Exercise 1: Abbott 7.2.5

Suppose $\{f_n\}$ are a sequence of functions uniformly convergent on f, and suppose that $f_n \in R[a,b]$. Choose $\epsilon > 0$. Define $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [a,b]$, $|f_n(x) - f(x)| < \alpha = \epsilon/(4(b-a))$. Define $P \in P[a,b]$ such that $U(f_N, P) - L(f_N, P) < \beta = \epsilon/2$. Define M_k and m_k to be the suppremum and infimum for f_N in the kth interval of P. Define n to be the number of partitions in P. Define Δx_k to be the width of the kth interval in P. Note that $U(f_N, P) - L(f_N, P) = \sum_{k=1}^{n}(M_k - m_k)\Delta x_k < \beta$. Note that $|f_N(x) - f(x)| < \alpha$ or $f_N(x) - \alpha < f(x) < f_N(x) + \alpha$. Consider a perticular interval, I_k. Note that in this interval $m_k - \alpha \leq f_N(x) - \alpha < f(x) < f_N(x) + \alpha \leq M_k + \alpha$. We can now see $U(f, P) \leq \sum_{k=1}^{n}(M_k + \alpha)\Delta x_k = U(f_N, P) + \alpha(b-a)$ and $L(f, P) \geq \sum_{k=1}^{n}(m_k - \alpha)\Delta x_k = L(f_N, P) - \alpha(b-a)$ thus $U(f, P) - L(f, P) \leq U(f_N, P) + 2\alpha(b-a) - L(f_N, P) < \beta + 2\alpha(b-a) = \epsilon$. We conclude $f \in R[a,b]$.

Exercise 2: Abbott 7.2.7

Suppose $f : [a,b] \to \mathbb{R}$ is a increasing function. Choose $\epsilon > 0$. Define $n \in \mathbb{N}$ such that $1/n < \gamma = \epsilon/(f(b) - f(a))(b-a)$. Define $\Delta x = (b-a)/n$. Define $x_0 = a$, $x_k = x_{k-1} + \Delta x$ for all $k \in [1,n]$. Note that $x_n = x_0 + n\Delta x = b$. We can define $P \in P[a,b]$ as the partition using $\{x_k\}_{k=0}^{n}$. Note that $f(x_{k-1}) \leq f(x) \leq f(x_k)$ for $x \in I_k$ the kth interval in P. Thus $f(x_k) \geq \sup(f(I_k))$ and $f(x_{k-1}) \leq \inf(f(I_k))$ for all $k \in [1,n]$. Note that $\sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = f(x_n) - f(x_0) = f(b) - f(a)$. Note that $U(f,P) - L(f,P) = \sum_{k=1}^{n}(\sup(f(I_k)) - \inf(f(I_k)))\Delta x \leq \Delta x \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = \Delta x(f(b) - f(a)) = (f(b) - f(a))(b-a)/n < (f(b) - f(a))(b-a)\gamma = \epsilon$. We conclude $f \in R[a,b]$.

Exercise 3: Abbott 7.3.4