (\mathbf{T}\tilde{K}_\mu - \tilde{K}_\mu)$. *Use the fact that* $\tilde{K}_1(x) \lesssim \exp(-|x|)$ *uniformly in* $x \in \mathbb{R}^d$ *to show that* $\|\mathbf{T}\tilde{K}_\mu - \tilde{K}_\mu\|_{L^p(\mathbb{T})} \lesssim [\mu]^q$ *for any* $p \in [1,\infty]$ *and* $q > 0$.

5 Uniform bounds for cumulants

In this section we establish uniform bounds for the joint cumulants of the effective force coefficients. The proof is by induction and is based on the flow equation for the cumulants derived in Lemma 4.14. The cumulants that coincide with the expected values of the coefficients $(F^{i,1}_{\kappa,\mu})_{i\in\{1,\ldots,i_\sharp\}}$ will require a special treatment. More specifically, to prove bounds for $\mathbb{E}F^{i,1}_{\kappa,\mu}$ we will decompose $\mathbb{E}F^{i,1}_{\kappa,\mu}(x;\mathrm{d}y)$ into a local part proportional to $\delta_x(\mathrm{d}y)$ and a certain remainder using the maps introduced in the following definition.

Definition 5.1. For $m\in\mathbb{N}_+$ we define $\delta^{[m]}\in\mathscr{S}'(\mathbb{M}^{1+m})$ by the equality

$$\langle\delta^{[m]},\psi\otimes\varphi_1\otimes\ldots\otimes\varphi_m\rangle := \int_{\mathbb{M}}\psi(x)\varphi_1(x)\ldots\varphi_m(x)\,\mathrm{d}x$$

for all $\psi,\varphi_1,\ldots,\varphi_m\in\mathscr{S}(\mathbb{M})$. Let $\mathcal{X}^a(x;y):=(x-y)^a$ for $a\in\mathbb{N}_0^d$ and $x,y\in\mathbb{M}$ and let $V\in\mathcal{V}^1_{\mathrm{t}}$ be such that $V(x;y)=V(x-y;0)$, $V(x;y)=V(-x;-y)$ and $\mathcal{X}^aV\in\mathcal{V}^1_{\mathrm{t}}$ for all $a\in\mathbb{N}_0^d$. We define $\mathbf{I}V:=\int_{\mathbb{M}}V(x;y)\mathrm{d}y\in\mathbb{R}$. For $a\in\mathbb{N}_0^d$ we define $\mathbf{R}^aV\in\mathcal{V}^1_{\mathrm{t}}$ by the equality

$$(\mathbf{R}^aV)(x;y):=\frac{|a|}{a!}\int_0^1(1-\tau)^{|a|-1}/\tau^d\;(\mathcal{X}^aV)(x;x+(y-x)/\tau)\,\mathrm{d}\tau$$

for all $x,y\in\mathbb{M}$.

Exercise 5.1. *Prove that the following equality* $V=(\mathbf{I}V)\,\delta^{[1]}+\sum_{|a|=2}\partial^a\mathbf{R}^aV$ *holds in* $\mathscr{S}'(\mathbb{M}^2)$*, where the sum is over* $a\in\mathbb{N}_0^d$ *and* ∂^a *denotes the derivative with respect to the second argument. Show that* $|\mathbf{I}V|\leq\|V\|_{\mathcal{V}^1_{\mathrm{t}}}$ *and* $\|\mathbf{R}^aV\|_{\mathcal{V}^1_{\mathrm{t}}}\leq\|\mathcal{X}^aV\|_{\mathcal{V}^1_{\mathrm{t}}}$*. Hint: Use the integral form of the Taylor remainder.*

Remark 5.2. The proof of the following theorem uses the idea from the a very simple proof of perturbative renormalizability of QFT models given by Polchinski [Pol84] based on the renormalization group flow equation (see [Mul03] for a review). For non-perturbative applications of the flow equation see [BK87, BB21].

Theorem 5.3. *There exists a choice of the counterterms* $(c^{(i)}_\kappa)_{i\in\{1,\ldots,i_\sharp\}}$ *in Eq. (1.3) such that for all list of indices* $\mathtt{I}=((i_1,m_1,s_1,0),\ldots,(i_n,m_n,s_n,0))$ *the bound*

$$\|\tilde K_\mu^{\otimes(n+m)}*E^{\mathtt{I}}_{\kappa,\mu}\|_{\mathcal{V}^{\mathfrak{m}}_{\mathrm{t}}}\lesssim[\mu]^{\varrho_\varepsilon(\mathtt{I})-\sigma s(\mathtt{I})+d(n-1)} \tag{5.1}$$

holds uniformly in $\kappa\in(0,1]$, $\mu\in(0,1/2]$*, where* $\mathfrak{m}=(m_1,\ldots,m_n)$, $m=m_1+\ldots+m_n$.

Remark 5.4. Note that the equation we want to solve, $\varPhi_\kappa=G*F_\kappa[\varPhi_\kappa]$, is invariant under the transformations $(\varPhi_\kappa,\xi_\kappa)\mapsto-(\varPhi_\kappa,\xi_\kappa)$ and $(\varPhi_\kappa,\xi_\kappa)\mapsto(\varPhi_\kappa(-\bullet),\xi_\kappa(-\bullet))$. Using the fact that the law of ξ_κ is invariant under the transformations $\xi_\kappa\mapsto-\xi_\kappa$ and $\xi_\kappa\mapsto\xi_\kappa(-\bullet)$ one shows that $E^{\mathtt{I}}_{\kappa,\mu}=0$ unless $n(\mathtt{I})+m(\mathtt{I})\in2\mathbb{N}_0$ and $E^{\mathtt{I}}_{\kappa,\mu}$ is invariant under the inversion through the origin $0\in\mathbb{M}^{n(\mathtt{I})+m(\mathtt{I})}$.

Proof. We first note that the theorem is trivially true for all list of indices $\mathtt{I}$ such that $m(\mathtt{I}) > 3i(\mathtt{I})$ since then $E^{\mathtt{I}}_{\kappa,\mu} = 0$. The rest of the proof is by induction.

The base case: Consider a list of indices $\mathtt{I}$ such that $i(\mathtt{I}) = 0$. In this case the cumulants $E^{\mathtt{I}}_{\kappa,\mu}$ coincide with the cumulants of the white noise ξ_κ. The only non-vanishing cumulant is the covariance corresponding to $n(\mathtt{I}) = 2$, $\mathfrak{m}(\mathtt{I}) = (0,0)$, $m(\mathtt{I}) = 0$ and $s(\mathtt{I}) = 0$. It holds

$$\|\mathbb{E}(\tilde{K}_\mu * \xi_\kappa, \tilde{K}_\mu * \xi_\kappa)\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}} \leq \sup_{x_1\in\mathbb{T}} \int_{\mathbb{T}} |\mathbb{E}(\xi(x_1)\xi(\mathrm{d}x_2))| = 1.$$

This finishes the proof of the base case.

Induction step: Fix $i \in \mathbb{N}_+$ and $m \in \mathbb{N}_0$. Assume that the theorem is true for all lists of indices $\mathtt{I}$ such that either $i(\mathtt{I}) < i$, or $i(\mathtt{I}) = i$ and $m(\mathtt{I}) > m$. We shall prove the theorem for all $\mathtt{I}$ such that $i(\mathtt{I}) = i$ and $m(\mathtt{I}) = m$.

Consider the case $s(\mathtt{I}) > 0$. Then we use the flow equation for cumulants introduced in the previous section. More precisely, the bound (5.1) follows from the inductive assumption and Lemma 4.14 (B).

It remains to prove the statement for lists of indices $\mathtt{I} = ((i_1, m_1, 0), \ldots, (i_n, m_n, 0))$ such that $s(\mathtt{I}) = 0$. It follows from Def. 4.3 of the cumulants $E^{\mathtt{I}}_{\kappa,\mu}$ that

$$E^{\mathtt{I}}_{\kappa,\mu} = E^{\mathtt{I}}_{\kappa,0} + \sum_{q=1}^{n} \int_0^\mu E^{\mathtt{I}_q}_{\kappa,\eta}\,\mathrm{d}\eta, \tag{5.2}$$

where

$$\mathtt{I}_q = ((i_1, m_1, 0, 0), \ldots, (i_q, m_q, 1, 0), \ldots, (i_n, m_n, 0, 0)).$$

Note that $s(\mathtt{I}_q) = 1$, hence the bound (5.1) has already been established for $E^{\mathtt{I}_q}_{\kappa,\eta}$.

First, let us analyse the *irrelevant cumulants*, i.e. those with $\mathtt{I}$ such that

$$\varrho(\mathtt{I}) + (n(\mathtt{I}) - 1)d > 0.$$

Let us recall the non-zero force coefficients

$$F^{0,0}_\kappa = \xi_\kappa, \quad F^{1,3}_\kappa = \delta^{[3]}, \quad F^{i,1}_\kappa = c^{(i)}_\kappa\,\delta^{[1]}, \quad i \in \{1, \ldots, i_\sharp\}.$$

In particular $F^{i,m}_{\kappa,0} = F^{i,m}_\kappa$ is deterministic if $i \in \mathbb{N}_+$. If $n(\mathtt{I}) > 1$, then $E^{\mathtt{I}}_{\kappa,0}$ is a joint cumulant of a list of at least two random distributions. Since $i(\mathtt{I}) = i > 0$ one of these distributions is deterministic and the cumulant vanishes. If $n(\mathtt{I}) = 1$ and $\varrho(\mathtt{I}) > 0$, then $E^{\mathtt{I}}_{\kappa,0}$ coincides with $F^{i,m}_\kappa$ for some $i, m \in \mathbb{N}_0$ such that $\varrho(i,m) > 0$. However, $F^{i,m}_\kappa$ vanishes for $i, m \in \mathbb{N}_0$ such that $\varrho(i,m) > 0$. Hence, we conclude that $E^{\mathtt{I}}_{\kappa,0} = 0$ for all irrelevant cumulants. Using this fact and Eq. (5.2) we arrive at

$$\|\tilde{K}^{\otimes(n+m)}_\mu * E^{\mathtt{I}}_{\kappa,\mu}\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}} \leq \sum_{q=1}^{n} \int_0^\mu \|\tilde{K}^{\otimes(n+m)}_\eta * E^{\mathtt{I}_q}_{\kappa,\eta}\|_{\mathcal{V}^{\mathfrak{m}}_{\mathfrak{t}}}\,\mathrm{d}\eta,$$

which follows from the estimate proved in Exercise 5.4. Using the induction hypothesis we arrive at

$$\|\tilde{K}_\mu^{\otimes(n+m)} * E^{\mathtt{I}}_{\kappa,\mu}\|_{\mathcal{V}^{\mathtt{m}}_{\mathtt{t}}} \lesssim \int_0^\mu [\eta]^{\varrho_\varepsilon(\mathtt{I})-\sigma+d(n-1)} \lesssim [\mu]^{\varrho_\varepsilon(\mathtt{I})+d(n-1)}.$$

Note that to get the last bound we crucially used the fact that $\varrho_\varepsilon(\mathtt{I}) + d(n-1) > 0$ for sufficiently small $\varepsilon \in (0,\infty)$. This finishes the proof of the induction step for the irrelevant cumulants.

