Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified. Also note that I am operating under a convention that $\sum = \sum_{n=1}^{\infty}$ and the convention that $\sum_i^j = \sum_{n=i}^j$.

Exercise : Prove the alternating series theorem. If $\{a_n\}$ is monotone decreasing sequence, $a_n \to 0$, and $a_n \geq 0$. Then

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

converges.

Proof. Let's start with some notation. Define $b_n = (-1)^{n+1}a_n$. Define $s_n = \sum_{i=1}^n b_i$. Note that $|b_n| \geq |b_{n+1}|$, since $|b_n| = a_n$. Note that $b_n \to 0$ since $|b_n| = a_n \to 0$. Note that $b_n \leq 0$ if n is even and $b_n \geq 0$ if n is odd.
Note that Consider the sub sequence s_{2j+1}. Note that $s_{2(j+1)+1} = s_{2j+3} = s_{2j+1} + b_{2j+2} + b_{2j+3}$. Note that $2j + 2$ is even so $|b_{2j+2}| = -b_{2j+2}$. Note that $2j + 3$ is odd so $|b_{2j+3}| = b_{2j+3}$. Note that $|b_{2j+2}| \geq |b_{2j+3}|$ so $-b_{2j+2} \geq b_{2j+3}$ so $b_{2j+2} + b_{2j+3} \leq 0$ thus $s_{2j+1} + b_{2j+2} + b_{2j+3} \leq s_{2j+1}$ and so $s_{2(j+1)+1} \leq s_{2j+1}$. Thus this sequence is monotone decreasing.
Consider the sub sequence s_{2j}. Note that $s_{2(j+1)} = s_{2j+2} = s_{2j} + b_{2j+1} + b_{2j+2}$. Note that $2j + 2$ is even so $|b_{2j+2}| = -b_{2j+2}$. Note that $2j + 1$ is odd so $|b_{2j+1}| = b_{2j+1}$. Note that $|b_{2j+2}| \leq |b_{2j+1}|$ so $-b_{2j+2} \leq b_{2j+1}$ so $b_{2j+2} + b_{2j+1} \geq 0$ thus $s_{2j} + b_{2j+1} + b_{2j+2} \geq s_{2j+1}$ and so $s_{2(j+1)} \geq s_{2j}$. Thus this sequence is monotone increasing.
Note that $s_1 \leq s_1$. Suppose $s_{2j+1} \leq s_1$, noting that $s_{2(j+1)+1} \leq s_{2j+1} \leq s_1$, we conclude by induction on j that $s_{2j+1} \leq s_1$. Note that $s_2 = b_1 + b_2 \geq 0$. Suppose $s_{2j} \geq 0$, noting that $s_{2(j+1)} \geq s_{2j} \geq 0$, we conclude by induction on j that $s_{2j} \geq 0$. Note that $s_{2j+1} = s_{2j} + b_{2j+1} \geq s_{2j}$. We can now see the following inequality $s_1 \geq s_{2j+1} \geq s_{2j} \geq 0$. And conclude s_1 is a upper bound on $\{s_{2j}\}_{j=1}^{\infty}$ and 0 is a lower bound on $\{s_{2j+1}\}_{j=0}^{\infty}$.
We now see that both of these sequences are bounded and monotone, therefore they both converge. Define f and g so that $s_{2j} \to f$ and $s_{2j+1} \to g$. Note that $s_{2j+1} - s_{2j} \to g - f$. Note that $s_{2j+1} - s_{2j} = b_{2j+1} \to 0$. Conclude $g - f = 0$ so $g = f$. By the shuffle sequence therm I conclude s_n converges. □

Exercise : 2.7.2
Determine if the following converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$
This converges. Define $s_k = \sum_{n=1}^k \frac{1}{2^n+n}$. We can see that s_n is monotone increasing, since $\frac{1}{2^n+n} > 0$. We can also see that $\sum_{n=1}^k \frac{1}{2^n+n} < \sum_{n=1}^k \frac{1}{2^n} < \sum_{n=1}^{\infty}(\frac{1}{2})^n = l$, and so s_n is bounded above by l. Thus s_n converges.

