Exercise 1.2.5: Use the triangle inequality to establish the following inequalities:

(a) $|a - b| \leq |a| + |b|$.

Proof. Note that $|a-b| = |a+(-b)|$. By the triangle inequality we note that $|a+(-b)| \leq |a| + |-b|$. There are three possibilities ether $b < 0$, $b = 0$, or $b > 0$. In the case that $b < 0$ we know that $-b > 0$. From the definition of absolute value $|b| = -b$ and $|-b| = -b$ thus in this case $|b| = |-b|$. In the case that $b = 0$ we know that $-b = 0$. From the definition of absolute value $|b| = b$ and $|-b| = b$ thus in this case $|b| = |-b|$. In the case that $b > 0$ we know that $-b < 0$. From the definition of absolute value $|b| = b$ and $|-b| = -(-b) = b$ thus in this case $|b| = |-b|$. Thus in all cases $|b| = |-b|$ and so $|a| + |-b| = |a| + |b|$ thus $|a - b| \leq |a| + |b|$. □

(b) $||a| - |b|| \leq |a - b|$.

Proof. Note that $|c| = |c - d + d|$ witch by the triangle inequality means $|c| \leq |c - d| + |d|$ so $|c| - |d| \leq |c - d|$ for any c and d in $\mathbb{R}$. Consider $||a| - |b||$ noting that there are two possibilities, eater $||a| - |b|| = |a| - |b|$ or $||a| - |b|| = -(|a| - |b|) = |b| - |a|$, by the definition of absolute value. In the case that $||a| - |b|| = |a| - |b|$ we see from the first statement that $||a| - |b|| = |a| - |b| \leq |a - b|$. In the second case $||a| - |b|| = |b| - |a| \leq |b - a|$ and we proved in the previous question that $|b - a| = |-(b - a)| = |a - b|$, thus in this case $||a| - |b|| \leq |a - b|$. Thus in all cases $||a| - |b|| \leq |a - b|$. □

Exercise 1.2.6(b), (d): Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(a) = \{f(x) : x \in A\}$.

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
Suppose we let $A = \{1\}$ and $B = \{-1\}$. Consider the function $f(x) = x^2$. Note that $A \cap B = \varnothing$ and thus $f(A \cap B) = \varnothing$. Also note that $f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\}$ and since $\varnothing \neq \{1\}$ in this case $f(A \cap B) \neq f(A) \cap f(B)$.

(d) Form and prove a conjecture concerning $f(A \cup B)$ and $f(A) \cup f(B)$.
Conjecture $f(A \cup B) \subseteq f(A) \cup f(B)$

Proof. Choose some element y from the set $f(A \cup B)$. By our definition of evaluating a function on a set there must exist some element x in $A \cup B$ such that $f(x) = y$. By the definition of union $x \in A$ or $x \in B$. Thus $f(x) \in f(A)$ or $f(x) \in f(B)$ and so $y \in f(A)$ or $y \in f(B)$ witch means by definition $y \in f(A) \cup f(B)$. Since we chose y arbitrarily from $f(A \cup B)$ and showed that y is in $f(A) \cup f(B)$ we can say by the definition of subset $f(A \cup B) \subseteq f(A) \cup f(B)$. □

Exercise 1.2.8: Form the logical negation of each claim. Do not use the easy way out: "It is not the case that..." is not permitted

(a) For all real numbers satisfying $a < b$, there exists $n \in \mathbb{N}$ such that $a + (1/n) < b$.
There exists real numbers satisfying $a < b$, that for all $n \in \mathbb{N}$, $a + (1/n) \geq b$.

(b) Between every two distinct real numbers there is a rational number.
There exists two distinct real numbers where there is no rational number between them.

(c) For all natural numbers $n \in \mathbb{N}$, $\sqrt{n}$ is either a natural number or is an irrational number.
There exists some natural number $n \in \mathbb{N}$ where $\sqrt{n}$ is not a natural number or an irrational number.

(d) Given any real number $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ satisfying $n > x$.
There exists a real number $x \in \mathbb{R}$ where there is no $n \in \mathbb{N}$ satisfying $n > x$.

