Exercise 15.42: Determine all ring homomorphisms from $\mathbb{Q}$ to $\mathbb{Q}$.
Suppose $\phi : \mathbb{Q} \to \mathbb{Q}$ is a homomorphism. Suppose $\phi(1) = a$. Note that $a^2 = a$ and thus eather $a = 1$ or $a = 0$.
In the case that $a = 0$, $\phi(n/m) = \phi(n) * \phi(1/m) = n\phi(1) * \phi(1/m) = n * 0 * \phi(1/m) = 0$.
In the case where $a = 1$, note that for any integer m, $\phi(m^{-1}) * \phi(m) = \phi(1) = 1$ and thus $\phi(m^{-1}) = \phi(m)^{-1}$. Note that $\phi(n/m) = \phi(n)[\phi(m)]^{-1} = n\phi(1)[m\phi(1)]^{-1} = n1[m1]^{-1} = n[m]^{-1} = n/m$.
We have now found the only two possible homomorphisms, note that both are actually homomorphisms:

$$x \to 0$$

$$x \to x$$

Exercise 15.44: Let R be a commutative ring of prime characteristic p. Show that the Frobenius map $\phi : x \to x^p$ is a ring homomorphism from R to R.
Note that $\phi(a + b) = (a + b)^p = \sum_{k=0}^{p} \alpha_k a^k b^{p-k}$ for some integers α_k. Note that the number of copys of $a^k b^{p-k}$ appering here is exactly how many ways we can choose witch term to take the a's from in $(a + b) * \cdots * (a + b)$ witch is $\binom{p}{k}$ (for more info read about Pascal's triangle), thus $\alpha_k = \binom{p}{k} = \frac{p!}{(p-k)!k!}$.
Consider α_k when $k \neq 0$ and $k \neq p$. In this case note that p will be one of the multiples in $p!$ but not in $(p - k)!k!$ in other words p apears in the prime factorization of the top but not in the prime factorization of the bottom of $\frac{p!}{(p-k)!k!}$ (here we are using the fact that p is prime). Note that in the reduced form of $\frac{p!}{(p-k)!k!}$, p must still appear in the top thus noting that in the reduced form the botom must be 1 since α_k is a integer we conclude that $p \mid \alpha_k$.
Define β_k for $2 \leq k \leq p - 1$ such that $p\beta_k = \alpha_k$.
Now note that $\phi(a+b) = \sum_{k=0}^{p} \alpha_k a^k b^{p-k} = a^p+b^p+\sum_{k=1}^{p-1} \alpha_k a^k b^{p-k} = a^p+b^p+\sum_{k=1}^{p-1} \beta_k p(a^k b^{p-k}) = a^p + b^p + \sum_{k=1}^{p-1} \beta_k 0_R = a^p + b^p = \phi(a) + \phi(b)$, thus ϕ preserves addition.
Note that $\phi(ab) = [ab]^p = a^p b^p = \phi(a)\phi(b)$ thus ϕ is operation preserving under multiplication. Note that ϕ is a homomorphism.

Exercise 15.46: Show that a homomorphism from a field onto a ring with more than one element must be an isomorphism.
Let F be a field and $R \neq \{0_R\}$ be a ring. Suppose ϕ is a homomorphism from F onto R. Note that $\ker(\phi)$ must be a ideal in F. Note that the only ideals in a field are $\{0_F\}$ and F. Choose $a \in R - \{0_R\}$. Note that there must exist $a' \in F$ such that $\phi(a') = a$ thus $a' \notin \ker(\phi)$ and thus $\ker(\phi) \neq F$. Note that $\ker(\phi) = \{0_F\}$, thus ϕ is a isomorphism.

Exercise 15.48: A principal ideal ring is a ring with the property that every ideal has the form $< a >$. Show that the homomorphic image of a principal ideal ring is a principal ideal ring.
Suppose R is a principal ideal ring and ϕ is a homomorphisim to some other ring, let $\phi(R) = H$, thus ϕ becomes a homomorphisim from R onto H. Let $K = \phi(< a >) = \phi(\{r * a \mid r \in R\}) = \{\phi(r * a) \mid r \in R\} = \{\phi(r) * \phi(a) \mid r \in R\} = \{h * \phi(a) \mid h \in \phi(R)\} = \{h * \phi(a) \mid h \in H\} =< \phi(a) >$ in the ring H. Thus the homomorphic image of a principal ideal ring is

a principal ideal ring.

Exercise 15.53: Determine all ring homomorphisms from $\mathbb{R}$ to $\mathbb{R}$.
Note that $\mathbb{R}$ is a field and thus the ideals are $\{0\}$ and $\mathbb{R}$. Suppose ϕ is a homomorphisms from $\mathbb{R}$ to $\mathbb{R}$. Note that $\ker(\phi)$ is ideal and thus $\ker(\phi) = \{0\}$ or $\ker(\phi) = \mathbb{R}$.
In the case that $\ker(\phi) = \mathbb{R}$, we only have one possibility for ϕ and that is $\phi(x) = 0$.

In the case that $\ker(\phi) = \{0\}$ note that $\phi(1) = 1$ and ϕ is a isomorphisim. Note that if $n \in \mathbb{N} \subseteq \mathbb{R}$, $\phi(n) = \phi(n * 1) = n\phi(1) = n$ thus ϕ acts as identity on $\mathbb{N}$. Note that if $(-n) \in (-1) * \mathbb{N} \subseteq \mathbb{R}$ then $\phi(-n) + n = \phi(-n) + \phi(n) = \phi(0) = 0$ thus $\phi(-n) = -n$ and we know that ϕ acts as identity on all of $\mathbb{Z}$. If $q/p \in \mathbb{Q} \subseteq \mathbb{R}$ then $\phi(q/p) = \phi(q)\phi(p^{-1}) = \phi(q)\phi(p)^{-1} = qp^{-1} = q/p$ thus ϕ acts as identity on all of $\mathbb{Q}$.

Exercise 15.56: