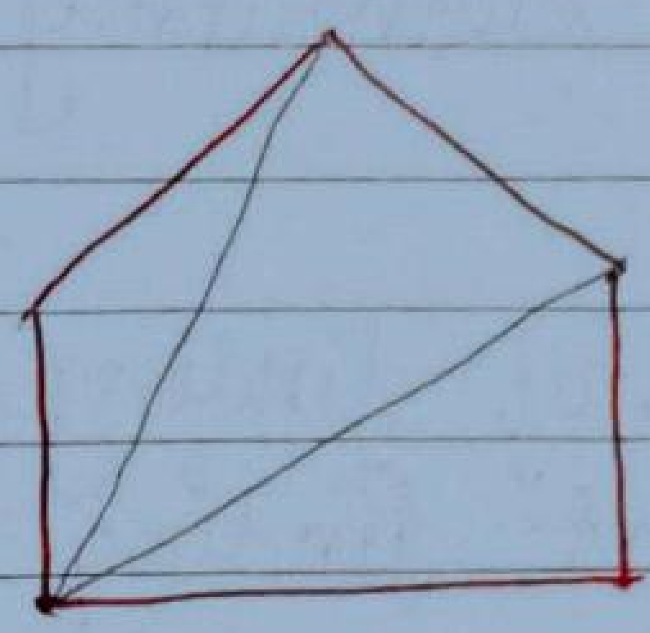
Lecture 6-2

CROSS PRODUCT:-

Area ?:-

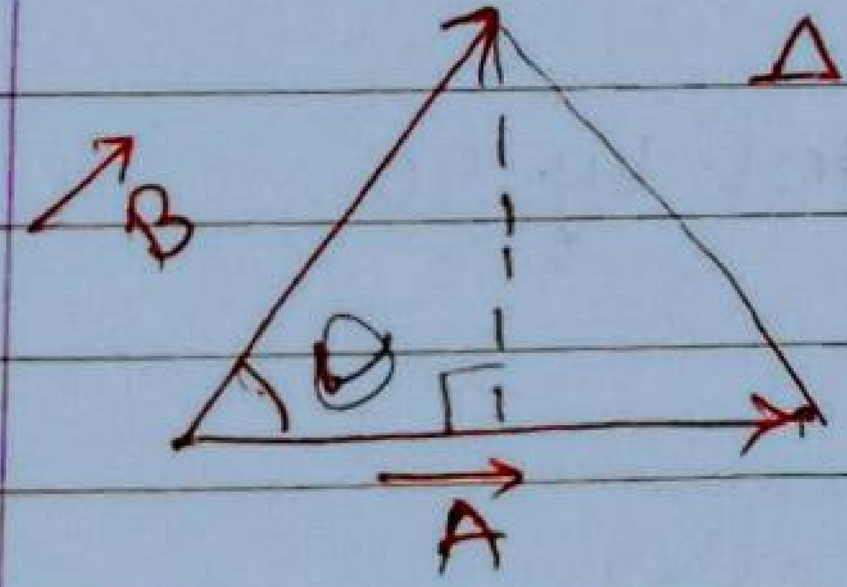
Find area of pentagon:
compute using 'vector'



Sol'. ~~① Simplify~~
① Cut into three triangles
② Sum the areas of triangle.

⇒ area of triangle ?

triangle in plane:
→ we need two vectors to describe it.



$\Delta = \frac{1}{2}$ base × height

$$\boxed{A = \frac{1}{2}\,|\vec{A}|\,|\vec{B}|\,\sin\theta} \quad — ①$$

↳ is similar to $|\vec{A}||\hat{u}|\cos\theta$; except for one little catch
this is a ~~sign~~ sin instead of a 'cos'

↳ How do we deal with that ?

Using Elementary geometry & dot products.

↳ 1st find cosine of the angle using dot product
(using $\boxed{\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}}$,

↳ 2nd :- Using dot product:
Solve for sine using $\sin^2\theta + \cos^2\theta = 1$.