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$
This converges. Define $s_k = \sum_{n=1}^k |\frac{\sin(n)}{n^2}|$. We can see that s_n is monotone increasing, since $|\frac{\sin(n)}{n^2}| \geq 0$. We can also see that $\sum_{n=1}^k |\frac{\sin(n)}{n^2}| \leq \sum_{n=1}^k \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2} = l$, and so s_n is bounded above by l. Thus our original sum is absolutely convergent and therefore converges.

(c) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{2n}$
Noting that the terms $(-1)^{n-1}\frac{n+1}{2n} \nrightarrow 0$ we conclude the series does not converge.

(d) $\sum_{n=0}^{\infty} \frac{1}{1+3n} + \frac{1}{2+3n} - \frac{1}{3+3n}$
Note that $\frac{1}{1+3n} + \frac{1}{2+3n} - \frac{1}{3+3n} \geq \frac{1}{1+3n} \geq \frac{1}{3(n+1)}$ so $\sum_{n=0}^{\infty} \frac{1}{1+3n} + \frac{1}{2+3n} - \frac{1}{3+3n} \geq \sum_{n=0}^{\infty} \frac{1}{3(n+1)} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n}$. Recalling that $\sum_{n=1}^{k}\frac{1}{n} \to \infty$ we can see that the series diverges towards ∞.

(e) $\sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{(2n)^2}$
Note that $\frac{1}{2n-1} - \frac{1}{(2n)^2} \geq \frac{1}{2n} - \frac{1}{4n} = \frac{1}{4}\frac{1}{n}$. So recalling that $\sum_{n=1}^{k}\frac{1}{n} \to \infty$ and that $\sum_{n=1}^{k}\frac{1}{2n-1} - \frac{1}{(2n)^2} \geq \frac{1}{4}\sum_{n=1}^{k}\frac{1}{n}$ we see that our series diverges towards infinity.

Exercise : 2.7.4
Give a example or explain why it is impossible.

(a) Two sequences a_n and b_n where $\sum a_n$ and $\sum b_n$ diverge and $\sum a_n b_n$ converges.
We have already dealt with this, define $a_n = b_n = 1/n$.

(b) Two sequences a_n and b_n where $\sum a_n$ converges and b_n is bounded and $\sum b_n a_n$ diverges.
Define $a_n = (-1)^n 1/n$ and $b_n = (-1)^n$. Note that all properties are fulfilled, $\sum(-1)^n 1/n$ converges and $(-1)^n$ is bounded and $\sum 1/n$ diverges.

(c) Two sequences a_n and b_n where $\sum a_n$ converges and $\sum a_n + b_n$ converges and $\sum b_n$ diverges.
Define $s_n = \sum^n a_i$, $t_n = \sum^n a_i + b_i$, $u_n = \sum^n b_i$. Suppose $\sum a_n$ converges and $\sum a_n + b_n$ converges. Define l,m as $s_n \to l$ and $t_n \to m$. Note that $u_n = t_n - s_n \to m - l$. Thus the desired a_n and b_n do not exist.

(d) A sequence a_n where $0 \leq a_n \leq 1/n$ and $\sum(-1)^{n+1}a_n$ diverges.
Define

$$a_n = \begin{cases} 1/n & 2 \nmid n \\ 0 & 2 \mid n \end{cases}$$

Note that $\sum(-1)^{n+1}a_n = \sum a_n$ witch behaves like $\sum 1/n$ with every other term removed and therefore diverges.

Exercise : Consider the series $\sum_{k=1}^{\infty} a_k$. Let

$$c_k = \begin{cases} a_k & a_k \geq 0 \\ 0 & a_k < 0 \end{cases}$$

and

$$d_k = \begin{cases} -a_k & a_k \leq 0 \\ 0 & a_k > 0 \end{cases}$$

Let's define $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} c_k$, $u_n = \sum_{k=1}^{n} d_k$. Note that $a_k = c_k - d_k$ also note that $|a_k| = |c_k| + |d_k|$ since eater $c_k = 0$ or $d_k = 0$. Note that $s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} c_k - d_k = t_n - u_n$.

a) Prove that s_n is absolutely convergent if and only if t_n and u_n are both convergent.

