Construction of fractional Φ^4_3 model of Euclidean QFT using flow equation approach to singular SPDEs

(Lecture notes based on a joint work with M. Gubinelli and P. Rinaldi)

Paweł Duch
École Polytechnique Fédérale de Lausanne
1015 Lausanne, Switzerland
pawel.duch@epfl.ch

June 17, 2024

Abstract

We present a construction of the Gibbs measure of the fractional Φ^4 model of Euclidean quantum field theory in three-dimensions. The measure is obtained as a perturbation of the Gaussian measure with covariance given by the inverse of a fractional Laplacian. Since the Gaussian measure is supported in the space of Schwartz distributions and the quartic interaction potential of the model involves pointwise products, to construct the measure it is necessary to solve the so-called renormalization problem. To this end, we study the stochastic quantization equation, which is a nonlinear parabolic PDE driven by the white noise. We prove a certain a priori estimate for solutions of this equation using the flow equation approach to singular stochastic PDEs and the maximum principle. We consider the entire range of powers of the fractional Laplacian for which the model is subcritical (i.e. super-renormalizable).

Contents

1 Introduction

We outline the construction of the fractional Φ^4 model of Euclidean quantum field theory given in [DGR23]. We first define the model on a finite lattice and subsequently study the continuum and infinite volume limits. Let $\mathbb{T}^d_{\varepsilon,\tau}$ be a d-dimensional toroidal lattice with spacings $\varepsilon \in \mathcal{A} := \{2^{-n} : n \in \mathbb{N}_0\}$ and length $\tau \in \mathbb{N}_+$. Define the probability measure $\nu_{\varepsilon,\tau}$ on field configurations $\{\phi : \mathbb{T}^d_{\varepsilon,\tau} \to \mathbb{R}\}$ by

$$\nu_{\varepsilon,\tau}(\mathrm{d}\phi) := Z^{-1}_{\varepsilon,\tau} \exp(-S_{\varepsilon,\tau}(\phi)) \prod_{x \in \mathbb{T}^d_{\varepsilon,\tau}} \mathrm{d}\phi(x),$$

where $Z_{\varepsilon,\tau}$ is the normalization constant,

$$S_{\varepsilon,\tau}(\phi) := \varepsilon^d \sum_{x \in \mathbb{T}^d_{\varepsilon,\tau}} \left(\phi(x)((-\Delta_\varepsilon)^{\sigma/2}\phi)(x) + \phi(x)^2 + \frac{1}{2}\phi(x)^4 - r_{\varepsilon,\tau}\, \phi(x)^2 \right)$$

is called we action, $(-\Delta_\varepsilon)^{\sigma/2}$ is the fractional discrete Laplacian of order $\sigma \in (0,\infty)$ and $r_{\varepsilon,\tau} \in \mathbb{R}$ is the mass counterterm. In order to make sense of the continuum and infinite volume limit we have to identify the measure $\nu_{\varepsilon,\tau}$ on $\{\phi : \mathbb{T}^d_{\varepsilon,\tau} \to \mathbb{R}\}$ with a measure on $\mathscr{S}'(\mathbb{R}^d)$. To this end, we extend $\phi : \mathbb{T}^d_{\varepsilon,\tau} \to \mathbb{R}$ to $\mathbb{R}^d$ using the following extension map

$$(\mathbf{E}_\varepsilon \phi)(x) := \sum_{y \in \mathbb{R}^d_\varepsilon} \chi((x-y)/\varepsilon)\, \phi(y), \qquad x \in \mathbb{R}^d.$$

Here and in what follows, we identify a function $\phi : \mathbb{T}^d_{\varepsilon,\tau} \to \mathbb{R}$ with a periodic function $\phi : \mathbb{R}^d_\varepsilon \to \mathbb{R}$, where $\mathbb{R}^d_\varepsilon$ is an infinite d-dimensional lattice with spacings $\varepsilon \in \mathcal{A}$. The radially symmetric function $\chi \in \mathscr{S}(\mathbb{R}^d)$ appearing in the above formula is chosen in such a way that $\mathbf{F}\chi$ is non-negative, $\mathbf{F}\chi = 1$ on a ball in $\mathbb{R}^d$ of radius $\pi/2$ centered at the origin and $\sum_{n\in\mathbb{Z}^d} \mathbf{F}\chi(\bullet + 2\pi n) = 1$, where $\mathbf{F}$ denotes the Fourier transform in $\mathbb{R}^d$ defined by

$$(\mathbf{F}\chi)(p) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \chi(x) \exp(\mathrm{i}p \cdot x) \,\mathrm{d}x, \qquad p \in \mathbb{R}^d.$$

One shows that $\mathbf{E}_\varepsilon$ is indeed an extension map, that is, $\mathbf{E}_\varepsilon\phi(x) = \phi(x)$ for all $x \in \mathbb{R}^d_\varepsilon$. Moreover, we also have $\mathbf{E}_\varepsilon\phi \in C^\infty_{\mathrm{b}}(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)$ for all $\phi : \mathbb{T}^d_{\varepsilon,\tau} \to \mathbb{R}$. We define the measure $\hat\nu_{\varepsilon,\tau}$ on $\mathscr{S}'(\mathbb{R}^d)$ as the pushforward of $\nu_{\varepsilon,\tau}$ by $\mathbf{E}_\varepsilon$. We are interested in the limit $\varepsilon \to 0, \tau \to \infty$ of $\hat\nu_{\varepsilon,\tau}$.

It turns out that the proof of existence of the continuum limit gets more complicated as the order of the fractional Laplacian σ decreases. In order to build some intuitions about this limit let us for a moment assume that the length of the torus $\tau \in \mathbb{N}+$ is fixed. If $\sigma \in (d, \infty)$, then the measure obtained in the continuum limit has the following form

$$\hat\nu_\tau(\mathrm{d}\phi) = Z^{-1} \exp(-V(\phi))\, \hat\mu_\tau(\mathrm{d}\phi),$$

where

$$V(\phi) = \frac{1}{2} \int_{\mathbb{T}^d_\tau} \phi(x)^4 \,\mathrm{d}x$$

is the interaction potential, $\mathbb{T}^d_\tau$ is the torus of length τ and $\hat\mu_\tau$ is the Gaussian measure on $\mathscr{S}'(\mathbb{T}^d_\tau)$ with covariance $((-\Delta)^{\sigma/2} + 1)^{-1}/2$. In order to prove that the measure $\hat\nu_\tau$ is well defined one uses crucially the fact that the Gaussian measure $\hat\mu_\tau$ is concentrated in the space of continuous functions and consequently powers of the field ϕ distributed according to this measure are well-defined. This is not the case if $\sigma \le d$ as then the field ϕ distributed according to $\hat\mu_\tau$ is a genuine distribution. In order to make sense of the potential one has to renormalize it by subtracting appropriate mass counterterm. If $\sigma \in (3d/4, d]$, then the so-called Wick renormalization is sufficient. In this regime the interacting measure still has an explicit density with respect to the Gaussian measure

$$\hat\nu_\tau(\mathrm{d}\phi) = Z^{-1} \exp(-\,{:}V(\phi){:})\, \hat\mu_\tau(\mathrm{d}\phi),$$

where ${:}V(\phi){:}$ denotes the so-called Wick renormalization formally defined by

$${:}V(\phi){:} := \frac{1}{2} \int_{\mathbb{T}^d_\tau} \phi(x)^4 \,\mathrm{d}x - 3\hat\mu_\tau(\phi^2) \int_{\mathbb{T}^d_\tau} \phi(x)^2 \,\mathrm{d}x + \frac{3}{2}\hat\mu_\tau(\phi^2), \qquad \hat\mu_\tau(\phi^2) \equiv \hat\mu_\tau(\phi(x)^2) = \infty.$$

If $\sigma \le 3d/4$, then some further renormalization beyond Wick renormalization is necessary. The renormalization problem is tractable if $\sigma > d/2$. In this regime the fractional Φ^4 model is scaling subcritical. That is, after a scale transformation with a scale factor δ the interaction term in the potential acquires a prefactor proportional to a positive power of δ. This suggests that the short distance behavior of a model should be governed by the free theory. If $\sigma \le d/2$, then it is expected and in some cases known [ADC21] that the continuum limit does not exist or is Gaussian. In what follows, we study the subcritical regime beyond Wick renormalization. The main result of these notes is the following theorem.

Theorem 1.1. *Suppose that $d = 3$ and $\sigma \in (3/2, 2)$. There exists a choice of the mass counterterm $(r_{\varepsilon,\tau})_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+}$ such that:*

(A) $\hat\nu := \lim_{n\to\infty} \hat\nu_{\varepsilon_n,\tau_n}$ exists in the sense of weak convergence of measures on $\mathscr{S}'(\mathbb{R}^d)$ for some sequences $(\varepsilon_n)_{n\in\mathbb{N}_+}$, $(\tau_n)_{n\in\mathbb{N}_+}$ such that $\lim_{n\to\infty}\varepsilon_n = 0$, $\lim_{n\to\infty}\tau_n = \infty$,

(B) $\hat\nu$ is invariant under translations and reflection positive,

(C) $\hat\nu$ has sub-Gaussian tails.

Remark 1.2. The limit in Item (A) of the above theorem exists only if $r_{\varepsilon,\tau}$ diverges at a particular rate as $\varepsilon \searrow 0$. It is expected that $\hat\nu$ is invariant under all Euclidean transformations of $\mathbb{R}^d$. Item (C) implies that $\hat\nu$ is non-Gaussian.

1.1 Strategy of the proof

In order to establish Item (A) of Theorem 1.1 it suffices to prove the following bound

$$\sup_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+} \int \|\phi\|_{\mathscr{B}_\varepsilon}\, \nu_{\varepsilon,\tau}(\mathrm{d}\phi) < \infty, \tag{1.1}$$

where $\|\bullet\|_{\mathscr{B}_\varepsilon}$ is some weighted Hölder-Besov norm on $\mathbb{R}^d_\varepsilon$ of negative regularity. Let $\|\bullet\|_{\mathscr{B}}$ be an analogous weighted Hölder-Besov norm on $\mathbb{R}^d$. By the uniform boundedness of the family of the extension maps $(\mathbf{E}_\varepsilon : \mathscr{B}_\varepsilon \to \mathscr{B})_{\varepsilon\in\mathcal{A}}$, proved in [MP19, Lemma 2.24], we conclude that

$$\sup_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+} \int \|\mathbf{E}_\varepsilon\phi\|_{\mathscr{B}}\, \nu_{\varepsilon,\tau}(\mathrm{d}\phi) < \infty.$$

The above bound implies tightness of the family of measures $(\hat\nu_{\varepsilon,\tau})_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+}$ on $\tilde{\mathscr{B}}$, where $\tilde{\mathscr{B}}$ is any weighted Hölder-Besov norm on $\mathbb{R}^d$ of negative regularity such that $\mathscr{B}$ is compactly embedded in $\tilde{\mathscr{B}}$. In consequence, Item (A) follows from the Prokhorov theorem.

In order to prove the bound (1.1) we use the parabolic stochastic quantization technique. We view $\nu_{\varepsilon,\tau}$ as an invariant measure of a certain parabolic stochastic PDE called the stochastic quantization equation. The stochastic quantization equation becomes singular, that is classically ill-posed, in the continuum limit $\varepsilon \searrow 0$. To address this problem, we use the approach to singular stochastic PDEs based on the renormalization group flow equation. We rewrite stochastic quantization equation as a certain system of equations that involves the so-called effective force and remains well-posed in the continuum limit provided the effective force is chosen appropriately. We derive a coercive estimate for the above-mentioned system of equations using the maximum principle. The proof of the coercive estimate relies on the estimates for the effective force that hold only for suitable choices of the counterterms and are proved by induction using the flow equation for cumulants of the kernels of the effective force functional.

The proof of Item (B) of Theorem 1.1 is completely standard and is already contained in [GH21]. To prove Item (C) we use the trick introduced in [HS22]. In what follows, we outline the proof of Item (A). We stress that $d = 3$ and $\sigma \in (3/2, 2)$ are fixed.

Remark 1.3. The technique of the proof can be adapted with considerable simplifications to the case $\sigma = 2$. The method does not work for $\sigma > 2$ because in this regime the operator $(-\Delta_\varepsilon)^{\sigma/2}$ does not satisfy the maximum principle. However, let us mention that for $\sigma > 2$ Theorem 1.1 (except the reflection positivity of $\hat{\nu}$, which is not expected to hold if $\sigma > 2$) can be quite easily proved using the (iterated) Da Prato-Debussche trick and the energy method.

1.2 Difficulties to overcome

- We study full subcritical regime and need a systematic procedure to prove the so-called stochastic estimates.
- In contrast to the standard heat kernel that is smooth outside origin and decays exponentially at infinity, the fractional heat kernel is not smooth at the entire time-zero hypersurface and decays only polynomially at infinite. The above-mentioned properties of the fractional heat kernel lead to many technical issues. Let us mention that problems of a similar type would also appear in the elliptic stochastic quantization approach. The difficulties related to badly behaved kernels could potentially be avoided in the approach developed in [DFG22, GM24].
- It is not sufficient to establish a bound for the parabolic spacetime Besov norm of the solution of the stochastic quantization equation. We need to prove a bound for the Besov norm in space for the solution evaluated at a fixed time. To this end, we have to pay close attention to the regularity in time of various objects.

1.3 Literature

The construction of the fractional Φ^4 model presented in these lecture notes was developed in [DGR23]. The approach to singular SPDEs based on the flow equation was originally proposed in [Duc22, Duc21] and was inspired by the renormalization group [Wil71] approach to singular SPDEs [Kup16]. For a pedagogical introduction to the flow equation approach to singular SPDEs we refer the reader to the lecture notes [Duc23]. The fractional Φ^4 model was previously studied in [Abd07, BMS03], where the trajectory connecting the Gaussian and non-Gaussian fixed points of the renormalization group transformation was constructed in this model for $\sigma > d/2$ sufficiently close to the critical value $d/2$. A very general local in time solution theory for singular SPDEs on compact spatial domains in full subcritical regime was developed in [Hai14, CH16, BHZ19, BCCH21, HS23], see also [OSSW21, LOTT21]. In order to prove global in time existence or establish well-posedness on non-compact spatial domain one has to take into account some specific properties of the equation. In particular, in the case of the stochastic quantization equations of the Φ^4 models the coercive cubic term plays a crucial role in the proof of existence of global solutions. The case of the standard Laplacian corresponding to $\sigma = 2$ is by now well understood and useful a priori estimates were establish in [MW17, GH19, MW20, GH21] using the energy method or the maximum principle. Let us also mention the work [CMW23], where an a priori estimate for a singular SPDE with the standard Laplacian and a cubic non-linearity was established in full-subcritical regime by adjusting the regularity of the noise term. Note that such an equation with a colored noise does not arise naturally in the context of stochastic quantization of Euclidean QFTs. Establishing a priori bounds for singular SPDEs without a damping term is significantly more challenging. See [HR23, BC24, CLFW24, SZZ24] for recent results about such equations.

1.4 Organization of the notes

In Sec. 2 we briefly introduce the parabolic stochastic quantization method. Next, in Sec. 3 we give an overview of the flow equation approach to singular SPDEs. The main technical result of these notes, the proof of a deterministic a priori estimate for the stationary solution of the stochastic quantization equation, is contained Sec. 4. In Sec. 5 we outline the proof of the so-called stochastic estimates for the enhanced noise. Note that to conclude the bound (1.1) implying tightness of the sequence of measures one has to combine the a priori estimate for the solution with the stochastic estimates for the enhanced noise.

