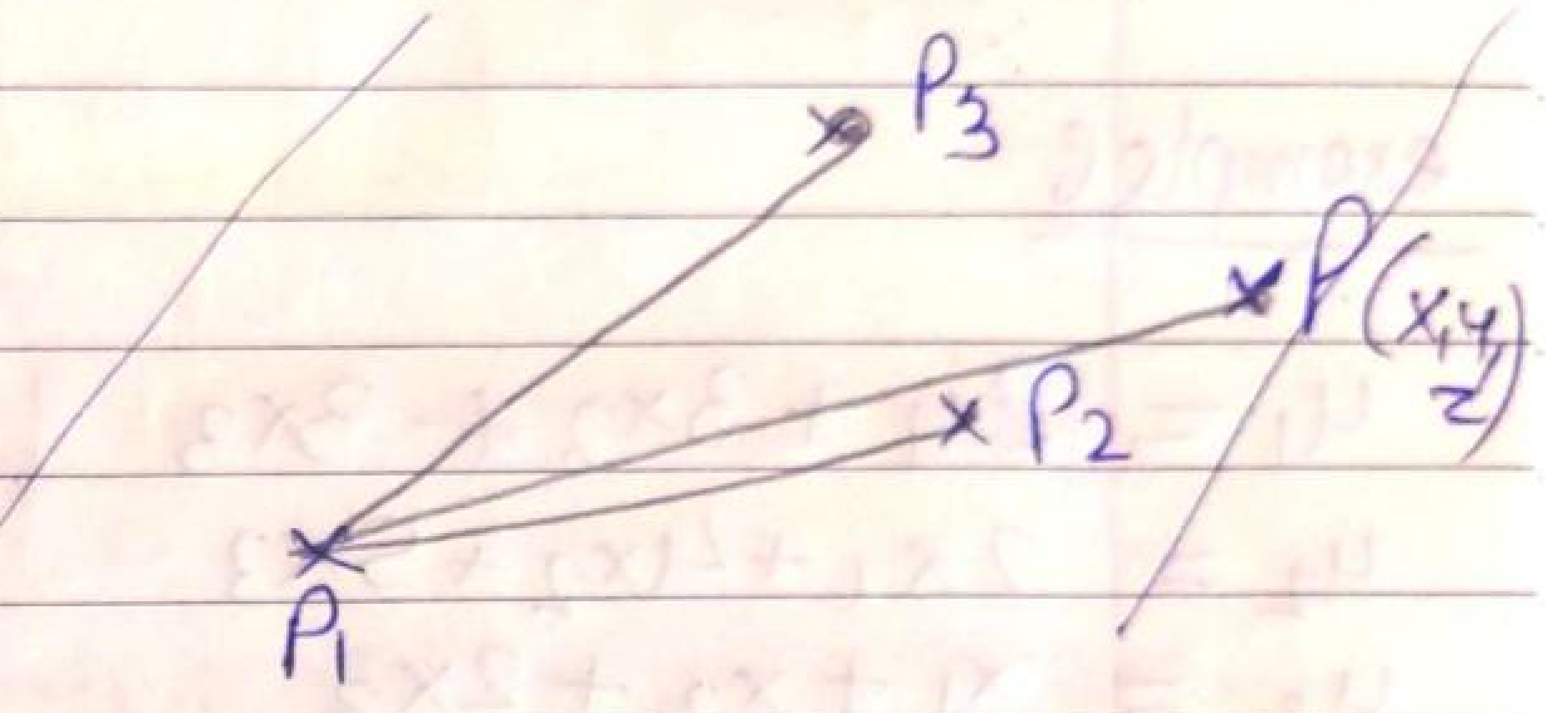
Lecture: #3

Matrices; Inverse matrices :

Application of cross product :

⇒ Let's say that I am given three points in space, and I want to find the equation of the plane that contains them.

