Note that I am operating under the convention $V_\epsilon(d) = (d-\epsilon, d+\epsilon)$.

Exercise 1: Suppose (x_n) and (y_n) are sequences such that $\lim_{n\to\infty} x_n = L$ and $lim_{n\to\infty} y_n = \infty$. Show that $lim_{n\to\infty} x_n/y_n = 0$.

Proof. Suppose (x_n) and (y_n) are sequences such that $\lim_{n\to\infty} x_n = L$ and $lim_{n\to\infty} y_n = \infty$. Note that since x_n is a convergent sequence it must be bounded thus there exists $M \in (0,\infty)$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. Choose $\epsilon > 0$. Define $k \in \mathbb{N}$ such that $1/k < \epsilon/M$. Define $N \in \mathbb{N}$ such that $y_n > k$ for all $n \in [N,\infty] \cap \mathbb{N}$. Choose $n \in \mathbb{N}$ where $n \geq N$. Note that $|x_n/y_n - 0| = |x_n|/|y_n| = |x_n|/y_n < |x_n|/k < M/k < \epsilon$. We conclude that $x_n/y_n \to 0$. □

Exercise 2: A number is algebraic if it is a solution of a polynomial equation $a_n x^n + a_{n1}x^{n1} + \cdots + a_1 x + a_0 = 0$ where each a_k is an integer, $n \geq 1$, and $a_n \neq 0$. Show that the collection of all algebraic numbers is countable.

First let us prove that polynomials have finitely many zeros.

Proof. Note that a polynomial as described above of order 1 takes the form $P_1(x) = a_1x+a_0$. Suppose that there were infinitely many solution to $P_1(x) = 0$, choose two of these, $x_1 < x_2$. Note that $P_1'(x) = a_1 \neq 0$ for all $x \in \mathbb{R}$. By the mean value theorem there exists some x_3 such that $x_3 \in (x_1, x_2)$ and $P_1'(x_3) = 0$. This is a contradiction and thus we conclude that $P_1(x)$ has finitely many solutions.

Suppose that all polynomials of degree $n-1$ have finitely many solutions. Consider $P_n(x) = \sum_{k=0}^{n} a_k x^k$ a arbitrary polynomial of degree n. Note that $P_n'(x) = \sum_{k=1}^{n} k a_k x^{k-1}$ a polynomial of degree $n-1$. We conclude that $P_n'(x) = 0$ has finitely many solutions, define l to be the number of solutions to $P_n'(x) = 0$. Suppose $P_n(x) = 0$ has infinitely many solutions. Select $l+2$ solutions to $P_n(x) = 0$ and arrange them in a list $\{x_k\}_{k=0}^{l+1}$ such that $x_k < x_{k+1}$ for all $k \in [0, l+1] \cap \mathbb{N}$. Now construct intervals $I_k = (x_{k-1}, x_k)$ for all $k \in [1, l+1] \cap \mathbb{N}$. Note that there are no shared elements between any two intervals. By the mean value theorem there exists a value $y_k \in I_k$ such that $P'(y_k) = 0$. Noting that there are $l+1$ non-identical y_k's we conclude that we have found $l+1$ solutions to $P_n'(x) = 0$, a contradiction, we are forced to conclude that there are finitely many solution to $P_n(x) = 0$.

By induction conclude that there are finitely many solutions to any polynomial as constructed above. □

Now to prove that there are countably infinite algebraic numbers.

