Exercise 15.12: Let $Z_3[i] = \{a + bi \mid a, b \in Z_3\}$ (see Example 9 in Chapter 13). Show that the field $Z_3[i]$ is ring isomorphic to the field $Z_3[x]/ < x^2 + 1 >$.
Define $H =< x^2+1 >$. Define $\phi : Z_3[i] \to Z_3[x]/ < x^2+1 >$ as $\phi(a+bi) = a+bx+H$. Note that any element in $Z_3[x]/ < x^2+1 >$ can be written in the form $g(x) = f(x)*(x^2+1)+r(x)$ where $r(x) = a+bx$ is of order at most x, simply by noting that $g(x) \in a+bx+H = \phi(a+bi)$ we can say that ϕ is onto. Note that $\phi((a+bi)(c+di)) = \phi(ac-bd+(cb+da)i) = ac-bd+(cb+da)x+H$. Note that $\phi(a+bi)\phi(c+di) = (a+bx+H)(c+dx+H) = ac+(cb+da)x+bdx^2+H = ac + (cb + da)x + bdx^2 - bd(x^2 + 1) + H = ac - bd + (cb + da)x + H$, thus ϕ preserves multiplication. Note that $\phi((a + bi) + (c + di)) = \phi(a + bi + c + di) = a + c + (b + d)x + H$. Note that $\phi(a + bi) + \phi(c + di) = (a + bx + H) + (c + dx + H) = a + c + (b + d)x + H$ thus ϕ preserves addition and so ϕ is a homomorphisim. Note that if $a + bi \in \ker(\phi)$ then $\phi(a + bi) = H$ thus $a + bx \in H$ witch is only true if $a = 0$ and $b = 0$ thus $\ker(\phi) = 0$ and we know that ϕ is one-to-one and thus ϕ is a isomorphisim and thus $Z_3[i]$ is ring isomorphic to the field $Z_3[x]/ < x^2 + 1 >$.

Exercise 15.14: Let $Z[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Z\}$ and

$$H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \middle| a, b \in Z \right\}$$

Show that $Z[\sqrt{2}]$ and H are isomorphic rings.
Define $\phi : Z[\sqrt{2}] \to H$ as $\phi(a+b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$. Note that $\phi((a+b\sqrt{2})(c+d\sqrt{2})) = \phi(ac+2bd+(ad+bc)\sqrt{2})$ and $\phi(a+b\sqrt{2})\phi(c+d\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}\begin{bmatrix} c & 2d \\ d & c \end{bmatrix} = \begin{bmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{bmatrix} = \phi(ac + 2bd + (ad + bc)\sqrt{2})$ thus ϕ is multiplication preserving. Note that $\phi((a + b\sqrt{2}) + (c + d\sqrt{2})) = \phi((a + c) + (b + d)\sqrt{2}) = \begin{bmatrix} a+c & 2(b+d) \\ b+d & a+c \end{bmatrix} = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} + \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} = \phi(a + b\sqrt{2}) + \phi(c + d\sqrt{2})$ and thus ϕ is addition preserving. Note that ϕ is a homomorphisim. By inspection it is clear that ϕ is onto and also that ϕ is one-to-one, thus ϕ is a isomorphisim and so $Z[\sqrt{2}]$ and H are isomorphic rings.

Exercise 15.16: Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a, b, c \in Z \right\}$. Prove or disprove that the mapping $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \to a$ is a ring homomorphisim.
Note that $\phi(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}) = \phi(\begin{bmatrix} ad & ? \\ 0 & ? \end{bmatrix}) = ad = \phi(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix})\phi(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix})$ where ? represents something that is not solved here, thus ϕ is multiplication preserving. Note that $\phi(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}) = \phi(\begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}) = a + d = \phi(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}) + \phi(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix})$ thus ϕ is addition preserving and thus a homomorphisim.

Exercise 15.20: Recall that a ring element a is called an idempotent if $a^2 = a$. Prove that a ring homomorphism carries an idempotent to an idempotent.

Suppose $\phi : A \to B$ where A, B are rings and ϕ is a homomorphisim. Suppose $a \in A$ is a idempotent. Note that $\phi(a)^2 = \phi(a^2) = \phi(a)$ thus $\phi(a)$ is a idempotent. We conclude that a ring homomorphism carries an idempotent to an idempotent.

Exercise 15.32: Let n be an integer with decimal representation $a_k a_{k-1} \cdots a_1 a_0$. Prove that n is divisible by 11 iff $a_0 - a_1 + a_2 - \cdots (-1)^k a_k$ is divisible by 11.
Let $\phi : Z \to Z_{11}$ be the natural homomorphisim $\phi(a) = a \bmod 11$. Note that $11 \mid a$ iff $\phi(a) = 0$. Note that $\phi(\sum_{i=0}^k a_k * 10^k) = \sum_{i=0}^k a_k * \phi(10)^k = \sum_{i=0}^k a_k * \phi(-1)^k = \phi(\sum_{i=0}^k a_k * (-1)^k)$ thus $11 \mid \sum_{i=0}^k a_k * 10^k$ iff $\phi(\sum_{i=0}^k a_k * (-1)^k) = 0 \longleftrightarrow 11 \mid \sum_{i=0}^k a_k * (-1)^k$.

Exercise 15.36: Let n be an integer with decimal representation $a_k a_{k-1} \cdots a_1 a_0$. Prove that n is divisible by 4 iff $a_1 a_0$ is divisible by 4.
Let $\phi : Z \to Z_4$ be the natural homomorphisim $\phi(a) = a \bmod 4$. Note that $4 \mid a$ iff $\phi(a) = 0$. Note that $\phi(\sum_{i=0}^k a_k * 10^k) = \phi((\sum_{i=2}^k a_k * 10^k) + a_1 a_0) = \phi(\sum_{i=2}^k a_k * 10^k) + \phi(a_1 a_0) = \sum_{i=2}^k (\phi(a_k) * \phi(10)^{k-2} * \phi(100)) + \phi(a_1 a_0) = \sum_{i=2}^k (\phi(a_k) * \phi(10)^{k-2} * 0) + \phi(a_1 a_0) = \phi(a_1 a_0)$. Note that $4 \mid \sum_{i=0}^k a_k * 10^k$ iff $\phi(a_1 a_0) = 0 \longleftrightarrow 4 \mid a_1 a_0$.