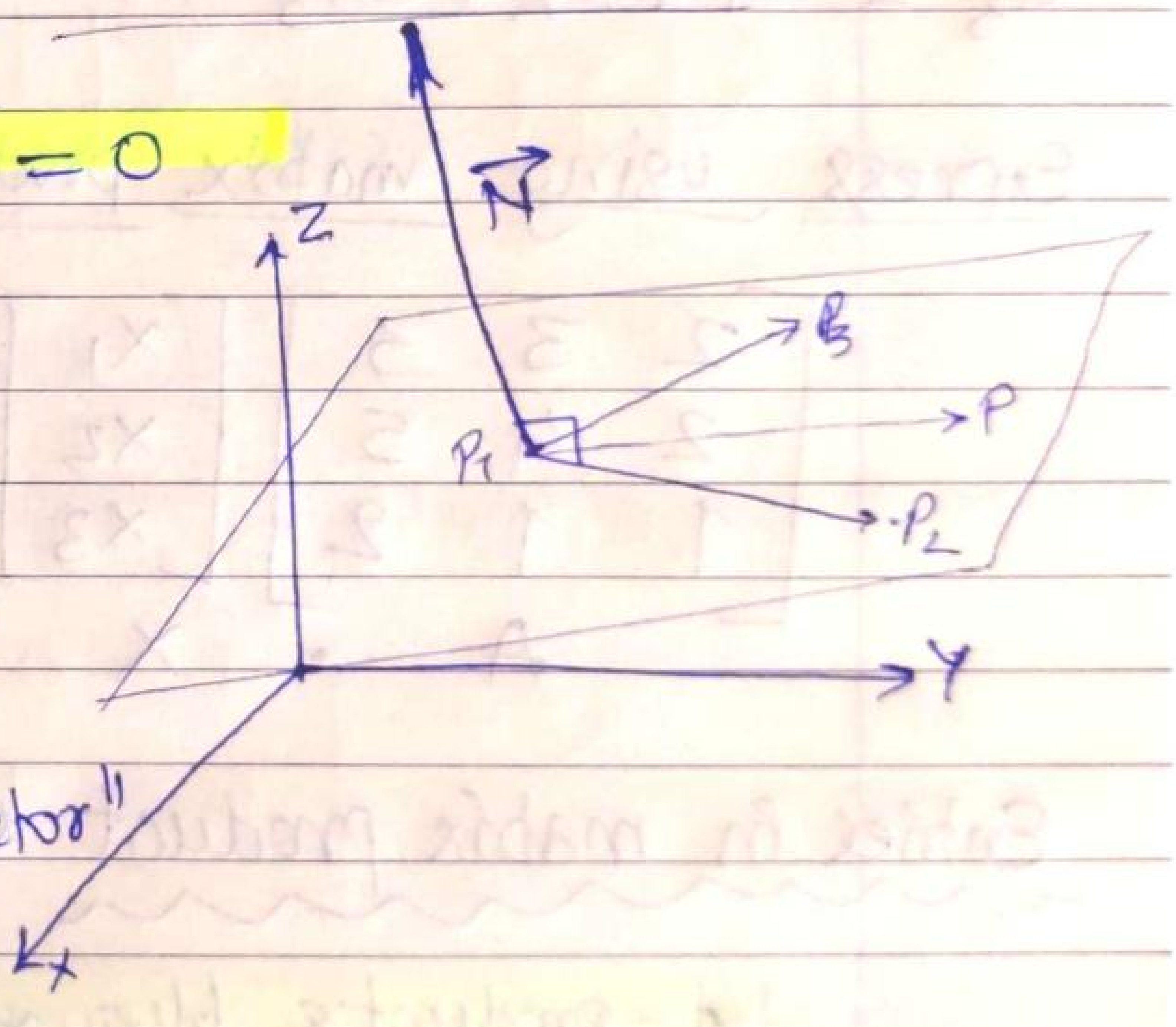


equation of plane : $P_1P_2P_3$

= find condition on (x,y,z) telling us whether P is in the plane?

1st way

- $\det(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 0$



2nd way (faster)

P is in the plane when

$\Leftrightarrow \overrightarrow{P_1P} \perp \vec{N}$ ← some vector ⊥ plane "normal vector"

$\Leftrightarrow \overrightarrow{P_1P} \cdot \vec{N} = 0$;
↳ ⊥r when there dot product is zero.

How to find $\vec{N} \perp$ to plane? Ans:- $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$
Cross product

So; $\overrightarrow{P_1P} \cdot \vec{N} = 0$

$\overrightarrow{P_1P} \cdot (\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}) = 0$ (triple product) = determinant