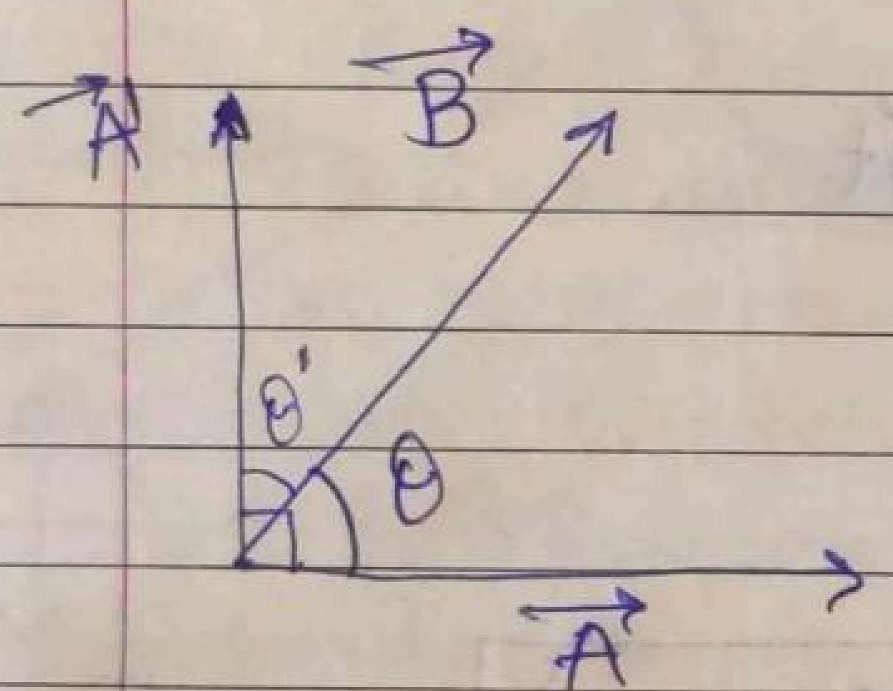
↳ 3rd: plug that into eq-①.

easier ways: Using determinants

Solving with elementary geometry & dot products first:

↳ Instead of finding the $\sin(\theta)$; well we're not good at finding ~~signs~~ sin of angles.

but we are very good in finding cosines of angles so maybe we can find another angle whose cosine is the same as the sin of theta. (complementary angles)



→ I rotate my vector $\vec{A}$ by 90° to get new vector $\vec{A}'$

$$\vec{A}' = \vec{A} \text{ rotated by } 90^\circ$$

$$\theta' = \frac{\pi}{2} - \theta$$

$$\boxed{\cos(\theta') = \sin(\theta)}$$

↳ means that Length $|\vec{A}| \cdot |\vec{B}| \sin\theta$

$\because |\vec{A}| = |\vec{A}'|$

$$= \boxed{|\vec{A}'|\,|\vec{B}| \cos(\theta')} \quad \text{—} (2)$$

multiplication

$$= \vec{A}' \cdot \vec{B}$$

? But How to find $\underline{\vec{A}'}$? check example 1(a)

