Lecture: 12

Gradient, directional derivative, tangent Plane.

 # Chain Rule: $w = w(x, y, z)$; $x = x(t)$, $y = y(t)$, $z = z(t)$

$$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} + w_z \frac{dz}{dt}$$

$$= \nabla w \cdot \frac{d\vec{r}}{dt}$$ → velocity vector

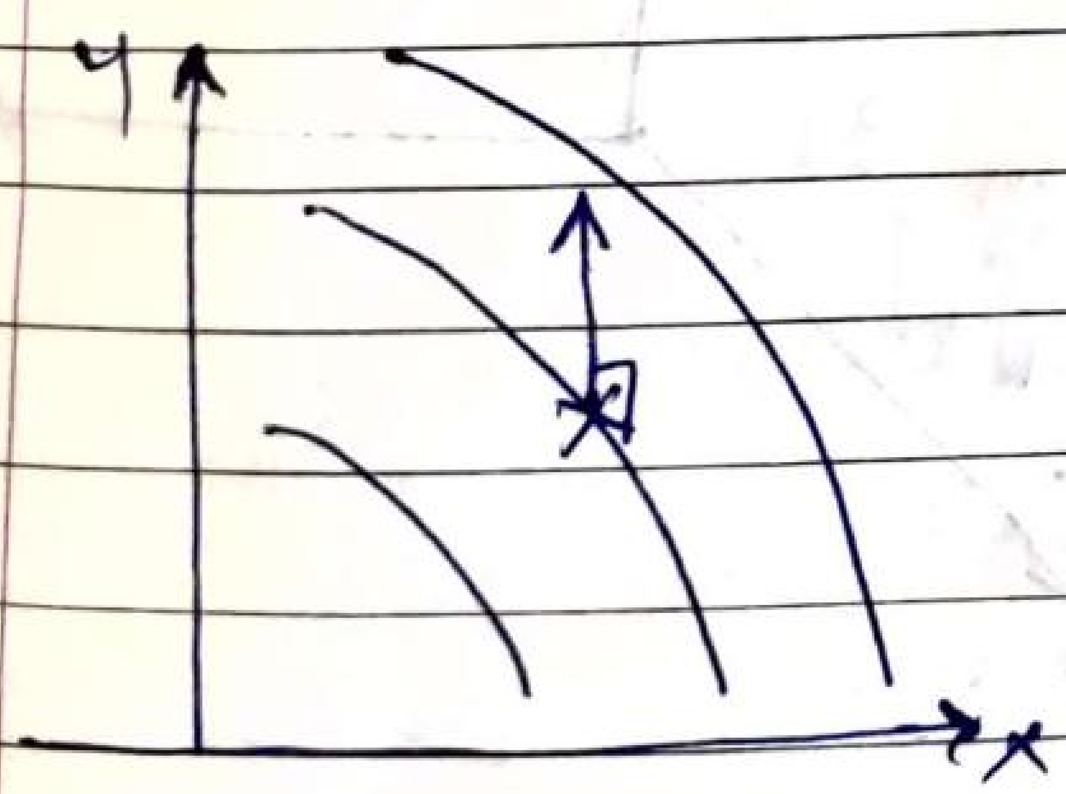
$$\boxed{\nabla w = \langle w_x, w_y, w_z \rangle}$$

GRADIENT OF W AT SOME POINT

⇒ putting together all the partial derivatives.

$$\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$ (Velocity Vector)

Theorem: $\nabla w \perp$r level surface $\{w = \text{constant}\}$



← (POINT TOWARDS HIGHER VALUES OF W.)

example 1: $w = a_1 x + a_2 y + a_3 z$

$$\nabla w = \langle a_1, a_2, a_3 \rangle$$
$$ \frac{\partial w}{\partial x} \quad \frac{\partial w}{\partial y} \quad \frac{\partial w}{\partial z}$$

level surface of ∇w?