2 Parabolic stochastic quantization

The basic idea behind the stochastic quantization is to view a probability measure of interest $\nu_{\varepsilon,\tau}$ as the law of a measurable function $F_{\varepsilon,\tau}$ of a certain Gaussian random variable $\xi_{\varepsilon,\tau}$, that is $\nu_{\varepsilon,\tau} = \mathrm{Law}(F_{\varepsilon,\tau}(\xi_{\varepsilon,\tau})$. In the parabolic stochastic quantization approach one chooses the Gaussian random variable to be the space-time white noise $\xi_{\varepsilon,\tau} \in \mathscr{S}'(\mathbb{R}\times\mathbb{T}^d_{\varepsilon,\tau})$, which is defined uniquely by the conditions

$$\mathbb{E}(\xi_{\varepsilon,\tau}(\varphi)) = 0, \qquad \mathbb{E}(\xi_{\varepsilon,\tau}(\varphi)\xi_{\varepsilon,\tau}(\psi)) = \varepsilon^d \sum_{x\in\mathbb{R}^d_\varepsilon}\sum_{n\in\mathbb{Z}^2}\int_{\mathbb{R}} \varphi(t,x)\psi(t,x+\tau n)\,\mathrm{d}t,$$

that is, the covariance of $\xi_{\varepsilon,\tau}$ is the periodization in space of the Dirac delta. In order to define the map $F_{\varepsilon,\tau}$ one studies the following parabolic PDE

$$\big(\partial_t + (-\Delta_\varepsilon)^{\sigma/2} + 1\big)\varPhi_{\varepsilon,\tau} = \xi_{\varepsilon,\tau} - \varPhi^3_{\varepsilon,\tau} + r_{\varepsilon,\tau}\,\varPhi_{\varepsilon,\tau} \tag{2.1}$$

posed on $\Lambda_{\varepsilon,\tau} := \mathbb{R}\times\mathbb{T}^3_{\varepsilon,\tau}$, which is called the stochastic quantization equation of the fractional $\varPhi^4_3$ model or the dynamical fractional $\varPhi^4_3$ model. Since $\mathbb{T}^3_{\varepsilon,\tau}$ is a finite set, the above equation is actually a finite-dimensional stochastic differential equation in a gradient form. It is easy to prove that for every $\varepsilon\in\mathcal{A}$ and $\tau\in\mathbb{N}_+$ the above SDE has a global stationary solution $\varPhi_{\varepsilon,\tau}\in C(\Lambda_{\varepsilon,\tau})$ and $\nu_{\varepsilon,\tau} = \mathrm{Law}(\varPhi_{\varepsilon,\tau}(t,\bullet))$ for all $t\in\mathbb{R}$. Then the bound (1.1), which we want to prove, is equivalent to the bound

$$\sup_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+} \mathbb{E}\|\Phi_{\varepsilon,\tau}(t,\bullet)\|_{\mathscr{B}_\varepsilon} < \infty \tag{2.2}$$

for any $t \in \mathbb{R}$. We can hope to be able to use PDE tools to prove the above bound. The difficulty is related to the fact that Eq. (2.1) becomes singular in the continuum limit $\varepsilon \searrow 0$. Indeed, in the continuum limit in the parabolic Hölder-Besov scale the white noise noise has regularity $-\dim(\xi) - \kappa$ for all $\kappa \in (0,\infty)$, where $\dim(\xi)$ is introduced in Def. 2.1 below. Taking into account the regularizing effect of the inverse of the parabolic differential operator that appears in Eq. (2.1) we expect that in the continuum limit the solution $\Phi_{\varepsilon,\tau}$ has regularity $-\dim(\Phi) - \kappa$. In particular, the expected regularity of the solution is negative since $d = 3$ and $\sigma \in (d/3, 2)$. Hence, in the continuum the cubic term in the equation is not well-defined and the equation is singular.

Definition 2.1. We define the following constants:

$$\dim(\xi) = (d+\sigma)/2, \qquad \dim(\Phi) = (d-\sigma)/2, \qquad \dim(\lambda) = 2\sigma - d.$$

Remark 2.2. The condition $\dim(\lambda) > 0$ is equivalent to the sub-criticality condition.

3 Flow equation approach to singular SPDEs

In this section we introduce main ideas behind the flow equation approach to singular SPDEs. In order to prove the bound (2.2) uniform in the lattice spacing $\varepsilon \in \mathcal{A}$ we need some control of the continuum limit $\varepsilon \searrow 0$ of the stochastic quantization equation (2.1). We start by rewriting Eq. (2.1) in the following compact form

$$\mathcal{L}_\varepsilon \Phi_{\varepsilon,\tau} = F_{\varepsilon,\tau}[\Phi_{\varepsilon,\tau}], \tag{3.1}$$

where:

- $\mathcal{L}_\varepsilon := (\partial_t + (-\Delta_\varepsilon)^{\sigma/2} + 1)$,
- $F_{\varepsilon,\tau}[\varphi] := \xi_{\varepsilon,\tau} - \varphi^3 + r_{\varepsilon,\tau}\,\varphi$ – force,
- $\xi_{\varepsilon,\tau}$ – spacetime white noise,
- $r_{\varepsilon,\tau}$ – mass counterterm.

The above equation is posed on $\Lambda_\varepsilon := \mathbb{R} \times \mathbb{R}^3_\varepsilon$ and we are interested in solutions that are periodic in space with period $\tau \in \mathbb{N}_+$. In Sec. 3.2 we show that the stochastic quantization equation (3.1) can be rewritten as a certain system of equations that remains well-posed in the continuum limit. The system of equations involves the so-called coarse-grained process and an effective force, which are introduced in Sec. 3.1. In Sec. 3.3 we present a construction of a suitable effective force functional and in Sec. 3.4 we state the stochastic estimates for the kernels of this functional.

3.1 Coarse-grained process and effective force

The basic idea of the approach is to replace the stochastic quantization equation (3.1) by a certain equivalent equation that is expressed in terms of the so-called coarse-grained process. Note that in contrast to the solution of stochastic quantization equation, the coarse-grained process remains smooth in the continuum limit.

Definition 3.1. Let $\varepsilon \in \mathcal{A}$. We define the fractional Laplacian $(-\Delta_\varepsilon)^{\sigma/2}$ as the Fourier multiplier with the symbol

$$(-\pi/\varepsilon,\pi/\varepsilon)^d \ni \bar{p} \mapsto (\omega_\varepsilon(\bar{p}))^\sigma := \Big(\textstyle\sum_{i=1}^d \sin(\varepsilon\bar{p}_i)^2/\varepsilon^2\Big)^{\sigma/2} \in \mathbb{R}.$$

Definition 3.2. Let $\varepsilon \in \mathcal{A}$ and $\mu \in (0,1]$. We fix $j \in C^\infty(\mathbb{R})$ such that $j(\eta) = 1$ for $\eta \in [-1/2,1/2]$ and $j(\eta) = 0$ for $\eta \in \mathbb{R} \setminus (-1,1)$ and define $J_{\varepsilon;\mu} \in C^\infty(\Lambda_\varepsilon)$ as the Fourier transform on Λ_ε of the function

$$\mathbb{R} \times (-\pi/\varepsilon,\pi/\varepsilon)^d \ni p \equiv (\mathring{p},\bar{p}) \mapsto j(\mathring{p}/\mu^\sigma)j(\omega_\varepsilon(\bar{p})/\mu).$$

Remark 3.3. The function $J_{\varepsilon;\mu}$ is a smooth approximation of characteristic length scale μ of the Dirac delta at the origin $\delta^{(0)}_{\Lambda_\varepsilon} \in \mathscr{S}'(\Lambda_\varepsilon)$.

Remark 3.4. To simplify the notation we write $\|\bullet\|$ for $\|\bullet\|_{L^\infty(\Lambda_\varepsilon)}$. Moreover, we use the symbol $*$ to denote the convolution $*_\varepsilon$ on Λ_ε.

Definition 3.5. Let $\varepsilon \in \mathcal{A}$. The fractional heat kernel $G_\varepsilon \in L^1(\Lambda_\varepsilon)$ is defined by the equation

$$(\partial_t + (-\Delta_\varepsilon)^{\sigma/2} + 1)G_\varepsilon = \delta^{(0)}_{\Lambda_\varepsilon},$$

where $\delta^{(0)}_{\Lambda_\varepsilon} \in \mathscr{S}'(\Lambda_\varepsilon)$ is the Dirac delta at the origin. For $\mu \in (0,1]$ we set

$$G_{\varepsilon;\mu} := J_{\varepsilon;\mu} * G_\varepsilon, \qquad \dot{G}_{\varepsilon;\mu} := \partial_\mu J_{\varepsilon;\mu} * G_\varepsilon.$$

Definition 3.6. For $a = (\mathring{a},\bar{a}) \in \mathfrak{M} := \mathbb{N}_0 \times \mathbb{N}_0^d$ we define the differential operator ∂^a_ε on $\Lambda_\varepsilon = \mathbb{R} \times \mathbb{R}^d_\varepsilon$ by

$$\partial^a_\varepsilon := \partial_t^{\mathring{a}} \prod_{i=1}^d \partial_{\varepsilon,i}^{\bar{a}_i},$$

where ∂_t is the time derivative and $\partial_{\varepsilon,i}$ is the right discrete derivative on the lattice $\mathbb{R}^d_\varepsilon$ in the i-th direction. We set $|a|_\sigma := \sigma\mathring{a} + a_1 + \ldots + a_d$.

Definition 3.7. The parabolic distance $|\bullet|_\sigma$ compatible with the operator $\mathcal{L}_0$ is defined by $|z|_\sigma := (|\mathring{z}|^{2/\sigma} + |\bar{z}|^2)^{1/2}$ for $z = (\mathring{z},\bar{z}) \in \mathbb{R} \times \mathbb{R}^d = \Lambda_0$.

Remark 3.8. Note that $\Lambda_0 \ni (\mathring{z},\bar{z}) \to |z|^2_\sigma \in \mathbb{R}$ is continuously differentiable in $\mathring{z}$ and smooth in $\bar{z}$. The regularity of the parabolic distance plays a role in the definitions of various weights introduced below. The fact that

Lemma 3.9. *For $\mu \in (0,1]$ let $\tilde{w}_\mu \in C(\Lambda_0)$ be defined by*

$$\tilde{w}_\mu(z) := \big(1+\mu^{-2}|z|_\sigma^2\big)^{1/2}.$$

For all spacetime multi-indices $a \in \mathfrak{M}$ and all $\beta \in [0,\sigma)$ the following bounds

$$\|\tilde{w}_1^\beta \partial_\varepsilon^a G_{\varepsilon;1}\|_{L^1(\Lambda_\varepsilon)} \lesssim 1, \qquad \|\tilde{w}_\mu^\beta \partial_\varepsilon^a \dot{G}_{\varepsilon;\mu}\|_{L^1(\Lambda_\varepsilon)} \lesssim \mu^{\sigma-1-|a|_\sigma},$$
$$\|\mathbf{T}_\tau(\tilde{w}_\mu^\beta |\partial_\varepsilon^a \dot{G}_{\varepsilon;\mu}|)\| \lesssim \mu^{-d-1-|a|_\sigma}$$

hold uniformly in $\varepsilon \in \mathcal{A}$, $\tau \in \mathbb{N}_+$ and $\mu \in (0,1]$, where $\mathbf{T}_\tau$ denotes the periodization in space.

Remark 3.10. If $\sigma \in 2\mathbb{N}_+$, then the bounds stated in the above theorem hold for all $\beta \geq 0$. If $\sigma \notin 2\mathbb{N}_+$, then the bounds stated in the above theorem are not true for $\beta \geq \sigma$. The fact that the fractional heat kernel has only limited polynomial decay causes many technical problems. The reason behind the slow decay is the fact that for $\sigma \notin 2\mathbb{N}_+$ the symbol of the operator $\mathcal{L}_\varepsilon$ is not smooth.

Definition 3.11. Let $w \in C(\Lambda_0)$ be a weight defined by $w(z) = (1+|z|_\sigma^2)^{-1/6}$ for all $z \in \Lambda_0$. For $\varepsilon \in \mathcal{A}$ and $\tau \in \mathbb{N}_+$ we define $C_\flat(\Lambda_{\varepsilon,\tau})$ to be the Fréchet space

$$\big\{\varphi \in C(\Lambda_{\varepsilon,\tau}) \,:\, \forall_{n\in\mathbb{N}_+} \|w^{1/n}\,\varphi\| < \infty\big\}.$$

Remark 3.12. Since $\varphi \in C(\Lambda_{\varepsilon,\tau})$ is periodic in space, the fact that the weight w decays in space does not play any role in the above definition. It is easy to show that $\Psi_{\varepsilon,\tau} \in C_\flat(\Lambda_{\varepsilon,\tau})$ almost surely, where $\Psi_{\varepsilon,\tau}$ is the stationary solution of $\mathcal{L}_\varepsilon \Psi_{\varepsilon,\tau} = \xi_{\varepsilon,\tau}$. Note that since the domain $\Lambda_{\varepsilon,\tau} = \mathbb{R} \times \mathbb{T}^d_{\varepsilon,\tau}$ is not compact, $\Psi_{\varepsilon,\tau} \notin C_\mathrm{b}(\Lambda_{\varepsilon,\tau})$ almost surely. One proves that almost surely $\|w^\alpha(J_{\varepsilon;\mu} * \Psi_{\varepsilon,\tau})\| \lesssim \mu^\beta$ uniformly in $\varepsilon \in \mathcal{A}$, $\tau \in \mathbb{N}_+$ and $\mu \in (0,1]$ for all $\alpha \in (0,1]$ and $\beta < -\dim(\Phi)$, where $\dim(\Phi)$ is introduced in Def. 2.1. Here we used crucially the fact that the weight w decays in space as otherwise the estimate would not be uniform in $\tau \in \mathbb{N}_+$. Let $\Phi_{\varepsilon,\tau} \in C(\Lambda_{\varepsilon,\tau})$ be the stationary solution of the stochastic quantization equation (3.1). By applying the maximum principle to the equation for $\Phi_{\varepsilon,\tau} - \Psi_{\varepsilon,\tau}$ one shows that $\Phi_{\varepsilon,\tau} \in C_\flat(\Lambda_{\varepsilon,\tau})$ almost surely. We expect that $\|w^\alpha \Phi_{\varepsilon,\tau;\mu}\| \lesssim \mu^\beta$ uniformly in $\varepsilon \in \mathcal{A}$, $\tau \in \mathbb{N}_+$ and $\mu \in (0,1]$ for all $\alpha \in (0,1]$ and $\beta < -\dim(\Phi)$, where $\Phi_{\varepsilon,\tau;\mu}$ is the coarse-grained process introduced below.

Definition 3.13. Let $\varepsilon \in \mathcal{A}$, $\tau \in \mathbb{N}_+$ and $\Phi_{\varepsilon,\tau} \in C_\flat(\Lambda_{\varepsilon,\tau})$ be the stationary solution of the stochastic quantization equation (3.1). The coarse-grained process at the scale $\mu \in (0,1]$ is defined by the equality

$$\Phi_{\varepsilon,\tau;\mu} := J_{\varepsilon;\mu} * \Phi_{\varepsilon,\tau} \in C_\flat(\Lambda_{\varepsilon,\tau}).$$

Remark 3.14. Informally, the coarse-grained process is obtained by averaging the solution of the original equations over regions of size μ. The coarse-grained process captures the behavior of the solution of the original equation at spatial scales larger than μ and is essentially constant at smaller scales. In particular, for all strictly positive scale parameters μ the coarse grained process is smooth even in the continuum limit. However, since in the continuum limit the solution of the original equation becomes a distribution the L^∞ norm of the coarse-grained process must blow up in the limit $\mu \searrow 0$ as indicated in Remark 3.12.

