**Exercise :** Write down the 4th order Taylor polynomial of $\sqrt{x}$ centered at $x = 1$. Let $P(x)$ denote this polynomial. If $1 \le x \le 2$, what can you say about the size of $|\sqrt{x} - P(x)|$? Hint: Use the remainder term!

Lets start by getting the derivatives of $f(x) = \sqrt{x}$.

$$f(x) = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$$

$$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$$

$$f^{(4)}(x) = -\frac{15}{16}x^{-\frac{7}{2}}$$

$$f^{(5)}(x) = \frac{105}{32}x^{-\frac{9}{2}}$$

Now we can construct $P(x)$ as

$$P(x) = 1^{\frac{1}{2}} + \frac{1}{2}1^{-\frac{1}{2}}(x-1) + -\frac{1}{2}\frac{1}{4}1^{-\frac{3}{2}}(x-1)^2 + \frac{1}{6}\frac{3}{8}1^{-\frac{5}{2}}(x-1)^3 + -\frac{1}{24}\frac{15}{16}1^{-\frac{7}{2}}(x-1)^4$$

$$P(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

Note that the remainder term would be

$$R(x) = \frac{1}{5!}\frac{105}{32}\xi^{-\frac{9}{2}}(x-1)^5 = \frac{7}{256}\xi^{-\frac{9}{2}}(x-1)^5$$

Where $f(x) = P(x) + R(x)$. We are now asked to compute the error, $E$, in the case that $1 \le x \le 2$. We see that $E = |f(x) - P(x)| = |R(x)| = |\frac{7}{256}\xi^{-\frac{9}{2}}(x-1)^5|$ and noting that $\xi, x \in [1, 2]$ we get $E = \frac{7}{256}\xi^{-\frac{9}{2}}(x-1)^5$. The maximum value E could have would be $x = 2$ and $\xi = 1$ so $E_{max} = \frac{7}{256}$ and of course the smallest value is at $x = 1$ where by our construction $f(x) = p(x)$ so $0 \le E \le \frac{7}{256}$.

**Exercise :** Chapter 4: 2 (b)

This is my code for newton's method.

```
#newton(guess,f,df,tol_y,N)
#guess=initial x
#f=a function of one peramiter that gives the value of f(x)
#df=a function giving the derivitive of f at x
#tol_y=the tolerance in y the user wants to end up with
#N=maximum number of steps
function [guess x]=newton(guess,f,df,tol_y,N)
step=@(x) x-f(x)/df(x);
err_y=@(x) abs(f(x));
x=[guess];
disp("        x           f(x)       error");
#    "   4.9663e+00  -1.6564e-01   1.6564e-01"
disp([guess, f(guess), err_y(guess)]);
while(N>0 && err_y(guess)>tol_y)
    N=N-1;
    guess=step(guess);
    x=[x guess];
    disp([guess, f(guess), err_y(guess)]);
end
endfunction
```

And this is my driver for newton's method

```
format short e
f=@(x) (5-x) .* exp(x) -5;
df=@(x) 4*exp(x)-x.*exp(x);
[xfin x]=newton(5,f,df,1e-8,1000);

#now I need to estimate the error of future steps of newtons method
#first i will get the errors assuming x_j-x_f=x_j-x_*.
#this is approximently true assuming that we are approching the zero and j>f
error=[];
for n=[1:length(x)-1]
    error=[error x(n)-xfin];
end
#assuming we are close to the zero we know that e_(k+1)/e_k^2 is a constant
#here we find those constants
constant=[];
for n=[1:length(error)-1]
    constant=[constant error(n+1)/error(n)^2];
end
disp("these valuse shuld be the same:");
disp(constant);
C=mean(constant);
Eo=error(1);
#this give a approximate error in f given a error in x
Ef=@(Ex) Ex*df(xfin);
f_error=[abs(Ef(Eo))];
for n=0:7
    Eo=C*Eo^2;
    f_error=[f_error abs(Ef(Eo))];
end
disp("approximate errors in y");
disp(f_error')
```

```
>> question_2b
         x           f(x)        error
   5.0000e+00  -5.0000e+00   5.0000e+00
   4.9663e+00  -1.6564e-01   1.6564e-01
   4.9651e+00  -2.0120e-04   2.0120e-04
   4.9651e+00  -2.9789e-10   2.9789e-10
these valuse shuld be the same:
   9.8276e-01   1.0168e+00
approximate errors in y
   4.8256e+00
   1.6831e-01
   2.0475e-04
   3.0300e-10
   6.6358e-22
   3.1827e-45
   7.3216e-92
  3.8745e-185
   0.0000e+00
>> diary off
```

Noting that the expected errors and the actual errors appear to be very similar I conclude that the method for estimating error in y is valid and conclude that it will take one more iteration to have a error less than $10^{-}16$.

