Lecture: 23

Flux; normal form of Green's Theorem

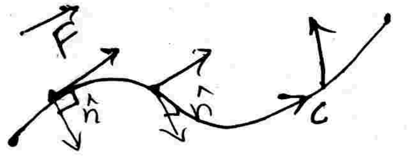
Flux: another kind of line Integral

C plane curve

$\vec{F}$ vector field

flux of $\vec{F}$ across C is

$$\boxed{\int_C \vec{F}\cdot\hat{n}\,ds}$$



$\hat{n}$ = unit normal to C, 90° clockwise from $\hat{T}$

If break C into small pieces

$\Delta s \Rightarrow$ flux $= \lim_{\Delta s \to 0}\left(\sum \vec{F}\cdot\hat{n}\,ds\right)$

Work

$$\int_C \vec{F}\,d\vec{r} = \int_C \vec{F}\cdot\hat{T}\,ds.$$

summing tangential component of $\vec{F}$.

flux is integral $\vec{F} \Rightarrow \int_C \vec{F}\cdot\hat{n}\,ds$

summing normal component of $\vec{F}$.

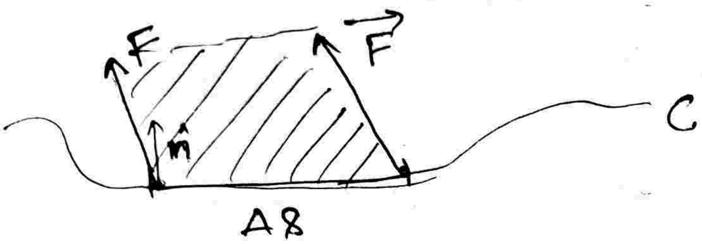
Interpretation:

Let's say $\vec{F}$ a velocity field

↳ represent fluid moment

flux measures how much fluid passes C per unit time.

(II) What passes through a portion of C in unit time.



Area = base · height

$A = \Delta s \cdot (\vec{F}\cdot\hat{n})$

⟹ what flows across C from left-to-right counted positively, what flows right to left counted negatively.
(total net flux)

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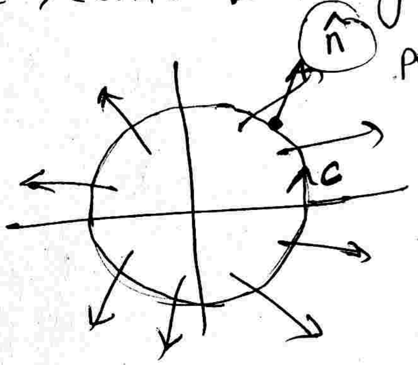
example: C = circle of radius 'a', centred at origin going counterclockwise

$$\boxed{\vec{F} = x\hat{i} + y\hat{j}}$$



Along C: $\vec{F} \parallel \hat{n}$

$\vec{F}\cdot\hat{n} = |\vec{F}| = \cancel{|\hat{n}|}^{1}$

$= a$

$$\int_C \vec{F}\cdot\hat{n}\, ds = \int_C a\, ds = a\cdot \underbrace{\text{length}(C)}_{2\pi a} = 2\pi a^2$$

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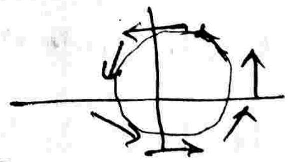
ex: Same C; $\vec{F} = \langle -y, x\rangle$



Sol: things are flowing along the circle, not inside out

flux = 0; $\vec{F}$ tangent to C

$\vec{F}\cdot\hat{n} = 0$; Flux = 0.

How to do calculation using components.

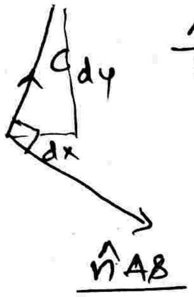
<u>Remember:</u> $d\vec{r} = \hat{T}\, ds = \langle dx, dy \rangle$

$\hat{n}$ is $\hat{T}$ rotated $90°$ clockwise

$$\boxed{\hat{n}\, ds = \langle +dy, -dx \rangle}$$

↳ clockwise

<u>ex:</u>



$\hat{T}\, ds = \Delta\vec{r} = \langle \Delta x, \Delta y \rangle$

$\cancel{\hat{n}\, ds} = \hat{n}\Delta s = \langle \Delta y, -\Delta x \rangle$

So, if $\vec{F} = \langle P, Q \rangle$, then

$$\int_C \vec{F}\cdot\hat{n}\, ds = \int_C \langle P, Q \rangle \cdot \langle dy, -dx \rangle$$

$$= \boxed{\int_C -Q\,dx + P\,dy}$$

$\Rightarrow$ If $\vec{F} = \langle M, N \rangle$ $\int_C \vec{F}\cdot\hat{n}\, ds = \int_C -N\,dx + M\,dy.$

GREEN'S THEOREM FOR FLUX

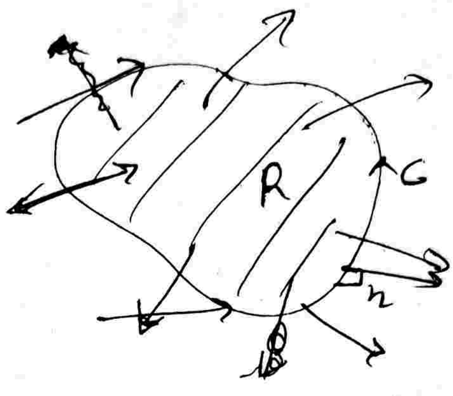
If 'C' is the ~~grea~~ curve, encloses region R, clockwise and $\vec{F}$ defined $\vec{F} = \langle P, Q \rangle$ in R.

then

$$\oint_C \vec{F}\cdot\hat{n}\, ds = \iint_R \underbrace{\operatorname{div}\vec{F}}_{\text{divergence of } \vec{F}}\, dA$$

flux out of R through C

$$\boxed{\operatorname{div}\langle P, Q\rangle = P_x + Q_y}$$



This is Green's theorem in normal form.
(vs. Green in tangential form)

Proof: $\oint_C \underbrace{-Q}_{M}\,dx + \underbrace{P}_{N}\,dy = \iint_R (\underset{\downarrow\, N_x}{P_x} + \underset{\downarrow\, -M_y}{Q_y})\,dA$

$\Rightarrow$ Let $M = -Q,\ N = P$

L.H.S. $\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$ — using Green's Th for work

R.H.S. $\iint_R (N_x - M_y)\,dA$ //

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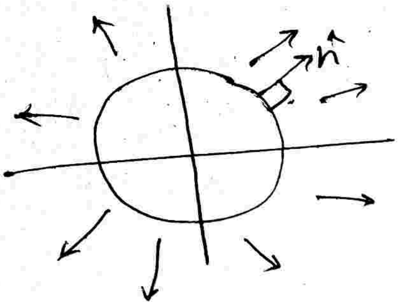
example: $\vec{F} = x\hat{i} + y\hat{j}$;
C = circle of radius a.



Sol^n Divergence $\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y)$
$= 1 + 1 = 2$

$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R 2\,dA = 2\,\text{area}(R) = \underline{\underline{2\pi a^2}}$.

Curl $\Rightarrow$ measures rotation

divergence: (1) measures how much the flow is "expanding"
(2) "source rate" = amount of fluid added to the system per unit time per unit Area.