Exercise 11.6: Show that there are two Abelian groups of order 108 that have exactly one subgroup of order 3.
Note that $108 = 2^2 3^3$. Thus any Abelian group of order 108 is isomorphic to one of the following:

group	exactly one subgroup of order 3	and it is
$Z_{3^3} \bigoplus Z_{2^2}$	yes	$< (9,0) >$
$Z_{3^3} \bigoplus Z_2 \bigoplus Z_2$	yes	$< (9,0,0) >$
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_{2^2}$	no	
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	no	
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_{2^2}$	no	
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	no	

Exercise 11.7: Show that there are two Abelian groups of order 108 that have exactly four subgroups of order 3.
Note that $108 = 2^2 3^3$. Thus any Abelian group of order 108 is isomorphic to one of the following:

group	has?	and they are
$Z_{3^3} \bigoplus Z_{2^2}$	no	
$Z_{3^3} \bigoplus Z_2 \bigoplus Z_2$	no	
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_{2^2}$	yes	$< (3,0,0) >, < (3,1,0) >, < (0,1,0) >, < (3,2,0) >$
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	yes	the above with a additional 0
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_{2^2}$	no	
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	no	

Exercise 11.8: Show that there are two Abelian groups of order 108 that have exactly 13 subgroups of order 3.
Note that $108 = 2^2 3^3$. Thus any Abelian group of order 108 is isomorphic to one of the following:

group	has?	and they are
$Z_{3^3} \bigoplus Z_{2^2}$	no	
$Z_{3^3} \bigoplus Z_2 \bigoplus Z_2$	no	
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_{2^2}$	no	
$Z_{3^2} \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	no	
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_{2^2}$	yes	see below
$Z_3 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2$	yes	the above with a additional 0

For the above section $< (1,0,0,0) >, < (0,1,0,0) >, < (0,0,1,0) >, < (1,1,0,0) >, < (1,2,0,0) >, < (1,0,1,0) >, < (1,0,2,0) >, < (0,1,1,0) >, < (0,1,2,0) >, < (1,1,1,0) >, < (1,1,2,0) >, < (1,2,1,0) >, < (1,2,2,0) >$.

Exercise 11.10: Find all Abelian groups (up to isomorphism) of order 360.
Note that $360 = 5 \cdot 3^2 \cdot 2^3$ thus all Abelian groups of order 360 are isomorphic to one of the following:

$Z_5 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_2 \bigoplus Z_2 \bigoplus Z_2$
$Z_5 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_{2^2} \bigoplus Z_2$
$Z_5 \bigoplus Z_3 \bigoplus Z_3 \bigoplus Z_{2^3}$
$Z_5 \bigoplus Z_{3^2} \bigoplus Z_2 \bigoplus Z_2 \bigoplus Z_2$
$Z_5 \bigoplus Z_{3^2} \bigoplus Z_{2^2} \bigoplus Z_2$
$Z_5 \bigoplus Z_{3^2} \bigoplus Z_{2^3}$

Exercise 11.12: Suppose that the order of some finite Abelian group is divisible by 10. Prove that the group has a cyclic subgroup of order 10.
Note that there must exist some isomorphisim ϕ that maps our Abelian group to something of the form $Z_{(p_1)^{n_1}} \bigoplus Z_{(p_2)^{n_2}} \cdots$. Note that the primes 2 and 5 must divide the order of the group and thus there are $p_k = 5$ and $p_l = 2$. WLoG let $k = 1$, $l = 2$. Note that the element $(5^{n_1-1}, 2^{n_2-1}, 0, 0, 0 \cdots)$ is of order 10, thus if we move back by ϕ^{-1} we have found a element in our original group of order 10.