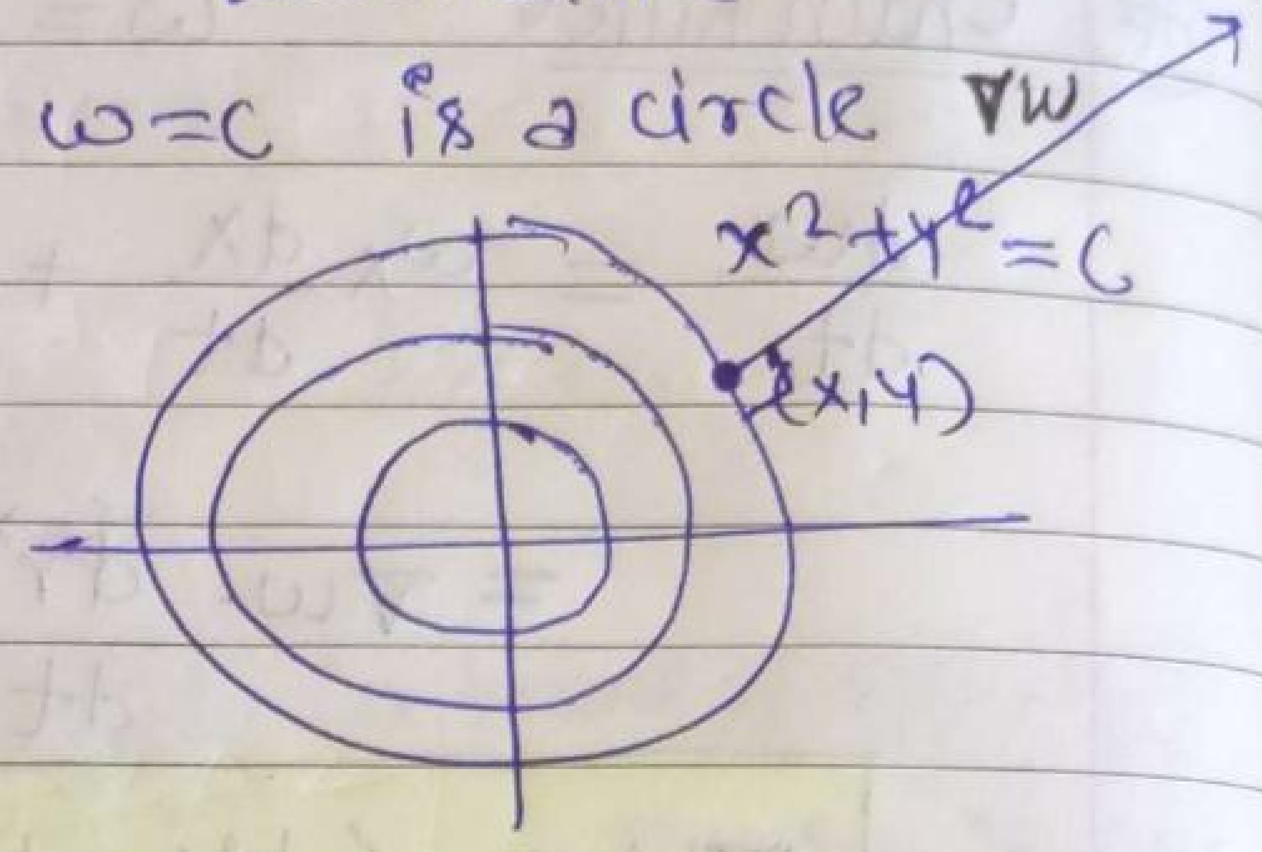
$a_1 x + a_2 y + a_3 z = c$

plane with normal $\langle a_1, a_2, a_3 \rangle$

example 2: $\quad w = x^2 + y^2$

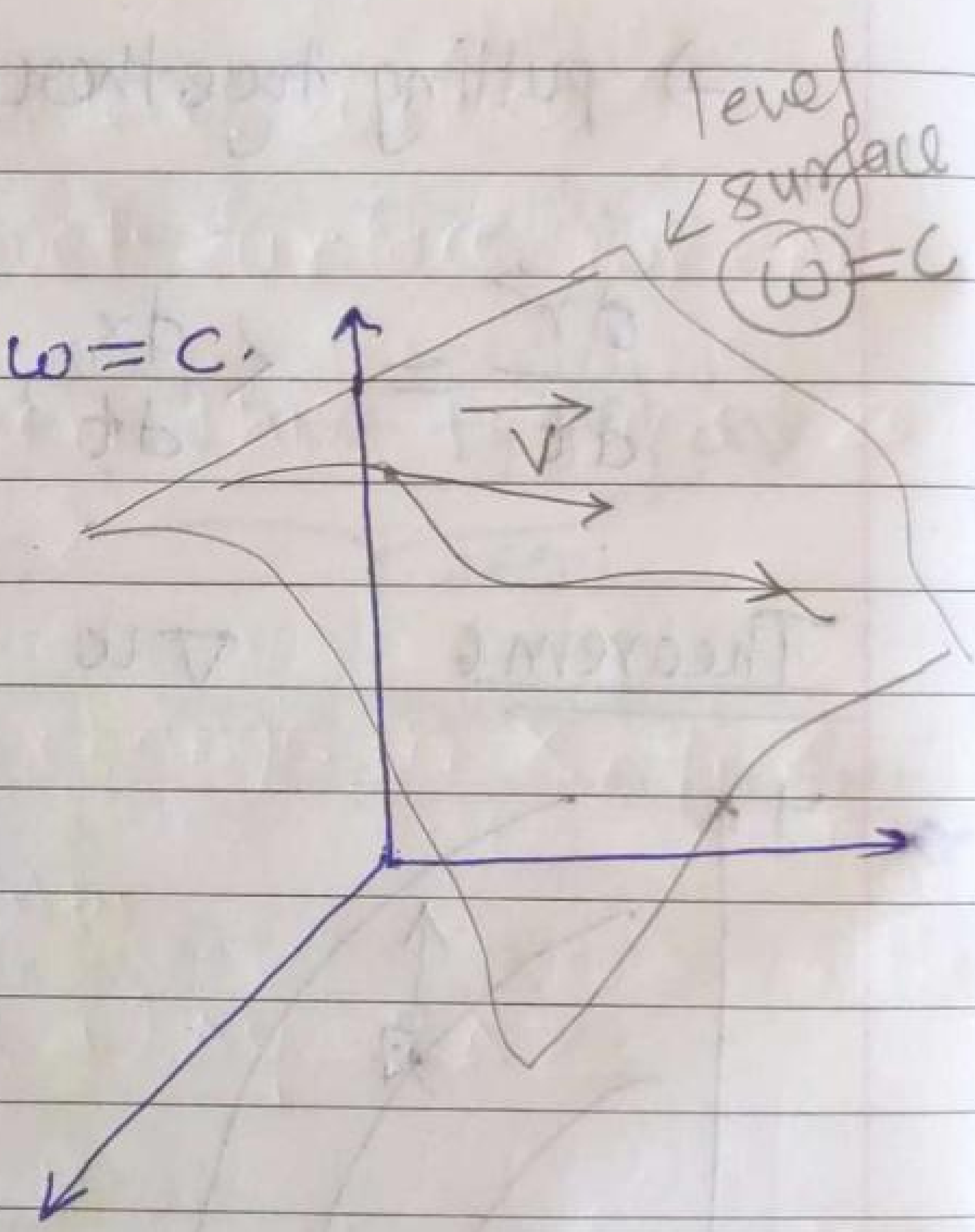
Level curve

$\nabla w = \langle 2x, 2y \rangle \qquad w = c$ is a circle



Proof: How the gradient vector & the motion on the level surface relate.

Take a curve $\vec{r} = \vec{r}(t)$ that stays on the level $w = c$.



Observation:
At an given time, that the velocity, ~~what~~ what can I say about velocity vector of this motion?

It's going to be tangent to the level surface as well as tangent to curve. Because curve is inside the surface.

Velocity $\vec{v} = \dfrac{d\vec{r}}{dt}$ is tangent to the level $w = c$.

By chain Rule:

