Wyznacz wartości i wektory własne macierzy:

DODATKOWA PRACA DOMOWA Z MATEMATYKI

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OZE 102

$$A=\begin{bmatrix}0 & 3 & -2 & -1\\ 6 & 6 & -11 & 3\\ 6 & 7 & -12 & 3\\ 8 & 7 & -15 & 6\end{bmatrix}$$

Rozwiązanie:

$$A-\lambda\cdot J=\begin{bmatrix}0 & 3 & -2 & -1\\ 6 & 6 & -11 & 3\\ 6 & 7 & -12 & 3\\ 8 & 7 & -15 & 6\end{bmatrix}-\lambda\cdot\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}-\lambda & 3 & -2 & -1\\ 6 & 6-\lambda & -11 & 3\\ 6 & 7 & -12-\lambda & 3\\ 8 & 7 & -15 & 6-\lambda\end{bmatrix}$$

Równanie charakterystyczne:

$$\det(A-\lambda\cdot J)=0 \Rightarrow \begin{vmatrix}-\lambda & 3 & -2 & -1\\ 6 & 6-\lambda & -11 & 3\\ 6 & 7 & -12-\lambda & 3\\ 8 & 7 & -15 & 6-\lambda\end{vmatrix}=0 \Rightarrow$$

$$\Rightarrow -\lambda\cdot\begin{vmatrix}6-\lambda & -11 & 3\\ 7 & -12-\lambda & 3\\ 7 & -15 & 6-\lambda\end{vmatrix}-3\cdot\begin{vmatrix}6 & -11 & 3\\ 6 & -12-\lambda & 3\\ 8 & -15 & 6-\lambda\end{vmatrix}+(-2)\cdot\begin{vmatrix}6 & 6-\lambda & 3\\ 6 & 7 & 3\\ 8 & 7 & 6-\lambda\end{vmatrix}-(-1)\cdot$$

$$\begin{vmatrix}6 & 6-\lambda & -11\\ 6 & 7 & -12-\lambda\\ 8 & 7 & -15\end{vmatrix}=0 \Rightarrow -\lambda(-\lambda^3+7\lambda+6)-3(6\lambda^2-6\lambda+12)$$

$$-2(-6\lambda^2+6\lambda+12)+1(8\lambda^2-8)=0 \Rightarrow$$

$$\Rightarrow \lambda^4-7\lambda^2-6\lambda-18\lambda^2+18\lambda+36+12\lambda^2-12\lambda-24+8\lambda^2-8=0 \overset{*}{\Rightarrow}$$

$$\begin{vmatrix} 6-\lambda & -11 & 3 \\ 7 & -12-\lambda & 3 \\ 7 & -15 & 6-\lambda \end{vmatrix} = (6-\lambda)(-12-\lambda)(6-\lambda) + (7)(-15)(3) + (7)(-11)(3) -$$

$$\begin{matrix} 6-\lambda & -11 & 3 \\ 7 & -12-\lambda & 3 \end{matrix}$$

$$\left[(3)(-12-\lambda)(7) + (3)(-15)(6-\lambda) + (6-\lambda)(-11)(7)\right] =$$

$$= (-\lambda^3 + 108\lambda - 432) + (-315) + (-231) -$$

$$\left[(-252-21\lambda) + (-270+45\lambda) + (-462+77\lambda)\right] =$$

$$= -\lambda^3 \underline{+108\lambda} - 432 - 315 - 231 + 252 \underline{+21\lambda} + 270 \underline{-45\lambda} +$$

$$+ 462 \underline{-77\lambda} =$$

$$= \underline{\underline{-\lambda^3 + 7\lambda + 6}}$$

$$\begin{vmatrix} 6 & -11 & 3 \\ 6 & -12-\lambda & 3 \\ 8 & -15 & 6-\lambda \end{vmatrix} = (6)(-12-\lambda)(6-\lambda) + (6)(-15)(3) + (8)(-11)(3) -$$

$$\begin{matrix} 6 & -11 & 3 \\ 6 & -12-\lambda & 3 \end{matrix}$$

$$\left[(3)(-12-\lambda)(8) + (3)(-15)(6) + (6-\lambda)(-11)(6)\right] =$$

$$= (6\lambda^2 + 36\lambda - 432) + (-270) + (-264) -$$

$$\left[(-288-24\lambda) + (-270) + (-396+66\lambda)\right] =$$

$$= 6\lambda^2 \underline{+36\lambda} - 432 - 270 - 264 + 288 \underline{+24\lambda} + 270 + 396 \underline{-66\lambda} =$$

