Exercise 4.66: Note that $U(2^n)$ is the set of all odd naturals (relatively prime to 2) less than 2^n. Note that $U(2^n) \subseteq U(2^{n+1})$. Note that $U(2^3) = U(8)$ is non cyclic, this is left as a exercise to the reader and is simply a bit of brute force.
Suppose $U(2^n)$ is non-cyclic.
Suppose $U(2^{n+1})$ is cyclic.
There exists a element call it a that is a generator of $U(2^{n+1})$. Define a' using the division algorithm (deviding a by 2^n) $a = j2^n + a'$. Note that $a' = a - j2^n$ and that since $j2^n$ is even and a is odd a' must be odd and thus a element of $U(2^n)$. Choose $x \in U(2^n)$. Note that $x \in U(2^{n+1})$, thus there exists a natural k such that $a^k \mod 2^{n+1} = x$. Note that we could write $x = a^k - i2^{n+1} = (j2^n + a')^k - i2^{n+1} = j^k(2^n)^k + 2a'j2^n + a'^k - i2^{n+1} = a'^k - (2i - j^k(2^n)^{k-1} - 2a'j)2^n$ thus $a'^k \mod 2^n = x$. Noting that a' is a generator for $U(2^n)$ we conclude that $U(2^n)$ is cyclic, a contradiction, thus $U(2^{n+1})$ is non cyclic.
By induction $U(2^n)$ is non cyclic for all $n \geq 3$.

Exercise 4.72: For each of these it would simply be the greatest common divisor between 48 and the power of a.

1. $< a^3 >$
2. $< a^{24} >$
3. $< a^6 >$

Exercise 4.74: Note that all elements of H have determinate of 1 and thus $H \subseteq GL(2, \mathbb{R})$. Note that $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in H$. Also note that its inverse $a^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \in H$. Note that a increments the upper right hand value of a matrix in H, $a\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$, thus a along with a^{-1} can generate all elements in H. Noting that the identity is in H we can conclude H is a cyclic sub group of $GL(2, \mathbb{R})$.

Exercise 4.81: For all cases there is the set of all rotations, a cyclic sub group of order n. If n is odd there are no other sub groups of order n. If n is even we can inscribe it (see attached) inside of two $D_{n/2}$ and there set of operations will give us two additional sub groups of order n. Thus if n is odd there is one sub group of order n and if n is even there are three sub groups of order n.

Exercise 4.82: Note that G is clearly closed as the addition of two integers mod 3 will be in $\{0, 1, 2\}$. Also note that there is a identity namely 0. The inverse is trivial $a_1x^2 + a_2x + a_3$ has a inverse of $b_1x^2 + b_2x + b_3$ where b_k is the inverse of a_k in Z_3. Also addition is associative and thus we know this is a group. By simply observing permutations we can see that there are 3 possibilities for each coefficient, thus $|G| = 3 * 3 * 3 = 27$. Suppose $a_1x^2 + a_2x + a_3$ is a generator for G. Note that $a_k \neq 0$, since if a a_k were 0 we would never obtain the elements where that coefficient was non-zero ie $0x^2 + a_2x + a_3$ can never generate x^2 witch is in G. We can now conclude that two of our coefficients are the same since there are three coefficients and two possibilities (pigeonhole principal), without loss of generality assume these are a_1 and a_2. Since $a_1x^2 + a_2x + a_3$ is a generator we know that there exists a k such that $(a_1x^2 + a_2x + a_3)^k = x^2$, however since $a_1 = a_2$ we know that those two coefficients will always be the same and thus the first coefficient being one implies that the second coefficient will be one, in other words no such k could exist. We conclude the negation of our supposition and conclude that G has no generator and thus is non cyclic.

Exercise 4.83: Note that is m and n are relatively prime the only shared element between $< a >$ and $< b >$ will be the identity, since any shared element must have a order that divides both m and n. Thus $a^k = b^k$ implies that $a^k = b^k = e$. Note that $m \mid k$ and $n \mid k$, thus $\text{gcm}(m, n) \mid k$. Note that $\text{gcm}(m, n) = mn$ and thus $mn \mid k$.