Exercise 1.3.4: Assume that A and B are nonempty, bounded above, and satisfy $B \subseteq A$. Show that $\sup B \leq \sup A$.

Proof. Assume to the contrary, namely that there exists sets A and B that are nonempty, bounded above, and satisfy $B \subseteq A$. Further suppose that $\sup B > \sup A$. Lets define $\alpha = \sup A$. Suppose there is no element in B grater than α. By the definition of upper bound, α would be a upper bound to B, however $\sup B > \alpha$, a contradiction, thus our assumption that there is no element of B grater than α must be false, and there is some element of B grater than α. Lets take one of these elements with the property $\gamma \in B$ and $\gamma > \alpha$. Note that since $B \subseteq A$, $\gamma \in A$. Since α is a upper bound to A we know that every element of A is less than or equal to α thus $\gamma \leq \alpha$. Contradiction $\gamma > \alpha$ and $\gamma \leq \alpha$, thus our initial supposition that $\sup B > \sup A$ must be false and so we are forced to conclude $\sup B \leq \sup A$. □

Exercise 1.3.5: Let A be bounded above and let $c \in \mathbb{R}$. Define the sets $c + A = \{a + c : a \in A\}$ and $cA = \{ca : a \in A\}$.

(a) Show that $\sup(c + A) = c + \sup(A)$.

(b) If $c \geq 0$, show that $\sup(cA) = c \sup(A)$.

(c) Postulate a similar statement for $\sup(cA)$ when $c < 0$.

Proof (a). Lets start by defining $\alpha = \sup(A), \beta = c + \sup(A)$. We will proceed by showing that β must have the two properties defining $\sup(c + A)$.

Suppose that there existed some $\gamma \in c + A$ where $\gamma > \beta$. Note that $\gamma - c \in A$ and that $\gamma - c > \beta - c = \alpha$. Contradiction, we have found a element in A, $\gamma - c$, that is greater than $\sup(A)$. We are forced to conclude the negation of our supposition and so conclude that there is no element in $c + A$ that is greater than β, and so β is a upper bound on $c + A$, the first condition on $\sup(c + A)$.

Suppose that there is a upper bound to $c + A$, lets call it λ, that is smaller than β. Note that $\lambda - c < \beta - c = \alpha$. Since α is larger than $\lambda - c$ we know from the definition of sup that $\lambda - c$ is not a upper bound on A, therefore there must be at least one element in A greater than $\lambda - c$, lets call it τ. Since τ is in A $c + \tau$ is in $c + A$, and since $\tau > \lambda - c$, $\tau + c > \lambda$.

Contradiction, λ is a upper bound on $c + A$ but there is a element in $c + A$, namely $\tau + c$, that is grater than λ. Thus we are forced to conclude the negation of our supposition, that all upper bounds on $c + A$ are grater than or equal to β.

β meets the definition of $\sup(c + A)$ and so $\beta = \sup(c + A)$ and $c + \sup(A) = \sup(c + A)$. □

Proof (b). Firstly let me eliminate a special case, $c = 0$. In this case $cA = \{0\}$, by inspection $\sup(cA) = 0$ and also $c\sup(A) = 0 * \sup(A) = 0$. In this degenerate case it is clearly true that $c\sup(A) = \sup(cA)$. From here on I will work with $c > 0$. Note, in this proof I am taking advantage of the fact that dividing over a positive number across a inequality does not affect the inequality, that is why $c > 0$ is necessary for this proof.

Lets start by defining $\alpha = \sup(A)$, $\beta = c\sup(A)$. We will proceed by showing that β must have the two properties defining $\sup(cA)$.

Suppose that there existed some $\gamma \in cA$ where $\gamma > \beta$. Note that $\gamma/c \in A$ and that $\gamma/c > \beta/c = \alpha$. Contradiction, we have found a element in A, γ/c, that is greater than $\sup(A)$. We are forced to conclude the negation of our supposition and so conclude that there is no element in cA that is greater than β, and so β is a upper bound on cA, the first condition on $\sup(cA)$.

