

**Exercise 1.2.5:** Use the triangle inequality to establish the following inequalities:

(a)  $|a - b| \leq |a| + |b|$ .

*Proof.* Note that  $|a - b| = |a + (-b)|$ . By the triangle inequality we note that  $|a + (-b)| \leq |a| + |-b|$ . There are three possibilities either  $b < 0$ ,  $b = 0$ , or  $b > 0$ . In the case that  $b < 0$  we know that  $-b > 0$ . From the definition of absolute value  $|b| = -b$  and  $|-b| = -b$  thus in this case  $|b| = |-b|$ . In the case that  $b = 0$  we know that  $-b = 0$ . From the definition of absolute value  $|b| = b$  and  $|-b| = b$  thus in this case  $|b| = |-b|$ . In the case that  $b > 0$  we know that  $-b < 0$ . From the definition of absolute value  $|b| = b$  and  $|-b| = -(-b) = b$  thus in this case  $|b| = |-b|$ . Thus in all cases  $|b| = |-b|$  and so  $|a| + |-b| = |a| + |b|$  thus  $|a - b| \leq |a| + |b|$ .  $\square$

(b)  $||a| - |b|| \leq |a - b|$ .

*Proof.* Note that  $|c| = |c - d + d|$  which by the triangle inequality means  $|c| \leq |c - d| + |d|$  so  $|c| - |d| \leq |c - d|$  for any  $c$  and  $d$  in  $\mathbb{R}$ . Consider  $||a| - |b||$  noting that there are two possibilities, either  $||a| - |b|| = |a| - |b|$  or  $||a| - |b|| = -( |a| - |b| ) = |b| - |a|$ , by the definition of absolute value. In the case that  $||a| - |b|| = |a| - |b|$  we see from the first statement that  $||a| - |b|| = |a| - |b| \leq |a - b|$ . In the second case  $||a| - |b|| = |b| - |a| \leq |b - a|$  and we proved in the previous question that  $|b - a| = |-(a - b)| = |a - b|$ , thus in this case  $||a| - |b|| \leq |a - b|$ . Thus in all cases  $||a| - |b|| \leq |a - b|$ .  $\square$

**Exercise 1.2.6(b), (d):** Given a function  $f$  and a subset  $A$  of its domain, let  $f(A)$  represent the range of  $f$  over the set  $A$ ; that is,  $f(A) = \{f(x) : x \in A\}$ .

(b) Find two sets  $A$  and  $B$  for which  $f(A \cap B) \neq f(A) \cap f(B)$ .

Suppose we let  $A = \{1\}$  and  $B = \{-1\}$ . Consider the function  $f(x) = x^2$ . Note that  $A \cap B = \emptyset$  and thus  $f(A \cap B) = \emptyset$ . Also note that  $f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\}$  and since  $\emptyset \neq \{1\}$  in this case  $f(A \cap B) \neq f(A) \cap f(B)$ .

(d) Form and prove a conjecture concerning  $f(A \cup B)$  and  $f(A) \cup f(B)$ .

Conjecture  $f(A \cup B) \subseteq f(A) \cup f(B)$

*Proof.* Choose some element  $y$  from the set  $f(A \cup B)$ . By our definition of evaluating a function on a set there must exist some element  $x$  in  $A \cup B$  such that  $f(x) = y$ . By the definition of union  $x \in A$  or  $x \in B$ . Thus  $f(x) \in f(A)$  or  $f(x) \in f(B)$  and so  $y \in f(A)$  or  $y \in f(B)$  which means by definition  $y \in f(A) \cup f(B)$ . Since we chose  $y$  arbitrarily from  $f(A \cup B)$  and showed that  $y$  is in  $f(A) \cup f(B)$  we can say by the definition of subset  $f(A \cup B) \subseteq f(A) \cup f(B)$ .  $\square$

**Exercise 1.2.8:** Form the logical negation of each claim. Do not use the easy way out: "It is not the case that..." is not permitted

(a) For all real numbers satisfying  $a < b$ , there exists  $n \in \mathbb{N}$  such that  $a + (1/n) < b$ .

There exists real numbers satisfying  $a < b$ , that for all  $n \in \mathbb{N}$ ,  $a + (1/n) \geq b$ .

- (b) Between every two distinct real numbers there is a rational number.  
There exists two distinct real numbers where there is no rational number between them.
- (c) For all natural numbers  $n \in \mathbb{N}$ ,  $\sqrt{n}$  is either a natural number or is an irrational number.  
There exists some natural number  $n \in \mathbb{N}$  where  $\sqrt{n}$  is not a natural number or an irrational number.
- (d) Given any real number  $x \in \mathbb{R}$  there exists  $n \in \mathbb{N}$  satisfying  $n > x$ .  
There exists a real number  $x \in \mathbb{R}$  where there is no  $n \in \mathbb{N}$  satisfying  $n > x$ .

**Exercise 1.3.4:** Assume that  $A$  and  $B$  are nonempty, bounded above, and satisfy  $B \subseteq A$ . Show that  $\sup B \leq \sup A$ .