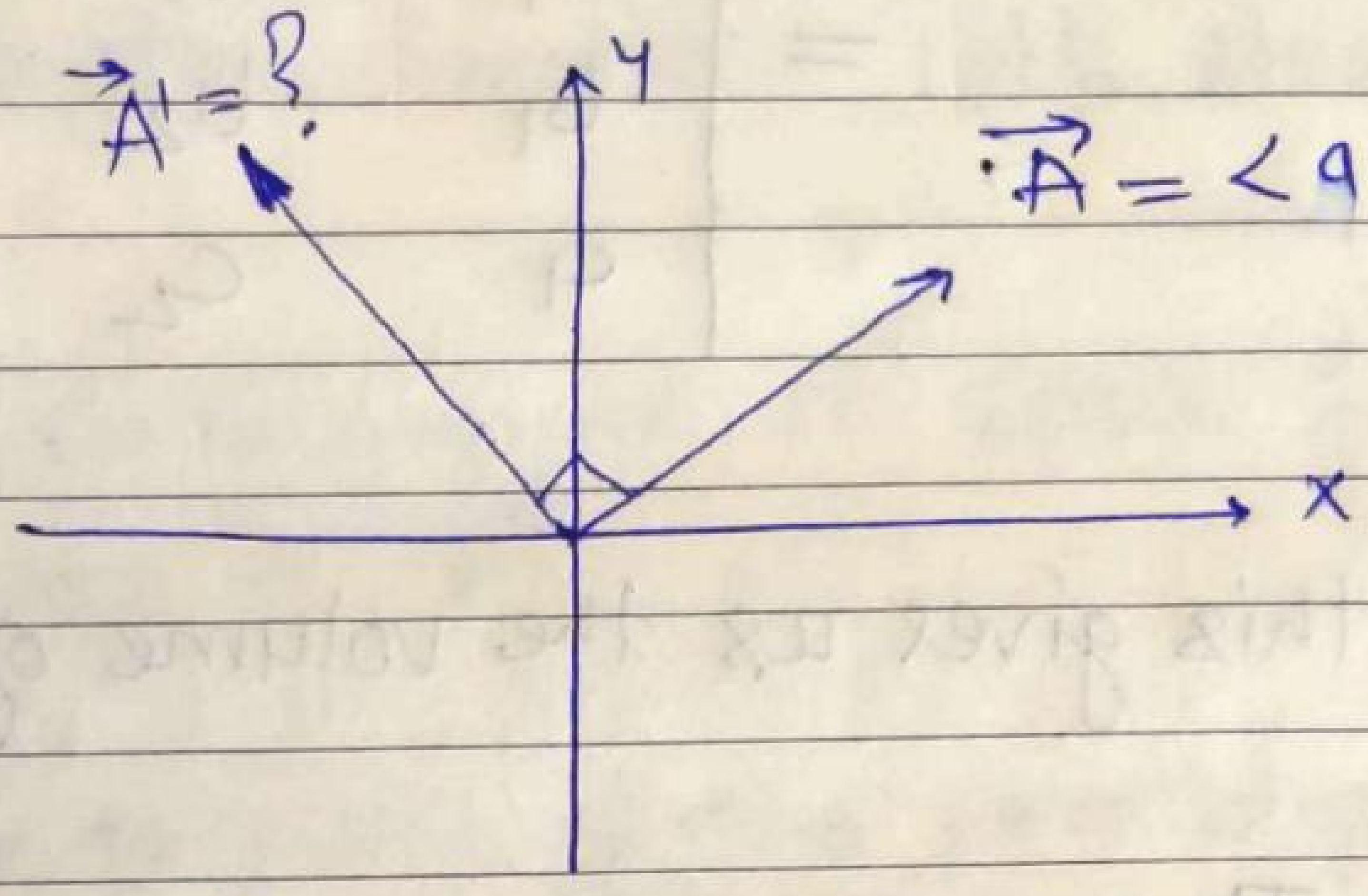
$$= \langle -a_2, a_1 \rangle \cdot \langle b_1, b_2 \rangle$$

