

Exercise 1.4.1: Recall that \mathbb{I} stands for the set of irrational numbers.

- (a) Show that if $a, b \in \mathbb{Q}$ then ab and $a + b \in \mathbb{Q}$ as well.
- (b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$ and if $a \neq 0$ then $at \in \mathbb{I}$ as well.
- (c) Part (a) says that the rational numbers are closed under multiplication and addition. What can be said about st and $s + t$ when $s, t \in \mathbb{I}$?

- (a) *Proof.* Select two arbitrary elements from the rational numbers, since they are rational we can represent them as i/j and m/n where $i, j, m, n \in \mathbb{Z}$ and $j \neq 0$ and $n \neq 0$.

Note that $i/j * m/n = \frac{im}{jn}$, from the definition of multiplication of rational numbers. Since the multiple of any two non-zero numbers is non-zero and since the multiple of any two integers is a integer so $jn \in \mathbb{Z} - \{0\}$ and $im \in \mathbb{Z}$ therefore $\frac{im}{jn} \in \mathbb{Q}$.

Note that $i/j + m/n = \frac{in+mj}{jn}$, from the definition of addition of rational numbers. Since the multiple of any two non-zero numbers is non-zero and since the multiple of any two integers is a integer so $jn \in \mathbb{Z} - \{0\}$ and $in, mj \in \mathbb{Z}$ and so also $in + mj \in \mathbb{Z}$ therefore $\frac{in+mj}{jn} \in \mathbb{Q}$. \square

- (b) *Proof.* Proof by contradiction.

Suppose that there exists $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ where $a + t = b \notin \mathbb{I}$. Since the reals are closed under addition we can say $b \in \mathbb{R}$. Note that $b \in \mathbb{R} - \mathbb{I} = \mathbb{Q}$. We can do a little math and see $a + t = b$ means $t = b + (-a)$. Since the additive inverse of a rational is a rational and since the sum of two rationals is rational we conclude $t \in \mathbb{Q}$. A contradiction has been reached since the rationals and irrationals are, by definition, mutually exclusive, t cannot be a element of both.

Proof by contradiction.

Suppose that there exists $a \in \mathbb{Q} - \{0\}$ and $t \in \mathbb{I}$ where $at = b \notin \mathbb{I}$. Since the reals are closed under multiplication, and since the multiple of any two non zero numbers is itself non zero, we can say $b \in \mathbb{R} - \{0\}$. Note that $b \in \mathbb{R} - \{0\} - \mathbb{I} = \mathbb{Q} - \{0\}$. We can do a little math and see $at = b$ means $t = b(a^{-1})$. Since the multiplicative inverse of a non zero rational is a rational (informally $(\frac{i}{m})(\frac{m}{i}) = 1$) and since the multiple of two rationals is rational we conclude $t \in \mathbb{Q}$. A contradiction has been reached since the rationals and irrationals are, by definition, mutually exclusive, t cannot be a element of both. \square

- (c) All we can conclude is that $st \in \mathbb{R} - \{0\}$ and that $s + t \in \mathbb{R}$. As a example that the irrationals are not closed with respect to multiplication or addition note that $\sqrt{2}\sqrt{2} = 2$ and that $\pi + (-\pi) = 0$.

Exercise 1.4.2: Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + (1/n)$ is an upper bound for A but $s - (1/n)$ is not an upper bound for A . Show that $s = \sup A$.

Proof. Take some $a \in \mathbb{R}$ where $s < a$. Note $a - s \in \mathbb{R}^+$ and so by the proof we previously completed in class there is a natural number n with the property $\frac{1}{n} < a - s$. So $s + \frac{1}{n} < a$ which means a is bigger than an upper bound on A and so is itself not in A . Since we chose an arbitrary real greater than s and showed it is not in A we can conclude no real bigger than s is in A , in other words all elements of A are less than or equal to s . By definition s is an upper bound on A .

