Exercise 1: Abbott 7.2.5 **(W) (Hand this one in to David.)**

Suppose $\{f_n\}$ are a sequence of functions uniformly convergent on f, and suppose that $f_n \in R[a,b]$. Choose $\epsilon > 0$. Define $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [a,b]$, $|f_n(x) - f(x)| < \alpha = \epsilon/(4(b-a))$. Define $P \in P[a,b]$ such that $U(f_N, P) - L(f_N, P) < \beta = \epsilon/2$. Define M_k and m_k to be the suppremum and infimum for f_N in the kth interval of P. Define n to be the number of partitions in P. Define Δx_k to be the width of the kth interval in P. Note that $U(f_N, P) - L(f_N, P) = \sum_{k=1}^{n}(M_k - m_k)\Delta x_k < \beta$. Note that $|f_N(x) - f(x)| < \alpha$ or $f_N(x) - \alpha < f(x) < f_N(x) + \alpha$. Consider a particular interval, I_k. Note that in this interval $m_k - \alpha \leq f_N(x) - \alpha < f(x) < f_N(x) + \alpha \leq M_k + \alpha$. We can now see $U(f, P) \leq \sum_{k=1}^{n}(M_k + \alpha)\Delta x_k = U(f_N, P) + \alpha(b-a)$ and $L(f, P) \geq \sum_{k=1}^{n}(m_k - \alpha)\Delta x_k = L(f_N, P) - \alpha(b-a)$ thus $U(f, P) - L(f, P) \leq U(f_N, P) + 2\alpha(b-a) - L(f_N, P) < \beta + 2\alpha(b-a) = \epsilon$. We conclude $f \in R[a,b]$.

Exercise 2: Abbott 7.2.7

Suppose $f : [a, b] \to \mathbb{R}$ is a increasing function. Choose $\epsilon > 0$. Define $n \in \mathbb{N}$ such that $1/n < \gamma = \epsilon/(f(b) - f(a))(b - a)$. Define $\Delta x = (b - a)/n$. Define $x_0 = a$, $x_k = x_{k-1} + \Delta x$ for all $k \in [1, n]$. Note that $x_n = x_0 + n\Delta x = b$. We can define $P \in P[a, b]$ as the partition using $\{x_k\}_{k=0}^n$. Note that $f(x_{k-1}) \leq f(x) \leq f(x_k)$ for $x \in I_k$ the kth interval in P. Thus $f(x_k) \geq \sup(f(I_k))$ and $f(x_{k-1}) \leq \inf(f(I_k))$ for all $k \in [1, n]$. Note that $\sum_{k=1}^n f(x_k) - f(x_{k-1}) = f(x_n) - f(x_0) = f(b) - f(a)$. Note that $U(f, P) - L(f, P) = \sum_{k=1}^n (\sup(f(I_k)) - \inf(f(I_k)))\Delta x \leq \Delta x \sum_{k=1}^n f(x_k) - f(x_{k-1}) = \Delta x(f(b) - f(a)) = (f(b) - f(a))(b - a)/n < (f(b) - f(a))(b - a)\gamma = \epsilon$. We conclude $f \in R[a, b]$.

Exercise 3: Abbott 7.3.4

Let f and g be functions defined on (possibly different) closed intervals, and assume the range of f is contained in the domain of g so that the composition $g \circ f$ is properly defined.

(a) Show, by example, that is not the case that if f and g are integrable, then $g \circ f$ is integrable.
Since a increasing bounded function is integrable see part c for a counterexample.

Now decide on the validity of each of the following conjectures, supplying a proof or counterexample as appropriate.

(b) If f is increasing and g is integrable, then $g \circ f$ is integrable.

(c) If f is integrable and g is increasing, then $g \circ f$ is integrable.
Let $f : [0, 1] \to [0, 1]$ where $f(x) = t(x)$, Thomae's function. Let $g : [0, 1] \to [0, 1]$ where $g(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$. Note that f is integrable by the proof in the following section, also note that g is increasing, $g(x) \geq g(y)$ if $x \geq y$. Note that $g \circ f = \begin{cases} 0 & x \notin \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$ witch is non-integrable.

Exercise 4: Abbott 7.3.2

Recall that Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} - \{0\} \text{ is in the lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

has a countable set of discontinuities occurring at precisely every rational number. Follow these steps to prove $t(x)$ is integrable on $[0, 1]$ with $\int_0^1 t = 0$

(a) First argue that $L(t, P) = 0$ for any partition P of $[0, 1]$.
Suppose P is a partition on $[0, 1]$. Take I_k to be the kth interval in P. Note that in any

interval I_k there exists a irrational number thus $\inf(t(I_k)) \leq 0$. Note that $t(x) \geq 0$ for all $x \in [0,1]$ thus $\inf(t(I_k)) \geq 0$. We conclude that $\inf(t(I_k)) = 0$. Note that if there are n intervals in P then $L(t,P) = \sum_{k=1}^{n} \inf(t(I_k))\Delta x_k = \sum_{k=1}^{n} 0 = 0$.

