Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified. I am also operating under the convention $v_a(b) = \{x \in \mathbb{R} : |x - b| < a\}$

Exercise 1: Suppose $f : A \to \mathbb{R}$ and c is a limit point of A. Suppose $f(x) \geq 0$ for all $x \in A$ and that $lim_{x \to c} f(x)$ exists. Show that the limit is non-negative. Provide two proofs, one ϵ - δ style, and the other using the sequential characterization of limits.

Proof. Suppose $f : A \to \mathbb{R}$ and c is a limit point of A. Suppose $f(x) \geq 0$ for all $x \in A$ and that $lim_{x \to c} f(x) = L$ exists.
Suppose $L < 0$. Define $\epsilon = -L/2 > 0$. There must exist a $\delta > 0$ such that for all $x \in A$ where $0 < |x - c| < \delta$, $|f(x) - L| < \epsilon$. Note that c is a limit point of A, thus $v_\delta(c) \cap A - \{c\} \neq \emptyset$. Take one of the elements of this set $a \in v_\delta(c) \cap A - \{c\}$. Note that $a \neq c$. Note that $a \in A$. Note $|a - c| < \delta$. Thus $|f(a) - L| < \epsilon$ and so $L - \epsilon < f(a) < L + \epsilon = L/2 < 0$. A contradiction, we know that $f(a) \geq 0$ and now we have $f(a) < 0$. We now conclude $L \geq 0$. □

Proof. Suppose $f : A \to \mathbb{R}$ and c is a limit point of A. Suppose $f(x) \geq 0$ for all $x \in A$ and that $lim_{x \to c} f(x) = L$ exists.
Note that since c is a limit point of A there exists a sequence in A converging on c. Take a_n to be a arbitrary sequence in A converging on c. Note that by the sequential characterization of limits, $f(a_n) \to L$. Note that $f(a_n) \geq 0$ thus by the limit order theorem $L \geq 0$. □

Exercise 2: Let a_n be a sequence of numbers such that for some $M \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n M^n$ converges. Suppose that $|x| < M$. Show that $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely. Give an example to show that divergence is possible if $|x| = |M|$. Hint: $(a_n M^n)$ converges to zero, and is hence bounded.

Proof. Let a_n be a sequence of numbers such that for some $M \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n M^n$ converges. Suppose that $|x| < M$.
Note that by our supposition $|x| < M$ we can conclude $0 < M$. Define $-1 < 0 \leq r = \frac{|x|}{M} < 1$. Note that $\lim_{n\to\infty} a_n M^n = 0$ since $\sum_{n=1}^{\infty} a_n M^n$ converges. Note that a convergent sequence is bounded so we can define $N \in \mathbb{R}$ such that $|a_n M^n| < N$ for all n. Note $|r| < 1$, thus $\sum_{n=1}^{\infty} N r^n$ converges lets call it's limit L. Noting that $0 \leq N$ and $0 \leq r$ we can conclude $0 \leq N r^n$ for $n \in \mathbb{N}$ and thus $\sum_{n=1}^{k} N r^n$ is monotone increasing. Note $\sum_{n=1}^{k} N r^n \leq \sum_{n=1}^{\infty} N r^n = L$. Define $S_k = \sum_{n=1}^{k} |a_n x^n|$. Note that $S_k \leq S_k + |a_{k+1} x^{k+1}| = S_{k+1}$ for $k \in \mathbb{N}$ and thus S_k is monotone increasing. Note that $|a_n| < \frac{N}{M^n}$, recall $|a_n M^n| < N$ and $0 \leq M$. Note that $S_k = \sum_{n=1}^{k} |a_n x^n| < \sum_{n=1}^{k} \frac{N}{M^n} |x^n| = \sum_{n=1}^{k} N r^n \leq L$. By the MCT S_k converges and thus $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely.

□

For a example where $|x| = |M|$ and $\sum_{n=1}^{\infty} a_n x^n$ does not converges absolutely, consider the case $a_n = (-1)^n 1/n$, $M = 1$, $x = 1$. Note that $\sum_{n=1}^{\infty} a_n M^n = \sum_{n=1}^{\infty} (-1)^n 1/n$ converges. However $\sum_{n=1}^{\infty} |a_n x^n| = \sum_{n=1}^{\infty} 1/n$ does not converge.

Exercise 3: Suppose $f : (0, 1] = A \to \mathbb{R}$ is uniformly continuous. Show that $lim_{x \to 0} f(x)$

exists.

