Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified.

Exercise : Let A and B be nonempty sets that are bounded above. Suppose $\sup A < \sup B$. Prove that there is an element of B that is an upper bound for A.

Proof. Suppose A and B are nonempty sets that are bounded above. Further suppose $\sup A < \sup B$. Define $a = \sup A$ and $b = \sup B$. Note that a is less than the suppremum of B thus a is not a upper bound on B. Since a is not a upper bound on B there must exist at least one element of B grater than a, take one of these elements lets call it k, $k \in B$, $a < k$. Choose a arbitrary element $c \in A$. Since $a = \sup A$ we know that a is a upper bound on A therefore $c \le a < k$. Since we chose a arbitrary element from A and showed that it is less than k we can say that all elements in A are less then k thus k is a upper bound on A. □

Exercise : In class we proved that $\mathbb{N}^2$ is countably infinite. Use this fact and a proof by induction to show that $\mathbb{N}^n$ is countably infinite for every $n \in \mathbb{N}$.

Proof. We want to show that for every $n \in \mathbb{N}$, $\mathbb{N}^n$ is countably infinite. I will proceed with a proof by induction.
Base case $n = 1$. There is a bijective map, the identity map, mapping $\mathbb{N}^1 \to \mathbb{N}$. So the statement holds in the $n = 1$ case.
Suppose $\mathbb{N}^m$ is countably infinite for all $m \le n$ where $n \ge 1$. There must exist a bijective map from $\mathbb{N}^n \to \mathbb{N}$, the definition of countably infinite. Note that $\mathbb{N}^{n+1}$ can trivially be bijectively mapped to $\mathbb{N}^n \times \mathbb{N}$, by mapping the first term to $\mathbb{N}$ and the next terms to $\mathbb{N}^n$. Note that there exists a bijective map from $\mathbb{N}^2 \to \mathbb{N}$ since $\mathbb{N}^2$ is countably infinite. Note that we can bijectively map $\mathbb{N}^{n+1} \to \mathbb{N}^n \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}^2 \to \mathbb{N}$. Thus $\mathbb{N}^{n+1}$ is countably infinite.
By induction we can conclude that for every $n \in \mathbb{N}$, $\mathbb{N}^n$ is countably infinite. □

Exercise : Compute

$$\lim_{n\to\infty} \frac{3^n}{n!}$$

.

Define $a_n = 0$,$b_n = \frac{3^n}{n!}$,$c_n = \frac{3^n}{4^{n-4}}$. Note that for $1 \le n \le 4$, $b_n = \frac{3^n}{n!} \le 3^n \le \frac{3^n}{4^{n-4}} = c_n$. Note that for $n = 4$, $b_n \le c_n$. Suppose for some $n \ge 4$, $b_n \le c_n$. Note that $b_{n+1} = \frac{3^{n+1}}{(n+1)!} = \frac{3^n}{n!}\frac{3}{n+1} = b_n\frac{3}{n+1} \le c_n\frac{3}{n+1} \le c_n\frac{3}{4} = \frac{3^n}{4^{n-4}}\frac{3}{4} = \frac{3^{n+1}}{4^{(n+1)-4}} = c_{n+1}$. By induction I conclude that for all $n \ge 4$, $b_n \le c_n$. So for all $n \in \mathbb{N}$, $b_n \le c_n$. Also note that b_n is always positive and thus $a_n \le b_n$. Note that $a_n \to 0$. Note that c_n is bounded bellow by 0. Also note that $c_{n+1} = \frac{3}{4}c_n \le c_n$, so c_n is monotone decreasing and bounded below, by the MCT it must converge. Define l as the limit of c_n. Note that $c_{n+1} \to l$ also note that $c_{n+1} = \frac{3}{4}c_n \to \frac{3}{4}l$. So $l = \frac{3}{4}l$ therefore $l = 0$. By the squeeze theorem $\frac{3^n}{n!} \to 0$.

Exercise : Suppose F is a collection of open intervals such that if $I, J \in F$ and $I \neq J$, then $I \cap J = \emptyset$. Prove that F is countable.