Proof. Define s_n^a to be $\sum_{k=1}^{n} |a_k|$ and t_n^a and u_n^a in similar fashion.
Suppose $s_n^a \to l$. Note that $|c_k| \leq |a_k|$ thus $t_n^a \leq s_n^a \leq l$ also note that t_n^a is a sum of positive terms and therefore monotonic increasing, it is bounded above and monotonic increasing therefore convergent.
Note that $|d_k| \leq |a_k|$ thus $u_n^a \leq s_n^a \leq l$ also note that u_n^a is a sum of positive terms and therefore monotonic increasing, it is bounded above and monotonic increasing therefore convergent.
Suppose $t_n^a \to l$ and $u_n^a \to k$. Note that $s_n^a = t_n^a + u_n^a \to l + k$. Therefore s_n is absolutely convergent if and only if t_n and u_n are both convergent. □

b) Prove that if s_n^a is convergent and s_n is divergent, then t_n and u_n are both divergent.

Proof. Suppose that if s_n^a is convergent and s_n is divergent.
Further suppose t_n is convergent. Note that $c_k \geq 0$ thus $t_n = t_n^a$. Note that $d_k \geq 0$ thus $u_n = u_n^a$. Note that $s_n^a - t_n^a = u_n^a = u_n$, by the arithmetic limit therm u_n converges. Note that $s_n = t_n - u_n$, by the arithmetic limit therm s_n converges, a contradiction, thus t_n is divergent.
Suppose u_n is convergent. Note that $s_n^a - u_n^a = t_n^a = t_n$, by the arithmetic limit therm t_n converges. Note that $s_n = t_n - u_n$, by the arithmetic limit therm s_n converges, a contradiction, thus u_n is divergent. □

c) If $\sum c_n$ and $\sum d_n$ are divergent, is it true that $\sum a_n$ is conditionally convergent.
No, examine the following counterexample.
Define $a_n = (-1)^n$ note that $c_n = \begin{cases} 1 & 2 \mid n \\ 0 & 2 \nmid n \end{cases}$ and $d_n = \begin{cases} 0 & 2 \mid n \\ 1 & 2 \nmid n \end{cases}$. Note that $\sum |a_n|$ is divergent and $\sum c_n$, $\sum d_n$ are divergent. This is clearly a case where $\sum c_n$, $\sum d_n$ are divergent and $\sum a_n$ is not conditionally convergent.

Exercise : 2.7.7

(a) Show that if $a_n > 0$ and $na_n \to l \neq 0$ then $\sum a_n$ diverges.

Proof. Suppose $a_n > 0$ and $na_n \to l \neq 0$.
Note that since $n \geq 0$ and $a_n \geq 0$ we can say that $l \geq 0$ and since $l \neq 0$ we note that $l > 0$ thus $l/2 > 0$. There must exist a N such that for all $n \geq N$, $|na_n - l| < l/2$. Therefore $-l/2 < na_n - l < l/2$ and $l/2 * 1/n < a_n$. Note that we can break up the sum to $\sum a_n = \sum_1^N a_n + \sum_N^\infty a_n$. Note that $\sum_1^N a_n$ is the sum of finitely many finite terms and thus is finite. However we see that $\sum_N^\infty a_n \geq l/2 \sum_N^\infty 1/n$. Since $\sum_N^\infty 1/n$ tends towards infinity we conclude $\sum_N^\infty a_n$ tends towards infinity and therefore $\sum a_n$ tends towards infinity and thus diverges. □

(b) Assume $a_n > 0$ and $n^2 a_n \to l$, show that $\sum a_n$ converges.