Proof. Note that 0 is a limit point of $(0, 1]$. Take a arbitrary sequence $a_n \in A$ where $a_n \to 0$. Define $f_n = f(a_n)$. Choose $\epsilon > 0$. Note that there exists $\delta > 0$ such that for all $x, y \in A$ if $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon$. There exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|a_n - a_m| < \delta$, Caushey criterion for a sequence. Choose $m, n \geq N$. Note that $|f_n - f_m| < \epsilon$. Thus f_n is a Caushey sequence and thus converges. We now have established that for any sequence $a_n \in A$ where $a_n \to 0$, $f(a_n)$ converges to some limit.

Consider two sequences $a_n \in A$ where $a_n \to 0$ and $b_n \in A$ where $b_n \to 0$. Note that $f(a_n)$ converges and $f(b_n)$ converges, define there limits to be L_a and L_b. Define a new sequence c_n to be the shuffled sequence $a_1, b_1, a_2, b_2, \cdots$. Note that by the shuffled sequence theorem $c_n \to 0$. We conclude $f(c_n)$ converges. Note that $f(c_n)$ is the shuffled sequence of $f(a_n)$ and $f(b_n)$ thus by the shuffled sequence theorem $L_a = L_b = L$. We selected two sequences in A converging on 0 and showed that the sequences made of the functional evaluations of those sequences converge to the same value. We can now conclude all sequences made this way converge to the same value, call this value L. We have now proven that for a arbitrary sequence $a_n \in A$ where $a_n \to 0$, $f(a_n)$ converges to L. By the sequential characterization of limits we can conclude that $lim_{x\to 0} f(x)$ exists and is L □

Exercise 4: Abbott 4.3.11

Let f be a function defined on all of $\mathbb{R}$, and assume there exists a c where $0 < c < 1$ such that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous on all $\mathbb{R}$.
Choose $x \in \mathbb{R}$. Choose $\epsilon > 0$. Define $\delta = \epsilon$. Choose $y \in \mathbb{R}$ such that $|x - y| < \delta$. Note that $|f(x) - f(y)| \leq c|x - y| \leq |x - y| < \delta = \epsilon$. Thus f is continuous on all $\mathbb{R}$.

(b) Show that for all $y_1 \in \mathbb{R}$ the sequence $y_n = f(y_{n-1})$ converges to some y.
Choose $y_1 \in \mathbb{R}$. Note that $|y_{n+1} - y_{n+2}| \leq c|y_n - y_{n+1}|$.

Define $k = \frac{1}{c}|y_2 - y_1|$. For $n = 1$, $|y_{n+1} - y_n| \leq c^n k$ would mean $|y_2 - y_1| \leq |y_2 - y_1|$, clearly true. Suppose $|y_{n+1} - y_n| \leq c^n k$. Note that $|y_{n+2} - y_{n+1}| \leq c|y_{n+1} - y_n|$, so $|y_{n+2} - y_{n+1}| \leq c^{n+1}k$. By induction $|y_{n+1} - y_n| \leq c^n k$ for all natural numbers n.

Note that for $m = 1$, $|y_{n+m} - y_n| \leq c^{n-1}k \sum_{i=1}^m c^i$ would mean $|y_{n+1} - y_n| \leq c^n k$, clearly true. Suppose for some m, $|y_{n+m} - y_n| \leq c^{n-1}k \sum_{i=1}^m c^i$. Note that $|y_{n+m+1} - y_{n+m}| \leq c^{n+m}k$. Note that $|y_{n+m+1} - y_n| = |y_{n+m+1} - y_{n+m} + y_{n+m} - y_n| \leq |y_{n+m+1} - y_{n+m}| + |y_{n+m} - y_n| \leq c^{n-1}k \sum_{i=1}^m c^i + c^{n+m}k = c^{n-1}k(\sum_{i=1}^m c^i + c^{m+1}) = c^{n-1}k \sum_{i=1}^{m+1} c^i$. By induction on m I conclude $|y_{n+m} - y_n| \leq c^{n-1}k \sum_{i=1}^m c^i$ for all natural numbers m.

Note that $\sum_{i=1}^m c^i \leq \sum_{i=1}^\infty c^i = L$ (see geometric series Pg. 73). Thus $|y_{n+m} - y_n| \leq c^{n-1}kL$ for all m and n.

