Lecture :- 10

Second derivative test, boundaries & infinity.

(How to decide whether critical point is max. min or saddle)

Critical points : of $f(x,y)$: where $f_x = 0$ & $f_y = 0$;

How do we decide b/w diff possibility:

$$\begin{cases} \text{local minimum} \\ \text{local max.} \\ \text{saddle pt} \end{cases}$$

⇒ How do we find global min or max. pt.

↳ these occur either at a critical pt

or on boundary / at infinity.

Second derivative tests

first consider $w = ax^2 + bxy + cy^2$

↳ critical pt at origin

example: $w = x^2 + 2xy + 3y^2$

$= (x+y)^2 + 2y^2$

In general: ; if $a \neq 0$

$$w = a\left(x^2 + \frac{b}{a}xy\right) + cy^2$$

$$= a\left(x + \frac{b}{2a}y\right)^2 + \left(c - \frac{b^3}{4a}\right)y^2$$

$$w = \frac{1}{4a}\left[4a^2\left(x + \frac{b}{2a}y\right)^2 + (4ac - b^2)y^2\right]$$

↳ sum of squares.

3 cases: ① $4ac - b^2 < 0$ (negative)

one term ≥ 0, the other ≤ 0;

⇒ Saddle point.

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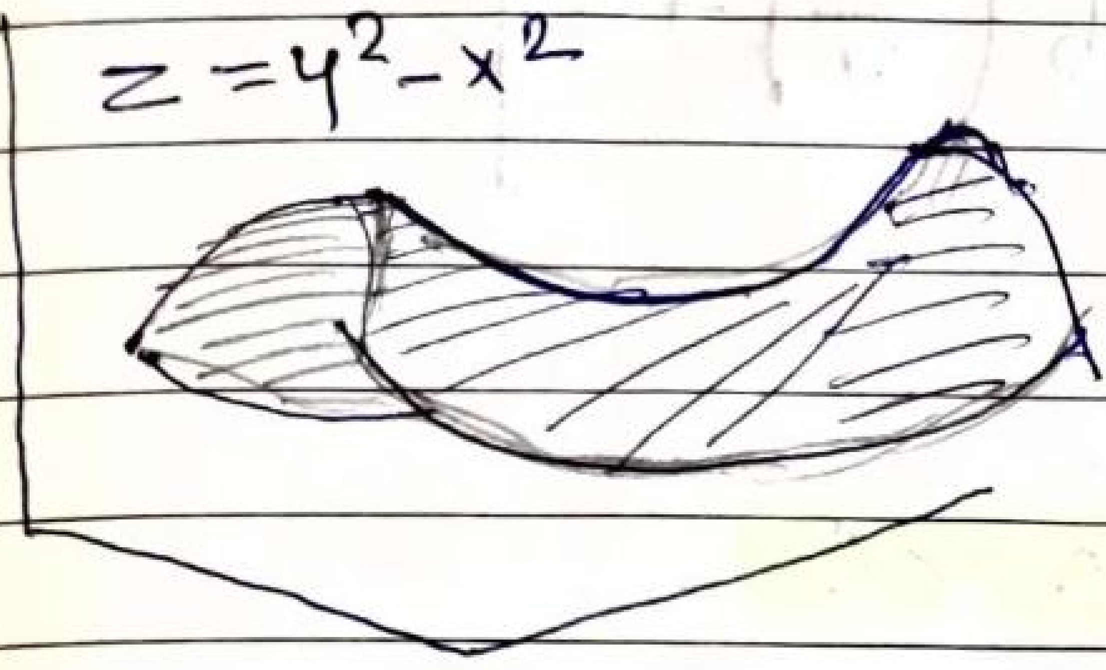
Note:

- sum of squares :→ f^n will always take non-negative value so origin will be a minimum.

