PHYS 351 #7

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We have shown in class that the Carnot cycle has the four stages: isothermal absorption of heat from a hot reservoir, adiabatic expansion, isothermal rejection of heat to a cold reservoir, adiabatic compression. The ideal gas equation of state $PV = nRT$ tells us how the gas behaves during the isothermal changes. During the adiabatic changes, PV^γ is constant. We have a Carnot engine which uses an ideal gas for the working substance.

1.1

Suppose that the hot reservoir has temperature T_h and the cold reservoir has temperature T_c. What is the efficiency of the Carnot engine?

This was a extreamly confusing question... Are we supposed to assume that the thermodynamic temperature is the ideal gas temperature and $\eta = 1 - T_c/T_h$? Are we supposed to procede from the ideal gas equation and show that the equation $\eta = 1-T_c/T_h$ holds? I tried this method but was unable to procede beyond a point without making the assumption that T is a thermodynamic temperature. I give up on this question I have no idea of how to show that ideal gas temperature is a thermodynamic temperature. In fact I don't eaven think this is posible because it would allow me to define $PV/nR = T$ for all gases and thus all gases regardless of there properties could be treated as ideal - this fails.

1.2

The isothermal absorption step takes the volume from V_1 to V_2. What is the work done on the system during one cycle of the engine? (Express your answer in terms of the given parameters: n, R, T_h, T_c, V_1, and V_2.)

We have four steps in our carnot cycle.
Isothermal: $PV = nRT_h$
Adiabatic: $PV^\gamma = PV^\gamma$
Isothermal: $PV = nRT_c$
Adiabatic: $PV^\gamma = PV^\gamma$

Along each of these paths $\Delta w = -\int PdV$. Let's add these up.

$$\Delta w_1 = -\int \frac{nRT_h}{V}dV = nRT_h \ln(\frac{V_1}{V_2})$$

$$\Delta w_2 = -\int PV^\gamma \cdot V^{-\gamma}dV = -P_2V_2^\gamma \int V^{-\gamma}dV = -\frac{P_2V_2^\gamma}{1-\gamma}(V_3^{1-\gamma} - V_2^{1-\gamma}) = -\frac{P_3V_3 - P_2V_2}{1-\gamma}$$

Noting that $P_2V_2^\gamma = P_3V_3^\gamma$. Also note that $P_3V_3 = P_4V_4$ so:

$$\Delta w_2 = -\frac{P_4V_4 - P_2V_2}{1-\gamma}$$

$$\Delta w_3 = -\int \frac{nRT_c}{V} dV = nRT_c \ln(\frac{V_3}{V_4})$$

$$\Delta w_4 = -\int PV^\gamma \cdot V^{-\gamma} dV = -P_4V_4^\gamma \int V^{-\gamma} dV = -\frac{P_4V_4^\gamma}{1-\gamma}(V_1^{1-\gamma} - V_4^{1-\gamma}) = -\frac{P_1V_1 - P_4V_4}{1-\gamma}$$

Noting that $P_4V_4^\gamma = P_1V_1^\gamma$. Also note that $P_1V_1 = P_2V_2$ so:

$$\Delta w_4 = -\frac{P_2V_2 - P_4V_4}{1-\gamma}$$

Adding all of these up we get (I will drop Δw_2 and Δw_4 since they add to nothing):

$$\sum w = nRT_h \ln(\frac{V_1}{V_2}) + nRT_c \ln(\frac{V_3}{V_4})$$

Note:

$$\frac{V_3}{V_4} = \sqrt[\gamma-1]{\frac{V_4V_3^\gamma}{V_3V_4^\gamma}} = \sqrt[\gamma-1]{\frac{P_4V_4P_3V_3^\gamma}{P_3V_3P_4V_4^\gamma}}$$

Noting that: $P_4V_4 = P_3V_3$ and that $P_3V_3^\gamma = P_2V_2^\gamma$ and $P_4V_4^\gamma = P_1V_1^\gamma$ we see

$$\frac{V_3}{V_4} = \sqrt[\gamma-1]{\frac{P_2V_2^\gamma}{P_1V_1^\gamma}} = \sqrt[\gamma-1]{\frac{P_2V_2V_2^{\gamma-1}}{P_1V_1V_1^{\gamma-1}}}$$

