Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified.

Exercise : 2.2.6
The limit of a sequence, if it exists is unique

Proof. Suppose to the contrary that there exists a sequence $\{a_x\}_{x=1}^{\infty}$ that converges to two values, a and b where $a \neq b$. Without loss of generality assume $a > b$. Define $2\epsilon = a - b > 0$ and note that $\epsilon > 0$. By the definition of limit of a sequence we know that there exists a N_a such that for all $n \geq N_a$, $|a_n - a| < \epsilon$. Also note that by the definition of limit of a sequence we know that there exists a N_b such that for all $n \geq N_b$, $|a_n - b| < \epsilon$. Take $N = \max(N_a, N_b)$ note that for all $n \geq N$, $|a_n - a| < \epsilon$ and $|a_n - b| < \epsilon$. Now we see that $|a_N - a| < \epsilon$ and $|a_N - b| < \epsilon$ so $|a_N - a| + |a_N - b| < 2\epsilon$ or $|a - a_N + a_N - b| \leq |a - a_N| + |a_N - b| < 2\epsilon$ via the triangle inequality. Thus $|a - b| < 2\epsilon$, and noting that $a - b > 0$ we get $a - b < 2\epsilon = a - b$, a contradiction. We are forced to conclude the negation of our supposition, that there is no sequence with two limits, or that a limit to a sequence, if it exists is unique. □

Exercise : 2.3.1(a)
Let $x_n \geq 0$ for all $n \in \mathbb{N}$. If $x_n \to 0$ show $\sqrt{x_n} \to 0$.

Proof. Choose a $\epsilon > 0$. Define $\omega = \epsilon^2$. Note that $x_n \to 0$ implies that there exists a N such that for all $n \geq N$, $|x_n| < \omega$. Note that $|x_n| < \omega = \epsilon^2$ implies $\sqrt{|x_n|} < \epsilon$ witch means $|\sqrt{x_n} - 0| < \epsilon$ for all $n \geq N$. Thus by definition $\sqrt{x_n} \to 0$. □

Exercise : 2.3.1(b)
Let $x_n \geq 0$ for all $n \in \mathbb{N}$. If $x_n \to x$ show $\sqrt{x_n} \to \sqrt{x}$.

Proof. Assume $x \neq 0$ since we have already proven the statement true in that case, further note that since the sequence is bounded below by 0, $x \geq 0$
Choose a $\epsilon > 0$. Define $\omega = \epsilon\sqrt{x} > 0$. Note that $x_n \to x$ implies that there exists a N such that for all $n \geq N$, $|x_n - x| < \omega$. Note that $|x_n - x| < \omega = \epsilon\sqrt{x}$ implies $|x - x_n| < \epsilon\sqrt{x}$, $|\sqrt{x} + \sqrt{x_n}||\sqrt{x} - \sqrt{x_n}| < \epsilon\sqrt{x}$. Noting that $|\sqrt{x} + \sqrt{x_n}| = \sqrt{x} + \sqrt{x_n} \geq \sqrt{x}$ implies $\sqrt{x}|\sqrt{x} - \sqrt{x_n}| \leq |\sqrt{x} + \sqrt{x_n}||\sqrt{x} - \sqrt{x_n}| < \epsilon\sqrt{x}$. Thus $|\sqrt{x} - \sqrt{x_n}| < \epsilon$ for all $n \geq N$. By definition $\sqrt{x_n} \to \sqrt{x}$. □

Exercise : 2.3.3
Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and $\lim x_n = \lim z_n = l$, then $\lim y_n = l$.

For this proof I need the therm that $|a| < b \Leftrightarrow -b < a < b$ where $b > 0$.

Proof. There are two cases $a \geq 0$ or $a < 0$.
Case $a \geq 0$. In this case $|a| = a$ and our statement becomes $0 \leq a < b \Leftrightarrow -b < a < b$ witch is clearly true.
Case $a < 0$. In this case $|a| = -a$ and our statement becomes $0 \leq -a < b \Leftrightarrow -b < a < b$ witch,noting that $0 \leq -a < b$ is equivalent to $0 \geq a > -b$, is clearly true. □

