Exercise : Problem 8.12

Show that

$$s(x) = \begin{cases} 1 + x - x^3 & 0 \le x < 1 \\ 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3 & 1 \le x < 2 \\ 4(x-2) + 9(x-2)^2 - 3(x-2)^3 & 2 \le x < 3 \end{cases}$$

is a cubic spline through $(0, 1), (1, 1), (2, 0), (3, 10)$.

For notation 1^+ means a little more than 1 and 1^- is a little less than 1, for dealing with the peace wise function.
First lets make sure the values link up.
$s(x = 0) = 1 + x - x^3 = 1$ good on $(0, 1)$
$s(x = 1^-) = 1 + x - x^3 = 1$ and $s(x = 1^+) = 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3 = 1$ good on $(1, 1)$
$s(x = 2^-) = 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3 = 0$ and $s(x = 2^+) = 4(x-2) + 9(x-2)^2 - 3(x-2)^3 = 0$ good on $(2, 0)$
$s(x = 3) = 4(x-2) + 9(x-2)^2 - 3(x-2)^3 = 10$ good on $(3, 10)$
Now check that the derivatives mach.
$s'(x = 1^-) = 1 - 3x^2 = -2$ and $s'(x = 1^+) = -2 - 6(x-1) + 12(x-1)^2 = -2$ good.
$s'(x = 2^-) = -2 - 6(x-1) + 12(x-1)^2 = 4$ and $s'(x = 2^+) = 4 + 18(x-2) - 9(x-2)^2 = 4$ good.
Now check that the second derivatives mach.
$s''(x = 1^-) = -6x = -6$ and $s''(x = 1^+) = -6 + 24(x-1) = -6$ good.
$s''(x = 2^-) = -6 + 24(x-1) = 18$ and $s''(x = 2^+) = 18 - 18(x-2) = 18$ good.
Therefore it is a cubic spline. Note that $s''(x = 0) = -6x = 0$ and $s''(x = 3) = 18 - 18(x-2) = 0$ thus s is a natural cubic spline.

Exercise : Problem 8.13

For what constants is

$$s(x) = \begin{cases} ax^2 + b(x-1)^3 & x \in (-\infty, 1] \\ cx^2 + d & x \in [1, 2] \\ ex^2 + f(x-2)^3 & x \in [1, 2] \end{cases}$$

a cubic spline?
Note

$$s'(x) = \begin{cases} 2ax + 3b(x-1)^2 & x \in (-\infty, 1] \\ 2cx & x \in [1, 2] \\ 2ex + 3f(x-2)^2 & x \in [1, 2] \end{cases}$$

$$s''(x) = \begin{cases} 2a + 6b(x-1) & x \in (-\infty, 1] \\ 2c & x \in [1, 2] \\ 2e + 6f(x-2) & x \in [1, 2] \end{cases}$$

We require that up to the second derivative link so our constraints are:
values
$a = c + d$
$4c + d = 4e$
derivatives
$2a = 2c$
$4c = 4e$
second derivatives
$2a = 2c$
$2c = 2e$
These equations hold true if and only if $d = 0$, $a = c = e$. Note that this allows us 3 degrees of freedom in choosing our constants, b, f and one of $a = c = e$.

Exercise : Find n so that degree n polynomial interpolation of $f(x) = \cos(3x)$, using equally spaced points on $[0, 2]$, gives a maximum approximation error $|f(x) - p(x)|$ which is less than 10^{-6} on $[0, 2]$.

