Exercise 1.4.7: Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ contradicts the assumption that $\alpha = \sup A$.

Proof. Suppose for the purpose of contradiction that $\alpha^2 > 2$. Note that $2\alpha \in \mathbb{R}^+$ and $\alpha^2 - 2 \in \mathbb{R}^+$ thus $\frac{\alpha^2-2}{2\alpha} \in \mathbb{R}^+$. There exists a natural number n such that $\frac{1}{n} < \frac{\alpha^2-2}{2\alpha}$.

$$\frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha}$$

$$\frac{2\alpha n - 1}{n^2} < \frac{2\alpha n}{n^2} < \alpha^2 - 2$$

Noting that $2\alpha n - 1 > 0$ and $n^2 > 0$ we conclude $\frac{2\alpha n-1}{n^2} > 0$.

$$0 < \frac{2\alpha n - 1}{n^2} < \alpha^2 - 2$$

$$0 < \frac{2\alpha}{n} - \frac{1}{n^2} < \alpha^2 - 2$$

$$0 > -\frac{2\alpha}{n} + \frac{1}{n^2} > -\alpha^2 + 2$$

$$\alpha^2 > \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > 2$$

$$\alpha^2 > (\alpha - 1/n)^2 > 2$$

Note that $(\alpha - 1/n)^2 > 2 > t^2$ where $t \in A$ thus $\alpha - 1/n > t$ and so $\alpha - 1/n$ is a upper bound on A. Also note that $\alpha > \alpha - 1/n$ a contradiction we have found a upper bound on A less than $\sup A$. □

Exercise Supplemental 1: Suppose for each $k \in \mathbb{N}$ that A_k is at most countable. Use the fact that $\mathbb{N} \times \mathbb{N}$ is countably infinite to show that $\cup_{k=1}^{\infty} A_k$ is at most countable. Hint: take advantage of surjections.

Proof. First let me define a new set B where $B = \{X \in A; X \neq \emptyset\}$. Note that $\cup B = \cup_{k=1}^{\infty} A_k$. Lets next deal with the case that $B = \emptyset$ in this case $\cup B = \emptyset$ and so is at most countable infinite. Next lets consider the case that B has a finite number of elements, we proved this case in class, a union of a finite number of at most countably infinite sets is at most countably infinite. Now we know that we are dealing with B a infinite set of at most countably infinite non-empty sets. Now I will introduce the notation $B_{k,l}$ where $B_{k,l}$ is the l element of B_k. Consider the function $f : \mathbb{N} \times \mathbb{N} \to B_{k,l}$ where

$$f(a,b) = \begin{cases} B_{a,b} & B_a \text{ has a } b\text{th element} \\ B_{1,1} & \text{otherwise} \end{cases}$$

Note that this function is surjective, since given a $B_{j,k}$ we see that (j,k) maps to it. There is also a surjection between each of our $B_{j,k}$ and $\cup B$ simply map the element $B_{j,k}$ to itself in $\cup B$. From knowing that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality, I conclude that there is a surjection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. Thus I can surjectively map $\mathbb{N} \to \mathbb{N} \times \mathbb{N} \to B_{j,k} \to \cup B$. Thus $\cup B$ is at most countably infinite. □

Exercise Supplemental 2: **(W) (Hand this one in to David.)**

Suppose B is finite and $A \subseteq B$. Show that A is empty or finite.

Consider the case where $A \neq \emptyset$. There must exist a bijective function mapping $f : S_m \to B$, the definition of finite. Since $A \subseteq B$ there must be a subset of S_m, lets call it $S_m|_A$ that has the property $f(S_m|_A) = A$. Lets now consider the function $g : S_m|_A \to A$ where g(x)=f(x). Note that by construction g is onto, since $g(S_m|_A) = f(S_m|_A) = A$ and since f is one-to-one on B we can see $g(a) = g(b)$ implies $f(a) = f(b)$ implies $a = b$, and so g is one-to-one. Thus g is bijective. Since $S_m|_A \in \mathbb{N}$ it will have a least element. Construct a map $h : S_m|_A \to S_l$ where the minimum of $S_m|_A$ gets mapped to 1 and the next smallest gets mapped to 2 and so on until the last element maps to l. We can say that this procedure is possible since at most it could take m steps and m is finite. By construction this function is onto and one-to-one. We now have a bijection between $S_m|_A$ and S_l, notice that we can now bijectively map A to $S_m|_A$ and $S_m|_A$ to S_l thus by definition A is finite, or empty.

