Exercise 10.31: Suppose that ϕ is a homomorphism from $U(30)$ to $U(30)$ and that $\ker\phi = \{1, 11\}$. If $\phi(7) = 7$, find all elements of $U(30)$ that map to 7.
The elements that map onto 7 are precicely $a\ker\phi$ where $a \in \phi^{-1}(7)$ thus these elements are $\{7, 17\}$.

Exercise 10.35: Prove that the mapping $\phi : Z \bigoplus Z \to Z$ given by $(a, b) \to a - b$ is a homomorphism. What is the kernel of ϕ? Describe the set $\phi^{-1}(3)$ (that is, all elements that map to 3).
Clearly ϕ is well defined and thus we need only show that operation are perserved. Note that $\phi((a, b))\phi((c, d)) = a-b+c-d = (a+c)-(b+d) = \phi(((a+c), (b+d))) = \phi((a, b)+(c, d))$, and thus ϕ is a homomorphism.
Note that $(a, b) \in \ker\phi$ iff $a-b = 0$ in other words where $a = b$ thus $\ker\phi = \{(a, a) \mid a \in \mathbb{Z}\}$.
Note that $(3, 0) \in \phi^{-1}(3)$ thus $\phi^{-1}(3) = (3, 0)\ker\phi = \{(a + 3, a) \mid a \in \mathbb{Z}\}$.

Exercise 10.40: For each pair of positive integers m and n, we can define a homomorphism from Z to $Z_m \bigoplus Z_n$ by $x \to (x \mod m, x \mod n)$. What is the kernel when $(m, n) = (3, 4)$? What is the kernel when $(m, n) = (6, 4)$? Generalize.
Note that $x \in \ker\phi$ iff $\phi(x) = (0, 0)$. Thus $\ker\phi = \{x \in Z \mid \phi(x) = (0, 0)\}$ or in our case $\ker\phi = \{x \in Z \mid (m \mid x) \wedge (n \mid x)\}$ or $\ker\phi = \{x\,\mathrm{lcm}(m, n) \mid x \in Z\}$. If $(m, n) = (3, 4)$ then $\ker\phi = \{12x \mid x \in Z\}$. If $(m, n) = (6, 4)$ then $\ker\phi = \{12x \mid x \in Z\}$.

Exercise 10.43: Let $\phi(d)$ denote the Euler phi function of d (see page 85). Show that the number of homomorphisms from Z_n to Z_k is $\sum \phi(d)$, where the sum runs over all common divisors d of n and k. [It follows from number theory that this sum is actually $\gcd(n, k)$.]
First let's break up all of the homomorphisms by the size of the image. Note that the size of the image call it d must devide n since the homomorphism associated with this image devides the group Z_n into d chunks of equal size. Note that the size of the image call it d must devide m since $\psi(Z_n)$ is a subgroup of size d in Z_m.
How many homomorphisms have a size of there image equal to d? Well as descussed above if $d \nmid m$ or $d \nmid n$ then there are no homomorphisms associated with it. However if $d \mid n$ and $d \mid m$ then we will have homomorphisms associated with it and these homomorphisms map onto $< m/d >$ the only subgroup of Z_m with d elements. If we know where the genorator 1 in Z_n gets mapped to in $< m/d >$ we know were eavery item gets maped to. Noting that 1 must get maped to a genorator we know that there are exactly as many homomorphisms onto $< m/d >$ as $< m/d >$ has genorators. Note that $< m/d >$ has $\phi(| < m/d > |) = \phi(d)$ genorators.
Now simply add up the number of homomorphisms associated with any d value and we have the total number of homomorphisms. The result of this sum is exacty the sum described in the question.

Exercise 10.48: Suppose that Z_{10} and Z_{15} are both homomorphic images of a finite group G. What can be said about $|G|$? Generalize.
We know that $|Z_{10}| \mid |G|$ and that $|Z_{15}| \mid |G|$ thus $\mathrm{lcm}\,(10, 15) \mid |G|$. In general if groups $a_1 \cdots a_n$ are homomorphic images of a finite group G then $\mathrm{lcm}\,(|a_1| \cdots |a_n|) \mid |G|$.

Exercise 10.59: Suppose that H and K are distinct subgroups of G of index 2. Prove that $H \cap K$ is a normal subgroup of G of index 4 and that $G/(H \cap K)$ is not cyclic.
Note that $H \not\subset K$ and $K \not\subset H$. Let $a \in H \cap K^c$ and let $b \in K \cap H^c$. Suppose H is not normal in G. in this case there exists $a \in G$ and $h \in H$ such that $aha^{-1} \notin H$. Note that $a \notin H$, since H has closure. Note that $H \cap K$ is a normal subgroup of G