Suppose that there is a upper bound to cA, lets call it λ, that is smaller than β. Note that $\lambda/c < \beta/c = \alpha$. Since α is larger than λ/c we know from the definition of sup that λ/c is not a upper bound on A, therefore there must be at least one element in A greater than λ/c, lets call it τ. Since τ is in A $c\tau$ is in cA, and since $\tau > \lambda/c$, $\tau c > \lambda$. Contradiction, λ is a upper bound on cA but there is a element in cA, namely τc, that is grater than λ. Thus we are forced to conclude the negation of our supposition, that all upper bounds on cA are grater than or equal to β.

β meets the definition of $\sup(cA)$ and so $\beta = \sup(cA)$ and $c\sup(A) = \sup(cA)$. □

Statement for part (c):
The region A would be flipped across 0 and be magnified by a factor of $|c|$, thus $\sup(cA) = c\inf(A)$.

Exercise 1.3.6: Compute, without proof, the suprema and infima of the following sets.

(a) $\{n \in \mathbb{N} : n^2 < 10\}$.

(b) $\{n/(n + m) : n, m \in \mathbb{N}\}$.

(c) $\{n/(2n + 1) : n \in \mathbb{N}\}$.

(d) $\{n/m : m, n \in \mathbb{N} \text{ with } m + n \leq 10\}$.

Solution:

(a) sup = 3, inf = 1

(b) sup = 1, inf = 0

(c) sup = 1/2, inf = 1/3

(d) sup = 1/9, inf = 9

Exercise 1.3.7: Prove that if a is an upper bound for A and if a is also an element of A, then $a = \sup A$.

Proof. Suppose b is a upper bound of A and that $b < a$. Since b is a upper bound on A and $a \in A$ we know that $a \leq b$. We have arrived at a contradiction, thus there is no upper bound on A that is less than a, and so all upper bounds on A are grater than or equal to a. We now can say that a meets both of the elements of the definition of $\sup A$ thus $a = \sup A$ □

Exercise 1.3.8: If $\sup A < \sup B$ then show that there exists an element $b \in B$ that is an upper bound for A.

Proof. Let's begin with a short contradiction, suppose there is no element in B greater than $\sup A$. By the definition of upper bound $\sup A$ is a upper bound for B. Thus by the definition of $\sup B$ we conclude $\sup B \leq \sup A$. This is a contradiction, and so we are forced to conclude that there is at least one element of B greater than $\sup A$. Cause one of these elements, $\beta \in B$, $\sup A < \beta$. The definition of $\sup A$ gives us that for all $\alpha \in A$, $\alpha \leq \sup A \Rightarrow \alpha \leq \beta$. We then see that β must be a upper bound on A and thus there is a element in B that is a upper bound on A. □

Authors note: Is it necessary that $\sup A < \sup B$, or only that $\sup A \leq \sup B$? Consider $A = [0, 1]$ and $B = [0, 1)$, here $\sup A \leq \sup B$ but there is no element of B that is a upper bound for A.

Exercise 1.2.9: Show that the sequence $(x_1, x_2, x_3, \ldots)$ defined in Example 1.2.7 is bounded above by 2. That is, show that for every $i \in \mathbb{N}$, $x_i \leq 2$.

Proof. We will proceed with a proof by induction on i.

In the base case $i = 1$ we are given $x_i = 1$ since $1 \leq 2$ the statement $i \in \mathbb{N}$, $x_i \leq 2$ holds in the base case.

Suppose $x_i \leq 2$. Consider the next step x_{i+1}, note that by definition $x_{i+1} = (1/2)x_i + 1$. Note that $x_i \leq 2 \Rightarrow (1/2)x_i \leq (1/2)2 = 1 \Rightarrow (1/2)x_i + 1 \leq 1 + 1 = 2 \Rightarrow x_{i+1} \leq 2$. Thus by induction we conclude that for all $i \in \mathbb{N}$, $x_i \leq 2$. □

Exercise 1.3.7: Prove that if a is an upper bound for A and if a is also an element of A, then $a = \sup A$.

Proof. Suppose b is a upper bound of A and that $b < a$. Since b is a upper bound on A and $a \in A$ we know that $a \leq b$. We have arrived at a contradiction, thus there is no upper bound on A that is less than a, and so all upper bounds on A are grater than or equal to a. We now can say that a meets both of the elements of the definition of $\sup A$ thus $a = \sup A$ □