Lecture :- 4

Square Systems ; equations of plane

Page No.

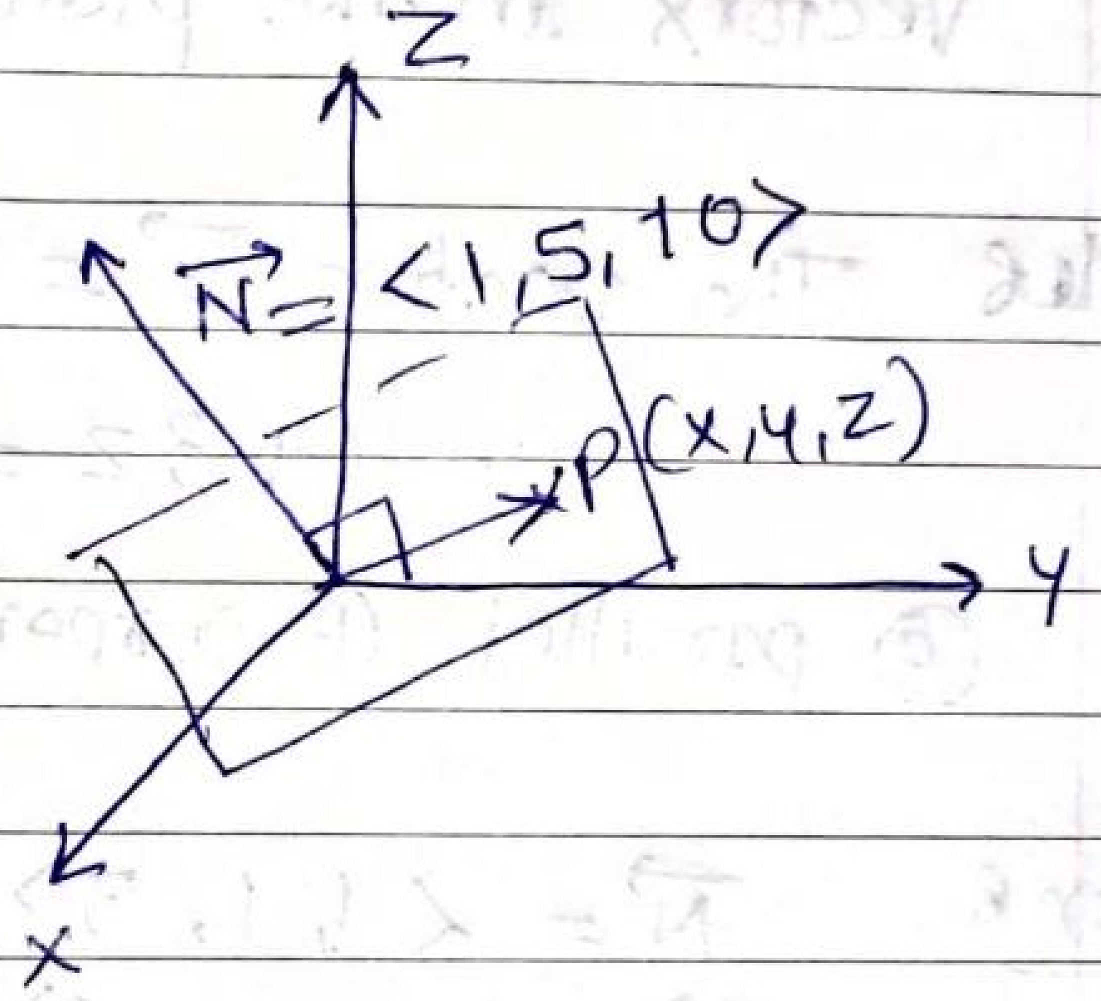
(1) equation of plane :

