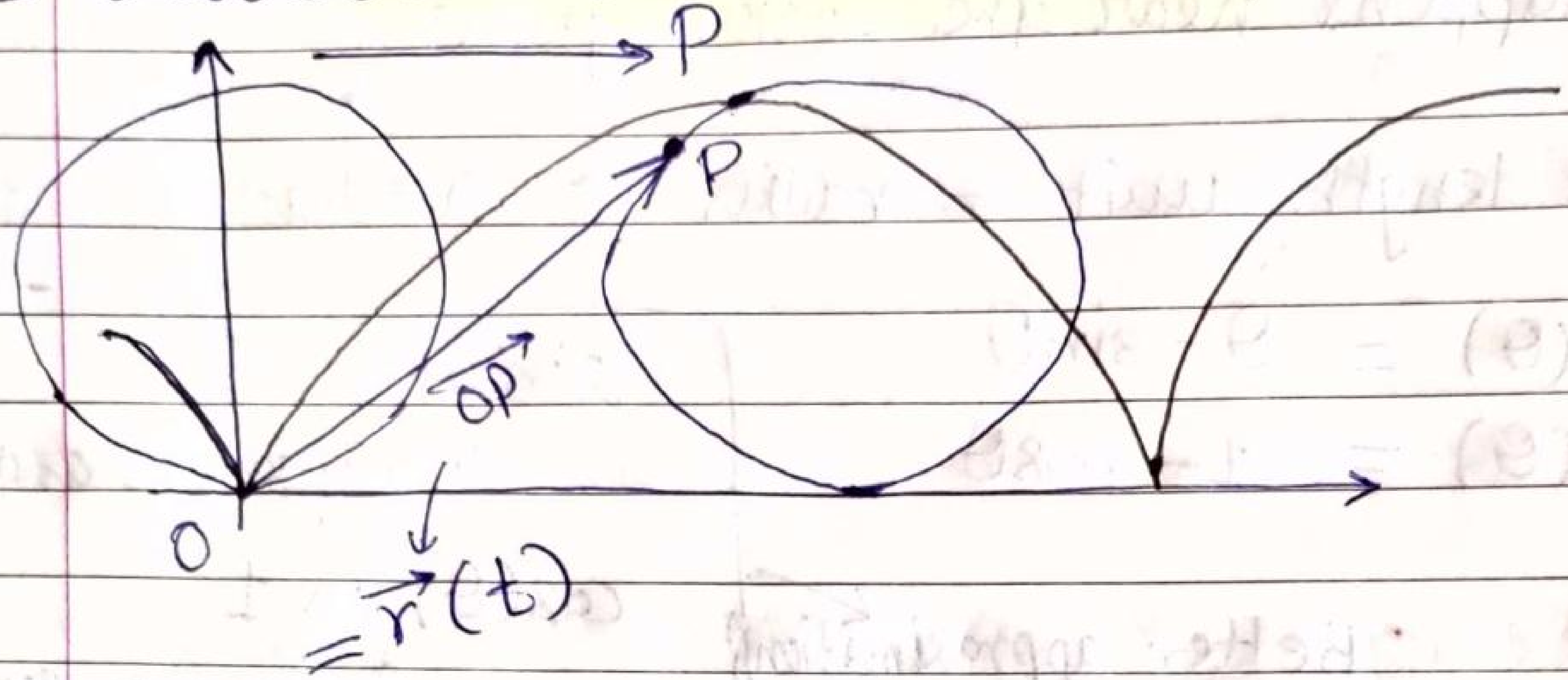
Lecture 5

Velocity, acceleration, Kepler's second law.

parametric equation :-



$(x(t), y(t), z(t))$ position of a moving point.

Position vector : $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Examples :- Cycloid (wheel radius 1, at unit speed)

$$\vec{r}(t) = \langle t - \sin t,\ 1 - \cos t \rangle$$

Will study how it varies in the speed & acceleration.