Proof. Recall from proofs that each natural number has a unique prime factorization and also that there are countably infinite primes. Now we can define a bijective mapping from a polynomial of the described form to the naturals. Define p_n to be the n'th prime number. Associate a arbitrary polynomial $P_n(x) = \sum_{k=0}^{N} a_k x^k$ where $N \in \mathbb{N}$ with the natural $\sum_{k=1}^{N} p_n^{a_n}$. Note that each polynomial is associated with one and only one natural, via the uniqueness of prime factorization. Note that given a natural number there is at most one polynomial associated with it, via the uniqueness of prime factorization, thus this mapping is one-to-one. Call this mapping M_1 witch takes a polynomial to a natural in a one-to-one manner. Note that there exists a bijective map from $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ call this map M_2. Consider a arbitrary algebraic number k. Note that there exists a non-empty set of polynomials S where $P(k) = 0$ for all $P(x) \in S$. Define $P(x)$ to be the polynomial in S with the smallest mapped value under M_1. Take $X = \{x \in \mathbb{R} : P(x) = 0\}$. Note that there are finitely many elements in X as proved in the previous proof. Ordering X by increasing value we can associate k with a integer i where i is k's position in the ordered X. Note that I have now defined a mapping, let's call it M_3 that maps a algebraic number to a polynomial and a natural number. Note that M_3 is one-to-one, given a polynomial $P(x)$ a a natural n there is at most one number that is the n'th solution to $P(x) = 0$. Note that now I can construct a one-to-one mapping from the algebraic numbers to the naturals, use M_3 to map a algebraic to a polynomial and a natural, use M_1 to map the polynomial to a natural, use M_2 to map the resulting two naturals to one natural, since all of these steps are individually one-to-one the entire process is one-to-one. Now we can conclude that the algebraic numbers are at most countably infinite. Noting that for every $n \in \mathbb{N}$, n will be a solution to $x-n = 0$ we conclude the naturals are algebraic and thus that the algebraic numbers are at least countably infinite. We are forced to conclude that the algebraic numbers are countably infinite. □

Exercise 3: Let p be a fifth order polynomial, so $p(x) = \sum_{k=0}^{5} a_k x^k$ where each $a_k \in \mathbb{R}$, and $a_5 \neq 0$. Prove that there is a solution of $p(x) = 0$.

Proof. Suppose $p(x) = \sum_{k=0}^{5} a_k x^k$ where each $a_k \in \mathbb{R}$, and $a_5 \neq 0$. Suppose $a_5 > 0$. Define $a = a_5/5$. Note that I can now re-write $p(x) = \sum_{k=0}^{5} a_k x^k = 5ax^5 + \sum_{k=0}^{4} a_k x^k = \sum_{k=0}^{4} ax^5 + a_k x^k$. Note that $a > 0$. Choose some $x_M > \max(1, \{|a_n|/a : n \in \{0,1,2,3,4\}\})$. Note that $ax_M^5 = ax_M x_M^4 > |a_n|x_M^4 = |a_n x_M^n||x_M^{4-n}| \geq |a_n x_M^n| \geq -a_n x_M^n$, or $ax_M^5 + a_n x_M^n > 0$ for all $n \in \{0,1,2,3,4\}$. Now note that $p(x_M) = \sum_{k=0}^{4} ax_M^5 + a_k x_M^k > 0$. Choose some $x_m < \min(-1, \{-|a_n|/a : n \in \{0,1,2,3,4\}\})$. Note that $ax_m^5 = ax_m x_m^4 < -|a_n|x_m^4 = -|a_n x_m^4| = -|a_n x_m^n||x_m^{4-n}| \leq -|a_n x_m^n| \leq -a_n x_m^n$, or $ax_m^5 + a_n x_m^n < 0$ for all $n \in \{0,1,2,3,4\}$. Now note that $p(x_m) = \sum_{k=0}^{4} ax_m^5 + a_k x_m^k < 0$. By the intermediate value theorem there exists some $x' \in [x_m, x_M]$ such that $p(x') = 0$. We now conclude that there is a solution to a fifth order polynomial with $a_5 > 0$.
Suppose $p(x) = \sum_{k=0}^{5} a_k x^k$ where each $a_k \in \mathbb{R}$, and $a_5 \neq 0$. Suppose $a_5 < 0$. Note that $p_2(x) = -p(x)$ is a fifth order polynomial with $a_5 > 0$ and thus there exist some x' such that $p_2(x') = 0$, note that $p(x') = -p_2(x') = 0$. We now can conclude that there is a solution to $p(x) = 0$ for any fifth order polynomial, $a_5 \neq 0$. □

Exercise 4: We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is periodic if there is a number L such that $f(x) = f(x + L)$ for all $x \in \mathbb{R}$. Show that a continuous, periodic function is uniformly continuous.

Proof. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is periodic with periodicity L and continuous. Define $A = [-L, L]$. Note that A is closed and bounded thus compact. Note that f is uniformly convergent on A. Choose $\epsilon > 0$. Note that there exists $\delta > 0$ such that if $x, y \in A$ and $|x - y| < \delta$

then $|f(x) - f(y)| < \epsilon$. Choose $x, y \in \mathbb{R}$ such that $|x - y| < \min(\delta, L)$. Without loss of generality suppose $x \geq y$.