Definition 3.15. Let $\varepsilon \in \mathcal{A}$ and $\tau \in \mathbb{N}_+$. An effective force is a family of functionals $F_{\varepsilon,\tau;\mu} : C_\flat(\Lambda_{\varepsilon,\tau}) \to \mathscr{S}'(\Lambda_{\varepsilon,\tau})$ parameterized by $\mu \in [0,1]$ such that:

(1) $F_{\varepsilon,\tau;\mu=0} = F_{\varepsilon,\tau}$, where $F_{\varepsilon,\tau}$ is the force,

(2) for all $\varphi \in C_\flat(\Lambda_{\varepsilon,\tau})$ the function $[0,1] \ni \mu \to F_{\varepsilon,\tau;\mu}[\varphi] \in \mathbb{R}$ is continuous and piecewise differentiable.

We call $\zeta_{\varepsilon,\tau;\mu} := F_{\varepsilon,\tau}[\Phi_{\varepsilon,\tau}] - F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon,\tau;\mu}] \in \mathscr{S}'(\Lambda_{\varepsilon,\tau})$ the remainder, where $\Phi_{\varepsilon,\tau}$ and $\Phi_{\varepsilon,\tau;\mu}$ are as in Def. 3.13.

Remark 3.16. In practical application one has to choose an effective force in such a way that the remainder vanishes or is small in a sense discussed below.

3.2 System of equations for coarse-grained process and remainder

In this section we show that the stochastic quantization equation (3.1) can be rewritten as a certain system of equations for the coarse-grained process and the remainder. To this end, observe that Eq. (3.1) and Def. 3.13, 3.15 imply that the coarse-grained process satisfies the following equation

$$\mathcal{L}_\varepsilon \Phi_{\varepsilon,\tau;\mu} = J_{\varepsilon;\mu} * F_{\varepsilon,\tau}[\Phi_{\varepsilon,\tau}] = J_{\varepsilon;\mu} * (F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon,\tau;\mu}] + \zeta_{\varepsilon,\tau;\mu}).$$

Consequently,

$$\partial_\mu \Phi_{\varepsilon,\tau;\mu} = \dot{G}_{\varepsilon;\mu} * (F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon,\tau;\mu}] + \zeta_{\varepsilon,\tau;\mu}). \tag{3.2}$$

Moreover,

$$\partial_\mu \zeta_{\varepsilon,\tau;\mu} = \partial_\mu (F_{\varepsilon,\tau}[\Phi_{\varepsilon,\tau}] - F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon,\tau;\mu}]) = -\partial_\mu F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon,\tau;\mu}]. \tag{3.3}$$

Using the identities (3.2) and (3.3) one easily derives an equation for the remainder that involves only the remainder and the coarse-grained process. We arrive at the following system of equations

$$\begin{cases} \mathcal{L}_\varepsilon \Phi_{\varepsilon,\tau;\mu} = J_{\varepsilon;\mu} * (F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon,\tau;\mu}] + \zeta_{\varepsilon,\tau;\mu}) \\ \zeta_{\varepsilon,\tau;\mu} = -\int_0^\mu (H_{\varepsilon,\tau;\eta}[\Phi_{\varepsilon,\tau;\eta}] + \mathrm{D}F_{\varepsilon,\tau;\eta}[\Phi_{\varepsilon,\tau;\eta}] \cdot (\dot{G}_{\varepsilon;\eta} * \zeta_{\varepsilon,\tau;\eta}))\,\mathrm{d}\eta\,, \end{cases} \tag{3.4}$$

where

$$H_{\varepsilon,\tau;\eta}[\varphi] := \partial_\eta F_{\varepsilon,\tau;\eta}[\varphi] + \mathrm{D}F_{\varepsilon,\tau;\eta}[\varphi] \cdot (\dot{G}_{\varepsilon;\eta} * F_{\varepsilon,\tau;\eta}[\varphi]). \tag{3.5}$$

We view (3.4) as a system of equations for

$$(0,\bar{\mu}] \ni \mu \mapsto (\Phi_{\varepsilon,\tau;\mu},\zeta_{\varepsilon,\tau;\mu}) \in C_\flat(\Lambda_{\varepsilon,\tau})^2, \tag{3.6}$$

where $\bar{\mu} \in (0,1]$ is an arbitrary terminal scale.

Remark 3.17. It will play an important role that (3.4) is a closed system of equations for any choice of the terminal scale $\bar{\mu} \in (0,1]$. As we will see, in order to derive an a priori estimate for the stochastic quantization equation (3.1) using the system of equations (3.4) we will choose $\bar{\mu} \in (0,1]$ small and random.

The advantage of the system of equations (3.4) over the stochastic quantization equation (3.1) is that the system of equations remains meaningful in the continuum limit $\varepsilon \searrow 0$ provided an effective force $F_{\varepsilon,\tau;\mu}$ is chosen appropriately. In informal terms, we have to choose an effective force $F_{\varepsilon,\tau;\mu}$ so that it admits a continuum limit for all $\mu \in (0,1]$ and $H_{\varepsilon,\tau;\eta}$, the source term in the equation for the remainder, is in some sense small so that $H_{\varepsilon,\tau;\eta}[\Phi_{\varepsilon,\tau;\eta}]$ remains integrable in $\eta \in [0,1]$ at $\eta = 0$ in the continuum limit. Since $H_{\varepsilon,\tau;\eta}$ is a function of $F_{\varepsilon,\tau;\mu}$, the above smallness condition for $H_{\varepsilon,\tau;\eta}$ is a constraint for $F_{\varepsilon,\tau;\mu}$. A natural choice for the effective force $F_{\varepsilon,\tau;\mu}$ is to define it so that $H_{\varepsilon,\tau;\mu} = 0$, i.e. the following flow equation is satisfied

$$\partial_\mu F_{\varepsilon,\tau;\mu}[\varphi] + \mathrm{D}F_{\varepsilon,\tau;\mu}[\varphi] \cdot (\dot{G}_{\varepsilon;\mu} * F_{\varepsilon,\tau;\mu}[\varphi]) = 0.$$

Then the unique solution of the equation for the remainder is $\zeta_{\varepsilon,\tau;\mu} = 0$. Constructing an exact solution $F_{\varepsilon,\tau;\mu}[\varphi]$ of the flow equation is quite complicated and is typically only possible if a small parameter is available. For this reason, we choose instead $F_{\varepsilon,\tau;\mu}$ that satisfies the flow equation up to some small error term $H_{\varepsilon,\tau;\mu}$.

3.3 Construction of effective force

Definition 3.18. We define $i_\sharp, i_\flat \in \mathbb{N}_+$ as the largest integers such that

$$\dim(\xi) + i_\sharp \dim(\lambda) \leq 0, \qquad \dim(\xi) - \dim(\Phi) + i_\flat \dim(\lambda) \leq 0,$$

where the dimensions $\dim(\xi)$, $\dim(\Phi)$, $\dim(\lambda)$ are introduced in Def. 2.1. We set $m_\flat := 3i_\flat$.

Remark 3.19. Note that for $\dim(\lambda) > 0$ is small, that is close to criticality, the constants $i_\sharp, i_\flat, m_\flat$ are big.

In this section we construct a suitable effective force functional $F_{\varepsilon,\tau;\mu}$. As we mentioned in the previous section, roughly speaking, an effective force $F_{\varepsilon,\tau;\mu}$ has to be chosen in such a way that it admits a continuum limit for all $\mu \in (0,1]$ and the functional $H_{\varepsilon,\tau;\eta}$, defined by Eq. (3.5), is such that $H_{\varepsilon,\tau;\eta}[\Phi_{\varepsilon,\tau;\eta}]$ remains integrable in $\eta \in [0,1]$ at $\eta = 0$ in the continuum limit. It turns out that a suitable choice of the mass counterterm is essential for the existence of the continuum limit. This will become apparent in Sec. 5. In order to satisfy the integrability condition for $H_{\varepsilon,\tau;\eta}$ we proceed as follows. We introduce a book-keeping parameter $\lambda \in \mathbb{R}$ in the expression for the force

$$F_{\varepsilon,\tau}[\varphi] = F_{\varepsilon,\tau;\mu=0}[\varphi] = \xi_{\varepsilon,\tau} - \lambda\,\varphi^3 + \sum_{i=1}^{i_\sharp} \lambda^i\, r^{(i)}_{\varepsilon,\tau}\,\varphi.$$

Subsequently, we demand that the effective force $F_{\varepsilon,\tau;\mu}[\varphi]$ satisfies the flow equation

$$\partial_\mu F_{\varepsilon,\tau;\mu}[\varphi] + \mathrm{D}F_{\varepsilon,\tau;\mu}[\varphi]\cdot(\dot{G}_{\varepsilon;\mu} * F_{\varepsilon,\tau;\mu}[\varphi]) = H_{\varepsilon,\tau;\mu}[\varphi] = O(\lambda^{i_\flat+1}) \tag{3.7}$$

up to an error term $H_{\varepsilon,\tau;\mu}[\varphi]$ of order $\lambda^{i_\flat+1}$.

Remark 3.20. The parameter λ plays the role of the strength of the nonlinear term and will be set to 1 later. Note that we assumed that the mass counterterm is a polynomial in λ.

Remark 3.21. The effective force constructed in this section satisfies the crucial bounds formulated in Theorem 4.6. However, the constant $C^F_{\varepsilon,\tau}(\kappa,\alpha,\beta)$ that appears in these bounds has moments uniformly bounded in the lattice spacing $\varepsilon \in \mathcal{A}$ only if the mass counterterms $(r^{(i)}_{\varepsilon,\tau})_{i\in\{1,\ldots,i_\sharp\}}$ are chosen appropriately.

In order to construct an approximate solution of the flow equation we make the following ansatz for the effective force

$$F_{\varepsilon,\tau;\mu}[\varphi](z) = \sum_{i=0}^{i_\flat} \lambda^i \sum_{m=0}^{3i} \int_{\Lambda^m_\varepsilon} F^{i,m}_{\varepsilon,\tau;\mu}(z;\mathrm{d}z_1,\ldots,\mathrm{d}z_m)\;\varphi(z_1)\ldots\varphi(z_m), \tag{3.8}$$

where $F^{i,m}_{\varepsilon,\tau;\mu} \in \mathscr{S}'(\Lambda^{1+m}_\varepsilon)$ are called the effective force coefficients.

Remark 3.22. The non-vanishing force coefficients $F^{i,m}_{\varepsilon,\tau} = F^{i,m}_{\varepsilon,\tau;\mu=0}$ take the following form

$$F^{0,0}_{\varepsilon,\tau}(z) = \xi_{\varepsilon,\tau}(z), \qquad F^{1,3}_{\varepsilon,\tau}(z;\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3) = -\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_1)\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_2)\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_3),$$
$$F^{i,1}_{\varepsilon,\tau}(z;\mathrm{d}z_1) = r^{(i)}_{\varepsilon,\tau}\,\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_1), \quad i \in \{0,\ldots,i_\sharp\},$$

where $(r^{(i)}_{\varepsilon,\tau})_{i\in\{1,\ldots,i_\sharp\}}$ are the counterterms and $\delta^{(z)}_{\Lambda_\varepsilon} \in \mathscr{S}'(\Lambda_\varepsilon)$ is the Dirac delta at $z \in \Lambda_\varepsilon$.

The above ansatz (3.8) for the effective force and Eq. (3.7) imply that the effective force coefficients satisfy the following flow equation

$$\partial_\mu F^{i,m}_{\varepsilon,\tau;\mu} = -\sum_{j=0}^{i}\sum_{k=0}^{m} (k+1)\,\mathbf{B}(\dot{G}_{\varepsilon;\mu}, F^{j,k+1}_{\varepsilon,\tau;\mu}, F^{i-j,m-k}_{\varepsilon,\tau;\mu}), \tag{3.9}$$

where the map $\mathbf{B}$ is defined by

$$\mathbf{B}(G,W,U)(z;\mathrm{d}z_1,\ldots,\mathrm{d}z_m) := \frac{1}{m!}\sum_{\pi\in\mathcal{P}_m}\int_{\Lambda^2_{\varepsilon,\tau}} W(z;\mathrm{d}z_0,\mathrm{d}z_{\pi(1)},\ldots,\mathrm{d}z_{\pi(k)})$$
$$\times\, G(z_0-w)\,U(w;\mathrm{d}z_{\pi(k+1)},\ldots,\mathrm{d}z_{\pi(m)})\,\mathrm{d}w.$$

The effective force coefficients are constructed recursively as follows:

(0) $F^{0,0}_{\varepsilon,\tau;\mu} := \xi_{\varepsilon,\tau}$ and $F^{i,m}_{\varepsilon,\tau;\mu} := 0$ if $m > (1+2i) \wedge 3i$.

(1) Assuming that all $F^{i,m}_{\varepsilon,\tau;\mu}$ with $i < i_\circ$, or $i = i_\circ$ and $m > m_\circ$ were constructed we define $\partial_\mu F^{i,m}_{\varepsilon,\tau;\mu}$ with $i = i_\circ$ and $m = m_\circ$ by Eq. (3.9).

(2) $F^{i,m}_{\varepsilon,\tau;\mu} := F^{i,m}_{\varepsilon,\tau} + \int_0^\mu \partial_\eta F^{i,m}_{\varepsilon,\tau;\eta}\,\mathrm{d}\eta$.

Remark 3.23. The effective force coefficients $F^{i;m}_{\varepsilon,\tau;\mu}$ are multi-linear functionals of the white noise $\xi_{\varepsilon,\tau}$ and in addition depend on the counterterms $(r^{(i)}_{\varepsilon,\tau})_{i\in\{1,\dots,i_\sharp\}}$ and the book-keeping parameter λ. The parameter λ was introduced only to motivate the ansatz (3.8) for the effective force. In what follows, we set $\lambda = 1$.

Remark 3.24. Using the flow equation (3.9) and the condition $F^{i,m}_{\varepsilon,\tau;\mu} = 0$ if $m > (1+2i)\wedge 3i$ one shows that

$$\partial_\mu F^{0,0}_{\varepsilon,\tau;\mu} = 0, \qquad \partial_\mu F^{1,3}_{\varepsilon,\tau;\mu} = 0.$$

Hence, at any scale $\mu \in [0,1]$ the sum in the ansatz (3.8) for the effective force $F_{\varepsilon,\tau;\mu}[\varphi](z)$ includes the noise term $\xi_{\varepsilon,\tau}(z)$ and the cubic term $-\varphi(z)^3$. The fact that the prefactor of the cubic term does not depend on the scale μ plays an important role in the proof of the a priori estimate presented in Sec. 4, cf. the second of the bounds in (4.3).

Definition 3.25. The finite collection $\{F^{i,m}_{\varepsilon,\tau;\mu} \,|\, m \in \{0,\dots,(1+2i)\wedge 3i\}, i \in \{0,\dots,i_\flat\}\}$ of effective force coefficients that appear in the ansatz for the effective force $F_{\varepsilon,\tau;\mu}$ is called the enhanced noise.

Remark 3.26. The enhanced noise introduced above plays an analogous role to the model in the regularity structure framework.

3.4 Stochastic estimates

In this section we present bounds for moments of elements of the enhanced noise called the stochastic estimates. The main result of this section is Theorem 3.38.

Definition 3.27. Let $\varepsilon \in \mathcal{A}$. We define $K_{\varepsilon;\mu} \in \mathscr{S}'(\Lambda_\varepsilon)$ to be the solution of the equation

$$(1+\mu^\sigma\partial_t)(1-\mu^2\Delta_\varepsilon)^2 K_{\varepsilon;\mu} = \delta^{(0)}_{\Lambda_\varepsilon},$$

where $\delta^{(0)}_{\Lambda_\varepsilon} \in \mathscr{S}'(\Lambda_\varepsilon)$ is the Dirac delta at the origin. For $n,m \in \mathbb{N}_0$ we set

$$K^{n,m}_{\varepsilon;\mu} := (\delta^{(0)}_{\Lambda_\varepsilon})^{\otimes n} \otimes (K_{\varepsilon;\mu})^{\otimes m}, \qquad \tilde{K}^{n,m}_{\varepsilon;\mu} := (K_{\varepsilon;\mu})^{\otimes n} \otimes (K_{\varepsilon;\mu} * K_{\varepsilon;\mu})^{\otimes m}.$$

Remark 3.28. Even though, in contrast to $J_{\varepsilon;\mu}$, the kernel $K_{\varepsilon;\mu}$ is not smooth, the convolution of a distribution with $K_{\varepsilon;\mu}$ has some smoothing effect.