$$\frac{dw}{dt} = \nabla w \cdot \frac{d\vec{r}}{dt}$$

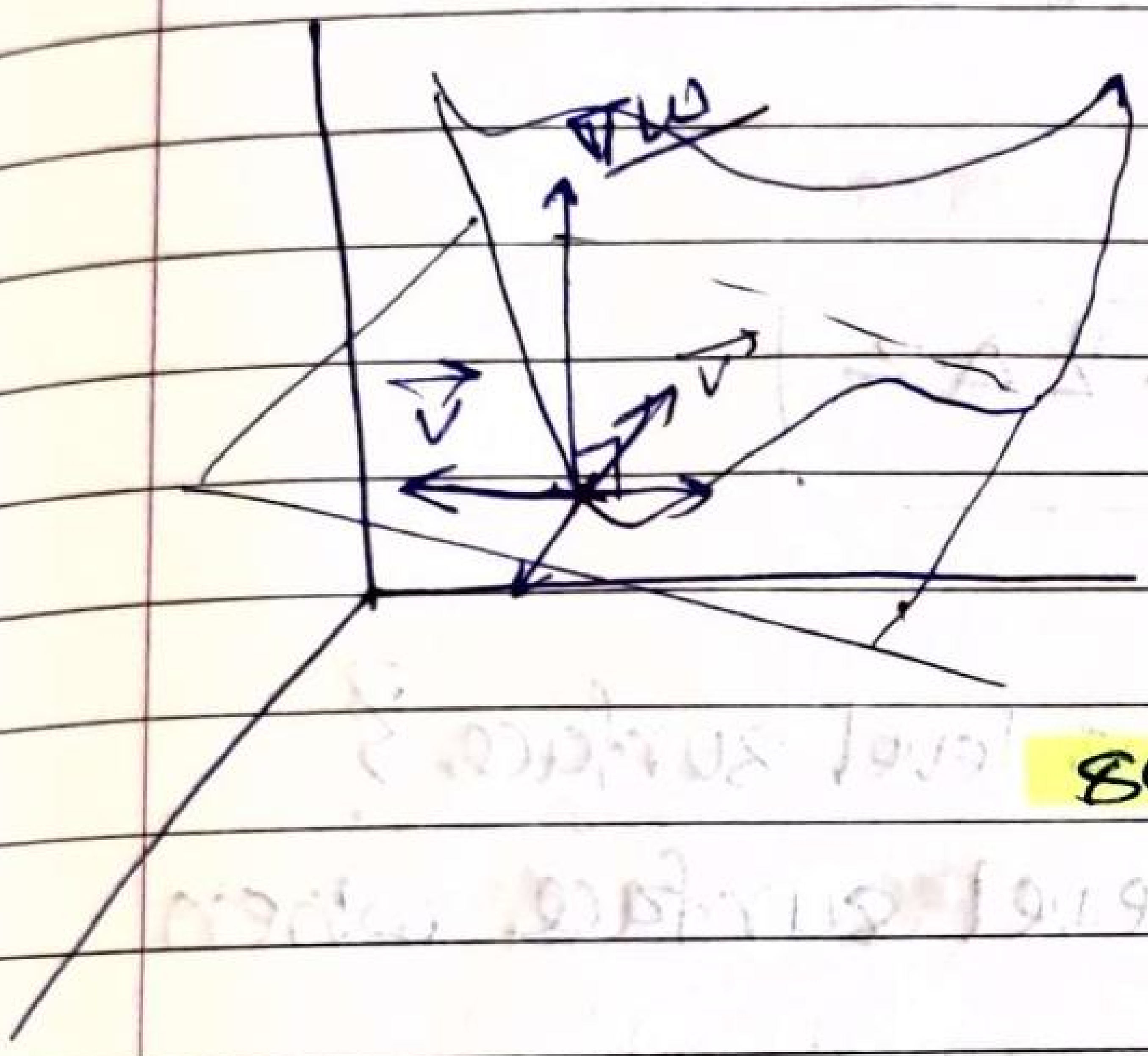
$$= \nabla w \cdot \vec{v} = 0$$

since $w(t) = c =$ constant.

So $\boxed{\nabla w \perp \vec{v}}$

This is true for any motion on surface $w=c$

$\vec{v}$ can be any vector tangent to $w=c$.



Given any vector $\vec{v}$ tangent to level,

$\nabla w \perp \vec{v}$

so $\nabla w \perp$ tangent plane to the level.//

example: find the tangent plane to surface $x^2+y^2-z^2=4$ at $(2,1,1)$. ?

Solution: Level set $w=4$, where $w = x^2+y^2-z^2$

gradient: $\nabla w = \langle 2x, 2y, -2z \rangle$

at $(2,1,1)$

$\hookrightarrow \nabla w = \langle 4, 2, -2 \rangle$

normal vector to the surface or to the tangent plane.

that's the one way to define the tangent plane.

→ it has the same normal vector as the surface.

→ Being ⊥r to surface means you are ⊥r to tangent plane.

So equation: $\boxed{4x+2y-2z=8}$ (2,1,1) at L.H.S.

(Another way):

$$dw = 2x\,dx + 2y\,dy - 2z\,dz$$

change into approx. formula

$$dw = 4\,dx + 2\,dy - 2\,dz$$

at $(2,1,1)$

$$\boxed{\Delta w \simeq 4\Delta x + 2\Delta y - 2\Delta z}$$

so, when do we stay on the level surface?

