Exercise 13.15: Let a belong to a ring R with unity and suppose that $a^n = 0$ for some positive integer n. (Such an element is called nilpotent.) Prove that $1-a$ has a multiplicative inverse in R. [Hint: Consider $(1-a)(1+a+a^2+\cdots+a^{n-1})$.]
Note that $b = (1+a+a^2+\cdots+a^{n-1}) \in R$ and that $(1-a)b = (1+a+a^2+\cdots+a^{n-1}) - (a+a^2+a^3+\cdots+a^n) = 1-a^n = 1$ thus $b = (1-a)^{-1}$.

Exercise 13.18: A ring element a is called an idempotent if $a^2 = a$. Prove that the only idempotents in an integral domain are 0 and 1.
Suppose R is a intagable domain. Suppose that $a \notin \{0, 1\}$ and that $a^2 = a$. Note that a^{-1} exists. Note that $1 = aa^{-1} = a^2a^{-1} = aaa^{-1} = a$ a contradiction we now conclude that the only idempotents in an integral domain are 0 and 1.

Exercise 13.22: Prove that if a is a ring idempotent, then $a^n = a$ for all positive integers n.
I will procede with proof by induction.
Note that the statement is true for $n = 1$ and true for $n = 2$.
Suppose that the statement $a^n = a$ holds for $n \geq 2$. Note that $a = a^n = aa^{n-1} = a^2a^{n-1} = a^{n+1}$. By induction we conclude that if a is a ring idempotent, then $a^n = a$ for all positive integers n.

Exercise 13.25: Find an idempotent in $Z_5[i] = \{a+bi \mid a, b \in Z_5\}$.
Note that $(3+i)^2 = 8+6i = 3+i$, thus $3+i$ is a idempotent in $Z_5[i] = \{a+bi \mid a, b \in Z_5\}$.

Exercise 13.28: Let R be the set of all real-valued functions defined for all real numbers under function addition and multiplication.

a) Determine all zero-divisors of R.
The function f is a zero-divisors of R if there exists some $a \in \mathbb{R}$ such that $f(a) = 0$.
Suppose $f \in R - \{0\}$ and there exists $a \in \mathbb{R}$ such that $f(a) = 0$. Define

$$g(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$$

Note that $g \neq 0$. Note that $f * g = 0$ thus f is a zero-divisors of R.
Suppose f is a never zero function. Suppose f is a zero-divisors of R. There exists $g \in R - \{0\}$ such that $f * g = 0$. There exists $a \in \mathbb{R}$ such that $g(a) \neq 0$. Note that $f * g(a) = f(a) * g(a) \neq 0$ thus we have a contradiction and we conclude that no never zero functions are zero-divisors of R.

b) Determine all nilpotent elements of R.
Suppose $f \in R - \{0\}$ and f is a nilpotent element of R. There exists some n such that $f^n = 0$. Note that there exists $a \in \mathbb{R}$ such that $f(a) \neq 0$. Note that $f^n(a) = (f(a))^n \neq 0$ a contradiction conclude that only 0 is a nilpotent element of R.

c) Show that every nonzero element is a zero-divisor or a unit.
Suppose f is a nonzero element. Suppose f is not a zero-divisor. Note that f is a never

zero function. Define a function $g(x) = 1/f(x)$. Note that $f * g = 1$. Conclude f is a unit. Conclude that every nonzero element is a zero-divisor or a unit.

Exercise 13.31: Let R be a ring with unity 1. If the product of any pair of nonzero elements of R is nonzero, prove that $ab = 1$ implies $ba = 1$.
Suppose $ab = 1$. Note that $(ba)b = b(ab) = b$ and thus $ba = 1$.

Exercise 13.35: Let F be a field of order 2^n. Prove that char $F = 2$.
Suppose that $(\nexists a \neq 0)a = -a$. In this case we can pair off non-zero elements each with there addative inverse. This means that there are a even number of non-zero elements, or that the total number of elemets is odd, a contradiction with the total number of elemets being 2^n.
We know there must exist some $a \in F - \{0\}$ such that $a = -a$. Note that $a(1 + 1) = a + a = a + (-a) = 0$ implies that $1 + 1 = 0$ and thus char $F = 2$.

Exercise 13.51: Show that any finite field has order p^n , where p is a prime. Hint: Use facts about finite Abelian groups. (This exercise is referred to in Chapter 22.)