Next, let us analyse the *relevant cumulants*, i.e. those with $\mathtt{I}$ such that

$$\varrho(\mathtt{I}) + (n(\mathtt{I}) - 1)d \leq 0.$$

Note that for $\mathrm{i}(\mathtt{I}) \geq 1$ the above inequality implies that $n(\mathtt{I}) = 1$. Consequently, by Remark 5.4 the non-trivial cases are $\mathtt{I} = (1,3,0)$ or $\mathtt{I} = (i,1,0)$ with $i \in \{1,\ldots,i_\sharp\}$. Thus, we have to prove bounds for the following cumulants

$$\mathbb{E}F^{1,3}_{\kappa,\mu}, \qquad \mathbb{E}F^{i,1}_{\kappa,\mu}, \quad i \in \{1,\ldots,i_\sharp\}.$$

Recall that $F^{i,m}_{\kappa,0} = F^{i,m}_\kappa$. Since $\partial_\mu F^{1,3}_{\kappa,\mu} = 0$ we have $\mathbb{E}F^{1,3}_{\kappa,\mu} = \delta^{[3]}$. Consequently,

$$\|\tilde{K}_\mu^{\otimes 4} * \mathbb{E}F^{1,3}_{\kappa,\mu}\|_{\mathcal{V}^3_{\mathtt{t}}} = 1 \leq [\mu]^{-\varepsilon} = [\mu]^{\varrho_\varepsilon(1,3)}.$$

Let us now study the cumulants $\mathbb{E}F^{i,1}_{\kappa,\mu}$ with $i \in \{1,\ldots,i_\sharp\}$. By Exercise 5.1 the bound for the cumulant

$$E^{\mathtt{I}_1}_{\kappa,\mu} = \mathbb{E}\partial_\mu F^{i,1}_{\kappa,\mu} =: \dot{E}^i_{\kappa,\mu},$$

which follows from the induction hypothesis, implies that

$$\|\tilde{K}_\mu^{\otimes 2} * \dot{E}^i_{\kappa,\mu}\|_{\mathcal{V}^1_{\mathtt{t}}} \lesssim [\mu]^{\varrho_\varepsilon(i,1)-\sigma}, \qquad |\mathbf{I}(\tilde{K}_\eta^{\otimes 2} * \dot{E}^i_{\kappa,\eta})| \lesssim [\mu]^{\varrho_\varepsilon(i,1)-\sigma}. \tag{5.3}$$

Using Exercise 5.1 we introduce the following decomposition

$$\mathbb{E}F^{i,1}_{\kappa,\mu} = \hat{E}^i_{\kappa,\mu}\,\delta^{[1]} + \check{E}^i_{\kappa,\mu} + \tilde{E}^i_{\kappa,\mu} \in \mathscr{S}'(\mathbb{M}^2), \tag{5.4}$$

where $\hat{E}^i_{\kappa,\mu} \in \mathbb{R}$ and $\check{E}^i_{\kappa,\mu}, \tilde{E}^i_{\kappa,\mu} \in \mathscr{S}'(\mathbb{M}^2)$ are defined by the equalities

$$\begin{aligned}
&\partial_\eta \hat{E}^i_{\kappa,\eta} := \mathbf{I}(\tilde{K}_\eta^{\otimes 2} * \dot{E}^i_{\kappa,\eta}), && \hat{E}^i_{\kappa,0} := c^{(i)}_\kappa\\
&\partial_\eta \check{E}^i_{\kappa,\eta} := \textstyle\sum_{|a|=2} \partial^a \mathbf{R}^a(\tilde{K}_\eta^{\otimes 2} * \dot{E}^i_{\kappa,\eta}), && \check{E}^i_{\kappa,0} := 0,\\
&\partial_\eta \tilde{E}^i_{\kappa,\eta} := \dot{E}^i_{\kappa,\eta} - \tilde{K}_\eta^{\otimes 2} * \dot{E}^i_{\kappa,\eta}, && \tilde{E}^i_{\kappa,0} := 0
\end{aligned}$$

The motivation behind the above decomposition is that, as we shall see below, $\partial_\eta \check{E}^i_{\kappa,\eta}$ and $\partial_\eta \tilde{E}^i_{\kappa,\eta}$ satisfy the following bound

$$\|K_\mu^{\otimes 2} * \partial_\eta \check{E}^i_{\kappa,\eta}\|_{\mathcal{V}^1_{\mathtt{t}}} \vee \|K_\mu^{\otimes 2} * \partial_\eta \tilde{E}^i_{\kappa,\eta}\|_{\mathcal{V}^1_{\mathtt{t}}} \lesssim [\eta/\mu]^2\, \|\tilde{K}_\eta^{\otimes 2} * \dot{E}^i_{\kappa,\eta}\|_{\mathcal{V}^1_{\mathtt{t}}} \lesssim [\mu]^{-2}\, [\eta]^{\varrho_\varepsilon(i,1)-\sigma+2}.$$

It turns out that for $d \in \{1, \ldots, 6\}$, $\sigma \in (d/3, d/2]$ and sufficiently small $\varepsilon \in (0, \infty)$ it holds $\varrho_\varepsilon(i, 1) + 2 > 0$ for all $i \in \mathbb{N}_+$. Consequently, the RHS of the above estimate is integrable in η at $\eta = 0$. On the other hand, the bound for $\partial_\eta \hat{E}^i_{\kappa,\eta}\, \delta^{[1]}$, given in (5.3), is not integrable in η at $\eta = 0$ but this contribution is local and, as we will prove in a moment, $\hat{E}^i_{\kappa,\mu} \in \mathbb{R}$ can be bounded by making a suitable choice of the counterterm $c^{(i)}_\kappa \in \mathbb{R}$.

Let us first study the local term $\hat{E}^i_{\kappa,\mu}\, \delta^{[1]}$. We start by fixing the counterterm to be

$$c^{(i)}_\kappa := - \int_0^{1/2} \mathbf{I}(\tilde{K}^{\otimes 2}_\eta * \dot{E}^i_{\kappa,\eta})\, \mathrm{d}\eta.$$

Then it holds

$$\hat{E}^i_{\kappa,\mu} = c^{(i)}_\kappa + \int_0^\mu \mathbf{I}(\tilde{K}^{\otimes 2}_\eta * \dot{E}^i_{\kappa,\eta})\, \mathrm{d}\eta = - \int_\mu^{1/2} \mathbf{I}(\tilde{K}^{\otimes 2}_\eta * \dot{E}^i_{\kappa,\eta})\, \mathrm{d}\eta.$$

Consequently, since $\varrho_\varepsilon(i, 1) < 0$ for $i \in \{1, \ldots, i_\sharp\}$ we obtain

$$|\hat{E}^i_{\kappa,\mu}| \lesssim \int_\mu^{1/2} [\eta]^{\varrho_\varepsilon(i,1)-\sigma}\, \mathrm{d}\eta \lesssim [\mu]^{\varrho_\varepsilon(i,1)}.$$

This implies the desired bound

$$\|\tilde{K}^{\otimes 2}_\mu * (\hat{E}^i_{\kappa,\mu}\, \delta^{[1]})\|_{\mathcal{V}^1_{\mathfrak{t}}} \lesssim [\mu]^{\varrho_\varepsilon(i,1)}.$$

Next, let us proceed to the estimates for the non-local terms $\check{E}^i_{\kappa,\eta}$ and $\tilde{E}^i_{\kappa,\eta}$. Using the boundary condition $\check{E}^i_{\kappa,0} = 0$ and the bound $\|\partial^a \tilde{K}_\mu\|_{\mathcal{K}} \lesssim [\mu]^{-|a|}$ proved in Lemma 4.19 (A) we estimate

$$\|\tilde{K}^{\otimes 2}_\mu * \check{E}^i_{\kappa,\mu}\|_{\mathcal{V}^1_{\mathfrak{t}}} \lesssim [\mu]^{-2} \sup_{|a|=2} \int_0^\mu \|\mathbf{R}^a(\tilde{K}^{\otimes 2}_\eta * \dot{E}^i_{\kappa,\eta})\|_{\mathcal{V}^1_{\mathfrak{t}}}\, \mathrm{d}\eta.$$

By Exercise 5.1 the RHS is bounded up to a constant by

$$\sup_{|a|=2}\, [\mu]^{-2} \int_0^\mu \|\mathcal{X}^a(\tilde{K}^{\otimes 2}_\eta * \dot{E}^i_{\kappa,\eta})\|_{\mathcal{V}^1_{\mathfrak{t}}}\, \mathrm{d}\eta$$

Recall that by Lemma 3.18 there exists $c \in (0, \infty)$ such that for all $\eta \in (0, 1/2]$ the coefficient $\partial_\eta F^{i,1}_{\kappa,\eta}$ as well as its expected value $\dot{E}^i_{\kappa,\eta} = \mathbb{E}\partial_\eta F^{i,1}_{\kappa,\eta}$ are supported in the set

$$\{(x, y) \in \mathbb{M}^2 \,|\, |x - y| \le c\, [\eta]\}.$$

Pretending that there exist $c \in (0, \infty)$ such that for all $\eta \in (0, 1/2]$ the kernel $\tilde{K}_\eta$ is supported in

$$\{x \in \mathbb{M} \,|\, |x| < c\, [\eta]\}$$

we obtain that $\tilde{K}^{\otimes 2}_\eta * \dot{E}^i_{\kappa,\eta}$ has the same support property as $\dot{E}^i_{\kappa,\eta}$. Consequently,

$$\|\mathcal{X}^a(\tilde{K}^{\otimes 2}_\eta * \dot{E}^i_{\kappa,\eta})\|_{\mathcal{V}^1_{\mathfrak{t}}} \lesssim [\eta]^2\, \|\tilde{K}^{\otimes 2}_\eta * \dot{E}^i_{\kappa,\eta}\|_{\mathcal{V}^1_{\mathfrak{t}}}.$$

Since the kernel $\tilde{K}_\eta$ does not have the above-mentioned support property the rigorous proof of the above estimate is slightly more complicated and is given in Lemma 5.6 below. Taking into account the induction hypothesis, we obtain the bound

$$\|\tilde{K}_\mu^{\otimes 2} * \hat{E}^i_{\kappa,\mu}\|_{\mathcal{V}^1_t} \lesssim [\mu]^{-2} \int_0^\mu [\eta]^2 \, \|\tilde{K}_\eta^{\otimes 2} * \dot{E}^i_{\kappa,\eta}\|_{\mathcal{V}^1_t} \, \mathrm{d}\eta \lesssim [\mu]^{-2} \int_0^\mu [\eta]^{\varrho_\varepsilon(i,1)+2} \, \mathrm{d}\eta.$$

Similarly, by Exercise 5.5 we have

$$\|\tilde{K}_\mu^{\otimes 2} * \tilde{E}^i_{\kappa,\mu}\|_{\mathcal{V}^1_t} \lesssim [\mu]^{-2} \int_0^\mu [\eta]^2 \, \|\tilde{K}_\eta^{\otimes 2} * \dot{E}^i_{\kappa,\eta}\|_{\mathcal{V}^1_t} \, \mathrm{d}\eta \lesssim [\mu]^{-2} \int_0^\mu [\eta]^{\varrho_\varepsilon(i,1)+2} \, \mathrm{d}\eta.$$

Finally, we use the fact that $\varrho_\varepsilon(i,1)+2>0$ to show

$$[\mu]^{-2} \int_0^\mu [\eta]^{\varrho_\varepsilon(i,1)+2} \, \mathrm{d}\eta \lesssim [\mu]^{\varrho_\varepsilon(i,1)},$$

which concludes the proof. □

Exercise 5.2. *Verify Eq. (5.4) using Remark 5.4.*

Exercise 5.3. *Check that in the Da Prato-Debussche regime corresponding to $\sigma \in (2d/5, d/2]$ the above proof simplifies as the use of the maps $\mathbf{I}$ and $\mathbf{R}$ is redundant.*

Remark 5.5 (♠). Observe that the counterterms $(c_\kappa^{(i)})_{i\in\{1,\ldots,i_\sharp\}}$ are implicitly fixed by the following renormalization conditions

$$\int_{\mathbb{M}} \mathbb{E}F^{i,1}_{\kappa,\mu=1/2}(x;\mathrm{d}y) = 0, \qquad i \in \{1,\ldots,i_\sharp\}.$$

The procedure of fixing the counterterms works because of the property of the effective force coefficients mentioned in Remark 3.8.

Exercise 5.4 (♠). *Let $n \in \mathbb{N}_+$, $\mathfrak{m} = (m_1,\ldots,m_n) \in \mathbb{N}_0^n$ and $m = m_1 + \ldots + m_n$. Using Exercise 2.1 (3) prove that for all $0 \leq \eta \leq \mu \leq 1$ and $V \in \mathcal{V}^{\mathfrak{m}}_t$ it holds*

$$\|\tilde{K}_\mu^{\otimes(n+m)} * V\|_{\mathcal{V}^{\mathfrak{m}}_t} \leq \|\tilde{K}_\eta^{\otimes(n+m)} * V\|_{\mathcal{V}^{\mathfrak{m}}_t}.$$

Exercise 5.5 (♠). *Show that it holds*

$$\|(\tilde{K}_\mu^{\otimes 2} - \tilde{K}_\mu^{\otimes 2} * \tilde{K}_\eta^{\otimes 2}) * V\|_{\mathcal{V}^1_t} \lesssim [\eta/\mu]^2 \, \|\tilde{K}_\eta^{\otimes 2} * V\|_{\mathcal{V}^1_t}$$

uniformly over $\mu,\eta \in (0,1]$ and $V \in \mathcal{V}^1_t$. Hint: Let $\hat{K}_\mu \in \mathcal{K}$ be the solution of $\hat{\mathbf{P}}_\mu \hat{K}_\mu = \delta_0$, where $\hat{\mathbf{P}}_\mu := (1-[\mu]^2\Delta)$. Verify that $\hat{K}_\mu - \hat{K}_\mu * \hat{K}_\eta = [\eta/\mu]^2 \, (\hat{K}_\eta - \hat{K}_\eta * \hat{K}_\mu)$.

