Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified. I am also operating under the convention $v_a(b) = \{x \in \mathbb{R} : b - a < x < b + a\}$

Exercise : Abbott 5.2.3 (a,b)

(a) Find from definition the derivative of $h(x) = \frac{1}{x}$.
Note that $\frac{1}{x}$ is defined on $\mathbb{R} - \{0\}$. Choose $c \in \mathbb{R} - \{0\}$. Consider the function $g(x) = \frac{h(x)-h(c)}{x-c}$. Note that $g(x)$ is defined on $A = \mathbb{R} - \{c\}$ and thus c is a limit point of the domain A. Note that $g(x) = \frac{1/x-1/c}{x-c}$. Define $d(x) = x - c$, note that $d(x) \neq 0$ for $x \in A$. Note that $g(x) = \frac{1/x-1/(x-d(x))}{d(x)} = \frac{x-d(x)-x}{x(x-d(x))d(x)} = \frac{-1}{x(x-d(x))}$. Note that as $x \to c$ $d(x) \to 0$ and thus by the arithmetic limit therm $g(x) \to \frac{-1}{c^2}$. Thus by definition $h'(c) = \frac{-1}{c^2}$.

(b) Suppose $g(c) \neq 0$. Find $(f/g)'(c)$, assuming that f and g are differentiable at c.
Note that $(f/g)(x) = f(x) * 1/g(x)$. Define $h(x) = 1/x$. Note $(f/g)(x) = f(x) * h(g(x))$, everywhere that f/g is defined. Note that $(f/g)(c)$ is defined. Note that $(f/g)'(c) = f'(c)h(g(x)) + f(c)h'(g(c))g'(c)$ by the chain rule and product rule. Note that $(f/g)'(c) = \frac{f'(c)}{g(c)} + \frac{-f(c)g'(c)}{g(c)^2} = \frac{f'(c)g(c)-f(c)g'(c)}{g(c)^2}$.

Exercise : Abbott 5.3.1

(a) Suppose f' exists and is continuous on $[a, b]$. Note that f is continuous on $[a, b]$. Noting that $[a, b]$ is compact and f' is a continuous mapping $f' : [a, b] \to \mathbb{R}$ we can say that f' achieves a minimum and a maximum in [a,b], lets call them a and b respectively. Define $M = \max(-a, b)$. Note that for all $x \in [a, b]$, $-M \leq a \leq f'(x) \leq b \leq M$ and thus $|f'(x)| \leq M$. Choose $x \neq y \in [a, b]$. By the mean value theorem there exists a $c \in [a, b]$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$. Note that $\frac{f(x)-f(y)}{x-y} = f'(c) \leq M$. Conclude that f is Lipschitz on $[a, b]$.

(b) Suppose f' exists and is continuous on $[a, b]$. Suppose that $|f'(x)| < 1$ for all $x \in [a, b]$. Note that f' achieves a maximum and a minimum in [a,b], take the one with the largest absolute value, lets call it M with associated value x_M. Note that $|M| = |f'(x_M)| < 1$, and $|f'(c)| \leq M$ for all $c \in [a, b]$. Choose $x, y \in [a, b]$. If $x = y$ note that $|f(x) - f(y)| = 0 = |M||x - y|$. Suppose $x \neq y$. Note that $\frac{|f(x)-f(y)|}{|x-y|} = |f'(c)|$ for some $c \in [a, b]$. Thus $\frac{|f(x)-f(y)|}{|x-y|} \leq |M|$ and so $|f(x) - f(y)| \leq |M||x - y|$. Thus f is a contraction function.

