Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified. I am also operating under the convention $v_a(b) = \{x \in \mathbb{R} : b - a < x < b + a\}$

Exercise : Prove the following

(a) $\lim_{x\to 2}(3x + 4) = 10$

Proof. Choose $\epsilon > 0$. Define $\delta = \epsilon/3 > 0$. Choose a x such that $0 < |x - 2| < \delta$. Note that $-\delta < x-2 < \delta$ so $2-\delta < x < 2+\delta$ or $6-3\delta < 3x < 6+3\delta$ so $10-\epsilon < 3x+4 < 10+\epsilon$ or $|(3x + 4) - 10| < \epsilon$. □

(b) $\lim_{x\to 0} x^3 = 0$

Proof. Choose $\epsilon > 0$. Define $\delta^3 = \epsilon$. Note that $\delta > 0$. Choose a x such that $0 < |x| < \delta$. Note that $-\delta < x < \delta$ so $-\delta^3 < x^3 < \delta^3$ or $|x^3| < \epsilon$. □

Exercise : 4.2.1 a,b

(a) Show how Corollary 4.2.4 (ii) follows from the sequential criterion for limits in therm 4.2.3 and the algebraic limit therm.

Proof. Suppose as $x \to c$, $f(x) \to L$ and $g(x) \to M$. Where f and g have domain A. Choose a sequence a_n where $a_n \in A - \{c\}$ and $a_n \to c$. Define $h(x) = f(x) + g(x)$ and define the sequences $f_n = f(a_n)$, $g_n = g(a_n)$, and $h_n = h(a_n)$. Note that $g_n \to M$ and $f_n \to L$. Note that $h_n = g_n + f_n$. by the arithmetic limit therm $h_n \to L + M$. Since a_n was chosen arbitrarily we can say that $h(x) \to L + M$ and $x \to c$. □

(b) Prove again from definition.

Proof. Suppose as $x \to c$, $f(x) \to L$ and $g(x) \to M$. Where f and g have domain A. Choose $\epsilon > 0$. There must exist $\delta_1 > 0$ such that for all $0 < |x-c| < \delta_1$, $|f(x)-L| < \epsilon/2$. There must exist $\delta_2 > 0$ such that for all $0 < |x - c| < \delta_2$, $|g(x) - M| < \epsilon/2$. Define $h(x) = f(x) + g(x)$. Define $\delta = \min(\delta_1, \delta_2)$. Choose $0 < |x - c| < \delta$. Note that $0 < |x - c| < \delta_1$ and $0 < |x - c| < \delta_2$ thus $|f(x) - L| < \epsilon/2$ and $|g(x) - M| < \epsilon/2$. Note that $|f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M| < \epsilon$ so $|h(x) - (M + L)| < \epsilon$. □

Exercise : 4.2.7
Let $g : A \to \mathbb{R}$ and assume f is a bounded function on A, in the since that there exists $M > 0$ such that $f(x) < M$ for all $x \in A$. Show that if $g(x) \to 0$ as $x \to c$ that $g(x)f(x) \to 0$.

Define $h(x) = g(x)f(x)$. Choose $\epsilon > 0$. There must exist $\delta_0 > 0$ such that for all $0 < |x - c| < \delta_0$, $|g(x)| < \epsilon/M$. Choose x such that $0 < |x - c| < \delta_0$. Note that $|h(x)| = |g(x)||f(x)| < (\epsilon/M)(M) = \epsilon$, thus $h(x) \to 0$ as $x \to c$.

Exercise : 4.2.11
Let $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If as $x \to c$, $f(x) \to L$ and $h(x) \to L$ show that $g(x) \to L$ as well.

Choose a arbitrary sequence $a_n \to c$ in $A - \{c\}$. Define $f_n = f(a_n)$, $g_n = g(a_n)$, and $h_n = h(a_n)$. Note that $f_n \to L$ and $h_n \to L$, and that $f_n \leq g_n \leq h_n$. By the squeeze therm on sequences $g_n \to L$. Since a_n was chosen arbitrarily we can say that $g(x) \to L$ as $x \to c$.