Proof. Suppose F is a collection of open intervals such that if $I, J \in F$ and $I \neq J$, then $I \cap J = \emptyset$.
If F is finite then it is at most countable.
Suppose F is non-finite. Select one element of F, let's call it d (for default). Consider the mapping $f : \mathbb{Q} \to G$ where $G = P(F)$, the power set of F, so that G is the set of all subsets of F.

$$f(q) = \begin{cases} \{f \in F : q \in f\} & \{f \in F : q \in f\} \neq \emptyset \\ \{d\} & \text{otherwise} \end{cases}$$

Suppose there existed a q such that $f(q)$ did not have cardinality 1. Note that $f(q) \neq \emptyset$, since we map anything that would have mapped to the empty set to the set containing the default set. Thus $f(q)$ must have at least two elements $I, J \in f(q)$. Also note that $f(q) \neq \{d\}$ since that has cardinality of one, $f(q) = \{f \in F : q \in f\}$. Note that by our above construction $I, J \in F$ and $I \neq J$ and $q \in I$, $q \in J$, since $f(q) \neq \{d\}$. Thus $q \in I \cap J$ so $I \cap J \neq \emptyset$. This is a contradiction since our initial supposition tells us $I \cap J = \emptyset$, we conclude the negation of our supposition, that $f(q)$ has one element for all $q \in \mathbb{Q}$.
We can now construct a function $g : \mathbb{Q} \to F$, where $g(q)$ is the one element in $f(q)$, noting that $f(q) \in G$ means that $f(q) \subseteq F$ and thus the one element in $f(q)$ is a element of F.
Choose $I \in F$. Note that by the density of the rationals there is a rational in the open interval I, select one of these elements and call it q, $q \in I$ and $q \in \mathbb{Q}$. Note that $g(q)$ is the element in $f(q)$, and $I \in \{f \in F : q \in f\} \neq \emptyset$ thus $f(q) = \{f \in F : q \in f\}$ and $I \in f(q)$ thus $g(q) = I$. Since we chose a arbitrary element in F and found a $q \in \mathbb{Q}$ that maps to it via g we can say g is onto. We know that there is a onto map $h : \mathbb{N} \to \mathbb{Q}$ since $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality. Consider the map $m : \mathbb{N} \to F$ where $m(n) = g(h(n))$. Note that m is onto. Since there is a onto map from $\mathbb{N} \to F$ we know that F is at most countably infinite. □

Exercise : Let (x_n) be a sequence converging to L. Define

$$y_n = \frac{x_1 + \cdots + x_n}{n}$$

. That is, y_n is the average of the first n terms of the sequence of x_n. Show that $\lim y_n = L$ as well.

Proof. Suppose (x_n) be a sequence converging to L. Define

$$y_n = \frac{x_1 + \cdots + x_n}{n}$$

.
Choose $\epsilon > 0$. There exists a $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $|x_n - L| < \epsilon/2$. There are a finite number of terms of x_n with $n \leq N_0$, thus we can find a $n_0 \in [1, N_0]$ that makes $|x_{n_0} - L|$ a maximum. Define $b = |x_{n_0} - L|$. There exists a natural number $C \geq 2 * b * N_0/\epsilon$. Define

$N = \max(C, N_0 + 1)$, note that $N \in \mathbb{N}$ and $N \geq 2 * b * N_0/\epsilon$ and $N \geq N_0 + 1$. Choose $n > N$. Note that

$$|y_n - L| =$$

$$|\frac{x_1 + \cdots + x_n}{n} - L| \leq$$

(triangle inequality)

$$1/n \sum_{k=1}^{n} |x_k - L| =$$

(noting that $n \geq N \geq N_0 + 1$)

$$1/n \sum_{k=1}^{N_0} |x_k - L| + 1/n \sum_{k=N_0+1}^{n} |x_k - L| <$$

(in the range $N_0 \leq k$, $|x_k - L| < \epsilon/2$)

$$1/n \sum_{k=1}^{N_0} |x_k - L| + 1/n \sum_{k=N_0+1}^{n} \epsilon/2 \leq$$

($n \geq N \geq (2bN_0)/\epsilon$, $1/n \leq \epsilon/(2bN_0)$)

$$\epsilon/(2bN_0) \sum_{k=1}^{N_0} |x_k - L| + 1/n \sum_{k=N_0+1}^{n} \epsilon/2 \leq$$

(in the range $N_0 > k$,$|x_k - L| \leq b$)

$$\epsilon/(2bN_0) \sum_{k=1}^{N_0} b + 1/n \sum_{k=N_0+1}^{n} \epsilon/2 =$$

$$\epsilon/(2bN_0) * N_0 b + 1/n * (n - N_0)\epsilon/2 \leq$$

($n - N_0 < n$)

$$\epsilon/2 + \epsilon/2 = \epsilon$$

Thus $y_n \to L$. □

Exercise : Suppose that (a_n) is a sequence of positive numbers and that $\lim_{n\to\infty} a_n = L > 0$. Prove that there exists an $m > 0$ such that $a_n \geq m$ for all $n \in \mathbb{N}$.