$$\boxed{ax + by + cz = d.}$$

(2) Plane through origin with normal vector :-

$$\vec{N} = \langle 1, 5, 10 \rangle \text{ ?}$$

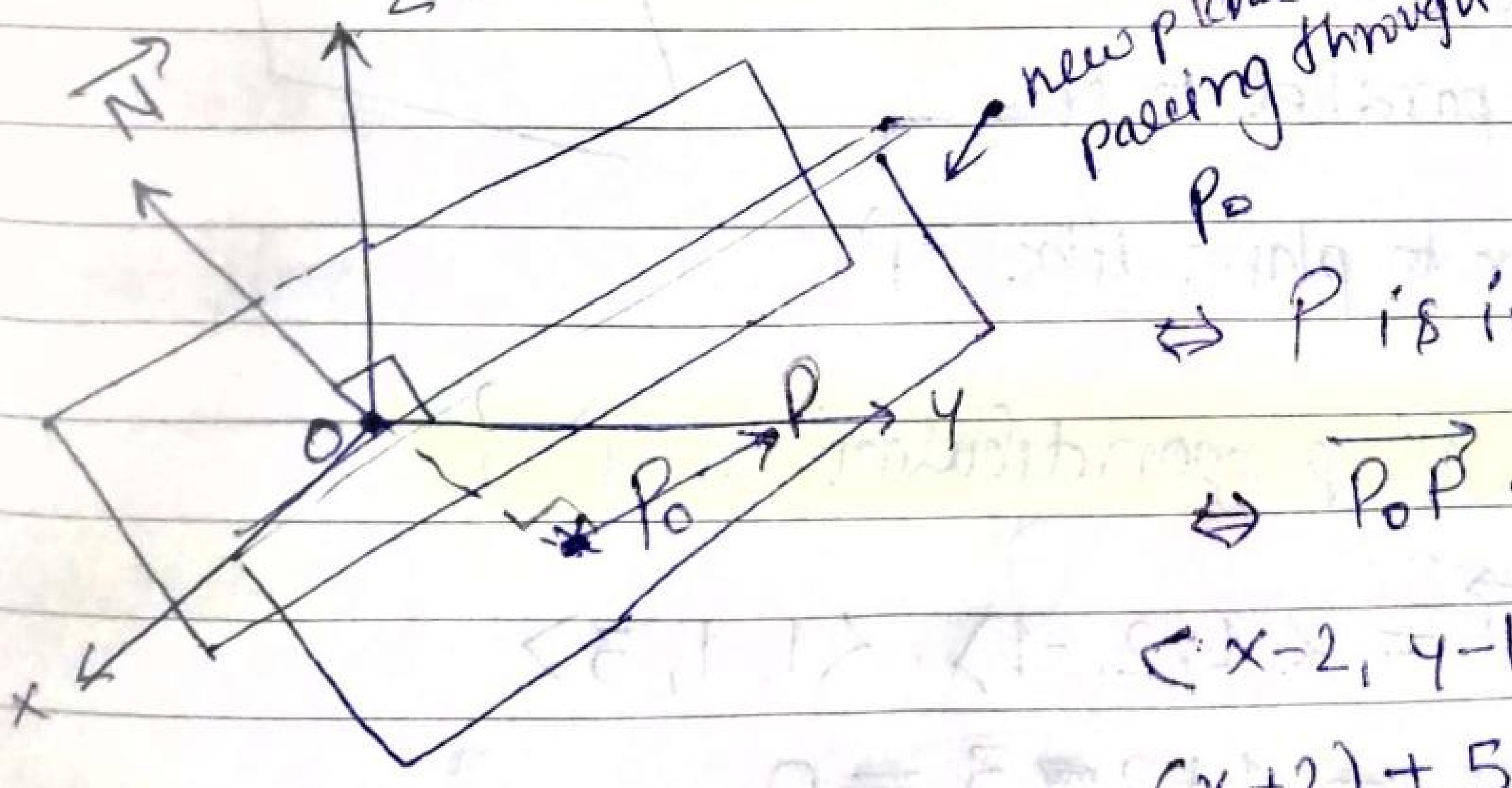
P is in the plane



$\Leftrightarrow \vec{OP} \cdot \vec{N} = 0$

$\Leftrightarrow \boxed{x + 5y + 10z = 0}$ — (a)

(3) eqⁿ of plane through $P_0(2, 1, -1)$ and $\perp \vec{N} = \langle 1, 5, 10 \rangle$?



$\Leftrightarrow$ P is in plane

$\Leftrightarrow \vec{P_0P} \cdot \vec{N} = 0$

$\langle x-2, y-1, z+1 \rangle \cdot \langle 1, 5, 10 \rangle = 0$

$(x+2) + 5(y-1) + 10(z+1) = 0$

$\Leftrightarrow \boxed{x + 5y + 10z = -3}$ — (b)

Note :- In eq - (a) & (b), only thing changes is constant term.
Common feature :- coefficient of x, y, z, correspond exactly to the normal vector

