Exercise 6.12: Find two groups G and H such that $G \not\approx H$, but $\text{Aut}(G) \approx \text{Aut}(H)$.
Let $G = Z_4$ and $H = Z_6$. Note that they have different numbers of elements thus $G \not\approx H$. Note that they each have two auto-morphisims, the identity and the exchange of generators, and all groups with two elements are isomorphic, thus $\text{Aut}(G) \approx \text{Aut}(H)$.

Exercise 6.14: Find $\text{Aut}(Z_6)$.
As descused in the previous question there are two elements to this group,

$$\text{The identity } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\text{The exchange of generators } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

Exercise 6.20: Show that Z has infinitely many subgroups isomorphic to Z.
Choose $n \in \mathbb{N}$. Define $H \subseteq Z$ as $H = \{k \in Z : n \mid k\}$.
Choose $a, b \in H$. Note that $a = nc$ and $b = nd$ for some integers c, d. Note that $ab^{-1} = nc - nd = n(c - d) \in H$. Thus H is a group, and so H is a subgroup of Z.
Define $\phi(x) = nx$. Note that $\phi : Z \to H$ bijectively. Note that $\phi(ab) = \phi(a+b) = n(a+b) = na + nb = \phi(a) + \phi(b) = \phi(a)\phi(b)$, thus $H \approx Z$.
Since H is unique depending on our choice of n and there are a infinite number of possible n's we can say that Z has infinitely many subgroups isomorphic to Z.

Exercise 6.23: Give an example of a cyclic group of smallest order that contains a subgroup isomorphic to Z_{12} and a subgroup isomorphic to Z_{20}. No need to prove anything, but explain your reasoning.
Essentially we are looking for a Z_n witch contains a element of order 12 and a element of order 20. This must mean $12 \mid n$ and $20 \mid n$, the smallest n with this property is 60. Note that in Z_{60}, $| < 5 > | = 12$ and $| < 3 > | = 20$, thus, do to all cyclic groups of the same order being isomorphic, $< 5 > \approx Z_{12}$ and $< 3 > \approx Z_{20}$. I have demonstrated that Z_{60} is the smallest that could have this property and that it does have this property.

Exercise 6.24: Suppose that $\phi : Z_{20} \to Z_{20}$ is an automorphism and $\phi(5) = 5$. What are the possibilities for $\phi(x)$?
To require that ϕ is a automorphism is exactly to require that it map at least one generator to another generator. Note that $5 = \phi(5) = \phi(1^5) = \phi(1)^5$, noting that $\phi(1)$ is a generator it is only left to check witch generators have this property.

```
>> a=[1,3,7,9,11,13,17,19];
>> [a;mod(a*5,20)]
ans =

     1     3     7     9    11    13    17    19
     5    15    15     5    15     5     5    15

>> diary off
```

We now see that the automorphisms that work are $\phi(1) = 1$, $\phi(1) = 9$, $\phi(1) = 13$, $\phi(1) = 17$. Note that defining where the generator goes defines a entire automorphism and thus what I have given are the complete descriptions of the 4 automorphisims with the property $\phi(5) = 5$.

Exercise 6.26: Prove that the mapping from $U(16)$ to itself given by $x \to x^3$ is an automorphism. What about $x \to x^5$ and $x \to x^7$? Generalize.
Take $\phi_n : x \to x^n$. Note that $U(16)$ is a abelian group thus $\phi_n(ab) = (ab)^n = a^n b^n = \phi_n(a)\phi_n(b)$. Now all we need to show isomorphisim is bijectivity. Since this is a finite group it is sufficient to show that ϕ_n is onto.

```
>> x=[1,3,5,7,9,11,13,15];
>> mod(x.^3,16)
ans =

     1    11    13     7     9     3     5    15

>> mod(x.^5,16)
ans =

     1     3     5     7     9    11    13    15

>> mod(x.^7,16)
ans =

     1    11    13     7     9     3     5    15

>> mod(x.^2,16)
ans =

   1   9   9   1   1   9   9   1

>> diary off
```

As demonstrated above ϕ_3, ϕ_5, ϕ_7 work, however note that ϕ_2 will not work as a automorphism.