*Proof.* Assume to the contrary, namely that there exists sets  $A$  and  $B$  that are nonempty, bounded above, and satisfy  $B \subseteq A$ . Further suppose that  $\sup B > \sup A$ . Let's define  $\alpha = \sup A$ . Suppose there is no element in  $B$  greater than  $\alpha$ . By the definition of upper bound,  $\alpha$  would be an upper bound to  $B$ , however  $\sup B > \alpha$ , a contradiction, thus our assumption that there is no element of  $B$  greater than  $\alpha$  must be false, and there is some element of  $B$  greater than  $\alpha$ . Let's take one of these elements with the property  $\gamma \in B$  and  $\gamma > \alpha$ . Note that since  $B \subseteq A$ ,  $\gamma \in A$ . Since  $\alpha$  is an upper bound to  $A$  we know that every element of  $A$  is less than or equal to  $\alpha$  thus  $\gamma \leq \alpha$ . Contradiction  $\gamma > \alpha$  and  $\gamma \leq \alpha$ , thus our initial supposition that  $\sup B > \sup A$  must be false and so we are forced to conclude  $\sup B \leq \sup A$ .  $\square$

**Exercise 1.3.5:** Let  $A$  be bounded above and let  $c \in \mathbb{R}$ . Define the sets  $c + A = \{a + c : a \in A\}$  and  $cA = \{ca : a \in A\}$ .

- (a) Show that  $\sup(c + A) = c + \sup(A)$ .
- (b) If  $c \geq 0$ , show that  $\sup(cA) = c \sup(A)$ .
- (c) Postulate a similar statement for  $\sup(cA)$  when  $c < 0$ .

*Proof (a).* Let's start by defining  $\alpha = \sup(A)$ ,  $\beta = c + \sup(A)$ . We will proceed by showing that  $\beta$  must have the two properties defining  $\sup(c + A)$ .

Suppose that there existed some  $\gamma \in c + A$  where  $\gamma > \beta$ . Note that  $\gamma - c \in A$  and that  $\gamma - c > \beta - c = \alpha$ . Contradiction, we have found an element in  $A$ ,  $\gamma - c$ , that is greater than  $\sup(A)$ . We are forced to conclude the negation of our supposition and so conclude that there is no element in  $c + A$  that is greater than  $\beta$ , and so  $\beta$  is an upper bound on  $c + A$ , the first condition on  $\sup(c + A)$ .

Suppose that there is an upper bound to  $c + A$ , let's call it  $\lambda$ , that is smaller than  $\beta$ . Note that  $\lambda - c < \beta - c = \alpha$ . Since  $\alpha$  is larger than  $\lambda - c$  we know from the definition of  $\sup$  that  $\lambda - c$  is not an upper bound on  $A$ , therefore there must be at least one element in  $A$  greater than  $\lambda - c$ , let's call it  $\tau$ . Since  $\tau$  is in  $A$   $c + \tau$  is in  $c + A$ , and since  $\tau > \lambda - c$ ,  $c + \tau > \lambda$ .

Contradiction,  $\lambda$  is a upper bound on  $c + A$  but there is a element in  $c + A$ , namely  $\tau + c$ , that is grater than  $\lambda$ . Thus we are forced to conclude the negation of our supposition, that all upper bounds on  $c + A$  are grater than or equal to  $\beta$ .

$\beta$  meets the definition of  $\sup(c + A)$  and so  $\beta = \sup(c + A)$  and  $c + \sup(A) = \sup(c + A)$ .  $\square$

*Proof (b).* Firstly let me eliminate a special case,  $c = 0$ . In this case  $cA = \{0\}$ , by inspection  $\sup(cA) = 0$  and also  $c \sup(A) = 0 * \sup(A) = 0$ . In this degenerate case it is clearly true that  $c \sup(A) = \sup(cA)$ . From here on I will work with  $c > 0$ . Note, in this proof I am taking advantage of the fact that dividing over a positive number across a inequality does not affect the inequality, that is why  $c > 0$  is necessary for this proof.

Lets start by defining  $\alpha = \sup(A)$ ,  $\beta = c \sup(A)$ . We will proceed by showing that  $\beta$  must have the two properties defining  $\sup(cA)$ .

Suppose that there existed some  $\gamma \in cA$  where  $\gamma > \beta$ . Note that  $\gamma/c \in A$  and that  $\gamma/c > \beta/c = \alpha$ . Contradiction, we have found a element in  $A$ ,  $\gamma/c$ , that is greater than  $\sup(A)$ . We are forced to conclude the negation of our supposition and so conclude that there is no element in  $cA$  that is greater than  $\beta$ , and so  $\beta$  is a upper bound on  $cA$ , the first condition on  $\sup(cA)$ .

