Lecture : 24

Simply connected Regions, Review.

More About validity of Green's theorem :

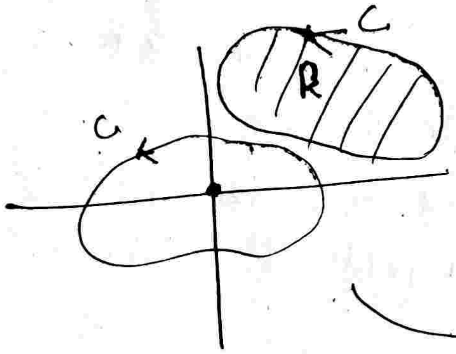
Green's thm:
$$\boxed{\oint \vec{F}\cdot\hat{T}\,ds = \iint_R \text{curl}\,\vec{F}\,dA}$$

$$\boxed{\oint \vec{F}\,\hat{n}\,ds = \iint_R \text{div}\,\vec{F}\,dA}$$

$\Rightarrow$ Only works if $\vec{F}$ & its derivatives are defined everywhere in R.

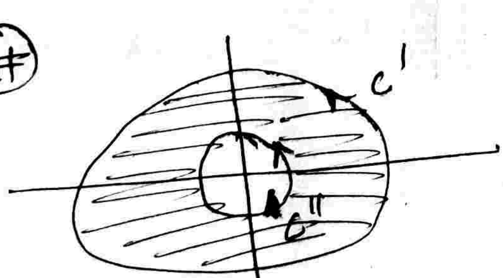
$\vec{F} = \dfrac{-y\hat{i} + x\hat{j}}{x^2+y^2}$: $\vec{F}$ not defined at origin.

curl $\vec{F} = 0$ everywhere else.



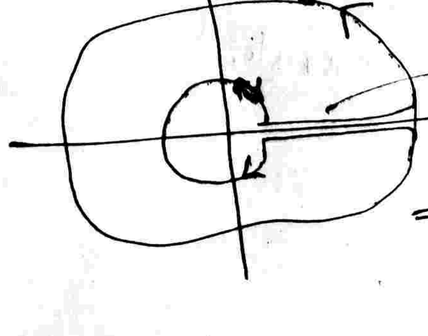
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \underbrace{\text{curl}\,\vec{F}}_{0}\,dA = 0$$

$\oint \vec{F}\cdot d\vec{r} = ?$ can't use Green directly.



$$\oint_{C'} \vec{F}\cdot d\vec{r} - \oint_{C''} \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\cdot dA$$

(in our case $= 0$)

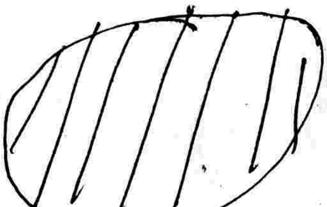


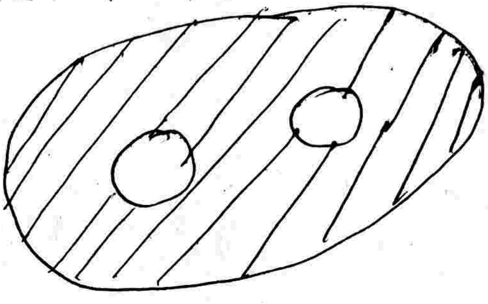
Total line Integral

$= \oint_{C'} - \oint_{C''}$ (+ white segment cancel out)

Definition :-

a connected region in the plane is simply connected if ~~any closed curve in R boon~~ interior of any closed curve in R is also contained in R.





Simply connected

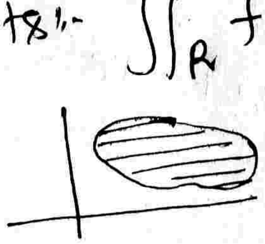
→ If domain where $\vec{F}$ is defined (+ differentiable) is simply connected, then can always apply Green's theorem.

$\vec{F} = \frac{-y\hat{i} + x\hat{j}}{x^2+y^2}$ ↳ domain : plane minus origin : NOT simply connected.

(#) If curl $\vec{F} = 0$ & domain of definition where $\vec{F}$ is defined is simply connected, then $\vec{F}$ is a conservative & a gradient field.

(#) 2 main objects :- $\iint_R f \, dA$ || $\int_C \vec{F} \cdot \hat{T} \, ds$

$\hat{n} \, ds$

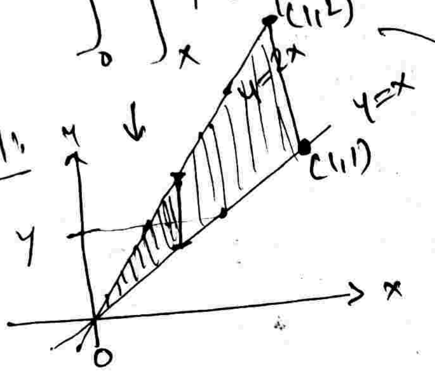


- Set up $\iint_R$.

draw picture of R & take slice ↔ integrate $\int$

(##) $\int_0^1 \int_x^{2x} f\, dy\, dx \longrightarrow$ Exchange $\iint f\, dx\, dy$

Sol:



$$\int_0^1 \int_{y/2}^{y} f\, dx\, dy + \int_1^2 \int_{y/2}^{1} f\, dx\, dy.$$

- Polar Coordinates $x \Rightarrow r\cos\theta$; $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

Remember area(R) $= \iint_R 1\, dA$

mass = avg. value of f (in particular $\bar{x}, \bar{y}$ = center of mass)

polar moment of inertia

$$I_0 = \iint (x^2+y^2)\,\delta\, dA \;;\; I_x, I_y$$

$\Rightarrow$ Evaluation

(##) Changes of variables $u = u(x,y)$
$v = v(x,y)$

(1) Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \qquad du\,dv = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\,dy$$

(2) Substitute x,y's in the integrand

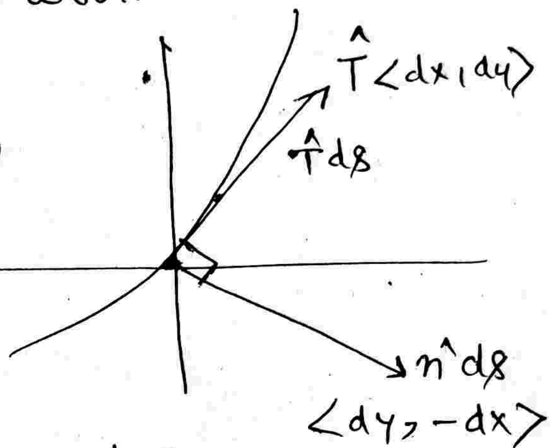
(3) Setting in bounds.

Line Integrals: $\vec{F} = \langle M, N \rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy \longrightarrow \text{Work}$$

$$\int_C \vec{F}\cdot \hat{n}\,ds = \int_C -N\,dx + M\,dy$$

↳ flux



Evaluation: by reducing to single parameter.

$x = x(t)$; $y = y(t)$

express $\int \underline{\quad t \quad}\, dt$

Only for Work

if $\operatorname{curl}(\vec{F}) = N_x - M_y = 0$

(& domain simply-connected) : $\vec{F}$ is a gradient.

$$\begin{cases} f_x = M \\ f_y = N \end{cases} \Rightarrow \vec{F} = \nabla f$$

Green's thm: $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \underbrace{\operatorname{curl}\vec{F}}_{N_x - M_y}\, dA$

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \underbrace{\operatorname{div}\vec{F}}_{M_x + N_y}\, dA$$