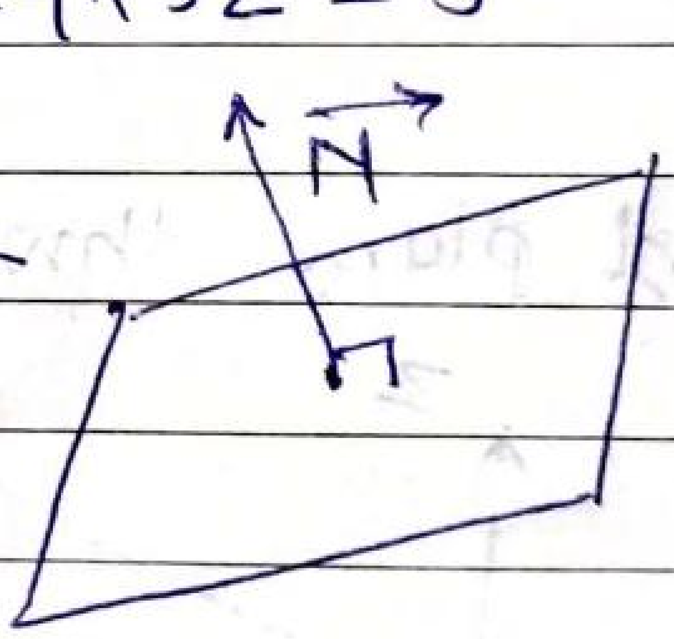
$$\underbrace{1x + 5y + 10z}_{\text{NORMAL VECTOR}} = -3 \leftarrow \text{value of L.H.S. at } P_0$$

<u>Conclusion</u>: In equation $ax+by+cz=d$,
$\langle a, b, c\rangle$ = normal vector $\vec{N}$.

<u>for example</u>:- get $\vec{N}$ by cross-product of 2 vectors in the plane.

<u>example</u>: The vector $\vec{v} = \langle 1, 2, -1\rangle$ and the plane $x+y+3z=5$ are
(a) parallel (b) perpendicular (c) neither

<u>Solution</u>: $\vec{N} = \langle 1, 1, 3\rangle \leftarrow x+y+3z=5.$
$\vec{v} = \langle 1, 2, -1\rangle$



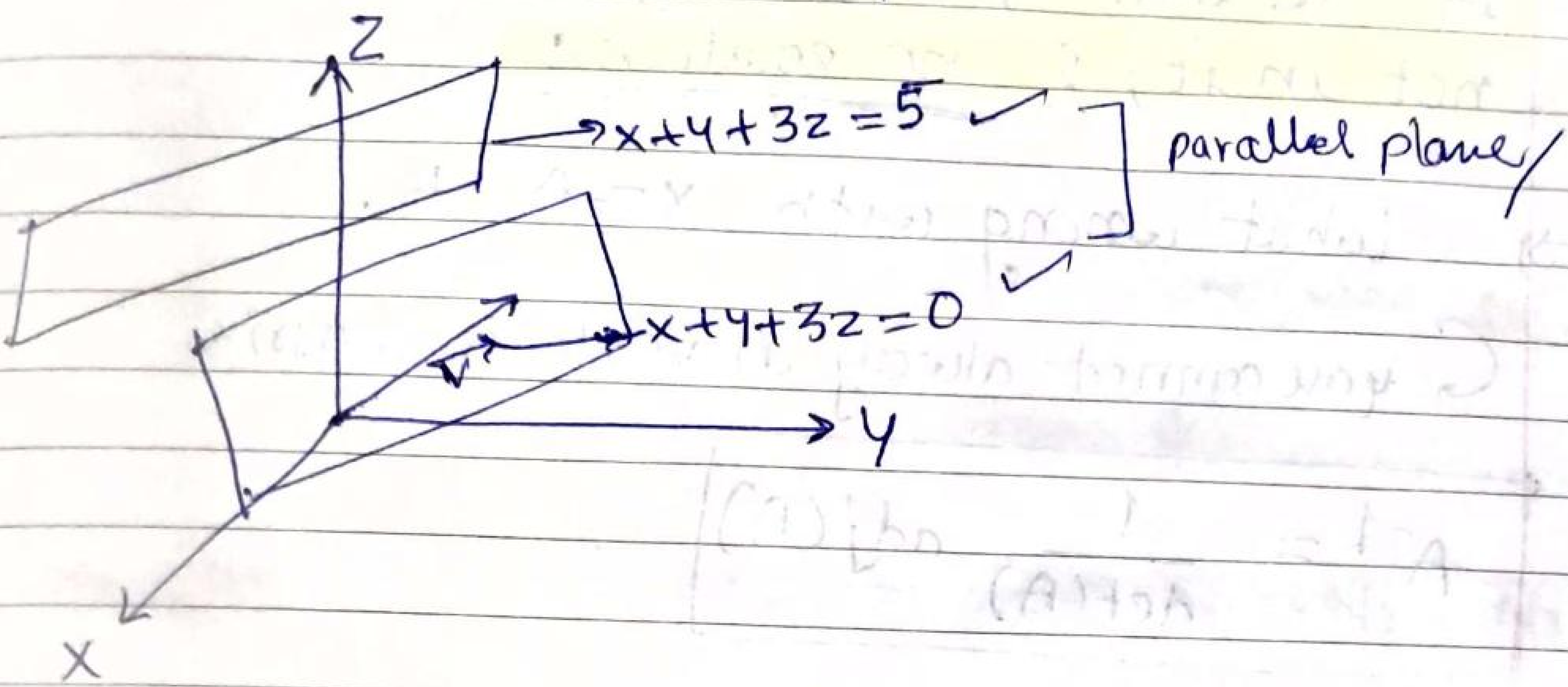
$\vec{v}$ is not multiple to $\vec{N}$
mean its not parallel to $\vec{N}$.
↳ means not ⊥r to plane like $\vec{N}$

