Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified. I am also operating under the convention $v_a(b) = \{x \in \mathbb{R} : b - a < x < b + a\}$

Exercise : IVT

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function with $f(a) < f(b)$. Choose $v \in \mathbb{R}$ such that $f(a) < v < f(b)$. Define for $Y \subseteq \mathbb{R}$, $f^{-1}(Y) = \{a \in A : f(a) \in Y\}$. Define $A_v = f^{-1}((-\infty, v))$. Note that $f(a) < v$ and so $a \in A_v$. Note that for all $x \in A_v$, $x \in A$ and thus $x \leq b$ and so b is a upper bound on A_v. Since A_v is bounded and non-empty it has a suppremum. Define $x = \sup(A_v)$. We have previously proven there is a sequence A_v that converges to x, This can be easily proven since $[\sup(S) - 1/n, \sup(S)] \cap S \neq \emptyset$ for all $n \in \mathbb{N}$, call this sequence $\{a_n\}$. Note that $f(a_n) \in (-\infty, v)$ since $a_n \in A_v$, thus $f(a_n) < v$. By the limit Order theorem $f(x) \leq v$. Note that $x < \frac{x^n+b}{n+1} = z_n < b$ for all n, and $z_n \to x$. Since $x = \sup(A_v)$ we know that $z_n \notin A_v$ thus $\neg f(z_n) \in (-\infty, v)$ and so $f(z_n) \geq v$. By the limit order theorem $f(x) \geq v$. We thus conclude $f(x) = v$. □

Exercise : Abbott 4.2.10

(a) Define sided neighborhoods as $V_\epsilon^+(c) = \{x \in \mathbb{R} : 0 \leq x - c < \epsilon\}$ and $V_\epsilon^-(c) = \{x \in \mathbb{R} : \epsilon < x - c \leq 0\}$. We can now define sided limit points of A, c is a positive limit point of A if $\forall \epsilon > 0, V_\epsilon^+(c) \cap A - \{c\} \neq \emptyset$, and c is a negative limit point of A if $\forall \epsilon > 0, V_\epsilon^-(c) \cap A - \{c\} \neq \emptyset$.
Let $f : A \to \mathbb{R}$, and let c be a positive limit point of A. We say that $\lim_{x \to c^+} f(x) = L$ if $\forall \epsilon > 0, \exists \delta > 0$ such that if $0 < x - c < \delta$ then $|f(x) - L| < \epsilon$.
Let $f : A \to \mathbb{R}$, and let c be a negative limit point of A. We say that $\lim_{x \to c^-} f(x) = L$ if $\forall \epsilon > 0, \exists \delta > 0$ such that if $0 > x - c > -\delta$ then $|f(x) - L| < \epsilon$.

(b) Suppose $f : A \to \mathbb{R}$ and c is both a positive and negative limit point of A.

Suppose $\lim_{x \to c} f(x) = L$. Choose $\epsilon > 0$. There must exist a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$. Choose a x where $0 < x - c < \delta$, note that $0 < |x - c| < \delta$, thus $|f(x) - L| < \epsilon$. Conclude $\lim_{x \to c^+} f(x) = L$. Choose a x where $0 > x - c > -\delta$, note that $0 < |x - c| < \delta$, thus $|f(x) - L| < \epsilon$. Conclude $\lim_{x \to c^-} f(x) = L$.

Suppose $\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$. Choose $\epsilon > 0$. There must exist a $\delta_1 > 0$ such that if $0 < x - c < \delta$ then $|f(x) - L| < \epsilon$. There must exist a $\delta_2 > 0$ such that if $0 > x - c > \delta$ then $|f(x) - L| < \epsilon$. Define $\delta = \min(\delta_1, \delta_2)$. Choose a x where $0 < |x - c| < \delta$. Note that eater $0 < x - c < \delta \leq \delta_1$ or $0 > x - c > -\delta \geq -\delta_2$. Conclude $|f(x) - L| < \epsilon$, thus $\lim_{x \to c} f(x) = L$.