Suppose $x > 0$. Define sequences $x_0 = x$ and $y_0 = y$ and $x_n = x_{n-1} - L$ and $y_n = y_{n-1} - L$ for $n \in \mathbb{N}$. Note that $|f(x_n)-f(y_n)| = |f(x_{n-1})-f(y_{n-1})|$ and thus $|f(x_n)-f(y_n)| = |f(x_0)-f(y_0)| = |f(x) - f(y)|$ Define $n' \in \mathbb{Z}$ to be the floor of x/L. Note that $x/L \geq n' > x/L - 1 \geq -1$, thus $n' \in \mathbb{N} + \{0\}$. Note that $x \geq Ln' > x - L$ and $0 \geq Ln' - x > -L$ and $0 \leq x - Ln' < L$. Note that $x - Ln' = x_{n'}$. Note that $x_{n'} - y_{n'} = x - y = |x - y| < L$ thus $-L \leq x_{n'} - L < y_{n'} < x_{n'} < L$. Thus $x_{n'}, y_{n'} \in A$ and $|x_{n'} - y_{n'}| = |x - y| < \delta$ so $|f(x) - f(y)| = |f(x_{n'}) - f(y_{n'})| < \epsilon$.

The exercise for this in the case the $x < 0$ is a copy paste with a few sign changes and inequality flips (I was instructed on the midterm to refrain from doing this repetition) however the conclusion is the same. This allows us to say $|f(x) - f(y)| < \epsilon$. We conclude that the function is uniform convergent. $\square$

Exercise 5: Use the Nested Interval Property to deduce the Axiom of Completeness without using any other form of the Axiom of Completeness. Hint: Look at the proof of the Bolzano Weierstrass Theorem.

Proof. Suppose S is a bounded set. There exists M such that $x < M$ for all $x \in S$. There exists $x \in S$. Define a sequence by $a_0 = x - 1$ and

$$a_n = \begin{cases} a_{n-1} & \frac{a_{n-1}+b_{n-1}}{2} \text{ is a upper bound on } S \\ \frac{a_{n-1}+b_{n-1}}{2} & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$. Define a sequence by $b_0 = M$ and

$$b_n = \begin{cases} \frac{a_{n-1}+b_{n-1}}{2} & \frac{a_{n-1}+b_{n-1}}{2} \text{ is a upper bound on } S \\ b_{n-1} & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$. Note that $b_n - a_n = \frac{b_0-a_0}{2^n} \leq \frac{b_0-a_0}{n}$ for all $n \in \mathbb{N}$. Note that a_n is not a upper bound on S for all $n \in \mathbb{N}$. Note that b_n is a upper bound on S for all $n \in \mathbb{N}$. Consider the sequence of intervals $I_n = [a_n, b_n]$. Note that $I_n \subseteq I_{n-1}$ for all $n \in \mathbb{N}$. By NIP we can conclude that $K = \cap_{n=0}^{\infty} I_n \neq \emptyset$. Suppose that $a \neq b \in K$. Note that there exists a $n_1 \in \mathbb{N}$ such that $\frac{b_0-a_0}{n_1} < |b - a|$. Note that $a, b \in I_{n_1}$ and thus $|b - a| \leq b_{n_1} - a_{n_1} < \frac{b_0-a_0}{n_1}$, a contradiction conclude that there is one and only one element in K, call this number c. Suppose that there exists $d \in S$ such that $c < d$. Note that there exists a $n_2 \in \mathbb{N}$ such that $\frac{b_0-a_0}{n_2} < |d - c|$. Note that $a_{n_2} \leq c < d \leq b_{n_2}$ and thus $|c - d| \leq b_{n_2} - a_{n_2} < \frac{b_0-a_0}{n_2}$, a contradiction conclude that c is a upper bound on S. Suppose that e is a upper bound on S and $e < c$. Note that there exists a $n_3 \in \mathbb{N}$ such that $\frac{b_0-a_0}{n_3} < |e - c|$. Note that $a_{n_3} \leq e < c \leq b_{n_3}$ and thus $|c - e| \leq b_{n_3} - a_{n_3} < \frac{b_0-a_0}{n_3}$, a contradiction conclude that c is less than or equal to every upper bound on S. We have proven $c = \sup(S)$, thus we conclude AoC. $\square$

Exercise 6: Let A be a subset of $\mathbb{R}$. The closure of A is the union of A together with its limit points. We denote it by $\bar{A}$.