Note that c^n is a monotone decreasing sequence and is bounded below by 0, thus by

MCT it will converge to l. Also note that c^{n+1} will converge to the same value, thus $l = cl$ and so $l = 0$. Thus $c^n \to 0$.

Choose $\epsilon > 0$. Noting that $\frac{c\epsilon}{KL} > 0$ we can say that there exists a natural number N such that for all $n \geq N$, $|c^n| < \frac{c\epsilon}{KL}$. Choose $m > n \geq N$ Define N this way. Define $d = m - n$, note that $d \in \mathbb{N}$. Note that $c^{n-1}kL < \epsilon$. Note that $|y_m - y_n| = |y_{n+d} - y_n| \leq c^{n-1}kL < \epsilon$. We now know the sequence is Caushey and therefore it will converge.

(c) Prove that any y obtained in the manner above will be a fixed point, then prove that y will be unique.
Define $y' = f(y)$. Suppose $y' \neq y$. Note $\epsilon = |y' - y| > 0$. Note that there exists $N \in \mathbb{N}$ such that if $n \geq N$, $|y_n - y| < \epsilon/3$. Note that $|y_{N+1} - y'| = |f(y_N) - f(y)| \leq c|y_N - y| < \epsilon/3$. Note that $|y - y_{N+1}| < \epsilon/3$. Note $\epsilon = |y - y_{N+1} + y_{N+1} - y'| \leq |y - y_{N+1}| + |y_{N+1} - y'| < 2\epsilon/3$, a contradiction, negate our supposition to conclude $y = f(y)$.

Suppose that y is not unique. There must exist y_1 and y_2 with the properties of y and $y_1 \neq y_2$. Note that $|f(y_1) - f(y_2)| = |y_1 - y_2| > c|y_1 - y_2|$. We also know that $|f(y_1) - f(y_2)| \leq c|y_1 - y_2|$, a contradiction and so we must conclude that y is unique.

(d) We have proven that an arbitrary element of the reals will converge to some y under repeated applications of f. We have also proved the uniqueness of y, and can now conclude that any element of the reals will converge under repeated applications of f to some real number y.

Exercise 5: Suppose that $f : (0, 1) \to \mathbb{R}$ is continuous and that $lim_{x\to 0} f(x) = \infty$ and $lim_{x\to 1} f(x) = \infty$. Show that f obtains a minimum on (0,1).

Proof. Suppose that $f : (0, 1) \to \mathbb{R}$ is continuous and that $lim_{x\to 0} f(x) = \infty$ and $lim_{x\to 1} f(x) = \infty$. Define $c_0 = f(1/3)$ and $c_1 = f(2/3)$. There exists a $\delta_0 > 0$ such that if $|x| < \delta_0$, $f(x) > c_0 + 1$. There exists a $\delta_1 > 0$ such that if $|x - 1| < \delta_1$, $f(x) > c_1 + 1$. Suppose $\delta_0 > 1/3$, we would then conclude $c_0 = f(1/3) > c_0 + 1$, a contradiction, thus $\delta_0 \leq 1/3$. Suppose $\delta_1 > 1/3$, we would then conclude $c_1 = f(2/3) > c_1 + 1$, a contradiction, thus $\delta_1 \leq 1/3$. We can now conclude $0 < \delta_0 \leq 1/3 < 2/3 \leq 1 - \delta_1 < 1$. We now conclude $[\delta_0, 1 - \delta_1] \subset (0, 1)$ and $[\delta_0, 1 - \delta_1] \neq \emptyset$ also note that $[\delta_0, 1 - \delta_1]$ is compact. By the extreme value theorem there exists a $m \in [\delta_0, 1 - \delta_1]$ such that $f(m) \leq f(y)$ for all $y \in [\delta_0, 1 - \delta_1]$. Note that $f(m) \leq c_0$ and $f(m) \leq c_1$.
Choose $x \in (0, 1)$. There are three possibilities, $x \in (0, \delta_0)$ or $x \in [\delta_0, 1 - \delta_1]$ or $x \in (1 - \delta_1, 1)$.
Suppose $x \in (0, \delta_0)$. Note that $|x| < \delta_0$ and thus $f(m) \leq c_0 < c_0 + 1 < f(x)$.
Suppose $x \in [\delta_0, 1 - \delta_1]$. Note that $f(m) \leq f(x)$.
Suppose $x \in (0, \delta_0)$. Note that $|x| < \delta_0$ and thus $f(m) \leq c_1 < c_1 + 1 < f(x)$.
We conclude that f achieves a minimum at m in $(0, 1)$. □