Exercise : Abbott 5.3.2

Suppose f is differentiable on some interval A. Suppose further that $f'(x) \neq 0$ for all $x \in A$. Suppose $f(x) = f(y)$ for some $x \neq y \in A$. Note that there exists a $c \in A$ such that $f'(c) = \frac{f(x)-f(y)}{x-y} = \frac{0}{x-y} = 0$. We have reached a contradiction and thus conclude $f(x) \neq f(y)$ for all $x \neq y \in A$, or that the function f is one-to-one.
The converse is not true, consider $f(x) = x^3$ on $A = [-1, 1]$. Clearly this function is differentiable with derivative $f'(x) = 3x^2$ and also one-to-one. However note that $f'(0) = 0$.

Exercise : Abbott 5.3.6 (a,b)

(a) Let $g : A = [0, a] \to \mathbb{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in A$. Choose $x \in A$. If $x = 0$ then $|g(x)| = 0 = Mx$. Suppose $x \neq 0$. Note that there exists a $c \in [0, a]$ such that $g'(c) = \frac{g(x)-g(0)}{x-0} = \frac{g(x)}{x}$. Note that $\frac{|g(x)|}{x} = |\frac{g(x)}{x}| \leq M$ and thus $|g(x)| \leq Mx$.

(b) Let $h : A = [0, a] \to \mathbb{R}$ be twice differentiable, $h'(0) = h(0) = 0$, and $|h''(x)| \leq M$ for all $x \in A$. Define $g : A \to \mathbb{R}$ as $g(x) = h'(x)$. Note that $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in A$, thus $|g(x)| \leq Mx$. Note $|h'(x)| \leq Mx$ for all $x \in A$. Define $f(x) = Mx^2/2$. Note that $f'(x) = Mx$. Note that $|h'(x)|/|f'(x)| \leq 1$ for all $x \in (0, a]$. Choose $x \in [0, a]$. If $x = 0$ clearly $|h(x)| \leq Mx^2/2$. Suppose $x \neq 0$. By the general mean value theorem there exists a $c \in (0, x)$ such that $\frac{h'(c)}{f'(c)} = \frac{h(x)-h(0)}{f(x)-f(0)} = \frac{h(x)}{f(x)}$ thus $|\frac{h(x)}{f(x)}| \leq 1$ or $|h(x)| \leq Mx^2/2$.

Exercise : Abbott 5.3.7

Proof. Suppose f is differentiable on a interval A and that $f'(x) \neq 0$. Further suppose f has at least two fixed points, a, b. Note that there exists a $c \in A$ such that $f'(c) = \frac{f(a)-f(b)}{a-b} = \frac{a-b}{a-b} = 1$. We have a contradiction and so conclude that there is at most one fixed point. □

Exercise : Abbott 6.2.1 (a,b)

Let $f_n(x) = \frac{nx}{1+nx^2}$.

1. Find the point-wise limit.
   Choose $x \in (0, \infty)$. Consider the sequence $f_n(x)$. Note that $\frac{nx}{1+nx^2} = \frac{x}{1/n+x^2} \to \frac{x}{x^2} = \frac{1}{x}$.
2. Suppose uniform convergence on $(0, \infty)$. There exists $N \in \mathbb{N}$ such that if $n \geq N$ then for all $x \in (0, \infty)$, $|f_n(x) - 1/x| < 1$. Choose $x = \min(1/2, 1/\sqrt{N})$. Note that $|f_n(x) - 1/x| < 1$ so $\frac{1}{x(1+nx^2)} < \epsilon$ or $1 < \epsilon x(1 + nx^2) < \epsilon x(1 + 1) < \epsilon = 1$, a contradiction thus f is not uniformly convergent.

