Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified. I am also operating under the convention $v_a(b) = \{x \in \mathbb{R} : b - a < x < b + a\}$

Exercise 1: Abbott 6.3.5 (c)

Let $f_n(x) = \frac{nx^2+1}{2n+x}$. Note that $f : [0, \infty) \to \mathbb{R}$.

(a) Note that $f_n(x) = \frac{x^2+1/n}{2+x/n}$. Note $f_n(x) \to f(x) = x^2/2$ and thus $f'(x) = x$.

(b) Note that $f_n'(x) = \frac{4n^2x+nx^2-1}{4n^2+4nx+x^2}$. Choose $M \in \mathbb{R}^+$. Choose $\epsilon > 0$. Let $N_1 \in \mathbb{N}$ such that $1/(4N_1^2) < \epsilon/3$. Let $N_2 \in \mathbb{N}$ such that $M^4/(4N_2^2) < \epsilon/3$. Let $N_3 \in \mathbb{N}$ such that $3M^2/(4N_3) < \epsilon/3$. Let $N = \max(N_1, N_2, N_3)$. Choose $n \geq N$. Choose $x \in [0, M]$. Note that $|f_n'(x) - x| = |\frac{-3nx^2-1-x^3}{4n^2+4nx+x^2}| = \frac{3nx^2+1+x^3}{4n^2+4nx+x^2} \leq \frac{3nx^2+1+x^3}{4n^2} \leq \frac{3nM^2+1+M^3}{4n^2} \leq 3\epsilon/3 = \epsilon$. We conclude that $f_n'(x) \to x$ uniformly on any domain $[0, M]$ and thus $f'(x) = x$.

Exercise 2: Abbott 6.4.2

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly then (g_n) converges to zero. True since uniform convergence implies point-wise convergence and that implies convergence for any value of x, if, for a particular x, $\sum_{n=1}^{\infty} g_n(x)$ then $g_n(x) \to 0$. We conclude $g_n(x) \to 0$ for all x.

(b) Suppose $0 \leq f_n \leq g_n$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly. Choose $\epsilon > 0$. Note that there exists a $N \in \mathbb{N}$ such that $\forall n > m \geq N$, $\sum_{k=m}^{n} g_k = |\sum_{k=m}^{n} g_k| < \epsilon$. Define N in this manner. Choose $n > m \geq N$. Note that $|\sum_{k=m}^{n} f_k| = \sum_{k=m}^{n} f_k \leq \sum_{k=m}^{n} g_k < \epsilon$, therefore $\sum_{k=m}^{n} f_k$ converges uniformly.

(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

False, as a counterexample let $f_n(x) = \begin{cases} e^x & x = 1 \\ -e^x & x = 2 \\ 0 & \text{otherwise} \end{cases}$. Note that $\sum_{n=1}^{\infty} f_n$ converges uniformly on $\mathbb{R}$, however there is no upper bound on $f_1(x) = e^x$.

Exercise 3: Abbott 6.4.3

(a) Show that

$$g(x) = \sum_{n=0}^{\infty} \frac{cos(2^n x)}{2^n}$$

is continuous on all of $\mathbb{R}$.

Note that $|\frac{cos(2^n x)}{2^n}| \leq 1/2^n$ for all x, n. Also note that $\sum_{n=0}^{\infty} 1/2^n$ is a geometric series and thus converges. By the Weierstrass M-test we note that $\sum_{n=0}^{\infty} \frac{cos(2^n x)}{2^n}$ converges uniformly.

(b) The function g was cited in Section 5.4 as an example of a continuous nowhere differentiable function. What happens if we try to use Theorem 6.4.3 to explore whether g is differentiable?
The problem with using this theorem is that $\sum_{n=0}^{\infty} g'(x) = \sum_{n=0}^{\infty} -sin(2^n x)$ witch does not converge for some x values, and thus fails one of our assumptions for that theorem.

Exercise 4: Abbott 6.4.7

Let

$$f(x) = \sum_{k=1}^{\infty} \frac{sin(kx)}{k^3} = f_k(x).$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.
Choose $x \in \mathbb{R}$. Note that $|\frac{sin(kx)}{k^3}| \leq \frac{1}{k^3} \leq \frac{1}{k^2}$, for all $k \in \mathbb{N}$. Noting that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges we can say via the comparison test that $\sum_{k=1}^{\infty} \frac{sin(kx)}{k^3}$ converges point-wise to some function f.

Note that $f_k'(x) = fraccos(kx)k^2$. Note that $f_k'(x) \leq 1/k^2$, and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges so $f_k'(x) = fraccos(kx)k^2$ converges uniformly and f' exists. Note That since each of our finite sums $\sum_{k=1}^{\infty} \frac{sin(kx)}{k^3}$ are the continuous and they converge uniformly on $f'(x)$ we can say $f'(x)$ is continuous.