So; Let's check $\vec{v}$ perpendicular to $\vec{N}$?

$$\vec{v}\cdot\vec{N} = \langle 1, 2, -1\rangle \cdot \langle 1, 1, 3\rangle$$
$$= 1+2-3 = 0$$

Yes. means $\vec{v}$ is $\parallel$ to plane.

$\Rightarrow$ $(1, 2, -1)$ is not in the plane.



3x3 linear system?

examples

$$\begin{cases} x + z = 1 \\ x + y = 2 \\ x + 2y + 3z = 3 \end{cases}$$

what do you mean by solving this?

It means we want to find x, y, z which ~~sa~~ satisfy all of these.

$\rightarrow$ these two planes; intersect in a ~~PLANE~~ LINE

$\rightarrow$ third plane

$\Rightarrow$ the line $P_1 \cap P_2$ intersects P_3 in a point = the solution

Solution: to $AX = B$ is given by $X = A^{-1}B$

Unless . . . the 3rd plane is parallel to the line when P_1 & P_2 intersect.

$\Rightarrow$ if line $P_1 \cap P_2$ is ~~contine~~ contained in P_3 $\therefore$ infinity many solutions!

(any point on the line is a solution).

⇒ If the line $P_1 \cap P_2$ is parallel to P_3 (and not in it) : no solutions.

↳ What wrong with $X = A^{-1}B$.

↳ you cannot alway invert a matrix.

$$\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$$

Note : A is invertible when its determinant is not zero

$$\boxed{A \text{ is invertible} \Leftrightarrow \det(A) \neq 0}$$

↳ ~~there~~ ~~it is not invertible~~

Start with : HOMOGENEOUS CASE

$$\boxed{AX = 0}$$

e.g.

$$\begin{cases} x + z = 0 \\ x + y = 0 \\ x + 2y + 3z = 0 \end{cases}$$

There's always an obvious solⁿ

$(0,0,0) \leftarrow$ TRIVIAL SOLⁿ

(origin is a solⁿ because the 3 planes pass through 0)

→ if $\det(A) \neq 0$: can invert A

$$AX = 0 \Leftrightarrow X = A^{-1}0 = 0.$$

no other solⁿ

CASE: if $\det(A) = 0$;

$\Leftrightarrow \det(\vec{N_1}, \vec{N_2}, \vec{N_3}) = 0$

$\Leftrightarrow \vec{N_1}, \vec{N_2}, \vec{N_3}$ are in same plane, coplaner

(parallelepiped has volume 0)

Line through origin $\perp$r to plane $\vec{N_1}, \vec{N_2}, \vec{N_3}$

~~is $\parallel$ to all 3 planes.~~

and in fact contained in them.

$\Rightarrow$ ∞ many solutions.

Ex: $\vec{N_1} \times \vec{N_2}$ is $\perp$ $\vec{N_1}, \vec{N_2}$ — but also to $\vec{N_3}$!

so its a nontrivial solution.

GENERAL CASE: $AX = B$

→ if $\det(A) \neq 0$; then unique sol^n $X = A^{-1}B$

→ if $\det(A) = 0$; then either no sol. or ∞ many.