Lemma 5.6 (♠). *Fix some $m \in \mathbb{N}_+$ and $c \in \mathbb{R}$. There exists $C \in (0,\infty)$ such that if for some $\mu \in (0,1/2]$ and $V \in \mathscr{S}'(\mathbb{T}\times\mathbb{M}^m)$ it holds*

$$\operatorname{supp} V \subset \{(x,y_1,\ldots,y_m) \,|\, |x-y_1| \vee \ldots \vee |x-y_m| \leq c\,[\mu]\}$$

then

$$\|\mathcal{X}^a(\tilde{K}_\mu^{\otimes(1+m)} * V)\|_{\mathcal{V}^m_t} \leq C\,[\mu]^{|a|}\,\|\tilde{K}_\mu^{\otimes(1+m)} * V\|_{\mathcal{V}^m_t}.$$

Remark 5.7. Note that the lemma would be obvious if $\tilde{K}_\mu \in C(\mathbb{M})$ was compactly supported in a ball $\{x \in \mathbb{M} \,|\, |x| < c\,[\mu]\}$ for some $c \in (0,\infty)$ independent of $\mu \in (0,1/2]$. Even though this is not exactly true, one can easily proved the above result by leveraging the fact that the kernel $\tilde{K}_\mu$ decays exponentially with the rate $1/[\mu]$.

Proof (♠). A simple rescaling reduces the proof to the case $\mu = 1$. Let $v \in C^\infty(\mathbb{M})$ be such that $\operatorname{supp} v \subset \{x \in \mathbb{M} \,|\, |x| < 1\}$ and $v = 1$ on $\{x \in \mathbb{M} \,|\, |x| \leq 1/2\}$. For $\tau \in [1,\infty)$ let $L_\tau(x) := \tilde{K}_1(x)\, v(x/\tau)$. It holds

$$\begin{aligned}\|\mathcal{X}^a(\tilde{K}_1^{\otimes(1+m)} * V)\|_{\mathcal{V}^m_{\mathfrak{t}}} &\leq \|\mathcal{X}^a(L_1^{\otimes(1+m)} * V)\|_{\mathcal{V}^m_{\mathfrak{t}}} + \int_1^\infty \|\mathcal{X}^a \partial_\tau (L_\tau^{\otimes(1+m)} * V)\|_{\mathcal{V}^m_{\mathfrak{t}}}\, \mathrm{d}\tau \\ &\lesssim \|L_1^{\otimes(1+m)} * V\|_{\mathcal{V}^m_{\mathfrak{t}}} + \int_1^\infty \tau^{|a|}\, \|\partial_\tau (L_\tau^{\otimes(1+m)} * V)\|_{\mathcal{V}^m_{\mathfrak{t}}}\, \mathrm{d}\tau,\end{aligned}$$

where to get the last estimate we used the fact that $\operatorname{supp} L_\tau \subset \{x \in \mathbb{M} \,|\, |x| < \tau\}$. Next, we observe that

$$L_\tau = \tilde{\mathbf{P}}_1 L_\tau * \tilde{K}_1, \qquad \partial_\tau L_\tau = \tilde{\mathbf{P}}_1 \partial_\tau L_\tau * \tilde{K}_1$$

and $\|\tilde{\mathbf{P}}_1 L_\tau\|_{\mathcal{K}} \lesssim 1$ and $\|\tilde{\mathbf{P}}_1 \partial_\tau L_\tau\|_{\mathcal{K}} \lesssim \tau^{-N}$ uniformly in $\tau \in [1,\infty)$ for any $N \in \mathbb{N}_+$ because of the exponential decay of the kernel $\tilde{K}_1$. Consequently, we have

$$\|\mathcal{X}^a(\tilde{K}_1^{\otimes(1+m)} * V)\|_{\mathcal{V}^m_{\mathfrak{t}}} \lesssim \|\tilde{K}_1^{\otimes(1+m)} * V\|_{\mathcal{V}^m_{\mathfrak{t}}} + \int_1^\infty \tau^{|a|-N}\, \|\tilde{K}_1^{\otimes(1+m)} * V\|_{\mathcal{V}^m_{\mathfrak{t}}}\, \mathrm{d}\tau,$$

which finishes the proof. □

Exercise 5.6 (♠). *Prove that with the choice of the counterterms $(c_\kappa^{(i)})_{i\in\{1,\ldots,i_\sharp\}}$ made in the proof of Theorem 5.3 for all list of indices $\mathtt{I} = ((i_1,m_1,s_1,r_1),\ldots,(i_n,m_n,s_n,r_n))$ the bound*

$$\|\tilde{K}_\mu^{\otimes(n+m)} * E^{\mathtt{I}}_{\kappa,\mu}\|_{\mathcal{V}^{\mathtt{m}}_{\mathfrak{t}}} \lesssim [\kappa]^{(\varepsilon-\sigma)r(\mathtt{I})}\, [\mu]^{\varrho_\varepsilon(\mathtt{I})-\sigma s(\mathtt{I})+d(n-1)}$$

holds uniformly in $\kappa \in (0,1]$, $\mu \in (0,1/2]$. Hint: First generalize appropriately Lemma 4.13. Then follow the proof of Theorem 5.3. To prove the base case of the induction verify that

$$\sup_{x\in\mathbb{T}} \int_{\mathbb{T}} |\mathbb{E}((\tilde{K}_\mu * \partial_\kappa^{r_1}\xi_\kappa)(x)\, (\tilde{K}_\mu * \partial_\kappa^{r_2}\xi_\kappa)(y))|\, \mathrm{d}y \lesssim [\kappa]^{(\varepsilon-\sigma)(r_1+r_2)}\, [\mu]^{-2\varepsilon}.$$

A Kolmogorov-type argument

The following lemma proves the stochastic estimates for the enhanced noise assuming bounds for the cumulants of the effective force, which were established in Theorem 5.3 and Exercise 5.6. The idea is to first infer the bounds for the moments of the effective force coefficients from the bounds for the cumulants and subsequently use a Kolmogorov-type argument.

Lemma A.1. *Fix $n \in 2\mathbb{N}_+$ such that $d/n < \varepsilon$ and $i, m \in \mathbb{N}_0$ such that $\varrho(i,m) \leq 0$. For $s, r \in \{0,1\}$ we define the list of indices $\mathtt{I} \equiv \mathtt{I}(s,r) = ((i,m,s,r),\ldots,(i,m,s,r))$, $n(\mathtt{I}) = n$. Assume that for $s, r \in \{0,1\}$ the following bound*

$$\|\tilde{K}_\mu^{\otimes(n+nm)} * E^{\mathtt{I}}_{\kappa,\mu}\|_{\mathcal{V}_{\mathrm{t}}^{\mathfrak{m}(\mathtt{I})}} \lesssim [\mu]^{\varrho_\varepsilon(\mathtt{I})-\sigma s(\mathtt{I})+d(n-1)}$$

holds uniformly in $\kappa \in (0,1]$ and $\mu \in (0,1/2]$. Then there exists a random variable $R \in [1,\infty]$ such that $\mathbb{E}R^n < \infty$ and the following bound

$$\|K_\mu^{\otimes(1+m)} * F^{i,m}_{\kappa,\mu}\|_{\mathcal{V}^m} \leq R\,[\mu]^{\varrho_{3\varepsilon}(i,m)}.$$

holds for all $\kappa \in (0,1]$ and $\mu \in (0,1/2]$.

Proof (♠). By the assumption, Exercise A.1, the relation between the expectation of a product of random variables and their joint cumulants expressed by Eq. (4.4) as well as the equalities $\varrho_\varepsilon(\mathtt{I}) = n\varrho_\varepsilon(i,m)$ and $s(\mathtt{I}) = ns$ we have

$$\mathbb{E}(\tilde{K}_\mu^{\otimes(1+m)} \otimes \tilde{K}_\mu^{\otimes(1+m)} * \partial_\mu^s \partial_\kappa^r F^{i,m}_{\kappa,\mu}(x,y_1,\ldots,y_m))^n \lesssim [\kappa]^{n(\varepsilon-\sigma)r}[\mu]^{n(\varrho_\varepsilon(i,m)-\sigma s-dm)}$$

uniformly in $\kappa \in (0,1]$ and $\mu \in (0,1/2]$ and $x, y_1, \ldots, y_m \in \mathbb{M}$. Using the Fubini theorem and the argument from the proof of Lemma 5.6 one shows that

$$\mathbb{E}\|\tilde{K}_\mu^{\otimes(1+m)} \otimes \tilde{K}_\mu^{\otimes(1+m)} * \partial_\mu^s \partial_\kappa^r F^{i,m}_{\kappa,\mu}\|^n_{L^n(\mathbb{T}\times\mathbb{M}^m)} \lesssim [\kappa]^{n(\varepsilon-\sigma)r}[\mu]^{n(\varrho_\varepsilon(i,m)-\sigma s-dm)+dm}.$$

Taking into account the fact that $K_\mu = \tilde{K}_\mu * \tilde{K}_\mu * \tilde{K}_\mu$ we conclude by Lemma A.2 that

$$\mathbb{E}\|K_\mu^{\otimes(1+m)} * \partial_\mu^s \partial_\kappa^r F^{i,m}_{\kappa,\mu}\|^n_{L^\infty(\mathbb{T}\times\mathbb{M}^m)} \lesssim [\kappa]^{n(\varepsilon-\sigma)r}[\mu]^{n(\varrho_\varepsilon(i,m)-\sigma s-dm)-d},$$

Employing again the strategy from the proof of Lemma 5.6 we obtain

$$\mathbb{E}\|K_\mu^{\otimes(1+m)} * \partial_\mu^s \partial_\kappa^r F^{i,m}_{\kappa,\mu}\|^n_{\mathcal{V}^m} \lesssim [\kappa]^{n(\varepsilon-\sigma)r}[\mu]^{n(\varrho_\varepsilon(i,m)-\sigma s)-d}.$$

Using Lemma 4.19 (B) we arrive at

$$\mathbb{E}\|\partial_\mu^s \partial_\kappa^r (K_\mu^{\otimes(1+m)} * F^{i,m}_{\kappa,\mu})\|^n_{\mathcal{V}^m} \lesssim [\kappa]^{n(\varepsilon-\sigma)r}[\mu]^{n(\varrho_\varepsilon(i,m)-\sigma s)-d}.$$

Finally, we apply the result stated in Exercise A.2 with

$$\zeta_{\kappa,2\mu} = K_\mu^{\otimes(1+m)} * F^{i,m}_{\kappa,\mu}, \qquad \rho = \varrho_\varepsilon(i,m) - 2\varepsilon.$$

Since $F^{i,m}_{\kappa,\mu} = 0$ for $m > 3i$ and $\rho \geq \varrho_{3\varepsilon}(i,m)$ for $m \leq 3i$ this finishes the proof. □

Exercise A.1 (♠). *Let $n \in \mathbb{N}_+$, $\mathfrak{m} = (m_1,\ldots,m_n) \in \mathbb{N}_0^n$ and $m = m_1 + \ldots + m_n$. Show that it holds*

$$\|\tilde{K}_\mu^{\otimes(n+m)} * V\|_{L^\infty(\mathbb{M}^{n+m})} \lesssim [\mu]^{-d(n+m-1)}\,\|V\|_{\mathcal{V}_{\mathrm{t}}^{\mathfrak{m}}}$$

uniformly in $\mu \in (0,1]$ and $V \in \mathcal{V}_{\mathrm{t}}^{\mathfrak{m}}$. Hint: Use Lemma 4.19 (C) and (D) with $p = \infty$.