Note that by inspection our polynomial will be at least order 2 since a line is a terrible approximation for cos and thus 0 order and 1 order won't work.
The max error on a polynomial interpolation would be

$$e = |\frac{f^{(n+1)}(\xi)}{(n+1)!} * \Pi_{i=0}^{n}(\eta - x_i)|$$

where ξ and η are in $[0, 2]$. Note that for equally spaced points $\Pi_{i=0}^{n}|\eta - x_i|$ achieves its maximum with $\eta \in (x_0, x_1)$ and again at $\eta \in (x_{n-1}, x_n)$. The proof for this is trivial just think about the symmetry and what happens when you shift left or right by $x_1 - x_0$ (one of those two shifts is guaranteed to give you a increase). Let η_0 be the $\eta_0 \in (x_0, x_1)$ that makes $\Pi_{i=0}^{n}|\eta - x_i|$ a maximum. Thus $\Pi_{i=0}^{n}|\eta - x_i| \leq \Pi_{i=0}^{n}|\eta_0 - x_i| = (\eta_0 - x_0)\Pi_{i=1}^{n}(x_i - \eta_0) \leq (x_1 - x_0)\Pi_{i=1}^{n}(x_i - x_0) = \frac{2}{n}\Pi_{i=1}^{n}(i\frac{2}{n}) = (\frac{2}{n})^{n+1}\Pi_{i=1}^{n} i = (\frac{2}{n})^{n+1} n!$. Note that $f^{(n+1)}(x) = 3^{n+1}S(3x)$ where $S(x)$ is a $\pm\sin(x)$ or $\pm\cos(x)$. Noting that $|S(x)| \leq 1$ conclude $|f^{(n+1)}(\xi)| \leq 3^{n+1}$. Now we can conclude

$$e = |\frac{f^{(n+1)}(\xi)}{(n+1)!} * \Pi_{i=0}^{n}(\eta - x_i)| \leq \frac{3^{n+1}}{(n+1)!} * (\frac{2}{n})^{n+1} n! = \frac{6^{n+1}}{n^{n+1}(n+1)}$$

So we can guarantee a $e < 10^{-6}$ if $\frac{6^{n+1}}{n^{n+1}(n+1)} < 10^{-6}$.

```
>> n=[1:20];
>> e=(6.^(n+1)./(n.^(n+1).*(n+1)));
>> [n',e']
ans =

   1.0000e+00   1.8000e+01
   2.0000e+00   9.0000e+00
   3.0000e+00   4.0000e+00
   4.0000e+00   1.5188e+00
```

```
    5.0000e+00   4.9766e-01
    6.0000e+00   1.4286e-01
    7.0000e+00   3.6420e-02
    8.0000e+00   8.3427e-03
    9.0000e+00   1.7342e-03
    1.0000e+01   3.2982e-04
    1.1000e+01   5.7799e-05
    1.2000e+01   9.3900e-06
    1.3000e+01   1.4216e-06
    1.4000e+01   2.0149e-07
    1.5000e+01   2.6844e-08
    1.6000e+01   3.3735e-09
    1.7000e+01   4.0121e-10
    1.8000e+01   4.5284e-11
    1.9000e+01   4.8632e-12
    2.0000e+01   4.9811e-13

>> diary off
```

From this test we can see that the error will be guaranteed to be under 10^{-6} for a polynomial of degree 14. My suspicion would be that a polynomial of order 13 will work since it is guaranteed to have a error less than $1.4216e{-}06$, but based on this math I can only guarantee that a polynomial of degree 14 will work.
We can now run some code to see what will work.

```
function n=errfind()
    n=2;
    while(true)
        x=linspace(0,2,n+1);
        p=polyfit(x,cos(3.*x),n);
        tx=rand(1,100000);
        ty=polyval(p,tx);
        em=max(abs(ty-cos(3*tx)))
        if(em<10^-6)
            return;
        end
        n+=1;
    end
endfunction
```

Witch ends up telling us that a polynomial of degree 12 will work.

Exercise : At the bottom of page 198 is an inequality that describes the error from the piece wise linear interpolate $l(x)$ for $f(x)$ on $[a, b]$. Suppose we have equally spaced points $a = x_0 < x_1 < \cdots < x_n = b$ with spacing $h = xi - x_{i-1}$. then:

$$|f(x) - l(x)| \leq \frac{Mh^2}{8}$$

for all $x \in [a, b]$. In this inequality we are assuming $f''(x)$ exists and is bounded by the number M, so that $|f''(x)| \leq M$ for all $x \in [a, b]$. Use this inequality to find n so that $|f(x) - l(x)| \leq 10^{-6}$ for $x \in [0, 2]$ if $f(x) = \cos(3x)$.