Definition 3.29. Let $\mu \in (0,1]$, $\alpha,\beta \in [0,\infty)$, $m \in \mathbb{N}_0$ and $\gamma := (\dim(\xi) + 3\kappa)/3$. The weight $w_\mu \in C(\Lambda_0)$ is defined by

$$w_\mu(z) := (1 + \mu^{6\gamma/\kappa}|z|^2_\sigma)^{-1/6},$$

where the parabolic distance $|\bullet|_\sigma$ was introduced in Def. 3.7. The weight $\tilde{w}_\mu^{m,\alpha,\beta} \in C(\Lambda_0^{1+m})$ is defined by

$$\tilde{w}_\mu^{m,\alpha,\beta}(z;z_1,\ldots,z_m) := w_\mu^\alpha(z)\,\left(1+\max_{i\in\{1,\ldots,m\}}|z_i-z|_\sigma^2\right)^{-\beta/2}.$$

We set $\tilde{w}_\mu^{m,\beta} := \tilde{w}_\mu^{m,0,\beta}$.

Remark 3.30. Note that in order to prove the bound (2.2) implying tightness we have to study a Besov norm in space of the solution $\Phi_{\varepsilon,\tau}$ of the stochastic quantization equation at fixed time. We shall prove that

$$\|w_1(K_{\varepsilon;\mu} * \mathcal{L}_\varepsilon \Phi_{\varepsilon,\tau})\| \lesssim \mu^{\sigma-3\gamma}.$$

Using the fact that the operator $\mathcal{L}_\varepsilon$ is of first order in the time direction and the kernel $K_{\varepsilon;\mu}$ is the inverse of the differential operator that is of first order in the time direction one shows that the above bound implies $\|\Phi_{\varepsilon,\tau}(t,\bullet)\|_{\mathscr{B}_\varepsilon} \lesssim 1$, where $\|\bullet\|_{\mathscr{B}_\varepsilon}$ is the Hölder-Besov norm on $\mathbb{R}^d_\varepsilon$ of regularity $\sigma-3\gamma$ with the weight $w_1(0,\bullet)$.

Lemma 3.31. *Let $\varepsilon\in\mathcal{A}$. For all $\alpha\in[0,\infty)$ the following bounds*

(A) $\|w_\eta^\alpha(J_{\varepsilon;\mu} * \varphi)\| \lesssim \|w_\eta^\alpha(K_{\varepsilon;\mu} * \varphi)\| \lesssim \|w_\eta^\alpha\varphi\|$,

(B) $\|w_\eta^\alpha(J_{\varepsilon;\mu} * \varphi)\| \lesssim \|w_\eta^\alpha(J_{\varepsilon;\nu/2} * \varphi)\| \lesssim \|w_\eta^\alpha\varphi\|$,

(C) $\|w_\eta^\alpha(K_{\varepsilon;\mu} * \varphi)\| \lesssim \|w_\eta^\alpha(K_{\varepsilon;\nu} * \varphi)\| \lesssim \|w_\eta^\alpha\varphi\|$,

hold uniformly in $\varepsilon\in\mathcal{A}$, $\tau\in\mathbb{N}_+$, $\eta,\mu\in(0,1]$, $\nu\in(0,\mu]$ and $\varphi\in L^\infty(\Lambda_{\varepsilon,\tau})$.

Proof. Note that $J_{\varepsilon;\mu} = J_{\varepsilon;\mu} * J_{\varepsilon;\nu/2}$ and

$$J_{\varepsilon;\mu} = (1+\mu^\sigma\partial_t)(1-\mu^2\Delta_\varepsilon)^2 J_{\varepsilon;\mu} * K_{\varepsilon;\mu}, \qquad K_{\varepsilon;\mu} = (1+\nu^\sigma\partial_t)(1-\nu^2\Delta_\varepsilon)^2 K_{\varepsilon;\mu} * K_{\varepsilon;\nu}.$$

The stated bounds follow from the Young inequality for convolution. □

Definition 3.32. For $\kappa\in[0,1]$ and $i,m\in\mathbb{N}_0$ we define

$$\varrho_\kappa(i,m) := -(\dim(\xi)+3\kappa) + i(\dim(\lambda)-9\kappa) + m(\dim(\Phi)+3\kappa) + \kappa$$

We omit κ if $\kappa=0$.

Remark 3.33. Note that $i_\flat, i_\sharp \in\mathbb{N}_+$, introduced in Def. 3.18, are the smallest positive integers such that $\varrho(i_\flat+1,0)>0$, $\varrho(i_\sharp+1,1)>0$, respectively.

Remark 3.34. Recall that $F^{i,m}_{\varepsilon,\tau;\mu}$ are the effective force coefficients constructed using the recursive procedure presented in Sec. 3.3 and $r_{\varepsilon,\tau} = \sum_{i=1}^{i_\sharp} r^{(i)}_{\varepsilon,\tau} \in\mathbb{R}$ is the mass counterterm.

Definition 3.35. Let $\varepsilon \in \mathcal{A}, \tau \in \mathbb{N}_+$ and $m \in \mathbb{N}_0$. The vector space $\mathcal{V}^m_{\varepsilon,\tau}$ consists of kernels $V \in \mathscr{S}'(\Lambda_\varepsilon^{1+m})$ of operators $C_\mathrm{b}(\Lambda^m_{\varepsilon,\tau}) \mapsto C_\mathrm{b}(\Lambda_{\varepsilon,\tau})$ such that the following norm

$$\|V\|_{\mathcal{V}^m_{\varepsilon,\tau}} := \sup_{x\in\Lambda_\varepsilon} \int_{\Lambda_\varepsilon^m} |V(x;\mathrm{d}y_1 \dots \mathrm{d}y_m)|$$

is finite.

Definition 3.36. For $\kappa, \alpha \in (0,1]$, $\beta \in [0,\sigma)$ and $\varepsilon \in \mathcal{A}, \tau \in \mathbb{N}_+$ the random variable $C^F_{\varepsilon,\tau}(\kappa,\alpha,\beta) \in \mathbb{R}$ is defined by

$$C^F_{\varepsilon,\tau}(\kappa,\alpha,\beta) := 1 + \sum_{i=0}^{i_\flat} \sum_{m=0}^{3i} \sup_{\mu\in(0,1]} \mu^{-\varrho_\kappa(i,m)} \|\tilde{w}^{m,\alpha,\beta}_\mu(\tilde{K}^{1,m}_{\varepsilon;\mu} * F^{i,m}_{\varepsilon,\tau;\mu})\|_{\mathscr{V}^m_{\varepsilon,\tau}}. \tag{3.10}$$

Remark 3.37. Note that $C^F_{\varepsilon,\tau}(\kappa,\alpha,\beta)$ is monotonically decreasing in $\kappa, \alpha \in (0,1]$ and increasing in $\beta \in [0,\sigma)$.

Theorem 3.38. *There exists a choice of the counterterms* $(r^{(i)}_{\varepsilon,\tau})_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+,i\in\{1,\dots,i_\sharp\}}$ *such that for all* $\kappa, \alpha \in (0,1]$, $\beta \in [0,\sigma)$ *and* $p \in [1,\infty)$ *it holds*

$$\sup_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+} \mathbb{E}(C^F_{\varepsilon,\tau}(\kappa,\alpha,\beta))^p < \infty. \tag{3.11}$$

Remark 3.39. The above theorem implies, for example, that the white noise $\xi_{\varepsilon,\tau} = F^{0,0}_{\varepsilon,\tau;\mu}$ satisfies the bound

$$\sup_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+} \mathbb{E}\left(\sup_{\mu\in(0,1]} \mu^{-(\dim(\xi)+2\kappa)} \|w^\alpha_\mu(K_{\varepsilon;\mu} * \xi_{\varepsilon,\tau})\|\right)^p < \infty.$$

Note that the above bound is only true if $\kappa > 0$ and $\alpha > 0$.

Remark 3.40. Because of the presence of the weight $\tilde{w}^{m,\alpha,\beta}_\mu$ in Eq. (3.10) the bound (3.11) with $\beta > 0$ says that the effective force coefficients $F^{i,m}_{\varepsilon,\tau;\mu}$ are in some sense localized close to the diagonal $\{(z,\dots,z) \in \Lambda_\varepsilon^{1+m}\}$. The assumption $\beta < \sigma$ is related to the fact that for $\sigma \notin 2\mathbb{N}_+$ the fractional heat kernel G_ε decays only polynomially at infinity and satisfies the estimates stated in Lemma 3.9 only for $\beta < \sigma$.

Remark 3.41. In order to prove the above theorem we first establish uniform bounds for the cumulants of the effective force coefficients, which are stated in stated in Theorem 5.6. The rest of the proof is a combination of a Kolmogorov-type argument and a certain deterministic argument based on the flow equation for the effective force coefficients.

4 Coercive estimate and tightness

In this section we use the system of equations (3.4) for the coarse grained process and the remainder to derive an a priori estimate for the stochastic quantization equation that implies tightness of $(\nu_{\varepsilon,\tau})_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+}$. We first discuss the general idea of the proof by presenting an elementary application of the maximum principle in Sec. 4.1. Next, in Sec. 4.2 we show how to adapt the argument to the weighted setting and obtain an estimate applicable to the system of equations (3.4). In Sec. 4.3 we derive auxiliary estimates for the effective force and the source term in the equation for the remainder. Finally, in Sec. 4.4 we present the a priori estimate. The main result of this section is Theorem 4.14.

4.1 Application of maximum principle

In the lemma below we present a simple but not directly useful application of the maximum principle. In the remark below we argue how to obtain an interesting estimate using the idea of the proof of this lemma.

Lemma 4.1. *Let $\sigma \in (0,2]$. Suppose that $\varPsi \in C_0^\infty(\Lambda_{\varepsilon,\tau})$ and*

$$f := (\partial_t + (-\Delta_\varepsilon)^{\sigma/2})\varPsi + \varPsi^3. \tag{4.1}$$

Then $\|\varPsi\|^3 \leq \|f\|$.

Proof. Since $\varPsi$ is continuous and vanishes at infinity, $\varPsi$ attains a maximum at some point $z_\star \in \Lambda_{\varepsilon,\tau} = \mathbb{R}\times\mathbb{T}^d_{\varepsilon,\tau}$. We have $(\partial_t\varPsi)(z_\star) = 0$ and $(-\Delta_\varepsilon\varPsi)(z_\star) \geq 0$. Moreover, for $\sigma \in (0,2)$ it holds

$$((-\Delta_\varepsilon)^{\sigma/2}\varPsi)(z_\star) = C_\sigma \int_0^\infty (\varPsi(z_\star) - (\mathrm{e}^{s\Delta_\varepsilon}\varPsi)(z_\star))s^{-1-\sigma/2}\mathrm{d}s \geq 0$$

The inequality follows from the positivity of kernel of $\mathrm{e}^{s\Delta_\varepsilon}$ and the identity $\mathrm{e}^{s\Delta_\varepsilon}1 = 1$. Consequently, we obtain

$$\sup_{z\in\mathbb{R}\times\mathbb{T}^d_{\varepsilon,\tau}} \varPsi(z)^3 \leq \varPsi(z_\star)^3 \leq ((\partial_t + (-\Delta_\varepsilon)^{\sigma/2})\varPsi)(z_\star) + \varPsi(z_\star)^3 = f(z_\star) \leq \|f\|.$$

To complete the proof we apply the above reasoning to $-\varPsi$. □

Remark 4.2. The usefulness of the above elementary lemma is limited by the assumption that $\varPsi$ vanishes at infinity. In typical applications $\varPsi$ can grow polynomially at infinity and one has to apply the argument from the proof of the above lemma to $w\varPsi$, where w is a suitable weight decaying sufficiently fast at infinity. To this end, we multiply both sides of Eq. (4.1) by w^3 and rewrite the resulting equation in the form

$$w^3 f + w^2\big[\partial_t + (-\Delta_\varepsilon)^{\sigma/2}, w\big]\varPsi = w^2(\partial_t + (-\Delta_\varepsilon)^{\sigma/2})(w\varPsi) + (w\varPsi)^3,$$

where $[\bullet,\bullet]$ denotes the commutator. In order to get a useful estimate one has to bound the commutator term, which is nontrivial if $\sigma \in (0,2)$. It is useful to introduce the following map

$$\mathcal{D}^\sigma_\varepsilon(\varphi,\psi) := (-\Delta_\varepsilon)^{\sigma/2}(\varphi\psi) - \varphi(-\Delta_\varepsilon)^{\sigma/2}\psi - \psi(-\Delta_\varepsilon)^{\sigma/2}\varphi$$

and show that it satisfies the bounds

$$|\mathcal{D}^\sigma_\varepsilon(\varphi,\psi)| \leq \mathcal{D}^\sigma_\varepsilon(\varphi,\varphi)^{1/2}\mathcal{D}^\sigma_\varepsilon(\psi,\psi)^{1/2}, \qquad \|\mathcal{D}^\sigma_\varepsilon(\varphi,\varphi)\|^{1/2} \lesssim \|\varphi\|^{1-\sigma/2}\|\nabla\varphi\|^{\sigma/2}. \tag{4.2}$$

Using the first of the above bounds we estimate

$$\begin{aligned}\big\|w^2\big[\partial_t+(-\Delta_\varepsilon)^{\sigma/2},w\big]\varPsi\big\| &= \|w^4\varPsi(-\Delta_\varepsilon)^{\sigma/2}(1/w) - w^3\mathcal{D}^\sigma_\varepsilon(1/w,w\varPsi)\| \\ &\leq \|w^3(-\Delta_\varepsilon)^{\sigma/2}(1/w)\|\,\|w\varPsi\| + \|w^3\mathcal{D}^\sigma_\varepsilon(1/w)^{1/2}\|\,\|\mathcal{D}^\sigma_\varepsilon(w\varPsi)^{1/2}\|.\end{aligned}$$

The second of the bound (4.2) implies that

$$\begin{aligned}&\big\|w^2\big[\partial_t+(-\Delta_\varepsilon)^{\sigma/2},w\big]\varPsi\big\| \\ &\quad\lesssim \big(\|w^3(-\Delta_\varepsilon)^{\sigma/2}(1/w)\| + \|w^3\mathcal{D}^\sigma_\varepsilon(1/w)^{1/2}\|\,\|\nabla w\|^{\sigma/2}\big)\,\|w\varPsi\| \\ &\qquad + \|w^3\mathcal{D}^\sigma_\varepsilon(1/w)^{1/2}\|\,\|w\varPsi\|^{1-\sigma/2}\,\|w\nabla\varPsi\|^{\sigma/2}.\end{aligned}$$

In consequence, for suitable weights w we have

$$\big\|w^2\big[\partial_t+(-\Delta_\varepsilon)^{\sigma/2},w\big]\varPsi\big\| \lesssim \|w\varPsi\| + \|w\varPsi\|^{1-\sigma/2}\,\|w\nabla\varPsi\|^{\sigma/2}.$$

As a result, by the argument from the proof of Lemma 4.1 we obtain a potentially useful estimate

$$\|w\varPsi\|^3 \lesssim \|w^3 f\| + \|w\varPsi\| + \|w\varPsi\|^{1-\sigma/2}\,\|w\nabla\varPsi\|^{\sigma/2}.$$

Note that the proof of the above bound in the case $\sigma=2$ is elementary.