$z=x^2+y^2$ →

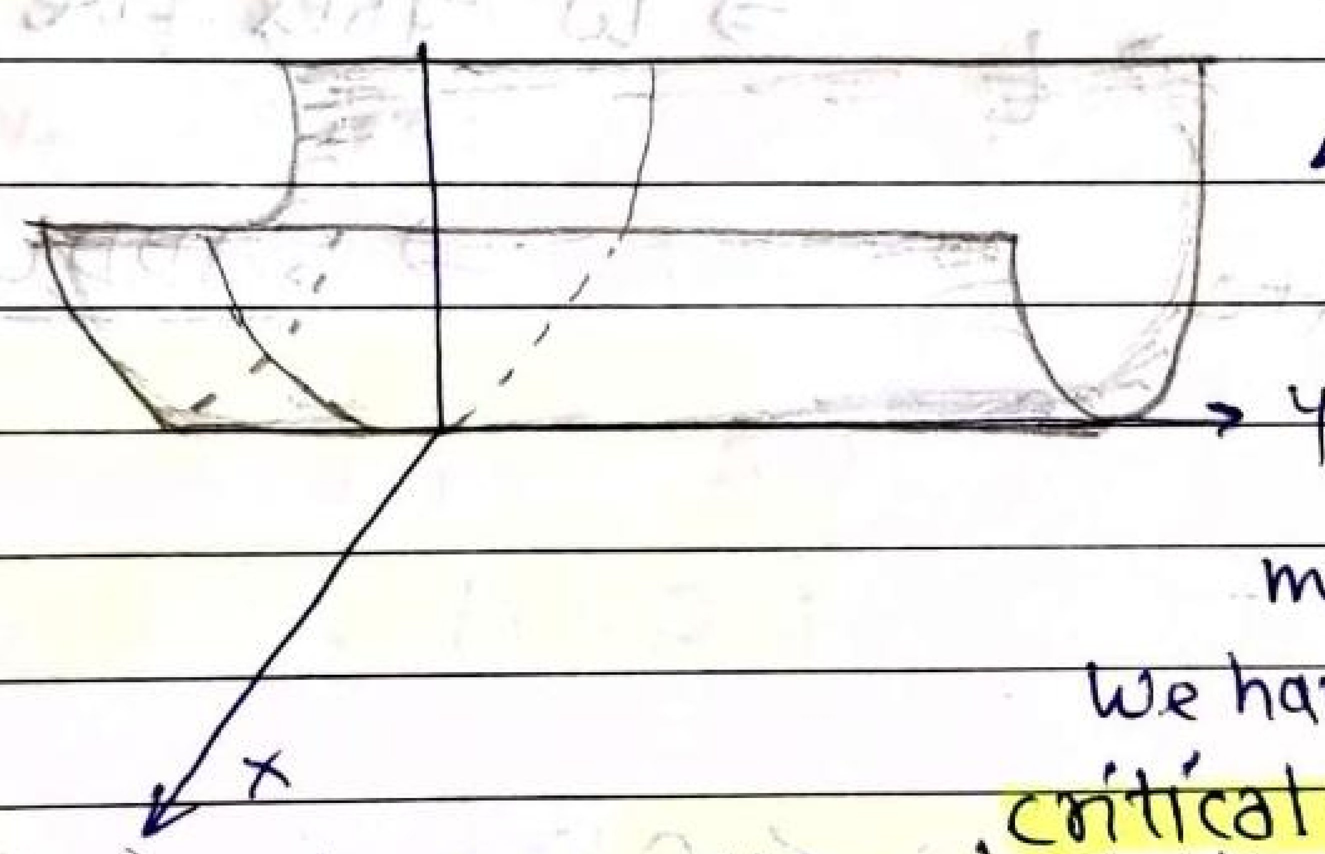
- difference of two squares :→ saddle point



② $4ac-b^2=0$; means f^n depend on one direction.

ex: $w=x^2$

$w=a\left(x+\frac{b}{2a}y\right)^2$



valley (bottom is completely flat)
↓
means:
We have a degenerate critical point.
because there is a direction in which nothing happens. Infact, you have critical points everywhere along the y-axis.

③ $4ac-b^2>0$ (

$$w=\frac{1}{4a}\Big[\underbrace{+(\dots)^2+(\dots)^2}_{\geq 0}\Big]$$

→ if $a>0$; minimum.
if $a<0$; maximum.

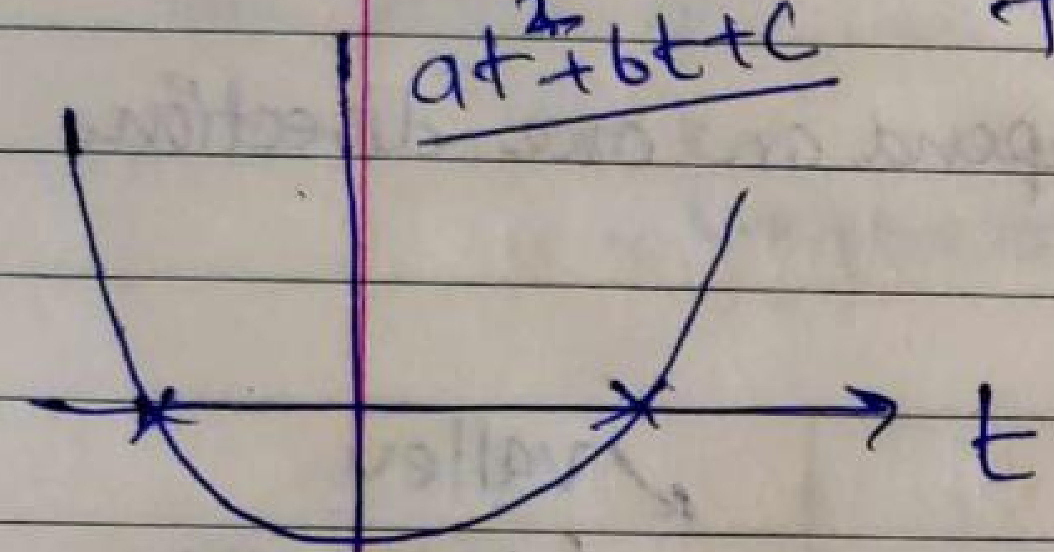
b^2-4ac : quadratic formula ??

