Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified. I am also operating under the convention $v_a(b) = \{x \in \mathbb{R} : b - a < x < b + a\}$

Exercise : 4.3.7

(a) Show that Dirichlet's function is not continus for all points in $\mathbb{R}$.
Recall that Dirichlet's function is defined as

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Suppose Dirichlet's function is continus at c. Note that we can construct a series of rational numbers q_n, by selecting rational numbers from $(c - 1/n, c + 1/n)$. Noting that $q_n \to c$ and that $g(x)$ is continus at c we can see that $g(q_n) \to g(c)$ and sinse all $g(q_n) = 1$ we know that $g(c) = 1$. Note that we can construct a series of non-rational numbers a_n, by selecting non-rational numbers from $(c - 1/n, c + 1/n)$. Noting that $a_n \to c$ and that $g(x)$ is continus at c we can see that $g(a_n) \to g(c)$ and sinse all $g(a_n) = 0$ we know that $g(c) = 0$. We conclude $1 = 0$, a contradiction, thus Dirichlet's function is not continus for all points in $\mathbb{R}$.

(b) Define

$$h(x) = \begin{cases} 1 & x = 0 \\ 1/n & x = m/n \in \mathbb{Q} - \{0\} \text{ is in lowest terms} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Demonstrate that $h(x)$ is discontinus at eavery rational point.
Suppose $h(x)$ is continus at some point $c \in \mathbb{Q}$. Note that we can construct a series of non-rational numbers a_n, by selecting non-rational numbers from $(c - 1/n, c + 1/n)$. Noting that $a_n \to c$ and that $h(x)$ is continus at c we can see that $h(a_n) \to h(c)$ and sinse all $h(a_n) = 0$ we know that $h(c) = 0$. Note that $h(0) \neq 0$ an so $c \neq 0$. since c is a non-zero rational it can be written in its lowest terms with $n > 0$, m/n. Note that $h(c) = 1/n \neq 0$. A contradiction thus we conclude there are no rational numbers for witch $h(x)$ is continus.

(c) Demonstrate $h(x)$ is continus at eavery irrational point.
Consider a arbitrary irrational point c.
Consider a arbitrary sequence a_n, where $a_n \to c$.
Choose a $\epsilon > 0$. Note that there exists a natural number i such that $1/i < \epsilon$. Consider the set $S = \{|m/n - c| : m \in [-i*(|c|+1), i*(|c|+1)] \cap \mathbb{Z} \text{ and } n \in [0, i] \cap \mathbb{N}\}$. Note S is finite since there are a finite number of posible n and m values. Note that all elements of S are irrational. Note that for all $x \in S$, $x > 0$. Define $\epsilon' = min(S, 1)$, this can be done since S has finitely many elements. Note that $\epsilon' > 0$, Therfore there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - c| < \epsilon'$, select this N. Choose $n \geq N$. Suppose $a_n \notin \mathbb{Q}$. In this case $|h(a_n) - h(c)| = |0 - 0| = 0 < \epsilon$. Suppose $a_n \in \mathbb{Q}$. Cosider the reduced form of $a_n = j/k$.

Suppose $k < i$. Note $|a_n - c| < \epsilon' \leq 1$ thus $-(|c| + 1) \leq c - 1 < j/k < c + 1 \leq |c| + 1$ so $-i(|c|+1) \leq -k(|c|+1) < j < k(|c|+1) \leq i(|c|+1)$ and so $j \in [-i*(|c|+1), i*(|c|+1)] \cap \mathbb{Z}$ and thus $|a_n - c| \in S$. We now have $|a_n - c| < \epsilon' \leq |a_n - c|$, a contradiction. We conclude $k \geq i$. Note that $|h(a_n) - h(c)| = |h(j/k) - 0| = |1/k| = 1/k < 1/i < \epsilon$. We conclude that for all $n \geq N$, $|h(a_n) - h(c)| < \epsilon$ and thus $h(a_n) \to h(c)$. Since a_n is a arbitrary sequence converging on c and we showed $h(a_n)$ converges on $h(c)$ we can conclude $h(x)$ is continus at $x = c$. Since c was a arbitrary irrational we can say $h(x)$ is continus at eavery irrational.

Exercise : Suppose $K \subseteq \mathbb{R}$ is compact. Show that there exists $x_M \in K$ such that $x_M \geq x$ for all $x \in K$. Then, with very little work, show that there exists $x_m \in K$ such that $x_m \leq x$ for all $x \in K$.