Note $P_2V_2 = P_1V_1$

$$\frac{V_3}{V_4} = \sqrt[\gamma-1]{\frac{V_2^{\gamma-1}}{V_1^{\gamma-1}}} = \frac{V_2}{V_1}$$

$$\sum w = nRT_h \ln(\frac{V_1}{V_2}) + nRT_c \ln(\frac{V_2}{V_1})$$

1.3

How much heat energy is drawn from the hot reservoir? How much heat energy is rejected to the cold reservoir? (Express your answer in terms of the given parameters: n, R, T_h, T_c, V_1, and V_2.)

$$\eta = -w/Q_h \Rightarrow Q_h = -w/\eta = -\frac{nRT_h \ln(\frac{V_1}{V_2}) + nRT_c \ln(\frac{V_2}{V_1})}{1 - T_c/T_h}$$

$$Q_c = Q_h \frac{T_c}{T_h} = -\frac{nRT_h \ln(\frac{V_1}{V_2}) + nRT_c \ln(\frac{V_2}{V_1})}{T_h/T_c - 1}$$

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A paramagnetic salt has a magnetic susceptibility $\chi_m = b/T$, where b is a constant; recall that $B = \mu_0(H + M)$ and $M = \chi_m H$. Thus, isothermal curves in the BM plane are lines of constant slope. Adiabats are lines of constant M. We have a Carnot engine which uses a paramagnetic salt for the working substance. Assume that we have a unit volume of paramagnetic salt, and that all parts of the system are uniform.

2.1

Suppose that the hot reservoir has temperature T_h and the cold reservoir has temperature T_c. What is the efficiency of the Carnot engine?

Same argument as before $\eta = 1 - T_c/T_h$.

2.2

The isothermal absorption process takes the magnetic induction from B_1 to B_2. What is the work done on the system during one cycle of the engine? (Express your answer in terms of the given parameters: b, μ_0, T_h, T_c, B_1, and B_2.)

We have a four step carnot engine:
1. T_h
2. $m_\alpha = m_\beta$
3. T_c
4. $m_\alpha = m_\beta$

Along all legs of our carnot engine $dw = \mu(T/b + 1)mdmdV$. Note that since we are always dealing with a unit volume and none of the other aspects of dw depend on position we can simply integrate it out to get $dw = V\mu(T/b + 1)mdm$. Lets now compute the work done allong each leg:

$$\Delta w_1 = \int V\mu(T_h/b + 1)mdm = 2V\mu(T_h/b + 1)(m_2^2 - m_1^2)$$

$$\Delta w_2 = \int V\mu(T/b + 1)m_2 dm = \int_{m_2}^{m_3} = 0$$

since $m_2 = m_3$

$$\Delta w_3 = \int V\mu(T_c/b + 1)mdm = 2V\mu(T_c/b + 1)(m_3^2 - m_4^2) = 2V\mu(T_c/b + 1)(m_2^2 - m_1^2)$$

$$\Delta w_4 = \int V\mu(T/b + 1)m_4 dm = \int_{m_4}^{m_1} = 0$$

so:

$$\sum w = 2V\mu(T_h/b + T_c/b + 2)(m_2^2 - m_1^2)$$

$$m = B/(\mu T/b + 1)$$

$$\sum w = 2V\mu(T_h/b + T_c/b + 2)((B_2/(\mu T_h/b + 1))^2 - (B_1/(\mu T_h/b + 1))^2)$$

2.3

How much heat energy is drawn from the hot reservoir? How much heat energy is rejected to the cold reservoir? (Express your answer in terms of the given parameters: b, μ_0, T_h, T_c, B_1, and B_2.)

$$\eta = -w/Q_h \Rightarrow Q_h = -w/\eta = -\frac{2V\mu(T_h/b + T_c/b + 2)((B_2/(\mu T_h/b + 1))^2 - (B_1/(\mu T_h/b + 1))^2)}{1 - T_c/T_h}$$

$$Q_c = Q_h \frac{T_c}{T_h} = -\frac{2V\mu(T_h/b + T_c/b + 2)((B_2/(\mu T_h/b + 1))^2 - (B_1/(\mu T_h/b + 1))^2)}{T_h/T_c - 1}$$