4.2 Coercive estimate with weight

In this section we derive an estimate applicable to the system of equations (3.4). Recall that we view (3.4) as the system of equation for the pair (3.6) consisting of the coarse grained process and the remainder at scales $\mu\in(0,\bar\mu]$, where $\bar\mu\in(0,1]$ is a terminal scale that at this stage is arbitrary.

Definition 4.3. Let $\kappa,\bar\mu\in(0,1]$. We define the following norms

$$|||\mu\mapsto\varPhi_\mu|||_{\bar\mu} := \sup_{\mu\in(0,\bar\mu]}\mu^\gamma\,\|w_\mu\varPhi_\mu\|, \qquad |||\mu\mapsto f_\mu|||_{\sharp,\bar\mu} := \sup_{\mu\in(0,\bar\mu]}\mu^{3\gamma}\,\|w_\mu^3 f_\mu\|,$$

where $\gamma := (\dim(\xi)+3\kappa)/3$ and the weight w_μ is introduced in Def. 3.29.

Lemma 4.4. *Let* $\kappa\in(0,1]$ *and*

$$\varPhi_{\varepsilon;\mu} := J_{\varepsilon;\mu}\ast\varPhi, \qquad f_{\varepsilon;\mu} := \mathcal{L}_\varepsilon\varPhi_{\varepsilon;\mu} + \varPhi^3_{\varepsilon;\mu}$$

for $\varPhi\in C_\flat(\Lambda_{\varepsilon,\tau})$ *and* $\mu\in(0,1]$. *Then there exists* $C\in(0,\infty)$ *such that for all* $\varepsilon\in\mathcal{A}$, $\tau\in\mathbb{N}_+$, $\bar\mu\in(0,1]$ *and* $\varPhi\in C_\flat(\Lambda_{\varepsilon,\tau})$ *the following bound is true*

$$|||\mu\mapsto\varPhi_{\varepsilon;\mu}|||^3_{\bar\mu} \vee |||\mu\mapsto\mathcal{L}_\varepsilon\varPhi_{\varepsilon;\mu}|||_{\sharp,\bar\mu} \leq C\,\big(\bar\mu^\kappa + |||\mu\mapsto f_{\varepsilon;\mu}|||_{\sharp,\bar\mu}\big).$$

Remark 4.5. In order to prove that above lemma we use the argument from Remark 4.2. Note that by the support property of $J_{\varepsilon;\mu}$ it holds $\|w_\mu \nabla \Psi_{\varepsilon;\mu}\|^{\sigma/2} \lesssim \mu^{-\sigma/2}\,\|w\Psi_{\varepsilon;\mu}\|^{\sigma/2}$. Alternatively, one could also bound $\|w_\mu \nabla\Psi_{\varepsilon;\mu}\|$ in terms of $\|w_\mu \mathcal{L}_\varepsilon \Psi_{\varepsilon;\mu}\|$ and then use Lemma 4.11 to account for the fact that the norm $|||\mu \mapsto \mathcal{L}_\varepsilon \Phi_{\varepsilon;\mu}|||_{\sharp,\bar\mu}$ involves the weight w_μ^3 and not w_μ. Finally, we apply the Young inequality for products and the bound $3(2\gamma - \sigma/2)/2 \geq \kappa$.

4.3 Auxiliary bounds

In this section we show that the effective force functional $F_{\varepsilon,\tau;\mu}$ and the source term $H_{\varepsilon,\tau;\mu}$ in the equation for the remainder can be controlled in terms of the constant $C^F_{\varepsilon,\tau}(\kappa,\kappa,1)$ measuring the size of the enhanced noise introduced in Def. (3.36).

Theorem 4.6. *For all $\kappa \in (0,1]$ sufficiently small there exists $c \in (0,\infty)$ such that for all $\varepsilon \in \mathcal{A}$, $\tau \in \mathbb{N}_+$, $\bar\mu \in (0,1]$, $\mu \in (0,\bar\mu]$, $\eta \in (0,\mu]$ and $\Phi,\zeta \in C_\flat(\Lambda_{\varepsilon,\tau})$ it holds*

$$\begin{aligned} \mu^{3\gamma}\,\|w_\mu(J_{\varepsilon;\mu} * \Phi^3_{\varepsilon;\mu} - \Phi^3_{\varepsilon;\mu})\| &\leq c\,\bar\mu^\kappa\,|||\Phi_{\varepsilon;\bullet}|||^2_{\bar\mu}\,|||\mathcal{L}_\varepsilon \Phi_{\varepsilon;\bullet}|||_{\sharp,\bar\mu} \\ \mu^{3\gamma}\,\|w_\mu(J_{\varepsilon;\mu} * (F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon;\mu}] + \Phi^3_{\varepsilon;\mu}))\| &\leq c\,C^F_{\varepsilon,\tau}\,\bar\mu^\kappa(1 + |||\Phi_{\varepsilon;\bullet}|||_{\bar\mu})^{m_\flat} \end{aligned} \tag{4.3}$$

and

$$\mu^{3\gamma}\,\|w_\mu(K_{\varepsilon;\mu} * F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon;\mu}])\| \leq c\,C^F_{\varepsilon,\tau}\,(1 + |||\Phi_{\varepsilon;\bullet}|||_{\bar\mu})^{m_\flat} \tag{4.4}$$

as well as

$$\begin{aligned} \|w^3_\mu(K_{\varepsilon;\eta} * (\mathrm{D}F_{\varepsilon,\tau;\eta}[\Phi_{\varepsilon;\eta}]\cdot(\dot G_{\varepsilon;\eta} * \zeta)))\| &\leq c\,C^F_{\varepsilon,\tau}\,\eta^{\kappa-1}(1 + |||\Phi_{\varepsilon;\bullet}|||_{\bar\mu})^{m_\flat}\,\|w^2_\mu(K_{\varepsilon;\eta} * \zeta)\| \\ \|w^3_\mu(K_{\varepsilon;\eta} * H_{\varepsilon,\tau;\eta}[\Phi_{\varepsilon;\eta}])\| &\leq c\,(C^F_{\varepsilon,\tau})^2\,\eta^{\kappa-1}(1 + |||\Phi_{\varepsilon;\bullet}|||_{\bar\mu})^{2m_\flat}, \end{aligned} \tag{4.5}$$

where $\Phi_{\varepsilon;\mu} = J_{\varepsilon;\mu} * \Phi$, the functional $F_{\varepsilon,\tau;\mu}$ is the effective force constructed in Sec. 3.3, the functional $H_{\varepsilon,\tau;\eta}$ is defined by Eq. (3.5), the constant $C^F_{\varepsilon,\tau} := C^F_{\varepsilon,\tau}(\kappa,\kappa,1)$ is introduced in Def. (3.36) and $m_\flat$ is introduced in Def. 3.18.

Remark 4.7. The proof of all of the bounds relies crucially on Lemma 4.11 stated below, which shows that the weight in the expressions for the norms $|||\bullet|||_{\bar\mu}$, $|||\bullet|||_{\sharp,\bar\mu}$ can be replaced with a weight decaying at infinity at a slower rate.

Remark 4.8. The bounds (4.3) say that the size of $J_{\varepsilon;\mu} * F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon;\mu}]$ measured with the use of the $|||\bullet|||_{\sharp,\bar\mu}$ is a small perturbation of order $\bar\mu^\kappa$ of the cubic term $\Phi_{\varepsilon;\mu}$ if the terminal scale $\bar\mu \in (0,1]$ is chosen small and the norms $|||\Phi_{\varepsilon;\bullet}|||_{\bar\mu}$, $|||\mathcal{L}_\varepsilon\Phi_{\varepsilon;\bullet}|||_{\sharp,\bar\mu}$ are bounded.

Remark 4.9. In the proof of the second of the bounds (4.3) we use the fact that at all scales $\mu \in [0,1]$ the sum in the ansatz (3.8) for the effective force $F_{\varepsilon,\tau;\mu}[\varphi]$ contains the cubic term $\varphi(z)$ with the prefactor -1. Note that the sum in the ansatz (3.8) contains also the white noise term $\xi_{\varepsilon,\tau}$. Observe that for term we have the following estimate

$$\mu^{3\gamma}\,\|w_\mu(J_{\varepsilon;\mu} * \xi_{\varepsilon,\tau}))\| \leq c\,\bar\mu^\kappa\mu^{-\varrho_\kappa(0,0)}\,\|w^\kappa_\mu(K_{\varepsilon;\mu} * \xi_{\varepsilon,\tau}))\| = c\,C^F_{\varepsilon,\tau}\,\bar\mu^\kappa,$$

where we used the identity $\varrho_\kappa(0,0) = 3\gamma + \kappa$, Lemma 3.31 (A), Def. (3.36) of $C^F_{\varepsilon,\tau}(\kappa,\kappa,1)$ and the fact that $\|\bullet\|_{\mathscr{V}^0_{\varepsilon,\tau}} = \|\bullet\|_{L^\infty(\Lambda_\varepsilon)}$.

Remark 4.10. The last of the bound (4.5) relies crucially on the fact that the parameter $i_\flat \in \mathbb{N}+$ in Eq. (3.7) is chosen big enough, that is the effective force $F_{\varepsilon,\tau;\mu}$ solves the flow equation up to a sufficiently small error term.

Sketch of the proof. The fact that $\varPhi_{\varepsilon;\mu} \in C((0,1], C_\flat(\Lambda_{\varepsilon,\tau}))$ is of the form $\varPhi_{\varepsilon;\mu} = J_{\varepsilon;\mu} * \varPhi$ is only used in the proof of the first of the bounds (4.3). In order to prove this bound we use the following decomposition

$$\varPhi_{\varepsilon;\mu} = \varPhi_{\varepsilon;6\mu} + (\varPhi_{\varepsilon;\mu} - \varPhi_{\varepsilon;6\mu})$$

of $\varPhi_{\varepsilon;\mu}$ into low- and high-frequency parts. By Def. 3.2 the Fourier transform of $J_{\varepsilon;\mu}$ is equal to 1 on the support of $\varPhi^3_{\varepsilon;6\mu}$. As a consequence,

$$J_{\varepsilon;\mu} * \varPhi^3_{\varepsilon;6\mu} - \varPhi^3_{\varepsilon;6\mu} = 0$$

and

$$\|w_\mu(J_{\varepsilon;\mu} * \varPhi^3_{\varepsilon;\mu} - \varPhi^3_{\varepsilon;\mu})\| \leq c\left(\|w^\kappa_\mu \varPhi_{\varepsilon;\mu}\|^2 + \|w^\kappa_\mu \varPhi_{\varepsilon;6\mu}\|^2\right)\|w^\kappa_\mu(\varPhi_{\varepsilon;\mu} - \varPhi_{\varepsilon;3\mu})\|$$

provided $\kappa \in (0,1/3]$, where $c \in (0,\infty)$ is a universal constant. By Lemma 4.11 and the fact that the support of the Fourier transform of $\varPhi_{\varepsilon;\mu} - \varPhi_{\varepsilon;3\mu}$ is contained in a shell and $\varPhi_{\varepsilon;6\mu} = J_{\varepsilon;6\mu} * \varPhi_{\varepsilon;\mu}$ we obtain

$$\begin{aligned}\mu^\gamma\,\|w^\kappa_\mu \varPhi_{\varepsilon;6\mu}\| \leq c\,\mu^\gamma\,\|w^\kappa_\mu \varPhi_{\varepsilon;\mu}\| \leq c\,\|\!|\varPhi_{\varepsilon;\bullet}|\!\|_{\bar{\mu}},\\ \mu^\gamma\,\|w^\kappa_\mu(\varPhi_{\varepsilon;\mu} - \varPhi_{\varepsilon;3\mu})\| \leq c\,\mu^{\sigma-2\gamma}\,\|\!|\mathcal{L}_\varepsilon \varPhi_{\varepsilon;\bullet}|\!\|_{\sharp,\bar{\mu}},\end{aligned}$$

where $c \in (0,\infty)$ is a universal constant. The first of the bounds (4.3) follows since $\sigma - 2\gamma \geq \kappa$ provided $\kappa \in (0,1]$ is small enough.

Let us proceed to the proof of the remaining bounds. By Remark 3.24 we have

$$F^{1,3}_{\varepsilon,\tau;\mu}(z;\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3) = F^{1,3}_{\varepsilon,\tau}(z;\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3) = -\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_1)\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_2)\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_3). \tag{4.6}$$

Consequently, $F_{\varepsilon,\tau;\mu}[\varPhi_{\varepsilon;\mu}] + \varPhi^3_{\varepsilon;\mu}$ is given by the sum (3.8) with the term $i = 0$, $m = 3$ removed. The second of the bounds (4.3) with $J_{\varepsilon;\mu}$ replaced by $K_{\varepsilon;\mu}$ follows then directly from Def. 3.36 of $C^F_{\varepsilon,\tau}(\kappa,\alpha,\beta)$ and the fact that

$$\varrho_\kappa(i,m) + (\dim(\xi) + 3\kappa) - m(\dim(\xi) + 3\kappa)/3 \geq \kappa$$

for all $i, m \in \mathbb{N}_0$ such that $F^{i,m}_{\varepsilon,\tau;\mu} \neq 0$ and $(i,m) \neq (1,3)$ provided $\kappa \in (0,1]$ is small enough. The second of the bounds (4.3) is now a consequence of Lemma 3.31 (A). The bounds (4.4) follows from the second of the bounds (4.3) with $J_{\varepsilon;\mu}$ replaced by $K_{\varepsilon;\mu}$ and Eq. (4.6). To prove the first of the bounds (4.5) we use Def. 3.36 of $C^F_{\varepsilon,\tau}(\kappa,\alpha,\beta)$, the first of the bounds stated in Lemma 3.9, Eq. (4.6) and the fact that

$$\varrho_\kappa(i,m)+\sigma-(m-1)(\dim(\xi)+3\kappa)/3\geq\kappa$$

for all $i,m\in\mathbb{N}_0$ such that $F^{i,m}_{\varepsilon,\tau;\mu}\neq 0$ and $(i,m)\neq(1,3)$ provided $\kappa\in(0,1]$ is small enough. Finally, the proof of the second the bounds (4.5) relies on the fact that $\varrho_\kappa(i,m)\geq\kappa$ for all $i\in\{i_\flat+1,i_\flat+2,\ldots\}$ and for all $m\in\mathbb{N}_0$ such that $F^{i,m}_{\varepsilon,\tau;\mu}\neq 0$ provided $\kappa\in(0,1]$ is small enough. $\square$

Lemma 4.11. *For all $\kappa\in(0,1]$ there exists $C\in(0,\infty)$ such that*

$$\sup_{\mu\in(0,\bar\mu]}\mu^\gamma\,\|w^\kappa_\mu\varPhi_{\varepsilon;\mu}\|\leq C\,|||\mu\mapsto\varPhi_{\varepsilon;\mu}|||_{\bar\mu},\qquad\sup_{\mu\in(0,\bar\mu]}\mu^{3\gamma}\,\|w^{3\kappa}_\mu\varPhi_{\varepsilon;\mu}\|\leq C\,|||\mu\mapsto\varPhi_{\varepsilon;\mu}|||_{\sharp,\bar\mu}\tag{4.7}$$

for all $\varepsilon\in\mathcal{A}$, $\tau\in\mathbb{N}_+$ and $\varPhi\in C_\flat(\Lambda_{\varepsilon,\tau})$, where $\varPhi_{\varepsilon;\mu}=J_{\varepsilon;\mu}\ast\varPhi$.

Remark 4.12. Note that the weight $w_\mu(z)=(1+\mu^{6\gamma/\kappa}|z|^2_\sigma)^{-1/6}$ introduced in Def. 3.29 depends implicitly on κ.