(b) Let $\epsilon > 0$ and consider the set of points $D_{\epsilon/2} = \{x \in [0,1] : t(x) \geq \epsilon/2\}$. How big is $D_{\epsilon/2}$?
This is a very challenging problem since there is a multiplicity problem ie $1/2 = 2/4 = 3/6$, for the next step all I will need is that there are finitely many points, so I will try to bound the number of elements in $D_{\epsilon/2}$. Note that there exists $N \in \mathbb{N}$ such that $1/N < \epsilon/2$. Note that if $x \in [0,1]$ and $t(x) \geq \epsilon/2$ then $x = 1$ or $x = m/n$ where $m < n \in \mathbb{N} \cap [1,N]$. Note that in the previous representation we will have N possibilities for n and fewer than N possibilities for m thus I can say that there are fewer than N^2 possible m/n values and so there are fewer than $N^2 + 1$ elements in $D_{\epsilon/2}$.

(c) To complete the argument, explain how to construct a partition P_ϵ of $[0,1]$ so that $U(t,P_\epsilon) < \epsilon$.
Define $N \in \mathbb{N}$ such that $1/N < \epsilon$. Define $\gamma = \epsilon/(8N^2)$ Let P be the partition defined by $\{x \in [0,1] : x - \gamma \in D_{\epsilon/2} \text{ or } x + \gamma \in D_{\epsilon/2}\} + \{0,1\}$, witch we note has finitely many elements. Define n to be the number of intervals in P. Define Δx_k to be the width of the kth interval. Note that $U(t,P) - L(t,P) = U(t,P) = \sum_{k=1}^{n} \sup(t(I_k))\Delta x_k$. Define I^a to be the set of intervals in P that contain a element of $D_{\epsilon/2}$ and I^b to be the rest of the intervals of P. Define the shorthand $\sum_{I_k \in I}$ to mean sum over all of the I_k in I, defined only if I has finitely many elements. Note that $U(t,P) - L(t,P) = \sum_{I_k \in I^a} \sup(t(I_k))\Delta x_k + \sum_{I_k \in I^b} \sup(t(I_k))\Delta x_k$. Note that there are fewer than N^2 elements in $D_{\epsilon/2}$ and so there are fewer than N^2 elements in I^a. Note that if $I_k \in I^a$ then $\Delta x_k \leq 2\gamma$. Note that $\sup(t(I_k)) \leq 1$. Note that $\sum_{I_k \in I^a} \sup(t(I_k))\Delta x_k \leq \sum_{I_k \in I^a} 2\gamma \leq 2N^2\gamma < \epsilon/2$. Note that if $I_k \in I^b$ then $\sup(t(I_k)) \leq \epsilon/2$ since I_k contains no points in $D_{\epsilon/2}$. Note that $\sum_{I_k \in I^b} \sup(t(I_k))\Delta x_k \leq \sum_{I_k \in I^b} \epsilon/2\Delta x_k = \epsilon/2 \sum_{I_k \in I^b} \Delta x_k \leq \epsilon/2 \sum_{I_k \in I} \Delta x_k = \epsilon/2$. Conclude that $U(t,P) - L(t,P) = \sum_{I_k \in I^a} \sup(t(I_k))\Delta x_k + \sum_{I_k \in I^b} \sup(t(I_k))\Delta x_k < \epsilon/2 + \epsilon/2 = \epsilon$, and thus that $t(x)$ is integrable.

Exercise 5: Abbott 7.4.1

Let f be a bounded function on a set A, and set

$$M(A) = \sup\{f(x) : x \in A\}, m(A) = \inf\{f(x) : x \in A\}$$

$$M'(A) = \sup\{|f(x)| : x \in A\}, \text{ and } m'(A) = \inf\{|f(x)| : x \in A\}$$

(a) Show that $M(A) - m(A) \geq M'(A) - m'(A)$.
Let's consider three cases.

(1) Suppose $M(A) \geq m(A) \geq 0$. Note that $f = |f|$ on all of A, thus $M(A) = M'(A)$ and $m(A) = m'(A)$, therefore $M(A) - m(A) = M'(A) - m'(A)$.

(2) Suppose $0 \geq M(A) \geq m(A)$. Note that $-f = |f|$ on all of A, thus $-M(A) = m'(A)$ and $-m(A) = M'(A)$, therefore $M(A) - m(A) = M'(A) - m'(A)$.

(3) Suppose $M(A) \geq 0 \geq m(A)$. Note that $\max(M(A), -m(A)) = M'(A)$ and $M'(A) \geq m'(A) \geq 0$, therefore $M'(A) - m'(A) \leq M'(A) = \max(M(A), -m(A)) \leq M(A) - m(A)$.

Note that in all three cases $M(A) - m(A) \geq M'(A) - m'(A)$.