(a) Explain why it is not obvious that $\bar{A}$ is a closed set.
It is non-obvious since we are not guaranteed that $\bar{A}$ has the same limit points as A. Perhaps adding in all of A's limit points generated a new limit point, witch would not be included in $\bar{A}$.

(b) Suppose c is a limit point of A. Show that for every $\epsilon > 0$ there are at least two points $x \in A$ with $x \neq c$ and $|x - c| < \epsilon$.
By the definition of limit point $V_\epsilon(c) \cap A - \{c\} \neq \emptyset$. Choose one on these elements and call it x_1. Note that $x_1 \neq c$ and $|x_1 - c| < \epsilon$. Define $\epsilon_2 = |x_1 - c|$. By the definition of limit point $V_{\epsilon_2}(c) \cap A - \{c\} \neq \emptyset$. Choose one on these elements and call it x_2. Note that $x_2 \neq c$ and $x_2 \neq x_1$ and $|x_2 - c| < \epsilon_2 < \epsilon$.

(c) Suppose d is a limit point of $\bar{A}$. Show that there exists a sequence in A, with no terms equal to c, that converges to d.
Suppose d is not a limit point of A. There must exist $\epsilon > 0$ such that $V_\epsilon(d) \cap A - \{d\} = \emptyset$. Note that since d is a limit point of $\bar{A}$, $V_{\epsilon/2} \cap \bar{A} - \{d\} \neq \emptyset$. Let $a \in V_{\epsilon/2} \cap \bar{A} - \{d\}$. Define $\delta = \min(|a - d|, \epsilon/2)$. Suppose $x \in V_\delta(a) \cap A - \{a\}$. Note that $x \neq d$ since $|a - x| < \delta \leq |a - d|$. Note that $|a - d| < \epsilon/2$ also note that $|x - d| = |x - a + a - d| \leq |x - a| + |a - d| < \epsilon/2 + \epsilon/2$ thus $x \in V_\epsilon(d)$. Note that $x \in V_\epsilon(d) \cap A - \{d\} = \emptyset$ we can now conclude that no such x could exist and thus $V_\delta(a) \cap A - \{a\} = \emptyset$. This means that a is not a limit point of A and thus since it ended up in $\bar{A}$ we can conclude $a \in A$. Thus $a \in V_\epsilon(d) \cap A - \{d\} = \emptyset$, a contradiction, thus d is a limit point of A. Since d is a limit point of A it must also be a limit point of $A - \{c\}$ thus there exists a sequence in $A - \{c\}$ converging on d.

(d) Conclude that $\bar{A}$ is closed.
Suppose d is a limit point of $\bar{A}$. Note that d is a limit point of A. Thus $d \in \bar{A}$.

Exercise 7: Let (r_n) be an enumeration of the rational numbers. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/n & x = r_n \\ 0 & x \neq \mathbb{Q} \end{cases}$$

Determine, with proof, where f is continuous.

Proof. Choose $x \in \mathbb{Q}$. Suppose f is continuous at x. Define $n \in \mathbb{N}$ such that $r_n = x$. Define $\epsilon = 1/n$. There exists $\delta > 0$ such that if $y \in V_\delta(x)$ then $|f(x) - f(y)| < \epsilon$. Define $y \notin \mathbb{Q}$ such that $y \in V_\delta(x)$. Note that $|f(x) - f(y)| = |f(x)| = 1/n = \epsilon < \epsilon$, a contradiction. We are forced to conclude that f is discontinuous at every rational.
Choose $x \notin \mathbb{Q}$. Choose $\epsilon > 0$. Define $n \in \mathbb{N}$ such that $1/n < \epsilon$. Define $\delta = \min(\{|x - r_k| : k \in \mathbb{N} \text{ and } k \leq n\})$. Note that $\delta > 0$. Choose $y \in V_\delta(x)$. Note that $y \notin \{|x - r_k| : k \in \mathbb{N} \text{ and } k \leq n\}$, thus $f(y) < 1/n$. Note that $|f(x) - f(y)| = |f(y)| < 1/n < \epsilon$. Conclude that f is continuous on the irrationals only. □

Exercise 8: Abbott 5.3.5 a)

Prove the generalized mean value theorem.