$$w = ax^2 + bxy + cy^2$$

$$= \underbrace{y^2}_{\geq 0}\left[a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c \right]$$

$\longrightarrow$ if $b^2-4ac > 0$

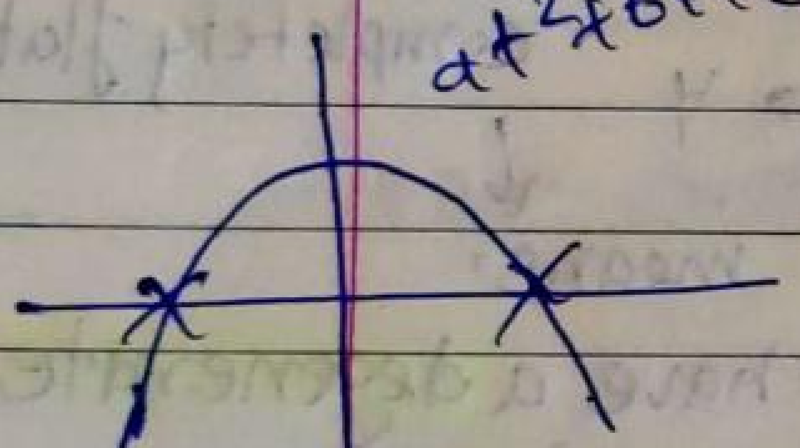
at^2+bt+c then this takes + & -ve values.



$\Rightarrow$ w takes +ve & -ve values

$\Rightarrow$ <u>SADDLE</u>.

at^2+bt+c

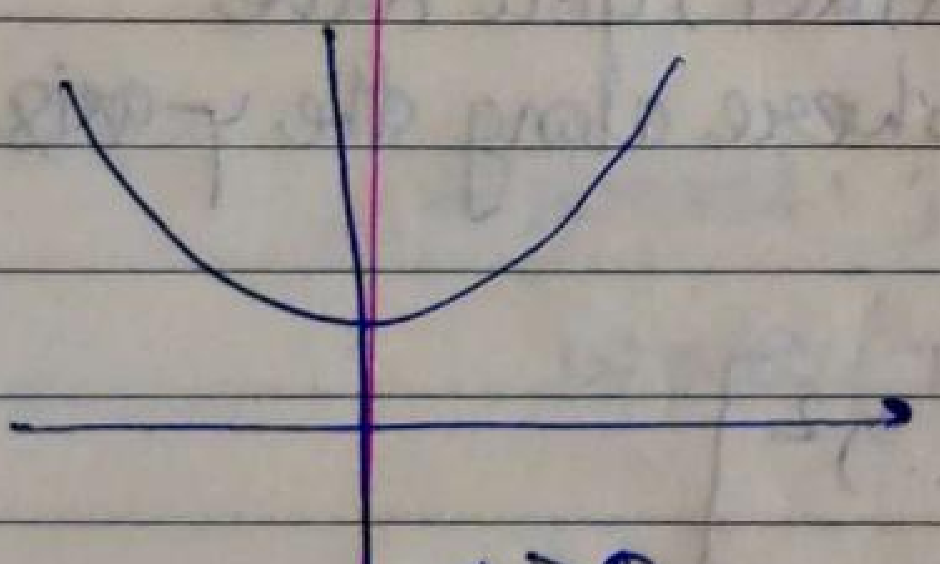


$\longrightarrow$ if $b^2-4ac < 0$: $a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c$