Exercise : Suppose $f : [a, b] \to \mathbb{R}$ is increasing. Show that for each $c \in (a, b]$, $lim_{x \to c^-} f(x)$ exists. State, but do not prove, a similar result for limits from the right.

Proof. Suppose $f : [a,b] \to \mathbb{R}$ is increasing. Choose $c \in (a,b]$.
Choose $\epsilon > 0$. Note that $\max(a, c-\epsilon/2) \in V^-_\epsilon(c) \cap A - \{c\}$ and thus c is a negative limit point of $[a,b]$.
Define $L = \sup(f([a,c)))$, note that $f([a,c))$ is bounded, by $f(c)$, and non-empty, contains $f(a)$, and thus admits a suppremum. Choose $\epsilon > 0$. Define $A_\epsilon = f^{-1}((L-\epsilon/2, L])$. We know that $L-\epsilon/2$ is not a upper bound on $f([a,c))$ thus there exists $f(d) \in f([a,c))$ where $f(d) > L-\epsilon/2$. Note that $f(d) \leq L$, $d \in A_\epsilon$. Also note that A_ϵ is bounded below by a thus A_ϵ admits a infimum. Note that $\inf(A_\epsilon) \leq d < c$. Define $\delta = c - \inf(A_\epsilon) > 0$. Choose a x where $0 > x-c > -\delta$. Note that $c > x > \inf(A_\epsilon)$, thus $f(c) \geq f(x) \geq f(\inf(A_\epsilon)) \geq L-\epsilon/2$. Note that $f(x) \in f([a,c))$ thus $f(x) \leq L$. Conclude $-\epsilon < x-L \leq 0 < \epsilon$ thus $|x-L| < \epsilon$, and $lim_{x\to c^-} f(x) = L$ exists. □

Suppose $f : [a,b] \to \mathbb{R}$ is increasing. For each $c \in [a,b)$, $lim_{x\to c^+} f(x)$ exists.

Exercise : Suppose that $f : [a,b] \to \mathbb{R}$ is increasing. Show that for each $c \in (a,b)$, $lim_{x\to c^-} f(x) \leq f(c) \leq lim_{x\to c^+} f(x)$.

Proof. Suppose that $f : A = [a,b] \to \mathbb{R}$ is increasing. Choose $c \in (a,b)$. Note that by the above proof $lim_{x\to c^-} f(x) = L_-$ and $lim_{x\to c^+} f(x) = L_+$ exist.
Suppose $f(c) < L_-$. Let $\epsilon = L_- - f(c) > 0$. There exists a $\delta > 0$ such that if $0 > x-c > -\delta$ then $|f(x) - L_-| < \epsilon$. Note that $V^-_\delta(c) \cap A - \{c\} \neq \emptyset$, take $x \in V^-_\delta(c) \cap A - \{c\}$. Note that $\delta < x-c < 0$. Note that $x < c$, so $f(x) \leq f(c)$. Note that $|L_- - f(x)| < \epsilon$, $L_- - f(x) < L_- - f(c)$, $-f(x) < -f(c)$,$f(x) > f(c)$ a contradiction, we thus conclude $L_- \leq f(c)$.
Suppose $f(c) > L_+$. Let $\epsilon = f(c) - L_+ > 0$. There exists a $\delta > 0$ such that if $0 < x-c < \delta$ then $|f(x) - L_+| < \epsilon$. Note that $V^+_\delta(c) \cap A - \{c\} \neq \emptyset$, take $x \in V^-_\delta(c) \cap A - \{c\}$. Note that $\delta > x-c > 0$. Note that $x > c$, so $f(x) \geq f(c)$. Note that $|f(x) - L_+| < \epsilon$, $f(x) - L_+ < f(c) - L_+$, $f(x) < f(c)$ a contradiction, we thus conclude $L_+ \geq f(c)$.