$$= \underline{\underline{6\lambda^2 - 6\lambda - 12}}$$

$$\begin{vmatrix} 6 & 6-\lambda & 3 \\ 6 & 7 & 3 \\ 8 & 7 & 6-\lambda \end{vmatrix} = (6)(7)(6-\lambda) + (6)(7)(3) + (8)(6-\lambda)(3) -$$

$$\begin{matrix} 6 & 6-\lambda & 3 \\ 6 & 7 & 3 \end{matrix} \qquad \left[(3)(7)(8) + (3)(7)(6) + (6-\lambda)(6-\lambda)(6)\right] =$$

$$= (252-42\lambda) + (126) + (144-24\lambda) -$$

$$\left[(168) + (126) + (6x^2-72x+216)\right] =$$

$$= 252 - 42x + 126 + 144 - 24x - 168 - 126 - 6x^2 + 72x - 216 =$$

$$\cancel{216}$$

$$= \underline{\underline{-6x^2 + 6x + 12}}$$

$$\begin{vmatrix} 6 & 6-\lambda & -11 \\ 6 & 7 & -12-\lambda \\ 8 & 7 & -15 \end{vmatrix} = (6)(7)(-15) + (6)(7)(-11) + (8)(6-\lambda)(-12-\lambda) -$$

$$\begin{matrix} 6 & 6-\lambda & -11 \\ 6 & 7 & -12-\lambda \end{matrix} \qquad \left[(-11)(7)(8) + (6)(7)(-12-\lambda) + (-15)(6-\lambda)(6)\right] =$$

$$= (-630) + (-462) + (8\lambda^2 + 48\lambda - 576) -$$

$$\left[(-616) + (-504 - 42\lambda) + (-540 + 90\lambda)\right] =$$

$$= -630 - 462 + 8\lambda^2 + 48\lambda - 576 + 616 + 504 + 42\lambda + 540 - 90\lambda =$$

$$= \underline{\underline{8\lambda^2 - 8}}$$

$$\overset{*}{\Longrightarrow} \boxed{\lambda^4 - 5\lambda^2 + 4 = 0}$$

Uzyskane równanie:

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

$$t = \lambda^2$$

$$t^2 - 5t + 4 = 0$$

$$\Delta = (-5)^2 - 4 \cdot 1 \cdot 4 = 25 - 16 = 9$$

$$\sqrt{\Delta} = 3$$

$$t_1 = \frac{5-3}{2} = \frac{2}{2} = 1$$

$$t_2 = \frac{8}{2} = 4$$

$$\Rightarrow \lambda^2 = 1 \ \vee \ \lambda^2 = 4$$

$$\lambda = 1 \vee \lambda = -1 \vee \lambda = 2 \vee \lambda = -2$$

Wartości własne: $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = 2$

Wektory własne V_1, V_2, V_3, V_4 odpowiadające $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ wyznaczone rozpatrując układy równań: $\underline{(A - \lambda_i \cdot J) \cdot V_i = 0}$

Dla $\lambda_1 = -2$ mamy:

$$A - \lambda \cdot J = \begin{bmatrix} -\lambda & 3 & -2 & -1 \\ 6 & 6-\lambda & -11 & 3 \\ 6 & 7 & -12-\lambda & 3 \\ 8 & 7 & -15 & 6-\lambda \end{bmatrix} \Rightarrow A - \lambda_1 \cdot J = A + 2J = \begin{bmatrix} 2 & 3 & -2 & -1 \\ 6 & 8 & -11 & 3 \\ 6 & 7 & -10 & 3 \\ 8 & 7 & -15 & 8 \end{bmatrix}$$

$$V_1 = [x, y, z, t]^T, \ (A - \lambda_1 \cdot J) \cdot V_1 = 0 \Rightarrow \begin{cases} 2x + 3y - 2z - t = 0 \\ 6x + 8y - 11z + 3t = 0 \\ 6x + 7y - 10z + 3t = 0 \\ 8x + 7y - 15z + 8t = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 3 & -2 & -1 & 0 \\ 6 & 8 & -11 & 3 & 0 \\ 6 & 7 & -10 & 3 & 0 \\ 8 & 7 & -15 & 8 & 0 \end{array}\right] \overset{w_4 := w_4 - 4w_1}{\Longleftrightarrow} \left[\begin{array}{cccc|c} 2 & 3 & -2 & -1 & 0 \\ 6 & 8 & -11 & 3 & 0 \\ 6 & 7 & -10 & 3 & 0 \\ 0 & -5 & -7 & 12 & 0 \end{array}\right] \overset{w_3 := w_3 - 3w_1}{\Longleftrightarrow} \left[\begin{array}{cccc|c} 2 & 3 & -2 & -1 & 0 \\ 6 & 8 & -11 & 3 & 0 \\ 0 & -2 & -4 & 6 & 0 \\ 0 & -5 & -7 & 12 & 0 \end{array}\right] \Leftrightarrow$$