(b) Can we determine if f is twice-differentiable?
No, at least not using this procedure, the failure occurs due to $f_k''(x) = frac{-}sin(kx)k$. Witch does not have a limiting function.

Exercise 5: Abbott 6.5.4

Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is defined on $(-R, R)$ and satisfies $F'(x) = f(x)$.
Choose $x \in (-R, R)$. By the algebraic limit theorem for series we know that $\sum_{n=0}^{\infty} R|a_n x^n|$ converges. Since $|\frac{a_n}{n+1} x^{n+1}| \leq R|a_n x^n|$ we can say that $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is defined on $(-R, R)$. Note that $f_n = F_n'$. Also note that f_n and F_n are power series that converge point-wise, thus they each converge uniformly and $F'(x) = f(x)$.

(b) Anti-derivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.
Suppose $g_n = \sum_{k=0}^{n} b_k x^k$ and $g_n \to g$ point-wise. Note that $g_n \to g$ uniformly. Further suppose $g' = f$. Note that $g^{(n+1)}(0) = b_{n+1}(n+1)! = f^{(n)} = a_n(n)!$ or $b_{n+1} = a_n/(n+1)$. This fixes all b_n except b_0. Suppose the most general case b_0 is a arbitrary real. Note that $g'(x) = \sum_{n=1}^{\infty} b_n n x^{n-1} = \sum_{n=0}^{\infty} b_{n+1}(n+1)x^n = f(x)$ so in the most general case $b_{n+1} = a_n/(n+1)$ and b_0 is a arbitrary real.

Exercise 6: Abbott 6.5.5

Theorem 6.5.6 states that if $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on compact sets contained in $(-R, R)$.

(a) If s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $n \geq 1$.
Observe that the difference between successive terms is $(n+1)s^n - ns^{n-1} = (n+1)s^n - ns^{n-1} = (sn+s-n)s^{n-1}$. Note that the sequence ns^{n-1} will be decreasing if $(sn+s-n)s^{n-1}$, in other words if $\frac{s}{1-s} < n$. Since $0 < s < 1$ $\frac{s}{1-s}$ is some number, after witch we will be guaranteed to be decreasing. Since this sequence is bounded below by 0 and eventually decreasing it converges. Recalling that the terms of a convergent sequence are bounded we can say that the terms ns^{n-1} are bounded.

(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use this start to construct a proof for Theorem 6.5.6

Proof. Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$. Choose $x \in (-R, R)$. Note that there exists $t > 0$ such that $|x| < t < R$. Note that $|na_n x^{n-1}| = n|a_n|(|x|/t)^{n-1}t^{n-1}$. Let l be a upper bound on $n(|x|/t)^{n-1}$. Note $|na_n x^{n-1}| \leq l/t|a_n|t^n$. Since $\sum_{n=0}^{\infty} |a_n t^n|$ converges we can say that $\sum_{n=0}^{\infty} |na_n x^{n-1}|$ converges and thus $\sum_{n=0}^{\infty} na_n x^{n-1}$ converges. □

Exercise 7: Abbott 6.5.6

Previous work on geometric series (Example 2.7.5) justifies the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \text{ for all } |x| < 1.$$

Use the results about power series proved in this section to find values for $\sum_{n=1}^{\infty} n/2^n$ and $\sum_{n=1}^{\infty} n^2/2^n$. The discussion in Section 6.1 may be helpful.
Consider the power series $f(x) = \sum_{n=0}^{\infty} x^n$. Recall that this converges for all $x \in (-1, 1)$. Note that it's derivative is $g(x) = \sum_{n=0}^{\infty}(n/x)x^n$ and it's second derivative is $h(x) = \sum_{n=0}^{\infty}(n(n-1)/x^2)x^n$. Noting that power series give us uniform convergence we see that $f'(x) = g(x)$ and $f''(x) = h(x)$. In other words $\frac{1}{(1-x)^2} = g(x)$ and $\frac{2(1-x)}{(1-x)^4} = h(x)$, when $x \in (-1, 1)$. Note that $4 = \frac{1}{(1-1/2)^2} = g(1/2) = \sum_{n=0}^{\infty} 2n/2^n = 2\sum_{n=0}^{\infty} n/2^n$, thus $2 = \sum_{n=0}^{\infty} n/2^n$. Note that $2^4 = \frac{2(1-1/2)}{(1-1/2)^4} = h(1/2) = sum_{n=0}^{\infty}4(n^2 - n)(1/2)^n = sum_{n=0}^{\infty}4n^2(1/2)^n - 4sum_{n=0}^{\infty}n(1/2)^n = 4sum_{n=0}^{\infty}n^2(1/2)^n - 8$ thus $6 = sum_{n=0}^{\infty}n^2(1/2)^n$.