Note that K is bounded. Define $x_M = \sup(K)$. Suppose $x_M \notin K$. Choose $\epsilon > 0$. Note that $x_M - \epsilon$ is not a upper bound on K thus there exist a element $x \in K$ such that $x > x_M - \epsilon$. Note that $x < x_M$, so $x \in v_\epsilon(x_M) \cap (K - \{x_M\})$. Therfore x_M is a limit point of K, since K is closed $x_M \in K$, a contradiction with our supposition, thus $x_M \in K$. We have found a $x_M \in K$ such that $x_M \geq x$ for all $x \in K$.
Note that K is bounded. Define $x_m = \inf(K)$. Suppose $x_m \notin K$. Choose $\epsilon > 0$. Note that $x_M + \epsilon$ is not a lower bound on K thus there exist a element $x \in K$ such that $x < x_m + \epsilon$. Note that $x > x_m$, so $x \in v_\epsilon(x_m) \cap (K - \{x_M\})$. Therfore x_m is a limit point of K, since K is closed $x_m \in K$, a contradiction with our supposition, thus $x_m \in K$. We have found a $x_m \in K$ such that $x_m \leq x$ for all $x \in K$.

Exercise : Suppose $f : \mathbb{R} \to \mathbb{R}$ and $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$. Show that either f achieves at least one of a minimum or a maximum. Give an example to show that f need not achieve both.

This is clearly false. I will present a counterexample. Consider the function

$$f(x) = \begin{cases} 1/(x+1) & x > 0 \\ 0 & x = 0 \\ -1/(-x+1) & x < 0 \end{cases}$$

This function clearly has the property $f : \mathbb{R} \to \mathbb{R}$ and $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$, however note that the suppremum of 1 is not a possible output and the infimum of -1 is not a possible output, thus f never acceves a minimum or maximum.
This example shows that f need not achieve both.

Exercise : 4.4.6

(a) A continus function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence a_n where $f(a_n)$ is not Cauchy.
Define $f(x) = 1/x$. Define $a_n = 1/n$. Note that $f(x)$ is continus. Note that a_n is convergent and therfore Cauchy. Note that $f(a_n) = n$ is divergent and thus not Cauchy.

(b) A uniformly continus function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence a_n where $f(a_n)$ is not Cauchy.
Impossible. Choose $\epsilon > 0$. There must exist a $\delta > 0$ such that for all $x, y \in (0, 1)$ where $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$. Note that there exists a N such that for $n, m \geq N$, $|a_n - a_m| < \delta$. Note that for $n, m \geq N$, $|f(a_n) - f(a_m)| < \epsilon$. Therfore $f(a_n)$ is Cauchy.

(c) A continus function $f : [0, \infty) \to \mathbb{R}$ and a Cauchy sequence a_n where $f(a_n)$ is not Cauchy.
Impossible. Note that a_n is a convergent sequence define l to be its limit. Note that $a_n \geq 0$ for all n and thus $0 \leq l$ so $l \in [0, \infty)$ and $f(l)$ is defined. Choose $\epsilon > 0$. There must exist a $\delta > 0$ such that for all $x \in [0, \infty)$ where $|x - l| < \delta$, $|f(x) - f(l)| < \epsilon$. Note that there exists a N such that for $n \geq N$, $|a_n - l| < \delta$. Note that for $n \geq N$, $|f(a_n) - f(l)| < \epsilon$. Therfore $f(a_n)$ is convergent and therfore Cauchy.

Exercise : 5

a) Assume $f : [0, \infty) \to \mathbb{R}$ is continuous and is uniformly continuous on $[b, \infty)$ for some $b > 0$. Show that f is uniformly continuous.
Note that $[0, b]$ is compact, thus f is uniformly continuous on $[0, b]$. Choose $\epsilon > 0$. There exists a $\delta_0 > 0$ such that $x, y \in [0, b]$ where $|x - y| < \delta_0$, $|f(x) - f(y)| < \epsilon$. There exists a $\delta_1 > 0$ such that $x, y \in [b, \infty)$ where $|x - y| < \delta_1$, $|f(x) - f(y)| < \epsilon$. Define $\delta = \min(\delta_0, \delta_1)$. Note that for all $x, y \in [0, \infty)$ where $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$. Thus f is uniformly continuous.

b) Prove $f(x) = \sqrt{x}$ is uniformly continuous.
Note that $f : [0, \infty) \to \mathbb{R}$ is continuous. Choose $\epsilon > 0$. Define $\delta = \epsilon$. Choose $x, y \in [1, \infty)$ where $|x - y| < \delta$. Note that $\sqrt{x} \leq x$ and $\sqrt{y} \leq y$. Note that $|f(x) - f(y)| = \sqrt{(\sqrt{x} - \sqrt{y})^2} = \sqrt{(x + y - 2\sqrt{x}\sqrt{y}}$. Note that $x + y - 2\sqrt{x}\sqrt{y} \leq x + y - 2xy$. Note $|f(x) - f(y)| \leq \sqrt{x + y - 2xy} = \sqrt{(x - y)^2} = |x - y| < \delta = \epsilon$. Thus f is uniformly continuous on $[1, \infty)$ and by the above proof in part a), f is uniformly continuous.