Velocity : vector : $\boxed{\vec{v} = \frac{d\vec{r}}{dt}} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

derivative of $\vec{r}(t)$:-

$$\boxed{\vec{v} = \langle 1 - \cos t,\ \sin t \rangle} \checkmark$$

(at $t = 0$) : $\boxed{\vec{v} = 0}$!

Speed :- (scalar):

$$|\vec{v}| = \sqrt{(1-\cos t)^2 + \sin^2 t}$$

$$= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$$

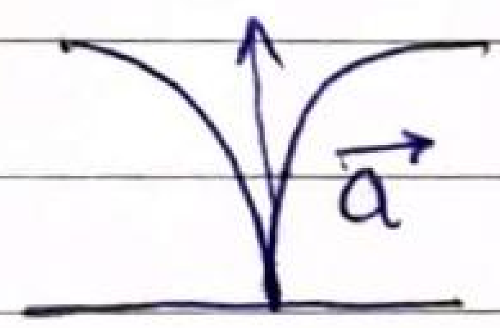
$$\boxed{|\vec{v}| = \sqrt{2 - 2\cos t}}$$ ✓

acceleration: a vector!

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt}}$$

e.g.: cycloid :- $\vec{a} = \langle \sin t, \cos t \rangle$ ✓

At $t = 0$; $\vec{a} = \langle 0, 1 \rangle$



Note:

$$\boxed{\left|\frac{d\vec{r}}{dt}\right| \neq \frac{d|\vec{r}|}{dt}}$$!

Arc length : s = distance travelled along trajectory.

Relate Arc length & time ??

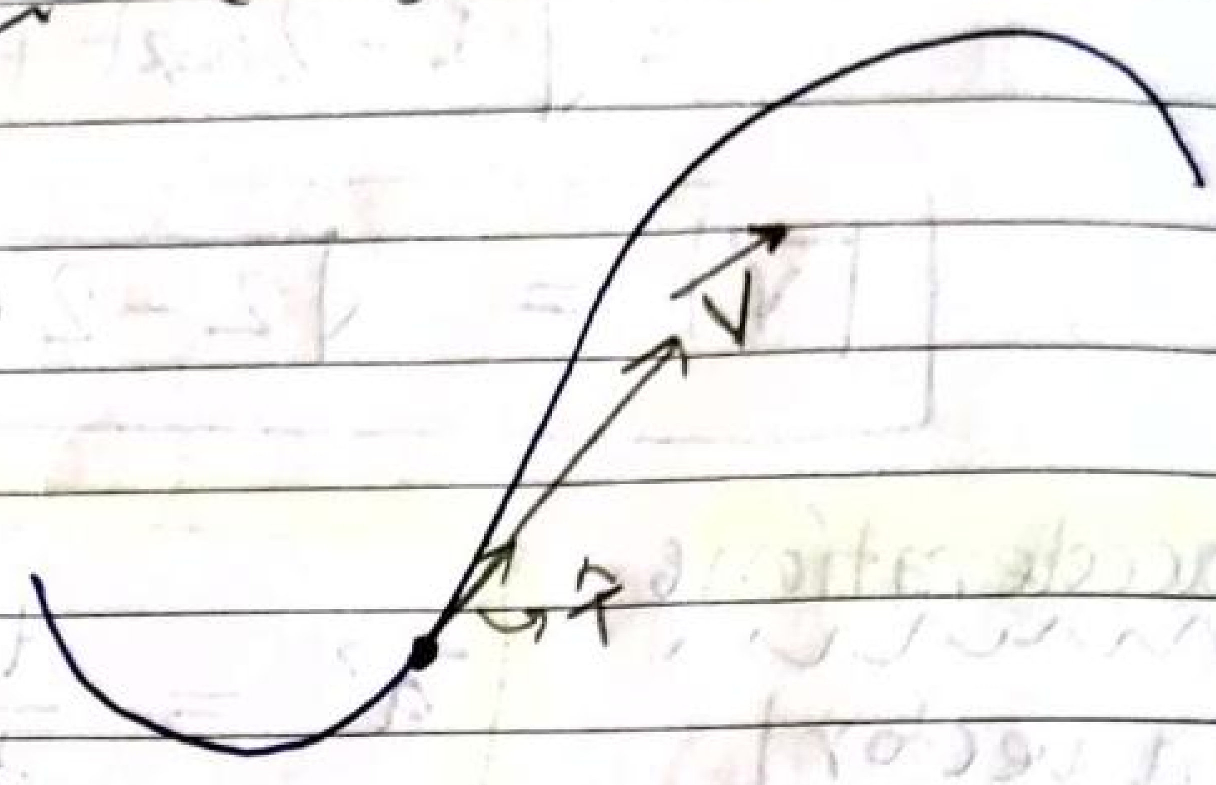
$$\left|\frac{ds}{dt}\right| = \text{speed} = |\vec{v}|$$

