**Exercise 1.4.1:** Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$ then $ab$ and $a + b \in \mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$ and if $a \neq 0$ then $at \in \mathbb{I}$ as well.

(c) Part (a) says that the rational numbers are closed under multiplication and addition. What can be said about $st$ and $s + t$ when $s, t \in \mathbb{I}$?

(a) *Proof.* Select two arbitrary elements from the rational numbers, since they are rational we can represent them as $i/j$ and $m/n$ where $i, j, m, n \in \mathbb{Z}$ and $j \neq 0$ and $n \neq 0$.

Note that $i/j * m/n = \frac{im}{jn}$, from the definition of multiplication of rational numbers. Since the multiple of any two non-zero numbers is non-zero and since the multiple of any two integers is a integer so $jn \in \mathbb{Z} - \{0\}$ and $im \in \mathbb{Z}$ therefore $\frac{im}{jn} \in \mathbb{Q}$.

Note that $i/j + m/n = \frac{in+mj}{jn}$, from the definition of addition of rational numbers. Since the multiple of any two non-zero numbers is non-zero and since the multiple of any two integers is a integer so $jn \in \mathbb{Z} - \{0\}$ and $in, mj \in \mathbb{Z}$ and so also $in + mj \in \mathbb{Z}$ therefore $\frac{in+mj}{jn} \in \mathbb{Q}$. □

(b) *Proof.* Proof by contradiction.
Suppose that there exists $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ where $a + t = b \notin \mathbb{I}$. Since the reals are closed under addition we can say $b \in \mathbb{R}$. Note that $b \in \mathbb{R} - \mathbb{I} = \mathbb{Q}$. We can do a little math and see $a + t = b$ means $t = b + (-a)$. Since the additive inverse of a rational is a rational and since the sum of two rationals is rational we conclude $t \in \mathbb{Q}$. A contradiction has been reached since the rationals and irrationals are, by definition, mutually exclusive, $t$ cannot be a element of both.

Proof by contradiction.
Suppose that there exists $a \in \mathbb{Q} - \{0\}$ and $t \in \mathbb{I}$ where $at = b \notin \mathbb{I}$. Since the reals are closed under multiplication, and since the multiple of any two non zero numbers is itself non zero, we can say $b \in \mathbb{R} - \{0\}$. Note that $b \in \mathbb{R} - \{0\} - \mathbb{I} = \mathbb{Q} - \{0\}$. We can do a little math and see $at = b$ means $t = b(a^{-1})$. Since the multiplicative inverse of a non zero rational is a rational (informally $(\frac{i}{m})(\frac{m}{i}) = 1$) and since the multiple of two rationals is rational we conclude $t \in \mathbb{Q}$. A contradiction has been reached since the rationals and irrationals are, by definition, mutually exclusive, $t$ cannot be a element of both. □

(c) All we can conclude is that $st \in \mathbb{R} - \{0\}$ and that $s + t \in \mathbb{R}$. As a example that the irrationals are not closed with respect to multiplication or addition note that $\sqrt{2}\sqrt{2} = 2$ and that $\pi + (-\pi) = 0$.

**Exercise 1.4.2:** Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + (1/n)$ is an upper bound for $A$ but $s - (1/n)$ is not an upper bound for $A$. Show that $s = \sup A$.

*Proof.* Take some $a \in \mathbb{R}$ where $s < a$. Note $a - s \in \mathbb{R}^+$ and so by the proof we previously completed in class there is a natural number $n$ with the property $\frac{1}{n} < a - s$. So $s + \frac{1}{n} < a$ witch means $a$ is bigger than a upper bound on $A$ and so is itself not in $A$. Since we chose a arbitrary real grater than $s$ and showed it is not in $A$ we can conclude no real bigger than $s$ is in $A$, in other words all elements of $A$ are less than or equal to $s$. By definition $s$ is a upper bound on $A$.
Next take some $b \in \mathbb{R}$ where $b < s$. Note $s - b \in \mathbb{R}^+$ and so by the proof we previously completed in class there is a natural number $n$ with the property $\frac{1}{n} < s - b$. So $b < s - \frac{1}{n}$ witch means $b$ is smaller than a number that is not a upper bound on $A$. Noting that if $b$ were upper bound on $A$ we would be forced to conclude $s - \frac{1}{n}$ is a upper bound on $A$, witch we know to be false, we are forced to conclude $b$ is not a upper bound on A. and since $b$ was chosen arbitrarily we can conclude all upper bounds on $A$ are grater than or equal to $s$. Noting that $s$ has both properties of a sup we conclude $s = \sup A$. □