Exercise 6: Abbott 4.4.13

(a) *Proof.* Suppose $f : A \to \mathbb{R}$ is uniformly continuous, and $\{x_n\}$ is a Cauchy sequence in A. Choose $\epsilon > 0$. Note that there exists $\delta > 0$ such that for $x, y \in A$ if $|x - y| < \delta$ then

$|f(x) - f(y)| < \epsilon$. There exists a $N \in \mathbb{N}$ such that if $n, m \geq N$, $|x_n - x_m| < \delta$. Choose $n, m \geq N$. Note that $|x_n - x_m| < \delta$, thus $|f(x_n) - f(x_m)| < \epsilon$. Conclude that $f(x_n)$ is a Cauchy sequence. □

(b) *Proof.* Let g be a continuous function on the open interval (a, b).

Suppose it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g' is continuous on $[a, b]$. Note that $[a, b]$ is a compact set, thus g' is continuous on a compact set. By Theorem 4.4.7 g' is uniformly continuous on $[a, b]$. Choose $\epsilon > 0$. There exists a $\delta > 0$ such that for all $x, y \in [a, b]$, if $|x - y| < \delta$ then $|g'(x) - g'(y)| < \epsilon$. Choose $x, y \in (a, b)$. Note that $x, y \in [a, b]$ and $g'(x) = g(x)$, $g'(y) = g(y)$. Note that if $|x - y| < \delta$ then $|g(x) - g(y)| = |g'(x) - g'(y)| < \epsilon$. Conclude g is uniformly continuous on (a, b).

Note that since uniformly continuous functions preserve causheyness they will also preserve convergence of a sequence, if x_n converges $f(x_n)$ will converge if f is uniformly continuous.
Suppose that g is uniformly continuous on (a, b). Select a sequence a_n in (a, b) where $a_n \to a$. Define $g'(a)$ as $g(a_n) \to g'(a)$. Select a sequence b_n in (a, b) where $b_n \to a$. Define a new sequence c_n to be the shuffled sequence $a_1, b_1, a_2, b_2, \cdots$. Note that by the shuffled sequence theorem $c_n \to a$. We conclude $g(c_n)$ converges. Note that $g(c_n)$ is the shuffled sequence of $g(a_n)$ and $g(b_n)$ thus by the shuffled sequence theorem $g(b_n) \to g'(a)$. By the sequential characterization of limits $\lim_{x \to a} g(x) = g'(a)$.
Select a sequence d_n in (a, b) where $d_n \to b$. Define $g'(b)$ as $g(d_n) \to g'(b)$. Select a sequence e_n in (a, b) where $e_n \to b$. Define a new sequence f_n to be the shuffled sequence $d_1, e_1, d_2, e_2, \cdots$. Note that by the shuffled sequence theorem $f_n \to b$. We conclude $g(f_n)$ converges. Note that $g(f_n)$ is the shuffled sequence of $g(d_n)$ and $g(e_n)$ thus by the shuffled sequence theorem $g(e_n) \to g'(b)$. By the sequential characterization of limits $\lim_{x \to b} g(x) = g'(b)$.
Note that the function

$$g'(x) = \begin{cases} g(x) & x \neq a, b \\ g'(a) & x = a \\ g'(b) & x = b \end{cases}$$

is continuous at every point in (a, b) since $g(x)$ is continuous on (a, b) and is continuous at a and b since the value matches the limit at those points.

Now conclude g is uniformly continuous on (a, b) if and only if it is possible to define value $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. □

Exercise 7: Show that if $f : [a, b] \to \mathbb{R}$ is strictly increasing and continuous, then it has a continuous inverse function $f^{-1} : [f(a), f(b)] \to [a, b]$. Use this result to show that $x^{1/n}$ is continuous for each $n \in \mathbb{N}$. You may use any homework problems you have done to help with this. But your proof must give a careful demonstration that the domain of f^{-1} is the whole interval $[f(a), f(b)]$, that the image is exactly $[a, b]$, and that f^{-1} is increasing.