$v_4 := v_4 - \frac{5}{2} v_3$

$$\Longleftrightarrow \begin{array}{c} \left[\begin{array}{cccc|c} 2 & 3 & -2 & -1 & 0 \\ 6 & 8 & -11 & 3 & 0 \\ 0 & -2 & -4 & 6 & 0 \\ 0 & 0 & 3 & -3 & 0 \end{array}\right] \\ \begin{array}{cccc} x & y & z & t \end{array} \end{array} \Rightarrow \begin{array}{l} 3z - 3t = 0 \\ 3z = 3t \\ z = t, \end{array}$$

$$\begin{cases} 2x + 3y - 2z - t = 0 \\ 6x + 8y - 11z + 3t = 0 \\ -2y - 4z + 6t = 0 \end{cases}$$

$$\begin{cases} 2x + 3y - 2z - z = 0 \\ 6x + 8y - 11z + 3z = 0 \\ -2y - 4z + 6z = 0 \end{cases}$$

$$\begin{cases} 2x + 3y - 3z = 0 \\ 6x + 8y - 8z = 0 \\ -2y + 2z = 0 \end{cases}$$

$$\begin{cases} 2x + 3y - 3z = 0 \\ 6x + 8y - 8z = 0 \\ z = y \end{cases}$$

$$\underline{\underline{z = y = t}}$$

$$\begin{cases} 2x + 3y - 3z = 0 \\ 6x + 8y - 8z = 0 \end{cases}$$

$$\begin{cases} 2x + 3y - 3y = 0 \\ 6x + 8y - 8y = 0 \end{cases}$$

$$\begin{cases} 2x = 0 \\ 6x = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ x = 0 \end{cases}, \quad x = x$$

$$2x + 3y - 2z - t = 0$$
$$0 + 3z - 2z - t = 0$$
$$z - t = 0$$
$$t - t = 0$$
$$t = t; \ t \in \mathbb{R}$$

$$\begin{cases} x = 0 \\ y = z \\ z = t \\ t \in \mathbb{R} \end{cases} \Rightarrow V_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{przyjęto } t = 1$$

Dla $\lambda_2 = -1$ mamy:

$$A - \lambda \cdot J = \begin{bmatrix} -\lambda & 3 & -2 & -1 \\ 6 & 6-\lambda & -11 & 3 \\ 6 & 7 & -12-\lambda & 3 \\ 8 & 7 & -15 & 6-\lambda \end{bmatrix} \Rightarrow A - \lambda_2 \cdot J = A + J = \begin{bmatrix} 1 & 3 & -2 & -1 \\ 6 & 7 & -11 & 3 \\ 6 & 7 & -11 & 3 \\ 8 & 7 & -15 & 7 \end{bmatrix}$$

$$V_2 = [x, y, z, t]^T, \ (A - \lambda_2 \cdot J) \cdot V_2 = 0 \Rightarrow \begin{cases} x + 3y - 2z - t = 0 \\ 6x + 7y - 11z + 3t = 0 \\ 6x + 7y - 11z + 3t = 0 \\ 8x + 7y - 15z + 7t = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ 6 & 7 & -11 & 3 & 0 \\ 6 & 7 & -11 & 3 & 0 \\ 8 & 7 & -15 & 7 & 0 \end{array}\right] \overset{w_2 := w_3}{\Longleftrightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ 6 & 7 & -11 & 3 & 0 \\ 8 & 7 & -15 & 7 & 0 \end{array}\right] \overset{w_3 := w_3 - 8w_1}{\Longleftrightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ 6 & 7 & -11 & 3 & 0 \\ 0 & -17 & 1 & 15 & 0 \end{array}\right]$$

$$\overset{w_2 := w_2 - 6w_1}{\Longleftrightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ 0 & -11 & 1 & 9 & 0 \\ 0 & -17 & 1 & 15 & 0 \end{array}\right] \overset{w_3 := w_3 - \frac{17}{11} w_2}{\Longleftrightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ 0 & -11 & 1 & 9 & 0 \\ 0 & 0 & -\frac{6}{11} & \frac{12}{11} & 0 \end{array}\right] \Rightarrow -\frac{6}{11} z + \frac{12}{11} t = 0$$