Well, we stay on the level surface when 'w' doesn't change, so when this becomes zero, approximation sign mean?

Well, it means for small change in x, y, z

~~this guy will be close to Δw~~

Δw will be close to $4\Delta x + 2\Delta y - 2\Delta z$

↳ it also mean; these approximation formulas, they are linear approximations.

They mean that we replace the function, actually, by some closest linear formula that will be nearby. And so, in particular, If we set $\Delta w = 0$ instead of approximately zero;

it means we'll actually be moving on the tangent plane to the level set.

If you want strict equalities in approximations means that we replace the function by its tangent approximation

So, level; $\Delta w = 0$;

its tangent plane corresponds to:-

$$4\Delta x + 2\Delta y - 2\Delta z = 0$$

$$4(x-2) + 2(y-1) - 2(z-1) = 0$$

↳ eq^n of tangent plane.

Another application of gradient:

Directional Derivatives:

$w = w(x,y) \rightsquigarrow$ know $\underbrace{\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}}$

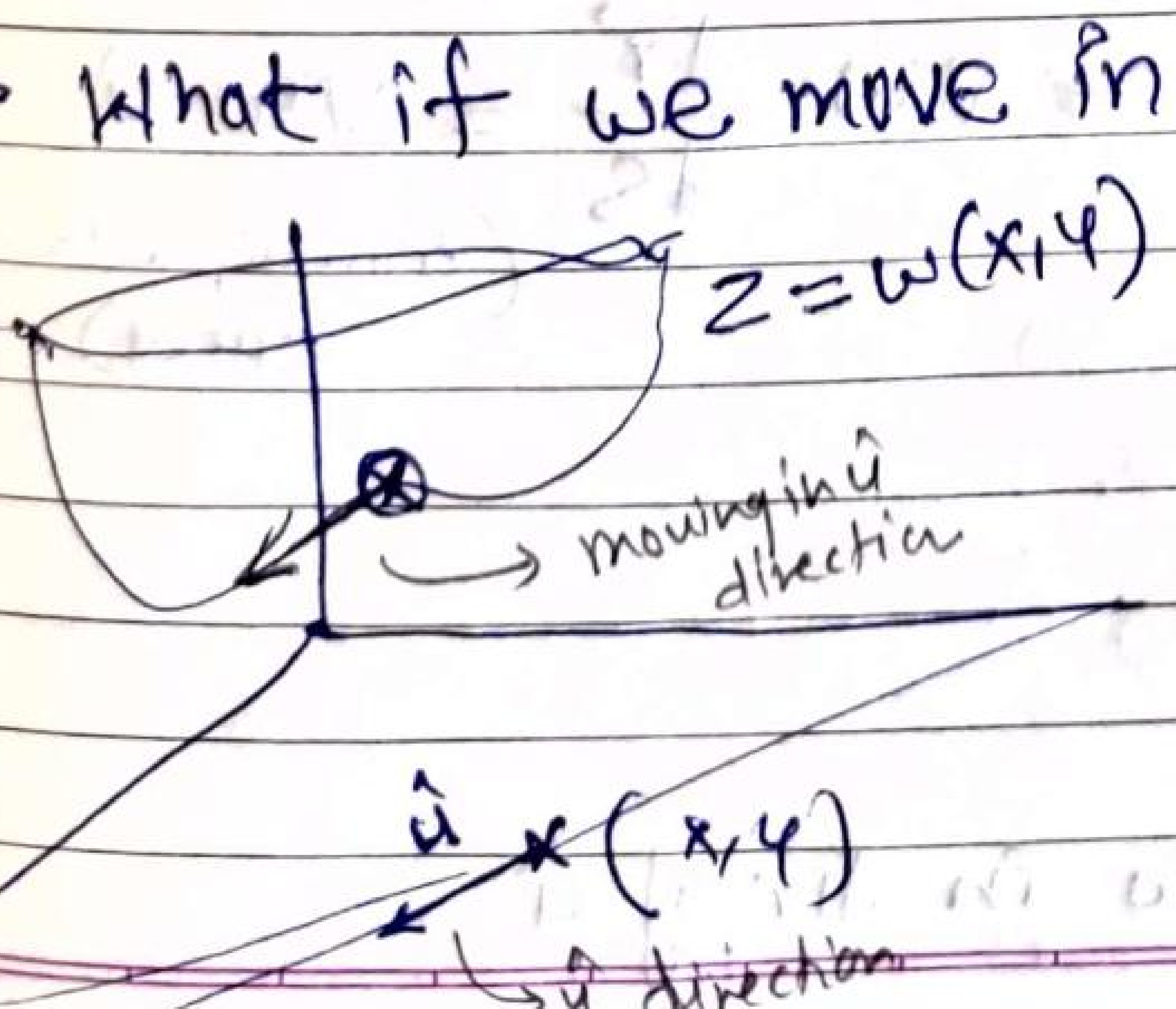
tells that rate of change in ~~x~~ & y-direction.

