Exercise 11.6: Show that there are two Abelian groups of order 108 that have exactly one subgroup of order 3.
Note that $108 = 2^2 3^3$. Thus any Abelian group of order 108 is isomorphic to one of the following:

group	exactly one subgroup of order 3	and it is
$Z_{3^3} \bigoplus Z_{2^2}$	yes	$<(9,0)>$
$Z_{3^3} \bigoplus Z_2 \bigoplus Z_2$	yes	$<(9,0,0)>$
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_{2^2}$	no	
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	no	
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_{2^2}$	no	
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	no	

Exercise 11.7: Show that there are two Abelian groups of order 108 that have exactly four subgroups of order 3.
Note that $108 = 2^2 3^3$. Thus any Abelian group of order 108 is isomorphic to one of the following:

group	has?	and they are
$Z_{3^3} \bigoplus Z_{2^2}$	no	
$Z_{3^3} \bigoplus Z_2 \bigoplus Z_2$	no	
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_{2^2}$	yes	$<(3,0,0)>, <(3,1,0)>, <(0,1,0)>, <(3,2,0)>$
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	yes	the above with a additional 0
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_{2^2}$	no	
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	no	

Exercise 11.8: Show that there are two Abelian groups of order 108 that have exactly 13 subgroups of order 3.
Note that $108 = 2^2 3^3$. Thus any Abelian group of order 108 is isomorphic to one of the following:

group	has?	and they are
$Z_{3^3} \bigoplus Z_{2^2}$	no	
$Z_{3^3} \bigoplus Z_2 \bigoplus Z_2$	no	
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_{2^2}$	no	
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	no	
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_{2^2}$	yes	see below
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	yes	the above with a additional 0

For the above section $<(1,0,0,0)>, <(0,1,0,0)>, <(0,0,1,0)>, <(1,1,0,0)>, <(1,2,0,0)>, <(1,0,1,0)>, <(1,0,2,0)>, <(0,1,1,0)>, <(0,1,2,0)>, <(1,1,1,0)>, <(1,1,2,0)>, <(1,2,1,0)>, <(1,2,2,0)>$.

Exercise 11.10: Find all Abelian groups (up to isomorphism) of order 360.
Note that $360 = 5 \cdot 3^2 \cdot 2^3$ thus all Abelian groups of order 360 are isomorphic to one of the following:

$Z_5 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2 \bigoplus Z_2$
$Z_5 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_{2^2} \bigoplus Z_2$
$Z_5 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_{2^3}$
$Z_5 \bigoplus Z_{3^2} \bigoplus Z_2 \bigoplus Z_2 \bigoplus Z_2$
$Z_5 \bigoplus Z_{3^2} \bigoplus Z_{2^2} \bigoplus Z_2$
$Z_5 \bigoplus Z_{3^2} \bigoplus Z_{2^3}$

Exercise 11.12: Suppose that the order of some finite Abelian group is divisible by 10. Prove that the group has a cyclic subgroup of order 10.
Note that there must exist some isomorphisim ϕ that maps our Abelian group to something of the form $Z_{(p_1)^{n_1}} \bigoplus Z_{(p_2)^{n_2}} \cdots$. Note that the primes 2 and 5 must divide the order of the group and thus there are $p_k = 5$ and $p_l = 2$. WLoG let $k = 1$, $l = 2$. Note that the element $(5^{n_1-1}, 2^{n_2-1}, 0, 0, 0 \cdots)$ is of order 10, thus if we move back by ϕ^{-1} we have found a element in our original group of order 10.

Exercise 12.12: Let a, b, and c be elements of a commutative ring, and suppose that a is a unit. Prove that b divides c if and only if ab divides c.
Suppose b divides c. In this case there exists a d such that $bd = c$. Note that $ab(a^{-1}d) = (aa^{-1})bd = c$ thus ab divides c.
Suppose ab divides c. In this case there exists a d such that $abd = c$. Note that $abd = b(ad) = c$ thus b divides c.

Exercise 12.36: Let m and n be positive integers and let k be the least common multiple of m and n. Show that $mZ \cap nZ = kZ$.
Suppose $a \in mZ \cap nZ$. Note that $m \mid a$ and $n \mid a$ thus $\operatorname{lcm}(m, n) \mid a$ so $k \mid a$ thus $a \in kZ$ so $mZ \cap nZ \subseteq kZ$.
Suppose $a \in kZ$. Note that $k \mid a$ thus $m \mid a$ and $n \mid a$ so $a \in mZ$ and $a \in nZ$ thus $a \in mZ \cap nZ$ and so $kZ \subseteq mZ \cap nZ$. By definition $mZ \cap nZ = kZ$.

Exercise 12.42: Let $R = \left\{ \begin{bmatrix} a & a \\ b & b \end{bmatrix} \middle| a, b \in Z \right\}$. Prove or disprove that R is a subring of $M_2(Z)$.
Suppose $A = \begin{bmatrix} a & a \\ b & b \end{bmatrix} \in R$ and $B = \begin{bmatrix} c & c \\ d & d \end{bmatrix} \in R$, Note that $A - B = \begin{bmatrix} a-c & a-c \\ b-d & b-d \end{bmatrix} \in R$ and that $A * B = \begin{bmatrix} ac+ad & ac+ad \\ bc+bd & bc+bd \end{bmatrix} \in R$ and thus by the two step subring test we know that R is a subring.

Exercise 12.46: Show that $2Z \cup 3Z$ is not a subring of Z.
Suppose $2Z \cup 3Z$ is a subring of Z. Note that $2 \in 2Z \cup 3Z$ and $3 \in 2Z \cup 3Z$ thus by closure $2 + 3 \in 2Z \cup 3Z$. Noting that $5 \notin 2Z$ and $5 \notin 3Z$ we note that $5 \notin 2Z \cup 3Z$, a contradiction thus we conclude $2Z \cup 3Z$ is not a subring of Z.

Exercise 12.48: Determine the smallest subring of $\mathbb{Q}$ that contains $2/3$.
In general the smallest subring of R that contains x is $H = \{\sum_{i=1}^{k} a_i x^i \mid k \in \mathbb{Z}^+, (\forall i \in \mathbb{Z}) a_i \in \mathbb{Z}\}$. This is trivial to show, simply note that any subring that contains x must contain any power of x and any sum of those powers or there negation and so we conclude that any subring containing x must also contain all of H. Then we note that H passes the two step subring test, since any polynomial with integer coefficients subtracted from another polynomial with integer coefficients will be a polynomial with integer coefficients and the multiple of any two polynomials with integer coefficients will be another polynomial with integer coefficients, thus H is a subring of R. Since H is a subring of R and it is contained in all subrings of R containing x we conclude it is the smallest subring of R containing x.
Thus the smallest subring of $\mathbb{Q}$ that contains $2/3$ is $\{\sum_{i=1}^{k} a_i 2^i/3^i \mid k \in \mathbb{Z}^+, (\forall i \in \mathbb{Z}) a_i \in \mathbb{Z}\}$.