

Exercise 1.2.5: Use the triangle inequality to establish the following inequalities:

(a) $|a - b| \leq |a| + |b|$.

Proof. Note that $|a - b| = |a + (-b)|$. By the triangle inequality we note that $|a + (-b)| \leq |a| + |-b|$. There are two possibilities either $b < 0$, $b = 0$, or $b > 0$. In the case that $b < 0$ we know that $-b > 0$ and from the definition of absolute value $|b| = -b$ and $|-b| = -b$ thus in this case $|b| = |-b|$. In the case that $b = 0$ we know that $-b = 0$ and from the definition of absolute value $|b| = b$ and $|-b| = b$ thus in this case $|b| = |-b|$. In the case that $b > 0$ we know that $-b < 0$ and from the definition of absolute value $|b| = b$ and $|-b| = -(-b) = b$ thus in this case $|b| = |-b|$. Thus in all cases $|b| = |-b|$ and so $|a| + |-b| = |a| + |b|$ thus $|a - b| \leq |a| + |b|$. \square

(b) $||a| - |b|| \leq |a - b|$.

Proof. Note that $|c| = |c - d + d|$ which by the triangle inequality means $|c| \leq |c - d| + |d|$ so $|c| - |d| \leq |c - d|$ for any c and d in \mathbb{R} . Consider $||a| - |b||$ noting that there are two possibilities, either $||a| - |b|| = |a| - |b|$ or $||a| - |b|| = -(|a| - |b|) = |b| - |a|$ by the definition of absolute value. In the case that $||a| - |b|| = |a| - |b|$ we see from the first statement that $||a| - |b|| = |a| - |b| \leq |a - b|$. In the second case $||a| - |b|| = |b| - |a| \leq |b - a|$ and we proved in the previous question that $|b - a| = |-(a - b)| = |a - b|$, thus in this case $||a| - |b|| \leq |a - b|$. Thus in all cases $||a| - |b|| \leq |a - b|$. \square

Exercise 1.2.6(b), (d): Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A ; that is, $f(A) = \{f(x) : x \in A\}$.

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

Suppose we let $A = \{1\}$ and $B = \{-1\}$. Consider the function $f(x) = x^2$. Note that $A \cap B = \emptyset$ and thus $f(A \cap B) = \emptyset$. Also note that $f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\}$ and since $\emptyset \neq \{1\}$ in this case $f(A \cap B) \neq f(A) \cap f(B)$.

(d) Form and prove a conjecture concerning $f(A \cup B)$ and $f(A) \cup f(B)$.

Conjecture $f(A \cup B) \subseteq f(A) \cup f(B)$

Proof. Choose some element y from the set $f(A \cup B)$. By our definition of evaluating a function on a set there must exist some element x in $A \cup B$ such that $f(x) = y$. By the definition of union $x \in A$ or $x \in B$. Thus $f(x) \in f(A)$ or $f(x) \in f(B)$ and so $y \in f(A)$ or $y \in f(B)$ which means by definition $y \in f(A) \cup f(B)$. Since we chose y arbitrarily from $f(A \cup B)$ and showed that y is in $f(A) \cup f(B)$ we can say by the definition of subset $f(A \cup B) \subseteq f(A) \cup f(B)$. \square

Exercise 1.2.8: Form the logical negation of each claim. Do not use the easy way out: "It is not the case that..." is not permitted

- (a) For all real numbers satisfying $a < b$, there exists $n \in \mathbb{N}$ such that $a + (1/n) < b$.
There exists real numbers satisfying $a < b$, that for all $n \in \mathbb{N}$, $a + (1/n) \geq b$.

- (b) Between every two distinct real numbers there is a rational number.
There exists two distinct real numbers where there is no rational number between them.
- (c) For all natural numbers $n \in \mathbb{N}$, \sqrt{n} is either a natural number or is an irrational number.
There exists some natural number $n \in \mathbb{N}$ where \sqrt{n} is not a natural number or an irrational number.
- (d) Given any real number $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ satisfying $n > x$.
There exists a real number $x \in \mathbb{R}$ where there is no $n \in \mathbb{N}$ satisfying $n > x$.

Exercise 1.2.9: Show that the sequence (x_1, x_2, x_3, \dots) defined in Example 1.2.7 is bounded above by 2. That is, show that for every $i \in \mathbb{N}$, $x_i \leq 2$.

Proof. We will proceed with a proof by induction on i .

In the base case $i = 1$ we are given $x_1 = 1$ since $1 \leq 2$ the statement $i \in \mathbb{N}$, $x_i \leq 2$ holds in the base case.

Suppose $x_i \leq 2$. Consider the next step x_{i+1} , note that by definition $x_{i+1} = (1/2)x_i + 1$. Note that $x_i \leq 2 \Rightarrow (1/2)x_i \leq (1/2)2 = 1 \Rightarrow (1/2)x_i + 1 \leq 1 + 1 = 2 \Rightarrow x_{i+1} \leq 2$. Thus by induction we conclude that for all $i \in \mathbb{N}$, $x_i \leq 2$. \square

Exercise 1.3.4: Assume that A and B are nonempty, bounded above, and satisfy $B \subseteq A$. Show that $\sup B \leq \sup A$.