Lemma A.2 (♠). *Let $n\in 2\mathbb{N}_+$, $m\in\mathbb{N}_0$. There exists a constant $C>0$ such that for all random fields $\zeta\in L^\infty(\mathbb{T}\times\mathbb{M}^m)$ and $\mu\in(0,1]$ it holds*

$$\mathbb{E}\|\tilde{K}_\mu^{\otimes(1+m)}\ast\zeta\|^n_{L^\infty(\mathbb{T}\times\mathbb{M}^m)}\leq C\,[\mu]^{-d(1+m)}\,\mathbb{E}\|\zeta\|^n_{L^n(\mathbb{T}\times\mathbb{M}^m)}$$

Proof. Note that

$$\tilde{K}_\mu^{\otimes(1+m)}\ast\zeta=(\mathbf{T}\tilde{K}_\mu\otimes\tilde{K}_\mu^{\otimes m})\star\zeta$$

where $\star$ is the convolution in $\mathbb{T}\times\mathbb{M}^m$ and $\mathbf{T}\tilde{K}_\mu$ is the periodization of $\tilde{K}_\mu$ (see Def. 4.16). Using the Young inequality for convolutions we obtain

$$\mathbb{E}\|\tilde{K}_\mu^{\otimes(1+m)}\ast\zeta\|^n_{L^\infty(\mathbb{T})}\leq\|\mathbf{T}\tilde{K}_\mu\|^n_{L^{n/(n-1)}(\mathbb{T})}\,\|\tilde{K}_\mu\|^{mn}_{L^{n/(n-1)}(\mathbb{M})}\,\mathbb{E}\|\zeta\|^n_{L^n(\mathbb{T})}.$$

The lemma follows now from Lemma 4.19 (C), (D). □

Exercise A.2 (♠). *Fix $n\in\mathbb{N}_+$, $m\in\mathbb{N}_0$. Prove that there exists a universal constant $c>0$ such that if*

$$\mathbb{E}\|\partial_\mu^s\partial_\kappa^r\zeta_{\kappa,\mu}\|^n_{\mathcal{V}^m}\leq C\,[\kappa]^{n(-\sigma r+\varepsilon r)}[\mu]^{n(\rho-\sigma s+\varepsilon s)},\qquad s,r\in\{0,1\},\ \kappa,\mu\in(0,1],$$

for some differentiable random function $\zeta:(0,1]^2\to\mathcal{V}^m$ and $C>0$, $\rho\leq 0$, then

$$\mathbb{E}\Big(\sup_{\kappa,\mu\in(0,1]}[\mu]^{-n\rho}\,\|\zeta_{\kappa,\mu}\|^n_{\mathcal{V}^m}\Big)\leq c\,C.$$

Hint: Use the estimate

$$[\mu]^{-\rho}\,\|\zeta_{\kappa,\mu}\|_{\mathcal{V}^m}\leq\|\zeta_{1,1}\|_{\mathcal{V}^m}+\int_\mu^1[\eta]^{-\rho}\,\|\partial_\eta\zeta_{1,\eta}\|_{\mathcal{V}^m}\,\mathrm{d}\eta+\int_\mu^1\int_\kappa^1[\eta]^{-\rho}\,\|\partial_\eta\partial_\nu\zeta_{\nu,\eta}\|_{\mathcal{V}^m}\,\mathrm{d}\nu\mathrm{d}\eta$$

and the Minkowski inequality.

Exercise A.3 (♠). *Under the assumptions of Lemma A.1 prove that for all $\mu\in(0,1/2]$ there exist random $F^{i,m}_{0,\mu}\in\mathscr{S}'(\mathbb{M}^{1+m})$ such that*

$$\lim_{\kappa\searrow 0}\sup_{\mu\in(0,1/2]}[\mu]^{-\varrho_{3\varepsilon}(i,m)}\|K_\mu^{\otimes(1+m)}\ast(F^{i,m}_{0,\mu}-F^{i,m}_{\kappa,\mu})\|_{\mathcal{V}^m}=0$$

almost surely. Hint: Use the argument from the proof of Lemma A.1.

Remark A.3 (♠). Using the result stated in the above exercise and the fact that every $\kappa\in(0,1]$ the coefficients $((0,1/2]\ni\mu\mapsto F^{i,m}_{\kappa,\mu}\in\mathcal{V}^m)_{i\in\{0,\dots,i_\flat\},m\in\mathbb{N}_0}$ satisfy the flow equation (3.7) one verifies that the coefficients $((0,1/2]\ni\mu\mapsto F^{i,m}_{0,\mu}\in\mathscr{S}'(\mathbb{M}^{1+m}))_{i\in\{0,\dots,i_\flat\},m\in\mathbb{N}_0}$ are almost surely continuously differentiable and satisfy the flow equation.

B Relation to original equation and convergence

The lemma below specifies the conditions under which a fixed point of the map $\mathbf{Q}$, introduced in Lemma 2.15, corresponds to a solution of $\varPhi_\kappa = G * F_\kappa[\varPhi_\kappa]$ for $\kappa \in (0,1]$.

Remark B.1 (♠). In the case of parabolic equations a natural way to proceed would be to first construct the maximal classical solution of the mild formulation of the SPDE with regularized noise ξ_κ by patching together solutions constructed in short time intervals using the contraction principle. Then it is straightforward to verify that the maximal solution restricted to a short time interval is a fixed point of an analog of the map $\mathbf{Q}$ introduced in Lemma 2.15. In the case of elliptic equations the classical solution of $\varPhi_\kappa = G * F_\kappa[\varPhi_\kappa]$ constructed using the contraction principle exists only for $\lambda \in [-\lambda_{\star,\kappa}, \lambda_{\star,\kappa}]$, where $\lim_{\kappa\searrow 0} \lambda_{\star,\kappa} = 0$. Because of this, we proceed in opposite direction and argue that for some $\lambda_\star \in (0,\infty)$ independent of $\kappa \in (0,1]$ and all $\lambda \in [-\lambda_\star, \lambda_\star]$ and $\kappa \in (0,1]$ a fixed point of the map $\mathbf{Q}$, under some extra assumptions, is also a solution of $\varPhi_\kappa = G * F_\kappa[\varPhi_\kappa]$. It seems impossible to construct the above $\varPhi_\kappa$ for all $\kappa \in (0,1]$ by directly solving $\varPhi_\kappa = G * F_\kappa[\varPhi_\kappa]$.

Lemma B.2 (♠). *Fix $\kappa \in (0,1]$. Assume that the family of functionals $(F_{\kappa,\mu})_{\mu\in[0,1]}$ of polynomial type depends continuously on $\mu \in [0,1]$ and is piecewise continuously differentiable in μ for $\mu \in (0,1]$. Moreover, suppose that $F_{\kappa,0} = F_\kappa$ and for some $R \in [1,\infty)$ and $m_\flat \in \mathbb{N}_0$ it holds*

$$\|\mathrm{D}^k(\partial_\mu^s F_{\kappa,\mu})[\varphi]\cdot\psi^{\otimes k}\| \leq R\,\|\psi\|^k\,(1+\|\varphi\|)^{m_\flat}$$

for all $k \in \mathbb{N}_0$, $s \in \{0,1\}$, $\mu \in [0,1]$, $\varphi,\psi \in C(\mathbb{T})$. Let the family of functionals $(H_{\kappa,\mu})_{\mu\in(0,1]}$ be defined by Eq. (2.6). If a continuous and bounded function

$$(0,1] \ni \mu \mapsto (\tilde{\varPhi}_{\kappa,\mu}, \tilde{\zeta}_{\kappa,\mu}) \in C(\mathbb{T}) \times C(\mathbb{T}) \tag{B.1}$$

is the fixed point of the map $\mathbf{Q}_\kappa$ defined by Eq. (2.14) in terms of $\tilde{F}_\mu[\varphi] := K_\mu * F_{\kappa,\mu}[K_\mu * \varphi]$ and $\tilde{H}_\mu[\varphi] := K_\mu * H_{\kappa,\mu}[K_\mu * \varphi]$, then the limit $\lim_{\mu\searrow 0} K_\mu * \tilde{\varPhi}_{\kappa,\mu} =: \varPhi_\kappa$ exists in $C(\mathbb{T})$ and satisfies the equation $\varPhi_\kappa = G * F_\kappa[\varPhi_\kappa]$.

Sketch of the proof. Using $\mathbf{P}_\mu K_\mu = \delta_0$ and $K_{\eta,\mu} * K_\mu = K_\eta$ we show that

$$(0,1] \ni \mu \mapsto (\varPhi_{\kappa,\mu}, \zeta_{\kappa,\mu}) := (K_\mu * \tilde{\varPhi}_{\kappa,\mu}, \mathbf{P}_\mu \tilde{\zeta}_{\kappa,\mu}) \in C(\mathbb{T}) \times \mathscr{S}'(\mathbb{M})$$

satisfies the system of equations

$$\begin{cases} \varPhi_{\kappa,\mu} = -\int_\mu^1 \dot{G}_\eta * (F_{\kappa,\eta}[\varPhi_{\kappa,\eta}] + \zeta_{\kappa,\eta})\,\mathrm{d}\eta \\ \zeta_{\kappa,\mu} = -\int_0^\mu (H_{\kappa,\eta}[\varPhi_{\kappa,\eta}] + \mathrm{D}F_{\kappa,\eta}[\varPhi_{\kappa,\eta}]\cdot(\dot{G}_\eta * \zeta_{\kappa,\eta}))\,\mathrm{d}\eta \end{cases} \tag{B.2}$$

Using the assumptions about the effective force and the fixed point we show that the integrands above are continuous and bounded. Hence, we conclude that

$$(0,1] \ni \mu \mapsto (\varPhi_{\kappa,\mu}, \zeta_{\kappa,\mu}) \in C(\mathbb{T}) \times C(\mathbb{T})$$

and the above function is bounded, differentiable and has a limit at $\mu = 0$. Next, we show that $\partial_\eta(F_{\kappa,\eta}[\Phi_{\kappa,\eta}] + \zeta_{\kappa,\eta}) = 0$ by following the argument from the beginning of Sec. 2. As a result, we obtain $F_{\kappa,\eta}[\Phi_{\kappa,\eta}] + \zeta_{\kappa,\eta} = F_\kappa[\Phi_\kappa]$. Consequently, the first of the equations (B.2) implies that Φ_κ satisfies the equation $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$. □

In order to use the above lemma one has to verify that for $\kappa \in (0,1]$ the unique fixed point $(\tilde{\Phi}_{\kappa,\bullet}, \tilde{\zeta}_{\kappa,\bullet})$ of the map $\mathbf{Q}_\kappa : \mathscr{B}_R \to \mathscr{B}_R$ constructed in Lemma 2.15 is such that the map (B.1) is bounded. The proof of this fact is the subject of the following exercise.

Exercise B.1 (♠). *Fix $\kappa \in (0,1]$. Assume that $\|\xi_\kappa\| \lesssim 1$ and $|c_\kappa^{(i)}| \lesssim 1$ for all $i \in \{1, \ldots, i_\sharp\}$. Using the flow equation (3.7) show by induction that*

$$\|F^{i,m}_{\kappa,\mu}\| \lesssim 1 \wedge [\mu]^{\sigma(m-3)/2}$$

uniformly in $\mu \in (0,1]$, where the constants of proportionality depend only on $i \in \{0, \ldots, i_\flat\}$ and $m \in \{0, \ldots, 3i\}$. Conclude that there exists $R \in [1, \infty)$ such that for every $\delta \in [-\sigma/2, 0]$ it holds

$$[\mu]^{-3\delta}\,\|F_{\kappa,\mu}[\varphi]\| \le R\,(1 + [\mu]^{-\delta}\,\|\varphi\|)^{3i_\flat}, \tag{B.3}$$

where $F_{\kappa,\bullet}$ is the effective force defined by Eq. (3.9). Let $(\tilde{\Phi}_{\kappa,\bullet}, \tilde{\zeta}_{\kappa,\bullet}) \in \mathscr{B}_R$ be the unique fixed point of $\mathbf{Q}_\kappa$ constructed in Lemma 2.15 applied with $\alpha \in (-\sigma/2, \sigma - d/2)$. Using iteratively the bound (B.3) and the fact that $(\tilde{\Phi}_{\kappa,\bullet}, \tilde{\zeta}_{\kappa,\bullet})$ is a fixed point of $\mathbf{Q}_\kappa$ show that $\tilde{\Phi}_{\kappa,\bullet} : (0,1] \to C(\mathbb{T})$ is bounded.

To conclude the proof of Theorem 1.1 one has to solve the following exercise in which we study the existence of the limit $\kappa \searrow 0$.