**Exercise :** Chapter 4: 3

Estimating $\frac{1}{3}$ by newtons method.

$$f = x^{-1} - 3$$

$$f' = -x^{-2}$$

$$x_{k+1} = x_k + \frac{x_k^{-1} - 3}{x_k^{-2}} = x_k + x_k - 3x_k^2 = 2x_k - 3x_k^2$$

Now using $x_0 = .5$ I can calculate an estimate for $\frac{1}{3}$.

$$x_0 = .5$$

$$x_1 = .25$$

$$x_2 = .3125$$

$$x_3 = .3203$$

$$x_4 = .3333$$

This method definitely approaches the true value of $\frac{1}{3}$.
Lets try again with $x_0 = 1$

$$x_0 = 1$$

$$x_1 = -1$$

$$x_2 = -5$$
$$x_3 = -85$$

Clearly not approaching $\frac{1}{3}$.

**Exercise :** Chapter 4: 4

Note that $f(x) = x^2 - 2$ has a root at $x = \sqrt{2}, -\sqrt{2}$. So if we do newtons method we should find a approximation for $\sqrt{2}$.

```
>> f=@(x) x^2-2;
>> df=@(x) 2*x;
>> sqrttwo=newton(1.5,f,df,1e-10,1000)
         x            f(x)        error
   1.5000e+00   2.5000e-01   2.5000e-01
   1.4167e+00   6.9444e-03   6.9444e-03
   1.4142e+00   6.0073e-06   6.0073e-06
   1.4142e+00   4.5106e-12   4.5106e-12
sqrttwo =    1.4142e+00
>> format long e
>> sqrttwo
sqrttwo =    1.41421356237469e+00
```

**Exercise :** Chapter 4: 6

(a) Assuming $h(x)$ is differentiable, witch by inspection in this case it is, all extrema occur at zeros of $h'(x) = x^3 - 3$. Noting that $h''(x) = 3x^2$ we get newtons method as:

$$x_{k+1} = x_k - \frac{x_k^3 - 3}{3x_k^2}$$

(b) Taking one newtons method step with $x_0 = 1$ I get:

$$x_2 = 1 - \frac{1-3}{3} = 1 + 2/3 \approx 1.66$$

(c) First note that $h'(0) = -3$ and $h'(4) > 0$, now we know where to replace negative and positive values. Now the midpoint value would be $h'(2) = 5$, so our new interval would be $[0, 2]$. The midpoint value of these two would be $h'(1) = -2$. So after two midpoint steps our interval would be $[1, 2]$.

**Exercise :** Chapter 4: 9

(a) The bisection method is not usable in this case. For the bisection method we need to start with two $x$ values one of witch has a negative $f(x)$ and one that has a positive $f(x)$. The function $f(x) = \sin(x) + 1$ is never negative, thus we can't choose two points to use the bisection method on.

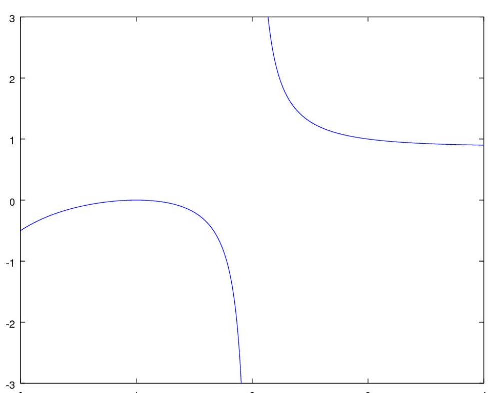
(b) Newtons method will work however since at any point where $f(x) = 0$ we know that $f'(x) = 0$ the method should not approach quadratically but only linearly.

