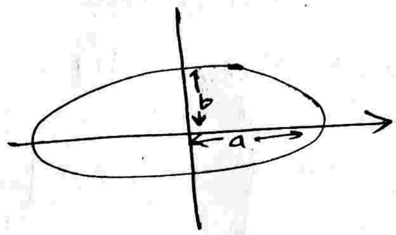
Lecture: 18

[Change of variables] (Jacobian)

Example 1: Area of ellipse with semi axes a, b.



eq^n of ellipse:-

$$\boxed{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1}$$

$$\iint_{\left(\frac{x}{a}\right)^2+\left(y/b\right)^2<1} dx\,dy = \iint_{u^2+v^2<1} ab\, du\,dv$$

set $\frac{x}{a}=u \longrightarrow du=\frac{1}{a}dx$

$\frac{y}{b}=v \longrightarrow dv=1/b\, dy$

$\left.\right\}$ $du\,dv=\frac{1}{ab}dx\,dy$

$\Rightarrow du\,dv=\frac{1}{ab}dx\,dy$

$\Rightarrow \underline{dx\,dy=ab\,du\,dv}$

$$\Leftrightarrow ab\iint_{u^2+v^2<1} du\,dv \Rightarrow ab\cdot\text{area (unit disk)}$$

$$=\underline{\pi ab}$$

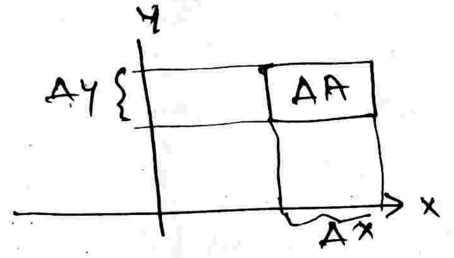
Note: So general problem, when we try to do this, is to figure out what is the scale factor? What's the relation b/w dxdy & dudv.

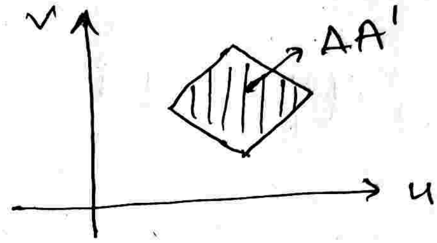
In general: find scaling factor. (dxdy vs dudv).

Example 2: $u = 3x - 2y$, $v = x + y$ { to simplify the integrand or bounds }.

Relation b/w $dA = dx\,dy$ & area element in uv co-ordinates, (dA') $dA' = du\,dv$.

$\Rightarrow$



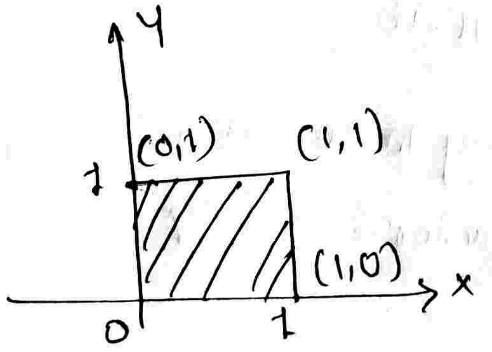


So, we have to figure out how dA & dA' are related.

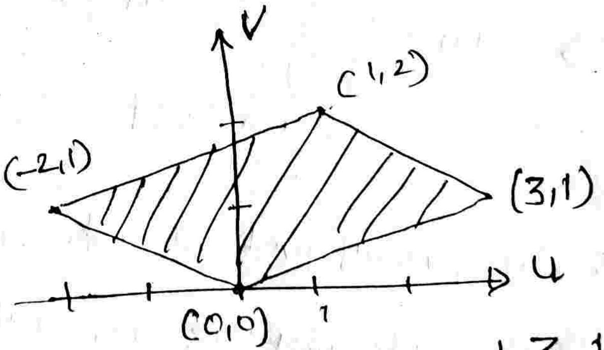
so that we can decide what conversion factor, b/w two currencies for area.

$\Rightarrow$ Area scaling factor here doesn't depend on the choice of recantangular, because we are doing linear change of variable.