Exercise 1.5.10 (a) (c):

(a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(c) Determine, with proof, if the same statement remains true replacing uncountable with infinite.

Proof (a). We will proceed with a proof by contradiction. Suppose all sets $C \cap [a, 1]$ where $a \in (0, 1)$ is at most countably infinite. This implies $C \cap [1/n, 1]$ where $n \in \mathbb{N}$ is at most countably infinite, since $1/n \in (0, 1)$. Consider a new set $A = \cup_n [C \cap [1/n, 1]]$. As previously shown a union of at most countably infinite sets is at most countably infinite thus A is at most countably infinite. Consider an arbitrary element $b \in C - \{0\}$. Note that there exists $i \in \mathbb{N}$ such that $1/i \leq b$ since $b \in \mathbb{R}^+$. Thus $b \in C \cap [1/i, 1]$ and so $b \in \cup_n [C \cap [1/n, 1]] = A$. Since b was chosen arbitrarily from $C - \{0\}$ and shown to be in A we can conclude $C - \{0\} \subseteq A$. Noting that $A + \{0\}$ is at most countably infinite we conclude C is at most countably infinite, a contradiction. Thus there exists a $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable. □

Proof (b). authors note: By inspection it is definitely not true if we replace uncountable with infinite, but initial it is hard to see where the first proof breaks down. However examine the statement "a union of at most countably infinite sets is at most countably infinite". To transition this proof to the infinites instead of the uncountables we would need "a union of at most finite sets is at most finite", witch is definitely not true. For example $\cup_n \{n\}$ is a union of at most finite sets and is definitely non-finite.

It is not true that if $C \subseteq [0, 1]$ is infinite that there exists a $a \in (0, 1)$ where $C \cap [a, 1]$ is infinite, I will prove this with a counterexample.
Consider the set $C = \{1/n : n \in \mathbb{N}\}$. Note that $C \subseteq [0, 1]$ and that C is infinite, since it clearly has the same number of elements as $\mathbb{N}$. Suppose for the purpose of contradiction that there existed a $a \in (0, 1)$ where $C \cap [a, 1]$ is infinite. Note that $1/a \in (1, \infty)$ thus the

minimum natural number grater than a, guaranteed to exist since there is a natural number bigger than a given real and the naturals are well ordered, is not 1 and so the number one less than the minimum natural is a natural. In other words there exists a $i \in \mathbb{N}$ such that $i \leq 1/a < i+1$ or $1/i \geq a > 1/(i+1)$. Now we see that $C \cap [a, 1]$ is nothing other than $C - \{1/n : n > i, n \in \mathbb{N}\}$ or $\{1/n : n \in [1, i], n \in \mathbb{N}\} = \{1/n : n \in S_i\}$. This clearly has a bijective map with S_i and is therefore finite, a contradiction, thus there is no a that makes $C \cap [a, 1]$ infinite. □

Exercise Supplemental 3: Show that the set of a finite subsets of $\mathbb{N}$ is countably infinite. Hint: Let A_k be the set of all subsets of $\mathbb{N}$ with no more than k elements. Show that each A_k is countably infinite.