example: length of an arch :- of cycloid is :

$$\boxed{\int_0^{2\pi} \sqrt{2 - 2\cos t}\; dt}$$

Unit tangent vector ($\hat{T}$)

$$\boxed{\hat{T} = \frac{\vec{v}}{|\vec{v}|}}$$



#

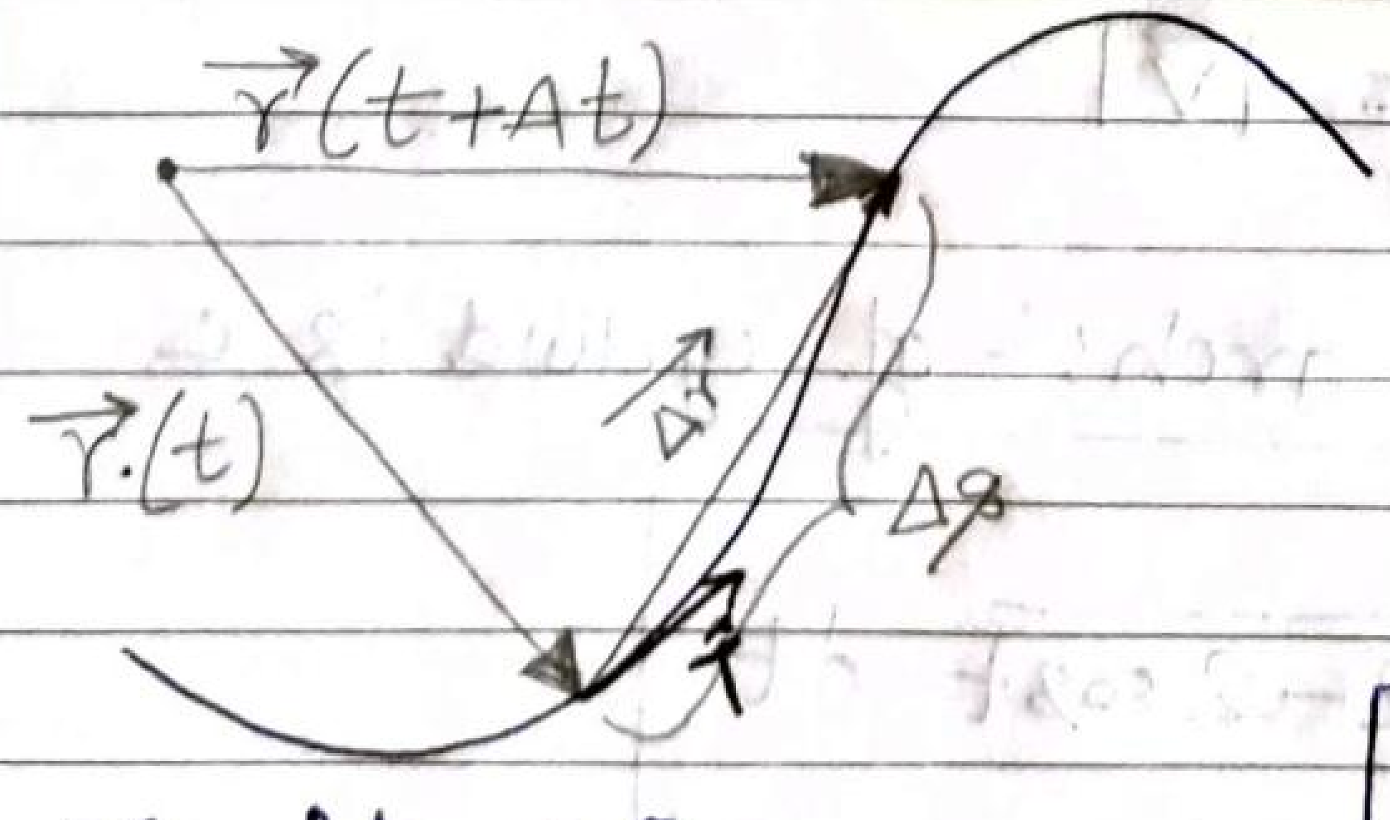
$$\vec{v} = \frac{d\vec{r}}{dt} = \underbrace{\frac{d\vec{r}}{ds}}_{=\hat{T}} \underbrace{\frac{ds}{dt}}_{=|\vec{v}|} = \hat{T}\cdot\frac{ds}{dt}$$