Proof. Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b). Define $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$ where $h(x)$ is defined on $[a, b]$. By the algebraic theorem for continuity and differentiability we can conclude that $h(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Note that $h'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x)$. By the mean value theorem there exists $c \in (a, b)$ such that $h'(c) = \frac{h(a)-h(b)}{a-b}$. Note that $[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = \frac{[f(b)-f(a)]g(a)-[g(b)-g(a)]f(a)-[f(b)-f(a)]g(b)-[g(b)-g(a)]f(b)}{a-b} = \frac{f(b)g(a)-f(a)g(a)-f(a)g(b)+f(a)g(a)-f(b)g(b)+f(a)g(b)+f(b)g(b)-f(b)g(a)}{a-b} = \frac{0}{a-b} = 0$. Now we see that $[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$, and so $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$. We have now concluded the generalized mean value theorem.
Suppose further that $g'(x) \neq 0$ for all $x \in (a, b)$. If $g(a) = g(b)$ we would conclude via the mean value theorem that for some $x \in (a, b)$, $g'(x) = 0$ we now can conclude $[g(b) - g(a)] \neq 0$. Now we can see in this case $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$ becomes $[f(b) - f(a)]/[g(b) - g(a)] = f'(c)/g'(c)$ for some $c \in (a, b)$. □

Exercise 9: Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} sin(kx).$$

Show that f is infinitely differentiable.

First recall that the derivative of $\sin(x)$ is $\cos(x)$ and the derivative of$\cos(x)$ is $-\sin(x)$. Next as a sub-proof let me show that $\sum_{k=1}^{n} \frac{k^\alpha}{2^k}$ converges for any $\alpha \in \mathbb{N} + \{0\}$.

Proof. Let us apply the ratio test to test for convergence. Note that $|\frac{2^k(k+1)^\alpha}{2^{k+1}k^\alpha}| = 1/2\frac{(k+1)^\alpha}{k^\alpha}$. Note that $\lim_{k\to\infty} \frac{(k+1)^\alpha}{k^\alpha} = \lim_{k\to\infty} \frac{(k+1)^{\alpha-1}}{k^{\alpha-1}}$ if $\alpha \neq 0$, noting that $\alpha \in \mathbb{N} + \{0\}$ we can continue doing this reduction α times to obtain $\lim_{k\to\infty} \frac{(k+1)^\alpha}{k^\alpha} = \lim_{k\to\infty} \frac{(k+1)^0}{k^0} = 1$. Thus $\lim_{k\to\infty} |\frac{2^k(k+1)^\alpha}{2^{k+1}k^\alpha}| = 1/2 < 1$ and so we conclude convergence. □

Now we are ready to begin the proof proper.

Proof. Define $g_{k,\alpha}(x) = \frac{k^\alpha}{2^k}\sin(kx)$. Define $f_{n,\alpha}(x) = \sum_{k=1}^{n} g_{k,\alpha}(x)$. Note that $|g_{k,\alpha}(x)| = |\frac{k^\alpha}{2^k}\sin(kx)| \leq \frac{k^\alpha}{2^k}$, By the Weierstrass M-Test we can conclude that $f_{n,\alpha}(x) \to f_\alpha(x) = \sum_{k=1}^{\infty} g_{k,\alpha}(x)$ uniformly.
Repeating this argument with cos we get exactly the same result.
We will begin a proof by induction, our induction hypothesis is the α derivative of $f(x)$ is

$$f^{(\alpha)}(x) = \begin{cases} \sum_{k=1}^{\infty} \frac{k^\alpha}{2^k}\sin(kx) & 4\alpha \in \mathbb{N} + \{0\} \\ \sum_{k=1}^{\infty} \frac{k^\alpha}{2^k}\cos(kx) & 4\alpha + 1 \in \mathbb{N} + \{0\} \\ \sum_{k=1}^{\infty} -\frac{k^\alpha}{2^k}\sin(kx) & 4\alpha + 2 \in \mathbb{N} + \{0\} \\ \sum_{k=1}^{\infty} -\frac{k^\alpha}{2^k}\cos(kx) & 4\alpha + 3 \in \mathbb{N} + \{0\} \end{cases}$$

In the base case this is true by inspection since the $f^{(0)}$ is the same as the f given in the problem.