**Exercise 1.4.3:** Show that $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

*Proof.* Proof by contradiction.
Suppose $a \in \cap_{n=1}^{\infty}(0, 1/n)$. Note that $a \in (0, 1) \subseteq \mathbb{R}^+$ so $a \in \mathbb{R}^+$. By the proof done in class there exists a natural i with the property $1/i < a$. Since a is in the intersect of all of the $(0, 1/n)$ sets $a \in (0, 1/i)$ so $a < 1/i$ a contradiction. We now conclude the negation of our supposition namely there is no $a$ with the property $a \in \cap_{n=1}^{\infty}(0, 1/n)$ or a logically equivalent statement $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$. □

**Exercise 1.4.5:** Use Exercise 1.4.1 to provide a proof of Corollary 1.4.4 by considering real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

*Proof.* We are asked to prove that for all $a, b \in \mathbb{R}$ where $a < b$ there is a $t \in (a, b)$ where $t \in \mathbb{I}$.
I will start by choosing a arbitrary set $a, b \in \mathbb{R}$ where $a < b$. Next let $c = a + \frac{1}{3}(b - a)$ and $d = a + \frac{2}{3}(b - a)$. Note that $a < c < d < b$ so $[c, d] \subseteq (a, b)$. Now consider two cases, ether $c, d \in \mathbb{Q}$ or one or more of them is irrational. In the case that one or more of them is irrational then it would be a irrational in $(a, b)$ and we're done. So now we only need to consider the case where $c, d \in \mathbb{Q}$. In this case consider $t = c + \frac{\sqrt{2}}{2}(d - c)$. Noting that $(d - c) \neq 0$ we conclude $\frac{\sqrt{2}}{2}(d - c) \in \mathbb{I}$ and thus $c + \frac{\sqrt{2}}{2}(d - c) \in \mathbb{I}$ also note that $0 < \frac{\sqrt{2}}{2} < 1$ so $c < c + \frac{\sqrt{2}}{2}(d - c) < d$ therefore $c + \frac{\sqrt{2}}{2}(d - c) \in (a, b)$ and we have found a irrational in $(a, b)$. □

**Exercise Supplemental 1:** Show that the sets $[0, 1)$ and $(0, 1)$ have the same cardinality.

Consider the function $f : [0, 1) \to (0, 1)$

$$f(x) = \begin{cases} 1/2 & x = 0 \\ (1/2)^{n+1} & x = (1/2)^n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

note that f has a inverse $g : (0, 1) \to [0, 1)$

$$g(x) = \begin{cases} 0 & x = 1/2 \\ (1/2)^{n-1} & x = (1/2)^n \text{ for some } n \in \mathbb{N} - \{1\} \\ x & \text{otherwise} \end{cases}$$

We now know $f$ is bijective and so we have found a bijective mapping from $[0, 1)$ to $(0, 1)$ so by definition they have the same cardinality.

**Exercise 1.4.4:** **(W) (Hand this one in to David.)**

Let $a < b$ be real numbers and let $T = [a, b] \cap \mathbb{Q}$. Show that $\sup T = b$.

*Proof.* First we note that $b$ is a upper bound on $T$, since $T \subseteq [a, b]$ and every element of $[a, b]$ is less than or equal to b.
Assume that there is a upper bound on $T$ less than $b$, lets call it $c$. We know that there is at least one element in $T$ since we proved in class that there is a rational between any two reals take one of these rationals and call it $d$. Note $d \geq a$. From that we can say $c \geq a$ as otherwise $c < d$ witch is not possible since $c$ is a upper bound. Next we note that there is a rational between $c$ and $b$, lets call it $f$ such that $a \leq c < f < b$ and $f \in \mathbb{Q}$. Note that $f \in T$. Contradiction there is a element of $T$ grater than $c$ and yet $c$ is a upper bound on $T$. Thus we conclude that there are no upper bounds on $T$ less than $b$.
We see that $b$ fulfills the definition of $\sup T$ and so $b = \sup T$. $\square$