□

Exercise : Suppose that $f : [a,b] \to \mathbb{R}$ is increasing and $f([a,b]) = [f(a), f(b)]$. Show that f is continuous.

Proof. Suppose that $f : A = [a,b] \to \mathbb{R}$ is increasing and $f([a,b]) = [f(a), f(b)]$.
Choose $c \in [a,b]$. Choose $\epsilon > 0$. Define $y^+ = \min(f(c)+\epsilon/2, f(b))$. Define $y^- = \max(f(c)-\epsilon/2, f(a))$. Note that $f(a) \leq y^- < y^+ \leq f(b)$, thus $y^-, y^+ \in [f(a), f(b)]$ and $y^-, y^+ \in f([a,b])$. Since $y^-, y^+ \in f([a,b])$ there must exist a $x^-, x^+ \in [a,b]$ such that $f(x^-) = y^-$ and $f(x^+) = y^+$. Note that $f(x^-) \leq f(c) \leq f(x^+)$ thus $x^- \leq c \leq x^+$. Define

$$\delta = \begin{cases} \min(c-x^-, x^+-c) & x^+ \neq c \wedge x^- \neq c \\ c-x^- & x^+ = c \wedge x^- \neq c \\ x^+-c & x^+ \neq c \wedge x^- = c \end{cases}$$

noting that $a \neq b$, $y^+ \neq y^-$, $x^+ \neq x^-$. Note $\delta > 0$. Choose $x \in [a, b]$ such that $|x - c| < \delta$. Note that $c - \delta < x < c + \delta$, also $a \geq x \leq b$. Note that if $x^+ = c$ then $c = b$ and if $x^- = c$ then $c = a$. Thus in all three cases for δ note that $x^- \leq x \leq x^+$. Note that $f(c) - \epsilon/2 \leq y^- \leq f(x) \leq y^+ \leq f(c) + \epsilon/2$. Note that $-\epsilon < f(x) - f(c) < \epsilon$. Thus f is continuous at all points in $[a, b]$. □

Exercise : Suppose that $f : [a, b] \to \mathbb{R}$ is increasing but discontinuous. Show that $f([a, b]) \subset [f(a), f(b)]$.

Proof. Suppose that $f : [a, b] \to \mathbb{R}$ is increasing but discontinuous. Choose $y \in f([a, b])$. Note that there exists a $x \in [a, b]$ where $f(x) = y$. Note that $a \leq x \leq b$ implies that $f(a) \leq f(x) \leq f(b)$ thus $y \in [f(a), f(b)]$. Hence $f([a, b]) \subseteq [f(a), f(b)]$.
Suppose $f([a, b]) = [f(a), f(b)]$. By the above proof f is continuous, a contradiction and thus we conclude $f([a, b]) \neq [f(a), f(b)]$.
Conclude $f([a, b]) \subset [f(a), f(b)]$. □

Exercise : 5.2.5
Let $f_a(x) = \begin{cases} x^a & x > 0 \\ 0 & x \leq 0 \end{cases}$.

(a) For what a is f_a continuous at 0?
We know that all functions of the form x^a are continuous everywhere that they can be evaluated, thus f_a will be continuous at 0 if and only if $0^a = 0$. This is true if and only if $a > 0$, as $a \leq 0$ gives us a undefined value of $f_a(0)$, (I think 0^0 is undefined).

(b) For what a is f_a differentiable at 0?
the function f_a is differentiable at 0 if and only if $f_a(x) = f(0) + \mu(x) * x = \mu(x) * x$ where $\mu(x)$ is continuous at 0. $f_a(x) = x^{a-1} * x$, note that $x^{a-1}$is continuous only when $a > 1$. Note that if f_a is differentiable $f_a'(0) = x^{a-1} = 0$ so $f_a'(x)$ is continuous if it exists.