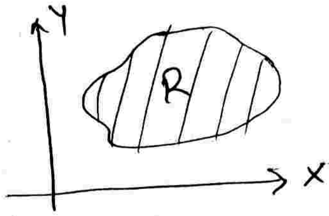
Lecture: 17 [Polar Coordinate]

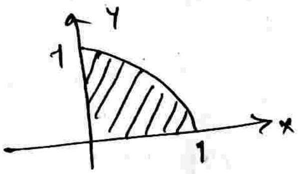
$$\iint_R f(x,y)\, dA$$

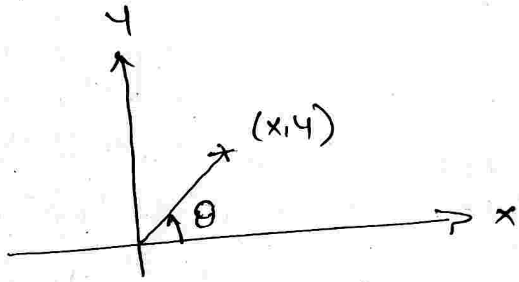


$$= \iint f(x,y)\, dy\, dx.$$

$$f = (1 - x^2 - y^2) \quad \text{—(1)}$$

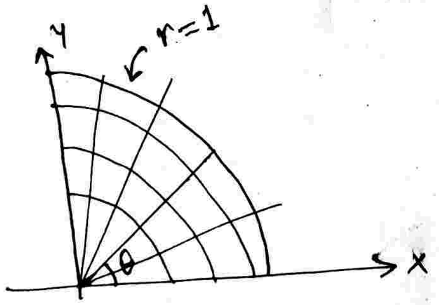
\# $\displaystyle\iint_{\substack{x^2+y^2 \le 1 \\ x,y \ge 0}} (1 - x^2 - y^2)\, dA = ?$ in polar coords?



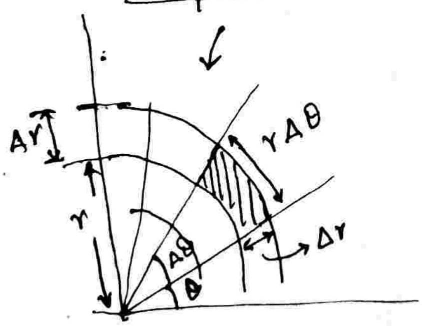


$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

\#



$$\int_0^{\pi/2}\int_0^1 f \cdot \underline{r\, dr\, d\theta}$$



$$f = 1 - x^2 - y^2$$
$$= 1 + (x^2 + y^2)$$
$$\{ f = 1 - r^2 \} \quad \text{—(2)}$$

$$\Delta A \simeq \Delta r \cdot r\, d\theta$$

$$\boxed{dA = r\, dr\, d\theta}$$

(#) $$\int_0^{\pi/2}\int_0^1 (1-r^2)\,r\,dr\,d\theta = \int_0^{\pi/2}\left(\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1\right)d\theta$$

$$= \int_0^{\pi/2}\frac{1}{4}\,d\theta = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}$$

Note: Switch to polar coordinate ! to make

- Boundaries simplier

or

- integeration simplier.

⇒ 99% of the cases you will integrate over 'r' first.

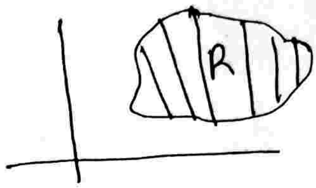
→ find bounds for 'r' in the region.

→ then put bounds of θ.

So; In polar coordinate, instead of slicing horizontally & vertically. We slice it radially.

Application of Double Integrals:

⇒ (1) To find the area of region R.



$$\text{Area}(R) = \iint_R 1 \, dA$$

~~measuring~~ volume with height 1 & area.

⇒ Mass of flat object with density : δ = mass per unit Area.

$$\Delta m = \delta \cdot \Delta A$$

$$\boxed{\text{Mass} = \iint_R \delta \cdot dA}$$

with varying density.

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(2) Average value of some quantity in a region.

ex: find average temperature in room.

Mathematically way to define continuous set of data is that you actually integrate the function over ~~than~~ the entire set of data, and then you divide by the size of the sample, which is area of the region.

$$\text{Average of 'f'} = \bar{f} = \frac{1}{\text{Area}(R)} \iint_R f \cdot dA$$

→ Where all the little points of the region are equally likely.

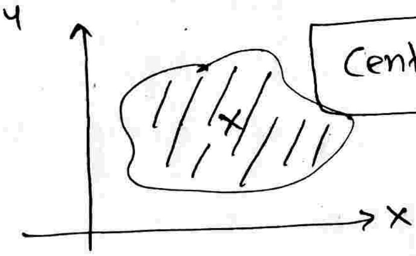
But, if want to do, an average of some solid with variable density or if you want to somehow give more importance to certain parts than to others then you actually do <u>weighted Average</u>.

$$\text{Weighted Average of 'f' with density } \delta = \frac{1}{\text{Mass}(R)} \iint_R \underbrace{f \cdot \delta \, dA}$$ ✓

(like replacing Area by Mass)

$$\left(\text{Mass}(R) = \iint_R \delta \cdot dA\right).$$

2a) Center of Mass of a (planar) object (with density δ):



$$\boxed{\text{Center} = (\bar{x}, \bar{y})}$$

$$\bar{x} = \frac{1}{\text{mass}} \iint_R x \, \delta \, dA \, ;$$

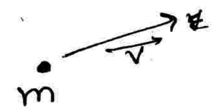
$$\bar{y} = \frac{1}{\text{Mass}} \iint y \, \delta \cdot dA.$$

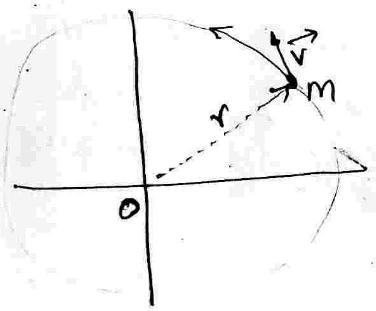
3) Moment of inertia : How hard to spin/push something; is given by its moment of inertia.

