Note that I am operating under the convention that N, n, m, i, j are natural numbers unless otherwise specified.

Exercise : Suppose $\{n_j\}_{j=1}^{\infty}$ is a sequence of natural numbers such that $n_j < n_{j+1}$ for all $j \in \mathbb{N}$. Show that $n_j \geq j$ for all $j \in \mathbb{N}$.

Proof. Suppose $\{n_j\}_{j=1}^{\infty}$ is a sequence of natural numbers such that $n_j < n_{j+1}$ for all $j \in \mathbb{N}$. I will prove that $n_j \geq j$ for all $j \in \mathbb{N}$ by induction on j.
Base case $j = 1$. Note that n_1 is a natural number, and by construction the smallest natural number is 1. Thus in the base case $n_j \geq j$.
Suppose $n_j \geq j$. Note that $n_{j+1} > n_j$, thus $n_{j+1} - n_j > 0$,noting that the integers are closed under subtraction I conclude that $n_{j+1} - n_j \in \mathbb{Z}$, thus $n_{j+1} - n_j \geq 1$ and $n_{j+1} \geq 1 + n_j \geq 1 + j$. □

Exercise : Show that a subsequence of a convergent sequence converges to the same limit. Be sure to use the previous problem in your proof!

Proof. Suppose $\{a_n\} \to l$ and it has a subsequence b_j. By the definition of subsequence we can express $b_j = a_{n_j}$ where $\{n_j\}$ is a sequence of natural numbers such that $n_j < n_{j+1}$. Choose $\epsilon > 0$. Since $\{a_n\} \to l$ there must exist a $N \in \mathbb{N}$ such that for any $n \geq N$, $|a_n - l| < \epsilon$. Note that for any $j \geq N$, $n_j \geq j$ thus $n_j \geq N$ and so $|b_j - l| = |a_{n_j} - l| < \epsilon$. Thus $|b_j - l| < \epsilon$ for all $j \geq N$ by the definition of convergence $b_j \to l$. □

Exercise : 2.4.4
Prove NIP using MCT.

Proof. Axium: if a sequence is monotone and bounded it converges.
Suppose we have sets $I_n = [a_n, b_n]$ defined for all $n \in \mathbb{N}$, where $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$.
Note that for $n = 1$, $a_1 \leq a_n \leq b_n \leq b_1$. Suppose $a_1 \leq a_n \leq b_n \leq b_1$. Note that $a_1 \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b_1$ and thus $a_1 \leq a_{n+1} \leq b_{n+1} \leq b_1$. By induction we are forced to conclude that $a_1 \leq a_n \leq b_n \leq b_1$ for all $n \in \mathbb{N}$.
Noting that a_n is bounded and monotonic we can conclude that it converges to some values $a_n \to a$.
Suppose there existed a $c \in \mathbb{N}$ where $a \notin I_c$. There are two sensibilities eater $a < a_c$ or $b_c < a$, I will proceed to prove both of these are impossible.
Suppose $a < a_c$. Choose $\epsilon = a_c - a > 0$. By the definition of limit there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$. Note that $N > c$ since if $N \leq c$ we could conclude $|a_c - a| < \epsilon$ and $|a_c - a| = |\epsilon| = \epsilon$ a contradiction. Note that $a_N \geq a_c > a$ since a_n is monotonic increasing, thus $\epsilon > |a_N - a| = a_N - a \geq a_c - a = \epsilon$. A contradiction thus it is impossible that $a < a_c$, and we conclude that $b_c < a$.
Choose $\epsilon = a - b_c > 0$. By the definition of limit there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$. Note that $N > c$ since if $N \leq c$ we could conclude $|a_c - a| < \epsilon$ and $a_c \leq b_c < a$, $|a_c - a| = a - a_c \geq a - b_c = \epsilon$ a contradiction. Note that $a_N \leq b_N \leq b_c < a$ since b_n is monotonic decreasing, thus $\epsilon > |a_N - a| = a - a_N \geq a - b_c = \epsilon$ a contradiction.
We are thus forced to conclude that for all $c \in \mathbb{N}$ $a \in I_c$. Thus $a \in \cap I_n$, proving that there

exists at least one value in $\cap I_n$, the nested interval property. $\square$

Exercise : 2.4.5(a)
Let $x_1 = 2$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

Show $(x_n)^2 \geq 2$ prove $x_n - x_{n+1} \geq 0$. Conclude $x_n \to \sqrt{2}$.