$$= a_1 b_2 - a_2 b_1 \checkmark$$

↳ this quantity is another name of determinant $(\vec{A}, \vec{B})$

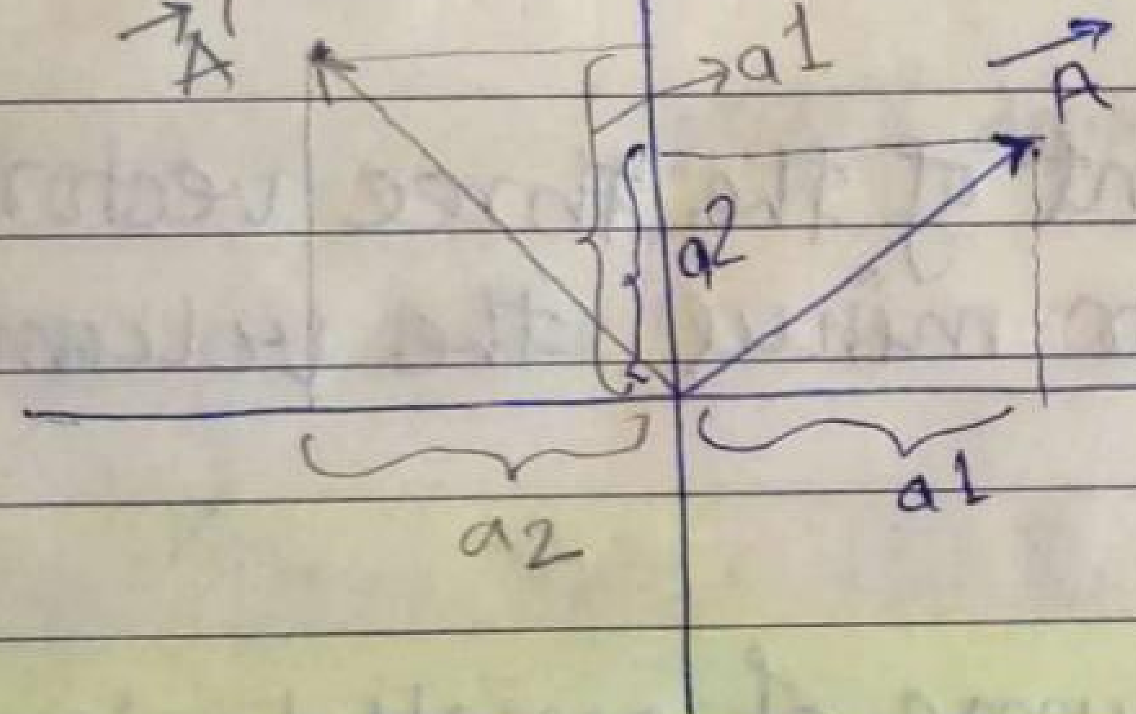
1(a) example to find A' //

let's say that I have a plane vector A with two component $\langle a_1, a_2 \rangle$ & I want to rotate it counterclockwise by 90°.



(a) $\langle a_2, a_1 \rangle$
(b) $\langle -a_2, a_1 \rangle$
(c) $\langle a_2, -a_1 \rangle$
(d) $\langle -a_1, a_2 \rangle$

Sol:-



$\vec{A} = \langle a_1, a_2 \rangle$

→ rotate this box, by 90°.

On rotation by 90°

$$\boxed{\vec{A}' = \cancel{\langle a a2,} \ \langle -a_2, a_1 \rangle}$$ ✓

(#) Geometrically, Determinant measures ??

is the area of parallelogram formed by $\vec{A}$ & $\vec{B}$

(#) $\det(\vec{A}, \vec{B})$ = Area of parallelogram formed by A & B

** absolute value of det.

so triangle area $= \frac{1}{2} \det(\vec{A}, \vec{B})$

→ Determinant is minus or plus the area.

$\pm$ area($\square$) $= |\vec{A}||\vec{B}| \sin\theta = \det(\vec{A}, \vec{B})$

$\pm$ area($\triangle$) $= \frac{1}{2} |\vec{A}||\vec{B}| \sin\theta = \frac{1}{2} \det(\vec{A}, \vec{B})$.

