

Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified.

Exercise : 2.2.6

The limit of a sequence, if it exists is unique

Proof. Suppose to the contrary that there exists a sequence $\{a_n\}_{n=1}^{\infty}$ that converges to two values, a and b where $a \neq b$. Without loss of generality assume $a > b$. Define $2\epsilon = a - b > 0$ and note that $\epsilon > 0$. By the definition of limit of a sequence we know that there exists a N_a such that for all $n \geq N_a$, $|a_n - a| < \epsilon$. Also note that by the definition of limit of a sequence we know that there exists a N_b such that for all $n \geq N_b$, $|a_n - b| < \epsilon$. Take $N = \max(N_a, N_b)$ note that for all $n \geq N$, $|a_n - a| < \epsilon$ and $|a_n - b| < \epsilon$. Now we see that $|a_N - a| < \epsilon$ and $|a_N - b| < \epsilon$ so $|a_N - a| + |a_N - b| < 2\epsilon$ or $|a - a_N + a_N - b| \leq |a - a_N| + |a_N - b| < 2\epsilon$ via the triangle inequality. Thus $|a - b| < 2\epsilon$, and noting that $a - b > 0$ we get $a - b < 2\epsilon = a - b$, a contradiction. We are forced to conclude the negation of our supposition, that there is no sequence with two limits, or that a limit to a sequence, if it exists is unique. \square

Exercise : 2.3.1(a)

Let $x_n \geq 0$ for all $n \in \mathbb{N}$. If $x_n \rightarrow 0$ show $\sqrt{x_n} \rightarrow 0$.

Proof. Chuse a $\epsilon > 0$. Define $\omega = \epsilon^2$. Note that $x_n \rightarrow 0$ implies that there exists a N such that for all $n \geq N$, $|x_n| < \omega$. Note that $|x_n| < \omega = \epsilon^2$ implies $\sqrt{|x_n|} < \epsilon$ wich means $|\sqrt{x_n} - 0| < \epsilon$ for all $n \geq N$. Thus by definition $\sqrt{x_n} \rightarrow 0$. \square

Exercise : 2.3.1(b)

Let $x_n \geq 0$ for all $n \in \mathbb{N}$. If $x_n \rightarrow x$ show $\sqrt{x_n} \rightarrow \sqrt{x}$.

Proof. Assume $x \neq 0$ since we have already proven the statement true in that case, furthur note that since the sequence is bounded below by 0, $x \geq 0$

Chuse a $\epsilon > 0$. Define $\omega = \epsilon \sqrt{x} > 0$. Note that $x_n \rightarrow x$ implies that there exists a N such that for all $n \geq N$, $|x_n - x| < \omega$. Note that $|x_n - x| < \omega = \epsilon \sqrt{x}$ implies $|x - x_n| < \epsilon \sqrt{x}$, $|\sqrt{x} + \sqrt{x_n}| |\sqrt{x} - \sqrt{x_n}| < \epsilon \sqrt{x}$. Noting that $|\sqrt{x} + \sqrt{x_n}| = \sqrt{x} + \sqrt{x_n} \geq \sqrt{x}$ implies $\sqrt{x} |\sqrt{x} - \sqrt{x_n}| \leq |\sqrt{x} + \sqrt{x_n}| |\sqrt{x} - \sqrt{x_n}| < \epsilon \sqrt{x}$. Thus $|\sqrt{x} - \sqrt{x_n}| < \epsilon$ for all $n \geq N$. By definition $\sqrt{x_n} \rightarrow \sqrt{x}$. \square

Exercise : 2.3.3

Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and $\lim x_n = \lim z_n = l$, then $\lim y_n = l$.

For this proof I need the therum that $|a| < b \Leftrightarrow -b < a < b$ where $b > 0$.

Proof. There are two cases $a \geq 0$ or $a < 0$.

Case $a \geq 0$. In this case $|a| = a$ and our statement becomes $0 \leq a < b \Leftrightarrow -b < a < b$ wich is clearly true.