Proof. Assume to the contrary, namely that there exists sets A and B that are nonempty, bounded above, and satisfy $B \subseteq A$. Further suppose that $\sup B > \sup A$. Let's define $\alpha = \sup A$. Suppose there is no element in B greater than α . By the definition of upper bound, α would be an upper bound to B , however $\sup B > \alpha$, a contradiction, thus our assumption that there is no element of B greater than α must be false, and there is some element of B greater than α . Let's take one of these elements with the property $\gamma \in B$ and $\gamma > \alpha$. Note that since $B \subseteq A$, $\gamma \in A$. Since α is an upper bound to A we know that every element of A is less than or equal to α thus $\gamma \leq \alpha$. Contradiction $\gamma > \alpha$ and $\gamma \leq \alpha$, thus our initial supposition that $\sup B > \sup A$ must be false and so we are forced to conclude $\sup B \leq \sup A$. \square

Exercise 1.3.5: Let A be bounded above and let $c \in \mathbb{R}$. Define the sets $c + A = \{a + c : a \in A\}$ and $cA = \{ca : a \in A\}$.

- (a) Show that $\sup(c + A) = c + \sup(A)$.
- (b) If $c \geq 0$, show that $\sup(cA) = c \sup(A)$.
- (c) Postulate a similar statement for $\sup(cA)$ when $c < 0$.

Proof (a). Let's start by defining $\alpha = \sup(A)$, $\beta = c + \sup(A)$. We will proceed by showing that β must have the two properties defining $\sup(c + A)$.

Suppose that there existed some $\gamma \in c + A$ where $\gamma > \beta$. Note that $\gamma - c \in A$ and that $\gamma - c > \beta - c = \alpha$. Contradiction, we have found a element in A , $\gamma - c$, that is greater than $\sup(A)$. We are forced to conclude the negation of our suposition and so conclude that there is no element in $c + A$ that is greater than β , and so β is a upper bound on $c + A$, the first condition on $\sup(c + A)$.

Suppose that there is a upper bound to $c + A$, lets call it λ , that is smaller than β . Note that $\lambda - c < \beta - c = \alpha$. Since α is larger than $\lambda - c$ we know from the definition of \sup that $\lambda - c$ is not a upper bound on A , therefore there must be at least one element in A greater than $\lambda - c$, lets call it τ . Since τ is in A $c + \tau$ is in $c + A$, and since $\tau > \lambda - c$, $\tau + c > \lambda$. Contradiction, λ is a upper bound on $c + A$ but there is a element in $c + A$, namely $\tau + c$, that is grater than λ . Thus we are forced to conclude the negation of our supposition, that all upper bounds on $c + A$ are grater than or equal to β .

β meets the definition of $\sup(c + A)$ and so $\beta = \sup(c + A)$ and $c + \sup(A) = \sup(c + A)$. \square

Proof (b). Firstly let me eliminate a special case, $c = 0$. In this case $cA = \{0\}$, by inspection $\sup(cA) = 0$ and also $c \sup(A) = 0 * \sup(A) = 0$. In this degenerate case it is clearly true that $c \sup(A) = \sup(cA)$. From here on I will work with $c > 0$. Note, in this proof I am taking advantage of the fact that deviding over a positive number across a inequality does not affect the inequality, that is why $c > 0$ is nessesary for this proof.

Lets start by defining $\alpha = \sup(A)$, $\beta = c \sup(A)$. We will procede by showing that β must have the two properties defining $\sup(cA)$.

Suppose that there existed some $\gamma \in cA$ where $\gamma > \beta$. Note that $\gamma/c \in A$ and that $\gamma/c > \beta/c = \alpha$. Contradiction, we have found a element in A , γ/c , that is greater than $\sup(A)$. We are forced to conclude the negation of our suposition and so conclude that there is no element in cA that is greater than β , and so β is a upper bound on cA , the first condition on $\sup(cA)$.

Suppose that there is a upper bound to cA , lets call it λ , that is smaller than β . Note that $\lambda/c < \beta/c = \alpha$. Since α is larger than λ/c we know from the definition of \sup that λ/c is not a upper bound on A , therefore there must be at least one element in A greater than λ/c , lets call it τ . Since τ is in A $c\tau$ is in cA , and since $\tau > \lambda/c$, $\tau c > \lambda$. Contradiction, λ is a upper bound on cA but there is a element in cA , namely τc , that is grater than λ . Thus we are forced to conclude the negation of our supposition, that all upper bounds on cA are grater than or equal to β .

β meets the definition of $\sup(cA)$ and so $\beta = \sup(cA)$ and $c \sup(A) = \sup(cA)$. \square

Statement for part (c):

The region A would be flipped across 0 and be magnified by a factor of $|c|$, thus $\sup(cA) = c \inf(A)$.

Exercise 1.3.6: Compute, without proof, the suprema and infima of the following sets.

- (a) $\{n \in \mathbb{N} : n^2 < 10\}$.
- (b) $\{n/(n+m) : n, m \in \mathbb{N}\}$.
- (c) $\{n/(2n+1) : n \in \mathbb{N}\}$.
- (d) $\{n/m : m, n \in \mathbb{N} \text{ with } m+n \leq 10\}$.

Solution:

- (a) $\sup = 3, \inf = 1$
- (b) $\sup = 1, \inf = 0$
- (c) $\sup = 1/2, \inf = 1/3$
- (d) $\sup = 1/9, \inf = 9$

Exercise 1.3.7: Prove that if a is an upper bound for A and if a is also an element of A , then $a = \sup A$.

Proof. □

Exercise 1.3.8: If $\sup A < \sup B$ then show that there exists an element $b \in B$ that is an upper bound for A .

Proof. □