Mass = how hard to it is to impact translation motion.

Moment of inertia about an axis = ________ rotation motion about that axis.

Idea: Kinectic Energy of a point mass $= \frac{1}{2}\boxed{m}v^2$; so instead of just push this mass, I am going to make it spin around something.





For a mass 'm' at distance r & angular velocity ω

rate of change the angle over time

$v = r\omega$ $\Leftrightarrow \frac{1}{2}mv^2 = \frac{1}{2}\underbrace{\boxed{mr^2}}_{M.I.}\omega^2$

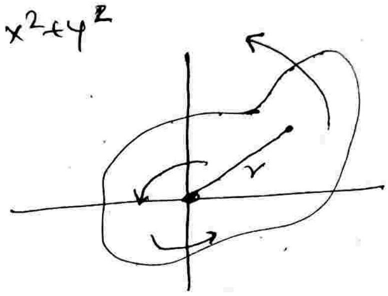
$$\boxed{\text{Moment of Inertia for point mass} = mr^2}$$

For a solid with density δ:

$\Delta m \approx \delta \cdot \Delta A$

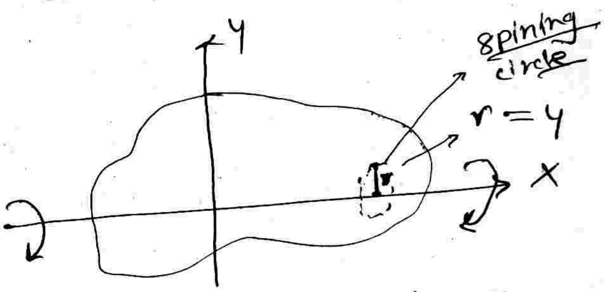
has moment of Inertia $\boxed{\Delta m \cdot r^2 = r^2 \cdot \delta \cdot \Delta A}$

Moment of inertia about the origin $I_o = \iint_R \underbrace{r^2}_{x^2+y^2} \delta \, dA$



[Rotational kinectic energy is $\frac{1}{2} I_o \omega^2$]

What about: rotation about x- axis ?



distance to x-axis $= |y|$

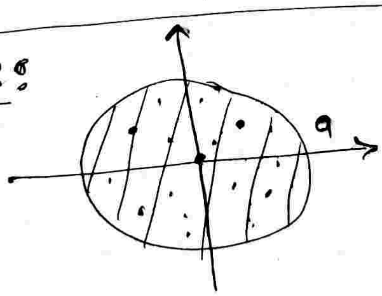
$(\text{distance of x-axis})^2 = y^2$

$$\boxed{I_x = \iint y^2 \cdot \delta \cdot dA}$$

y^2: square of a distance to the axis of rotation

→ y - tells me how far I am from the x-axis, so how hard it will to spin around the x-axis.

Example:



disk of radius :- a

$(\delta = 1)$

How hard to spin around its center?

$$I_o = \iint r^2 \, dA$$

Notes Any point inside a disk will have radius anything b/w 0 & 'a'.

$$I_o = \iint r^2 \, dA = \int_0^{2\pi}\int_0^{a} r^2 \cdot r\,dr \cdot d\theta = 2\pi \frac{a^4}{4}$$

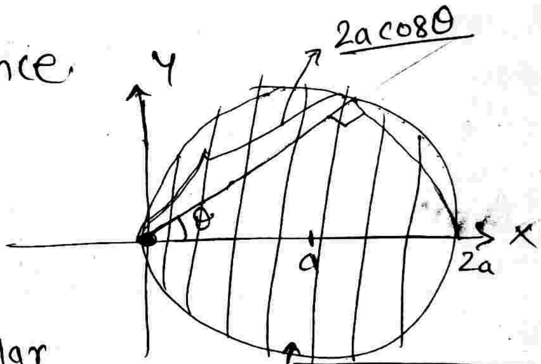
$r \rightarrow$ is a f^n:

$$\boxed{I_o = \frac{\pi a^4}{2}}$$

(#) How hard is to rotate ~~a~~ about ~~at~~ a point on circumference

→ move the origin to circumference point (make simpler to calculate

→ About point on circumference.



$$\boxed{r = 2a\cos\theta}$$

$$I_o = \iint r^2 \, dA.$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta} r^2 \cdot r\,dr\,d\theta$$ ↳ polar coordinate

Inner: $\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} = 4a^4\cos^4\theta$

$$\int_{-\pi/2}^{\pi/2} 4a^4\cos^4\theta\,d\theta = \boxed{\tfrac{3}{2}\pi a^4}$$

3-times harder.