

Exercise : Problem 8.12

Show that

$$s(x) = \begin{cases} 1 + x - x^3 & 0 \leq x < 1 \\ 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3 & 1 \leq x < 2 \\ 4(x-2) + 9(x-2)^2 - 3(x-2)^3 & 2 \leq x < 3 \end{cases}$$

is a cubic spline through $(0, 1), (1, 1), (2, 0), (3, 10)$.

For notation 1^+ means a little more than 1 and 1^- is a little less than 1, for dealing with the piece wise function.

First lets make sure the values link up.

$$s(x=0) = 1 + x - x^3 = 1 \text{ good on } (0, 1)$$

$$s(x=1^-) = 1 + x - x^3 = 1 \text{ and } s(x=1^+) = 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3 = 1 \text{ good on } (1, 1)$$

$$s(x=2^-) = 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3 = 0 \text{ and } s(x=2^+) = 4(x-2) + 9(x-2)^2 - 3(x-2)^3 = 0 \text{ good on } (2, 0)$$

$$s(x=3) = 4(x-2) + 9(x-2)^2 - 3(x-2)^3 = 10 \text{ good on } (3, 10)$$

Now check that the derivatives mach.

$$s'(x=1^-) = 1 - 3x^2 = -2 \text{ and } s'(x=1^+) = -2 - 6(x-1) + 12(x-1)^2 = -2 \text{ good.}$$

$$s'(x=2^-) = -2 - 6(x-1) + 12(x-1)^2 = 4 \text{ and } s'(x=2^+) = 4 + 18(x-2) - 9(x-2)^2 = 4 \text{ good.}$$

Now check that the second derivatives mach.

$$s''(x=1^-) = -6x = -6 \text{ and } s''(x=1^+) = -6 + 24(x-1) = -6 \text{ good.}$$

$$s''(x=2^-) = -6 + 24(x-1) = 18 \text{ and } s''(x=2^+) = 18 - 18(x-2) = 18 \text{ good.}$$

Therefore it is a cubic spline. Note that $s''(x=0) = -6x = 0$ and $s''(x=3) = 18 - 18(x-2) = 0$ thus s is a natural cubic spline.

Exercise : Problem 8.13

For what constants is

$$s(x) = \begin{cases} ax^2 + b(x-1)^3 & x \in (-\infty, 1] \\ cx^2 + d & x \in [1, 2] \\ ex^2 + f(x-2)^3 & x \in [1, 2] \end{cases}$$

a cubic spline?

Note

$$s'(x) = \begin{cases} 2ax + 3b(x-1)^2 & x \in (-\infty, 1] \\ 2cx & x \in [1, 2] \\ 2ex + 3f(x-2)^2 & x \in [1, 2] \end{cases}$$

$$s''(x) = \begin{cases} 2a + 6b(x-1) & x \in (-\infty, 1] \\ 2c & x \in [1, 2] \\ 2e + 6f(x-2) & x \in [1, 2] \end{cases}$$

We require that up to the second derivative link so our constraints are:

values

$$a = c + d$$

$$4c + d = 4e$$

derivatives

$$2a = 2c$$

$$4c = 4e$$

second derivatives

$$2a = 2c$$

$$2c = 2e$$

These equations hold true if and only if $d = 0$, $a = c = e$. Note that this allows us 3 degrees of freedom in choosing our constants, b , f and one of $a = c = e$.

Exercise : Find n so that degree n polynomial interpolation of $f(x) = \cos(3x)$, using equally spaced points on $[0, 2]$, gives a maximum approximation error $|f(x) - p(x)|$ which is less than 10^{-6} on $[0, 2]$.

Note that by inspection our polynomial will be at least order 2 since a line is a terrible approximation for \cos and thus 0 order and 1 order won't work.