For the induction step suppose the hypothesis in the α case. Note that

$$f_n^{(\alpha)}(x) = \begin{cases} \sum_{k=1}^n \frac{k^\alpha}{2^k} \sin(kx) & 4\alpha \in \mathbb{N} + \{0\} \\ \sum_{k=1}^n \frac{k^\alpha}{2^k} \cos(kx) & 4\alpha + 1 \in \mathbb{N} + \{0\} \\ \sum_{k=1}^n -\frac{k^\alpha}{2^k} \sin(kx) & 4\alpha + 2 \in \mathbb{N} + \{0\} \\ \sum_{k=1}^n -\frac{k^\alpha}{2^k} \cos(kx) & 4\alpha + 3 \in \mathbb{N} + \{0\} \end{cases}$$

converges uniformly to $f^{(\alpha)}(x)$ by the first part of this proof. Note that

$$f^{(\alpha)\prime}{}_n(x) = \begin{cases} \sum_{k=1}^n -\frac{k^{\alpha+1}}{2^k} \cos(kx) & 4\alpha \in \mathbb{N} + \{0\} \\ \sum_{k=1}^n \frac{k^{\alpha+1}}{2^k} \sin(kx) & 4\alpha + 1 \in \mathbb{N} + \{0\} \\ \sum_{k=1}^n \frac{k^{\alpha+1}}{2^k} \cos(kx) & 4\alpha + 2 \in \mathbb{N} + \{0\} \\ \sum_{k=1}^n -\frac{k^{\alpha+1}}{2^k} \sin(kx) & 4\alpha + 3 \in \mathbb{N} + \{0\} \end{cases}$$

converges uniformly on

$$h(x) = \begin{cases} \sum_{k=1}^\infty -\frac{k^{\alpha+1}}{2^k} \cos(kx) & 4\alpha \in \mathbb{N} + \{0\} \\ \sum_{k=1}^\infty \frac{k^{\alpha+1}}{2^k} \sin(kx) & 4\alpha + 1 \in \mathbb{N} + \{0\} \\ \sum_{k=1}^\infty \frac{k^{\alpha+1}}{2^k} \cos(kx) & 4\alpha + 2 \in \mathbb{N} + \{0\} \\ \sum_{k=1}^\infty -\frac{k^{\alpha+1}}{2^k} \sin(kx) & 4\alpha + 3 \in \mathbb{N} + \{0\} \end{cases}$$

so we can conclude that $f^{(\alpha+1)}(x) = f^{(\alpha)\prime}(x) = h(x)$. By doing a change of variable $\alpha' = \alpha+1$ we see

$$f_n^{(\alpha')}(x) = \begin{cases} \sum_{k=1}^n \frac{k^{\alpha'}}{2^k} \sin(kx) & 4\alpha' \in \mathbb{N} + \{0\} \\ \sum_{k=1}^n \frac{k^{\alpha'}}{2^k} \cos(kx) & 4\alpha' + 1 \in \mathbb{N} + \{0\} \\ \sum_{k=1}^n -\frac{k^{\alpha'}}{2^k} \sin(kx) & 4\alpha' + 2 \in \mathbb{N} + \{0\} \\ \sum_{k=1}^n -\frac{k^{\alpha'}}{2^k} \cos(kx) & 4\alpha' + 3 \in \mathbb{N} + \{0\} \end{cases}$$

Via induction we now conclude that the nth derivative of f is the $f^{(n)}(x)$ described above, and thus infinitely differentiable. □

Exercise 10: Suppose $f : (0, \infty) \to \mathbb{R}$ is twice differentiable.

(a) Suppose $|f(x)| \leq M_0$ and $|f''(x)| \leq M_1$ for all $x > 0$. Show that for all $h > 0$,

$$|f'(x)| \leq \frac{2}{h}M_0 + \frac{h}{2}M_1.$$

Choose $h > 0$, $x > 0$. Note that the Taylor remainder theorem tells us that $f(x + h) = f(x) + f'(x)h + f''(\xi)h^2/2$ for some ξ in the interval between x and h. After some manipulation we can see $|f'(x)| = |f(x + h)/h - f(x)/h - f''(\xi)h/2| \leq M_0/h + M_0/h + M_1h/2 = \frac{2}{h}M_0 + \frac{h}{2}M_1$.

