Exercise 1.4.7: Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ contradicts the assumption that $\alpha = \sup A$.

Proof. □

Exercise Supplemental 1: Suppose for each $k \in \mathbb{N}$ that A_k is at most countable. Use the fact that $\mathbb{N} \times \mathbb{N}$ is countably infinite to show that $\cup_{k=1}^{\infty} A_k$ is at most countable. Hint: take advantage of surjections.

Proof. First let me define a new set B where $B = \{X \in A; X \neq \emptyset\}$. Note that $\cup B = \cup_{k=1}^{\infty} A_k$. Lets next deal with the case that $B = \emptyset$ in this case $\cup B = \emptyset$ and so is at most countable infinate. Next lets consider the case that B has a finite number of elements, we proved this case in class, a union of a finite number of at most countably infinate sets is at most countably infinate. Now we know that we are dealing with B a infinate set of at most countably infinate non-empty sets. Now I will introduce the notation $B_{k,l}$ where $B_{k,l}$ is the l element of B_k. Consider the function $f : \mathbb{N} \times \mathbb{N} \to B_{k,l}$ where

$$f(a, b) = \begin{cases} B_{a,b} & B_a \text{ has a } b\text{th element} \\ B_{1,1} & \text{otherwise} \end{cases}$$

Note that this function is surjective, since given a $B_{j,k}$ we see that (j, k) maps to it. There is also a surjection between each of our $B_{j,k}$ and $\cup B$ simply map the element $B_{j,k}$ to itself in $\cup B$. From knowing that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality, I conclude that there is a surjection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. Thus I can surjectively map $\mathbb{N} \to \mathbb{N} \times \mathbb{N} \to B_{j,k} \to \cup B$. Thus $\cup B$ is at most countably infinate. □

Exercise Supplemental 2: **(W) (Hand this one in to David.)**

Suppose B is finite and $A \subseteq B$. Show that A is empty or finite.

Consider the case where $A \neq \emptyset$. There must exist a bijective function mapping $f : S_m \to B$, the definition of finite. Since $A \subseteq B$ there must be a subset of S_m, lets call it $S_m|_A$ that has the property $f(S_m|_A) = A$. Lets now consider the function $g : S_m|_A \to A$ where g(x)=f(x). Note that by construction g is onto, since $g(S_m|_A) = f(S_m|_A) = A$ and since f is one-to-one on B we can see $g(a) = g(b)$ implies $f(a) = f(b)$ implies $a = b$, and so g is one-to-one. Thus g is bijective. Since $S_m|_A \in \mathbb{N}$ it will have a least element. Construct a map $h : S_m|_A \to S_l$ where the minimum of $S_m|_A$ gets mapped to 1 and the next smallest gets maped to 2 and so on until the last element maps to l. We can say that this procedure is possiable since at most it could take m steps and m is finite. By construction this function is onto and one-to-one. We now have a bijection between $S_m|_A$ and S_l, notice that we can now bijectively map A to $S_m|_A$ and $S_m|_A$ to S_l thus by definition A is finite.

Exercise 1.5.10 (a) (c):

(a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(c) Determine, with proof, if the same statement remains true replacing uncountable with infinite.

Proof (a). □

Proof (b). □

Exercise Supplemental 3: Show that the set of a finite subsets of $\mathbb{N}$ is countably infinite. Hint: Let A_k be the set of all subsets of $\mathbb{N}$ with no more than k elements. Show that each A_k is countably infinite.

Proof. □

Exercise 2.2.2: Verify using the definition of convergence the following limits.

(a) $\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$.

(c) $\lim_{n\to\infty} \frac{\sin(n^2)}{\sqrt{3n}} = 0$.

Proof (a). □

Proof (b). □

Proof (c). □

Exercise Supplemental 4: **(W) (Hand this one in to David.)**Carefully prove that the sequence (x_n) given by $x_n = (-1)^n$ does not converge.

Proof. □