

Exercise 1.4.7: Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ contradicts the assumption that $\alpha = \sup A$.

Proof.

□

Exercise Supplemental 1: Suppose for each $k \in \mathbb{N}$ that A_k is at most countable. Use the fact that $\mathbb{N} \times \mathbb{N}$ is countably infinite to show that $\bigcup_{k=1}^{\infty} A_k$ is at most countable. Hint: take advantage of surjections.

Proof. First let me define a new set B where $B = \{X \in A; X \neq \emptyset\}$. Note that $\bigcup B = \bigcup_{k=1}^{\infty} A_k$. Lets next deal with the case that $B = \emptyset$ in this case $\bigcup B = \emptyset$ and so is at most countable infinite. Next lets consider the case that B has a finite number of elements, we proved this case in class, a union of a finite number of at most countably infinite sets is at most countably infinite. Now we know that we are dealing with B a infinite set of at most countably infinite non-empty sets. Now I will introduce the notation $B_{k,l}$ where $B_{k,l}$ is the l element of B_k . Consider the function $f : \mathbb{N} \times \mathbb{N} \rightarrow B_{k,l}$ where

$$f(a, b) = \begin{cases} B_{a,b} & B_a \text{ has a } b\text{th element} \\ B_{1,1} & \text{otherwise} \end{cases}$$

Note that this function is surjective, since given a $B_{j,k}$ we see that (j, k) maps to it. There is also a surjection between each of our $B_{j,k}$ and $\bigcup B$ simply map the element $B_{j,k}$ to itself in $\bigcup B$. From knowing that \mathbb{N} and $\mathbb{N} \times \mathbb{N}$ have the same cardinality, I conclude that there is a surjection between \mathbb{N} and $\mathbb{N} \times \mathbb{N}$. Thus I can surjectively map $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \rightarrow B_{j,k} \rightarrow \bigcup B$. Thus B is at most countably infinite. □

Exercise Supplemental 2: (W) (Hand this one in to David.)

Suppose B is finite and $A \subseteq B$. Show that A is empty or finite.

Exercise 1.5.10 (a) (c):

- (a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.
- (c) Determine, with proof, if the same statement remains true replacing uncountable with infinite.

Proof (a).

□

Proof (b).

□

Exercise Supplemental 3: Show that the set of a finite subsets of \mathbb{N} is countably infinite. Hint: Let A_k be the set of all subsets of \mathbb{N} with no more than k elements. Show that each A_k is countably infinite.

Proof.

□

Exercise 2.2.2: Verify using the definition of convergence the following limits.

(a) $\lim_{n \rightarrow \infty} \frac{2n+1}{5n+4} = \frac{2}{5}.$

(b) $\lim_{n \rightarrow \infty} \frac{2n^2}{n^3+3} = 0.$

(c) $\lim_{n \rightarrow \infty} \frac{\sin(n^2)}{\sqrt{3n}} = 0.$

Proof (a). □

Proof (b). □

Proof (c). □

Exercise Supplemental 4: (W) (Hand this one in to David.) Carefully prove that the sequence (x_n) given by $x_n = (-1)^n$ does not converge.

Proof. □