Exercise B.2 (♠). *For $\kappa \in [0,1]$ let*

$$(\tilde{F}_\mu)_{\mu\in(0,1]} \equiv (\tilde{F}_{\kappa,\mu})_{\mu\in(0,1]}, \qquad (\tilde{H}_\mu)_{\mu\in(0,1]} \equiv (\tilde{H}_{\kappa,\mu})_{\mu\in(0,1]} \tag{B.4}$$

be families of functionals such that:

(1) for all $\kappa \in (0,1]$ the assumptions of Lemma B.2 are satisfied,

(2) for all $\kappa \in [0,1]$ the assumptions of Lemma 2.15 are satisfied,

(3) for all $\kappa \in [0,1]$ there exists $r_\kappa \in \mathbb{R}$ such that

$$[\mu]^{\sigma-\alpha}\,\|\mathrm{D}^k(\tilde{F}_{0,\mu} - \tilde{F}_{\kappa,\mu})[\varphi]\cdot\psi^{\otimes k}\| \le r_\kappa\,(\lambda^{1/3}[\mu]^{-\alpha}\,\|\psi\|)^k\,(1/2 + \lambda^{1/3}[\mu]^{-\alpha}\,\|\varphi\|)^{m_\flat},$$
$$[\mu]^{\sigma-\beta}\,\|(\tilde{H}_{0,\mu} - \tilde{H}_{\kappa,\mu})[\varphi]\| \le r_\kappa\,(1/2 + \lambda^{1/3}[\mu]^{-\alpha}\,\|\varphi\|)^{m_\flat}$$

for all $k \in \{0,1\}$, $\kappa, \mu \in (0,1]$, $\varphi, \psi \in C(\mathbb{T})$, $\lambda \in [-1,1]$ and $\lim_{\kappa\searrow 0} r_\kappa = 0$.

For $\kappa \in [0,1]$ let $\mathbf{Q}_\kappa$ be defined by Eq. (2.14) in terms of the functionals (B.4). By Assumption (2) and Lemma 2.15 for every $\lambda \in [-\lambda_\star, \lambda_\star]$ and $\kappa \in [0,1]$ the map $\mathbf{Q}_\kappa : \mathscr{B}_R \to \mathscr{B}_R$ is well defined and is a contraction with the Lipschitz constant less than $1/2$. For $\kappa \in [0,1]$ let $(\tilde{\Phi}_{\kappa,\bullet}, \tilde{\zeta}_{\kappa,\bullet}) \in \mathscr{B}_R$ be the unique fixed point of $\mathbf{Q}_\kappa$. By Assumption (1), Lemma B.2 and Exercise B.1 the limit $\Phi_\kappa := \lim_{\mu\searrow 0}\tilde{\Phi}_{\kappa,\mu} \in C_\mathrm{b}(\mathbb{M})$ exists and solves $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$.

(i) Show that $\lim_{\kappa\searrow 0}\mathbf{Q}_\kappa[\tilde\Phi_\bullet,\tilde\zeta_\bullet]=\mathbf{Q}_0[\tilde\Phi_\bullet,\tilde\zeta_\bullet]$ *for all* $(\tilde\Phi_\bullet,\tilde\zeta_\bullet)\in\mathscr{B}_R$.

(ii) Using Item (i) conclude that $\lim_{\kappa\searrow 0}(\tilde\Phi_{\kappa,\bullet},\tilde\zeta_{\kappa,\bullet})=(\tilde\Phi_{0,\bullet},\tilde\zeta_{0,\bullet})$.

(iii) Verify the equality $K_\mu*\Phi_\kappa=K_\mu*K_\mu*\tilde\Phi_{\kappa,\mu}+(G-G_\mu)*(\tilde F_{\kappa,\mu}[\tilde\Phi_{\kappa,\mu}]+\tilde\zeta_{\kappa,\mu})$.

(iv) Prove that there exists $\Phi_0\in\mathscr{S}'(\mathbb{M})$ *such that*

$$\lim_{\kappa\searrow 0}\|\Phi_\kappa-\Phi_0\|_{\mathscr{C}^\alpha(\mathbb{M})}=\lim_{\kappa\searrow 0}\sup_{\mu\in(0,1]}[\mu]^{-\alpha}\,\|K_\mu*(\Phi_\kappa-\Phi_0)\|_{\mathscr{C}^\alpha(\mathbb{M})}=0.$$

Hints: To prove (iii) follow the proof of Lemma B.2. To prove (iv) use $\|G-G_\mu\|\lesssim[\mu]^\sigma$.

Proof of Theorem 1.1 (♠). First note that by Theorem 5.3 and Exercise A.3 the assumption of Lemma A.1 is satisfied. By Lemma A.1 and Exercise A.3 one shows along the lines of the proof of Corollary 3.17 that the families of functionals $(\tilde F_{\kappa,\mu})_{\mu\in(0,1]}$ and $(\tilde H_{\kappa,\mu})_{\mu\in(0,1]}$ defined by Eqs. (3.9), (3.2) and (3.1) fulfill all the assumptions formulated in Exercise B.2. This implies Theorem 1.1. □

C Alternative proof of stochastic estimates

In this appendix we give another proof of the stochastic estimates stated in Theorem 3.14, which is not based on the bounds for the joint cumulants of the effective force coefficients. The proof is an alternative to the proof contained in Sec. 4, Sec. 5 and Appendix A and can be read independently. The idea of the proof is to estimate separately the expected values of the coefficients and their covariances. To bound the covariances we use the following simple lemma, which is a consequence of the the support property of the decomposition of the Green function G stated in Remark 2.10.

Remark C.1 (♠). In order to generalize the proof of the stochastic estimates presented in this section to scale decompositions of the Green function G that do not posses the support property stated in Remark 2.10 one would have to estimate the Malliavin derivative of the effective force coefficients in an appropriate weighted space.

Lemma C.2. *For all* $i\in\{0,\ldots,i_\flat\}$ *and* $m\in\mathbb{N}_0$ *there exists* $c\in\mathbb{R}_+$ *such that for all* $s\in\{0,1\}$, $\kappa\in(0,1]$, $\mu\in[0,1/2]$ *and* $x\in\mathbb{M}$, $A\in\mathrm{Borel}(\mathbb{M}^m)$ *the random variable*

$$\partial_\mu^s F^{i,m}_{\kappa,\mu}(x;A+x)$$

is measurable with respect to the σ*-algebra generated by*

$$\{\langle\xi,\psi\rangle\,|\,\psi\in C^\infty(\mathbb{M}),\ \forall_{y\in\mathbb{M},|y-x|>c\,[\kappa\vee\mu]}\,\psi(y)=0\}.$$

Proof. Since $F^{i,m}_{\kappa,\mu}=0$ if $m>3i$, the lemma is clearly true for all $i\in\{0,\ldots,i_\flat\}$ and $m\in\mathbb{N}_0$ such that $m>3i$. To prove the lemma for $i=0$ it is enough to study $F^{0,0}_{\kappa,\mu}=\xi_\kappa=\vartheta_\kappa*\xi$. By assumption $\operatorname{supp}\vartheta_\kappa\subset\{x\in\mathbb{M}\,|\,|x|\leq[\kappa]\}$. This implies the statement. The rest of the proof is by induction and is based on the flow equation (3.7), the fact that $\dot G_\mu\subset\{x\in\mathbb{M}\,|\,|x|\leq[\mu]\}$ and the support property of the effective force coefficients stated in Lemma 3.18. □

We replace Definitions 3.10 and 2.7 by the following definitions, which are more convenient in the present setting.

Definition C.3. Let $\varepsilon\in[0,\infty)$ and $\alpha\equiv\alpha_\varepsilon:=\sigma-d/2-3\varepsilon$, $\gamma\equiv\gamma_\varepsilon:=3\sigma-d-9\varepsilon$. For $i,m\in\mathbb{N}_0$ we define

$$\varrho_\varepsilon(i,m):=\alpha_\varepsilon-\sigma-m\,\alpha_\varepsilon+i\,\gamma_\varepsilon+\varepsilon\in\mathbb{R}.$$

We omit ε if $\varepsilon=0$. We define $i_\flat,i_\sharp,i_\natural\in\mathbb{N}_+$ to be the smallest positive integers such that $\varrho(i_\flat+1,0)>0$, $\varrho(i_\sharp+1,1)>0$, $\varrho(i_\natural+1,2)>0$, respectively.

Definition C.4. Let $\mu\in[0,1]$ and $[\mu]=\mu^{1/\sigma}$. The kernel $\tilde K_\mu\in\mathcal{K}$ is the unique solution of $\tilde{\mathbf{P}}_\mu\tilde K_\mu=\delta_0$, where $\tilde{\mathbf{P}}_\mu:=(1-[\mu]^2\Delta)^{d+2}$. We define $K_\mu:=\tilde K_\mu*\tilde K_\mu\in\mathcal{K}$ and $\mathbf{P}_\mu:=\tilde{\mathbf{P}}_\mu^2$. Let $v\in C^\infty(\mathbb{R})$ such that $v(r)=1$ for $|r|\leq1$ and $v(r)=0$ for $|r|\geq2$. Define $\check K_\mu,\hat K_\mu\in\mathcal{K}$ by $\check K_\mu(x):=v(|x|/[\mu])\,\tilde K_\mu(x)$, $\hat K_\mu(x):=v(|x|/[\mu])\,K_\mu(x)$.

Remark C.5. Note that $\alpha_\varepsilon\leq0$ for all $\varepsilon\in[0,\infty)$. Moreover, $\gamma_\varepsilon>0$ for all $\varepsilon\in[0,\infty)$ in a sufficiently small neighbourhood of $\varepsilon=0$ by the condition of subcriticality. In particular, $i_\flat,i_\sharp\in\mathbb{N}_+$ are well defined. For arbitrary $\varepsilon\in(0,\infty)$ and $i,m\in\mathbb{N}_0$ such that $m\leq3i$ it holds $\varrho_\varepsilon(i,m)<\varrho(i,m)$.

To prove the bound for $\mathbb{E}F^{i,1}_{\kappa,\mu}$ we will decompose $\mathbb{E}F^{i,1}_{\kappa,\mu}(x;\mathrm{d}y)$ into a local part proportional to $\delta_x(\mathrm{d}y)$ and a certain remainder using the maps introduced in the following definition.

Definition C.6. For $m\in\mathbb{N}_+$ we define $\delta^{[m]}\in\mathscr{S}'(\mathbb{M}^{1+m})$ by the equality

$$\langle\delta^{[m]},\psi\otimes\varphi_1\otimes\ldots\otimes\varphi_m\rangle:=\int_{\mathbb{M}}\psi(x)\varphi_1(x)\ldots\varphi_m(x)\,\mathrm{d}x$$

for all $\psi,\varphi_1,\ldots,\varphi_m\in\mathscr{S}(\mathbb{M})$. Let $\mathcal{X}^a(x;y):=(x-y)^a$ for $a\in\mathbb{N}_0^d$ and $x,y\in\mathbb{M}$ and let $V\in\mathcal{V}^1$ be such that $V(x;\mathrm{d}y)=V(0;\mathrm{d}(y-x))$, $V(x;\mathrm{d}y)=V(-x;\mathrm{d}(-y))$ and $\mathcal{X}^aV\in\mathcal{V}^1$ for all $a\in\mathbb{N}_0^d$. We define $\mathbf{I}V:=\int_{\mathbb{M}}V(x;\mathrm{d}y)\in\mathbb{R}$. For $a\in\mathbb{N}_0^d$ we define $\mathbf{R}^aV\in\mathcal{V}^1$ by the equality

$$(\mathbf{R}^aV)(x;\mathrm{d}y):=\frac{|a|}{a!}\int_0^1(1-\tau)^{|a|-1}/\tau^d\;(\mathcal{X}^aV)(x;\mathrm{d}(x+(y-x)/\tau))\,\mathrm{d}\tau$$

for all $x,y\in\mathbb{M}$.

Exercise C.1. *Prove that the following equality $V=(\mathbf{I}V)\,\delta^{[1]}+\sum_{|a|=2}\partial^a\mathbf{R}^aV$ holds in $\mathscr{S}'(\mathbb{M}^2)$, where the sum is over $a\in\mathbb{N}_0^d$. Show that $|\mathbf{I}V|\leq\|V\|_{\mathcal{V}^1}$ and $\|\mathbf{R}^aV\|_{\mathcal{V}^1}\leq\|\mathcal{X}^aV\|_{\mathcal{V}^1}$. Hint: Use the integral form of the Taylor remainder.*

Lemma C.7. *Let $d\in\{1,\ldots,4\}$ and $\sigma\in(d/3,d/2]$. There exist a choice of the counterterms $(c_\kappa^{(i)})_{i\in\{1,\ldots,i_\sharp\}}$ in the expression (1.3) for the force F_κ such that for all $n\in\mathbb{N}_+$, $i\in\{0,\ldots,i_\flat\}$, $m\in\{0,\ldots,3i\}$, $r,s\in\{0,1\}$ it holds*

$$\big(\mathbb{E}\|K_\mu^{\otimes(1+m)}*\partial_\mu^s\partial_\kappa^rF^{i,m}_{\kappa,\mu}\|^n_{\mathcal{V}^m}\big)^{1/n}\lesssim[\kappa]^{(\varepsilon-\sigma)r}\,[\kappa\vee\mu]^{\varrho_\varepsilon(i,m)-(\sigma-\varepsilon)s-\sigma(3i-m-s)}\,[\mu]^{\sigma(3i-m-s)}$$

uniformly in $\kappa \in (0,1]$ and $\mu \in (0,1/2]$.