Proof. Define A_k to be the set of all subsets of $\mathbb{N}$ with no more than k elements.
I will proceed with induction on k.
In the base case $k = 1$. Note that $A_k = \{\{n\} : n \in \mathbb{N}\}$ in this case A_k is countably infinite since there is clearly a bijective map to $\mathbb{N}$, $\{n\} \to n$.
Suppose A_k is countably infinite. Define A_{kn} to be the nth element of A_k, we can order A_k since there exists a bijective map to $\mathbb{N}$. Define $B_{ni} = A_{kn} + \{i\}$, note that B_{ni} has at most $k+1$ elements since, by construction of A_k, A_{kn} has at most k elements. Note that $\{B_{ni} : n \in \mathbb{N}, i \in \mathbb{N}\}$ has a surjective map to $\mathbb{N} \times \mathbb{N}$ and thus a surjective map exists to $\mathbb{N}$ and so is at most countably infinite also note that $A_{k+1} = \{B_{ni} : n \in \mathbb{N}, i \in \mathbb{N}\}$, since all non-empty sets made up of $k+1$ or fewer elements are simply a non-empty set containing k or fewer elements adding in a new element (note that the new element could already exist in the set), thus A_{k+1} is at most countably infinite. Since $A_k \subseteq A_{k+1}$ we know A_{k+1} is not finite, thus A_{k+1} is countably infinite.
By induction all sets A_k are countably infinite □

Exercise 2.2.2: Verify using the definition of convergence the following limits.

(a) $\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$.

(c) $\lim_{n\to\infty} \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Proof (a). Chose an $\epsilon > 0$. Select $N \in \mathbb{N}$ such that $(\frac{3}{5\epsilon} - 4)/(5) < N$ this can be done since given a real number there is a natural bigger than it. Note that:

$$(\frac{3}{5\epsilon} - 4)/(5) < N$$

$$\frac{3}{5\epsilon} - 4 < 5N$$

$$\frac{3}{5\epsilon} < 5N + 4$$

$$\epsilon > \frac{3/5}{5N+4}$$

Chose a $n > N$. Note that $|\frac{2n+1}{5n+4} - \frac{2}{5}| = |\frac{2n+1-\frac{2}{5}(5n+4)}{5n+4}| = |\frac{1-\frac{8}{5}}{5n+4}| = |\frac{-3/5}{5n+4}| = \frac{3/5}{5n+4} \leq \frac{3/5}{5N+4} < \epsilon$. □

Proof (b). Chose an $\epsilon > 0$. Select $N \in \mathbb{N}$ such that $1/N < \epsilon/2$ this can be done since $\epsilon/2 > 0$. Chose a $n > N$. Note that $|\frac{2n^2}{n^3+3} - 0| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} < \frac{2}{N} < 2\epsilon/2 = \epsilon$. □

Proof (c). Chose an $\epsilon > 0$. Select $N \in \mathbb{N}$ such that $1/N < \epsilon^3$ this can be done since $\epsilon^3 > 0$. Chose a $n > N$. Note that $|\frac{\sin(n^2)}{\sqrt[3]{n}} - 0| \leq \frac{1}{\sqrt[3]{n}} < \frac{1}{\sqrt[3]{N}} = (\frac{1}{N})^{1/3} < (\epsilon^3)^{1/3} = \epsilon$. □

Exercise Supplemental 4: **(W) (Hand this one in to David.)**Carefully prove that the sequence (x_n) given by $x_n = (-1)^n$ does not converge.

Proof. Suppose that $x_n = (-1)^n$ converges to a. Let's let $\epsilon = .1$. from the definition of limit we know there exists a N such that for all $n > N$ $|(-1)^n - a| < \epsilon$. Take n_1 to be a odd natural number grater than N. Take $n_2 = n_1 + 1$. Note that $|(-1)^{n_1} - a| < \epsilon$ and $|(-1)^{n_2} - a| < \epsilon$, thus $|1 - a| < \epsilon$ and $|-1 - a| < \epsilon$. Consider two cases $a \geq 0$ and $a < 0$. In the case $a \geq 0$, $.1 = \epsilon > |-1-a| = 1 + a \geq 1$ in this case we have a contradiction $.1 > 1$. In the case $a < 0$, $.1 = \epsilon > |1 - a| = 1 - a \geq 1$ in this case we have a contradiction $.1 > 1$. Thus this series cannot converge. □