Lecture : 21

GRADIENT FIELDS AND POTENTIAL FUNCTIONS

if $\vec{F} = \nabla f$ gradient field;

then $\boxed{\int_C \vec{F} \cdot d\vec{r} = f(P_1) - f(P_0)}$ (path-independent) (conservative)

Testing whether $\vec{F} = \langle M, N \rangle$ is gradient field ?

$\Rightarrow$ If $\vec{F} = \nabla f$: $M = f_x$, $N = f_y$ then $f_{xy} = f_{yx}$ ⊛⊛ $\Rightarrow M_y = N_x$. ///

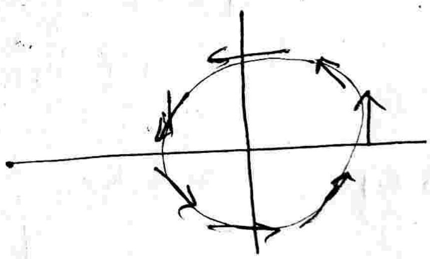
Conversely : if $\vec{F} = \langle M, N \rangle$ defined, differentiable everywhere.

and $M_y = N_x$

then $\vec{F}$ is a gradient field.

example :

$$\vec{F} = \underbrace{-y}_{M}\hat{i} + \underbrace{x}_{N}\hat{j}$$



$$\frac{\partial M}{\partial y} = -1 ; \quad \frac{\partial N}{\partial x} = 1$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$ $\vec{F}$ is not a gradient.

example : $\vec{F} = \underbrace{(4x^2 + axy)}_{M}\hat{i} + \underbrace{(3y^2 + 4x^2)}_{N}\hat{j}$

for which value 'a' ; $\vec{F}$ is gradient.

Soln $\dfrac{\partial M}{\partial y} = ax$; $\dfrac{\partial N}{\partial x} = 8x$

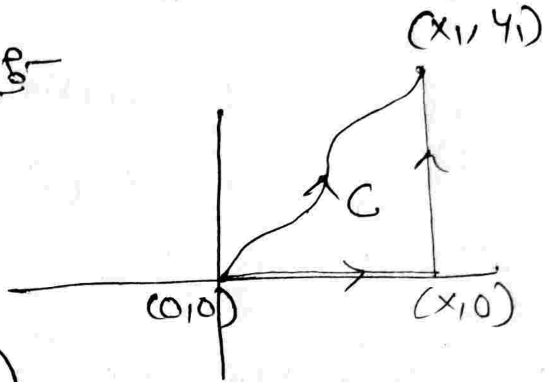
$$ax = 8x$$

$$\boxed{a = 8}$$ ✓

~~Ex~~ ⇒ ~~How do~~

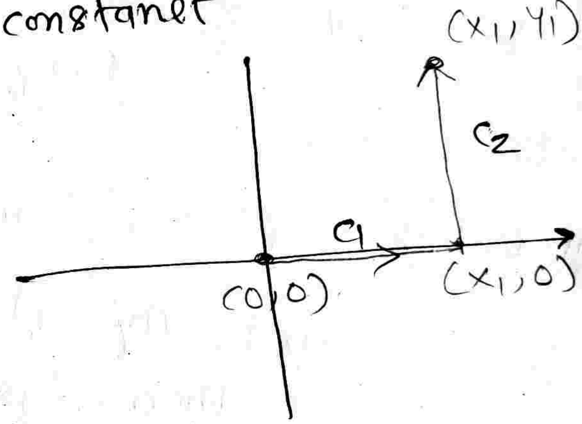
finding potential :- (if gradient field).

