Exercise : Chapter 4: 2 (c)

Here is my secant method:

```
function  [r,hist] = secant(f,x0,x1,ftol,xtol,Nmax)
m=@(xk,xk1) (f(xk)-f(xk1))/(xk-xk1);
xk2=@(xk,xk1) xk1-f(xk1)/m(xk,xk1);
xk2h=@(hist) xk2(hist(length(hist)-1),hist(length(hist)));
hist=[x0 x1];
r=x1;
exit1=@(hist) abs(f(hist(length(hist))))<=ftol;
exit2=@(hist) abs(hist(length(hist))-hist(length(hist)-1))<=xtol;
while(~(exit1(hist) || exit2(hist)))
    if(length(hist)-2>Nmax)
        error("more than Nmax iterations");
    end
    hist=[hist xk2h(hist)];
end
r=hist(length(hist));
endfunction
```

The driver code:

```
f=@(x) (5-x).*exp(x)-5;
disp("exit with error test");

try
    [r,hist] = secant(f,4,5,1e-8,0,2);
    fhist=f(hist);
    xerr=[inf];
    for i=1:(length(hist)-1)
        xerr=[xerr abs(hist(i)-hist(i+1))];
    end
    #this is the desplayed line
    disp(
    "    x history    y history      error in y    error in x")
    #    5.0000e+00  -5.0000e+00    5.0000e+00    1.0000e+00
    disp([hist' fhist' abs(fhist)' xerr']);
catch ME
    disp(["exited with the error <",ME.message,">"]);
end
disp("");
disp("");



disp("exit with xtol");
[r,hist] = secant(f,4,5,1e-8,1e-2,1000);
fhist=f(hist);
xerr=[inf];
for i=1:(length(hist)-1)
    xerr=[xerr abs(hist(i)-hist(i+1))];
```

```
end
#this is the desplayed line
disp(
"   x history   y history     error in y   error in x")
#   5.0000e+00  -5.0000e+00   5.0000e+00   1.0000e+00
disp([hist' fhist' abs(fhist)' xerr']);
disp("");
disp("");



disp("normal run");
[r,hist] = secant(f,4,5,1e-8,0,1000);
fhist=f(hist);
xerr=[inf];
for i=1:(length(hist)-1)
    xerr=[xerr abs(hist(i)-hist(i+1))];
end
#this is the desplayed line
disp(
"   x history   y history     error in y   error in x")
#   5.0000e+00  -5.0000e+00   5.0000e+00   1.0000e+00
disp([hist' fhist' abs(fhist)' xerr']);
disp("");
disp("");
#We know that for secant method the error goes as
#ek1=C ek^(1+sqrt(5))/2, so lets try to find C

#the error between x values and our r value is a
#good estimation for the true xerr

xerr=[];
for i=1:(length(hist)-1)
    xerr=[xerr abs(hist(i)-r)];
end
c=[];
alpha=(1+sqrt(5))/2;
for i=1:(length(xerr)-1)
    c=[c (xerr(i+1))/(xerr(i)^alpha)];
end
C=median(c);
#using meadian to avoid erronious values
#near the zero a good approximation is that the slope <m>
#is constant and that erry=m*errx assuming m is not close to zero

xerrpredic=[xerr(3)];
#i selected xerr(3) because the first couple of xerr are going to
#be far away from the region where xerr goes as ek1=C ek^(1+sqrt(5))/2

for i=1:10
    xerrpredic=[xerrpredic  C*xerrpredic(length(xerrpredic))^alpha];
end
dy=(f(hist(length(hist)))-f(hist(length(hist)-1)));
dx=(hist(length(hist))-hist(length(hist)-1));
```

```
m=dy/dx;
yerrpredic=abs(m*xerrpredic);
yerrpredic=[nan nan yerrpredic];
#added two nan for the two values i didn't predict
yerr=abs(fhist);
while(length(yerrpredic)~=length(yerr))
    yerr=[yerr nan];
end
disp("actual and predicted errors in y");
#NaN in y error is a indication that
#the error on that step was not evaluated

#NaN in predicted y error is a indication that
#the error on that step was not predicted

disp(
"     actual        predicted");
disp(
"      y error       y error");
#    7.4020e+00    7.8420e+00
disp([yerr' yerrpredic']);
```