$$\boxed{\vec{v} = \hat{T}\cdot\frac{ds}{dt}}$$

velocity has
- direction ← tangent to traj. $\hat{T}$
- length ← speed $\frac{ds}{dt}$

In time Δt:

distance travelled by time



$\frac{\Delta s}{\Delta t} \simeq$ speed.

$$\boxed{\Delta\vec{r} \simeq \hat{T}\cdot\Delta s}$$

$$\boxed{\frac{\Delta\vec{r}}{\Delta t} \simeq \hat{T}\cdot\frac{\Delta s}{\Delta t}}$$

Limit as $\Delta t \to 0$
gives

$$\boxed{\frac{d\vec{r}}{dt} = \hat{T}\,\frac{ds}{dt}}$$

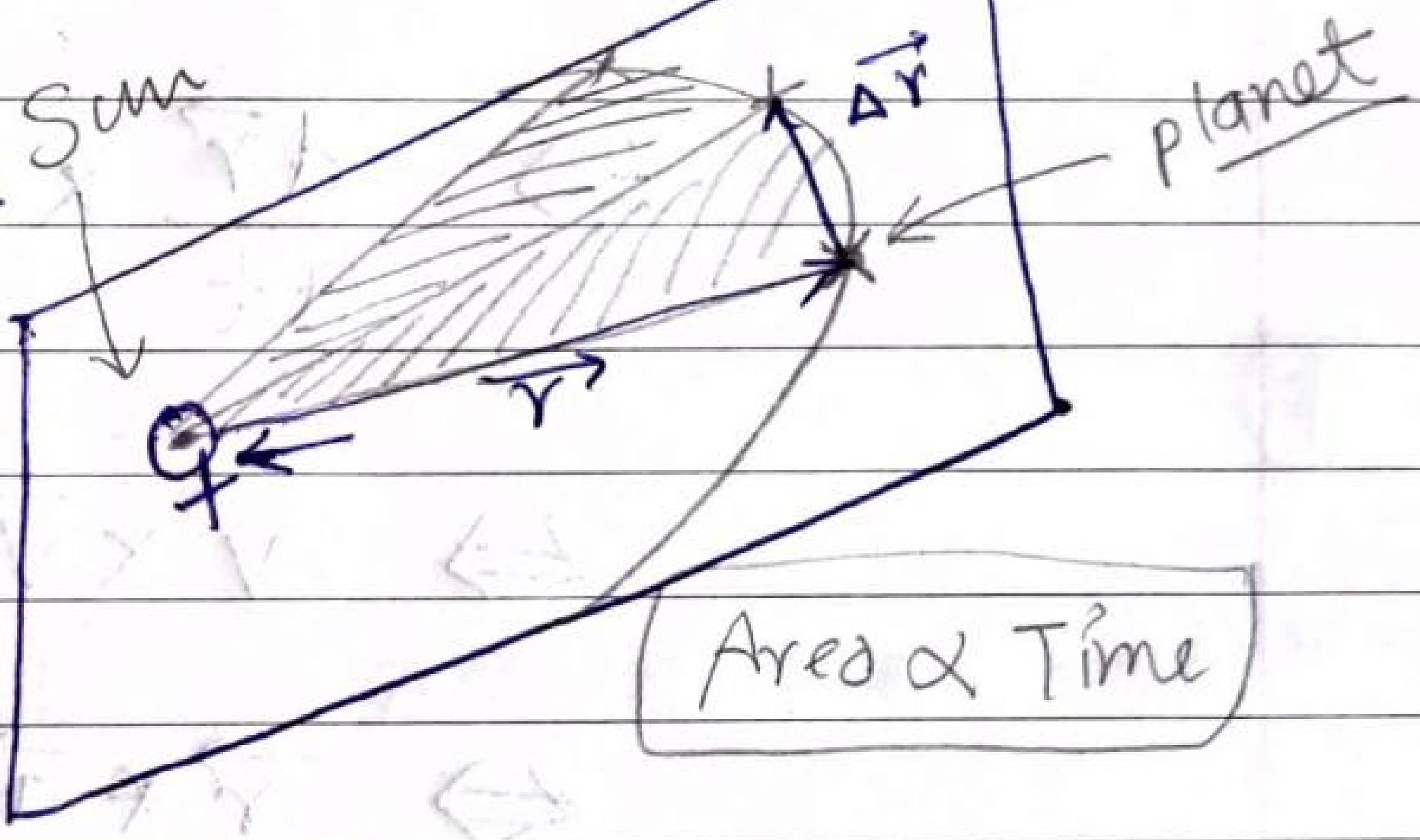
Kepler's second laws (1609)

it's an interesting example of why you might want to use vector methods to analyze motions.

→ kepler was trying to observe the motion of planets in the sky, and trying to come up general explanations of how they move.

↳ Motion of planets is in a plane, and the area is swept out by the line from sun to planet at a constant rate.

Newtons (later explained this) :- using formula for gravitational attraction.



Kepler's law in terms of vectors :-

① Area $\simeq \frac{1}{2} |\vec{r} \times \Delta\vec{r}|$ (Area swept in time Δt (small))