Exercise 8: Abbott 6.6.8

(a) First establish a lemma: if g and h are differentaible on $[0, x]$ with $g(0) = h(0)$ and $g'(t) \leq h'(t)$ for all $t \in [0, x]$, then $g(t) \leq h(t)$ for all $t \in [0, x]$.

Proof. Suppose g and h are differentaible on $[0, x]$ with $g(0) = h(0)$ and $g'(t) \leq h'(t)$ for all $t \in [0, x]$. Consider a new function $f(x) = h(x) - g(x)$. Note that $f'(x) = h'(x) - g'(x)$, and thus $f'(t) \geq 0$ for all $t \in [0, x]$. Choose $t \in [0, x]$. Note that $f(t) = f(0) + f'(\xi)t$ where $\xi \in [0, t]$. Since $f'(\xi)t \geq 0$, $f(t) \geq f(0) = 0$. Thus $h(t) \geq g(t)$. □

(b) Let f, S_N, and E_N, be as Theorem 6.6.3, and take $0 < x < R$. If $|f^{N+1}(t)| \leq M$ for all $t \in [0, x]$, show

$$|E_N(x)| \leq \frac{Mx^{N+1}}{(N+1)!}.$$

Note that $|E_N(x)| = \frac{|f^{(N+1)}(\xi)|x^{N+1}}{(N+1)!} \leq \frac{Mx^{N+1}}{(N+1)!}$.

(W) (Hand this one in to David.)

Exercise 9: Let $f(x) = \frac{1}{\sqrt{1+x}}$. Compute the Taylor series $S_\infty(x)$ for f and then use the remainder theorem to prove that in fact $f(x) = S_\infty(x)$ for all $x \in (-\frac{1}{2}, \frac{1}{2})$.

Let's start by establishing a rule for taking the derivative of $f_n(x) = \frac{1}{\sqrt{x}x^n}$ where $n \in \mathbb{N}$. Using product rule we get $f_n'(x) = [\frac{-1}{2\sqrt{x}xx^n} + \frac{-n}{\sqrt{x}x^{n+1}}] = \frac{1}{\sqrt{x}x^{n+1}}[-1/2 - n] = f_{n+1}(x)[-1/2 - n]$.

Using chain rule note that $f(x) = f_0(1 + x)$ and so $f'(x) = f_1(1 + x)[-1/2 - 0]$. It is trivial to show via induction that $f^{(n)}(x) = f_n(1 + x)\prod_{k=0}^{n-1}[-1/2 - k]$. Note that $f_n(1 + 0) = 1$ thus $f^{(n)}(0) = \prod_{k=0}^{n-1}[-1/2 - k]$.

We can now put down the Taylor series, $S_\infty(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} \prod_{k=0}^{i-1}[-1/2 - k]$.

Choose $x \in (-\frac{1}{2}, \frac{1}{2})$. What is the error in the nth Taylor estimation for $f(x)$? We can use the Lagrange remainder theorem to find out, $|E_n(x)| = |\frac{x^{(n+1)}}{(n+1)!} f_{(n+1)}(1 + \xi)\prod_{k=0}^{(n+1)-1}[-1/2 - k]|$ for some $\xi \in (-\frac{1}{2}, \frac{1}{2})$. Note that $|\prod_{k=0}^{(n+1)-1}[-1/2 - k]| = \prod_{k=0}^{n}[1/2 + k] \leq \prod_{k=0}^{n}[1 + k] = (n + 1)!$. Note that $|x^{(n+1)} f_{(n+1)}(1 + \xi)| = |\frac{x^{n+1}}{\sqrt{1+\xi}(1+\xi)^{n+1}}| = \frac{|x|^{n+1}}{\sqrt{1+\xi}(1+\xi)^{n+1}}$. Define $|x|/(1 + \xi) = b$, note that $|x| < 1/2$ and $(1 + \xi) > 1/2$ so $b \in [0, 1)$. Note that $\frac{|x|^{n+1}}{(1+\xi)^{n+1}} = (|x|/(1 + \xi))^{n+1} = b^{n+1}$. Thus $|E_n(x)| \leq \frac{b^{n+1}(n+1)!}{\sqrt{1+\xi}(n+1)!} = \frac{b^{n+1}}{\sqrt{1+\xi}}$. Note that $\sqrt{1 + \xi} \geq \sqrt{1/2} \geq 1/2$ thus $\frac{1}{\sqrt{1+\xi}} \leq 2$. We now know $|E_n(x)| < 2b^{n+1}$.

Simply note that as we take $n \to \infty$, $b^{n+1} \to 0$, and thus $|E_n(x)| \to 0$. We can now conclude our Taylor estimation is exact ($E_\infty(x) = 0$) for all $x \in (-\frac{1}{2}, \frac{1}{2})$.