The max error on a polynomial interpolation would be

$$e = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} * \prod_{i=0}^n (\eta - x_i) \right|$$

where ξ and η are in $[0, 2]$. Note that for equally spaced points $\prod_{i=0}^n |\eta - x_i|$ achieves its maximum with $\eta \in (x_0, x_1)$ and again at $\eta \in (x_{n-1}, x_n)$. The proof for this is trivial just think about the symmetry and what happens when you shift left or right by $x_1 - x_0$ (one of those two shifts is guaranteed to give you a increase). Let η_0 be the $\eta_0 \in (x_0, x_1)$ that makes $\prod_{i=0}^n |\eta - x_i|$ a maximum. Thus $\prod_{i=0}^n |\eta - x_i| \leq \prod_{i=0}^n |\eta_0 - x_i| = (\eta_0 - x_0) \prod_{i=1}^n (x_i - \eta_0) \leq (x_1 - x_0) \prod_{i=1}^n (x_i - x_0) = \frac{2}{n} \prod_{i=1}^n (i \frac{2}{n}) = (\frac{2}{n})^{n+1} \prod_{i=1}^n i = (\frac{2}{n})^{n+1} n!$. Note that $f^{(n+1)}(x) = 3^{n+1} S(3x)$ where $S(x)$ is a $\pm \sin(x)$ or $\pm \cos(x)$. Noting that $|S(x)| \leq 1$ conclude $|f^{(n+1)}(\xi)| \leq 3^{n+1}$. Now we can conclude

$$e = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} * \prod_{i=0}^n (\eta - x_i) \right| \leq \frac{3^{n+1}}{(n+1)!} * \left(\frac{2}{n}\right)^{n+1} n! = \frac{6^{n+1}}{n^{n+1}(n+1)}$$

So we can guarantee a $e < 10^{-6}$ if $\frac{6^{n+1}}{n^{n+1}(n+1)} < 10^{-6}$.

```

1 >> n=[1:20];
2 >> e=(6.^(n+1)./(n.^(n+1).*(n+1)));
3 >> [n',e']
4 ans =
5
6      1.0000e+00      1.8000e+01
7      2.0000e+00      9.0000e+00
8      3.0000e+00      4.0000e+00
9      4.0000e+00      1.5188e+00

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```

10      5.0000e+00      4.9766e-01
11      6.0000e+00      1.4286e-01
12      7.0000e+00      3.6420e-02
13      8.0000e+00      8.3427e-03
14      9.0000e+00      1.7342e-03
15      1.0000e+01      3.2982e-04
16      1.1000e+01      5.7799e-05
17      1.2000e+01      9.3900e-06
18      1.3000e+01      1.4216e-06
19      1.4000e+01      2.0149e-07
20      1.5000e+01      2.6844e-08
21      1.6000e+01      3.3735e-09
22      1.7000e+01      4.0121e-10
23      1.8000e+01      4.5284e-11
24      1.9000e+01      4.8632e-12
25      2.0000e+01      4.9811e-13
26
27 >> diary off

```

From this test we can see that the error will be guaranteed to be under 10^{-6} for a polynomial of degree 14. My suspicion would be that a polynomial of order 13 will work since it is guaranteed to have a error less than $1.4216e-06$, but based on this math I can only guarantee that a polynomial of degree 14 will work.

We can now run some code to see what will work.

```

1  function n=errfind()
2      n=2;
3      while(true)
4          x=linspace(0,2,n+1);
5          p=polyfit(x,cos(3.*x),n);
6          tx=rand(1,100000);
7          ty=polyval(p,tx);
8          em=max(abs(ty-cos(3*tx)))
9          if(em<10^-6)
10             return;
11         end
12         n+=1;
13     end
14 endfunction

```

Witch ends up telling us that a polynomial of degree 12 will work.

Exercise : At the bottom of page 198 is an inequality that describes the error from the piece wise linear interpolate $l(x)$ for $f(x)$ on $[a, b]$. Suppose we have equally spaced points $a = x_0 < x_1 < \dots < x_n = b$ with spacing $h = x_i - x_{i-1}$. then:

$$|f(x) - l(x)| \leq \frac{Mh^2}{8}$$

for all $x \in [a, b]$. In this inequality we are assuming $f''(x)$ exists and is bounded by the number M , so that $|f''(x)| \leq M$ for all $x \in [a, b]$. Use this inequality to find n so that $|f(x) - l(x)| \leq 10^{-6}$ for $x \in [0, 2]$ if $f(x) = \cos(3x)$.