(b) Suppose $|f(x)| \leq M_0$ and $|f''(x)| \leq M_1$ for all $x > 0$. Show that for all $x > 0$,

$$|f'(x)| \leq 2\sqrt{M_0 M_1}.$$

Suppose $M_1 = 0$, in this case $f''(x) = 0$ for all x, this means that $f'(x)$ is a constant, and $f(x) = f(0) + f'(x)h$, notice that in order for $f(x)$ to be bounded $f'(x) = 0$. In this case our inequality becomes $0 = |f'(x)| \leq 2\sqrt{M_0 M_1} = 0$, clearly true.
Suppose $M_0 = 0$, in this case we have a constant function and thus $f'(x) = 0$. In this case our inequality becomes $0 = |f'(x)| \leq 2\sqrt{M_0 M_1} = 0$, clearly true.
Suppose $M_1 \neq 0$ and $M_2 \neq 0$. In this case we can define $h = 2\sqrt{M_0}/\sqrt{M_1} > 0$. Notice that the first relation becomes $|f'(x)| \leq \frac{2}{h}M_0 + \frac{h}{2}M_1 = \sqrt{M_0 M_1} + \sqrt{M_0 M_1} = 2\sqrt{M_0 M_1}$.

(c) Suppose instead that $f''(x)$ is bounded for $x > 0$ and $\lim_{x\to\infty} f(x) = 0$. Show $\lim_{x\to\infty} |f'(x)| = 0$.
Define M_1 to be a upper bound on $f''(x)$. Choose $\epsilon > 0$. Define M such that if $x > M$, $f(x) < \epsilon^2/(4M_1)$. Note that the previous two proofs did not utilize the starting at zero property of f, thus they hold just as well for the section of $f : (M, \infty)$. In this range $|f(x)| < M_0 = \epsilon^2/(4M_1)$ and thus $|f'(x)| \leq 2\sqrt{M_0 M_1} < \epsilon$. We can now conclude that $\lim_{x\to\infty} |f'(x)| = 0$.

Exercise 11: Let g_n and g be uniformly bounded on $[0, 1]$, meaning that there exists a single $M > 0$ satisfying $|g(x)| \leq M$ and $|g_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in [0, 1]$. Assume $g_n \to g$ point-wise on $[0, 1]$ and uniformly on any set of the form $[0, \alpha]$, where $0 < \alpha < 1$.

If all the functions are integrable, show that $\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$.

Proof. Choose $\epsilon > 0$. Define $\alpha = \max(1 - \epsilon/(2M), 1/2)$. Note that $\lim_{n\to\infty} \int_0^\alpha g_n = \int_0^\alpha g$, by the integrable limit theorem. Define $N \in \mathbb{N}$ such that if $n \geq N$, $|\int_0^\alpha g_n - \int_0^\alpha g| < \epsilon/2$. Choose $n \geq N$. Note that $|\int_\alpha^1 g_n| \leq M(1 - \alpha)$ by Theorem 7.4.2, and $|\int_\alpha^1 g| \leq M(1 - \alpha)$. Note that $|\int_0^1 g_n - \int_0^1 g| \leq |\int_0^\alpha g_n - \int_0^\alpha g| + |\int_\alpha^1 g_n| + |\int_\alpha^1 g| < \epsilon/2 + 2M(1 - \alpha) \leq \epsilon$. $\square$

Exercise 12: Given a function f on $[a, b]$, define the *total variation* of f to be

$$Vf = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\right\},$$

where the supremum is taken over all partitions P of $[a, b]$.

(a) If f is continuously differentiable (f' exists as a continuous function), use the Fundamental Theorem of Calculus to show $Vf \leq \int_a^b |f'|$.
Consider a arbitrary partition of $[a, b]$ call it P. Note that $\int_a^b |f'| = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f'| \geq \sum_{k=1}^n |\int_{x_{k-1}}^{x_k} f'| = \sum_{k=1}^n |f(x_k) - f(x_{k-1})|$. Note that $\int_a^b |f'|$ is a upper bound on the set that Vf is taking the supremum of, thus $Vf \leq \int_a^b |f'|$.

(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $Vf = \int_a^b |f'|$.

Take P to be the partition using the extrema of the function f. Note that $\int_a^b |f'| = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f'| = \sum_{k=1}^n |\int_{x_{k-1}}^{x_k} f'| = \sum_{k=1}^n |f(x_k) - f(x_{k-1})|$.