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is strictly increasing and continuous. Define the function $G(y) = \{x \in [a, b] : f(x) = y\}$. Choose $y_- \in (-\infty, f(a))$. Suppose $x \in G(y_-)$, note that $f(x) = y_- < f(a)$ and $a \leq x$ so $f(a) \leq f(x)$, a contradiction thus $G(y_-) = \emptyset$. Choose $y_+ \in (f(b), \infty)$. Suppose $x \in G(y_+)$, note that $f(x) = y_+ > f(b)$ and $x \leq b$ so $f(x) \leq f(b)$, a contradiction thus $G(y_+) = \emptyset$. Choose $y \in [f(a), f(b)]$. By the IVT there exists a x such that $f(x) = y$, thus $G(y) \neq \emptyset$. Suppose $x_1 < x_2 \in G(y)$, note $f(x_1) < f(x_2)$ and $f(x_1) = f(x_2)$, a contradiction thus $G(y)$ only has one element.
We have now proven that $G(y)$ has one element if $y \in [f(a), f(b)]$ and no elements if $y \notin [f(a), f(b)]$. Noting that $G(y) \subseteq [a, b]$ we can now define $f^{-1} : [f(a), f(b)] \to [a, b]$ defined as $f(y)$ is the element in $G(y)$.
Choose $x \in [a, b]$. Note that $f(a) \leq f(x) \leq f(b)$, thus $f^{-1}(f(x))$ is defined. Note from the definition that $x \in G(f(x))$ therefore $f^{-1}(f(x)) = x$. Choose $y \in [f(a), f(b)]$. Note that $f^{-1}(y)$ is defined call it's value c. Note from the definition of G that $f(c) = y$, so $f(f^{-1}(y)) = y$. Conclude that $f^{-1}(y)$ is indeed the inverse function for $f(x)$.
Choose $y_1 < y_2 \in [f(a), f(b)]$. Suppose $f^{-1}(y_1) \geq f^{-1}(y_2)$. We would then conclude $f(f^{-1}(y_1)) \geq f(f^{-1}(y_2))$ or $y_1 \geq y_2$, a contradiction, conclude that f^{-1} is strictly increasing.
Choose $y \in (f(a), f(b))$. Choose $\epsilon > 0$. Define $x = f^{-1}(y)$, note $x \neq a$ and $x \neq b$. Define $x_+ = \min(b, x + \epsilon)$ and $x_- = \max(a, x - \epsilon)$. Note that $x_- < x < x_+$. Note that $f(x_-) < f(x) < f(x_+)$. Define $\delta = \min(f(x) - f(x_-), f(x_+) - f(x)) > 0$. Choose $y' \in [f(a), f(b)]$ such that $|y - y'| < \delta$. Note that $y = f(x)$. Note that $-\delta < y' - f(x) < \delta$ so $f(x_-) < y' < f(x_+)$ and $x - \epsilon \leq x_- < f^{-1}(y') < x_+ \leq x + \epsilon$ or $|f^{-1}(y') - f^{-1}(y)| < \epsilon$. So our inverse function is continuous for all $y \in (f(a), f(b))$.
Choose $\epsilon > 0$. Define $\delta = f(b) - f(b - \epsilon) > 0$. Choose $y \in [f(a), f(b)]$ such that $|f(b) - y| < \delta$. Note that $y \leq f(b)$. Note that $0 \leq f(b) - y < \delta$ or $f(b) \geq y > f(b) - \delta = f(b - \epsilon)$. Note $b \geq f^{-1}(y) > b - \epsilon$ or $|f^{-1}(y) - f^{-1}(f(b))| < \epsilon$. Thus f^{-1} is continuous at $f(b)$.
Choose $\epsilon > 0$. Define $\delta = f(a + \epsilon) - f(a) > 0$. Choose $y \in [f(a), f(b)]$ such that $|f(a) - y| < \delta$. Note that $y \geq f(a)$. Note that $0 \leq y - f(a) < \delta$ or $f(a) \leq y < f(a) + \delta = f(a + \epsilon)$. Note $a \leq f^{-1}(y) < a + \epsilon$ or $|f^{-1}(y) - f^{-1}(f(a))| < \epsilon$. Thus f^{-1} is continuous at $f(a)$.
Thus f^{-1} is continuous on $[f(a), f(b)]$.
Consider $f_n : [0, \infty) \to [0, \infty)$, $f(x) = x^n$ where $n \in \mathbb{N}$. Consider $g_n : [0, \infty) \to [0, \infty)$, $f(x) = x^{1/n}$ where $n \in \mathbb{N}$. Note that g_n and f_n are inverses, $g_n(f_n(x)) = (x^n)^{1/n} = x = (x^{1/n})^n = f_n(g_n(x))$ where $n \in \mathbb{N}$. Consider some point $c \in [0, \infty)$ and some $n \in \mathbb{N}$. Define $x = g_n(c)$. Note that $F[0, x+1] \to \mathbb{R}$ where $F(x) = f_n(x)$, is a strictly increasing continuous function. Note that it has a continuous inverse $G : [f_n(0), f_n(x + 1)] \to [0, x + 1]$. Note that $G(y) = g_n(y)$ for all $y \in [f(0), f(x + 1)]$. Note that $f_n(0) \leq c < f_n(x + 1)$. Conclude that g_n is continuous at c. Since c was chosen arbitrarily we can say that g_n is continuous along its domain. Since n was chosen arbitrarily we can say g_n is continuous for all $n \in \mathbb{N}$. Thus $x^{1/n}$ is continuous for all $n \in \mathbb{N}$. □