The output:

```
>> secdrvr
exit with error test
exited with the error <more than Nmax iterations>


exit with xtol
   x history   y history     error in y   error in x
   4.0000e+00   4.9598e+01   4.9598e+01          Inf
   5.0000e+00  -5.0000e+00   5.0000e+00   1.0000e+00
   4.9084e+00   7.4020e+00   7.4020e+00   9.1578e-02
   4.9631e+00   2.8089e-01   2.8089e-01   5.4658e-02
   4.9652e+00  -1.6750e-02   1.6750e-02   2.1560e-03


normal run
   x history   y history     error in y   error in x
   4.0000e+00   4.9598e+01   4.9598e+01          Inf
   5.0000e+00  -5.0000e+00   5.0000e+00   1.0000e+00
   4.9084e+00   7.4020e+00   7.4020e+00   9.1578e-02
   4.9631e+00   2.8089e-01   2.8089e-01   5.4658e-02
   4.9652e+00  -1.6750e-02   1.6750e-02   2.1560e-03
   4.9651e+00   3.4731e-05   3.4731e-05   1.2133e-04
   4.9651e+00   4.2810e-09   4.2810e-09   2.5106e-07


actual and predicted errors in y
     actual        predicted
      y error       y error
    4.9598e+01           NaN
```

```
     5.0000e+00           NaN
     7.4020e+00    7.8420e+00
     2.8089e-01    7.2703e-01
     1.6750e-02    1.5500e-02
     3.4731e-05    3.0635e-05
     4.2810e-09    1.2909e-09
            NaN    1.0751e-16
            NaN    3.7732e-28
            NaN    1.1029e-46
            NaN    1.1313e-76
            NaN   3.3920e-125
            NaN   1.0433e-203
>> diary off
```

Note that were I have a predicted a error in y and a actual error in y the two values are close (within a order of magnitude). This is a good indication that the model used to predict errors is valid and so we can use this model to predict that to get the y error under 10^{-16} will take two more steps.

Exercise : Chapter 4: 3 (Just the last sentence of the problem)

Newton's method in this problem iterates as $\rho(x) = 2x - x^2R$. Solving for stability we get $\rho(x) = x$ or $x = 2x - x^2R$, $0 = x(1 - xR)$, $x = 0, 1/R$. This method will converge in a interval around $1/R$ where $|\rho'(x)| < 1$, $|2 - 2xR| < 1$, $-1 < 2 - 2xR < 1$, $-3 < -2xR < -1$, $\frac{3}{2}1/R > x > \frac{1}{2}1/R$, note that the fixed point $1/R$ is in this range. So in the particular case where $1/R = 1/3$ we get convergence where $\frac{1}{6} < x < \frac{1}{2}$.

Exercise : Chapter 4: 5

Nowhere does it say "solve by hand"!

```
>> f=@(x) x^2-3;
>> [r,hist] = secant(f,0,1,1e-4,0,100);
>> hist'
ans =

   0.00000
   1.00000
   3.00000
   1.50000
   1.66667
   1.73684
   1.73196
   1.73205

>> diary off
```

Exercise : Chapter 4: 8

If we were attempting to find a root with secant method we would be using:

$$x_{k+1} = x_k - \frac{f(x_k)}{\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}}$$

so:

$$x_2 = x_1 - \frac{f(x_1)}{\frac{f(x_1)-f(x_0)}{x_1-x_0}}$$

$$-2 = -1 - \frac{4}{\frac{4-f(x_0)}{-1-2}}$$

$$-2 = -1 + \frac{3*4}{4-f(x_0)}$$

$$-1 = \frac{12}{4-f(x_0)}$$

$$-12 = 4 - f(x_0)$$

$$f(x_0) = 16$$

Exercise : Chapter 4: 12

(a) Fixed points are where $\rho(x) = x$ so $x^2 + 4 = 5x$, $x^2 - 5x + 4 = 0$, $(x-1)(x-4) = 0$, $x = 1, 4$.

(b) Basically is it true that $(\forall \xi \in [0, 2]), |\rho'(\xi)| < 1$. Well $|\rho'(\xi)| = |\frac{2}{5}\xi| = \frac{2}{5}\xi$ and $0 \leq \frac{2}{5}\xi \leq \frac{4}{5}$. Thus it is clearly true that $|\rho'(\xi)| < 1$ and so these iteration will converge to $x = 1$, $\forall x_0 \in [0, 2]$.