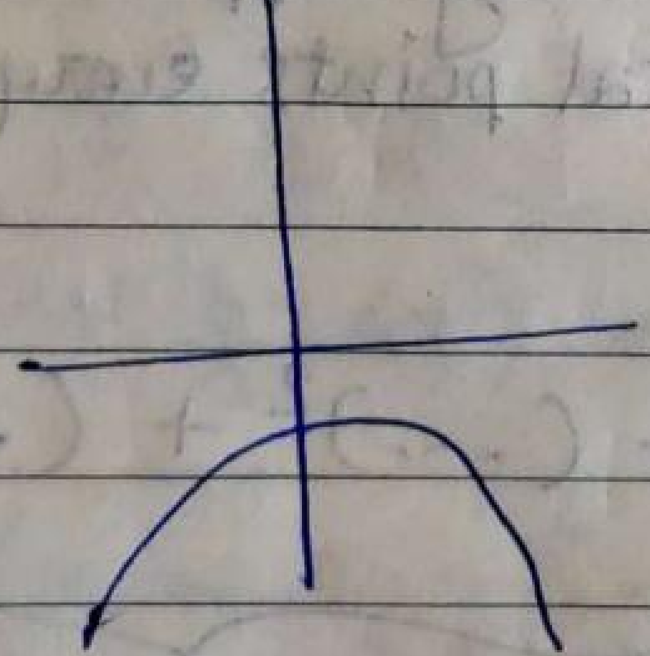
always $>0 \rightarrow w \geq 0$ (min)

or always $<0 \rightarrow w \leq 0$ (max)

~~$w \geq 0$~~



$a>0$



$a<0$

In general: look at second derivatives!

$$\boxed{\frac{\partial^2 f}{\partial x^2} = f_{xx}\,;} \quad \boxed{f_{xy} = \frac{\partial^2 f}{\partial x \partial y}\,; \quad = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}\,;}$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = f_{yy}}$$

⇒ Second derivative test:

At a critical point (x_0, y_0) of f & then compute part. derv.

Let $A = f_{xx}(x_0, y_0)\,;\quad B = f_{xy}(x_0, y_0)$

$C = f_{yy}(x_0, y_0)$

⇒ If $AC - B^2 > 0$ and $A > 0$; local minimum.

$AC - B^2 > 0$ and $A < 0$; local maximum

$AC - B^2 < 0$: Saddle

$AC - B^2 = 0$: Can't conclude

Verify in special case $W = ax^2 + bxy + cy^2$?

$W_x = 2ax + by \qquad W_y = bx + 2cy$

$\boxed{W_{xx} = 2a\,;} \quad \boxed{W_{xy} = b = W_{yx}} \quad \boxed{W_{yy} = 2c}$

$A = 2a\,; \quad B = b\,; \quad C = 2c.$

(i) $AC - B^2 = 4ac - b^2$

Quadratic approximations

approx. formula

$$\Delta f \simeq \underbrace{f_x}_{=0 \text{ at crit. pt}} \cdot (x-x_0) + \underbrace{f_y}_{=0 \text{ at crit pt}} (y-y_0)$$

$$+ \underbrace{\frac{1}{2} f_{xx}}_{\frac{1}{2}A = a} (x-x_0)^2 + \underbrace{f_{xy}}_{B=b} (x-x_0)(y-y_0)$$

$$+ \underbrace{\frac{1}{2} f_{yy}}_{\frac{1}{2}C = c} (y-y_0)^2$$

→ so general case reduces to quadratic case.

→ In degenerate case:, that depends on higher order derivatives.

example: $f(x,y) = x + y + \frac{1}{xy}$ $\quad x, y > 0$; min? Max?

Sol: look for critical pt :-

$$f_x = 1 - \frac{1}{x^2 y} = 0; \quad f_y = 1 - \frac{1}{xy^2} = 0$$

$$\begin{cases} x^2 y = 1 \\ xy^2 = 1 \end{cases} \quad \begin{matrix} x = y \\ y^3 = 1, \end{matrix} \quad \begin{matrix} \to x = 1 \\ \to y = 1 \end{matrix}$$

only one critical pt $\underline{(1,1)}$

Critical pt: (1) Local max, (2) local min, (3) Saddle.

$f_{xx} = \frac{2}{x^3 y}$	$f_{xy} = \frac{1}{x^2 y^2}$	$f_{yy} = \frac{2}{xy^3}$
$A = 2$	$B = 1$	$C = 2$

$$\left.\begin{matrix} AC - B^2 = 2 \cdot 2 - 1^3 = 3 > 0 \\ A > 0 \end{matrix}\right\} \to \text{local min}$$

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Max: $f \to \infty$ when $x \to \infty$, or $y \to \infty$
or $x, y \to 0$.

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(How to estimate the variation in arbitrary directions).

More tools to study functions:

Differentials: $y = f(x)$

Implicit differentiation: $dy = f'(x)\,dx$

ex: $y = \sin^{-1}(x)$

$x = \sin(y) \to dx = \cos(y)\,dy$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

Total differential: $f(x, y, z)$

$df = f_x\,dx + f_y\,dy + f_z + dz$

$$\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}$$

Importante

→ df is NOT Δf ↳ number

↳ ≠ number

(i) encode how change in x, y, z affect f.

(ii) placeholder for small variations $\Delta x, \Delta y, \Delta z$ to get approx. formula

$$\boxed{\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z}$$