

Exercise 1.4.1: Recall that \mathbb{I} stands for the set of irrational numbers.

- (a) Show that if $a, b \in \mathbb{Q}$ then ab and $a + b \in \mathbb{Q}$ as well.
- (b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$ and if $a \neq 0$ then $at \in \mathbb{I}$ as well.
- (c) Part (a) says that the rational numbers are closed under multiplication and addition. What can be said about st and $s + t$ when $s, t \in \mathbb{I}$?
- (a) *Proof.* Select two arbitrary elements from the rational numbers, since they are rational we can represent them as i/j and m/n where $i, j, m, n \in \mathbb{Z}$ and $j \neq 0$ and $n \neq 0$.

Note that $i/j * m/n = \frac{im}{jn}$, from the definition of multiplication of rational numbers. Since the multiple of any two non-zero numbers is non-zero and since the multiple of any two integers is a integer so $jn \in \mathbb{Z} - \{0\}$ and $im \in \mathbb{Z}$ therefore $\frac{im}{jn} \in \mathbb{Q}$.

Note that $i/j + m/n = \frac{in+mj}{jn}$, from the definition of addition of rational numbers. Since the multiple of any two non-zero numbers is non-zero and since the multiple of any two integers is a integer so $jn \in \mathbb{Z} - \{0\}$ and $in, mj \in \mathbb{Z}$ and so also $in + mj \in \mathbb{Z}$ therefore $\frac{in+mj}{jn} \in \mathbb{Q}$. \square

- (b) *Proof.* Proof by contradiction.

Suppose that there exists $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ where $a + t = b \notin \mathbb{I}$. Since the reals are closed under addition we can say $b \in \mathbb{R}$. Note that $b \in \mathbb{R} - \mathbb{I} = \mathbb{Q}$. We can do a little math and see $a + t = b$ means $t = b + (-a)$. Since the additive inverse of a rational is a rational and since the sum of two rationals is rational we conclude $t \in \mathbb{Q}$. A contradiction has been reached, the rationals and irrationals are, by definition, mutually exclusive and so t cannot be a element of both.

Proof by contradiction.

Suppose that there exists $a \in \mathbb{Q} - \{0\}$ and $t \in \mathbb{I}$ where $at = b \notin \mathbb{I}$. Since the reals are closed under multiplication, and since the multiple of any two non zero numbers is itself non zero, we can say $b \in \mathbb{R} - \{0\}$. Note that $b \in \mathbb{R} - \{0\} - \mathbb{I} = \mathbb{Q} - \{0\}$. We can do a little math and see $at = b$ means $t = b(a^{-1})$. Since the multiplicative inverse of a non zero rational is a rational (informally $(\frac{i}{m})(\frac{m}{i}) = 1$) and since the multiple of two rationals is rational we conclude $t \in \mathbb{Q}$. A contradiction has been reached, the rationals and irrationals are, by definition, mutually exclusive and so t cannot be a element of both. \square

- (c) All we can conclude is that $st \in \mathbb{R} - \{0\}$ and that $s + t \in \mathbb{R}$. As a example that the irrationals are not closed with respect to multiplication or addition note that $\sqrt{2}\sqrt{2} = 2$ and that $\pi + (-\pi) = 0$.

Exercise 1.4.2: Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + (1/n)$ is an upper bound for A but $s - (1/n)$ is not an upper bound for A . Show that $s = \sup A$.

Proof. Take some $a \in \mathbb{R}$ where $s < a$. Note $a - s \in \mathbb{R}^+$ and so by the proof we previously completed in class there is a natural number n with the property $\frac{1}{n} < a - s$. So $s + \frac{1}{n} < a$ which means a is bigger than an upper bound on A and so is itself not in A . Since we chose an arbitrary real greater than s and showed it is not in A we can conclude no real bigger than s is in A , in other words all elements of A are less than or equal to s . By definition s is an upper bound on A .

Next take some $b \in \mathbb{R}$ where $b < a$. Note $s - b \in \mathbb{R}^+$ and so by the proof we previously completed in class there is a natural number n with the property $\frac{1}{n} < s - b$. So $b < s - \frac{1}{n}$ which means b is smaller than a number that is not an upper bound on A . Noting that if b were an upper bound on A we would be forced to conclude $s - \frac{1}{n}$ is an upper bound on A , which we know to be false, we are forced to conclude b is not an upper bound on A . and since b was chosen arbitrarily we can conclude all upper bounds on A are greater than or equal to s . Noting that s has both properties of a supremum we conclude $s = \sup A$. \square

Exercise 1.4.3: Show that $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$.

Proof. Proof by contradiction.

Suppose $a \in \bigcap_{n=1}^{\infty} (0, 1/n)$. Note that $a \in (0, 1) \subseteq \mathbb{R}^+$ so $a \in \mathbb{R}^+$. By the proof done in class there exists a natural i with the property $1/i < a$. Since a is in the intersection of all of the $(0, 1/n)$ sets $a \in (0, 1/i)$ so $a < 1/i$ a contradiction. We now conclude the negation of our supposition namely there is no a with the property $a \in \bigcap_{n=1}^{\infty} (0, 1/n)$ or a logically equivalent statement $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$. \square

Exercise 1.4.4: (W) (Hand this one in to David.)

Let $a < b$ be real numbers and let $T = [a, b] \cap \mathbb{Q}$. Show that $\sup T = b$.

Proof. First we note that b is an upper bound on T , since $T \subseteq [a, b]$ and every element of $[a, b]$ is less than or equal to b .

Take some real number $c < b$. Note that $b - c \in \mathbb{R}^+$ and thus there exists a $n \in \mathbb{N}$ where $1/n < b - c$. \square

Exercise 1.4.5: Use Exercise 1.4.1 to provide a proof of Corollary 1.4.4 by considering real numbers $a = \sqrt{2}$ and $b = \sqrt{2}$.

Proof. \square

Exercise Supplemental 1: Show that the sets $[0, 1)$ and $(0, 1)$ have the same cardinality.