Taking a unit square:



A = 1



Area(◇) = $A' = \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} = 5$

for any other rectangle, area is also $*5$;

So $dA' = 5dA$

$dudv = 5\,dxdy$

$$\boxed{\iint \cdots dxdy = \iint \cdots \frac{1}{5}\,dudv.}$$

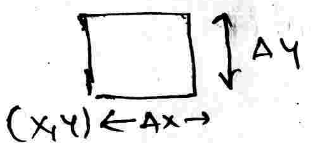
General case :-

$u = u(x,y)$ $\qquad \Delta u \simeq u_x \Delta x + u_y \Delta y$

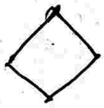
$v = v(x,y)$ $\qquad \Delta v \simeq v_x \Delta x + v_y \Delta y.$

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \simeq \begin{bmatrix} u_x & u_y \\ v_x & u_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

small xy-rectangle:



small ~~xy~~ uv-parallelogram



depend of $\partial u, \partial v$

$\Rightarrow$ Same argument $\Rightarrow$

if we move $\langle \Delta x, 0 \rangle \rightsquigarrow \langle \Delta u, \Delta v \rangle = \langle u_x \Delta x, v_x \Delta x \rangle$

$\langle 0, \Delta y \rangle \rightsquigarrow \langle \Delta u, \Delta v \rangle = \langle u_y \Delta y, v_y \Delta y \rangle$

sides of parallelogram

& determinant will be

area$' =$ det$(\cdots)\, \Delta x \Delta y$

Jacobian :

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Then $du\,dv = |J|\,dx\,dy = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx \cdot dy.$

for absolute value.

Example: polar coordinates :-

$x = r\cos\theta$; $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$

$y = r\sin\theta$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r\cos^2\theta + r\sin^2\theta$

$= r.$

So,

$dx\,dy = |r|\,dr\,d\theta$

$= r\,dr \cdot d\theta$

Note: • Conversion Ratio (Jacobian) works both ways,

$\frac{\partial(u,v)}{\partial(x,y)}$ and $\frac{\partial(x,y)}{\partial(u,v)}$; but product will be 1.

so we can compute the easiest one.!

• Ration b/w $\partial(u,v)$ & $\partial(x,y)$; will be f^n as well as scalar

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Example 2: Compute $\int_0^1 \int_0^1 x^2 y \, dx\,dy$; by changing to $u = x$, $v = xy$;

Solution:

$\Rightarrow$ ① area element :

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x.$$

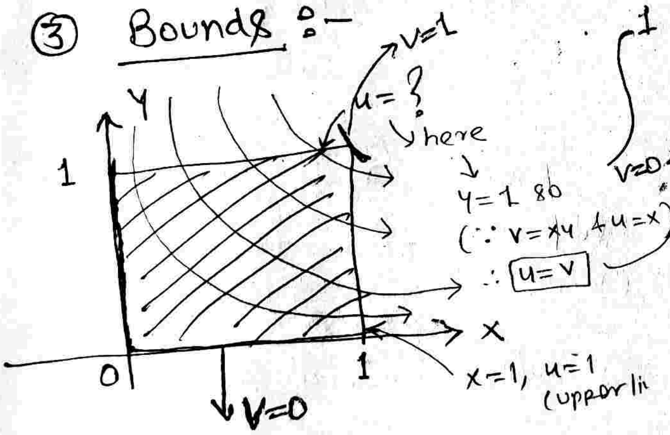
so; $\boxed{du\,dv = x\ dx \cdot dy}$ $(x \geq 0)$

② Integrand in terms of u,v :-

$$x^2y\ dx\,dy = x^2y \cdot \frac{1}{x} \cdot du\,dv = xy\ du\,dv$$
$$= v\,du\,dv$$

$$\iint_{??} v\,du\,dv \quad (\text{or } dv\,du\,?)$$

③ Bounds :-



$$\int_{v=0}^{1} \int_{v}^{1} v\ du\ dv$$

u changes, v – constant

xy = const.

Value of u here.