Suppose that there is a upper bound to  $cA$ , lets call it  $\lambda$ , that is smaller than  $\beta$ . Note that  $\lambda/c < \beta/c = \alpha$ . Since  $\alpha$  is larger than  $\lambda/c$  we know from the definition of  $\sup$  that  $\lambda/c$  is not a upper bound on  $A$ , therefore there must be at least one element in  $A$  greater than  $\lambda/c$ , lets call it  $\tau$ . Since  $\tau$  is in  $A$   $c\tau$  is in  $cA$ , and since  $\tau > \lambda/c$ ,  $c\tau > \lambda$ . Contradiction,  $\lambda$  is a upper bound on  $cA$  but there is a element in  $cA$ , namely  $c\tau$ , that is grater than  $\lambda$ . Thus we are forced to conclude the negation of our supposition, that all upper bounds on  $cA$  are grater than or equal to  $\beta$ .

$\beta$  meets the definition of  $\sup(cA)$  and so  $\beta = \sup(cA)$  and  $c \sup(A) = \sup(cA)$ .  $\square$

Statement for part (c):

The region  $A$  would be flipped across 0 and be magnified by a factor of  $|c|$ , thus  $\sup(cA) = c \inf(A)$ .

**Exercise 1.3.6:** Compute, without proof, the suprema and infima of the following sets.

- (a)  $\{n \in \mathbb{N} : n^2 < 10\}$ .
- (b)  $\{n/(n + m) : n, m \in \mathbb{N}\}$ .
- (c)  $\{n/(2n + 1) : n \in \mathbb{N}\}$ .
- (d)  $\{n/m : m, n \in \mathbb{N} \text{ with } m + n \leq 10\}$ .

**Solution:**

- (a)  $\sup = 3, \inf = 1$
- (b)  $\sup = 1, \inf = 0$
- (c)  $\sup = 1/2, \inf = 1/3$
- (d)  $\sup = 1/9, \inf = 9$

**Exercise 1.3.7:** Prove that if  $a$  is an upper bound for  $A$  and if  $a$  is also an element of  $A$ , then  $a = \sup A$ .

*Proof.* Suppose  $b$  is a upper bound of  $A$  and that  $b < a$ . Since  $b$  is a upper bound on  $A$  and  $a \in A$  we know that  $a \leq b$ . We have arrived at a contradiction, thus there is no upper bound on  $A$  that is less than  $a$ , and so all upper bounds on  $A$  are greater than or equal to  $a$ . We now can say that  $a$  meets both of the elements of the definition of  $\sup A$  thus  $a = \sup A$   $\square$

**Exercise 1.3.8:** If  $\sup A < \sup B$  then show that there exists an element  $b \in B$  that is an upper bound for  $A$ .

*Proof.* Let's begin with a short contradiction, suppose there is no element in  $B$  greater than  $\sup A$ . By the definition of upper bound  $\sup A$  is a upper bound for  $B$ . Thus by the definition of  $\sup B$  we conclude  $\sup B \leq \sup A$ . This is a contradiction, and so we are forced to conclude that there is at least one element of  $B$  greater than  $\sup A$ . Cause one of these elements,  $\beta \in B$ ,  $\sup A < \beta$ . The definition of  $\sup A$  gives us that for all  $\alpha \in A$ ,  $\alpha \leq \sup A \Rightarrow \alpha \leq \beta$ . We then see that  $\beta$  must be a upper bound on  $A$  and thus there is a element in  $B$  that is a upper bound on  $A$ .  $\square$

Authors note: Is it necessary that  $\sup A < \sup B$ , or only that  $\sup A \leq \sup B$ ? Consider  $A = [0, 1]$  and  $B = [0, 1)$ , here  $\sup A \leq \sup B$  but there is no element of  $B$  that is a upper bound for  $A$ .

**Exercise 1.2.9:** Show that the sequence  $(x_1, x_2, x_3, \dots)$  defined in Example 1.2.7 is bounded above by 2. That is, show that for every  $i \in \mathbb{N}$ ,  $x_i \leq 2$ .

*Proof.* We will proceed with a proof by induction on  $i$ .

In the base case  $i = 1$  we are given  $x_i = 1$  since  $1 \leq 2$  the statement  $i \in \mathbb{N}$ ,  $x_i \leq 2$  holds in the base case.

Suppose  $x_i \leq 2$ . Consider the next step  $x_{i+1}$ , note that by definition  $x_{i+1} = (1/2)x_i + 1$ . Note that  $x_i \leq 2 \Rightarrow (1/2)x_i \leq (1/2)2 = 1 \Rightarrow (1/2)x_i + 1 \leq 1 + 1 = 2 \Rightarrow x_{i+1} \leq 2$ . Thus by induction we conclude that for all  $i \in \mathbb{N}$ ,  $x_i \leq 2$ .  $\square$

**Exercise 1.3.7:** Prove that if  $a$  is an upper bound for  $A$  and if  $a$  is also an element of  $A$ , then  $a = \sup A$ .

*Proof.* Suppose  $b$  is a upper bound of  $A$  and that  $b < a$ . Since  $b$  is a upper bound on  $A$  and  $a \in A$  we know that  $a \leq b$ . We have arrived at a contradiction, thus there is no upper bound on  $A$  that is less than  $a$ , and so all upper bounds on  $A$  are greater than or equal to  $a$ . We now can say that  $a$  meets both of the elements of the definition of  $\sup A$  thus  $a = \sup A$   $\square$