Proof. Suppose that (a_n) is a sequence of positive numbers and that $\lim_{n\to\infty} a_n = L > 0$. Noting that $L/2 > 0$ we can say that there must exist a N such that for all $n > N$, $|a_n - L| < L/2$. There are a finite number of terms of a_n where $n < N$, so we can find the minimum of these terms, call it m_1. Noting that m_1 is a term in the sequence we can say $m_1 > 0$. Define $m = \min(m_1, L/2) > 0$. Choose an arbitrary element of the sequence a_n, call it a, and call its index n_0. If $n_0 < N$ then we know that $a \geq m_1 \geq m$. If $n_0 \geq N$ then we know that $|a - L| < L/2$ so $-L/2 < a - L$ so $L/2 < a$ thus $m \leq L/2 < a$. Since we chose a arbitrary element of the sequence and showed that it is grater than or equal to m we can say that m is less than or equal to all of the terms in the sequence. □

Exercise : Use the Bolzano-Weierstrass theorem to prove the Monotone Convergence Theorem without assuming any other form of the Axiom of Completeness.

Proof. Assume that every bounded sequence has a convergent sub sequence.
Consider a bounded monotone increasing sequence a_n. There exists a sub-sequence a_{m_j} that converges to l. Choose $\epsilon > 0$. There exists a $J \in \mathbb{N}$ such that for all $j \geq J$, $|a_{m_j} - l| < \epsilon$. Define $N = m_J$. Choose $n \geq N$. Note that we proved on a homework $n \leq m_n$. Note that $m_J \leq n \leq m_n$, since the sequence a_n is monotone increasing note that $a_{m_J} \leq a_n \leq a_{m_n}$. Note that $|a_{m_J} - l| < \epsilon$, $-\epsilon < a_{m_j} - l$. Also note that $n \geq N = m_J \geq J$ thus $|a_{m_n} - l| < \epsilon$, $a_{m_n} - l < \epsilon$. Note that $a_{m_J} \leq a_n \leq a_{m_n}$ means that $-\epsilon < a_{m_J} - l \leq a_n - l \leq a_{m_n} - l < \epsilon$ or $|a_n - l| < \epsilon$, for all $n \geq N$. Thus a_n converges on l. Since we chose a arbitrary bounded monotone increasing sequence and showed that it converged we can conclude that all bounded monotone increasing sequences converge.
Consider a bounded monotone decreasing sequence a_n. If a_n is a bounded monotone decreasing series note that $b_n = -a_n$ is monotone increasing and so $b_n \to l$. By the arithmetic limit theorem we can say $a_n = -b_n \to -l$. Thus all bounded monotone decreasing sequences converge.
We can now say all bounded monotone sequences converge (MCT). □

Exercise : Suppose (x_n) is a sequence and that for all $n \geq 2$,

$$|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$$

. Show that the sequence converges.

Proof. Suppose (x_n) is a sequence and that for all $n \geq 2$,

$$|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$$

Define $k = 2|x_2 - x_1|$. For $n = 1$, $|x_{n+1} - x_n| \leq (1/2)^n k$ would mean $|x_2 - x_1| \leq |x_2 - x_1|$, clearly true. Suppose $|x_{n+1} - x_n| \leq (1/2)^n k$. Note that $|x_{n+2} - x_{n+1}| \leq (1/2)|x_{n+1} - x_n|$, so $|x_{n+2} - x_{n+1}| \leq (1/2)^{n+1}k$. By induction $|x_{n+1} - x_n| \leq (1/2)^n k$ for all natural numbers n.

Note that for $m = 1$, $|x_{n+m} - x_n| \leq (1/2)^n k2 \sum_{i=1}^{m}(1/2)^i$ would mean $|x_{n+1} - x_n| \leq (1/2)^n k$, clearly true. Suppose for some m, $|x_{n+m} - x_n| \leq (1/2)^n k2 \sum_{i=1}^{m}(1/2)^i$. Note that $|x_{n+m+1} - x_{n+m}| \leq (1/2)^{n+m}k$. Note that $|x_{n+m+1} - x_n| = |x_{n+m+1} - x_{n+m} + x_{n+m} - x_n| \leq |x_{n+m+1} - x_{n+m}| + |x_{n+m} - x_n| \leq (1/2)^n k2 \sum_{i=1}^{m}(1/2)^i + (1/2)^{n+m}k = (1/2)^n k2(\sum_{i=1}^{m}(1/2)^i + (1/2)^{m+1}) = (1/2)^n k2 \sum_{i=1}^{m+1}(1/2)^i$. By induction on m I conclude $|x_{n+m} - x_n| \leq (1/2)^n k2 \sum_{i=1}^{m}(1/2)^i$ for all natural numbers m.

Note that $\sum_{i=1}^{m}(1/2)^i \leq \sum_{i=1}^{\infty}(1/2)^i = \sum_{i=0}^{\infty}(1/2)^i - 1 = 2 - 1 = 1$ (see geometric series Pg. 73). Thus $|x_{n+m} - x_n| \leq (1/2)^n k2$ for all m and n.