Exercise : Chapter 4: 13

Consider the equation $\rho(y) = a + \epsilon \sin(y)$. Note that the fixed points occur where $y = a + \epsilon sin(y)$, $y - \epsilon \sin(y) = a$. Assume there are two distinct fixed points $x \neq y$ and $\rho(x) = x$, $\rho(y) = y$. Note that $|x - y| = |\rho(x) - \rho(y)| = |\epsilon(\sin(x) - \sin(y)|$. So $\frac{|x-y|}{\epsilon} = |\sin(x) - \sin(y)|$ if we note that $|x - y| \neq 0$ and $\frac{1}{\epsilon} > 1$ we can see $|x - y| > \frac{|x-y|}{\epsilon}$ thus $|x - y| > |\sin(x) - \sin(y)|$.

Lets define $w = x - y$. Note that $\sin(x) = \sin(y + w)$. Doing a Taylor expansion about $w = 0$ we get $\sin(x) = \sin(y + w) = \sin(y) + \cos(y + \xi)\xi$ where ξ is between w and 0, even though we don't know the sign of w we can still say $|\xi| \leq |w|$. Now note that $|\sin(x) - \sin(y)| = |\sin(y) + \cos(y + \xi)\xi - \sin(y)| = |\cos(y + \xi)\xi| \leq |\xi| \leq |w| = |x - y|$. Thus $|\sin(x) - \sin(y)| \leq |x - y|$

Combining these two parts we get $|x - y| > |\sin(x) - \sin(y)| \geq |x - y|$ in other words $|x - y| > |x - y|$, a contradiction. We conclude the negation of our supposition, namely that there does not exist two fixed points to our $\rho(y)$ and thus if there is a solution it is unique. Noting that $a + \epsilon \sin(0) - 0 \geq 0$ and $a + \epsilon \sin(2\pi) - 2\pi = a - 2\pi < 0$ we conclude that there is a solution in $[0, 2\pi)$. Thus there is a unique fixed point.

Exercise : Chapter 4: 14

We are iterating $\rho(y) = \cos(y)$.

Fixed points must occur where $y = \cos(y)$. Note that fixed points must occur in $y \in [-1, 1]$ since $|y| = |\cos(y)| \leq 1$. Fixed points are also solutions to $\cos(y) - y = 0$, noting that $-\sin(y) - 1 < 0, y \in [-1, 1]$ we can say $\cos(y) - y$ is monotonically decreasing $y \in [-1, 1]$ and thus there is at most one zero, and so if there is a fixed point it is unique. Noting that $\cos(0) - 0 = 1 > 0$ and $\cos(1) - 1 \approx -.5 < 0$, we can say that there is a fixed point in $y \in [0, 1]$. Thus there is a unique fixed point and it is between 0 and 1.

What happens as we iterate on y, $y \in [-1, 1]$? In this range $|\rho'(y)| = |sin(y)| < 1$, thus in this range of y we converge to the fixed point. Note that if we start with a y outside of this range we are guaranteed by the range of cos that $\rho(y) \in [-1, 1]$, thus all values of y converge to the fixed point somewhere between 0 and 1. Experimentally this value is about 0.73909.

Exercise : Chapter 4: 15

We are asked to discuss how $\rho(x) = \frac{1}{2}(-x^2 + x + 2)$ behaves with $x_0 = .5$, given that it has a fixed point at $x = 1$. Note that $\rho'(x) = -x + .5$ and so in the interval $x \in (.5, 1.5)$ $|\rho(x)| < 1$, $\rho(x)$ converges to 1. Noting that $\rho(.5) \approx 1.1 \in (.5, 1.5)$ we can say that $\rho(x)$ converges to 1 if we start with $x_0 = .5$.