Case $a < 0$. In this case $|a| = -a$ and our statement becomes $0 \leq -a < b \Leftrightarrow -b < a < b$ wich, noting that $0 \leq -a < b$ is equivelent to $0 \geq a > -b$, is clearly true. \square

Proof. Suppose $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and $\lim x_n = \lim z_n = l$.

Chuse $\epsilon > 0$. By the definition of limit of a sequence we know that there exists a N_1 such that for all $n \geq N_1$, $|x_n - l| < \epsilon$. By the definition of limit of a sequence we know that there exists a N_2 such that for all $n \geq N_2$, $|z_n - l| < \epsilon$. Define $N = \max(N_1, N_2)$. Note that for all $n \geq N$, $|x_n - l| < \epsilon$ and $|z_n - l| < \epsilon$ or by the above theorem $-\epsilon < x_n - l < \epsilon$ and $-\epsilon < z_n - l < \epsilon$. Note that $-\epsilon < x_n - l \leq y_n - l \leq z_n - l < \epsilon$ thus $-\epsilon < y_n - l < \epsilon$ and so $|y_n - l| < \epsilon$ for all $n > N$. Thus by the definition of the limit of a sequence $\lim y_n = l$. \square

Exercise : 2.3.6

Find what $b_n = n - \sqrt{n^2 + 2n}$ converges to.

Proof. Note that $b_n = \frac{(n - \sqrt{n^2 + 2n})(n + \sqrt{n^2 + 2n})}{n + \sqrt{n^2 + 2n}} = \frac{-2n}{n + \sqrt{n^2 + 2n}} = \frac{-2}{1 + \sqrt{1 + 2/n}} = \frac{a_n}{c_n}$, where $a_n = -2$ and $c_n = 1 + \sqrt{1 + 2/n} = d_n + e_n$, where $d_n = 1$ and $e_n = \sqrt{1 + 2/n} = \sqrt{f_n}$, where $f_n = 1 + 2/n$. Noting that $1/n \rightarrow 0$ we see that $f_n \rightarrow 1$. Using 2.3.1 we see that $e_n \rightarrow \sqrt{1} = 1$. By inspection $d_n \rightarrow 1$ and so by the algebraic limit theorem $c_n \rightarrow 2$. Noting that $a_n \rightarrow -2$ and that $c_n \rightarrow 2$ we see by the algebraic limit theorem $b_n \rightarrow \frac{-2}{2} = -1$. \square

Exercise : 2.3.9(a)

Let (a_n) be a bounded sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the algebraic limit theorem to do this?

Firstly this is outside of the algebraic limit theorem entirely since we are not guaranteed that a_n has a limit.

Proof. Suppose (a_n) to be a bounded sequence, and $\lim b_n = 0$.

Chuse $\epsilon > 0$. Since (a_n) is bounded there exists a $M > 0$ such that $|a_n| \leq M$ for all n . By the definition of limit there exists a N such that for all $n \geq N$, $|b_n| < \epsilon/M$, since $\epsilon/M > 0$. Note that $|a_n b_n - 0| = |a_n b_n| = |a_n| |b_n| \leq M |b_n| < \epsilon$ for all $n \geq N$, thus by the definition of limit $\lim(a_n b_n) = 0$. \square

Exercise : 2.3.10(a)

If $\lim(a_n - b_n) = 0$ then $\lim a_n = \lim b_n$.

Counterexample, consider the case $a_n = b_n = n$. In this case $\lim(a_n - b_n) = \lim(0) = 0$. However $\lim a_n = \lim n$ which does not exist and so the statement $\lim a_n = \lim b_n$ cannot be true.

Exercise : 2.3.10(b)

If $\lim b_n = b$ then $\lim |b_n| = |b|$.

Proof. Chuse $\epsilon > 0$. By the definition of limit there exists a N such that for all $n \geq N$, $|b_n - b| < \epsilon$. Recall that we proved on the first homework that $||a| - |b|| \leq |a - b|$, thus $||b_n| - |b|| \leq |b_n - b| < \epsilon$ for all $n \geq N$. By the definition of limit $\lim |b_n| = |b|$. \square

Exercise : 2.3.10(c)

If $\lim a_n = a$ and $\lim(b_n - a_n) = 0$ then $\lim b_n = a$.

