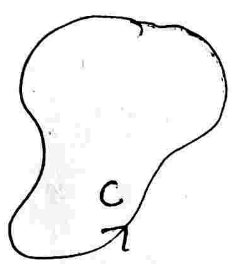
Lecture : 22

Green's Theorem

$\text{curl}(\vec{F}) = N_x - M_y$

$\vec{F} = \cancel{Mx} \langle M, N \rangle$

 $\oint_C \vec{F} \cdot d\vec{r} = ?$

GREEN'S THEOREM:

If 'c' is a closed curve, enclosing a region R, counterclockwise, $\vec{F}$ vector field defined & differentiable in R,

then $\boxed{\oint_c \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F}\, dA}$

line integral for work

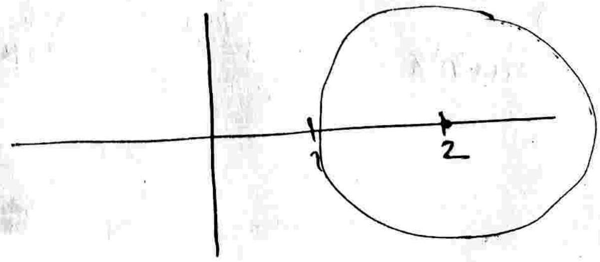
double intergal of some fn of x & y

$$\oint_c M dx + N dy = \iint_R (N_x - M_y)\, dA.$$

⇒ WARNING: only for closed curves.

Example: Let C = circle of radius 1 centered at (2,0) counterclockwise.

$$\oint_{C} \underbrace{y e^{-x}}_{M} dx + \underbrace{\left(\frac{1}{2}x^2 - e^{-x}\right)}_{N} dy$$



Sol: 1

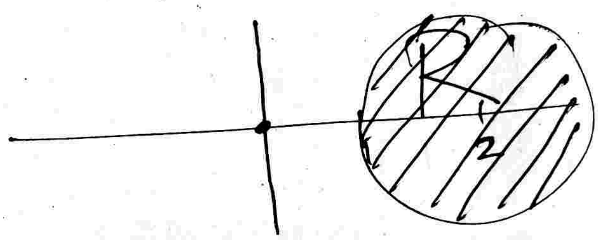
Doing directly: (Using Parametrization):

$$x = 2+\cos\theta \quad , \quad dx = -\sin\theta\, d\theta$$
$$y = \sin\theta \qquad\quad dy = \cos\theta\, d\theta$$

Sol 2 Green's theorem:

$\Rightarrow$ I will instead compute double integral.

$$\iint_R \text{curl}\,\vec{F}\, dA = \iint_R (N_x - M_y)\, dA$$



$$= \iint_R \underbrace{(x+e^{-x})}_{N_x} - \underbrace{e^{-x}}_{M_y}\, dA$$

$$= \underline{\underline{\iint_R x\, dA}} = \underbrace{\text{Area}(R)}_{\pi} \cdot \overset{\text{center of mass}}{\bar{x}} \rightarrow 2 \text{ by symmetry}$$

$$= \underline{2\pi}$$

$$\because \bar{x} = \frac{1}{\text{Area}} \iint x\, dA \quad \delta = 1$$
$$= \left(\frac{1}{\text{mass}} \iint x\delta\, dA\right)$$

Special Case where $\text{curl}\,\vec{F} = 0$;

If $\text{curl}\,\vec{F} = 0$ then $\vec{F}$ conservative {

$\Rightarrow$ Green's $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\, dA$

$= 0$ if level $\vec{F} = 0$

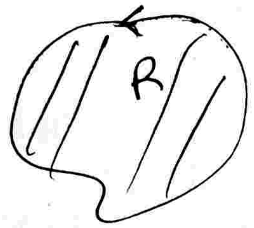
This proves: if $\text{curl}\,\vec{F} = 0$ everywhere in R

then $\oint \vec{F}\, d\vec{r} = 0$;

Consequences of Green's Theorem:

If $\vec{F}$ defined everywhere in the plane & curl $\vec{F}=0$ then $\vec{F}$ is everywhere/

proof:



$$\oint \vec{F}\cdot d\vec{r} = \iint_{R'} \text{curl}\,\vec{F}\,dA = \iint_{R} 0\,dA = 0$$

$\Rightarrow$ Cannot apply green's theorem to the vector field on ~~pretes~~ problem when encloses the origin.

Proof of Green's theorem:

Sta: $\oint M dx + N dy = \iint_R (N_x - M_y)\,dA$

observation: ① Prove $\oint M dx = \iint_R -M_y\,dA$ (special case $N=0$)

② A similar argument will show $\oint_c N dy = \iint_R N_x\,dA$

Summing, get the green's theorem.

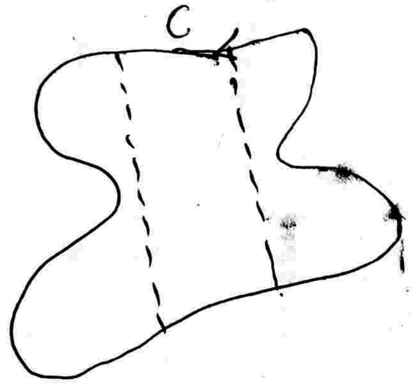
③ $\Rightarrow$ We can decompose R into simpler region!

If we prove $\oint_{C_1} M dx = \iint_R -M_y\,dA$

& $\oint_{C_2} M dx = \iint_R -M_y\,dA$

then $\oint_C M\,dx = \oint_{C_1} + \oint_{C_2} = \iint_{R_1} + \iint_{R_2} = \iint_R -M_y\,dA$.

because we go twice ~~through~~ along boundary b/w R_1 & R_2 with opp. orientation.

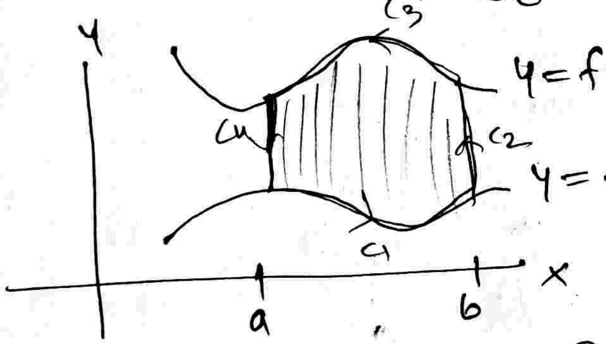


Cut R into "vertically simple" regions.

$a < x < b_1$, $f_1(x) < y < f_2(x)$

<u>Main step:</u> prove $\oint_C M\,dx = \iint_R -M_y\,dA$ if R vertically simple

C = boundary of R counterclockwise



<u>line integral along:</u> $\int_{C_1} M\,dx = \int_a^b M(x, f_1(x))\,dx$ with (x,y): $y = f_1(x)$, x from a to b

$\int_{C_2} M(x,y)\,dx = 0$. Similarly, $\int_{C_4} M\,dx = 0$;

$x = b$; $dx = 0$

$\int_{C_3} M\,dx = \int_b^a M(x, f_2(x))\,dx = -\int_a^b M(x, f_2(x))\,dx$

$y = f_2(x)$

x from b to a;

$$\oint_C M\,dx = \int_a^b M(x, f(x))\,dx - \int_a^b M(x, f_2(x))\,dx.$$

R.H.S. $\iint_R -M_y\,dA = -\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy\,dx = -\int_a^b M(x, f_2(x)) - M(x, f_1(x))$

Inner: $\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy = M(x, f_2(x)) - M(x, f_1(x))$

$= \text{L.H.S.}$