Assume that the convergence takes the form $\frac{|err_{k+1}|}{|err_k|^\alpha} \to C$ as $x_k \to 1$. Note that

$$\lim_{err \to 0} \frac{|err_{k+1}|}{|err_k|^\alpha} = \lim_{x \to 1} \frac{|\rho(x) - 1|}{|x - 1|^\alpha} = C$$

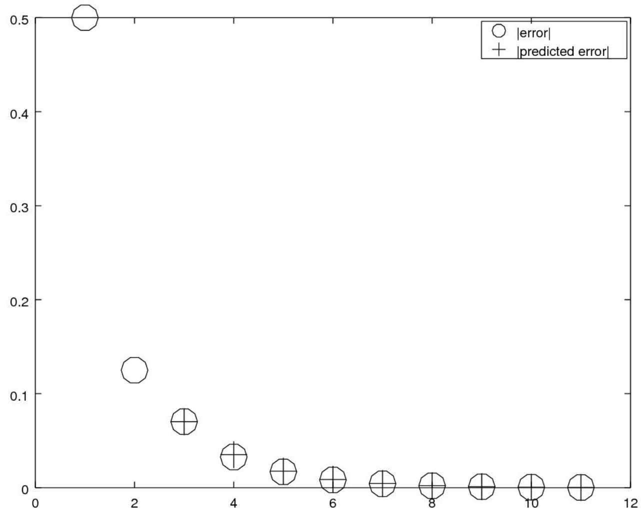
$$\lim_{x \to 1} \frac{|\rho(x) - 1|}{|x - 1|^\alpha} = \lim_{x \to 1} \frac{|-x^2/2 + x/2 + 1 - 1|}{|x - 1|^\alpha} = \lim_{x \to 1} \frac{|-x/2||x - 1|}{|x - 1|^\alpha}$$

assuming $\alpha = 1$

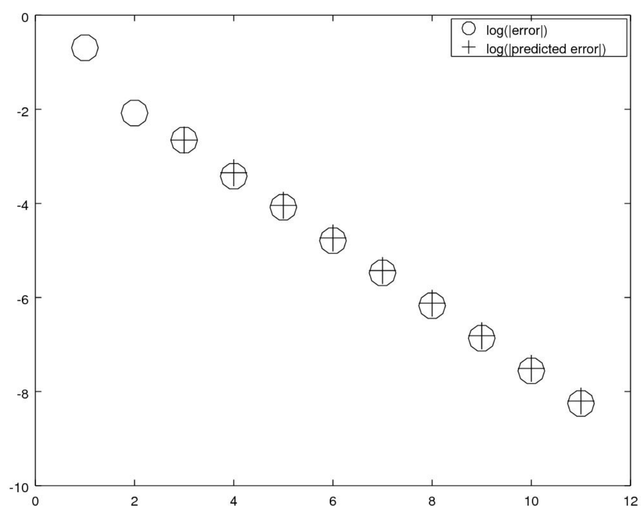
$$= \lim_{x \to 1} |-x/2| = 1/2$$

So we conclude $\frac{|err_{k+1}|}{|err_k|} \to 1/2$ as $x_k \to 1$, thus this converges linearly.

I wrote a script that illustrates this convergence by plotting the error and the predicted error using the above model.



The similarity is even more apparent in a log plot.



These graphs clearly show that the predicted end behavior is correct and that the absolute value of the error is halved with each iteration.

Driver code:

```
x=[.5];
n=10;
f=@(x) .5*(-x^2+x+2);
for i=1:n
    x=[x f(x(length(x)))];
end
x'
xerr=abs(1-x);
#assume we see end behavior start near the third itteration
xerrpredic=[nan,nan,xerr(3)];
while length(xerrpredic)~=length(xerr)
    xerrpredic=[xerrpredic abs(.5*xerrpredic(length(xerrpredic)))];
end
px=1:length(xerr);
figure(1);
plot(px,xerr,'ko','MarkerSize', 20,px,xerrpredic,'k+','MarkerSize', 20);
legend("|error|","|predicted error|");
saveas(gcf,"f1.jpg");
figure(2);
plot(px,log(xerr),'ko','MarkerSize', 20,px,log(xerrpredic),'k+','MarkerSize', 20);
legend("log(|error|)","log(|predicted error|)");
saveas(gcf,"f2.jpg");
```

Output:

```
>> q4_15
ans =

   0.50000
   1.12500
   0.92969
   1.03268
   0.98312
   1.00830
   0.99582
   1.00208
   0.99896
   1.00052
   0.99974

>> diary off
```