Proof. Note that for $n = 1$, $(x_n)^2 = 4 \geq 2$. Suppose $(x_n)^2 \geq 2$. Note that

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

$$\begin{aligned}
(x_{n+1})^2 &= \frac{1}{4}\left((x_n)^2 + 2 \cdot x_n \cdot \frac{2}{x_n} + \frac{4}{(x_n)^2}\right) \\
&= \frac{(x_n)^2}{4} + 1 + \frac{1}{(x_n)^2} \\
&= 1 + \frac{(x_n)^4 + 4}{4(x_n)^2}
\end{aligned}$$

Note that there exists some $\epsilon \geq 0$ where $2 + \epsilon = (x_n)^2$.

$$\begin{aligned}
&= 1 + \frac{(2+\epsilon)^2 + 4}{4(2+\epsilon)} \\
&= 1 + \frac{\epsilon^2 + 4\epsilon + 8}{4(2+\epsilon)} \\
&\geq 1 + \frac{4(2+\epsilon)}{4(2+\epsilon)} \\
&\geq 2
\end{aligned}$$

Thus by induction $(x_n)^2 \geq 2$.

Note

$$\begin{aligned}
(x_n)^2 &\geq 2 \\
x_n &\geq \frac{2}{x_n} \\
\frac{1}{x_n} &\leq \frac{x_n}{2} \\
0 &\leq \frac{x_n}{2} - \frac{1}{x_n} \\
0 &\leq x_n - \frac{x_n}{2} - \frac{1}{x_n}
\end{aligned}$$

$$0 \le x_n - \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big)$$
$$0 \le x_n - x_{n+1}$$

Since x_n is monotonic decreasing and bounded below we know it converges to some value, lets call it l. Note that $x_n \to l$ and $x_{n+1} \to l$ thus $x_n x_{n+1} \to l^2$ or in other words $(x_n^2 + 2)/2 \to l^2$. Thus $x_n^2 \to 2l^2 - 2$ however $x_n^2 \to l^2$ so $l^2 = 2l^2 - 2$ or $l^2 = 2$. Therefore this sequence converges on $\sqrt{2}$. □

Exercise : 2.4.5(b)
Modify the original sequence so it converges to $\sqrt{c}$. $(c > 0)$

Let $x_1 = \max(c, 1)$

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{c}{x_n}\Big)$$

Proof. Note that for $n = 1$, $(x_n)^2 = \max(c^2, 1) \ge c$. Suppose $(x_n)^2 \ge c$. Note that

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{c}{x_n}\Big)$$

$$\begin{aligned}
(x_{n+1})^2 &= \frac{1}{4}\Big((x_n)^2 + 2 \cdot x_n \cdot \frac{c}{x_n} + \frac{c^2}{(x_n)^2}\Big) \\
&= \frac{1}{4}\Big((x_n)^2 + 2c + \frac{c^2}{(x_n)^2}\Big) \\
&= \frac{1}{4}\Big(2c + (x_n)^2 + \frac{c^2}{(x_n)^2}\Big)
\end{aligned}$$

Note that there exists some $\epsilon \ge 0$ where $c + \epsilon = (x_n)^2$.

$$\begin{aligned}
&= \frac{1}{4}\Big(2c + \frac{(c+\epsilon)^2 + c^2}{c+\epsilon}\Big) \\
&= \frac{1}{4}\Big(2c + \frac{c^2 + 2c\epsilon + \epsilon^2 + c^2}{c+\epsilon}\Big) \\
&\ge \frac{1}{4}\Big(2c + \frac{2c^2 + 2c\epsilon}{c+\epsilon}\Big) \\
&\ge \frac{1}{4}(2c + 2c) = c
\end{aligned}$$

Thus by induction $(x_n)^2 \ge c$.