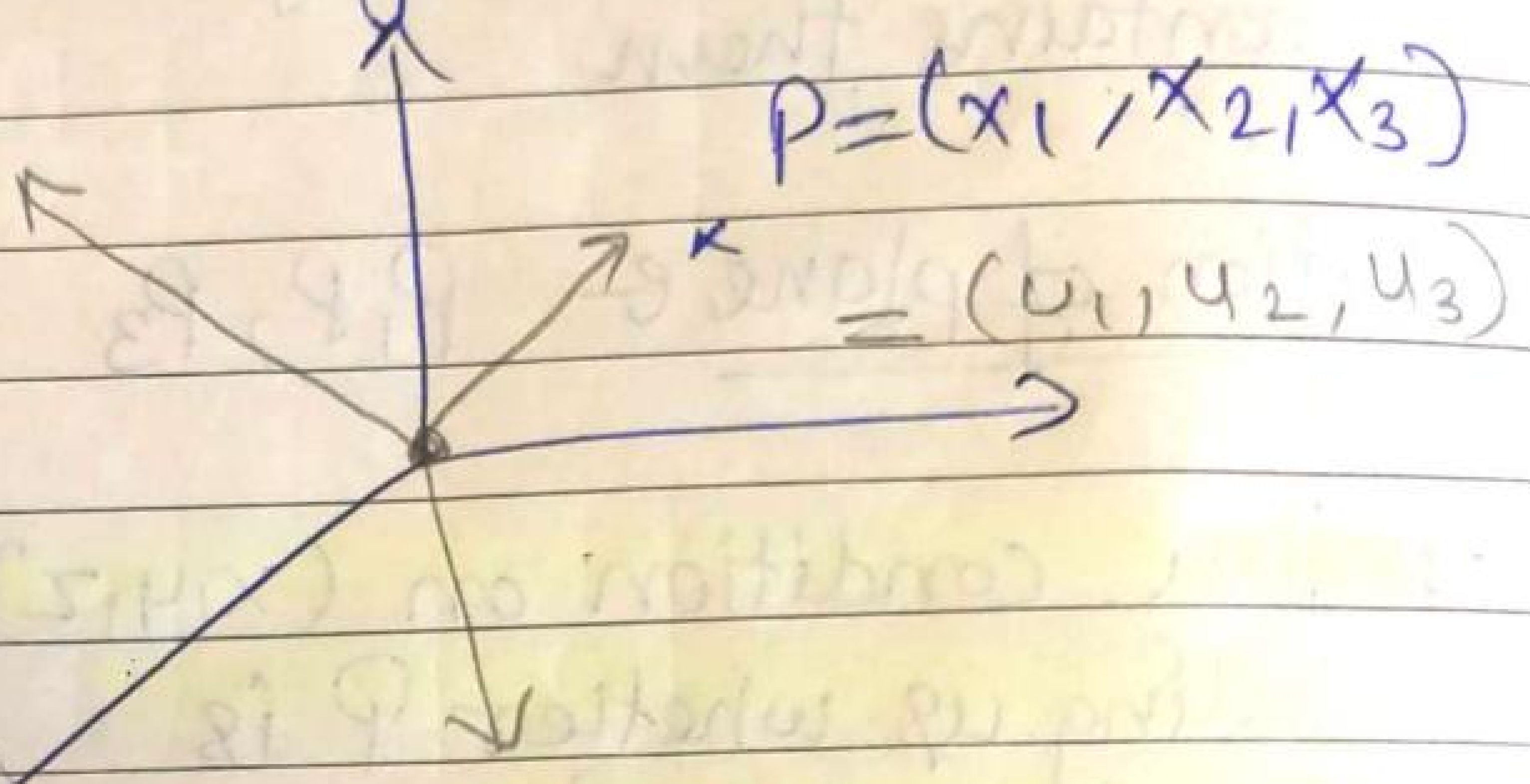
Notes: Triple product is same as determinant : Last Lecture

$\boxed{\det(\bar{A}, \bar{B}, \bar{C}) = \bar{A} \cdot (\bar{B} \times \bar{C})}$

MATRICES :-

often : linear relations b/w variables.

Ex: change of co-ordinate system.



example :

$$u_1 = 2x_1 + 3x_2 + 3x_3$$
$$u_2 = 2x_1 + 4x_2 + 5x_3$$
$$u_3 = x_1 + x_2 + 2x_3$$

Express using matrix product :

$$\underset{A}{\begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{u}{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}$$

Entries in matrix product :-

dot-products b/w rows of A & columns of X.

(A 3x3 matrix
X column matrix ⟷ 3x1 matrix)

Entries of AB :

$$\overset{A}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}} \overset{B}{\begin{bmatrix} \cdot & 0 \\ \cdot & 3 \\ \cdot & 0 \\ \cdot & 2 \end{bmatrix}} = \overset{AB}{\begin{bmatrix} \cdot & 14 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}}$$

Condition of multiply:

Width A equal to height of B.

What AB represents (meaning of multiplication):—

↳ doing first the transformation B, then transformation A.

you multiply things Right to left.

$$\boxed{(AB)X = A(BX)}$$ — associativity

X is some vector, that they want to transform.

Note: Matrix product is associative.

↳ BX means we apply the transformation B to X. then, multiplying by A means we apply the transform A.