Proof. Define $s_n = b_n - a_n$. Note that $s_n \rightarrow 0$ and $a_n \rightarrow a$. By the algebraic limit theorem $b_n = (s_n + a_n) \rightarrow a + 0 = a$ thus $b_n \rightarrow a$. \square

Exercise : 2.3.10(d)

If $a_n \rightarrow 0$ and $|b_n - b| \leq a_n$ for all n then $b_n \rightarrow b$.

Proof. Suppose $a_n \rightarrow 0$ and $|b_n - b| \leq a_n$ for all n . Note that $0 \leq |b_n - b| \leq a_n$ thus $|a_n| = a_n$. Choose $\epsilon > 0$. By the definition of limit there exists a N such that for all $n \geq N$, $|a_n| < \epsilon$. Note that $|b_n - b| \leq a_n = |a_n| < \epsilon$ for all $n \geq N$. Thus by the definition of limit $b_n \rightarrow b$. \square

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Exercise : 2.3.5 (W) (Hand this one in to David.)

Let (x_n) and (y_n) be given, and define (z_n) to be the sequence (x_1, y_1, x_2, \dots) . Prove that (z_n) is convergent if and only if $\lim x_n = \lim y_n$.

Proof. Note that we can formalize this definition as

$$z_n = \begin{cases} x_{(n+1)/2} & n \in \text{odds} \\ y_{n/2} & \text{otherwise} \end{cases}$$

We are asked in this proof to prove a "if and only if" statement, basically prove a double implication. I will break this up into proving two implications, first $\lim x_n = \lim y_n$ implies (z_n) is convergent, and second (z_n) is convergent implies $\lim x_n = \lim y_n$.

Suppose $\lim x_n = \lim y_n = l$.

Chuse $\epsilon > 0$. By the definition of limit of a sequence we know that there exists a N_1 such that for all $n \geq N_1$, $|x_n - l| < \epsilon$. By the definition of limit of a sequence we know that there exists a N_2 such that for all $n \geq N_2$, $|y_n - l| < \epsilon$. Define $N = 2 * (\max(N_1, N_2))$. Chuse $n \geq N$. There are two possibilities, either $n \in \text{odd}$ or $n \notin \text{odd}$.

Case $n \in \text{odd}$. In this case $z_n = x_{(n+1)/2}$. Note that $(n+1)/2 > n/2 \geq N/2 \geq N_1$ and so $|z_n - l| = |x_{(n+1)/2} - l| < \epsilon$.

Case $n \notin \text{odd}$. In this case $z_n = y_{n/2}$. Note that $n/2 \geq N/2 \geq N_2$ and so $|z_n - l| = |y_{n/2} - l| < \epsilon$. So $|z_n - l| < \epsilon$ for all $n \geq N$ and thus z_n will converge.

Suppose (z_n) is convergent.

let $l = \lim z_n$.

Chuse $\epsilon > 0$. By the definition of limit of a sequence we know that there exists a N such that for all $m \geq N$, $|z_m - l| < \epsilon$. Chuse a $n \geq N$. let $m = 2n - 1$. Note that $m \geq n \geq N$ and that $m \in \text{odds}$, so $z_m = x_{(m+1)/2} = x_n$. Since $m \geq N$, $|x_n - l| = |z_m - l| < \epsilon$. Thus x_n converges to l .

Chuse $\epsilon > 0$. By the definition of limit of a sequence we know that there exists a N such that for all $m \geq N$, $|z_m - l| < \epsilon$. Chuse a $n \geq N$. let $m = 2n$. Note that $m > n \geq N$ and that $m \notin \text{odds}$, so $z_m = y_{m/2} = y_n$. Since $m \geq N$, $|y_n - l| = |z_m - l| < \epsilon$. Thus y_n converges to l .

We conclude that $\lim x_n = \lim y_n$.

We can now conclude (z_n) is convergent if and only if $\lim x_n = \lim y_n$.

□