Proof. Suppose $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and $\lim x_n = \lim z_n = l$.
Choose $\epsilon > 0$. By the definition of limit of a sequence we know that there exists a N_1 such that for all $n \geq N_1$, $|x_n - l| < \epsilon$. By the definition of limit of a sequence we know that there exists a N_2 such that for all $n \geq N_2$, $|z_n - l| < \epsilon$. Define $N = \max(N_1, N_2)$. Note that for all $n \geq N$, $|x_n - l| < \epsilon$ and $|z_n - l| < \epsilon$ or by the above therm $-\epsilon < x_n - l < \epsilon$ and $-\epsilon < z_n - l < \epsilon$. Note that $-\epsilon < x_n - l \leq y_n - l \leq z_n - l < \epsilon$ thus $-\epsilon < y_n - l < \epsilon$ and so $|y_n - l| < \epsilon$ for all $n > N$. Thus by the definition of the limit of a sequence $\lim y_n = l$. □

Exercise : 2.3.6
Find what $b_n = n - \sqrt{n^2 + 2n}$ converges to.

Proof. Note that $b_n = \frac{(n-\sqrt{n^2+2n})(n+\sqrt{n^2+2n})}{n+\sqrt{n^2+2n}} = \frac{-2n}{n+\sqrt{n^2+2n}} = \frac{-2}{1+\sqrt{1+2/n}} = \frac{a_n}{c_n}$, where $a_n = -2$ and $c_n = 1 + \sqrt{1 + 2/n} = d_n + e_n$, where $d_n = 1$ and $e_n = \sqrt{1 + 2/n} = \sqrt{f_n}$, where $f_n = 1 + 2/n$. Noting that $1/n \to 0$ we see that $f_n \to 1$. Using 2.3.1 we see that $e_n \to \sqrt{1} = 1$. By inspection $d_n \to 1$ and so by the algebraic limit therm $c_n \to 2$. Noting that $a_n \to -2$ and that $c_n \nrightarrow 0$ we see by the algebraic limit therm $b_n \to \frac{-2}{2} = -1$. □

Exercise : 2.3.9(a)
Let (a_n) be a bounded sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the algebraic limit therm to do this?

Firstly this is outside of the algebraic limit therm entirely since we are not guaranteed that a_n has a limit.

Proof. Suppose (a_n) to be a bounded sequence, and $\lim b_n = 0$.
Choose $\epsilon > 0$. Since (a_n) is bounded there exists a $M > 0$ such that $|a_n| \leq M$ for all n. By the definition of limit there exists a N such that for all $n \geq N$, $|b_n| < \epsilon/M$, since $\epsilon/M > 0$. Note that $|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| \leq M|b_n| < \epsilon$ for all $n \geq N$, thus by the definition of limit $\lim(a_n b_n) = 0$. □

Exercise : 2.3.10(a)
If $\lim(a_n - b_n) = 0$ then $\lim a_n = \lim b_n$.

Counterexample, consider the case $a_n = b_n = n$. In this case $\lim(a_n - b_n) = \lim(0) = 0$. However $\lim a_n = \lim n$ witch does not exist and so the statement $\lim a_n = \lim b_n$ cannot be true.

Exercise : 2.3.10(b)
If $\lim b_n = b$ then $\lim |b_n| = |b|$.

Proof. Choose $\epsilon > 0$. By the definition of limit there exists a N such that for all $n \geq N$, $|b_n - b| < \epsilon$. Recall that we proved on the first homework that $||a| - |b|| \leq |a - b|$, thus $||b_n| - |b|| \leq |b_n - b| < \epsilon$ for all $n \geq N$. By the definition of limit $\lim |b_n| = |b|$. □

Exercise : 2.3.10(c)
If $\lim a_n = a$ and $\lim(b_n - a_n) = 0$ then $\lim b_n = a$.

Proof. Define $s_n = b_n - a_n$. Note that $s_n \to 0$ and $a_n \to a$. By the algebraic limit therm $b_n = (s_n + a_n) \to a + 0 = a$ thus $b_n \to a$. □

Exercise : 2.3.10(d)
If $a_n \to 0$ and $|b_n - b| \leq a_n$ for all n then $b_n \to b$.

Proof. Suppose $a_n \to 0$ and $|b_n - b| \leq a_n$ for all n. Note that $0 \leq |b_n - b| \leq a_n$ thus $|a_n| = a_n$. Choose $\epsilon > 0$. By the definition of limit there exists a N such that for all $n \geq N$, $|a_n| < \epsilon$. Note that $|b_n - b| \leq a_n = |a_n| < \epsilon$ for all $n \geq N$. Thus by the definition of limit $b_n \to b$. □

Exercise : 2.4.1(a)
Prove that the sequence $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

Proof. I will use the monotone convergence therm. So what I need to show is that our sequence is bounded and that our sequence is monotonic.