Exercise : Abbott 6.2.7

Suppose f is uniformly continuous on $\mathbb{R}$. Define $f_n(x) = f(x - 1/n)$. Choose $\epsilon > 0$. There exists a $\delta > 0$ such that for $x, y \in \mathbb{R}$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Define $N \in \mathbb{N}$ such that $1/N < \delta/2$. Choose $n, m \geq N$, $x \in \mathbb{R}$. Note that $0 < 1/n, 1/m < \delta/2$ and thus $|1/n - 1/m| \leq 1/n + 1/m < \delta$ or $|(x - 1/m) - (x - 1/n)| < \delta$. Thus $|f_n(x) - f_m(x)| < \epsilon$. We conclude that the Cauchy criterion is met and thus $f_n(x)$ converges uniformly. Also note that as $n \to \infty$, $f(x - 1/n) \to f(x)$. Thus $f_n \to f$ point-wise.
To the point that uniform continuity is necessary, consider the function $f(x) = x^2$. This function violates uniform continuity and also will not have the property described above. This can be demonstrated easily since $|f(x - 1/n) - f(x)| = | - 2x/n + 1/n^2|$ can be made large for any particular n by choosing a large x, in other words if you gave me a N that was supposed to work with a ϵ I could choose a huge x value and break the uniform convergence inequality.

Exercise : Abbott 6.3.5

Define $g_n(x) = \frac{nx+x^2}{2n}$ and $g(x)$ as the limit of the $g_n(x)$.

(a) Note that $g_n(x) = \frac{nx+x^2}{2n} = \frac{x+x^2/n}{2} \to x/2 = g(x)$. Noting that $x/2$ is a polynomial we can say $g(x)$ is differentiable and $g'(x) = 1/2$.

(b) Note that $g_n'(x) = \frac{n+2x}{2n} = \frac{1+2x/n}{2}$. Consider a interval $[-M, M]$. Choose $\epsilon > 0$. Note that there exists a $N \in \mathbb{N}$ such that $1/N < \epsilon/2M$. Choose $n, m \geq N$. Choose $x \in [-M, M]$. Note that $|g_n'(x) - g_m'(x)| = |x/n - x/m| \leq |x/n| + |x/m| < M/n + M/m < \epsilon$. Conclude that $g_n'(x)$ converges uniformly and note that it converges on $1/2$. Conclude $g'(x) = 1/2$.

(c) Define $f_n(x) = \frac{nx^2+1}{2n+x}$.
Note that $f_n(x) = \frac{x^2+1/n}{2+x/n} \to x^2/2 = f(x)$, thus $f'(x) = x$.

(d) Note that $f_n'(x) = \frac{4n^2x+2n+2nx^2+x-nx^2+1}{(2n+x)^2} = \frac{4x+2n+x^2/n+x/n^2+1/n^2}{4+4x/n+x^2/n^2}$.

(W) (Hand this one in to David.)

Exercise : Abbott 6.2.5

Proof. Suppose $f_n : A \to \mathbb{R}$.

Suppose for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that if $n, m \geq N$ and $x \in A$, $|f_n(x) - f_m(x)| < \epsilon$. Note that for a particular x, $f_n(x)$ is a Cauchy sequence and thus converges, thus $f_n(x)$ converges point-wise to some function $f(x)$. Choose $\epsilon > 0$. There exists a $N \in \mathbb{N}$ such that if $n, m \geq N$ and $x \in A$, $|f_n(x) - f_m(x)| < \epsilon/2$. Choose $n \geq N$. Note that $|f_n(x) - f_m(x)| < \epsilon/2$, $f_n(x) - \epsilon/2 < f_m(x) < f_n(x) + \epsilon/2$ for all $m \geq N$. By the limit order theorem $f_n(x) - \epsilon < f_n(x) - \epsilon/2 \leq f(x) \leq f_n(x) + \epsilon/2 < f_n(x) + \epsilon$, so $|f_n(x) - f(x)| < \epsilon$. Therefore $f_n \to f$ uniformly.

Suppose $f_n \to f$ uniformly. Choose $\epsilon > 0$. There exists a $N \in \mathbb{N}$ such that for all $x \in A$ and $n \geq N$, $|f_n(x) - f(x)| < \epsilon/2$. Choose $x \in A$, $n, m \geq N$. Note that $|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \epsilon/2 + \epsilon/2 = \epsilon$.

We have now demonstrated that a sequence converges uniformly if and only if it adheres to the Cauchy criterion for uniform convergence. □