1. computing line integrals :-



$$\int_C \vec{F}\cdot d\vec{r} = f(x_1, y_1) - f(0,0)$$

$$f(x_1, y_1) = \left(\int_C \vec{F}\cdot d\vec{r}\right) + \underbrace{f(0,0)}_{\text{constanet}}$$



$$\int_C \vec{F}\cdot d\vec{r}$$

find potentional for

$$\vec{F} = \langle 4x^2 + 8xy,\ 3y^2 + 4x^2 \rangle$$

$$= \int_C (4x^2 + 8xy)\,dx + (3y^2 + 4x^2)\,dy.$$

On C_1: ~~$0 \le x$~~ x from 0 to x_1: $y = 0$; $dy = 0$ $\Big\} \Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r}$

$$= \int_0^{x_1} 4x^2\,dx$$

$$= 4/3\, x_1^3$$

On C_2: y from 0 to y_1: $x = x_1$; $dx = 0$ $\Big\}$ $\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^{y_1} (3y^2 + 4x_1^2)\,dy$

$$= \left[y^3 + 4x_1^2\, y\right]_0^{y_1} = y_1^3 + 4x_1^2 y_1$$

So, $f(x_1, y_1) = \frac{4}{3} x_1^3 + y_1^3 + 4x_1^2 y_1 (+c)$

$f(x, y) = \frac{4}{3} x^3 + y^3 + 4x^2 y (+c)$ ////.

Method-(2): Anti-derivatives:-

ex: want to solve $\begin{cases} f_x = 4x^2 + 8xy & \text{— (i)} \\ f_y = 3y^2 + 4x^2 & \text{— (ii)} \end{cases}$

(i) $\Rightarrow f = \frac{4}{3}x^3 + 4x^2y + g(y)$ (On Integeration) (*) — (iii)

$\Downarrow$

$f_y = 4x^2 + g'(y)$ — match this eq-(ii)

$\cancel{4x^2} + g'(y) = 3y^2 + \cancel{4x^2}$

$g'(y) = 3y^2$

$\boxed{g(y) = y^3 + c}$

Plug into (iii), get,

$f = \frac{4}{3}x^3 + 4x^2y + y^3 (+c)$

$\vec{F} = \langle M, N \rangle$ is a gradient field in a region of the plane.

$\Updownarrow$ conservative

$\Updownarrow$ if $\vec{F}$ defined in entire plane (or simply connected region)

$N_x = M_y$ at every point ($\text{curl } \vec{F} = 0$)

$\oint_C \vec{F} \cdot d\vec{r} = 0$ for closed C.

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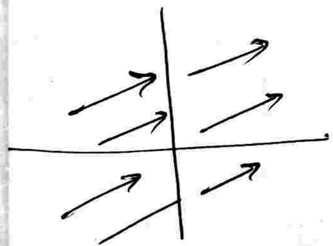
<u>Definition</u>: $\boxed{\text{curl}(\vec{F}) = N_x - M_y}$

(test for conservativeness :- curl $\vec{F} = 0$)

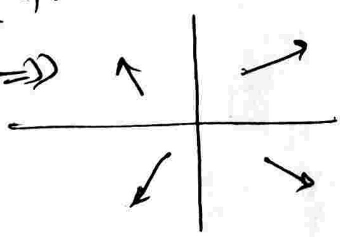
$\Rightarrow$ for a velocity field:
curl measures <u>rotation</u> component of motion.



fluid moving

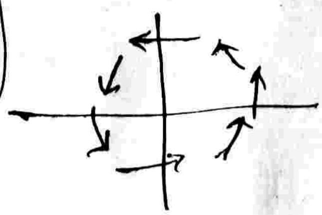
$\vec{F} = \langle a, b \rangle$ constants ; $\boxed{\text{curl } \vec{F} = 0}$

$\Rightarrow$



flows away from origin

$\vec{F} = \langle x, y \rangle$ $\quad$ $\text{curl } \vec{F} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0$



$\vec{F} = \langle -y, x \rangle$

$\text{curl } \vec{F} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2.$

Note: Curl actually measures twice the angular speed of a rotation part of a motion at any given point.

↳ Intuition: it measures how much rotation is taking place at any given point.

curl measures (2x) angular velocity of rotation component of velocity field.

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curl of a force field measures torque exerted on a test object in the field.

$$\frac{\text{torque}}{\text{moment of inertia}} = \frac{d}{dt}(\text{angular velocity})$$

$$\left(\frac{\text{force}}{\text{mass}}\right) = \frac{d}{dt}(\text{velocity})$$