Suppose $0 \leq x_n \leq 3$. Note that $-0 \geq -x_n \geq -3$, $4 \geq 4 - x_n \geq 1$, and since $4 - x_n \geq 1 > 0$ we see $0 \leq 1/4 \leq 1/(4 - x_n) \leq 1 \leq 3$. Thus $0 \leq x_{n+1} \leq 3$. Noting that $0 \leq x_1 = 3 \leq 3$, we conclude by induction that all x_n are in $0 \leq x_n \leq 3$. Thus $|x_n| \leq 3$ for all n and so the sequence is bounded.

I will prove the sequence is monotonic decreasing by induction.
In the base case is $x_n \geq x_{n+1}$? Well that would be, for $n = 1$, $3 \geq \frac{1}{4-3} = 1$. So it is monotonic decreasing in the base case.
Suppose $x_n \geq x_{n+1}$. Note that $x_n \geq x_{n+1}$, $-x_n \leq -x_{n+1}$, $4 - x_n \leq 4 - x_{n+1}$, and noting that $4 - x_n \geq 1 > 0$ since the sequence is bounded by 3, $\frac{1}{4-x_n} = x_{n+1} \geq \frac{1}{4-x_{n+1}} = x_{n+2}$. So I have shown that if $x_n \geq x_{n+1}$ we can conclude that $x_{n+1} \geq x_{n+2}$, and so by induction I conclude that $x_n \geq x_{n+1}$ for all n and thus the sequence is monotonic decreasing.

By the monotone convergence therm we can conclude that the sequence converges. □

Exercise : 2.4.1(b)
Given the sequence $x_n \to l$. Prove that the sequence $s_n = x_{n+1} \to l$

Proof. Choose $\epsilon > 0$. By the definition of limit there exists a N such that for all $m \geq N$, $|x_m - l| < \epsilon$. Choose $n \geq N$ let $m = n + 1 \geq N$. Note that $|x_m - l| < \epsilon$, $|x_{n+1} - l| < \epsilon$, $|s_n - l| < \epsilon$. Thus by the definition of limit $s_n \to l$. □

Exercise : 2.4.1(c)
Given the sequence $x_n \to l$ and $s_n = x_{n+1} \to l$, where $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

find l.

Proof. Note that $x_n - s_n \to l - l = 0$ by the arithmetic limit therm. Choose $\epsilon > 0$. By the definition of limit there exists a N such that for all $n \geq N$, $|x_n - s_n - 0| < 1/14\epsilon$. Note that

$$|x_n - s_n| =$$

$$|x_n - x_{n+1}| =$$

$$|x_n - \frac{1}{4 - x_n}| =$$

$$|\frac{x_n(4 - x_n) - 1}{4 - x_n}| =$$

$$\frac{| - x_n^2 + 4x_n - 1|}{|4 - x_n|}$$

Recalling that x_n is bounded by 3 we get

$$-3 \leq x_n \leq 3$$

$$4 + 3 = 7 \geq 4 - x_n \geq 4 - 3 = 1$$

and so $0 < |4 - x_n| \leq 7$ thus

$$\frac{| - x_n^2 + 4x_n - 1|}{|4 - x_n|} \geq 1/7| - x_n^2 + 4x_n - 1|$$

Note

$$1/7| - x_n^2 + 4x_n - 1| =$$

$$1/7| - (x_n - (2 + \sqrt{3}))(x_n - (2 - \sqrt{3}))| =$$

$$1/7|x_n - (2 + \sqrt{3})||x_n - (2 - \sqrt{3})|$$

via the quadratic equation.
Recalling that x_n is bounded by 3 we get

$$-3 \leq x_n \leq 3$$

$$6.5 \geq 3.5 - x_n \geq .5$$

Noting that $2 + \sqrt{3} > 3.5$ we get

$$|x_n - (2 + \sqrt{3})| = |2 + \sqrt{3} - x_n| = 2 + \sqrt{3} - x_n \geq 3.5 - x_n \geq 1/2$$

It follows that

$$1/7|x_n - (2 + \sqrt{3})||x_n - (2 - \sqrt{3})| \geq 1/14|x_n - (2 - \sqrt{3})|$$

Combining all of our inequalities we get $1/14|x_n - (2 - \sqrt{3})| \leq |x_n - s_n| < 1/14\epsilon$, or $|x_n - (2 - \sqrt{3})| < \epsilon$ for all $n \geq N$. By the definition of limit we can say $x_n \to 2 - \sqrt{3}$. □

Exercise : 2.4.3(a)
Show that

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},...$$

converges and find it's limit.