Determinant in Space

↳ tells volumes

→ $\det(\vec{A}, \vec{B}, \vec{C})$

$i+j$ = odd

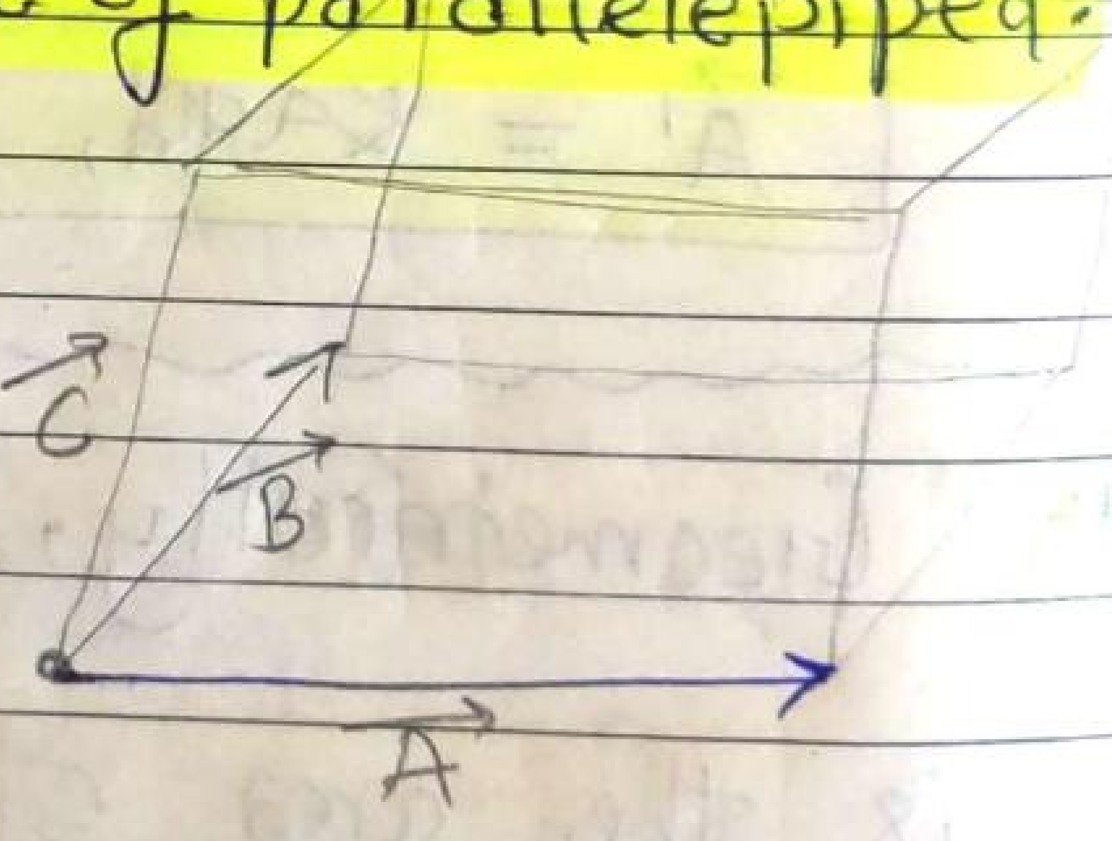
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

- this gives us the volume of the box with sides A, B, C.

Theorem:

Geometrically the determinant of the three vectors A, B, C is again plus or minus the volume of parallelepiped.

$$\det(\vec{A}, \vec{B}, \vec{C}) = \pm \text{ volume of parallelepiped.}$$



Cross Product:

→ You can apply to 2 vectors in space (3-D space)

Def: $\vec{A} \times \vec{B}$ is a vector

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Dot Product gives a scalar; Cross Product → vector.

$$\vec{A} \times \vec{B} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

↳ Geometric meaning of ~~so~~ this relation?

