

Exercise 8.26: The group $S_3 \oplus Z_2$ is isomorphic to one of the following groups: Z_{12} , $Z_6 \oplus Z_2$, A_4 , D_6 . Determine which one by elimination.

It cannot be isomorphic to Z_{12} since Z_{12} is cyclic and S_3 is not. Since S_3 is not abelian we know that it is not isomorphic to $Z_6 \oplus Z_2$. Note that A_4 has 8 elements of order 3 and $S_3 \oplus Z_2$ has 2 thus they are not isomorphic. By elimination $S_3 \oplus Z_2 \approx D_6$.

Exercise 8.32: What is the order of the largest cyclic subgroup of $Z_6 \oplus Z_{10} \oplus Z_{15}$? What is the order of the largest cyclic subgroup of $Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}$?

Noting that all of these Z_n are cyclic we see that the largest cyclic sub group will be generated by (g_1, g_2, \dots, g_k) where g_i is the generator for i th group in the product. Note that $|(g_1, g_2, \dots, g_k)| = \text{lcm}(|g_1|, |g_2|, \dots, |g_k|) = \text{lcm}(n_1, n_2, \dots, n_k)$. Thus for $Z_6 \oplus Z_{10} \oplus Z_{15}$ the largest cyclic sub group has order $\text{lcm}(2 * 3, 2 * 5, 3 * 5) = 2 * 3 * 5 = 30$.

Exercise 8.55: How many isomorphisms are there from Z_{12} to $Z_4 \oplus Z_3$?

Note that $Z_4 \oplus Z_3$ is cyclic and has twelve elements thus there is at least one isomorphism. Deciding where to map a generator uniquely defines a isomorphism, thus we need only examine what we map to the generator $(1, 1) \in Z_4 \oplus Z_3$. We know we can only map generators to that element and the generators in Z_{12} are 1,5,7,11. Thus we conclude there are exactly 4 isomorphisms

Exercise 8.58: Prove that $Z_5 \oplus Z_5$ has exactly six subgroups of order 5.

Note that

$$\langle (1, 0) \rangle = \{(1, 0), (2, 0), (3, 0), (4, 0), (0, 0)\}$$

$$\langle (1, 1) \rangle = \{(1, 1), (2, 2), (3, 3), (4, 4), (0, 0)\}$$

$$\langle (1, 2) \rangle = \{(1, 2), (2, 4), (3, 1), (4, 3), (0, 0)\}$$

$$\langle (1, 3) \rangle = \{(1, 3), (2, 1), (3, 4), (4, 2), (0, 0)\}$$

$$\langle (1, 4) \rangle = \{(1, 4), (2, 3), (3, 2), (4, 1), (0, 0)\}$$

$$\langle (0, 1) \rangle = \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 0)\}$$

thus there are at least 6 sub groups of order 5.

Suppose $Z_5 \oplus Z_5$ had 7 subgroups of order 5. Note that each of these sub groups must be cyclic, since 5 is prime. Note that each of the non identity elements are generators of there sub group since 5 is prime. Thus we see that each sub group has the identity and 4 elements in no other sub group thus there are at least $1 + 5 * 4 = 21$ elements, a contradiction since the group only has 25 elements, thus there are exactly 6 sub groups of order 5.

Exercise 8.66: Express $U(165)$ as an external direct product of cyclic groups of the form Z_n .

Note that $U(165) = U(5 * 3 * 11) \approx U(3) \oplus U(5) \oplus U(11) \approx Z_2 \oplus Z_4 \oplus Z_{10}$. (see pg 167)

Exercise 8.67: Express $U(165)$ as an external direct product of U-groups in four different ways.

$$U(165) = U(3 * 55) \approx U(3) \bigoplus U(55)$$

$$U(165) = U(5 * 33) \approx U(5) \bigoplus U(33)$$

$$U(165) = U(11 * 15) \approx U(11) \bigoplus U(15)$$

$$U(165) = U(5 * 3 * 11) \approx U(3) \bigoplus U(5) \bigoplus U(11)$$

Exercise 8.68: Without doing any calculations in $\text{Aut}(Z_{20})$, determine how many elements of $\text{Aut}(Z_{20})$ have order 4. How many have order 2?

Note that $\text{Aut}(Z_{20}) \approx U(20) \approx U(2^2) \bigoplus U(5) \approx Z_2 \bigoplus Z_4$. Note that $Z_2 \bigoplus Z_4$ has 4 elements of order 4 and 3 elements of order 2, thus $\text{Aut}(Z_{20})$ has 4 elements of order 4 and 3 elements of order 2.

Exercise 8.78: Find a subgroup of order 6 in $U(700)$. (see appendix)

From the table below $\langle 51 \rangle$ is a subgroup of order 6. $\langle 51 \rangle = \{51, 501, 351, 401, 151, 1\}$

table

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1 >> x=[1:1:699];
2 >> isDiv=(gcd(x,700)==1);
3 >> pow6=(mod(x.^6,700)==1);
4 >> not23=(mod(x.^3,700)~=1 & mod(x.^2,700)~=1);
5 >> [x;(isDiv & pow6 & not23)]'
6 ans =
7
8      1      0
9      2      0
10     3      0
11     4      0
12     5      0
13     6      0
14     7      0
15     8      0
16     9      0
17    10      0
18    11      0
19    12      0
20    13      0
21    14      0
22    15      0
23    16      0
24    17      0
25    18      0
26    19      0
27    20      0
28    21      0
29    22      0
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33    26      0
34    27      0
35    28      0
36    29      0
37    30      0
38    31      0
39    32      0
40    33      0
41    34      0
42    35      0
43    36      0
44    37      0
45    38      0
46    39      0
47    40      0
48    41      0
49    42      0
50    43      0
51    44      0
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52	45	0
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54	47	0
55	48	0
56	49	0
57	50	0
58	51	1
59	52	0
60	53	0
61	54	0
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63	56	0
64	57	0
65	58	0
66	59	0
67	60	0
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102	95	0
103	96	0
104	97	0
105	98	0

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