Exercise : 4.3.1
Define $g(x) = x^3$.

(a) Prove that $g(x)$ is continuous at $x = 0$.

Proof. Choose $\epsilon > 0$. Define $\delta = \sqrt[3]{\epsilon}$. Choose $|x| < \delta$. Note that $-\delta < x < \delta$ so $-\epsilon < x^3 - 0 < \epsilon$ or $|x^3 - 0| < \epsilon$. Note that $0^3 = 0$. □

(b) Prove that $g(x)$ is continuous at $c \neq 0$.

Proof. Choose $c \neq 0$. Choose $\epsilon > 0$. Define $0 < \delta = \min(\sqrt[3]{\epsilon/2}, \epsilon/(9c^2), |c/2|)$. Note that $-c/2 \leq \delta \leq c/2$. Choose $|x - c| < \delta$. Note that $|x^3 - c^3| < \delta|x^2 + xc + c^2| = \delta|x^2 - 2xc + c^2 + 3xc| \leq \delta(|x - c||x - c| + |3xc|) < \delta^3 + |3xc|\delta$. Note that $|x - c| < \delta$, $c - \delta < x < c + \delta$, $c/2 < x < 3c/2$, $0 < 3c^2/2 < 3xc < 9c^2/2$. Note $|x^3 - c^3| < \delta^3 + \delta 9c^2/2 \leq \epsilon/2 + \epsilon/2 = \epsilon$. □

Exercise : 4.3.3

(a) Prove therm 4.3.9.

Proof. Suppose $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Suppose $g(f(x))$ is defined for all $x \in A$. Suppose f is continuous at c and g is continuous at $f(c)$. Choose $\epsilon > 0$. There must exist a $\delta_1 > 0$ such that for all $|y - f(c)| < \delta_1$, $|g(y) - g(f(c))| < \epsilon$. There must exist a $\delta_2 > 0$ such that for all $|x - c| < \delta_2$, $|f(x) - f(c)| < \delta_1$. Choose $|x - c| < \delta_2$. Note that $|f(x) - f(c)| < \delta_1$, and thus $|g(f(x)) - g(f(c))| < \epsilon$. □

(b) prove again using sequential characterization.

Proof. Suppose $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Suppose $g(f(x))$ is defined for all $x \in A$. Suppose f is continuous at c and g is continuous at $f(c)$. Choose $a_n \to c$ where $a_n \in A$. Note that $b_n = f(a_n) \in B$ and since $f(x)$ is continuous at $x = c$, $b_n \to f(c)$. Note that $h_n = g(b_n) = g(f(a_n))$, since $g(y)$ is continuous at $y = f(c)$, $h_n \to g(f(c))$.

□

Exercise : 4.3.5
Prove that if c is a isolated point of A then $f : A \to \mathbb{R}$ is continuous at c.

Proof. Suppose c is a isolated point of A and $f : A \to \mathbb{R}$. Choose $\epsilon > 0$. Since c is a isolated point of A there exists a δ such that $v_\delta(c) \cap A = \{c\}$. Choose $x \in A$, $|x - c| < \delta$. Note that there is only one x with this property, thus $x = c$. Note that $|f(x) - f(c)| = 0 < \epsilon$. □

Exercise : 4.3.9
If $h : \mathbb{R} \to \mathbb{R}$ and h is continuous for all $\mathbb{R}$ then $\{x : h(x) = 0\}$ is a closed set.