Remark C.8. For simplicity we assume that $d \in \{1,\dots,4\}$. The case $d \in \{5,6\}$ requires one minor modification. For $d > 6$ other counterterms are needed to renormalize the nonlinearity.

Exercise C.2. *Show that $\varrho(i,m) - \sigma(3i-m) \le 0$ for all $i,m \in \mathbb{N}_0$ such that $m \le 2i+1$.*

Remark C.9. Since $F^{i,m}_{\kappa,\mu} = 0$ for $m > 2i+1$ by Exercise 3.2 the above lemma together with the result of the above exercise imply that for all $i \in \{0,\dots,i_\flat\}$, $m \in \{0,\dots,3i\}$, $r,s \in \{0,1\}$ it holds

$$\big(\mathbb{E}\|K^{\otimes(1+m)}_\mu * \partial^s_\mu \partial^r_\kappa F^{i,m}_{\kappa,\mu}\|^n_{\mathcal{V}^m}\big)^{1/n} \lesssim [\kappa]^{(\varepsilon-\sigma)r}\,[\mu]^{\varrho_\varepsilon(i,m)-\sigma s}$$

uniformly in $\kappa \in (0,1]$ and $\mu \in (0,1/2]$. Using Exercise A.2 and Lemma C.12 (B) we infer that there exists a random variable $\tilde{R} \in [1,\infty]$ such that $\mathbb{E}\tilde{R}^n < \infty$ for all $n \in \mathbb{N}_+$ and it holds

$$\sup_{\kappa\in(0,1]} \sup_{\mu\in(0,1/2]} [\mu]^{-\varrho_\varepsilon(i,m)}\,\|K^{\otimes(1+m)}_\mu * F^{i,m}_{\kappa,\mu}\|_{\mathcal{V}^m} \le \tilde{R}$$

for all $i \in \{0,\dots,i_\flat\}$, $m \in \{0,\dots,3i\}$. Furthermore, for all $i \in \{0,\dots,i_\flat\}$, $m \in \{0,\dots,3i\}$, $\mu \in (0,1/2]$ there exists random $F^{i,m}_{0,\mu} \in \mathscr{S}'(\mathbb{M}^{1+m})$ such that

$$\lim_{\kappa\searrow 0} \sup_{\mu\in(0,1/2]} [\mu]^{-\varrho_\varepsilon(i,m)}\|K^{\otimes(1+m)}_\mu * (F^{i,m}_{0,\mu} - F^{i,m}_{\kappa,\mu})\|_{\mathcal{V}^m} = 0.$$

This in particular proves Theorem 3.14.

Remark C.10. Note that the equation we want to solve, $\Phi_\kappa = G * F_\kappa[\Phi_\kappa]$, is invariant under the transformations $(\Phi_\kappa,\xi_\kappa) \mapsto -(\Phi_\kappa,\xi_\kappa)$ and $(\Phi_\kappa,\xi_\kappa) \mapsto (\Phi_\kappa(-\,\bullet),\xi_\kappa(-\,\bullet))$. Using the fact that the law of ξ_κ is invariant under the transformations $\xi_\kappa \mapsto -\xi_\kappa$ and $\xi_\kappa \mapsto \xi_\kappa(-\,\bullet)$ one shows that $\mathbb{E}F^{i,m}_{\kappa,\mu} = 0$ unless m is odd and $\mathbb{E}F^{i,m}_{\kappa,\mu}$ is invariant under the inversion through the origin $0 \in \mathbb{M}^{1+m}$.

Proof of Lemma C.7. For simplicity, we give the proof only for $r = 0$. Note that by the Jensen inequality it is enough to prove the lemma for sufficiently big $n \in \mathbb{N}_+$. The proof is by induction on $i,m \in \mathbb{N}_0$.

We start with the *base case* $i = 0$. Hence, $m = 0$ and $F^{i,m}_{\kappa,\mu} = \xi_\kappa$. It holds

$$\mathbb{E}((\tilde{K}_\mu * \xi_\kappa)(x)^2) \lesssim [\mu]^{-d}$$

by Lemma 4.19 (D) applied with $p = \infty$. By the Nelson estimate we obtain

$$\mathbb{E}((\tilde{K}_\mu * \xi_\kappa)(x)^n) \lesssim [\mu]^{-nd/2}$$

for all $n \in 2\mathbb{N}_+$. Since $K_\mu = \tilde{K}_\mu * \tilde{K}_\mu$ by Kolmogorov-type estimate stated in Lemma C.13 we obtain

$$\mathbb{E}\|K_\mu * \xi_\kappa\|^n_{L^\infty(\mathbb{T})} \lesssim [\mu]^{-nd/2-d},$$

which implies the statement of the lemma provided $n\varepsilon > d$. This finishes the proof of the base case.

We proceed to the proof of the *induction step.* Fix $i_\circ \in \{1,\ldots,i_\flat\}$ and $m_\circ \in \{0,\ldots,3i\}$ and assuming that the lemma is true for all $i < i_\circ$, and $i = i_\circ$ and $m > m_\circ$. The goal is to prove the lemma for $i = i_\circ$ and $m = m_\circ$. Let us observe that the flow equation (3.7) implies

$$K_\mu^{\otimes(1+m)} \ast \partial_\mu F_{\kappa,\mu}^{i,m} = -\sum_{j=0}^{i}\sum_{k=0}^{m}(1+k)\,\mathbf{B}(\tilde{G}_\mu, K_\mu^{\otimes(2+k)} \ast F_{\kappa,\mu}^{j,1+k}, K_\mu^{\otimes(1+m-k)} \ast F_{\kappa,\mu}^{i-j,m-k}),$$

where $\tilde{G}_\mu = \mathbf{P}_\mu^2 \dot{G}_\mu$. Consequently, the statement with $s = 1$ follows from the bound stated in Exercise 3.1, Lemma 2.11 as well as the identities

$$\begin{gathered}\varrho_\varepsilon(i,m) - \sigma + \varepsilon = \varrho_\varepsilon(j,1+k) + \varrho_\varepsilon(i-j,m-k),\\ \sigma\,(3i-m-1) = \sigma\,(3j-(1+k)) + \sigma\,(3(i-j)-(m-k)).\end{gathered}$$

It remains to establish the statement with $s = 0$. In order to prove the statement of the lemma for $s = 0$ and $i, m \in \mathbb{N}_0$ such that $\varrho(i,m) > 0$ we use the identity

$$F_{\kappa,\mu}^{i,m} = F_\kappa^{i,m} + \int_0^\mu \partial_\eta F_{\kappa,\eta}^{i,m}\,\mathrm{d}\eta.$$

We first observe that $F_{\kappa,0}^{i,m} = F_\kappa^{i,m} = 0$ if $\varrho(i,m) > 0$. Next, we note that

$$\|K_\mu^{\otimes(1+m)} \ast \partial_\eta F_{\kappa,\eta}^{i,m}\|_{\mathcal{V}^m} \leq \|K_\eta^{\otimes(1+m)} \ast \partial_\eta F_{\kappa,\eta}^{i,m}\|_{\mathcal{V}^m}$$

for $\eta \leq \mu$ by Exercise 2.1 (3). The statement of the lemma with $s = 0$ follows now from the statement with $s = 1$ and the bounds

$$\big(\mathbb{E}\|K_\mu^{\otimes(1+m)} \ast F_{\kappa,\mu}^{i,m}\|_{\mathcal{V}^m}^n\big)^{1/n} \leq \int_0^\mu \big(\mathbb{E}\|K_\eta^{\otimes(1+m)} \ast \partial_\eta F_{\kappa,\eta}^{i,m}\|_{\mathcal{V}^m}^n\big)^{1/n}\,\mathrm{d}\eta$$

and

$$\int_0^\mu [\kappa \vee \eta]^{\varrho_\varepsilon(i,m)-\sigma+\varepsilon-\sigma(3i-m-1)}\,[\eta]^{\sigma(3i-m-1)}\,\mathrm{d}\eta \lesssim [\kappa \vee \mu]^{\varrho_\varepsilon(i,m)+\sigma(3i-m)}[\eta]^{\sigma(3i-m)}.$$

We stress that the above bound is valid uniformly in $\kappa \in (0,1]$, $\mu \in (0,1/2]$ if $\varrho_\varepsilon(i,m) > 0$, which holds provided $\varrho(i,m) > 0$ and $\varepsilon \in (0,\infty)$ is sufficiently small. We also note that the bound is valid uniformly in $\kappa \in (0,1]$ and $\mu \in (0,\kappa]$ irrespective of the sign of $\varrho_\varepsilon(i,m)$. Consequently, it remains to prove the statement for $s = 0$, $\varrho(i,m) \leq 0$ and $\kappa \in (0,1]$, $\mu \in (\kappa,1]$.

To proceed, let us recall the non-zero force coefficients

$$F_\kappa^{0,0} = \xi_\kappa, \quad F_\kappa^{1,3} = \delta^{[3]}, \quad F_\kappa^{1,1} = c_\kappa^{(i)}\,\delta^{[1]}, \quad i \in \{1,\ldots,i_\sharp\}.$$

In particular $F^{i,m}_{\kappa,0} = F^{i,m}_{\kappa}$ is deterministic if $i \in \mathbb{N}_+$. Note that $\varrho(i,m) > 0$ if $m \geq 4$. For $m = 3$ the condition $\varrho(i,m) \leq 0$ implies $i = 1$. We have

$$\partial_\eta F^{1,3}_{\kappa,\eta} = 0, \qquad F^{1,3}_{\kappa,0} = \delta^{[3]}.$$

Consequently,

$$\|K^{\otimes 4}_\mu * F^{1,3}_{\kappa,\mu}\|_{\mathcal{V}^3} = 1,$$

which implies the statement. For $m \in \{0,1,2\}$ we decompose

$$\begin{aligned} F^{i,0}_{\kappa,\mu} &= \tilde{F}^{i,0}_{\kappa,\mu} + \check{F}^{i,0}_{\kappa,\mu}, && i \in \{1,\dots,i_\flat\}, \\ F^{i,1}_{\kappa,\mu} &= \hat{E}^i_{\kappa,\mu}\,\delta^{[1]} + \check{E}^i_{\kappa,\mu} + \tilde{F}^{i,1}_{\kappa,\mu} + \check{F}^{i,1}_{\kappa,\mu}, && i \in \{1,\dots,i_\sharp\}, \\ F^{i,2}_{\kappa,\mu} &= \tilde{F}^{i,2}_{\kappa,\mu} + \check{F}^{i,2}_{\kappa,\mu}, && i \in \{1,\dots,i_\natural\}, \end{aligned} \tag{C.1}$$

where

$$\begin{aligned} \partial_\eta \hat{E}^i_{\kappa,\eta} &:= \mathbf{I}(\mathbb{E}\mathring{F}^{i,1}_{\kappa,\eta}), & \hat{E}^i_{\kappa,0} &:= c^{(i)}_\kappa, \\ \partial_\eta \check{E}^i_{\kappa,\eta} &:= \textstyle\sum_{|a|=2} \partial^a \mathbf{R}^a(\mathbb{E}\mathring{F}^{i,1}_{\kappa,\eta}), & \check{E}^i_{\kappa,0} &:= 0, \\ \partial_\eta \tilde{F}^{i,m}_{\kappa,\eta} &:= \mathring{F}^{i,m}_{\kappa,\eta} - \mathbb{E}\mathring{F}^{i,m}_{\kappa,\eta}, & \tilde{F}^{i,m}_{\kappa,0} &:= 0, \\ \partial_\eta \check{F}^{i,m}_{\kappa,\eta} &:= \partial_\eta F^{i,m}_{\kappa,\eta} - \mathring{F}^{i,m}_{\kappa,\eta}, & \check{F}^{i,m}_{\kappa,0} &:= 0 \end{aligned}$$

and

$$\mathring{F}^{i,m}_{\kappa,\eta} := K^{\otimes(1+m)}_\eta * \partial_\eta F^{i,m}_{\kappa,\eta}.$$

The motivation behind the above decomposition is that we will be able to prove bounds for $\partial_\eta \check{E}^i_{\kappa,\eta}$, $\partial_\eta \tilde{F}^{i,m}_{\kappa,\eta}$ and $\partial_\eta \check{F}^{i,m}_{\kappa,\eta}$ that are integrable in η at $\eta = 0$. On the other hand, the bound for $\partial_\eta \hat{E}^i_{\kappa,\eta}\,\delta^{[1]}$ is not integrable in η at $\eta = 0$. The bound for $\hat{E}^i_{\kappa,\eta}\delta^{[1]}$ relies crucially on the fact that this term is local and is only valid for suitable choices of the counterterm $c^{(i)}_\kappa \in \mathbb{R}$.

