Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified. I am also operating under the convention $v_a(b) = \{x \in \mathbb{R} : |x - b| < a\}$

Exercise 1: Suppose $f : A \to \mathbb{R}$ and c is a limit point of A. Suppose $f(x) \geq 0$ for all $x \in A$ and that $lim_{x \to c} f(x)$ exists. Show that the limit is non-negative. Provide two proofs, one ϵ - δ style, and the other using the sequential characterization of limits.

Proof. Suppose $f : A \to \mathbb{R}$ and c is a limit point of A. Suppose $f(x) \geq 0$ for all $x \in A$ and that $lim_{x \to c} f(x) = L$ exists.
Suppose $L < 0$. Define $\epsilon = -L/2 > 0$. There must exist a $\delta > 0$ such that for all $x \in A$ where $0 < |x-c| < \delta$, $|f(x)-L| < \epsilon$. Note that c is a limit point of A, thus $v_\delta(c) \cap A - \{c\} \neq \emptyset$. Take one of the elements of this set $a \in v_\delta(c) \cap A - \{c\}$. Note that $a \neq c$. Note that $a \in A$. Note $|a-c| < \delta$. Thus $|f(a)-L| < \epsilon$ and so $L-\epsilon < f(a) < L+\epsilon = L/2 < 0$. A contradiction, we know that $f(a) \geq 0$ and now we have $f(a) < 0$. We now conclude $L \geq 0$. □

Exercise 2: Let a_n be a sequence of numbers such that for some $M \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n M^n$ converges. Suppose that $|x| < M$. Show that $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely. Give an example to show that divergence is possible if $|x| = |M|$. Hint: $(a_n M^n)$ converges to zero, and is hence bounded.

Proof. Let a_n be a sequence of numbers such that for some $M \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n M^n$ converges. Suppose that $|x| < M$.
Note that by our supposition $|x| < M$ we can conclude $0 < M$. Define $-1 < 0 \leq r = \frac{|x|}{M} < 1$. Note that $\lim_{n\to\infty} a_n M^n = 0$ since $\sum_{n=1}^{\infty} a_n M^n$ converges. Note that a convergent sequence is bounded so we can define $N \in \mathbb{R}$ such that $|a_n M^n| < N$ for all n. Note $|r| < 1$, thus $\sum_{n=1}^{\infty} Nr^n$ converges lets call it's limit L. Noting that $0 \leq N$ and $0 \leq r$ we can conclude $0 \leq Nr^n$ for $n \in \mathbb{N}$ and thus $\sum_{n=1}^{k} Nr^n$ is monotone increasing. Note $\sum_{n=1}^{k} Nr^n \leq \sum_{n=1}^{\infty} Nr^n = L$. Define $S_k = \sum_{n=1}^{k} |a_n x^n|$. Note that $S_k \leq S_k + |a_{k+1} x^{k+1}| = S_{k+1}$ for $k \in \mathbb{N}$ and thus S_k is monotone increasing. Note that $|a_n| < \frac{N}{M^n}$, recall $|a_n M^n| < N$ and $0 \leq M$. Note that $S_k = \sum_{n=1}^{k} |a_n x^n| < \sum_{n=1}^{k} \frac{N}{M^n} |x^n| = \sum_{n=1}^{k} Nr^n \leq L$. By the MCT S_k converges and thus $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely.

□

For a example where $|x| = |M|$ and $\sum_{n=1}^{\infty} a_n x^n$ does not converges absolutely, consider the case $a_n = (-1)^n 1/n$, $M = 1$, $x = 1$. Note that $\sum_{n=1}^{\infty} a_n M^n = \sum_{n=1}^{\infty} (-1)^n 1/n$ converges. However $\sum_{n=1}^{\infty} |a_n x^n| = \sum_{n=1}^{\infty} 1/n$ does not converge.

Exercise 3: Suppose $f : (0, 1] = A \to \mathbb{R}$ is uniformly continuous. Show that $lim_{x \to 0} f(x)$ exists.

Proof. Note that 0 is a limit point of $(0, 1]$. Take a arbitrary sequence $a_n \in A$ where $a_n \to 0$. Choose $\epsilon > 0$. Note that there exists $\delta > 0$ such that for all $x, y \in A$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon/2$ □