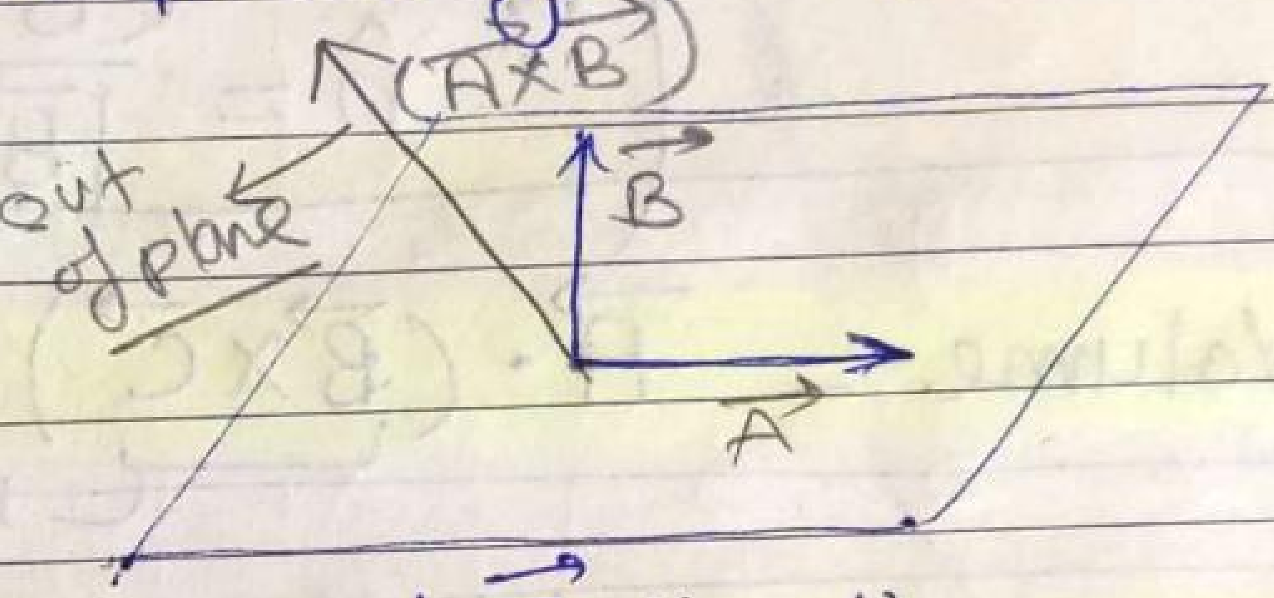
Theorem: Vector has length & direction. Let's start with length.

\# $|\vec{A} \times \vec{B}|$ = area of parallelogram in space formed by vector $\vec{A}$, $\vec{B}$.

\# direction $(\vec{A} \times \vec{B})$ ⊥r to plane of the parallelogram.

it can be ⊥r (upside & downside)

How to decide?



Right Hand Rule:
1. ~~R~~ Hand toward $\vec{A}$ direction
2. Curl ur finger toward $\vec{B}$
3. ~~#~~ Get the thumb straight out]

that tells ~~Acos~~ $(A \times \vec{B})$ will go up.