We shall first bound $\check{F}^{i,m}_{\kappa,\eta}$ and then proceed to $\hat{E}^i_{\kappa,\mu}$, $\check{E}^i_{\kappa,\mu}$ and $\tilde{F}^{i,m}_{\kappa,\mu}$. Note that

$$\partial_\eta \check{F}^{i,m}_{\kappa,\eta} = (\delta^{\otimes(1+m)}_0 - K^{\otimes(1+m)}_\eta) * \partial_\eta F^{i,m}_{\kappa,\eta}.$$

Using the above identity and the properties of the kernels K_η one shows that

$$\|K^{\otimes(1+m)}_\mu * \partial_\eta \check{F}^{i,m}_{\kappa,\eta}\|_{\mathcal{V}^m} \lesssim [\eta/\mu]^2\, \|K^{\otimes(1+m)}_\eta * \partial_\eta F^{i,m}_{\kappa,\eta}\|_{\mathcal{V}^m}.$$

Note that for $i \in \mathbb{N}_+$ and $m \in \mathbb{N}_0$ it holds $\varrho_\varepsilon(i,m)+2 > 0$ for all sufficiently small $\varepsilon \in (0,\infty)$. Here we used the assumption that $d \in \{1,\dots,4\}$. Consequently, we obtain

$$\|K^{\otimes(1+m)}_\mu * \check{F}^{i,m}_{\kappa,\mu}\|_{\mathcal{V}^m} \lesssim [\mu]^{-2} \int_0^\mu [\eta]^2\, \|K^{\otimes(1+m)}_\eta * \partial_\eta F^{i,m}_{\kappa,\eta}\|_{\mathcal{V}^m}\, \mathrm{d}\eta$$

and by the bound stated in the lemma with $s = 1$, which was proved above, we arrive at

$$\big(\mathbb{E}\|K^{\otimes(1+m)}_\mu * F^{i,m}_{\kappa,\mu}\|^n_{\mathcal{V}^m}\big)^{1/n} \lesssim [\mu]^{-2} \int_0^\mu [\eta]^2 \big(\mathbb{E}\|K^{\otimes(1+m)}_\eta * \partial_\eta F^{i,m}_{\kappa,\eta}\|^n_{\mathcal{V}^m}\big)^{1/n}\, \mathrm{d}\eta$$
$$[\mu]^{-2} \int_0^\mu [\eta]^{\varrho_\varepsilon(i,m)-\sigma+\varepsilon+2}\, \mathrm{d}\eta \lesssim [\mu]^{\varrho_\varepsilon(i,m)}.$$

Next, let us study the local and deterministic term $\hat{E}^i_{\kappa,\mu}\,\delta^{[1]}$. First, observe that by the bound stated in the lemma with $s=1$ we have

$$
\begin{aligned}
\big(\mathbb{E}\|\mathring{F}^{i,m}_{\kappa,\eta}\|^n_{\mathcal{V}^m}\big)^{1/n} &\lesssim [\kappa\vee\eta]^{\varrho_\varepsilon(i,m)-(\sigma-\varepsilon)-\sigma(3i-m-1)}\,[\eta]^{\sigma(3i-m-1)} \\
&\lesssim [\kappa\vee\eta]^{\varrho_\varepsilon(i,m)-\sigma+\varepsilon} \lesssim [\eta]^{\varrho_\varepsilon(i,m)-\sigma+\varepsilon}, \qquad \text{(C.2)}
\end{aligned}
$$

where the second to last estimate is true because $3i-m-1\geq 0$ unless $\partial_\eta F^{i,m}_{\kappa,\eta}=0$ and the last estimate is true because we study the case $\varrho_\varepsilon(i,m)\leq 0$. We fix the counterterm to be

$$
c^{(i)}_\kappa := -\int_0^{1/2} \mathbf{I}(\mathbb{E}\mathring{F}^{i,1}_{\kappa,\eta})\,\mathrm{d}\eta.
$$

Then the following identity

$$
\hat{E}^i_{\kappa,\mu} = c^{(i)}_\kappa + \int_0^\mu \mathbf{I}(\mathbb{E}\mathring{F}^{i,1}_{\kappa,\eta})\,\mathrm{d}\eta = -\int_\mu^{1/2} \mathbf{I}(\mathbb{E}\mathring{F}^{i,1}_{\kappa,\eta})\,\mathrm{d}\eta
$$

holds true. Consequently, since $\varrho_\varepsilon(i,1)<0$ for $i\in\{1,\ldots,i_\sharp\}$ using the bound (C.2) with $n=1$ we obtain

$$
|\hat{E}^i_{\kappa,\eta}| \lesssim \int_\mu^{1/2} [\eta]^{\varrho_\varepsilon(i,1)-\sigma}\,\mathrm{d}\eta \lesssim [\mu]^{\varrho_\varepsilon(i,1)}.
$$

This implies the desired bound

$$
\|K^{\otimes 2}_\mu * (\hat{E}^i_{\kappa,\mu}\,\delta^{[1]})\|_{\mathcal{V}^1} \lesssim [\mu]^{\varrho_\varepsilon(i,1)}.
$$

Let us proceed to the estimates for the non-local deterministic term $\check{E}^i_{\kappa,\eta}$. Using the boundary condition $\check{E}^i_{\kappa,0}=0$ and the bound $\|\partial^a\tilde{K}_\mu\|_{\mathcal{K}}\lesssim[\mu]^{-|a|}$ proved in Lemma C.12 (A) we estimate

$$
\|K^{\otimes 2}_\mu * \check{E}^i_{\kappa,\mu}\|_{\mathcal{V}^1} \lesssim [\mu]^{-2}\sup_{|a|=2}\int_0^\mu \|\mathbf{R}^a(\mathbb{E}\mathring{F}^{i,1}_{\kappa,\eta})\|_{\mathcal{V}^1}\,\mathrm{d}\eta.
$$

By Exercise C.1 the RHS is bounded up to a constant by

$$
\sup_{|a|=2}\,[\mu]^{-2}\int_0^\mu \|\mathcal{X}^a(\mathbb{E}\mathring{F}^{i,1}_{\kappa,\eta})\|_{\mathcal{V}^1}\,\mathrm{d}\eta.
$$

Recall that by Lemma 3.18 there exists $c\in(0,\infty)$ such that for all $\eta\in(0,1/2]$ it holds

$$
\operatorname{supp}\partial_\eta F^{i,1}_{\kappa,\eta} \subset \{(x,y)\in\mathbb{M}^2\,|\,|x-y|\leq c\,[\eta]\}.
$$

Pretending that there exist $c\in(0,\infty)$ such that for

$$
\operatorname{supp} K_\eta \subset \{x\in\mathbb{M}\,|\,|x|<c\,[\eta]\} \qquad \text{(C.3)}
$$

for all $\eta\in(0,1/2]$ we obtain that $\mathring{F}^{i,m}_{\kappa,\eta}=K^{\otimes(1+m)}_\eta * \partial_\eta F^{i,m}_{\kappa,\eta}$ has the same support property as $\partial_\eta F^{i,1}_{\kappa,\eta}$. Consequently, for $a\in\mathbb{N}^d_0$ such that $|a|=2$ it holds

$$
\|\mathcal{X}^a(\mathbb{E}\mathring{F}^{i,1}_{\kappa,\eta})\|_{\mathcal{V}^1} \lesssim [\eta]^2\,\|\mathbb{E}\mathring{F}^{i,1}_{\kappa,\eta}\|_{\mathcal{V}^1}.
$$

Since the kernel K_η does not have the above-mentioned support property the rigorous proof of the above estimate is slightly more complicated and is given in Lemma C.14 below. Taking into account the bound (C.2) we obtain the bound

$$\|K_\mu^{\otimes 2} * \hat{E}^i_{\kappa,\mu}\|_{\mathcal{V}^1} \lesssim [\mu]^{-2} \int_0^\mu [\eta]^2 \, \|\mathbb{E}\mathring{F}^{i,1}_{\kappa,\eta}\|_{\mathcal{V}^1} \,\mathrm{d}\eta \lesssim [\mu]^{-2} \int_0^\mu [\eta]^{\varrho_\varepsilon(i,1)+2} \,\mathrm{d}\eta.$$

To conclude we use the fact that $\varrho_\varepsilon(i,1) + 2 > 0$ to show

$$[\mu]^{-2} \int_0^\mu [\eta]^{\varrho_\varepsilon(i,1)+2} \,\mathrm{d}\eta \lesssim [\mu]^{\varrho_\varepsilon(i,1)}.$$

Finally, we discuss the proof of the bound for $\tilde{F}^{i,m}_{\kappa,\eta}$. By the Minkowski inequality we have

$$\begin{aligned}\sqrt{\mathbb{E}\big(\tilde{K}_\mu^{\otimes(1+m)} * \tilde{F}^{i,m}_{\kappa,\mu}(x;y_1,\ldots,y_m)\big)^2} &\leq \int_0^\mu \sqrt{\mathbb{E}\big(\tilde{K}_\mu^{\otimes(1+m)} * \partial_\eta \tilde{F}^{i,m}_{\kappa,\eta}(x;y_1,\ldots,y_m)\big)^2} \,\mathrm{d}\eta \\ &= \int_0^\mu \sqrt{\mathrm{Var}\big(\tilde{K}_\mu^{\otimes(1+m)} * \mathring{F}^{i,m}_{\kappa,\eta}(x;y_1,\ldots,y_m)\big)} \,\mathrm{d}\eta \qquad \text{(C.4)}\end{aligned}$$

Let us study the covariance $\mathrm{Cov}\big(\mathring{F}^{i,m}_{\kappa,\eta}, \mathring{F}^{i,m}_{\kappa,\eta}\big)$, where $\mathring{F}^{i,m}_{\kappa,\eta} = K_\eta^{\otimes(1+m)} * \partial_\eta F^{i,m}_{\kappa,\eta}$. By Lemma C.2 there exists $c \in (0,\infty)$ such that

$$\operatorname{supp} \mathrm{Cov}\big(\partial_\eta F^{i,m}_{\kappa,\eta}, \partial_\eta F^{i,m}_{\kappa,\eta}\big) \subset \{(x,y_1,\ldots,y_m,x',y'_1,\ldots,y'_m) \in \mathbb{M}^{2+2m} \,|\, |x-x'| \leq c\,[\kappa \vee \eta]\}.$$

Moreover, by the bound (C.2) we have

$$\|\mathrm{Cov}\big(\mathring{F}^{i,m}_{\kappa,\eta}, \mathring{F}^{i,m}_{\kappa,\eta}\big)\|_{L^\infty(\mathbb{M}^{2m+2m})} \lesssim [\kappa \vee \eta]^{2(\varrho_\varepsilon(i,m)-\sigma+\varepsilon)}.$$

As a result, pretending that the kernel K_η has the support property (C.3) we obtain

$$\int_{\mathbb{T}\times\mathbb{M}^{2m}} \big|\mathrm{Cov}\big(\mathring{F}^{i,m}_{\kappa,\eta}(x;y_1,\ldots,y_m), \mathring{F}^{i,m}_{\kappa,\eta}(x';y'_1,\ldots,y'_m)\big)\big| \,\mathrm{d}x'\mathrm{d}y_1\ldots\mathrm{d}y_m\mathrm{d}y'_1\ldots\mathrm{d}y'_m \lesssim [\kappa\vee\eta]^{2(\varrho_\varepsilon(i,m)-\sigma+\varepsilon)+d}.$$