Next take some $b \in \mathbb{R}$ where $b < s$. Note $s - b \in \mathbb{R}^+$ and so by the proof we previously completed in class there is a natural number n with the property $\frac{1}{n} < s - b$. So $b < s - \frac{1}{n}$ which means b is smaller than a number that is not an upper bound on A . Noting that if b were an upper bound on A we would be forced to conclude $s - \frac{1}{n}$ is an upper bound on A , which we know to be false, we are forced to conclude b is not an upper bound on A . and since b was chosen arbitrarily we can conclude all upper bounds on A are greater than or equal to s . Noting that s has both properties of a sup we conclude $s = \sup A$. \square

Exercise 1.4.3: Show that $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$.

Proof. Proof by contradiction.

Suppose $a \in \bigcap_{n=1}^{\infty} (0, 1/n)$. Note that $a \in (0, 1) \subseteq \mathbb{R}^+$ so $a \in \mathbb{R}^+$. By the proof done in class there exists a natural i with the property $1/i < a$. Since a is in the intersection of all of the $(0, 1/n)$ sets $a \in (0, 1/i)$ so $a < 1/i$ a contradiction. We now conclude the negation of our supposition namely there is no a with the property $a \in \bigcap_{n=1}^{\infty} (0, 1/n)$ or a logically equivalent statement $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$. \square

Exercise 1.4.5: Use Exercise 1.4.1 to provide a proof of Corollary 1.4.4 by considering real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Proof. We are asked to prove that for all $a, b \in \mathbb{R}$ where $a < b$ there is a $t \in (a, b)$ where $t \in \mathbb{I}$.

I will start by choosing an arbitrary set $a, b \in \mathbb{R}$ where $a < b$. Next let $c = a + \frac{1}{3}(b - a)$ and $d = a + \frac{2}{3}(b - a)$. Note that $a < c < d < b$ so $[c, d] \subseteq (a, b)$. Now consider two cases, either $c, d \in \mathbb{Q}$ or one or more of them is irrational. In the case that one or more of them is irrational then it would be an irrational in (a, b) and we're done. So now we only need to consider the case where $c, d \in \mathbb{Q}$. In this case consider $t = c + \frac{\sqrt{2}}{2}(d - c)$. Noting that $(d - c) \neq 0$ we conclude $\frac{\sqrt{2}}{2}(d - c) \in \mathbb{I}$ and thus $c + \frac{\sqrt{2}}{2}(d - c) \in \mathbb{I}$ also note that $0 < \frac{\sqrt{2}}{2} < 1$ so $c < c + \frac{\sqrt{2}}{2}(d - c) < d$ therefore $c + \frac{\sqrt{2}}{2}(d - c) \in (a, b)$ and we have found an irrational in (a, b) . \square

Exercise Supplemental 1: Show that the sets $[0, 1)$ and $(0, 1)$ have the same cardinality.

Consider the function $f : [0, 1) \rightarrow (0, 1)$

$$f(x) = \begin{cases} 1/2 & x = 0 \\ (1/2)^{n+1} & x = (1/2)^n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

note that f has an inverse $g : (0, 1) \rightarrow [0, 1)$

$$g(x) = \begin{cases} 0 & x = 1/2 \\ (1/2)^{n-1} & x = (1/2)^n \text{ for some } n \in \mathbb{N} - \{1\} \\ x & \text{otherwise} \end{cases}$$

We now know f is bijective and so we have found a bijective mapping from $[0, 1)$ to $(0, 1)$ so by definition they have the same cardinality.

Exercise 1.4.4: (W) (Hand this one in to David.)

Let $a < b$ be real numbers and let $T = [a, b] \cap \mathbb{Q}$. Show that $\sup T = b$.

Proof. First we note that b is a upper bound on T , since $T \subseteq [a, b]$ and every element of $[a, b]$ is less than or equal to b .

Assume that there is a upper bound on T less than b , lets call it c . We know that there is at least one element in T since we proved in class that there is a rational between any two reals take one of these rationals and call it d . Note $d \geq a$. From that we can say $c \geq a$ as otherwise $c < d$ witch is not possible since c is a upper bound. Next we note that there is a rational between c and b , lets call it f such that $a \leq c < f < b$ and $f \in \mathbb{Q}$. Note that $f \in T$. Contradiction there is a element of T grater than c and yet c is a upper bound on T . Thus we conclude that there are no upper bounds on T less than b .

We see that b fulfills the definition of $\sup T$ and so $b = \sup T$. □