Well $M = 9$ works since $|f''(x)| = 9|\cos(3x)| \leq 9$. Solving we get $\sqrt[3]{10^{-6} \cdot 8/9}^{-1} \cdot 2 = 2121.3$ so $n = 2122$ will work.

Exercise : 10.1

Derive the Newton-Cotes formula between $[0, 1]$ for a 3 regions.
Polynomial interpolation allows us to say

$$\begin{aligned}
 f(x) &\approx \sum_{i=0}^3 f(x_i) * \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \\
 \int_0^1 f(x) dx &\approx \int_0^1 \sum_{i=0}^3 f(x_i) * \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} dx = \\
 \int_0^1 f(0) \frac{x - 1/3}{-1/3} * \frac{x - 2/3}{-2/3} * \frac{x - 1}{-1} &+ f(1/3) \frac{x}{1/3} * \frac{x - 2/3}{-1/3} * \frac{x - 1}{-2/3} + \\
 f(2/3) \frac{x}{2/3} * \frac{x - 1/3}{1/3} * \frac{x - 1}{-1/3} &+ f(1) \frac{x}{1} * \frac{x - 1/3}{2/3} * \frac{x - 2/3}{1/3} dx = \\
 \int_0^1 f(0) \frac{(x - 1/3)(x - 2/3)(x - 1)}{-2/9} &+ f(1/3) \frac{(x)(x - 2/3)(x - 1)}{2/27} + \\
 f(2/3) \frac{(x)(x - 1/3)(x - 1)}{-2/27} &+ f(1) \frac{(x)(x - 1/3)(x - 2/3)}{2/9} dx = \\
 \int_0^1 f(0) \frac{(3x - 1)(3x - 2)(x - 1)}{-2} &+ f(1/3) \frac{(x)(3x - 2)(x - 1)}{2/9} + \\
 f(2/3) \frac{(x)(3x - 1)(x - 1)}{-2/9} &+ f(1) \frac{(x)(3x - 1)(3x - 2)}{2} dx = \\
 \int_0^1 f(0) \frac{(9x^2 - 9x + 2)(x - 1)}{-2} &+ f(1/3) \frac{(x)(3x^2 - 5x + 2)}{2/9} + \\
 f(2/3) \frac{(x)(3x^2 - 4x + 1)}{-2/9} &+ f(1) \frac{(x)(9x^2 - 9x + 2)}{2} dx = \\
 \int_0^1 f(0) \frac{9x^3 - 18x^2 + 11x - 2}{-2} &+ f(1/3) \frac{3x^3 - 5x^2 + 2x}{2/9} + \\
 f(2/3) \frac{(3x^3 - 4x^2 + x)}{-2/9} &+ f(1) \frac{9x^3 - 9x^2 + 2x}{2} dx = \\
 f(0) \frac{9/4 - 6 + 11/2 - 2}{-2} &+ f(1/3) \frac{3/4 - 5/3 + 1}{2/9} + f(2/3) \frac{3/4 - 4/3 + 1/2}{-2/9} + f(1) \frac{9/4 - 3 + 1}{2} = \\
 \frac{1}{8} f(0) &+ \frac{3}{8} f(1/3) + \frac{3}{8} f(2/3) + \frac{1}{8} f(1)
 \end{aligned}$$

Exercise : 10.2

Find

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f(1)$$

which is exact for all $f(x) = ae^x + b \cos(\pi x/2)$. If such a pair A_0, A_1 exist then they must be exact for e^x and $\cos(\pi x/2)$ in other words $e - 1 = A_0 + eA_1$ and $2/\pi = A_0$. Thus

$$\int_0^1 f(x)dx = \frac{2}{\pi} f(0) + \frac{e - 1 - 2/\pi}{e} f(1)$$

is exact for all $f(x) = ae^x + b \cos(\pi x/2)$.