To prove the above bound rigorously we use the same argument as in the proof of Lemma C.14. Consequently, by Lemma C.12 (C), (D) applied with $p = \infty$ it holds

$$\sqrt{\mathrm{Var}\big(\tilde{K}_\mu^{\otimes(1+m)} * \mathring{F}^{i,m}_{\kappa,\eta}(x;y_1,\ldots,y_m)\big)} \lesssim [\mu]^{-d/2-dm}\,[\kappa\vee\eta]^{\varrho_\varepsilon(i,m)-\sigma+\varepsilon+d/2}$$

and by the bound (C.4) we obtain

$$\sqrt{\mathbb{E}\big(\tilde{K}_\mu^{\otimes(1+m)} * \tilde{F}^{i,m}_{\kappa,\mu}(x;y_1,\ldots,y_m)\big)^2} \lesssim [\mu]^{-d/2-dm}\,[\kappa\vee\mu]^{\varrho_\varepsilon(i,m)+\varepsilon+d/2}.$$

From now on we assume that $\mu \in [\kappa,1]$. Recall that for $\mu \in (0,\kappa]$ the statement has already been proved. Since $F^{i,m}_{\kappa,\mu}$ is in a finite Wiener chaos we can use the Nelson estimate to conclude that

$$\mathbb{E}\big(\tilde{K}_\mu^{\otimes(1+m)} * \tilde{F}^{i,m}_{\kappa,\mu}(x;y_1,\ldots,y_m)\big)^n \lesssim [\kappa\vee\mu]^{n(\varrho_\varepsilon(i,m)+\varepsilon-dm)}.$$

Using the Fubini theorem and pretending again that the kernels $\tilde{K}_\mu$ and K_μ have the support property (C.3) one shows that

$$\mathbb{E}\|\tilde{K}_\mu^{\otimes(1+m)} * \tilde{F}_{\kappa,\mu}^{i,m}\|_{L^n(\mathbb{T}\times\mathbb{M}^m)}^n \lesssim [\mu]^{n(\varrho_\varepsilon(i,m)+\varepsilon-dm)+dm}$$

To make the above argument precise one can, for example, use weights to show that $\tilde{F}_{\kappa,\mu}^{i,m}$ is essentially localized in a neighbourhood of the diagonal of diameter of order $[\mu]$. Since $K_\mu = \tilde{K}_\mu * \tilde{K}_\mu$ by Kolmogorov-type estimate stated in Lemma C.13 we obtain

$$\mathbb{E}\big\|K_\mu^{\otimes(1+m)} * \tilde{F}_{\kappa,\mu}^{i,m}\big\|_{L^\infty(\mathbb{T}\times\mathbb{M}^m)}^n \lesssim [\mu]^{n(\varrho_\varepsilon(i,m)-dm)}$$

provided $n\varepsilon > d$. Employing again the fact that $\tilde{F}_{\kappa,\mu}^{i,m}$ is essentially localized in a neighbourhood of the diagonal of diameter of order $[\mu]$ we obtain

$$\mathbb{E}\big\|K_\mu^{\otimes(1+m)} * \tilde{F}_{\kappa,\mu}^{i,m}\big\|_{\mathcal{V}^m}^n \lesssim [\mu]^{n\varrho_\varepsilon(i,m)}.$$

Thus, we have estimated all the terms appearing in the decompositions (C.1). This finishes the proof. □

Definition C.11 (♠)**.** Let $\mathbb{T} := \mathbb{M}/(2\pi\mathbb{Z})^d$. For $K \in L^1(\mathbb{M})$ we define $\mathbf{T}K \in L^1(\mathbb{T})$ by

$$\mathbf{T}K(x) := \sum_{y\in(2\pi\mathbb{Z})^d} K(x+y).$$

Lemma C.12 (♠)**.** *Let $a \in \mathbb{N}_0^d$ and $p \in [1,\infty]$. The following is true:*

(A) If $|a| \leq d$, then $\|\partial^a \tilde{K}_\mu\|_{\mathcal{K}} \lesssim [\mu]^{-|a|}$ uniformly in $\mu \in (0,1]$.

(B) It holds $\|\tilde{\mathbf{P}}_\mu \partial_\mu \tilde{K}_\mu\|_{\mathcal{K}} \lesssim [\mu]^{-\sigma}$ uniformly in $\mu \in (0,1]$.

(C) $\|\tilde{K}_\mu\|_{L^p(\mathbb{M})} \lesssim [\mu]^{-d(p-1)/p}$ uniformly in $\mu \in (0,1]$.

(D) $\|\mathbf{T}\tilde{K}_\mu\|_{L^p(\mathbb{T})} \lesssim [\mu]^{-d(p-1)/p}$ uniformly in $\mu \in (0,1]$.

Lemma C.13 (♠)**.** *Let $n \in 2\mathbb{N}_+$, $m \in \mathbb{N}_0$. There exists a constant $C > 0$ such that for all random fields $\zeta \in L^\infty(\mathbb{T}\times\mathbb{M}^m)$ and $\mu \in (0,1]$ it holds*

$$\mathbb{E}\|\tilde{K}_\mu^{\otimes(1+m)} * \zeta\|_{L^\infty(\mathbb{T}\times\mathbb{M}^m)}^n \leq C\,[\mu]^{-d(1+m)}\,\mathbb{E}\|\zeta\|_{L^n(\mathbb{T}\times\mathbb{M}^m)}^n$$

Proof. Note that

$$\tilde{K}_\mu^{\otimes(1+m)} * \zeta = (\mathbf{T}\tilde{K}_\mu \otimes \tilde{K}_\mu^{\otimes m}) \star \zeta$$

where $\star$ is the convolution in $\mathbb{T}\times\mathbb{M}^m$ and $\mathbf{T}\tilde{K}_\mu$ is the periodization of $\tilde{K}_\mu$ (see Def. C.11). Using the Young inequality for convolutions we obtain

$$\mathbb{E}\|\tilde{K}_\mu * \zeta\|_{L^\infty(\mathbb{T})}^n \leq \|\mathbf{T}\tilde{K}_\mu\|_{L^{n/(n-1)}(\mathbb{T})}^n\,\|\tilde{K}_\mu\|_{L^{n/(n-1)}(\mathbb{T})}^{mn}\,\mathbb{E}\|\zeta\|_{L^n(\mathbb{T})}^n.$$

The lemma follows now from Lemma C.12 (B). □

Exercise C.3 (♠). *Fix $n \in \mathbb{N}_+$, $m \in \mathbb{N}_0$. Prove that there exists a universal constant $c > 0$ such that if*

$$\mathbb{E}\|\partial_\mu^s \partial_\kappa^r \zeta_{\kappa,\mu}\|_{\mathcal{V}^m}^n \leq C\,[\kappa]^{n(-\sigma r+\varepsilon r)}[\mu]^{n(\rho-\sigma s+\varepsilon s)}, \qquad s,r \in \{0,1\},\ \kappa,\mu \in (0,1],$$

for some differentiable random function $\zeta : (0,1]^2 \to \mathcal{V}^m$ and $C > 0$, $\rho \leq 0$, then

$$\mathbb{E}\Big(\sup_{\kappa,\mu\in(0,1]} [\mu]^{-n\rho}\,\|\zeta_{\kappa,\mu}\|_{\mathcal{V}^m}^n\Big) \leq c\,C.$$

Hint: Use the estimate

$$[\mu]^{-\rho}\,\|\zeta_{\kappa,\mu}\|_{\mathcal{V}^m} \leq \|\zeta_{1,1}\|_{\mathcal{V}^m} + \int_\mu^1 [\eta]^{-\rho}\,\|\partial_\eta \zeta_{1,\eta}\|_{\mathcal{V}^m}\,\mathrm{d}\eta + \int_\mu^1 \int_\kappa^1 [\eta]^{-\rho}\,\|\partial_\eta\partial_\nu \zeta_{\nu,\eta}\|_{\mathcal{V}^m}\,\mathrm{d}\nu\mathrm{d}\eta$$

and the Minkowski inequality.

Lemma C.14 (♠). *Fix some $m \in \mathbb{N}_+$ and $c \in \mathbb{R}$. There exists $C \in (0,\infty)$ such that if for some $\mu \in (0,1/2]$ and $V \in \mathscr{S}'(\mathbb{T}\times\mathbb{M}^m)$ it holds*

$$\operatorname{supp} V \subset \{(x,y_1,\ldots,y_m)\,|\,|x-y_1|\vee\ldots\vee|x-y_m| \leq c\,[\mu]\}$$

then

$$\|\mathcal{X}^a(K_\mu^{\otimes(1+m)} * V)\|_{\mathcal{V}_t^m} \leq C\,[\mu]^{|a|}\,\|K_\mu^{\otimes(1+m)} * V\|_{\mathcal{V}_t^m}.$$

Remark C.15. Note that the lemma would be obvious if $K_\mu \in C(\mathbb{M})$ was compactly supported in a ball $\{x \in \mathbb{M}\,|\,|x| < c\,[\mu]\}$ for some $c \in (0,\infty)$ independent of $\mu \in (0,1/2]$. Even though this is not exactly true, one can easily proved the above result by leveraging the fact that the kernel K_μ decays exponentially with the rate $1/[\mu]$.

Proof (♠). A simple rescaling reduces the proof to the case $\mu = 1$. Let $v \in C^\infty(\mathbb{M})$ be such that $\operatorname{supp} v \subset \{x \in \mathbb{M}\,|\,|x| < 1\}$ and $v = 1$ on $\{x \in \mathbb{M}\,|\,|x| \leq 1/2\}$. For $\tau \in [1,\infty)$ let $L_\tau(x) := K_1(x)\,v(x/\tau)$. It holds

$$\begin{aligned}\|\mathcal{X}^a(K_1^{\otimes(1+m)} * V)\|_{\mathcal{V}_t^m} &\leq \|\mathcal{X}^a(L_1^{\otimes(1+m)} * V)\|_{\mathcal{V}_t^m} + \int_1^\infty \|\mathcal{X}^a\partial_\tau(L_\tau^{\otimes(1+m)} * V)\|_{\mathcal{V}_t^m}\,\mathrm{d}\tau\\ &\lesssim \|L_1^{\otimes(1+m)} * V\|_{\mathcal{V}_t^m} + \int_1^\infty \tau^{|a|}\,\|\partial_\tau(L_\tau^{\otimes(1+m)} * V)\|_{\mathcal{V}_t^m}\,\mathrm{d}\tau,\end{aligned}$$

where to get the last estimate we used the fact that $\operatorname{supp} L_\tau \subset \{x \in \mathbb{M}\,|\,|x| < \tau\}$. Next, we observe that

$$L_\tau = \tilde{\mathbf{P}}_1 L_\tau * K_1, \qquad \partial_\tau L_\tau = \tilde{\mathbf{P}}_1 \partial_\tau L_\tau * K_1$$

and $\|\tilde{\mathbf{P}}_1 L_\tau\|_{\mathcal{K}} \lesssim 1$ and $\|\tilde{\mathbf{P}}_1\partial_\tau L_\tau\|_{\mathcal{K}} \lesssim \tau^{-N}$ uniformly in $\tau \in [1,\infty)$ for any $N \in \mathbb{N}_+$ because of the exponential decay of the kernel K_1. Consequently, we have

$$\|\mathcal{X}^a(K_1^{\otimes(1+m)} * V)\|_{\mathcal{V}_t^m} \lesssim \|K_1^{\otimes(1+m)} * V\|_{\mathcal{V}_t^m} + \int_1^\infty \tau^{|a|-N}\,\|K_1^{\otimes(1+m)} * V\|_{\mathcal{V}_t^m}\,\mathrm{d}\tau,$$

which finishes the proof. □

Lemma C.16 (Nelson's estimate)**.** *For every random variable X in an inhomogeneous Wiener chaos of order $n \in \mathbb{N}_+$ and every $p \in [2,\infty)$ it holds*

$$\left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} \leq (p-1)^{\frac{n}{2}} \left(\mathbb{E}X^2\right)^{\frac{1}{2}}.$$

Proof. The bound follows from the Nelson hypercontractivity of the Ornstein-Uhlenbeck operator (see e.g. [Nua06, Theorem 1.4.1]). □

Exercise C.4. *Complete the proof of Lemma C.7 by checking that all of the steps of the above proof generalize to the case $r = 1$. To prove the base case of the induction verify that*

$$\mathbb{E}((K_\mu * \partial_\kappa^r \xi_\kappa)(x)^2) \lesssim [\kappa]^{2(\varepsilon-\sigma)r}\, [\mu]^{-2(d/2+\varepsilon)}.$$

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