Well $M = 9$ works since $|f''(x)| = 9|\cos(3x)| \leq 9$. Solving we get $sqrt(10^{-6}*8/9)^{-1}*2 =$ 2121.3 so $n = 2122$ will work.

Exercise : 10.1

Derive the Newton-Cotes formula between [0, 1] for a 3 regions.
Polynomial interpolation allows us to say

$$f(x) \approx \sum_{i=0}^{3} f(x_i) * \prod_{j\neq i} \frac{x - x_j}{x_i - x_j}$$

$$\int_0^1 f(x)dx \approx \int_0^1 \sum_{i=0}^{n} f(x_i) * \prod_{j\neq i} \frac{x - x_j}{x_i - x_j} dx =$$

$$\int_0^1 f(0)\frac{x-1/3}{-1/3} * \frac{x-2/3}{-2/3} * \frac{x-1}{-1} + f(1/3)\frac{x}{1/3} * \frac{x-2/3}{-1/3} * \frac{x-1}{-2/3} +$$

$$f(2/3)\frac{x}{2/3} * \frac{x-1/3}{1/3} * \frac{x-1}{-1/3} + f(1)\frac{x}{1} * \frac{x-1/3}{2/3} * \frac{x-2/3}{1/3} dx =$$

$$\int_0^1 f(0)\frac{(x-1/3)(x-2/3)(x-1)}{-2/9} + f(1/3)\frac{(x)(x-2/3)(x-1)}{2/27} +$$

$$f(2/3)\frac{(x)(x-1/3)(x-1)}{-2/27} + f(1)\frac{(x)(x-1/3)(x-2/3)}{2/9} dx =$$

$$\int_0^1 f(0)\frac{(3x-1)(3x-2)(x-1)}{-2} + f(1/3)\frac{(x)(3x-2)(x-1)}{2/9} +$$

$$f(2/3)\frac{(x)(3x-1)(x-1)}{-2/9} + f(1)\frac{(x)(3x-1)(3x-2)}{2} dx =$$

$$\int_0^1 f(0)\frac{(9x^2-9x+2)(x-1)}{-2} + f(1/3)\frac{(x)(3x^2-5x+2)}{2/9} +$$

$$f(2/3)\frac{(x)(3x^2-4x+1)}{-2/9} + f(1)\frac{(x)(9x^2-9x+2)}{2} dx =$$

$$\int_0^1 f(0)\frac{9x^3-18x^2+11x-2}{-2} + f(1/3)\frac{3x^3-5x^2+2x}{2/9} +$$

$$f(2/3)\frac{(3x^3-4x^2+x)}{-2/9} + f(1)\frac{9x^3-9x^2+2x}{2} dx =$$

$$f(0)\frac{9/4-6+11/2-2}{-2} + f(1/3)\frac{3/4-5/3+1}{2/9} + f(2/3)\frac{3/4-4/3+1/2}{-2/9} + f(1)\frac{9/4-3+1}{2} =$$

$$\frac{1}{8}f(0) + \frac{3}{8}f(1/3) + \frac{3}{8}f(2/3) + \frac{1}{8}f(1)$$

Exercise : 10.2

Find

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f(1)$$

witch is exact for all $f(x) = ae^x + b\cos(\pi x/2)$. If such a pair A_0, A_1 exist then they must be exact for e^x and $\cos(\pi x/2)$ in other words $e - 1 = A_0 + eA_1$ and $2/\pi = A_0$. Thus

$$\int_0^1 f(x)dx = \frac{2}{\pi} f(0) + \frac{e - 1 - 2/\pi}{e} f(1)$$

is exact for all $f(x) = ae^x + b\cos(\pi x/2)$.