Max: $f \to \infty$ when $x \to \infty$, or $y \to \infty$
or $x, y \to 0$.

## Lecture: 11

(How to estimate the variation in arbitrary directions).

## # More tools to study functions:

Differentials: $y = f(x)$

Implicit differentiation: $dy = f'(x)\,dx$

ex: $y = \sin^{-1}(x)$

$x = \sin(y) \to dx = \cos(y)\,dy$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

## # Total differential: $f(x,y,z)$

$df = f_x\,dx + f_y\,dy + f_z + dz$

$$\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}$$

Importante

→ $df$ is NOT $\Delta f$ → number

↳ ≠ number

(i) encode how change in $x, y, z$ affect $f$.

(ii) placeholder for small variations $\Delta x, \Delta y, \Delta z$ to get approx. formula

$$\boxed{\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z}$$

(iii) divide by something like dt to get a rate of change. when $x = x(t), y = y(t), z = z(t)$

$$\boxed{\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} + f_z \cdot \frac{dz}{dt}}$$

chain Rule

Chain Rule: tells you, if you have $f^n$ that depends on something, and that something in turn depends on something else, how to find the rate of change of a $f^n$ on the new variable in terms of the derivatives of a $f^n$ & also the dependence b/w the various variables.

Why is this valid?

1st attempt: $df = f_x dx + f_y dy + f_z dz$

$dx = x'(t)dt$ ; $dy = y'(t)dt$, $dz = z'(t)dt$

So. $\boxed{df = f_x \cdot x'(t)dt + f_y\, y'(t)dt + f_z\, z'(t)dt}$

divide by dt ⇒ get chain rule.

Betters

approximation formula:-

$$\frac{\Delta f}{\Delta t} \simeq \frac{f_x \Delta x + f_y \Delta y + f_z \Delta z}{\Delta t}$$ (in time $\Delta t$) numbers

When $\Delta t \to 0$; $\frac{\Delta f}{\Delta t} \to \frac{df}{dt}$

$\hookrightarrow \frac{\Delta x}{\Delta t} \to \frac{dx}{dt}$; .....

Remeber; the def of $df/dt$ is the limit of this ratio when the time interval $\Delta t$ tends to zero. That means if I choose smaller & smaller value of $\Delta t$ then these ratios of number tends to value, & that value is derivative.

so $\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$

$\approx$ becomes $=$ in limit $\Delta t \to 0$

examples $w = x^2 y + z$, $x = t$, $y = e^t$, $z = \sin t$

Sol Chain Rule:

$$\frac{dw}{dt} = \overset{w_x}{2xy} \frac{dx}{dt} + \overset{w_y}{x^2} \frac{dy}{dt} + \overset{w_z}{1} \frac{dz}{dt}$$

$$= 2te^t + t^2 e^t + \cos t$$

$$\boxed{dw/dt = 2te^t + t^2 e^t + \cos t}$$

Substitute: $w = x^2 y + z$

$$w(t) = t^2 e^t + \sin(t)$$

$$\boxed{\frac{dw}{dt} = 2te^t + t^2 e^t + \cos(t)}$$

→ Justification of product rule for the derivative:

Application: Justify product rule.
quotient

(I)

$f = uv$ ; $u = u(t)$, $v = v(t)$

$$\boxed{\frac{d(uv)}{dt} = f_u \frac{du}{dt} + f_v \frac{dv}{dt} = v\frac{du}{dt} + u\frac{dv}{dt}}$$

(II) $g = \frac{u}{v}$ ; $u = u(t)$, $v = v(t)$

$$\boxed{\frac{d(u/v)}{dt} = \frac{1}{v} \frac{du}{dt} + \left(\frac{-u}{v^2}\right) \frac{dv}{dt}} \boxed{= \frac{u'v - v'u}{v^2}}$$

# Chain rule with more variables !!

$w = f(x, y)$ ; where $x = x(u, v)$
$y = y(u, v)$

$= f(x(u,v), y(u,v))$

Question:- What are $\frac{\partial w}{\partial u}$, $\frac{\partial w}{\partial v}$ in terms of $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$
$x_u, x_v, y_u, y_v$

$$dw = f_x\, dx + f_y\, dy$$

$$= f_x(x_u\, du + x_v\, dv) + f_y(y_u\, du + y_v\, dv)$$

$$= \underbrace{(f_x x_u + f_y y_u)}_{\partial f/\partial u} du + \underbrace{(f_x x_v + f_y y_v)}_{\partial f/\partial v} dv$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial u} &= \boxed{\frac{\partial f}{\partial x}} \cdot \frac{\partial x}{\partial u} + \boxed{\frac{\partial f}{\partial y}} \cdot \frac{\partial y}{\partial u} \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} \end{aligned}}$$

How to remember

□ → how f depend on x, y.

○ → how x, y depend on 'u'.

→ means:- how f change if I change 'u'.

→ Simplification like $\frac{\partial f}{\cancel{\partial x}} \cdot \frac{\cancel{\partial x}}{\partial u}$ are not legal incase of partial derivative.

→ if I chang 'u' how quickly 'x' changes.

→ if I change x at this rate $\frac{\partial x}{\partial u}$ ; changes in f $= \partial f/\partial x \cdot \frac{\partial x}{\partial u}$