Exercise 8: Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and that $f([0, 1]) \subseteq [0, 1]$. Prove that there is a solution of the equation $f(x) = x$.

Proof. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and that $f([0, 1]) \subseteq [0, 1]$. Suppose that there is no solution of the equation $f(x) = x$. In other words $f(x) \neq x$ for all $x \in [0,1]$. Note that $0 < f(0)$ and $f(1) < 1$ or $f(1)-1 < 0$, since we already knew that $f(0), f(1) \in [0,1]$. Define $g : [0,1] \to \mathbb{R}$ where $g(x) = f(x) - x$. Note that g is continuous. Note that $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$ thus by the IVT there exists a $c \in (0,1)$ such that $g(c) = 0$. Note that $g(c) = f(c) - c = 0$ thus $f(c) = c$, a contradiction. We conclude the negation of our supposition there is a solution to the equation $f(x) = x$. □

Exercise 9: Abbott 5.2.12

If $f : [a,b] \to \mathbb{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. In Exercise 4.5.8 we saw that if f is continuous on $[a,b]$, then f^{-1} is continuous on its domain. Let's add the assumption that f is differentiable on $[a,b]$ with $f'(x) \neq 0$ for all $x \in [a,b]$. Show f^{-1} is differentiable with

$$(f^{-1})'(y) = \frac{1}{f'(x)} \text{ where } y = f(x).$$

Proof. Suppose $f : [a,b] = X \to \mathbb{R}$ is one-to-one. Note there exists an inverse function f^{-1} defined on $f([a,b]) = Y$. Suppose f is continuous on X. Note that f^{-1} is continuous on its domain. Suppose f is differentiable on X with $f'(x) \neq 0$ for all $x \in X$. Choose $b \in Y$. Note that there exists $c \in X$ such that $f(c) = b$. Choose $y \in Y$. Note that there exists $x \in X$ such that $f(x) = y$. Note that f is differentiable at c thus $f(x) = f(c) + \mu(x)(x-c)$ where μ is continuous at c and $\mu(c) = f'(c)$. Note that $y = b + \mu(x)(f^{-1}(y) - f^{-1}(b))$ or $y - b = \mu(x)(f^{-1}(y) - f^{-1}(b))$. Consider two cases, $y = b$ and $y \neq b$. If $y = b$ then $x = c$ and $\mu(x) = \mu(c) = f'(c) \neq 0$. If $y \neq b$ then $y - b \neq 0$ and thus $\mu(x) \neq 0$. Thus $\mu(x) \neq 0$ in all cases. Note $f^{-1}(y) = f^{-1}(b) + \frac{1}{\mu(x)}(y-b)$ or $f^{-1}(y) = f^{-1}(b) + \frac{1}{\mu(f^{-1}(y))}(y-b)$. Since y was chosen arbitrarily from Y we can conclude that for all $y \in Y$, $f^{-1}(y) = f^{-1}(b) + \frac{1}{\mu(f^{-1}(y))}(y-b)$. Since $f^{-1}(y)$ is defined for all $y \in Y$ we know that $\frac{1}{\mu(f^{-1}(y))}$ is defined for all $y \in Y$ and by the algebraic continuity theorem and the composite continuity theorem $\frac{1}{\mu(f^{-1}(y))}$ is continuous at $y = b$. We can conclude that $(f^{-1})'(b) = \frac{1}{\mu(f^{-1}(b))} = \frac{1}{\mu(c)} = \frac{1}{f'(c)}$. Since b was chosen arbitrarily from Y we can conclude that f^{-1} is differentiable with $(f^{-1})'(y) = \frac{1}{f'(x)}$ where $y = f(x)$. □