Remark 4.13. In order to make the lemma plausible observe that

$$\|w_{\mu/2^i}\varPhi_{\varepsilon;\mu}\|=\|w_{\mu/2^i}(J_{\varepsilon;\mu}\ast\varPhi_{\varepsilon;\mu/2^i})\|\lesssim\|w_{\mu/2^i}\varPhi_{\varepsilon;\mu/2^i}\|\leq(\mu/2^i)^{-\gamma}\,|||\mu\mapsto\varPhi_{\varepsilon;\mu}|||_{\bar\mu}$$

uniformly in $\varepsilon\in\mathcal{A}$, $\tau\in\mathbb{N}_+$, $\mu\in(0,\bar\mu]$ and $i\in\mathbb{N}_+$ and $\varPhi\in C_\flat(\Lambda_{\varepsilon,\tau})$, where we used the identity $J_{\varepsilon;\mu}\ast J_{\varepsilon;\mu/2^i}=J_{\varepsilon;\mu}$ and Lemma 3.31 (B). The above estimate says that the weight w_μ in the trivial bound

$$\|w_\mu\varPhi_{\varepsilon;\mu}\|\leq\mu^{-\gamma}\,|||\mu\mapsto\varPhi_{\varepsilon;\mu}|||_{\bar\mu}$$

can be replaced by a flatter weight $w_{\mu/2^i}$ at a cost of extra factor $2^{\gamma i}$.

Proof. Let $(\chi_i)_{i\in\{-1\}\cup\mathbb{N}_0}$ be smooth dyadic decomposition of unity on Λ_0 such that for every $z\in\Lambda_0$ it holds $\chi_{i-1}(z)+\chi_i(z)+\chi_{i+1}(z)\geq 1$ for some $i\in\mathbb{N}_0$. Set $\chi_{i,\mu}(z)=\chi_i(\mu^{3\gamma/\kappa}z)$. We have

$$\begin{aligned}\mu^\gamma\,\|w^\kappa_\mu\varPhi_{\varepsilon;\mu}\|\leq 3\mu^\gamma\,\|\chi_{-1,\mu}w^\kappa_\mu\varPhi_{\varepsilon;\mu}\|+3\mu^\gamma\sup_{i\in\mathbb{N}_0}\|\chi_{i,\mu}w^\kappa_\mu\varPhi_{\varepsilon;\mu}\|\\ \lesssim\mu^\gamma\,\|w_\mu\varPhi_{\varepsilon;\mu}\|+\mu^\gamma\sup_{i\in\mathbb{N}_0}\|\chi_{i,\mu}w^\kappa_\mu\varPhi_{\varepsilon;\mu/2^i}\|,\end{aligned}$$

where the second estimate is a consequence of Lemma 3.31 (B). Consequently,

$$\begin{aligned}\mu^\gamma\,\|w^\kappa_\mu\varPhi_{\varepsilon;\mu}\|\lesssim\mu^\gamma\,\|w_\mu\varPhi_{\varepsilon;\mu}\|+\mu^\gamma\sup_{i\in\mathbb{N}_0}\|\chi_{i,\mu}w^\kappa_\mu/w_{\mu/2^i}\|\|w_{\mu/2^i}\varPhi_{\varepsilon;\mu/2^i}\|\\ \lesssim\mu^\gamma\,\|w_\mu\varPhi_{\varepsilon;\mu}\|+\mu^\gamma\sup_{i\in\mathbb{N}_0}2^{-i\gamma}\|w_{\mu/2^i}\varPhi_{\varepsilon;\mu/2^i}\|\lesssim C\,|||\mu\mapsto J_{\varepsilon;\mu}\ast\varPhi|||_{\bar\mu}.\end{aligned}$$

This proves the first of the bounds (4.7). The proof of the second bound follows the same lines. $\square$

4.4 A priori estimate

Let $\mathit{\Phi}_{\varepsilon,\tau}$ be the stationary solution of the stochastic quantization equation (3.1) and let $\mathit{\Phi}_{\varepsilon,\tau;\mu}$ be the corresponding coarse-grained process. Suppose that the effective force $F_{\varepsilon,\tau;\mu}$ is of the form (3.8), where the effective force coefficients are constructed as discussed in Sec. 3.3. Recall that the stochastic quantization equation (3.1) is equivalent to the following system of equations

$$\begin{cases} \mathcal{L}_\varepsilon \mathit{\Phi}_{\varepsilon,\tau;\mu} + \mathit{\Phi}^3_{\varepsilon,\tau;\mu} = f_{\varepsilon,\tau;\mu} \\ \zeta_{\varepsilon,\tau;\mu} = -\int_0^\mu (H_{\varepsilon,\tau;\eta}[\mathit{\Phi}_{\varepsilon,\tau;\eta}] + \mathrm{D}F_{\varepsilon,\tau;\eta}[\mathit{\Phi}_{\varepsilon,\tau;\eta}]\cdot(\dot{G}_{\varepsilon;\eta} * \zeta_{\varepsilon,\tau;\eta}))\,\mathrm{d}\eta \end{cases}$$

for the pair (3.6) consisting of the coarse-grained process and the remainder, where

$$f_{\varepsilon,\tau;\mu} := (J_{\varepsilon;\mu} * F_{\varepsilon,\tau;\mu}[\mathit{\Phi}_{\varepsilon,\tau;\mu}] + \mathit{\Phi}^3_{\varepsilon,\tau;\mu}) + J_{\varepsilon;\mu} * \zeta_{\varepsilon,\tau;\mu}. \tag{4.8}$$

The deterministic estimate stated in the theorem below is the main technical result of these notes. Below we argue how to combine this estimate with the bounds for moments of the effective force coefficients established in Theorem 3.38 to conclude the bound (2.2) implying tightness of $(\nu_{\varepsilon,\tau})_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+}$.

Theorem 4.14. *For all $\kappa\in(0,1]$ sufficiently small there exists $c\in[1,\infty)$ such that for all $\varepsilon\in\mathcal{A}$ and $\tau\in\mathbb{N}_+$ the following conditions*

$$|||\mu\mapsto\mathit{\Phi}_{\varepsilon,\tau;\mu}|||^3_{\bar\mu} \vee |||\mu\mapsto\mathcal{L}_\varepsilon\mathit{\Phi}_{\varepsilon,\tau;\mu}|||_{\sharp,\bar\mu} \leq 1, \qquad \bar\mu^{-\kappa} \leq c\,(C^F_{\varepsilon,\tau})^2$$

are satisfied for some $\bar\mu\equiv\bar\mu(\varepsilon,\tau)\in(0,1]$, where $C^F_{\varepsilon,\tau} := C^F_{\varepsilon,\tau}(\kappa,\kappa,1)$ is defined by Eq. (3.10).

Proof. Recall the stochastic process $\mathit{\Phi}_{\varepsilon,\tau;\mu}$ is the stationary solution the finite dimensional stochastic differential equation (2.1). By elementary estimates one shows that for every fixed $\varepsilon\in\mathcal{A}$ and $\tau\in\mathbb{N}_+$ the function

$$(0,1]\ni\bar\mu\mapsto C^F_{\varepsilon,\tau}\bar\mu^\kappa \vee |||\mu\mapsto\mathit{\Phi}_{\varepsilon,\tau;\mu}|||^3_{\bar\mu} \vee |||\mu\mapsto\mathcal{L}_\varepsilon\mathit{\Phi}_{\varepsilon,\tau;\mu}|||_{\sharp,\bar\mu} \in [0,\infty)$$

is continuous, increasing and vanishes in the limit $\bar\mu\searrow 0$. If the above function takes values in $[0,1]$, then we choose $\bar\mu=1$ and the proof is finished. Otherwise let us fix $\bar\mu\in(0,1]$ so that the above function evaluated at $\bar\mu$ takes the value 1. By the coercive estimate proved in Lemma 4.4 we obtain

$$\begin{aligned} 1 = C^F_{\varepsilon,\tau}\bar\mu^\kappa \vee |||\mu\mapsto\mathit{\Phi}_{\varepsilon,\tau;\mu}|||^3_{\bar\mu} \vee |||\mu\mapsto\mathcal{L}_\varepsilon\mathit{\Phi}_{\varepsilon,\tau;\mu}|||_{\sharp,\bar\mu} \\ \leq C^F_{\varepsilon,\tau}\bar\mu^\kappa \vee C\left(\bar\mu^\kappa + |||\mu\mapsto f_{\varepsilon,\tau;\mu}|||_{\sharp,\bar\mu}\right). \end{aligned} \tag{4.9}$$

Using the auxiliary bounds (4.3) we get

$$|||\mu\mapsto(J_{\varepsilon;\mu} * F_{\varepsilon,\tau;\mu}[\mathit{\Phi}_{\varepsilon,\tau;\mu}] + \mathit{\Phi}^3_{\varepsilon,\tau;\mu})|||_{\sharp,\bar\mu} \leq c\,C^F_{\varepsilon,\tau}\,\bar\mu^\kappa, \tag{4.10}$$

where $c \in (0,\infty)$ is a universal constant. Using the auxiliary bounds (4.4) and (4.5) together the Growall lemma we obtain

$$\|\!|\mu \mapsto J_{\varepsilon;\mu} * \zeta_{\varepsilon,\tau;\mu}\|\!|_{\sharp,\bar{\mu}} \leq c\,(C^F_{\varepsilon,\tau})^2 \bar{\mu}^\kappa, \tag{4.11}$$

Consequently, by the bound (4.9), the equality (4.8) and the triangle inequality together with the bounds (4.10), (4.11) we get

$$1 \leq c\,(C^F_{\varepsilon,\tau})^2 \bar{\mu}^\kappa,$$

where $c \in (0,\infty)$ is a universal constant. This finishes the proof. □

Remark 4.15. The presence of the term $C^F_{\varepsilon,\tau}\bar{\mu}^\kappa$ in the maximum in the first line of (4.9) is only used in the proof of the bound (4.11) for the remainder given in the remark below.

Remark 4.16. Let us provide some details about the proof of the bound (4.11) for the remainder claimed in the proof above. Observe that

$$\begin{aligned}\mu^{3\gamma}\|w_\mu^2(K_{\varepsilon;\eta} * \zeta_{\varepsilon,\tau;\eta})\| &\leq \mu^{3\gamma}\|w_\mu^2 w_\eta(K_{\varepsilon;\eta} * \zeta_{\varepsilon,\tau;\eta})\| + \mu^{3\gamma}\|w_\mu(1-w_\eta)\|\|w_\mu(K_{\varepsilon;\eta} * \zeta_{\varepsilon,\tau;\eta})\| \\ &\leq \mu^{3\gamma}\|w_\mu^2 w_\eta(K_{\varepsilon;\eta} * \zeta_{\varepsilon,\tau;\eta})\| + c\,\eta^{3\gamma}\,\|w_\eta(K_{\varepsilon;\eta} * (\mathcal{L}_\varepsilon \Phi_{\varepsilon,\tau;\eta} - F_{\varepsilon,\tau;\eta}[\Phi_{\varepsilon,\tau;\eta}]))\|\end{aligned}$$

for $0 < \eta < \mu \leq 1$, where $c \in (0,\infty)$ is a universal constant. Using Lemma 3.31 (A) and the auxiliary bound (4.4) we arrive at

$$\begin{aligned}\mu^{3\gamma}\|w_\mu^2 w_\eta(K_{\varepsilon;\eta} * \zeta_{\varepsilon,\tau;\eta})\| + c\,\eta^{3\gamma}\,\|w_\eta(J_{\varepsilon;\eta} * \mathcal{L}_\varepsilon \Phi_{\varepsilon,\tau;\eta})\| + \eta^{3\gamma}\,\|w_\eta(K_{\varepsilon;\eta} * F_{\varepsilon,\tau;\eta}[\Phi_{\varepsilon,\tau;\eta}])\| \\ \leq \mu^{3\gamma}\|w_\mu^2 w_\eta(K_{\varepsilon;\eta} * \zeta_{\varepsilon,\tau;\eta})\| + c + c\,C^F_{\varepsilon,\tau}\end{aligned}$$

for $0 < \eta < \mu \leq 1$, where $c \in (0,\infty)$ is a universal constant. Consequently, by the Gronwall lemma, the auxiliary estimates (4.5) and $C^F_{\varepsilon,\tau}\bar{\mu}^\kappa \leq 1$ we obtain the bound

$$\mu^{3\gamma}\,\|w_\mu^2 w_\eta(K_{\varepsilon;\eta} * \zeta_{\varepsilon,\tau;\eta})\| \leq c\,(C^F_{\varepsilon,\tau})^2\bar{\mu}^\kappa \exp(c\,C^F_{\varepsilon,\tau}\bar{\mu}^\kappa) \leq \tilde{c}\,(C^F_{\varepsilon,\tau})^2\bar{\mu}^\kappa$$

for $0 < \eta < \mu \leq 1$, where $c, \tilde{c} \in (0,\infty)$ are some universal constants. By Lemma 3.31 (A) the above bound implies

$$\|\!|\mu \mapsto J_{\varepsilon;\mu} * \zeta_{\varepsilon,\tau;\mu}\|\!|_{\sharp,\bar{\mu}} \leq \tilde{c}\,\|\!|\mu \mapsto K_{\varepsilon;\mu} * \zeta_{\varepsilon,\tau;\mu}\|\!|_{\sharp,\bar{\mu}} \leq c\,(C^F_{\varepsilon,\tau})^2\bar{\mu}^\kappa, \tag{4.12}$$

where $c, \tilde{c} \in (0,\infty)$ are some universal constants.

It is easy to prove the following estimate for the parabolic spacetime Besov norm of the solution of the stochastic quantization equation

$$\|\Phi_{\tau,\varepsilon}\|_{\mathscr{B}^{-\gamma}_{\infty,\infty}(\Lambda_0,w_1)} \leq c\;\|\!|\mu \mapsto \Phi_{\varepsilon,\tau;\mu}\|\!|_1 \leq c\;\bar{\mu}^{-\gamma}\;\|\!|\mu \mapsto \Phi_{\varepsilon,\tau;\mu}\|\!|_{\bar{\mu}},$$

where c is a universal constant. Applying the above bound with the scale $\bar{\mu}=\bar{\mu}(\varepsilon,\tau)$ chosen as in Theorem 4.14 we show that

$$\mathbb{E}\|\Phi_{\tau,\varepsilon}\|^p_{\mathscr{B}^{-\gamma}_{\infty,\infty}(\Lambda_0,w_1)} \leq c^p\,\mathbb{E}(C^F_{\varepsilon,\tau})^{2p\gamma/\kappa}<\infty$$

for all $p\in\mathbb{N}_+$, where the last bound is a consequence of Theorem 3.38. Unfortunately, the above bound does not allow to conclude the desired bound (2.2) implying tightness of $(\nu_{\varepsilon,\tau})_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+}$. In order to prove (2.2) we need an estimate for the solution $\Phi_{\varepsilon,\tau}$ at a fixed time. To this end, we choose the scale $\bar{\mu}=\bar{\mu}(\varepsilon,\tau)$ as in Theorem 4.14 and prove that

$$\begin{aligned}\|\!|K_{\varepsilon;\mu}*\mathcal{L}_\varepsilon\Phi_{\varepsilon,\tau}\|\!|_{\sharp,\bar{\mu}} &= \|\!|K_{\varepsilon;\mu}*(F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon,\tau;\mu}]+\zeta_{\varepsilon,\tau;\mu})\|\!|_{\sharp,\bar{\mu}}\\ &\leq \|\!|K_{\varepsilon;\mu}*F_{\varepsilon,\tau;\mu}[\Phi_{\varepsilon,\tau;\mu}]\|\!|_{\sharp,\bar{\mu}}+\|\!|K_{\varepsilon;\mu}*\zeta_{\varepsilon,\tau;\mu}\|\!|_{\sharp,\bar{\mu}}\leq c\,C^F_{\varepsilon,\tau},\end{aligned}\tag{4.13}$$

where the last bound follows from the auxiliary estimate (4.4), the bound for the coarse-grained process proved in Theorem 4.14 and the estimate (4.12) for the remainder established in Remark 4.16. It turns out that with some effort one can prove the following Schauder-type estimate

$$\|\Phi_{\tau,\varepsilon}(0,\bullet)\|_{\mathscr{B}^{\sigma-3\gamma}_{\infty,\infty}(\mathbb{R}^d,w_1(0,\bullet))}\leq c\;\bar{\mu}^{-3\gamma}\;\|\!|\mu\mapsto K_{\varepsilon;\mu}*\mathcal{L}_\varepsilon\Phi_{\varepsilon,\tau}\|\!|_{\sharp,\bar{\mu}}.\tag{4.14}$$

As a result, by the bounds (4.13), (4.14) and Theorem 4.14 we conclude

$$\mathbb{E}\|\Phi_{\tau,\varepsilon}(0,\bullet)\|^p_{\mathscr{B}^{\sigma-3\gamma}_{\infty,\infty}(\mathbb{R}^d,w_1(0,\bullet))}\leq c^{2p}\,\mathbb{E}(C^F_{\varepsilon,\tau})^{p(6\gamma/\kappa+1)}<\infty$$

for all $p\in\mathbb{N}_+$. We stress that the last bound, which is a consequence of Theorem 3.38, is only true for suitable choices of the mass counterterm. The above bound implies the desired bound (2.2) and proves Item (A) of Theorem 1.1.

