Exercise 6.30: Suppose that $\phi : Z_{50} \to Z_{50}$ is an automorphism with $\phi(11) = 13$. Determine a formula for $\phi(x)$.
Let us first determine $\phi(1)$. Note that:

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>> x=[1:50];
>> [x;mod(11.*x,50)]
ans =

 Columns 1 through 12:

    1    2    3    4    5    6    7    8    9   10   11   12
   11   22   33   44    5   16   27   38   49   10   21   32

 Columns 13 through 24:

   13   14   15   16   17   18   19   20   21   22   23   24
   43    4   15   26   37   48    9   20   31   42    3   14

 Columns 25 through 36:

   25   26   27   28   29   30   31   32   33   34   35   36
   25   36   47    8   19   30   41    2   13   24   35   46

 Columns 37 through 48:

   37   38   39   40   41   42   43   44   45   46   47   48
    7   18   29   40    1   12   23   34   45    6   17   28

 Columns 49 and 50:

   49   50
   39    0

>> diary off
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Now we can see that $11^{41} = 1$ and thus $\phi(1) = \phi(11^{41}) = 13^{41} = 33$. And now $\phi(x) = \phi(1^x) = \phi(1)^x = 33 * x \mod 50$.

Exercise 6.37: Prove that $\mathbb{Z}$ under addition is not isomorphic to $\mathbb{Q}$ under addition.
Suppose $q = \frac{i}{j}$ is a generator of $\mathbb{Q}$. Note that 0 cannot be the generator of $\mathbb{Q}$ and thus $i \neq 0$. Consider $q' = \frac{i}{2j} \in \mathbb{Q}$. Note that there must be some integer k such that $q * k = q'$, since q is a generator. Now we see that $k = j/i * q * k = j/i * q' = 1/2 \notin \mathbb{Z}$, a contradiction to k being a integer. We now conclude that $\mathbb{Q}$ has no generator, and thus is not cyclic. Since $\mathbb{Z}$ is cyclic we can conclude $\mathbb{Q} \not\approx \mathbb{Z}$.

Exercise 6.42: Suppose that G is a finite Abelian group and G has no element of order 2. Show that the mapping $\phi : g \to g^2$ is an automorphism of G. Show, by example, that there is an infinite Abelian group for which the mapping $g \to g^2$ is one-to-one and operation-preserving but not an automorphism.

Suppose $a \in G$ and $2 \mid n = |a|$. Note that $a^{n/2} \in G$ and $|a^{n/2}| = 2$, a contradiction conclude that no elements have order advisable by 2.
Suppose $|a| = n$ and $|a^2| = m$. Note that $(a^2)^n = (a^n)^2 = e$ thus $m \leq n$. Note that $a^{2m} = e$ thus $n \mid 2m$ and since n and 2 are relatively prime we see $n \mid m$ thus $m \geq n$ or $n = m$. We now know $|a| = |a^2|$ for all $a \in G$.
Suppose $a^2 = b^2$ where $a, b \in G$. Note that $|a| = |b| = 2k + 1$ for some integer k. Note that $a^{2k}a = e = b^{2k}b$ and that $a^{2k} = (a^2)^k = (b^2)^k = b^{2k}$ thus by cancellation we see that $a = b$.
We now conclude that ϕ is one-to-one, thus since G is finite ϕ is also onto. Note that $\phi(ab) = (ab)^2 = a^2b^2 = \phi(a)\phi(b)$, thus ϕ is a automorphism of G.

Let $G = \mathbb{Z}$ under addition. Note that $\phi : g \to g^2$ is one-to-one, since $2a = 2b \Rightarrow a = b$. And note that $\phi(ab) = 2 * (a + b) = 2 * a + 2 * b = \phi(a)\phi(b)$. Note that there is no element of $\mathbb{Z}$ that maps to 1 under ϕ, thus ϕ is not a automorphism.

Exercise 6.48: Let ϕ be an isomorphism from a group G to a group $\bar{G}$ and let a belong to G. Prove that $\phi(C(a)) = C(\phi(a))$.
Choose $\bar{x} \in \phi(C(a))$. Note that there exists a $x \in C(a)$ such that $\phi(x) = \bar{x}$. Note that $ax = xa$ since $x \in C(a)$ thus $\phi(a)\bar{x} = \phi(ax) = \phi(xa) = \bar{x}\phi(a)$, thus $\bar{x} \in C(\phi(a))$ and $\phi(C(a)) \subseteq C(\phi(a))$.

Choose $\bar{x} \in C(\phi(a))$. Note that there exists a $x \in G$ such that $\phi(x) = \bar{x}$. Note that $\phi(a)\bar{x} = \bar{x}\phi(a)$ since $\bar{x} \in C(\phi(a))$ thus $\phi(ax) = \phi(a)\bar{x} = \bar{x}\phi(a) = \phi(xa)$, thus due to bijectivity $ax = xa$ and so $x \in C(a)$ or $\bar{x} \in \phi(C(a))$ and $C(\phi(a)) \subseteq \phi(C(a))$. We conclude $\phi(C(a)) = C(\phi(a))$.

Exercise 6.51: Suppose that G is an Abelian group and ϕ is an automorphism of G. Prove that $H = \{x \in G \mid \phi(x) = x^{-1}\}$ is a subgroup of G.
Choose $a, b \in H$. Note that $\phi(ab^{-1}) = \phi(a)\phi(b^{-1}) = a^{-1}(\phi(b))^{-1} = a^{-1}b = (b^{-1}a)^{-1} = (ab^{-1})^{-1}$, thus $ab^{-1} \in H$. By the one step test we know that H is a subgroup of G.

Exercise 6.53: Let a belong to a group G and let $|a|$ be finite. Let ϕ a be the automorphism of G given by $\phi_a(x) = axa^{-1}$. Show that $|\phi_a|$ divides $|a|$. Exhibit an element a from a group for which $1 < |\phi_a| < |a|$.
Note that $\phi_a^{|a|}(x) = a^{|a|}xa^{-|a|} = exe = x$, thus $|\phi_a|$ divides $|a|$.
Recall in D_4 that R_{180} commutes with all elements but R_{90} does not. Let $a = R_{90}$. Note that $\phi_a(x) = axa^{-1} = R_{90}xR_{270}$ witch is not x for all x, in particular if $x = H$, $\phi_a(x) = V \neq H$, thus $|\phi_a| \neq 1$ or $|\phi_a| > 1$. Note that $\phi_a(\phi_a(x)) = R_{180}xR_{180} = xR_{180}R_{180} = x$ thus $|\phi_a| \leq 2$. Note that $1 < |\phi_a| \leq 2 < 4 = |a|$.