Exercise 10.31: Suppose that ϕ is a homomorphism from $U(30)$ to $U(30)$ and that $\ker\phi = \{1, 11\}$. If $\phi(7) = 7$, find all elements of $U(30)$ that map to 7.
The elements that map onto 7 are precisely $a\ker\phi$ where $a \in \phi^{-1}(7)$ thus these elements are $\{7, 17\}$.

Exercise 10.35: Prove that the mapping $\phi : Z \bigoplus Z \to Z$ given by $(a, b) \to a - b$ is a homomorphism. What is the kernel of ϕ? Describe the set $\phi^{-1}(3)$ (that is, all elements that map to 3).
Clearly ϕ is well defined and thus we need only show that operation are preserved. Note that $\phi((a, b))\phi((c, d)) = a-b+c-d = (a+c)-(b+d) = \phi(((a+c), (b+d))) = \phi((a, b)+(c, d))$, and thus ϕ is a homomorphism.
Note that $(a, b) \in \ker\phi$ iff $a-b = 0$ in other words where $a = b$ thus $\ker\phi = \{(a, a) \mid a \in \mathbb{Z}\}$.
Note that $(3, 0) \in \phi^{-1}(3)$ thus $\phi^{-1}(3) = (3, 0)\ker\phi = \{(a + 3, a) \mid a \in \mathbb{Z}\}$.

Exercise 10.40: For each pair of positive integers m and n, we can define a homomorphism from Z to $Z_m \bigoplus Z_n$ by $x \to (x \mod m, x \mod n)$. What is the kernel when $(m, n) = (3, 4)$? What is the kernel when $(m, n) = (6, 4)$? Generalize.
Note that $x \in \ker\phi$ iff $\phi(x) = (0, 0)$. Thus $\ker\phi = \{x \in Z \mid \phi(x) = (0, 0)\}$ or in our case $\ker\phi = \{x \in Z \mid (m \mid x) \wedge (n \mid x)\}$ or $\ker\phi = \{x \operatorname{lcm}(m, n) \mid x \in Z\}$. If $(m, n) = (3, 4)$ then $\ker\phi = \{12x \mid x \in Z\}$. If $(m, n) = (6, 4)$ then $\ker\phi = \{12x \mid x \in Z\}$.

Exercise 10.43: Let $\phi(d)$ denote the Euler phi function of d (see page 85). Show that the number of homomorphisms from Z_n to Z_k is $\sum \phi(d)$, where the sum runs over all common divisors d of n and k. [It follows from number theory that this sum is actually $\gcd(n, k)$.]
First let's break up all of the homomorphisms by the size of the image. Note that the size of the image call it d must divide n since the homomorphism associated with this image divides the group Z_n into d chunks of equal size. Note that the size of the image call it d must divide m since $\psi(Z_n)$ is a subgroup of size d in Z_m.
How many homomorphisms have a size of there image equal to d? Well as discussed above if $d \nmid m$ or $d \nmid n$ then there are no homomorphisms associated with it. However if $d \mid n$ and $d \mid m$ then we will have homomorphisms associated with it and these homomorphisms map onto $< m/d >$ the only subgroup of Z_m with d elements. If we know where the generator 1 in Z_n gets mapped to in $< m/d >$ we know were every item gets mapped to. Noting that 1 must get mapped to a generator we know that there are exactly as many homomorphisms onto $< m/d >$ as $< m/d >$ has generators. Note that $< m/d >$ has $\phi(|< m/d >|) = \phi(d)$ generators.
Now simply add up the number of homomorphisms associated with any d value and we have the total number of homomorphisms. The result of this sum is exactly the sum described in the question.

Exercise 10.48: Suppose that Z_{10} and Z_{15} are both homomorphic images of a finite group G. What can be said about $|G|$? Generalize.
We know that $|Z_{10}| \mid |G|$ and that $|Z_{15}| \mid |G|$ thus $\operatorname{lcm}(10, 15) \mid |G|$. In general if groups $a_1 \cdots a_n$ are homomorphic images of a finite group G then $\operatorname{lcm}(|a_1| \cdots |a_n|) \mid |G|$.

Exercise 10.59: Suppose that H and K are distinct subgroups of G of index 2. Prove that $H \cap K$ is a normal subgroup of G of index 4 and that $G/(H \cap K)$ is not cyclic.
Note that $H \not\subset K$ and $K \not\subset H$. Let $a \in H \cap K^c$ and let $b \in K \cap H^c$. Suppose H is not normal in G. In this case there exists $a \in G$ and $h \in H$ such that $aha^{-1} \notin H$. Note that $a \notin H$, since H has closure. Note that $aha^{-1} \notin H$ implies that $aha^{-1} \in aH$ since H is index 2. Note that $ha^{-1} \in H$ thus $(ha^{-1})^{-1} \in H$ thus $ah^{-1} \in H$ however $ah^{-1} \in aH$, a contradiction.
Note that H and K are normal subgroup of G. Note that $H \cap K = J$ is a normal subgroup of G. Let $a \in H$ and $a \notin K$, note that $a \notin J$ and thus aJ is a separate coset from J. Let $b \notin H$ and $b \in K$, note that $b \notin J$ and since $b \notin H = aH$, $b \notin aJ$, thus bJ is a separate coset from aJ and J. Let $c \notin H$ and $c \notin K$, note that $c \notin J$ and since $c \notin H = aH$, $c \notin aJ$, and since $c \notin K = bK$, $c \notin bJ$, thus cJ is a separate coset from bJ, aJ and J. Thus J is at least of index 4.
Note that if $d \in G$, d will fall in one of these 4 cosets. I will not prove all 4 cases but I will prove one case as example, Suppose $d \notin H$ and $d \in K$. Thus $d \in bH$ and $d \in K = bH$, so $d \in bJ$. Note that we now know that J is exactly of index 4.
Note that $G/(H \cap K) = \{J, aJ, bJ, cJ\}$. Suppose $G/(H \cap K)$ is cyclic. Note that it must have 2 generators. Note that J is not a generator since it is identity and thus aJ or bJ will be a generator. We can break symmetry at this point between K and H and say WLoG aJ is a generator. Note that $a^2 \in H$ (since $a \in H$) and that $a^2 \in K$ since if $a^2 \notin K$ we would know that $a^2 \in aJ$ and thus $(aJ)^2 = aJ$ a impossibility, thus $(aJ)^2 = J = e$. We have reached a contradiction since aJ is a generator, thus we negate our supposition and conclude that $G/(H \cap K)$ is not cyclic.