→ What about another direction?

Is there a derivative in every direction?

↳ that's the directional derivatives.

→ What if we move in direction of $\hat{u}$ = unit vector?



So, basically, I am asking myself how quickly does the value changes when I move on the graph in unit vector $\hat{u}$ direction?

Page No.
Date: / /

→ Straight line trajectory:

$\vec{r}(s)$, $\frac{d\vec{r}}{ds} = \hat{u}$

arc length
distance along line ⇒ so, what is dw/ds ?

(parametrizing by distance)

Def: $\boxed{\left.\frac{dw}{ds}\right|_{\hat{u}}}$

directional derivative in direction of $\hat{u}$.

→ If $\hat{u} = \langle a, b \rangle$

$$\begin{cases} x(s) = x_0 + as \\ y(s) = y_0 + bs \end{cases}$$

⇝ plug into w; $\frac{dw}{ds}$?

Note: Partial derivatives are slopes of slices of the graph by vertical planes that are parallel to the x or the y directions.

$\left.\frac{dw}{ds}\right|_{\hat{u}}$ = slope of slice of graph by a vertical plane $\parallel \hat{u}$

Chain Rule implies: $\frac{dw}{ds} = \nabla w \cdot \frac{d\vec{r}}{ds} = \nabla w \cdot \hat{u}$ (travelling with unit speed)

$$\boxed{\left.\frac{dw}{ds}\right|_{\hat{u}} = \nabla w \cdot \hat{u}}$$

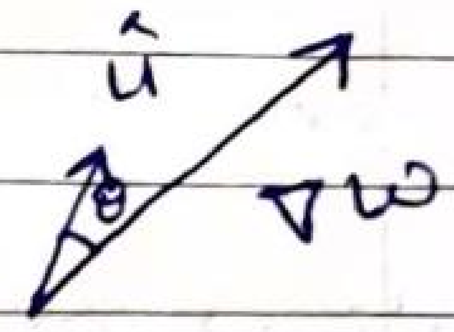
component of ∇w in dir of $\hat{u}$

same as $\left.\frac{dw}{ds}\right|_{\hat{i}} = \nabla w \cdot \hat{i}$

$= \frac{\partial w}{\partial x}$;

Geometry:

$$\left.\frac{dw}{ds}\right|_{\hat{u}} = \nabla w \cdot \hat{u} = |\nabla w| \overset{1}{|\hat{u}|} \cos(\theta)$$



In which direction w changes the ~~most~~ fastest, in which direction it increases the most or decreases the most, or doesn't actually change.

→ **maximal** :- when $\cos(\theta) = 1$.

$\Rightarrow \theta = 0; \Rightarrow \hat{u} = dir(\nabla w)$

The gradient is the direction in which the f^n increases the most quickly at that point. /

facing the mountain is fastest increase

so:. direction of ∇w = dir. of fastest increase of w.

↓

$|\nabla w|$ = directional derivative

↓

$\left.\frac{dw}{ds}\right|_{\hat{u}} = dir(\nabla w)$

opposite/ direction/ → **minimal** $\left.\frac{dw}{ds}\right|_{\hat{u}} \Rightarrow$ when $\cos(\theta) = -1$,

$\theta = 180°$

[$\hat{u}$ is in dir^n of $(-\nabla w)$]

→ No change: $\left.\frac{dw}{ds}\right|_{\hat{u}} = 0$; when $\cos\theta = 0$

$\theta = 90°$

$\hat{u} \perp \nabla w$

$\Leftrightarrow \hat{u}$ tangent to level.

↳ is direction that's tangent to the level.