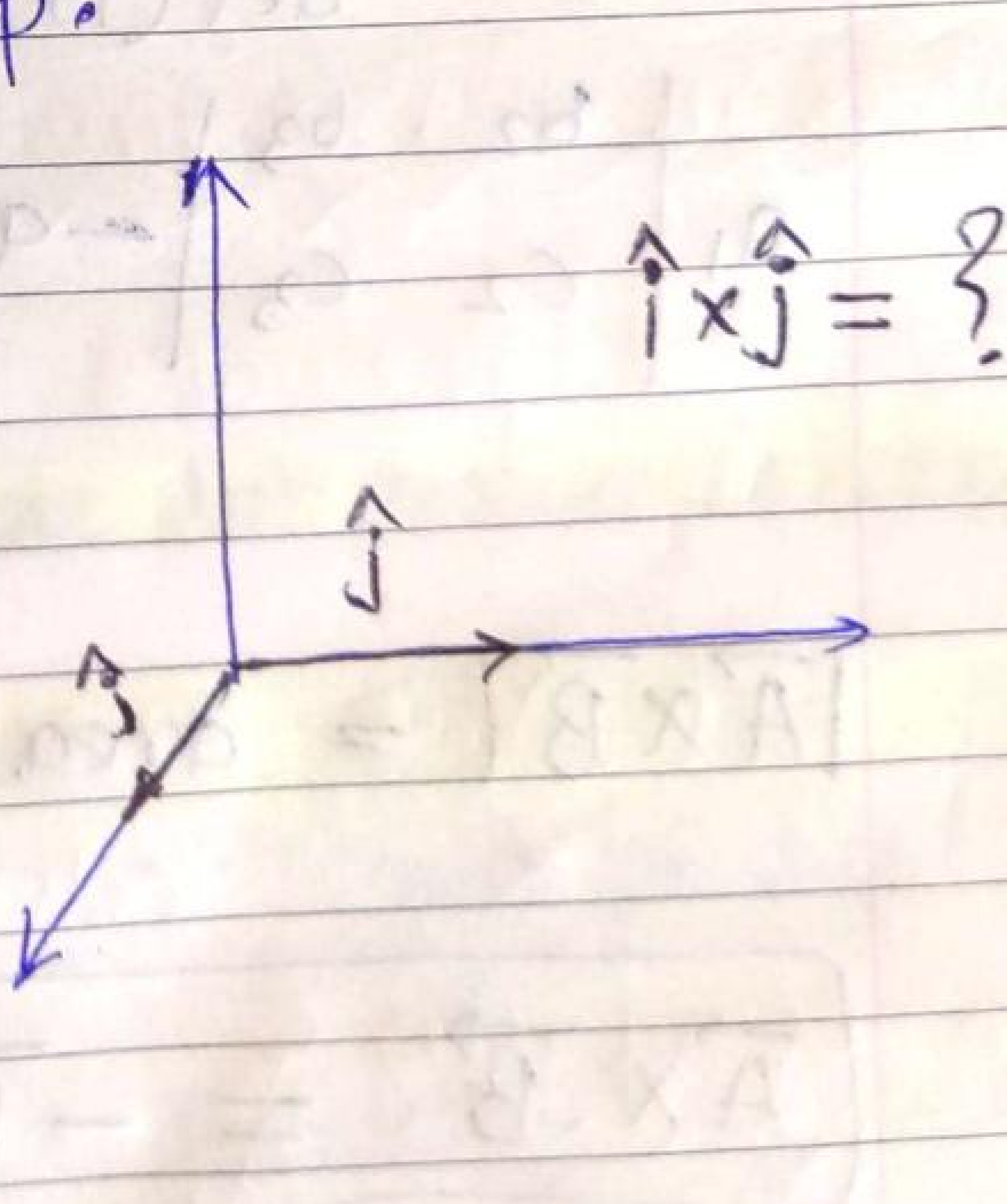
example: 2 (a)

$\hat{i} \times \hat{j} = \hat{k}$ (from R.H.R)

using determinant:

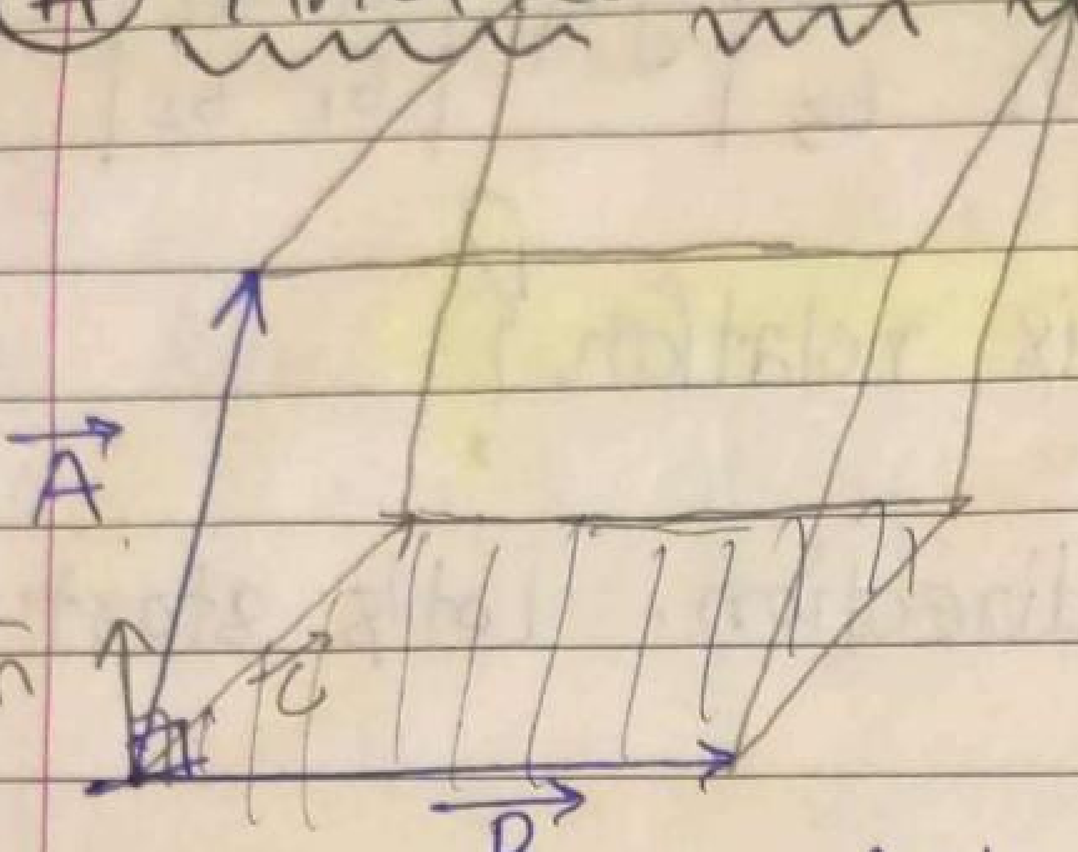
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 1\hat{k}$$

$$= \hat{k}$$



? 2D : How do I find volume if I don't want to know about determinants.

Another look at volume :-



Volume = area(base) $\times$ height

Area in space b/w two vector $\vec{B}$ & $\vec{C}$; $= |\vec{B} \times \vec{C}|$

$$\text{Volume} = |\vec{B} \times \vec{C}| \cdot \underbrace{(\vec{A} \cdot \hat{n})}_{\text{how much } \vec{A} \text{ goes in direction of } \hat{n}.}$$

$$\text{Volume} = |\vec{B} \times \vec{C}| \cdot \left(\vec{A} \cdot \frac{(\vec{B} \times \vec{C})}{|\vec{B} \times \vec{C}|}\right)$$

$$\left(\because \hat{n} = \frac{(\vec{B} \times \vec{C})}{|\vec{B} \times \vec{C}|}\right)$$

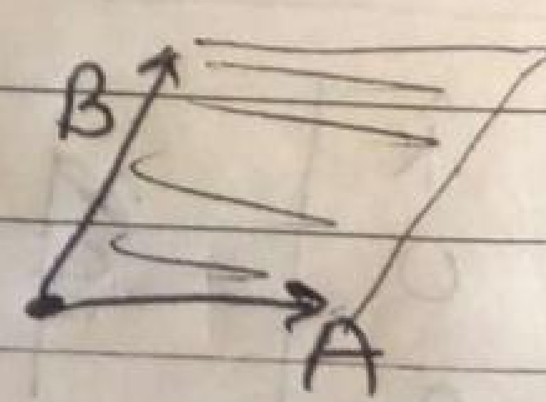
$$\boxed{\text{Volume} = \vec{A} \cdot (\underbrace{\vec{B} \times \vec{C}}_{\text{1st cross product}})}$$

$$\boxed{\det(\vec{A}, \vec{B}, \vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C})}$$

Verification: $\det(\vec{A}, \vec{B}, \vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C})$

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + a_3 \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \left(- \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}\right) + a_3 \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$$

- $|\vec{A} \times \vec{B}|$ = area of parallelogram



- $\underbrace{\boxed{\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}}}$ are not same as ~~Right~~ thumb direction change on applying R.H.R
- $\vec{A} \times \vec{A} = 0$. (area zero)