(b) Show that if f is integrable on the interval $[a, b]$, then $|f|$ is also integrable on this interval.
Choose $\epsilon > 0$. There exists a partition $P \in P[a, b]$ such that $U(f, P) - L(f, P) = \sum_{I_k \in P}(M(I_k) - m(I_k))\Delta x_k < \epsilon$ where Δx_k is defined in the same manner as the previous problems. Note that $U(|f|, P) - L(|f|, P) = \sum_{I_k \in P}(M'(I_k) - m'(I_k))\Delta x_k \leq \sum_{I_k \in P}(M(I_k) - m(I_k))\Delta x_k < \epsilon$, thus $|f|$ is integrable.

(c) Provide the details for the argument that in this case we have $|\int_a^b f| \leq \int_a^b |f|$.
Choose a partition $P \in P[a, b]$. Note that since $-|f| \leq f \leq |f|$, $-M'(A) \leq M(A) \leq M'(A)$. Note that $U(f, P) = \sum_{I_k \in P} M(I_k)\Delta x_k \leq \sum_{I_k \in P} M'(I_k)\Delta x_k = U(|f|, P)$ and that $-U(|f|, P) \leq U(f, P)$ in the same fashion. Note that $U(f) \leq U(f, P) \leq U(|f|, P)$ for all partitions P, thus $U(f)$ is a lower bound on $U(|f|, P)$ and so $U(f) \leq U(|f|)$. Note that $U(f) \leq U(f, P) \leq U(|f|, P)$ for all partitions P, thus $U(f)$ is a lower bound on $U(|f|, P)$ and so $U(f) \leq U(|f|)$. In the same fashion $-U(|f|) \leq U(f)$. Now we can say $|\int_a^b f| = |U(f)| \leq U(|f|) \int_a^b |f|$.

Exercise 6: Abbott 7.4.5

Let f and g be integrable functions on $[a, b]$.

(a) Show that if P is any partition of $[a, b]$, then

$$U(f + g, P) \leq U(f, P) + U(g, P)$$

Provide a specific example where the inequality is strict. What does the corresponding inequality for the lower sums look like?
Define $M(h, A) = \sup\{h(x) : x \in A\}$ and $m(h, A) = \inf\{h(x) : x \in A\}$ for some function h defined on a interval A. Consider I some closed interval in $[a, b]$. Note that $M(f + g, I) = sup\{f(x) + g(x) : x \in I\}$. Noting that $f(x) \leq M(f, I)$ and $g(x) \leq M(g, I)$ we can say that $f(x)+g(x) \leq M(f, I)+M(g, I)$, for all $x \in I$. Note now that $M(f+g, I) \leq M(f, I) + M(g, I)$. Note that $U(f + g, P) = \sum_{I_k \in P} M(f + g, I_k)\Delta x_k \leq \sum_{I_k \in P}(M(f, I_k) + M(g, I_k))\Delta x_k = \sum_{I_k \in P} M(f, I_k)\Delta x_k + \sum_{I_k \in P} M(g, I_k)\Delta x_k = U(f, P) + U(g, P)$.

(b) Review the proof of Theorem 7.4.2 (ii), and provide an argument for part (i) of this theorem.
Note that for any partition P and function h, where $U(h, P)$ and $L(h, P)$ exist, $L(h, P) = -U(-h, P)$. Note that since $U(f + g, P) \leq U(f, P) + U(g, P)$ for all $f, g \in R[a, b]$ then $U(-f - g, P) \leq U(-f, P) + U(-g, P)$, since $f, g \in R[a, b]$ implies $-f, -g \in R[a, b]$, note $L(f + g, P) = -U(-f - g, P) \geq -U(-f, P) - U(-g, P) = L(f, P) + L(g, P)$. Define a sequence of partitions P_n^1 such that $U(f, P_n^1) - L(f, P_n^1) \to 0$ and P_n^2 such that $U(g, P_n^2) - L(g, P_n^2) \to 0$. Define P_n to be the refinement of P_n^1 and P_n^2. Note that $L(f, P_n) + L(g, P_n) \leq L(f + g, P_n)$ and $U(f + g, P_n) \leq U(f, P_n) + U(g, P_n)$ thus

$0 \leq U(f+g, P_n) - L(f+g, P_n) \leq U(f, P_n) - L(f, P_n) + U(g, P_n) - L(g, P_n)$. Note now that $U(f+g, P_n) - L(f+g, P_n) \to 0$ by the squeeze theorem and thus $f+g$ is integrable. Note that $L(f, P) + L(g, P) \leq L(f+g, P) \leq L(f+g) \leq U(f+g) \leq U(f+g, P) \leq U(f, P) + U(g, P)$ for all partitions P. Note that $U(f+g)$ is a lower bound for $U(f, P) + U(g, P)$ and a upper bound for $L(f, P) + L(g, P)$ thus $\int_a^b f + \int_a^b g = L(f) + L(g) \leq U(f+g) = \int_a^b f + g \leq U(f) + U(g) = \int_a^b f + \int_a^b g$. We conclude that $\int_a^b f + \int_a^b g = \int_a^b f + g$.