$\frac{12}{11} t = \frac{6}{11} z \quad |\cdot 11$

$12t = 6z \quad |:6$

$\underline{2t = z}$

$$\begin{cases} x + 3y - 2z - t = 0 \\ -11y + z + 9t = 0 \end{cases}$$

$\cancel{x+3y-2z}$ $\begin{cases} x + 3y - 4t - t = 0 \\ -11y + 2t + 9t = 0 \end{cases}$

$$\begin{cases} x + 3y - 5t = 0 \\ -11y + 11t = 0 \end{cases}$$

$$\begin{cases} x + 3y - 5t = 0 \\ 11t = 11y \quad |:11 \end{cases}$$

$$\begin{cases} x + 3y - 5t = 0 \\ \underline{t = y} \end{cases}$$

$x + 3t - 5t = 0$

$x - 2t = 0$

$\underline{x = 2t}$

$x + 3y - 2z - t = 0$

$2t + 3t - 2\cdot 2t - t = 0$

$5t - 5t = 0$

$5t = 5t \quad |:5$

$t = t;\ t \in \mathbb{R}$

$$\begin{cases} x = 2t \\ y = t \\ z = 2t \\ t \in \mathbb{R} \end{cases} \Rightarrow V_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

przyjęto $t = 1$

Dla $\lambda_3 = 1$ mamy:

$$A - \lambda \cdot J = \begin{bmatrix} -\lambda & 3 & -2 & -1 \\ 6 & 6-\lambda & -11 & 3 \\ 6 & 7 & -12-\lambda & 3 \\ 8 & 7 & -15 & 6-\lambda \end{bmatrix} \Rightarrow A - \lambda_3 \cdot J = A - J = \begin{bmatrix} -1 & 3 & -2 & -1 \\ 6 & 5 & -11 & 3 \\ 6 & 7 & -13 & 3 \\ 8 & 7 & -15 & 5 \end{bmatrix}$$

$V_3 = [x, y, z, t]^T$, $(A - \lambda_3 \cdot J) \cdot V_3 = 0 \Rightarrow \begin{cases} -x+3y-2z-t=0 \\ 6x+5y-11z+3t=0 \\ 6x+7y-13z+3t=0 \\ 8x+7y-15z+5t=0 \end{cases}$

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & -1 & 0 \\ 6 & 5 & -11 & 3 & 0 \\ 6 & 7 & -13 & 3 & 0 \\ 8 & 7 & -15 & 5 & 0 \end{array}\right] \xLeftrightarrow{w_4 := w_4 + 8w_1} \left[\begin{array}{cccc|c} -1 & 3 & -2 & -1 & 0 \\ 6 & 5 & -11 & 3 & 0 \\ 6 & 7 & -13 & 3 & 0 \\ 0 & 31 & -31 & -3 & 0 \end{array}\right] \xLeftrightarrow{w_3 := w_3 - w_2}$$

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & -1 & 0 \\ 6 & 5 & -11 & 3 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 31 & -31 & -3 & 0 \end{array}\right] \xLeftrightarrow{w_4 := w_4 - \frac{31}{2} w_3} \left[\begin{array}{cccc|c} -1 & 3 & -2 & -1 & 0 \\ 6 & 5 & -11 & 3 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right] \Rightarrow -3t = 0$$

$$\underline{\underline{t = 0}}$$

$\begin{cases} -x+3y-2z-t=0 \\ 6x+5y-11z+3t=0 \\ 2y-2z=0 \end{cases}$

$\begin{cases} -x+3y-2z-t=0 \\ 6x+5y-11z+3t=0 \\ \underline{\underline{y = z}}, \end{cases}$

$\begin{cases} -x+3z-2z-0=0 \\ 6x+5z-11z+3\cdot 0=0 \end{cases}$

$\begin{cases} -x+z=0 \\ 6x-6z=0 \end{cases}$

$\begin{cases} x = z \\ \underline{\underline{x = z}}, \end{cases}$ $z = z;\ z \in \mathbb{R}$

$x = z = y;\ z \in \mathbb{R}$

$\begin{cases} x = z \\ y = z \\ z \in \mathbb{R} \\ t = 0 \end{cases} \Rightarrow V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ przyjęto $z = 1$