Note

$$(x_n)^2 \ge c$$

$$x_n \geq \frac{c}{x_n}$$

$$\frac{1}{x_n} \leq \frac{x_n}{c}$$

$$0 \leq \frac{x_n}{c} - \frac{1}{x_n}$$

$$0 \leq \frac{2x_n}{c} - \frac{x_n}{c} - \frac{1}{x_n}$$

$$0 \leq \frac{2x_n}{c} - \frac{2}{2c}\left(x_n + \frac{c}{x_n}\right)$$

$$0 \leq \frac{2}{c}\left(x_n - \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)\right)$$

$$0 \leq x_n - \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

$$0 \leq x_n - x_{n+1}$$

Since x_n is monotonic decreasing and bounded below we know it converges to some value, lets call it l. Note that $x_n \to l$ and $x_{n+1} \to l$ thus $x_n x_{n+1} \to l^2$ or in other words $(x_n^2 + c)/2 \to l^2$. Thus $x_n^2 \to 2l^2 - c$ however $x_n^2 \to l^2$ so $l^2 = 2l^2 - c$ or $l^2 = c$. Therefore this sequence converges on $\sqrt{c}$. □

Exercise : 2.5.6
Show that $a_n = b^{1/n} \to l_b$ if $b \geq 0$ and find l_b.

Proof. I will consider two cases $b \leq 1$ or $b > 1$.
In the case that $b \leq 1$.
Note that $b \leq b^{1/2} \leq b^{1/3} \leq \cdots \leq 1$. Note that a_n is monotone increasing and bounded above therefore it converges.
In the case that $b > 1$.
Note that $b > b^{1/2} > b^{1/3} > \cdots > 1$. Note that a_n is monotone decreasing and bounded below therefore it converges.
We conclude that this sequence will converge to some limit for any value $b \geq 0$. Let's consider this for a particular b. Note that $b^{1/n} = a_n \to l$ and $\sqrt{a_n} = b^{1/2n} = a_{2n} \to l$ thus $\sqrt{a_n} \to l$ and $\sqrt{a_n} \to \sqrt{l}$ so $\sqrt{l} = l$ and so $l = 1$ or $l = 0$. We can see above that if $b > 1$, then $a_n > 1$ and so our limit can not be 0 thus in this case $l_b = 1$. If $b \leq 1$ and $b \neq 0$ we can see that we have a monotone increasing sequence starting above 0 thus $0 < a_1 < a_n$ and we can not converge on 0 since we never get closer then a_1 and so $l_b = 1$. If $b = 0$ we get $a_n = 0$ and thus our sequence converges to 0.
In summery $a_n \to 1$ if $b \neq 0$ and $a_n \to 0$ if $b = 0$ □

Exercise : Suppose $|a_n| \to 0$. Show $a_n \to 0$.

Suppose $|a_n| \to 0$.
Choose $\epsilon > 0$. By the definition of limit there exists some N such that for all $n \geq N$, $||a_n| - 0| < \epsilon$. Note that $||a_n| - 0| = ||a_n|| = |a_n| = |a_n - 0|$. Choose $n \geq N$. Note that $|a_n - 0| = ||a_n| - 0| < \epsilon$. By the definition of limit $a_n \to 0$.

Exercise : 2.5.7

Proof. We know that $b^n \to 0$ for $0 \leq b < 1$. Note that $-(b^n) \to 0$ for $0 \leq b < 1$. Noting that $|b^n| = |b|^n$ for all b we see that $-|b|^n \leq b^n \leq |b|^n$. Thus if $-1 < b < 1$ we know that $0 \leq |b| < 1$ and so $-|b|^n \to 0$ and $|b|^n \to 0$. By the squeeze therm $b^n \to 0$ if $-1 < b < 1$. Suppose $b \notin (-1, 1)$. Further suppose $b^n \to 0$. Note that $|b^n| \to 0$ thus $|b|^n \to 0$. Note that $1 \leq |b| \leq |b|^n$. We have a contradiction all terms in the sequence are grater than 1 however they converge to 0. We are forced to conclude the negation of our supposition that $b^n \to 0$ only if $-1 < b < 1$. □

Exercise : 2.6.2

(a) $a_n = (-1)^n/n$
Note that this sequence is not monotone. Also we proved in class that this sequence converges and thus by the Cauchy criterion it is Cauchy.

(b) This is impossible. Any sequence with a unbounded sub sequence is unbounded and thus cannot converge, since all convergent sequences are bounded, and thus is not Cauchy.

(c) Suppose a_n is a monotone sequence and a_{n_j} is a Cauchy sub sequence of a_n. Choose $\epsilon > 0$ there exists N such that $i, j \geq N$, $|a_{n_j} - a_{n_i}| < \epsilon$. Choose $n, m \geq n_N \geq N$. Define $J = \max(n, m)$. Note that $n_J \geq \max(n, m) \geq n_N \geq N$ so $J \geq N$. Note $\epsilon > |a_{n_J} - a_{n_N}| \geq |a_n - a_m|$ since a_n is monotone. Thus a_n is Cauchy and so a_n is convergent. There are no monotone divergent sequences with a Cauchy sub sequence.