$\because$ $\boxed{\vec{\Delta r} \simeq \vec{v} \Delta t}$

$$\boxed{\text{Area} \simeq \frac{1}{2} |\vec{r} \times \vec{v}| \Delta t}$$

Law says :- $|\vec{r} \times \vec{v}| = \text{constant}$

② plane of motions contains $\vec{r}$ & $\vec{v}$?

Direction of $\vec{r} \times \vec{v}$ = normal to plane of motion.

Kepler's 2^{nd} Law $\Leftrightarrow$

$$\Leftrightarrow \vec{r} \times \vec{v} = \text{constant vector.}$$

$$\Leftrightarrow \frac{d}{dt}(\vec{r} \times \vec{v}) = 0$$

Product Rules ok:

for $\frac{d}{dt}(\vec{a} \cdot \vec{b})$; $\frac{d}{dt}(\vec{a} \times \vec{b})$

$$\Leftrightarrow \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = 0$$

$$\Leftrightarrow \underbrace{\vec{v} \times \vec{v}}_{=0} + \vec{r} \times \vec{a} = 0$$

$$\Leftrightarrow \vec{r} * \vec{a} = 0$$

$$\Leftrightarrow \vec{a} \parallel \vec{r}$$

Kepler's 2^{nd} Law $\Leftrightarrow$ Acceleration is parallel to the position vector.

$\Leftrightarrow$ gravitational force $\parallel \vec{r}$