Choose $\epsilon > 0$. Note that there exists a natural number $N > 2k/\epsilon$, choose one of these and set it equal to N. Choose $m > n \geq N$. Define $d = m - n$, note that $d \in \mathbb{N}$. Note that $2^n \geq n \geq N > 2k/\epsilon$ so $(1/2)^n = 1/2^n < \epsilon/(2k)$ and $(1/2)^n k2 < \epsilon$. Note that $|x_m - x_n| = |x_{n+d} - x_n| \leq (1/2)^n k2 < \epsilon$. We now know the sequence is Caushey and therefore it will converge. $\square$

Exercise : Let (a_n) and (b_n) be sequences with $b_n \geq 0$ for all n and $b_n \to 0$. We say that $a_n = O(b_n)$ if there is a constant C such that $|a_n| \leq Cb_n$ for all n. Roughly speaking, $a_n = O(b_n)$ if the sequence an converges to zero at least as fast as the sequence b_n. Suppose a_n and b_n are sequences with $b_n > 0$. Suppose also that $\frac{a_n}{b_n} \to L$ for some number L. Prove that $a_n = O(b_n)$.

There must exist a N such that for all $n \geq N$, $|\frac{a_n}{b_n} - L| < 1$. Noting that there are a finite number of elements of $\frac{a_n}{b_n}$ where $n \leq N$, we can find the minimum, call it $A_{\min}$, and maximum, call it $A_{\max}$, for these elements. Define $C_{\min} = \min(A_{\min}, L-1)$ and $C_{\max} = \max(A_{\max}, L+1)$. Choose a arbitrary element of $b = \frac{a_n}{b_n}$ with index n. If $n \leq N$ we know that $C_{\min} \leq A_{\min} \leq b \leq A_{\max} \leq C_{\max}$. If $n \geq N$ we know that $C_{\min} \leq L-1 \leq b \leq L+1 \leq C_{\max}$. So $C_{\min} \leq \frac{a_n}{b_n} \leq C_{\max}$ for all n. Note that $|\frac{a_n}{b_n}| \leq \max(C_{\max}, -C_{\min}) = C$, so $|a_n| \leq C|b_n| = Cb_n$ therefore $a_n = O(b_n)$.

Exercise : Suppose (a_n) and (b_n) are sequences with $b_n \geq 0$ and $a_n = O(b_n)$.

a) Suppose that $\sum b_n$ converges on l. Prove that $\sum a_n$ converges also.
Note that there exists a constant C such that $|a_n| \leq Cb_n$ for all n. Define $SAa_n = \sum_{k=1}^n |a_k|$ the partial sum of the absolute terms of a. Define $Sb_n = \sum_{k=1}^n b_k$. Note that SAa_n is monotonic increasing. Note that Sb_n is monotonic. Note that $SAa_n \leq Sb_n \leq l$. Thus SAa_n is bounded above and monotonic increasing, thus it converges. Since $\sum a_n$ converges absolutely it converges.

b) Suppose that $\sum a_n$ diverges. Prove that $\sum b_n$ diverges.
Suppose to the contrary that $\sum a_n$ diverges while that $\sum b_n$ converges. Since $\sum b_n$ converges we know that $\sum a_n$ converges, a contradiction. We are forced to conclude if $\sum a_n$, $\sum b_n$ diverges.

c) Determine if $\sum_{n=1}^{\infty} \sqrt{\frac{n^3-3n+2}{8n^4+n^2+22}}$ converges
Define $b_n = \sqrt{\frac{n^3-3n+2}{8n^4+n^2+22}}$. Note that b_n is the result of a square root, thus $b_n \geq 0$. Define $c_n = \frac{n^3-3n+2}{8n^4+n^2+22}$. Note that $c_n = \frac{1/n-3/n^3+2/n^4}{8+1/n^2+22/n^4} \to 0$ by the algebraic limit therm, and therefore $b_n \to 0$. Define $a_n = 1/n$. Note that assuming $n > 1$ ($b_n \neq 0$), $|\frac{a_n}{b_n}| = \sqrt{1/n^2 \frac{8n^4+n^2+22}{n^3-3n+2}} = \sqrt{\frac{8n^2+1+22/n^2}{n^3-3n+2}} = \sqrt{\frac{8/n+1/n^3+22/n^5}{1-3/n^2+2/n^3}} \to 0$. We now can say based on exercise 9 that $a_n = O(b_n)$. We also know that $\sum a_n$ diverges and so now can conclude that b_n will diverge, $\sum_{n=1}^{\infty} \sqrt{\frac{n^3-3n+2}{8n^4+n^2+22}}$ diverges.