(d)
$$a_n = \begin{cases} n & n \in \text{odds} \\ 0 & \text{otherwise} \end{cases}$$
Clearly a_n is unbounded but the sub sequence a_{2n} is a sequence of zeros and thus clearly converges and thus by the Cauchy criterion it is Cauchy.

Exercise : 2.6.5

(i) Definitely not. Consider $a_i = 1/i$ and $s_n = \sum_{i=1}^{n} a_i$. We have previously proven this sequence is unbounded. Choose $\epsilon > 0$. There exists N such that $1/N < \epsilon$. Choose $n \geq N$. $|s_{n+1} - s_n| = |a_{n+1}| = 1/(n+1) < 1/N < \epsilon$. This sequence is pseudo-Cauchy and unbounded.

(ii) Suppose a_n and b_n are pseudo-Cauchy. Define $c_n = a_n + b_n$.
Choose $\epsilon > 0$. There exists N_a such that for all $n > N_a$, $|a_{n+1} - a_n| < \epsilon/2$. There exists N_b such that for all $n > N_b$, $|b_{n+1} - b_n| < \epsilon/$. Choose $n \geq \max(N_a, N_b)$. Note that $|a_{n+1} - a_n| < \epsilon/2$ and $|b_{n+1} - b_n| < \epsilon/$. Note that $|c_{n+1} - c_n| = |a_{n+1} - a_n + b_{n+1} - b_n| \leq |a_{n+1} - a_n| + |b_{n+1} - b_n| < \epsilon$. Thus c_n is pseudo-Cauchy.

Exercise : 2.5.5**(W) (Hand this one in to David.)**
Assume (a_n) is a bounded sequence with the property that every convergent subsequence converges to a. Show that a_n must converge to a.

Proof. Suppose that $a_n \nrightarrow a$. That must mean that the statement $(\forall \epsilon > 0)$, $(\exists N \in \mathbb{N})$, $(\forall n \geq N)$, $|a_n - a| < \epsilon$ must be false, thus its negation is true. We now know $(\exists \epsilon > 0)$, $(\forall N \in \mathbb{N})$, $(\exists n \geq N)$, $|a_n - a| \geq \epsilon$, let's name one of the ϵ with this property call it ϵ_o so we know that $(\forall N \in \mathbb{N})$, $(\exists n \geq N)$, $|a_n - a| \geq \epsilon_o$.

author's note: the symbolic representation is necessary here to illustrate to the reader what I am doing. putting it all in paragraph form would needlessly complicate this proof for the reader.

To summarize I now have in hand ϵ_o a positive number with the property that for any natural number N there is a $n \geq N$ with the property that $|a_n - a| \geq \epsilon_o$.
Let's construct a subsequence, and call it a_{n_j}. How I will construct this sequence n_1 is the first $n \geq 1$ where $|a_n - a| \geq \epsilon_o$, this must exist since 1 is a natural number. I define n_{k+1} to be the first $n \geq n_k + 1$ with the property $|a_n - a| \geq \epsilon_o$, this must exist since $n_k + 1$ is a natural number. Thus for all j, $|a_{n_j} - a| \geq \epsilon_o$.
Since a_n is bounded we know that any subsequence of it will be bounded, thus a_{n_j} is bounded. By the Bolzano-Weierstrass theorem we conclude a_{n_j} has a convergent subsequence lets call it $a_{n_{j_i}}$. Note that $a_{n_{j_i}}$ is a subsequence of a_n. Since $a_{n_{j_i}}$ is a convergent subsequence of a_n we know $a_{n_{j_i}} \to a$.
Let $\epsilon = \epsilon_o > 0$. By the definition of convergence there exists a $N \in \mathbb{N}$ such that for all $i \geq N$, $|a_{n_{j_i}} - a| < \epsilon$. Thus $|a_{n_{j_N}} - a| < \epsilon$. Define $J = j_N$. Note that $|a_{n_J} - a| < \epsilon$. Recall from above that by construction $|a_{n_j} - a| \geq \epsilon_o$, for all j, including J! Thus we have arrived at a contradiction, $\epsilon > |a_{n_J} - a| \geq \epsilon_o$ or $\epsilon_o > \epsilon_o$. We conclude the negation of our supposition $a_n \to a$. □