Proof. Suppose $h : \mathbb{R} \to \mathbb{R}$ and h is continuous for all $\mathbb{R}$. Suppose $\{x : h(x) = 0\}$ is not a closed set. Since $H = \{x : h(x) = 0\}$ is not a closed set there must be a limit point l of H where $l \notin H$. Consider the set a_n where $a_1 \in v_1(l) \cap H - \{l\}$, note that $v_\epsilon(l) \cap H - \{l\} \neq \emptyset$ for any $\epsilon > 0$. And $a_n \in v_{|a_{n-1}-l|/2}(l) \cap H - \{l\}$. Note that $-1/2^{n-1} + l \leq a_n \leq 1/2^{n-1} + l$ by construction and thus $a_n \to l$ by the squeeze therm. Define $h_n = h(a_n)$. Note that $a_n \in H$ thus $h_n = 0$. Since $h(x)$ is continuous and $a_n \to l$ and $h_n \to 0$ we can say that $h(l) = 0$. Therefore $l \in H$, a contradiction. □

Exercise : 10

a) Show that a continuous function on all of $\mathbb{R}$ that equals zero on the rational numbers must be the zero function.

   Proof. Suppose H is closed set where $\mathbb{Q} \subseteq H \subseteq \mathbb{R}$. Choose $a \in \mathbb{R}$. Choose $\epsilon > 0$. Note that there exist a rational q such that $a < q < a + \epsilon$, by the density of the rationals. Note that $q \in H - \{a\}$ and that $q \in v_\epsilon(a)$, therefore a is a limit point of H and since H is closed $a \in H$ thus $\mathbb{R} \subseteq H$ and so $H = \mathbb{R}$.

   Suppose $h : \mathbb{R} \to \mathbb{R}$ that equals zero on the rational numbers and h is continuous for all $\mathbb{R}$. From the previous proof $H = \{x : h(x) = 0\}$ is a closed set. Note that H is closed set where $\mathbb{Q} \subseteq H \subseteq \mathbb{R}$. Conclude $H = \mathbb{R}$, $h(x) = 0$ for all $x \in \mathbb{R}$. □

b) Suppose f and g are two continuous functions on the real numbers. Is it true that if $f(q) = g(q)$ for all $q \in Q$, then f and g are the same function?
   Yes.

   Proof. Suppose f and g are two continuous functions on the real numbers where $f(q) = g(q)$ for all $q \in \mathbb{Q}$. Suppose $F = f(l) \neq g(l) = G$ for some $l \in \mathbb{R}$. Define $\epsilon = |F - G|/2 > 0$. There must exist a δ_1 such that for all $|x - l| < \delta_1$, $|f(x) - F| < \epsilon$. There must exist a δ_2 such that for all $|x - l| < \delta_2$, $|g(x) - G| < \epsilon$. Define $\delta = \min(\delta_1, \delta_2)$. Note that there exists a rational q such that $l - \delta < q < l + \delta$. Note that $|g(q) - G| < \epsilon$ and that $|f(q) - F| < \epsilon$, also note that $f(q) = g(q)$. Note that $2\epsilon = |F - G| = |g(q) - G + F - f(q)| \leq |g(q) - G| + |F - f(q)| < 2\epsilon$, a contradiction we conclude the negation of our supposition, that $f(l) = g(l)$ for all $l \in \mathbb{R}$. □

Exercise : 4.2.9
For infinite limits we replace the arbitrarily small $\epsilon > 0$ with the arbitrarily large $M > 0$.

(W) (Hand this one in to David.)

(a) Prove $\lim_{x \to 0} 1/x^2 = \infty$

Proof. Choose $M > 0$. Noting that $\sqrt{M} > 0$ there must exist a δ such that $1/\delta < \sqrt{M}$. Choose x such that $0 < |x| < \delta$. This means that $0 < x^2 < \delta^2 < 1/M$, so $1/x^2 > M$. □

(b) I would define $\lim_{x \to \infty} f(x) = L$ as for any $\epsilon > 0$ there exists a M such that if $x > M$, $|f(x) - L| < \epsilon$.
Show that $\lim_{x \to \infty} 1/x = 0$.

Proof. Choose $\epsilon > 0$. There must exist a $M > 0$ such that $1/M < \epsilon$. Choose $x > M$. Note that $x > M > 0$ means that $|1/x| = 1/x < 1/M < \epsilon$. □

(c) Define $\lim_{x \to \infty} f(x) = \infty$ as for any $K > 0$ there exists a M such that if $x > M$, $f(x) > K$.
As a example $\lim_{x \to \infty} x = \infty$.