Exercise 10.16: Prove that there is no homomorphism from $Z_8 \bigoplus Z_2$ onto $Z_4 \bigoplus Z_4$.
Suppose there exists ϕ a homomorphism from $Z_8 \bigoplus Z_2$ onto $Z_4 \bigoplus Z_4$. Noting that both $Z_8 \bigoplus Z_2$ and $Z_4 \bigoplus Z_4$ have 16 elements we can conclude ϕ is one-to-one and thus ϕ is a isomorphisim. Note that $(1, 0)$ in $Z_8 \bigoplus Z_2$ has order 8. There must exist some element in $Z_4 \bigoplus Z_4$ that is order 8. Note that every element of $Z_4 \bigoplus Z_4$ has order 4 or less. We have reached a contradiction and conclude there is no homomorphism from $Z_8 \bigoplus Z_2$ onto $Z_4 \bigoplus Z_4$.

Exercise 10.20: How many homomorphisms are there from Z_{20} onto Z_8? How many are there to Z_8?
The kernel of any homomorphisim from Z_{20} onto Z_8 must have $20/8$ elements since this is not a integer no such homomorphisim exists.
The kernel of a homomorphisim from Z_{20} onto $Z_4 \leq Z_8$ would have $20/4 = 5$ elements. Noting that if such a homomorphisim ϕ existed we would require the kernel to be a sub-group and noting that there is only one sub-group of order 5 in Z_{20} we can say that there is only one possible kernel $H = ker(\phi) = \{0, 4, 8, 12, 16\}$. We are now looking for isomorphisims from Z_{20}/H to Z_4. There are two, $1 + H \to 1$ and $3 + H \to 1$, thus there are two homomorphisims from Z_{20} onto Z_4.
Under a similar procedure regarding Z_2 we find another homomorphisim and there is the homomorphisim $\phi(x) = 0$, thus there are a total of 4 homomorphisims from Z_{20} to Z_8.

Exercise 10.22: Suppose that ϕ is a homomorphism from a finite group G onto $\bar{G}$ and that $\bar{G}$ has an element of order 8. Prove that G has an element of order 8. Generalize.
There exists $\bar{x} \in \bar{G}$ such that $|\bar{x}| = 8$. There exists $x \in G$ such that $\phi(x) = \bar{x}$. Let $n = |x|$. Note that $8|n$ thus $8k = n$ for some integer k. Consider x^k. Note that $(x^k)^r$ where $r < 8$ cannot be identity since $kr < n$. Note that $(x^k)^8 = x^n = e$ thus $|x^k| = 8$.

Exercise 10.24: Suppose that $\phi : Z_{50} \to Z_{15}$ is a group homomorphism with $\phi(7) = 6$.

a. Determine $\phi(x)$.
   Note that $\phi(x) = \phi(1^x) = \phi((7^{43})^x) = \phi(7^{43x}) = \phi(7)^{43x} = 6^{43x} = 6 * 43 * x \bmod 15 = 3 * x \bmod 15$.

b. Determine the image of ϕ.
   Noting that $\phi(x) = 3^x$ the image of ϕ will be a subset of $< 3 >$ since it can only generate things of the form 3^x. Note that $\phi(\{0, 1, 2, 3, 4\}) =< 3 >$ we can say that the image of ϕ is precisely $< 3 >= \{0, 3, 6, 9, 12\}$.

c. Determine the kernel of ϕ.
   Note that $x \in \ker(\phi)$ iff $3^x = e_{15}$. Thus $\ker(\phi) = \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\}$.

d. Determine $\phi^{-1}(3)$. That is, determine the set of all elements that map to 3.
   Note that $\phi(1) = 3$ thus $\phi^{-1}(3) = 1 + \ker(\phi)$.

Exercise 10.26: Determine all homomorphisms from Z_4 to $Z_2 \bigoplus Z_2$.
The order of the kernel divides 4 thus there are 3 possible orders for the kernel.
The order is 4. In this case all elements are in the kernel and we have the trivial homomorphism.
The order is 2. Noting that the kernel is a subgroup we can say $\ker(\phi) = \{0, 2\}$. Thus all that is left to decide is where we map $\{1, 3\}$ there are three possibilities, mapping to $(1, 0)$ or $(0, 1)$ or $(1, 1)$, and on inspection all three work.
The order is 1. in this case we would have a isomorphisim and noting that one group is cyclic and the other is not this is impossible.
We have described all 4 homomorphisms.

Exercise 10.28: Suppose that ϕ is a homomorphism from S_4 onto Z_2. Determine $\ker\phi$. Determine all homomorphisms from S_4 to Z_2.
The order of $\ker\phi = |S_4|/Z_2 = 12$. Noting that there is only one sub group of order 12 in S_4 and that is A_4. Thus we have the homomorphism $\phi(A_4) = 0$ and $\phi(S_4 - A_4) = 1$. The only other homomorphism would be the trivial homomorphism $\phi(S_4) = 0$.

Exercise 10.30: Suppose that ϕ is a homomorphism from a group G onto $Z_6 \bigoplus Z_2$ and that the kernel of ϕ has order 5. Explain why G must have normal subgroups of orders 5, 10, 15, 20, 30, and 60.
Recall that if $H \leq Z_6 \bigoplus Z_2$ then $\phi^{-1}(H) \leq G$. Note that $|\phi^{-1}(H)| = |\ker\phi| * |H|$.

H	$\|H\|$	$\|\phi^{-1}(H)\|$
$(0, 0)$	1	5
$< (0, 1) >$	2	10
$< (1, 0) >$	6	30
$< (2, 0) >$	3	15
$Z_6 \bigoplus Z_2$	2	10
$< 3 > \bigoplus Z_2$	4	20

From the table above we can see the groups in G of the orders described.