We can formalize this sequence as $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2+x_n}$.

In the case that $n = 1$, $1 \leq x_n \leq 2$ since $1 \leq \sqrt{2} \leq 2$
Suppose $1 \leq x_n \leq 2$. Note that

$$3 = 2+1 \leq 2+x_n \leq 2+2 = 4$$

$$1 \leq \sqrt{3} \leq \sqrt{2+x_n} \leq \sqrt{4} = 2$$

$$1 \leq x_{n+1} \leq 2$$

We can conclude by induction that $1 \leq x_n \leq 2$ for all n. And so our sequence is bounded.

In the case that $n = 1$, $x_n \leq x_{n+1}$ since $\sqrt{2} \leq \sqrt{2+\sqrt{2}}$.
Suppose $x_n \leq x_{n+1}$. Note that

$$1 \leq x_n \leq x_{n+1} \leq 2$$

$$3 \leq 2+x_n \leq 2+x_{n+1} \leq 4$$

$$\sqrt{3} \leq \sqrt{2+x_n} \leq \sqrt{2+x_{n+1}} \leq 2$$

$$\sqrt{2+x_n} \leq \sqrt{2+x_{n+1}}$$

$$x_{n+1} \leq x_{n+2}$$

We can conclude by induction that $x_n \leq x_{n+1}$ for all n. And so our sequence is monotone increasing.

This sequence is bounded and monotone therefore it has a limit lets call it l.

Define a new set $s_n = x_{n+1}$ note that $x_n \to l$ and $s_n \to l$ (from 2.4.1(b)).
Note that $(x_n - s_n) \to l - l = 0$. Choose $\epsilon > 0$. By the definition of limit there exists a N such that for all $n \geq N$, $|x_n - s_n - 0| < 1/4\epsilon$. Choose some $n > N$. Note that $1 \leq x_n \leq 2$ and $1 \leq s_n = x_{n+1} \leq 2$ thus $2 \leq x_n + s_n = |x_n + s_n| \leq 4$ and so $|x_n^2 - s_n^2| = |x_n - s_n||x_n + s_n| < \epsilon$. Note that

$$|x_n^2 - s_n^2| < \epsilon$$

$$|x_n^2 - 2 - x_n| < \epsilon$$

$$|x_n - 2||x_n + 1| < \epsilon$$

Note that $1 \leq x_n \leq 2$ thus $2 \leq x_n + 1 \leq 3$ and $x_n + 1 = |x_n + 1|$ so

$$2|x_n - 2| \leq |x_n - 2||x_n + 1|$$

$$2|x_n - 2| < \epsilon$$

$$|x_n - 2| < \epsilon$$

By the definition of limit $x_n \to 2$.

Exercise : 2.4.3(b)
Show that

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, ...$$

converges and find it's limit.

We can formalize this sequence as $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$.

In the case that $n = 1$, $1 \leq x_n \leq 2$ since $1 \leq \sqrt{2} \leq 2$
Suppose $1 \leq x_n \leq 2$. Note that

$$2 \leq 2x_n \leq 4$$

$$1 \leq \sqrt{2} \leq \sqrt{2x_n} \leq \sqrt{4} = 2$$

$$1 \leq x_{n+1} \leq 2$$

We can conclude by induction that $1 \leq x_n \leq 2$ for all n. And so our sequence is bounded.

In the case that $n = 1$, $x_n \leq x_{n+1}$ since $\sqrt{2} \leq \sqrt{2\sqrt{2}}$.
Suppose $x_n \leq x_{n+1}$. Note that

$$1 \leq x_n \leq x_{n+1} \leq 2$$

$$2 \leq 2x_n \leq 2x_{n+1} \leq 4$$

$$\sqrt{2} \leq \sqrt{2x_n} \leq \sqrt{2x_{n+1}} \leq 2$$

$$\sqrt{2x_n} \leq \sqrt{2x_{n+1}}$$

$$x_{n+1} \leq x_{n+2}$$

We can conclude by induction that $x_n \leq x_{n+1}$ for all n. And so our sequence is monotone increasing.