Proof. Suppose $a_n > 0$ and $n^2 a_n \to l$.
Define $0 < k = max(1, 1-l)$. There must exist a N such that for all $n \geq N$, $|n^2 a_n - l| < k$. Therefore $-k < n^2 a_n - l < k \leq 1 - l$ and $a_n < 1/n^2$. Note that we can break up the sum to $\sum a_n = \sum_1^N a_n + \sum_N^\infty a_n$. Note that $\sum_1^N a_n$ is the sum of finitely many finite terms and thus is finite. Define $s_n = \sum_N^n a_n$, where $n > N$. Note that $s_n \leq \sum_N^n 1/n^2 \leq \sum_1^n 1/n^2 \leq \sum_1^\infty 1/n^2 = f$ where f is a finite value. Also note $s_n = s_{n+1} - a_{n+1} \leq s_{n+1}$. Conclude that s_n is monotonic increasing and has a upper bound, so it converges to a finite value. The sum of two finite is finite thus $\sum a_n$ is finite and we can say it converges. □

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Exercise : 2.7.9**(W) (Hand this one in to David.)**
Given a series $\sum a_n$ with $a_n \neq 0$ and

$$\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$$

The series converges absolutely.

(a) Let $r < r' < 1$. Explain why there exists an N such that for all $n \geq N$, $|a_{n+1}| \leq |a_n|r'$.

Proof. Let $r < r' < 1$. Note that $0 < r' - r$, thus there must exist a N such that for all $n \geq N$, $||\frac{a_{n+1}}{a_n}| - r| < r' - r$, definition of limit. So $|\frac{a_{n+1}}{a_n}| - r < r' - r$ or $|\frac{a_{n+1}}{a_n}| < r'$ so $|a_{n+1}| < |a_n|r'$. □

(b) Why does $|a_N| \sum (r')^n$ converge?
Note that $|a_N| \sum (r')^n = \sum |a_N|(r')^n$ since $|a_N|$ is simply some finite value. We can see this is a geometric series and Example 2.7.5 tells us that it will converge if $|r'| < 1$ and since $0 \leq r < r' < 1$ we can say that the series converges.

(c) Show that $\sum |a_n|$ converges and that $\sum a_n$ converges.

Proof. Define $s_k = \sum_1^k |a_n|$. Note that for $k \geq N$, $s_k = \sum_1^N |a_n| + \sum_N^k |a_n|$.
Note that for $k = N$, $|a_k|(r')^N \leq |a_N|(r')^k$. Suppose $|a_k|(r')^N \leq |a_N|(r')^k$ for some $k \geq N$. Recall that $|a_{k+1}| < |a_k|r'$ and thus $|a_{k+1}|(r')^N \leq |a_N|(r')^{k+1}$. By induction we can conclude for all $k \geq N$, $|a_k|(r')^N \leq |a_N|(r')^k$.
Thus for $k \geq N$, $\sum_N^k |a_n| \leq \sum_N^k |a_N|(r')^n/(r')^N = |a_N|/(r')^N \sum_N^k (r')^n$ Noting that $(r')^n \geq 0$ we can say $\sum_N^k (r')^n \leq \sum_1^k (r')^n \leq \sum (r')^n = f$ where f is some finite value, since this geometric series converges. Thus for $k \geq N$, $\sum_N^k |a_n| \leq |a_N|/(r')^N f$ and so $s_k \leq \sum_1^N |a_n| + |a_N|/(r')^N f$. Noting that $\sum_1^N |a_n| + |a_N|/(r')^N f = g$ is finite we can say g is a upper bound on s_k while $k \geq N$. Note that $s_k = s_{k+1} - |a_{k+1}| \leq s_{k+1}$, so s_k is monotonic increasing. Thus for $k < N$, $s_k \leq s_N \leq g$, so g is a upper bound on s_k for all k. The sequence s_k is bounded above and monotonic increasing therefore it converges.
Since $\sum a_n$ is absolutely convergent, so we can conclude that $\sum a_n$ is convergent. □