Note that I am operating under the convention that $N, n, m, i, j$ are natural numbers unless otherwise specified.

**Exercise :** Let $A$ and $B$ be nonempty sets that are bounded above. Suppose $\sup A < \sup B$. Prove that there is an element of $B$ that is an upper bound for $A$.

*Proof.* Suppose $A$ and $B$ are nonempty sets that are bounded above. Furthur suppose $\sup A < \sup B$. Define $a = \sup A$ and $b = \sup B$. Note that $a$ is less than the suppremum of $B$ thus $a$ is not a upper bound on $B$. Since $a$ is not a upper bound on $B$ there must exist at least one element of $B$ grater than $a$, take one of these elements lets call it $k$, $k \in B$, $a < k$. Choose a arbitrary element $c \in A$. Since $a = \sup A$ we know that $a$ is a upper bound on $A$ therfore $c \leq a < k$. Since we chose a arbitrary element from $A$ and showed that it is less than $k$ we can say that all elements in $A$ are less then $k$ thus $k$ is a upper bound on $A$. □

**Exercise :** In class we proved that $\mathbb{N}^2$ is countably infinite. Use this fact and a proof by induction to show that $\mathbb{N}^n$ is countably infinite for every $n \in \mathbb{N}$.

*Proof.* We want to show that for every $n \in \mathbb{N}$, $\mathbb{N}^n$ is countably infinite. I will procede with a proof by induction.
Base case $n = 1$. There is a bijective map, the identity map, mapping $\mathbb{N}^1 \to \mathbb{N}$. So the statement holds in the $n = 1$ case.
Suppose $\mathbb{N}^m$ is countably infinite for all $m \leq n$ where $n \geq 1$. There must exist a bijective map from $\mathbb{N}^n \to \mathbb{N}$, the definition of countably infinate. Note that $\mathbb{N}^{n+1}$ can trivially be bijectively mapped to $\mathbb{N}^n \times \mathbb{N}$, by mapping the first term to $\mathbb{N}$ and the next terms to $\mathbb{N}^n$. Note that there exists a bijective map from $\mathbb{N}^2 \to \mathbb{N}$ since $\mathbb{N}^2$ is countably infinate. Note that we can bijectively map $\mathbb{N}^{n+1} \to \mathbb{N}^n \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}^2 \to \mathbb{N}$. Thus $\mathbb{N}^{n+1}$ is countably infinate.
By induction we can conclude that for every $n \in \mathbb{N}$, $\mathbb{N}^n$ is countably infinite. □

**Exercise :** Compute

$$\lim_{n\to\infty} \frac{3^n}{n!}$$

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Define $a_n = 0$,$b_n = \frac{3^n}{n!}$,$c_n = \frac{3^n}{4^{n-4}}$. Note that for $1 \leq n \leq 4$, $b_n = \frac{3^n}{n!} \leq 3^n \leq \frac{3^n}{4^{n-4}} = c_n$. Note that for $n = 4$, $b_n \leq c_n$. Suppose for some $n \geq 4$, $b_n \leq c_n$. Note that $b_{n+1} = \frac{3^{n+1}}{(n+1)!} = \frac{3^n}{n!}\frac{3}{n+1} = b_n\frac{3}{n+1} \leq c_n\frac{3}{n+1} \leq c_n\frac{3}{4} = \frac{3^n}{4^{n-4}}\frac{3}{4} = \frac{3^{n+1}}{4^{(n+1)-4}} = c_{n+1}$. By induction I conclude that for all $n \geq 4$, $b_n \leq c_n$. So for all $n \in \mathbb{N}$, $b_n \leq c_n$. Also note that $b_n$ is always positive and thus $a_n \leq b_n$. Note that $a_n \to 0$. Note that $c_n$ is bounded bellow by 0. Also note that $c_{n+1} = \frac{3}{4}c_n \leq c_n$, so $c_n$ is monotone decreasing and bounded below, by the MCT it must converge. Define $l$ as the limit of $c_n$. Note that $c_{n+1} \to l$ also note that $c_{n+1} = \frac{3}{4}c_n \to \frac{3}{4}l$. So $l = \frac{3}{4}l$ therfore $l = 0$. By the squeze teorem $\frac{3^n}{n!} \to 0$.

**Exercise :** Suppose $F$ is a collection of open intervals such that if $I, J \in F$ and $I \neq J$, then $I \cap J = \emptyset$. Prove that $F$ is countable.

*Proof.* Suppose $F$ is a collection of open intervals such that if $I, J \in F$ and $I \neq J$, then $I \cap J = \emptyset$.
If $F$ is finite then it is at most countable.
Suppose $F$ is non-finite. Select one element of $F$, let's call it $d$ (for default). Consider the mapping $f : \mathbb{Q} \to G$ where $G = P(F)$, the power set of $F$, so that $G$ is the set of all subsets of $F$.

$$f(q) = \begin{cases} \{f \in F : q \in f\} & \{f \in F : q \in f\} \neq \emptyset \\ \{d\} & \text{otherwise} \end{cases}$$

Suppose there existed a $q$ such that $f(q)$ did not have cardinality 1. Note that $f(q) \neq \emptyset$, since we map anything that would have maped to the emptyset to the set containing the default set. Thus $f(q)$ must have at least two elements $I, J \in f(q)$. Also note that $f(q) \neq \{d\}$ since that has cardinality of one, $f(q) = \{f \in F : q \in f\}$. Note that by our above construction $I, J \in F$ and $I \neq J$ and $q \in I$, $q \in J$, since $f(q) \neq \{d\}$. Thus $q \in I \cap J$ so $I \cap J \neq \emptyset$. This is a contradiction since our initial upposition tells us $I \cap J = \emptyset$, we conclude the negation of our supposition, that $f(q)$ has one element for all $q \in \mathbb{Q}$.
We can now construct a function $g : \mathbb{Q} \to F$, where $g(q)$ is the one element in $f(q)$, noting that $f(q) \in G$ means that $f(q) \subseteq F$ and thus the one element in $f(q)$ is a element of $F$.
Choose $I \in F$. Note that by the density of the rationals there is a rational in the open interval $I$, select one of these elements and call it $q$, $q \in I$ and $q \in \mathbb{Q}$. Note that $g(q)$ is the element in $f(q)$, and $I \in \{f \in F : q \in f\} \neq \emptyset$ thus $f(q) = \{f \in F : q \in f\}$ and $I \in f(q)$ thus $g(q) = I$. Since we chose a arbitrary element in $F$ and found a $q \in \mathbb{Q}$ that maps to it via $f$ we can say $f$ is onto. We know that there is a onto map $h : \mathbb{N} \to \mathbb{Q}$ since $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality. Consider the map $m : \mathbb{N} \to F$ where $m(n) = g(h(n))$. Note that $m$ is onto. Since there is a onto map from $\mathbb{N} \to F$ we know that $F$ is at most countably infinate. □