<u>Dla $\lambda_4 = 2$ mamy:</u>

$$A - \lambda \cdot J = \begin{bmatrix} -\lambda & 3 & -2 & -1 \\ 6 & 6-\lambda & -11 & 3 \\ 6 & 7 & -12-\lambda & 3 \\ 8 & 7 & -15 & 6-\lambda \end{bmatrix} \Rightarrow A - \lambda_3 \cdot J = A - 2J = \begin{bmatrix} -2 & 3 & -2 & -1 \\ 6 & 4 & -11 & 3 \\ 6 & 7 & -14 & 3 \\ 8 & 7 & -15 & 4 \end{bmatrix}$$

$$V_4 = [x, y, z, t]^T, \quad (A - \lambda_4 \cdot I) \cdot V_4 = 0 \Rightarrow \begin{cases} -2x + 3y - 2z - t = 0 \\ 6x + 4y - 11z + 3t = 0 \\ 6x + 7y - 14z + 3t = 0 \\ 8x + 7y - 15z + 4t = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} -2 & 3 & -2 & -1 & 0 \\ 6 & 4 & -11 & 3 & 0 \\ 6 & 7 & -14 & 3 & 0 \\ 8 & 7 & -15 & 4 & 0 \end{array}\right] \xleftrightarrow{r_4 := r_4 + 4r_1} \left[\begin{array}{cccc|c} -2 & 3 & -2 & -1 & 0 \\ 6 & 4 & -11 & 3 & 0 \\ 6 & 7 & -14 & 3 & 0 \\ 0 & 19 & -23 & 0 & 0 \end{array}\right] \xleftrightarrow{r_3 := r_3 + 3r_1}$$

$$\left[\begin{array}{cccc|c} -2 & 3 & -2 & -1 & 0 \\ 6 & 4 & -11 & 3 & 0 \\ 0 & 16 & -20 & 0 & 0 \\ 0 & 19 & -23 & 0 & 0 \end{array}\right] \xleftrightarrow{r_4 := r_4 - \frac{19}{16} r_3} \left[\begin{array}{cccc|c} -2 & 3 & -2 & -1 & 0 \\ 6 & 4 & -11 & 3 & 0 \\ 0 & 16 & -20 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 \end{array}\right] \Rightarrow \frac{3}{4} z = 0$$

$$\underline{\underline{z = 0}}$$

$$\begin{cases} -2x + 3y - 2z - t = 0 \\ 6x + 4y - 11z + 3t = 0 \\ 16y - 20z = 0 \end{cases}$$

$$\begin{cases} -2x + 3y - 2 \cdot 0 - t = 0 \\ 6x + 4y - 11 \cdot 0 + 3t = 0 \\ 16y = 20z \;/ : 2 \end{cases}$$

$$\begin{cases} -2x + 3y - t = 0 \\ 6x + 4y + 3t = 0 \\ 8y = 10z \end{cases}$$

$$\begin{cases} -2x + 3y - t = 0 \\ 6x + 4y + 3t = 0 \\ 8y = 0 \cdot 10 \end{cases}$$

$$\begin{cases} -2x + 3y - t = 0 \\ 6x + 4y + 3t = 0 \\ \underline{\underline{y = 0}} \end{cases}$$

$$\begin{cases} -2x + 0 - t = 0 \\ 6x + 0 + 3t = 0 \end{cases}$$

$$\begin{cases} -2x = t \\ 6x = -3t \;/ : 3 \end{cases}$$

$$\begin{cases} -2x = t \\ 2x = -t \;/ \cdot (-1) \end{cases}$$

$$\begin{cases} -2x = t \\ -2x = t \end{cases}$$

$$\underline{t = t, t \in R}$$

$$\begin{cases} x = -\frac{1}{2} t \\ y = 0 \\ z = 0 \\ t \in R \end{cases} \Rightarrow V_4 = \begin{bmatrix} -0{,}5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

przyjęto $t = 1$

$$-2x + 3y - 2z - t = 0$$
$$-2x + 0 - 0 - t = 0$$
$$-2x = t \;/ \cdot (-1)$$
$$2x = -t$$
$$\underline{\underline{x = -\frac{1}{2} t}}$$

Wartości $\overbrace{\text{macierzy}}^{\text{własne}}$: $\lambda_1 = -2$; $\lambda_2 = -1$; $\lambda_3 = 1$; $\lambda_4 = 2$

Wektory własne macierzy: $V_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$; $V_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$; $V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$; $V_4 = \begin{bmatrix} -0{,}5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$