Note: $\boxed{AB \neq BA!!}$

Identity Matrix: (transformation from itself).

$$\boxed{IX = X}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

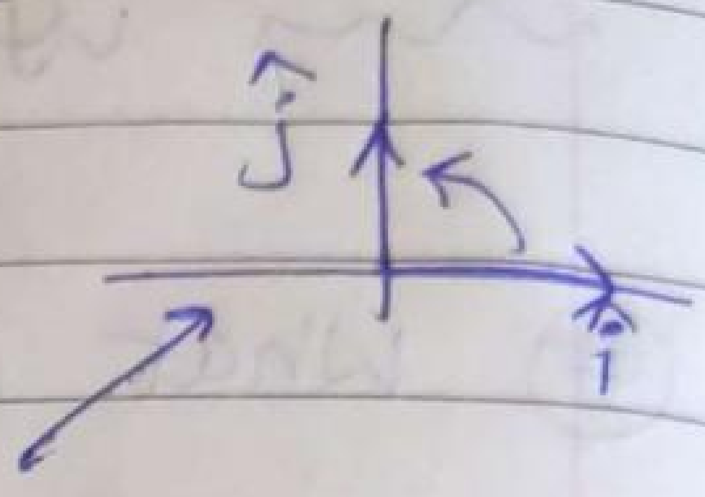
In general: $I_{n\times n} = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & \ddots 1 \end{bmatrix}$

example: In the plane:- ~~the~~ transformation that does rotate by 90° counter-clockwise.

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

to rotate 90°

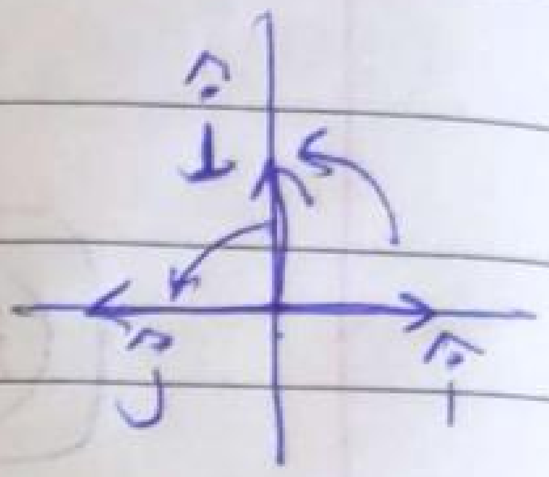
$$\begin{bmatrix} x \\ y \end{bmatrix} \longleftrightarrow \begin{bmatrix} -y \\ x \end{bmatrix}$$



if I apply that to the first vector,

$$R \cdot \hat{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{j}$$

$$R\hat{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\hat{i}$$



$$\boxed{R \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}}$$

→ co-ordinate change formula

example: Compute R^2

sol: $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_{2\times2}$

If I rotate something by 90° & then I rotate by 90° again, then I will rotate by 180°

That means I will actually just go to the opposite point around the origin.

If I apply R, 4 times, it would to Identity.

⑪ Invert Matrix: (Inverse Matrix):—

How to find inverse of matrix?

⇒ Inverse of A matrix M with property.

i.e. $AM = I$

$MA = I$

⇒ Need:- A → square matrix $(n \times n)$ ✓✓

⇒ $M = A^{-1}$

example Solution to $AX = B$. is:

$$\boxed{X = A^{-1}B}$$

$AX = B$

$\underline{A^{-1}(AX)} = A^{-1}B$

I

$$\boxed{X = A^{-1}B}$$

Formulae (good for small matrices):

$$\boxed{A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)}$$

"adjoint"

Steps:

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}$$

① MINORS:

$$\det\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} \rightarrow \begin{pmatrix} \overset{\det a_{11}}{3} & \overset{a_{12}}{-1} & \overset{a_{13}}{-2} \\ \overset{a_{21}}{3} & \overset{a_{22}}{1} & \overset{a_{23}}{-1} \\ 3 & 4 & 2 \end{pmatrix}$$

② COFACTOR:

flip signs in checkboard

leave alone ← $\textcircled{+}$

flip the sign ← $\textcircled{-}$

$$\begin{matrix} \textcircled{+} & - & + \\ \textcircled{-} & + & - \\ + & - & + \end{matrix} \rightarrow \begin{pmatrix} 3 & +1 & 2 \\ -3 & 1 & +1 \\ 3 & -4 & 2 \end{pmatrix}$$

③ transpose cofactors

switch rows & columns

$$\begin{pmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{pmatrix} \swarrow \underline{\underline{\text{adj}(A)}}$$

④ divide by determinant of A :-

$$\begin{vmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{vmatrix} = 3$$

$$\boxed{A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}} \checkmark$$

tells how to find x in terms of u's.