Exercise 1.4.1: Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$ then ab and $a + b \in \mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$ and if $a \neq 0$ then $at \in \mathbb{I}$ as well.

(c) Part (a) says that the rational numbers are closed under multiplication and addition. What can be said about st and $s + t$ when $s, t \in \mathbb{I}$?

(a) *Proof.* Select two arbitrary elements from the rational numbers, since they are rational we can represent them as i/j and m/n where $i, j, m, n \in \mathbb{Z}$ and $j \neq 0$ and $n \neq 0$.

Note that $i/j * m/n = \frac{im}{jn}$, from the definition of multiplication of rational numbers. Since the multiple of any two non-zero numbers is non-zero and since the multiple of any two integers is a integer so $jn \in \mathbb{Z} - \{0\}$ and $im \in \mathbb{Z}$ therfore $\frac{im}{jn} \in \mathbb{Q}$.

Note that $i/j + m/n = \frac{in+mj}{jn}$, from the definition of addition of rational numbers. Since the multiple of any two non-zero numbers is non-zero and since the multiple of any two integers is a integer so $jn \in \mathbb{Z} - \{0\}$ and $in, mj \in \mathbb{Z}$ and so also $in + mj \in \mathbb{Z}$ therfore $\frac{in+mj}{jn} \in \mathbb{Q}$. □

(b) *Proof.* Proof by contradiction.
Suppose that there exists $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ where $a + t = b \notin \mathbb{I}$. Since the reals are closed under addition we can say $b \in \mathbb{R}$. Note that $b \in \mathbb{R} - \mathbb{I} = \mathbb{Q}$. We can do a little math and see $a + t = b$ means $t = b + (-a)$. Since the addative inverse of a rational is a rational and since the sum of two rationals is rational we conclude $t \in \mathbb{Q}$. A contradiction has been reached, the rationals and irrationals are, by deffinition, mutualy exclusive and so t cannot be a element of both.

Proof by contradiction.
Suppose that there exists $a \in \mathbb{Q} - \{0\}$ and $t \in \mathbb{I}$ where $at = b \notin \mathbb{I}$. Since the reals are closed under multiplication, and since the multiple of any two non zero numbers is itself non zero, we can say $b \in \mathbb{R} - \{0\}$. Note that $b \in \mathbb{R} - \{0\} - \mathbb{I} = \mathbb{Q} - \{0\}$. We can do a little math and see $at = b$ means $t = b(a^{-1})$. Since the multiplicative inverse of a non zero rational is a rational (informaly $(\frac{i}{m})(\frac{m}{i}) = 1$) and since the multiple of two rationals is rational we conclude $t \in \mathbb{Q}$. A contradiction has been reached, the rationals and irrationals are, by deffinition, mutualy exclusive and so t cannot be a element of both. □

(c) All we can conclude is that $st \in \mathbb{R} - \{0\}$ and that $s + t \in \mathbb{R}$. As a example that the irrationals are not closed with respect to multiplication or addition note that $\sqrt{2}\sqrt{2} = 2$ and that $\pi + (-\pi) = 0$.

Exercise 1.4.2: Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + (1/n)$ is an upper bound for A but $s - (1/n)$ is not an upper bound for A. Show that $s = \sup A$.

Proof. □

Exercise 1.4.3: Show that $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

Proof. □

Exercise 1.4.4: **(W) (Hand this one in to David.)**
Let $a < b$ be real numbers and let $T = [a, b] \cap \mathbb{Q}$. Show that $\sup T = a$.

Proof. □

Exercise 1.4.5: Use Exercise 1.4.1 to provide a proof of Corollary 1.4.4 by considering real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Proof. □

Exercise Supplemental 1: Show that the sets $[0, 1)$ and $(0, 1)$ have the same cardinality.