This sequence is bounded and monotone therefore it has a limit lets call it l.

Define a new set $s_n = x_{n+1}$ note that $x_n \to l$ and $s_n \to l$ (from 2.4.1(b)).
Note that $(x_n - s_n) \to l - l = 0$. Choose $\epsilon > 0$. By the definition of limit there exists a N such that for all $n \geq N$, $|x_n - s_n - 0| < 1/4\epsilon$. Choose some $n > N$. Note that $1 \leq x_n \leq 2$ and $1 \leq s_n = x_{n+1} \leq 2$ thus $2 \leq x_n + s_n = |x_n + s_n| \leq 4$ and so $|x_n^2 - s_n^2| = |x_n - s_n||x_n + s_n| < \epsilon$. Note that

$$|x_n^2 - s_n^2| < \epsilon$$

$$|x_n^2 - 2x_n| < \epsilon$$

$$|x_n - 2||x_n| < \epsilon$$

Note that $1 \leq x_n$ so

$$|x_n - 2| \leq |x_n - 2||x_n|$$

$$|x_n - 2| < \epsilon$$

By the definition of limit $x_n \to 2$.

Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified.

Exercise : 2.3.5 **(W) (Hand this one in to David.)**
Let (x_n) and (y_n) be given, and define (z_n) to be the sequence $(x_1, y_1, x_2...)$. Prove that (z_n) is convergent if and only if $\lim x_n = \lim y_n$.

Proof. Note that we can formalize this definition as

$$z_n = \begin{cases} x_{(n+1)/2} & n \in \text{odds} \\ y_{n/2} & \text{otherwise} \end{cases}$$

We are asked in this proof to prove a "if and only if" statement, basically prove a double implication. I will break this up into proving two implications, first $\lim x_n = \lim y_n$ implies (z_n) is convergent, and second (z_n) is convergent implies $\lim x_n = \lim y_n$.

Suppose $\lim x_n = \lim y_n = l$.
Choose $\epsilon > 0$. By the definition of limit of a sequence we know that there exists a N_1 such that for all $n \geq N_1$, $|x_n - l| < \epsilon$. By the definition of limit of a sequence we know that there exists a N_2 such that for all $n \geq N_2$, $|y_n - l| < \epsilon$. Define $N = 2 * (\max(N_1, N_2))$. Choose $n \geq N$. There are two possibilities, eater $n \in$ odd or $n \notin$ odd.
Case $n \in$ odd. In this case $z_n = x_{(n+1)/2}$. Note that $(n+1)/2 > n/2 \geq N/2 \geq N_1$ and so $|z_n - l| = |x_{(n+1)/2} - l| < \epsilon$.
Case $n \notin$ odd. In this case $z_n = y_{n/2}$. Note that $n/2 \geq N/2 \geq N_2$ and so $|z_n - l| = |y_{n/2} - l| < \epsilon$.
So $|z_n - l| < \epsilon$ for all $n \geq N$ and thus z_n will converge.

Suppose (z_n) is convergent.
let $l = \lim z_n$.
Choose $\epsilon > 0$. By the definition of limit of a sequence we know that there exists a N such that for all $m \geq N$, $|z_m - l| < \epsilon$. Choose a $n \geq N$. let $m = 2n - 1$. Note that $m \geq n \geq N$ and that $m \in$ odds, so $z_m = x_{(m+1)/2} = x_n$. Since $m \geq N$, $|x_n - l| = |z_m - l| < \epsilon$. Thus x_n converges to l.
Choose $\epsilon > 0$. By the definition of limit of a sequence we know that there exists a N such that for all $m \geq N$, $|z_m - l| < \epsilon$. Choose a $n \geq N$. let $m = 2n$. Note that $m > n \geq N$ and that $m \notin$ odds, so $z_m = y_{m/2} = y_n$. Since $m \geq N$, $|y_n - l| = |z_m - l| < \epsilon$. Thus y_n converges to l.
We conclude that $\lim x_n = \lim y_n$.

We can now conclude (z_n) is convergent if and only if $\lim x_n = \lim y_n$.

□