5 Outline of the proof of stochastic estimates

In this section we present the proof of the uniform bounds for the cumulants of the effective force coefficients, which is the main ingredient of the proof of the stochastic estimates stated in Theorem 3.38. As mentioned in Remark 3.41, in order to conclude the stochastic estimates from the bounds for the cumulants one uses an argument in the spirit of the Kolmogorov continuity theorem and a certain deterministic argument based on the flow equation for the effective force coefficients. The main result of this section is Theorem 5.6 stated in Sec. 5.1. In Sec. 5.2 we a derive a certain flow equation for the cumulants of the effective force coefficients and in Sec. 5.3 we use it to prove uniform bounds for the cumulants. In particular, we discuss how to fix the mass counterterm. A simple inductive proof of the bounds for cumulants inspired by [Pol84] is one of the main advantages of the flow equation approach to singular SPDEs.

5.1 Cumulants of effective force coefficients

Definition 5.1. A list (i,m,s), where $i\in\{0,\ldots,i_\flat\}$, $m\in\mathbb{N}_0$ and $s\in\{0,1\}$ is called an index. For $n\in\mathbb{N}_+$ we call

$$\mathtt{I}=((i_1,m_1),\ldots,(i_n,m_n)) \tag{5.1}$$

a list of indices. We define $n(\mathtt{I}):=n$, $i(\mathtt{I}):=i_1+\ldots+i_n$, $\mathfrak{m}(\mathtt{I}):=(m_1,\ldots,m_n)$, $m(\mathtt{I}):=m_1+\ldots+m_n$ and

$$\varrho_\kappa(\mathtt{I}):=\varrho_\kappa(i_1,m_1)+\ldots+\varrho_\kappa(i_n,m_m),$$

where $\varrho_\kappa(i,m)$ is introduced in Def. 3.32. We omit κ if $\kappa=0$.

Definition 5.2. We use the following notation $\mathbb{E}(\zeta_1,\ldots,\zeta_n)$ for the joint cumulant of the random variables $\zeta_1,\ldots,\zeta_n$. Let $\mathtt{I}$ be a list of indices of the form (5.1). We denote by $E^{\mathtt{I}}_{\varepsilon,\tau;\mu}\in\mathscr{S}'(\Lambda_\varepsilon^{n(\mathtt{I})+m(\mathtt{I})})$ the joint cumulant of the effective force coefficients defined by

$$\langle E^{\mathtt{I}}_{\varepsilon,\tau;\mu},\psi_1\otimes\ldots\otimes\psi_n\otimes\varphi_1\otimes\ldots\otimes\varphi_n\rangle:=\mathbb{E}(\langle F^{i_1,m_1}_{\varepsilon,\tau;\mu},\psi_1\otimes\varphi_1\rangle,\ldots,\langle F^{i_n,m_n}_{\varepsilon,\tau;\mu},\psi_n\otimes\varphi_n\rangle)$$

for all $\psi_q\in\mathscr{S}(\Lambda_\varepsilon)$, $\varphi_q\in\mathscr{S}(\Lambda_\varepsilon^{m_q})$, $q\in\{1,\ldots,n\}$.

Remark 5.3. For the definition of cumulants and their properties see, for example, [PT11].

Definition 5.4. Let $\mu\in(0,1]$, $\beta\in[0,\infty)$, $n\in\mathbb{N}_0$, $\mathfrak{m}=(m_1,\ldots,m_n)\in\mathbb{N}_0^n$ and $m=m_1+\ldots+m_n$. The weight $\hat{w}^{\mathfrak{m},\beta}_\mu\in C(\Lambda_0^{n+m})$ is defined by

$$\tilde{w}^{\mathfrak{m},\beta}_\mu(z_1,\ldots,z_n;\mathrm{z}_1,\ldots,\mathrm{z}_n):=\tilde{w}^{m_1,\beta}_\mu(z_1;\mathrm{z}_1)\ldots\tilde{w}^{m_n,\beta}_\mu(z_n;\mathrm{z}_n)$$

for all $z_q\in\Lambda_0$, $\mathrm{z}_q\in\Lambda_0^{m_q}$, $q\in\{1,\ldots,n\}$, where the weight $\tilde{w}^{m,\beta}_\mu$ was introduced in Def.

Definition 5.5. Let $n\in\mathbb{N}_+$, $\mathfrak{m}=(m_1,\ldots,m_n)\in\mathbb{N}_0^n$ and $m=m_1+\ldots+m_n$. The vector space $\mathcal{V}^{\mathfrak{m}}_{\varepsilon,\tau;\mathrm{t}}$ consists of translationally invariant kernels $V\in\mathscr{S}'(\Lambda_\varepsilon^{n+m})$ of operators $C_{\mathrm{b}}(\Lambda^m_{\varepsilon,\tau})\mapsto C_{\mathrm{b}}(\Lambda^n_{\varepsilon,\tau})$ such that the following norm

$$\|V\|_{\mathcal{V}^{\mathfrak{m}}_{\varepsilon,\tau;\mathrm{t}}}:=\sup_{x_1\in\Lambda_\varepsilon}\int_{\Lambda^{n-1}_{\varepsilon,\tau}\times\Lambda^m_\varepsilon}|V(x_1,\mathrm{d}x_2,\ldots,\mathrm{d}x_n;\mathrm{d}y_1,\ldots,\mathrm{d}y_m)| \tag{5.2}$$

is finite.

Theorem 5.6. *There exists a choice of the counterterms $(r^{(i)}_{\varepsilon,\tau})_{\varepsilon\in\mathcal{A},\tau\in\mathbb{N}_+,i\in\{1,\ldots,i_\sharp\}}$ such that for all $\kappa\in(0,1]$, $\beta\in[0,\sigma)$ and all list of indices $\mathtt{I}$ the following bound*

$$\|\tilde{w}^{\mathfrak{m}(\mathtt{I}),\beta}_\mu(K^{n(\mathtt{I}),m(\mathtt{I})}_{\varepsilon;\mu}*E^{\mathtt{I}}_{\varepsilon,\tau;\mu})\|_{\mathcal{V}^{\mathfrak{m}}_{\varepsilon,\tau;\mathrm{t}}}\lesssim\mu^{\varrho_\kappa(\mathtt{I})+(\sigma+d)(n(\mathtt{I})-1)}$$

holds uniformly in $\varepsilon\in\mathcal{A}$, $\tau\in\mathbb{N}_+$ and $\mu\in(0,1]$, where $K^{n,m}_{\varepsilon;\mu}$ is introduced in Def. 3.27.

Remark 5.7. The above theorem is not true if one replaces cumulants by moments in the definition of $E^{\mathtt{I}}_{\varepsilon,\tau;\mu}$. The moments of the effective force coefficients satisfy a slightly different bound, which, however, cannot be easily proved by induction.

Remark 5.8. In order to prove the stochastic estimate (3.11) stated in Theorem 3.38 with $\kappa = \delta \in (0,1]$ one has to use the bound stated in the above theorem with $\kappa = \delta/2$. Actually, we only use the bound for the expected value and the covariance and infer bounds for higher moments using the Nelson hypercontractivity [Nua06, Lemma 1.1.1] and the fact that the effective force coefficients are multi-linear functionals of the white noise $\xi_{\varepsilon,\tau}$.

5.2 Flow equation for cumulants

In this section we show that the derivative of the cumulant of the effective force coefficients with respect to the scale parameter $\partial_\mu E^{\mathtt{I}}_{\varepsilon,\tau;\mu}$ is a linear combination of terms that can be expressed in terms of $E^{\mathtt{J}}_{\varepsilon,\tau;\mu}$ with lists of indices $\mathtt{J}$ such that $i(\mathtt{J}) < i(\mathtt{I})$, or $i(\mathtt{J}) = i(\mathtt{I})$ and $m(\mathtt{J}) > m(\mathtt{I})$. To this end, we first note that

$$\partial_\mu E^{\mathtt{I}}_{\varepsilon,\tau;\mu} := \sum_{l=1}^{n} \mathbb{E}(F^{i_1,m_1}_{\varepsilon,\tau;\mu},\ldots,\partial_\mu F^{i_l,m_l}_{\varepsilon,\tau;\mu},\ldots,F^{i_n,m_n}_{\varepsilon,\tau;\mu}).$$

The flow equation (3.9) for the effective force coefficients implies that

$$\begin{aligned}&\mathbb{E}(\partial_\mu F^{i_1,m_1}_{\varepsilon,\tau;\mu},F^{i_2,m_2}_{\varepsilon,\tau;\mu},\ldots,\ldots,F^{i_n,m_n}_{\varepsilon,\tau;\mu})\\ &\qquad= -\sum_{j=0}^{i}\sum_{k=0}^{m}(k+1)\,\mathbb{E}(\mathbf{B}(\dot G_\mu,F^{j,k+1}_{\varepsilon,\tau;\mu},F^{i_1-j,m_1-k}_{\varepsilon,\tau;\mu}),F^{i_2,m_2}_{\varepsilon,\tau;\mu},\ldots,\ldots,F^{i_n,m_n}_{\varepsilon,\tau;\mu}).\end{aligned} \tag{5.3}$$

Using the formula

$$\mathbb{E}\big((\zeta_q)_{q\in I},\Phi\Psi\big) = \mathbb{E}\big((\zeta_q)_{q\in I},\Phi,\Psi\big) + \sum_{\substack{I_1,I_2\subset I\\ I_1\cup I_2=I}} \mathbb{E}\big((\zeta_q)_{q\in I_1},\Phi\big)\;\mathbb{E}\big((\zeta_q)_{q\in I_2},\Psi\big)$$

one shows that the summands on the RHS of Eq. (5.3) are linear combinations of the expressions of the form

$$\mathbf{A}\Big(\dot G_{\varepsilon;\mu},\mathbb{E}\big(F^{j,1+k}_{\varepsilon,\tau;\mu},F^{i_1-j,m_1-k}_{\varepsilon,\tau;\mu},(F^{i_q,m_q}_{\varepsilon,\tau;\mu})_{q\in I}\big)\Big)$$

or

$$\mathbf{B}\Big(\dot G_{\varepsilon;\mu},\mathbb{E}\big(F^{j,1+k}_{\varepsilon,\tau;\mu},(F^{i_q,m_q}_{\varepsilon,\tau;\mu})_{q\in I_1}\big),\mathbb{E}\big(F^{i_1-j,m_1-k}_{\varepsilon,\tau;\mu},(F^{i_q,m_q}_{\varepsilon,\tau;\mu})_{q\in I_2}\big)\Big),$$

where $\mathbf{A},\mathbf{B}$ are certain multi-linear maps and the subsets $I_1,I_2 \subset I = \{1,\ldots,n\}$ are such that $I_1\cup I_2 = I$ and $I_1\cap I_2=\emptyset$. Using the above fact and Lemma 3.9 one proves that for all $\kappa\in(0,1]$ and all list of indices $\mathtt{J}$ the following bound

$$\|\tilde w^{\mathtt{m}(\mathtt{J}),\beta}_\mu(K^{n(\mathtt{J}),m(\mathtt{J})}_{\varepsilon;\mu} * \partial_\mu E^{\mathtt{J}}_{\varepsilon,\tau;\mu})\|_{\mathcal{V}^{\mathtt{m}}_{\varepsilon,\tau;\mathtt{t}}} \lesssim \mu^{\varrho_\kappa(\mathtt{J})+(\sigma+d)(n(\mathtt{J})-1)-1} \tag{5.4}$$

holds uniformly in $\varepsilon\in\mathcal{A}$, $\tau\in\mathbb{N}_+$ if the statement of Theorem 5.6 holds true for all lists of indices $\mathtt{I}$ such that $i(\mathtt{J}) < i(\mathtt{I})$, or $i(\mathtt{J}) = i(\mathtt{I})$ and $m(\mathtt{J}) > m(\mathtt{I})$.

5.3 Uniform bounds for cumulants

In this section we present main ideas of the proof of Theorem 5.6. The proof is by induction on $i(\mathtt{I})$ and $m(\mathtt{I})$. Since $\varrho_\kappa(\mathtt{I})$ is decreasing function of $\kappa\in(0,1]$ for list of indices $\mathtt{I}$ such that $E^{\mathtt{I}}_{\varepsilon,\tau;\eta}\neq 0$, without loss of generality we can assume that $\kappa\in(0,\kappa_\star]$ for some $\kappa_\star\in(0,1]$ to be fixed later. Moreover, let us observe that the bound (5.2) is trivially satisfied for all list of indices $\mathtt{I}$ such that $m(\mathtt{I})>3i(\mathtt{I})$ since then $E^{\mathtt{I}}_{\varepsilon,\tau;\mu}=0$.

Let us study first the case $i(\mathtt{I})=0$. In this case the cumulants $E^{\mathtt{I}}_{\varepsilon,\tau,\mu}$ coincide with the cumulants of the white noise $\xi_{\varepsilon,\tau}$. Hence, the only non-vanishing cumulant is the covariance corresponding to $n(\mathtt{I})=2$, $\mathfrak{m}(\mathtt{I})=(0,0)$ and the bound (5.2) is equivalent to

$$\|\mathbb{E}(\xi_{\varepsilon,\tau},\xi_{\varepsilon,\tau})\|_{\mathcal{V}^{\mathfrak{m}}_{\varepsilon,\tau;\mathrm{t}}}\leq \sup_{z_1\in\Lambda_{\varepsilon,\tau}}\int_{\Lambda_{\tau,\varepsilon}}|\mathbb{E}(\xi_{\varepsilon,\tau}(z_1)\xi_{\varepsilon,\tau}(\mathrm{d}z_2))|=1.$$

This proves the base case of the induction.

Now let us fix $i\in\mathbb{N}_+$, $m\in\mathbb{N}_0$ and assume that the theorem is true for all lists of indices $\mathtt{I}$ such that $i(\mathtt{I})<i$, or $i(\mathtt{I})=i$ and $m(\mathtt{I})>m$. Our goal is to prove the theorem for all $\mathtt{I}$ such that $i(\mathtt{I})=i$ and $m(\mathtt{I})=m$. We are going to study separately the following two cases:

(1) irrelevant cumulants $E^{\mathtt{I}}_{\varepsilon,\tau;\mu}$ with $\mathtt{I}$ such that $\varrho(\mathtt{I})+(\sigma+d)(n(\mathtt{I})-1)>0$,

(2) relevant cumulants $E^{\mathtt{I}}_{\varepsilon,\tau;\mu}$ with $\mathtt{I}$ such that $\varrho(\mathtt{I})+(\sigma+d)(n(\mathtt{I})-1)\leq 0$.