Exercise : Give a example or prove that such a function does not exist.

(a) A continuous function on $[0, 1]$ with the range $(0, 1)$.
Imposible, continuous functions map compact sets to compact sets, no continuous function can map $[0, 1]$, a compact set, to $(0, 1)$ a non-compact set.

(b) A continuous function on $(0, 1)$ with the range $[0, 1]$.
Consider $f : (0, 1) \to [0, 1]$ where $f(x) = \frac{1+sin(5000x)}{2}$. Note that $\frac{\pi}{2*5000} \in (0, 1)$ and $f(\frac{\pi}{2*5000}) = 1$. Note that $\frac{3\pi}{2*5000} \in (0, 1)$ and $f(\frac{3\pi}{2*5000}) = 0$. Note $f((0, 1)) = [0, 1]$ and f is continuous.

(c) A continuous function on $(0, 1]$ with the range $(0, 1)$.
Consider $f : (0, 1) \to [0, 1]$ where

$$f(x) = \frac{1 + \frac{\sin(\frac{1}{x})}{1+x}}{2}$$

. This function is continuous on $(0, 1]$ also eavery output will fall into the range $(0, 1)$. As x goes to zero this function begins ossilating rapidly from extreamly close to 1 to extreamly close to 0, thus it will cover all of $(0, 1)$.

Exercise : Give a example or prove that such a function does not exist.

(a) A continuous function defined on a open interval with a range of a closed interval.
See b on the previous question

(b) A continuous function defined on a closed interval with a range of a open interval.
Imposible, continuous functions map compact sets to compact sets, a closed interval is a compact set, and a open interval is not a compact set.

(c) A continuous function defined on a open interval with a range of a unbounded open set not equal to $\mathbb{R}$.
Consider $f : (0, 1) \to \mathbb{R}$ where $f(x) = 1/x$, The range on this set is the unbounded open set $S = \{x \in \mathbb{R} : x > 1\}$ witch is clearly not equal to the $\mathbb{R}$.

(d) A continuous function defined on $\mathbb{R}$ with a range of $\mathbb{Q}$.

(W) (Hand this one in to David.)

Exercise : A function $f : A \to \mathbb{R}$ is Lipschitz if there exists a $M > 0$ such that

$$|\frac{f(x) - f(y)}{x - y}| \leq M$$

for all $x \neq y \in A$.

(a) Show that if f is Lipschitz then f is uniform continuous on A.
Suppose $f : A \to \mathbb{R}$ is Lipschitz. There exists a $M > 0$ such that

$$|\frac{f(x) - f(y)}{x - y}| \leq M$$

for all $x \neq y \in A$. Choose $\epsilon > 0$. Define $\delta = \epsilon/M$. Choose $x, y \in A$ where $|x - y| < \delta$. If $x = y$ then $|f(x) - f(y)| = 0 < \epsilon$. If $x \neq y$ then $|f(x) - f(y)| \leq M * |x - y| < M\delta = \epsilon$. Therfore f is uniform continuous on A.

(b) It is not true that if f is uniform continuous on A then f is Lipschitz.
Consider $f : [0, \infty) \to \mathbb{R}$ where $f(x) = \sqrt{x}$. As previously proven in this homework f is uniform continuous. Suppose f is Lipschitz. There exists a $M > 0$ such that

$$|\frac{f(x) - f(y)}{x - y}| \leq M$$

for all $x \neq y \in A$. Define $x = \frac{1}{2M^2} < \frac{1}{M^2}$. Define $y = 0$. Note that $x \neq y \in A$. We can now conclude,

$$|\frac{f(x) - f(y)}{x - y}| \leq M$$

$$|\frac{\sqrt{x}}{x}| \leq M$$

$$|\frac{1}{\sqrt{x}}| \leq M$$

$$\frac{1}{\sqrt{x}} \leq M$$

$$\sqrt{x} \geq \frac{1}{M}$$

$$\frac{1}{M^2} > x \geq \frac{1}{M^2}$$

We have reached a contradiction and must conclude Suppose f is not Lipschitz. We now have a example of a function that is is uniform continuous but not Lipschitz.