**Exercise :** Chapter 4: 10

running the bisect method

```
f=@(x) (x.^2-2*x+1)./(x.^2-x-2);
findzero(0,3,1e-3,f)
x=[0:.001:4];
plot(x,f(x));
axis([-Inf,Inf,-3,3]);
saveas(gcf,"q20fig.jpg");
```

```
>> q10
   0.00000000000000e+00   3.00000000000000e+00
   1.50000000000000e+00   3.00000000000000e+00
   1.50000000000000e+00   2.25000000000000e+00
   1.87500000000000e+00   2.25000000000000e+00
   1.87500000000000e+00   2.06250000000000e+00
   1.96875000000000e+00   2.06250000000000e+00
   1.96875000000000e+00   2.01562500000000e+00
   1.99218750000000e+00   2.01562500000000e+00
   1.99218750000000e+00   2.00390625000000e+00
   1.99804687500000e+00   2.00390625000000e+00
   1.99804687500000e+00   2.00097656250000e+00
ans =    2.00024414062500e+00
>>
```




This method is clearly not going to the root. This is because the bisect method goes to the place where $f(x)$ changes sign, in this case that would be the asimtote at $x = 2$. In the range $[0, 1.5]$ the intermediate value therm does not guarantee a root since both 0 and 1.5 have negative associated $f(x)$ values. In the range $[1.5, 3]$ the intermediate value therm does not guarantee a root since $f(x)$ has a discontinuity at $x = 2$.

**Exercise :** Chapter 4: 11

(a) $f(x) = \sin(x)$, $f'(x) = \cos(x)$

$$x_{k+1} = x_k - \frac{\sin(x_k)}{\cos(x_k)}$$

NB This newton's method gives us $\pi$ but the evaluation of sin internally may use $\pi$, so I wouldn't necessarily think this is a way to find the value of $\pi$.

(b) $f(x) = x^3 - x^2 - 2x$, $f'(x) = 3x^2 - 2x - 2$

$$x_{k+1} = x_k - \frac{x_k^3 - x_k^2 - 2x_k}{3x_k^2 - 2x_k - 2}$$

(c) $f(x) = 1 - .01x$, $f'(x) = -.01$

$$x_{k+1} = x_k - \frac{1 - .01x}{-.01}$$

In this case one step of newtons method should get us to the exact root. Newtons method estimates a curve by a line that has the same slope and value as the curve at some point $x_k$ then uses the root of that line as a guess of the root of the curve. Since in this case the curve is a line newtons idea to estimate it as a line is exact and one run of newtons method, regardless of starting position will take us to the answer.

```
newtonRunner=@(f,df,x) newton(x,f,df,-1,5);

format long e
f=@(x) sin(x);
df=@(x) cos(x);
newtonRunner(f,df,3);

f=@(x) x^3-x^2-2*x;
df=@(x) 3*x^2-2*x-2;
newtonRunner(f,df,3);

f=@(x) 1-.01*x;
df=@(x) -.01;
newtonRunner(f,df,1);
```

```
>> q11
         x              f(x)          error
    3.00000000000000e+00   1.41120008059867e-01   1.41120008059867e-01
    3.14254654307428e+00  -9.53889339826441e-04   9.53889339826441e-04
    3.14159265330048e+00   2.89316249076218e-10   2.89316249076218e-10
    3.14159265358979e+00   1.22464679914735e-16   1.22464679914735e-16
    3.14159265358979e+00   1.22464679914735e-16   1.22464679914735e-16
    3.14159265358979e+00   1.22464679914735e-16   1.22464679914735e-16
         x              f(x)          error
    3.00000000000000e+00   1.20000000000000e+01   1.20000000000000e+01
    2.36842105263158e+00   2.93920396559265e+00   2.93920396559265e+00
    2.07716312466591e+00   4.93208927686069e-01   4.93208927686069e-01
    2.00452016332014e+00   2.72232316584615e-02   2.72232316584615e-02
    2.00001692963470e+00   1.01579241259309e-04   1.01579241259309e-04
    2.00000000023884e+00   1.43303235944359e-09   1.43303235944359e-09
         x              f(x)          error
    1.00000000000000e+00   9.90000000000000e-01   9.90000000000000e-01
    1.00000000000000e+02   0.00000000000000e+00   0.00000000000000e+00
    1.00000000000000e+02   0.00000000000000e+00   0.00000000000000e+00
    1.00000000000000e+02   0.00000000000000e+00   0.00000000000000e+00
    1.00000000000000e+02   0.00000000000000e+00   0.00000000000000e+00
    1.00000000000000e+02   0.00000000000000e+00   0.00000000000000e+00
>> diary off
```