We start with the case (1). First, note that there exists $\kappa_\star\in(0,1]$ such that the condition $\varrho(\mathtt{I})+(\sigma+d)(n(\mathtt{I})-1)>0$ implies that $\varrho_\kappa(\mathtt{I})+(\sigma+d)(n(\mathtt{I})-1)>0$ for all $\kappa\in(0,\kappa_\star]$ and lists of indices $\mathtt{I}$ for which $E^{\mathtt{I}}_{\varepsilon,\tau;\eta}\neq 0$. Next, observe that

$$\begin{aligned}\|\tilde w^{\mathfrak{m}(\mathtt{I}),\beta}_\mu(K^{n(\mathtt{I}),m(\mathtt{I})}_{\varepsilon;\mu}*E^{\mathtt{I}}_{\varepsilon,\tau;\mu})\|_{\mathcal{V}^{\mathfrak{m}}_{\varepsilon,\tau;\mathrm{t}}}&\lesssim\int_0^\mu\|\tilde w^{\mathfrak{m}(\mathtt{I}),\beta}_\mu(K^{n(\mathtt{I}),m(\mathtt{I})}_{\varepsilon;\mu}*\partial_\eta E^{\mathtt{I}}_{\varepsilon,\tau;\eta})\|_{\mathcal{V}^{\mathfrak{m}}_{\varepsilon,\tau;\mathrm{t}}}\,\mathrm{d}\eta\\&\lesssim\int_0^\mu\|\tilde w^{\mathfrak{m}(\mathtt{I}),\beta}_\eta(K^{n(\mathtt{I}),m(\mathtt{I})}_{\varepsilon;\eta}*\partial_\eta E^{\mathtt{I}}_{\varepsilon,\tau;\eta})\|_{\mathcal{V}^{\mathfrak{m}}_{\varepsilon,\tau;\mathrm{t}}}\,\mathrm{d}\eta\\&\lesssim\int_0^\mu\eta^{\varrho_\kappa(\mathtt{I})+(\sigma+d)(n(\mathtt{I})-1)-1}\lesssim\mu^{\varrho_\kappa(\mathtt{I})+(\sigma+d)(n(\mathtt{I})-1)}.\end{aligned}$$

The first of the above bounds follows from the Minkowski inequality. The second one is a consequence of the properties of the kernel $K^{n(\mathtt{I}),m(\mathtt{I})}_{\varepsilon;\mu}$ and the weight $\tilde w^{\mathfrak{m}(\mathtt{I}),\beta}_\mu$. To prove the third bound we used the induction hypothesis and the bound (5.4). The last bound relies crucially on the inequality $\varrho_\kappa(\mathtt{I})+(\sigma+d)(n(\mathtt{I})-1)>0$. This proves the theorem for irrelevant cumulants.

Let us proceed to the proof of the case (2). Note that for $i(\mathtt{I})>0$ the condition $\varrho(\mathtt{I})+(\sigma+d)(n(\mathtt{I})-1)\leq 0$ implies that $n(\mathtt{I})=1$. Hence, $\mathtt{I}=(i,m)$ for $i,m\in\mathbb{N}_0$ such that $\varrho(i,m)\leq 0$ and

$$E^{\mathtt{I}}_{\varepsilon,\tau;\mu}=\mathbb{E}F^{i,m}_{\varepsilon,\tau;\mu}.$$

The condition $\varrho(i,m) \leq 0$ implies that $m \in \{0,1,2,3\}$. Using the fact that the law of $\xi_{\varepsilon,\tau}$ is invariant under the transformations $\xi_{\varepsilon,\tau} \mapsto -\xi_{\varepsilon,\tau}$ one shows that $E^{\mathtt{I}}_{\varepsilon,\tau;\mu} = 0$ unless $n(\mathtt{I}) + m(\mathtt{I}) \in 2\mathbb{N}_0$. As a result, we can restrict attention to the cases $m = 3$ and $m = 1$. For $m = 3$ the condition $\varrho(i,3) \leq 0$ implies that $i = 1$ and

$$
\begin{aligned}
E^{\mathtt{I}}_{\varepsilon,\tau;\mu}(z,\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_2) &= \mathbb{E}F^{1,3}_{\varepsilon,\tau;\mu}(z,\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_2) \\
&= \mathbb{E}F^{1,3}_{\varepsilon,\tau;0}(z,\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_2) = -\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_1)\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_2)\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_3)
\end{aligned}
$$

by Remark 3.24, where $\delta^{(z)}_{\Lambda_\varepsilon} \in \mathscr{S}'(\Lambda_\varepsilon)$ is the Dirac delta at $z \in \Lambda_\varepsilon$. The proof of the bound (5.2) follows now from elementary estimates. For $m = 1$ the condition $\varrho(i,1) \leq 0$ implies that $i \in \{1,\ldots,i_\sharp\}$, where $i_\sharp$ was introduced in Def. 3.18. We note that the bound (5.2) in the case at hand takes the form

$$
\|\tilde{w}^{1,\beta}_\eta(K^{1,1}_{\varepsilon;\mu} * \mathbb{E}F^{i,1}_{\varepsilon,\tau;\mu})\|_{\mathcal{V}^1_{\varepsilon,\tau;\mathrm{t}}} \lesssim \mu^{\varrho_\kappa(i,1)}. \tag{5.5}
$$

To prove the above bound we would like to take advantage of the bound

$$
\|\tilde{w}^{1,\beta}_\eta(K^{1,1}_{\varepsilon;\eta} * \partial_\eta\mathbb{E}F^{i,1}_{\varepsilon,\tau;\eta})\|_{\mathcal{V}^1_{\varepsilon,\tau;\mathrm{t}}} \lesssim \eta^{\varrho_\kappa(i,1)-1}, \tag{5.6}
$$

which is a consequence of the induction hypothesis and the bound (5.4). The problem is that the RHS of the above bound is not integrable in $\eta \in [0,1]$ at $\eta = 0$. At this stage it is useful to realize that

$$
E^{\mathtt{I}}_{\varepsilon,\tau;0}(z,\mathrm{d}z_1) = \mathbb{E}F^{i,1}_{\varepsilon,\tau;0}(z,\mathrm{d}z_1) = r^{(i)}_{\varepsilon,\tau}\,\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_1), \tag{5.7}
$$

where $r^{(i)}_{\varepsilon,\tau} \in \mathbb{R}$ is the mass counterterm that has not yet been fixed and can be chosen arbitrarily. Now we idea is to use the following decomposition

$$
\mathbb{E}F^{i,1}_{\varepsilon,\tau;\mu} = \hat{E}^{i,1}_{\varepsilon,\tau;\mu} + \tilde{E}^{i,1}_{\varepsilon,\tau;\mu},
$$

where the local part $\hat{E}^{i,1}_{\varepsilon,\tau;\mu}$ is defined by

$$
\hat{E}^{i,1}_{\varepsilon,\tau;\mu}(z,\mathrm{d}z_1) := r^{(i)}_{\varepsilon,\tau;\mu}\,\delta^{(z)}_{\Lambda_\varepsilon}(\mathrm{d}z_1), \qquad r^{(i)}_{\varepsilon,\tau;\mu} := \int_{\Lambda_\varepsilon} \mathbb{E}F^{i,1}_{\varepsilon,\tau;\mu}(z,\mathrm{d}z_1) \in \mathbb{R}, \tag{5.8}
$$

and $\tilde{E}^{i,1}_{\varepsilon,\tau;\mu}$ is a certain non-local remainder. Observe that by translational invariance $r^{(i)}_{\varepsilon,\tau;\mu}$ does not depend on $z \in \Lambda_\varepsilon$. It turns out that the remainder $\tilde{E}^{i,1}_{\varepsilon,\tau;\mu}$ is in some sense irrelevant. More precisely, using the Taylor theorem and the bound (5.6) one shows with some effort that

$$
\|\tilde{w}^{1,\beta}_\eta(K^{1,1}_{\varepsilon;\mu} * \tilde{E}^{i,1}_{\varepsilon,\tau;\mu})\|_{\mathcal{V}^1_{\varepsilon,\tau;\mathrm{t}}} \lesssim \mu^{\varrho_\kappa(i,1)}.
$$

It remains to estimate the local part. By the second of Eqs. (5.8) and the bound (5.6) we have

$$
|\partial_\eta r^{(i)}_{\varepsilon,\tau;\eta}| \lesssim \eta^{\varrho_\kappa(i,1)-1}. \tag{5.9}
$$

We fix the counterterm $r^{(i)}_{\varepsilon,\tau} \in \mathbb{R}$ in Eq. (5.7) by the implicit condition $r^{(i)}_{\varepsilon,\tau;\mu=1} = 0$, which implies that

$$r^{(i)}_{\varepsilon,\tau} := -\int_0^1 \partial_\eta r^{(i)}_{\varepsilon,\tau;\eta}\,\mathrm{d}\eta, \qquad r^{(i)}_{\varepsilon,\tau;\mu} := -\int_\mu^1 \partial_\eta r^{(i)}_{\varepsilon,\tau;\eta}\,\mathrm{d}\eta.$$

Using the second of the above identities, the bound (5.9) and $\varrho_\kappa(i,1) < 0$ we obtain

$$|r^{(i)}_{\varepsilon,\tau;\mu}| \lesssim \mu^{\varrho_\kappa(i,1)}.$$

It follows that

$$\|\tilde{w}^{1,\beta}_\eta (K^{1,1}_{\varepsilon;\mu} * \hat{E}^{i,1}_{\varepsilon,\tau;\mu})\|_{\mathcal{V}^1_{\varepsilon,\tau;\mathrm{t}}} \lesssim \mu^{\varrho_\kappa(i,1)}.$$

This proves the bound (5.5) and finishes the proof of the inductive step.

References

[Abd07] A. Abdesselam, *A Complete Renormalization Group Trajectory Between Two Fixed Points*, Commun. Math. Phys. **276**, 727–772 (2007)

[ADC21] M. Aizenman, H. Duminil-Copin, *Marginal triviality of the scaling limits of critical 4D Ising and Φ^4_4 models*, Annals Math. **194**(1), 163–235 (2021) [arXiv:1912.07973]

[BC24] B. Bringmann, S. Cao, *Global well-posedness of the stochastic Abelian-Higgs equations in two dimensions*, [arXiv:2403.16878]

[BMS03] D. Brydges, P. Mitter, B. Scoppola, *Critical* $(\Phi^4)_{3,\epsilon}$, Comm. Math. Phys. **240**(1), 281–327 (2003)

[BCCH21] Y. Bruned, A. Chandra, I. Chevyrev, M. Hairer, *Renormalising SPDEs in regularity structures*, J. Eur. Math. Soc. **23**(3), 869–947 (2021) [arXiv:1711.10239]

[BHZ19] Y. Bruned, M. Hairer, L. Zambotti, *Algebraic renormalisation of regularity structures*, Invent. Math. **215**(3), 1039–1156 (2019) [arXiv:1610.08468]

[CH16] A. Chandra, M. Hairer, *An analytic BPHZ theorem for regularity structures*, [arXiv:1612.08138]

[CLFW24] A. Chandra, G. de Lima Feltes, H. Weber, *A priori bounds for 2-d generalised Parabolic Anderson Model*, [arXiv:2402.05544]

[CMW23] A. Chandra, A. Moinat, H. Weber, *A Priori Bounds for the Φ^4 Equation in the Full Sub-critical Regime*, Arch. Rational Mech. Anal. **247**(48) (2023) [arXiv:1910.13854]

[DPD03] G. Da Prato, A. Debussche, *Strong solutions to the stochastic quantization equations*, Ann. Probab. **31**(4), 1900–1916 (2003)

[DFG22] F. De Vecchi, L. Fresta, M. Gubinelli, *A stochastic analysis of subcritical Euclidean fermionic field theories*, [arXiv:2210.15047]

[Duc21] P. Duch, *Flow equation approach to singular stochastic PDEs*, [arXiv:2109.11380]

[Duc22] P. Duch, *Renormalization of singular elliptic stochastic PDEs using flow equation*, [arXiv:2201.05031]

[Duc23] P. Duch, *Lecture notes on flow equation approach to singular stochastic PDEs*, (2023) pawelduch.github.io/spde_flow_notes.pdf

[DGR23] P. Duch, M. Gubinelli, P. Rinaldi, *Stochastic quantisation of the fractional ϕ^4_3 model in the full subcritical regime*, [arXiv:2303.18112],

[GH19] M. Gubinelli, M. Hofmanová, *Global solutions to elliptic and parabolic Φ^4 models in Euclidean space*, Commun. Math. Phys. **368** 1201–1266 (2019) [arXiv:1804.11253]

[GH21] M. Gubinelli, M. Hofmanová, *A PDE construction of the Euclidean Φ^4_3 QFT*, Commun. Math. Phys. **382**, 1–75 (2021) [arXiv:1810.01700]

[GM24] M. Gubinelli, S.-J. Meyer, *The FBSDE approach to sine-Gordon up to* 6π, [arXiv:2401.13648]

[Hai14] M. Hairer, *A theory of regularity structures*, Invent. Math. **198**(2), 269–504 (2014) [arXiv:1303.5113]

[HR23] M. Hairer, T. Rosati, *Global existence for perturbations of the 2d stochastic navier-stokes equations with space-time white noise*, Ann. PDE **10**(3) (2024) [arXiv:2301.11059]

[HS22] M. Hairer, R. Steele, *The Φ^4_3 measure has sub-Gaussian tails*, Journ. Stat. Phys. **186**(3) (2022)

[HS23] M. Hairer, R. Steele, *The BPHZ theorem for regularity structures via the spectral gap inequality*, [arXiv:2301.10081]

[Kup16] A. Kupiainen, *Renormalization group and stochastic PDEs*, Ann. Henri Poincaré **17**(3), 497–535 (2016) [arXiv:1410.3094]

[LOTT21] P. Linares, F. Otto, M. Tempelmayr, P. Tsatsoulis, *A diagram-free approach to the stochastic estimates in regularity structures*, [arXiv:2112.10739]

[MP19] J. Martin, N. Perkowski, *Paracontrolled distributions on Bravais lattices and weak universality of the 2d parabolic Anderson model*, Ann. Inst. Henri Poincaré Probab. Stat. **55**(4), 2058–2110 (2019)

[MW17] J.-C. Mourrat, H. Weber, *The dynamic Φ^4_3 model comes down from infinity*, Comm. Math. Phys. **356**, 673-753 (2017) [arXiv:1601.01234]

[MW20] A. Moinat, H. Weber, *Space-time localisation for the dynamic Φ^4_3 model*, Commun. Pure Appl. Math. **73**(12), 2519—2555 (2020) [arXiv:1811.05764]

[Nua06] Nualart, D., *The Malliavin calculus and related topics*, (Springer, 2006)

[OSSW21] F. Otto, J. Sauer, S. Smith, H. Weber, *A priori estimates for quasi-linear SPDEs in the full sub-critical regime*, [arXiv:2103.11039]

[PT11] G. Peccati, M. Taqqu, *Wiener Chaos: Moments, Cumulants and Diagrams: A survey with computer implementation*, (Springer, 2011)

[Pol84] J. Polchinski, *Renormalization and effective lagrangians*, Nuclear Physics B, **231**(2), 269–295 (1984)

[SZZ24] H. Shen, R. Zhu, X. Zhu, *Global well-posedness for 2D generalized Parabolic Anderson Model via paracontrolled calculus*, [arXiv:2402.19137]

[Wil71] K. Wilson, *Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture*, Phys. Rev. B, **4** 3174–3183 (1971)