Lecture: 19

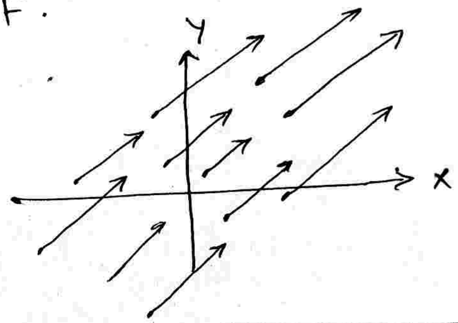
Vector fields:

$\vec{F} = M\hat{i} + N\hat{j}$, M and N are f^n of x, y.

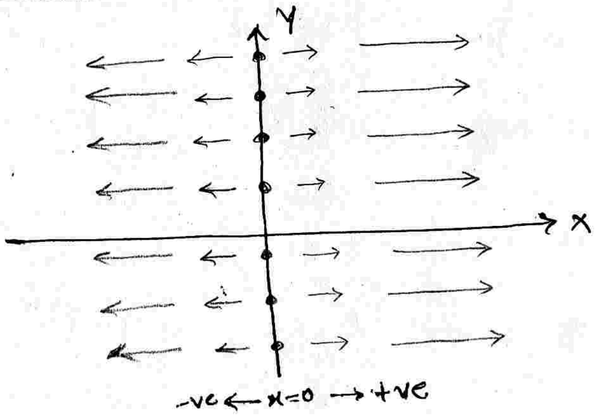
At each point (x, y), $\vec{F}$ a vector that depends on x & y.

Example: (i) velocity in a fluid $\vec{v}$
(ii) force field $\vec{F}$.

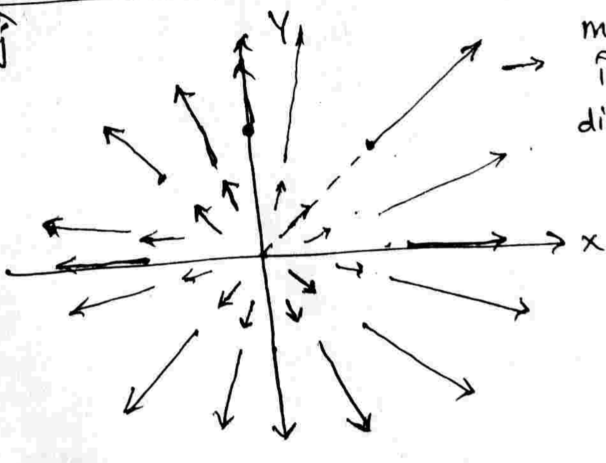
Example: $\vec{F} = 2\hat{i} + \hat{j}$



Example: $\vec{F} = x\hat{i}$

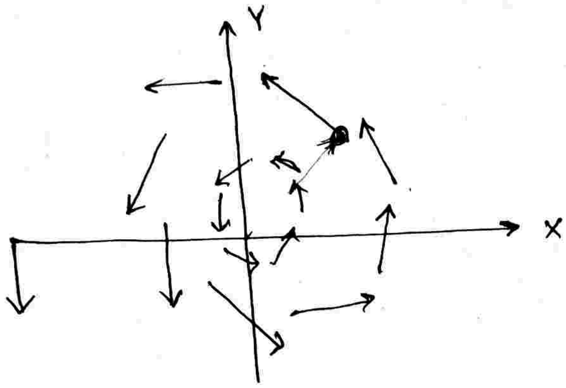


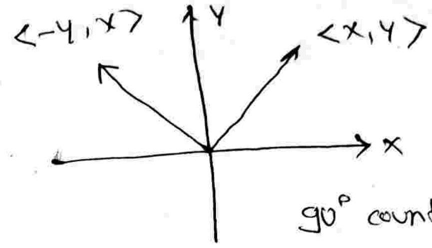
Example: $\vec{F} = x\hat{i} + y\hat{j}$



→ magnitude increases with distances from origin.

Example: $\vec{F} = -y\hat{i} + x\hat{j}$

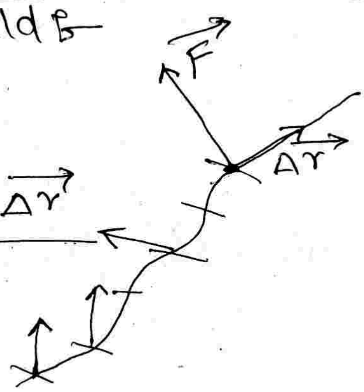




90° counterclock-wise

Velocity field for uniform rotation at unit angular velocity.

Compute work done by vector field:



Work & line integrals:

$W = (\text{Force}) \cdot (\text{distance}) = \vec{F} \cdot \vec{\Delta r}$

Total work along some trajectory: (C)

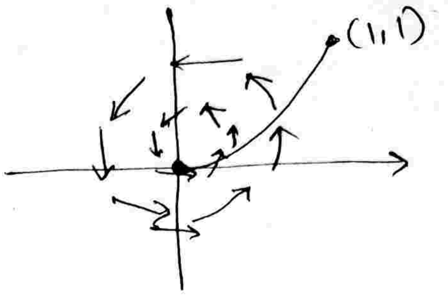
work adds up to

$$W = \int_C \vec{F} \cdot d\vec{r} \quad \left(= \cancel{\text{limit}} \lim_{\Delta r_i \to 0} \sum_i \vec{F} \cdot \vec{\Delta r_i} \right)$$

$$= \lim_{\Delta t \to 0} \sum_i \vec{F} \cdot \underbrace{\left(\frac{\vec{\Delta r}}{\Delta t} \right)}_{\text{velocity vector}} \Delta t$$

$$\boxed{W = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt}$$

②



Example 5— $\vec{F} = -y\hat{i} + x\hat{j}$;

c ; $x = t$ $\quad 0 \le t \le 1$

$y = t^2$

Work = ?

Sol5—

$$\int_C \vec{F} \cdot \vec{dr} = \int_0^1 \vec{F} \cdot \frac{\vec{dr}}{dt} \cdot dt = \int_0^1 \langle -t^2, t \rangle \cdot \langle 1, 2t \rangle dt$$

$\because \vec{F} = \langle -y, x \rangle \underset{\substack{x=t \\ y=t^2}}{=} \langle -t^2, t \rangle$

$\because dx/dt = 1; \quad dy/dt = 2t$

$$= \int_0^1 (-t^2 + 2t^2) dt$$

$$= \int_0^1 t^2 dt = 1/3.$$

Another Way: $\vec{F} = \langle M, N \rangle$

$\vec{dr} = \langle dx, dy \rangle$

$\vec{F} \cdot \vec{dr} = M dx + N dy$ $\quad \boxed{\int_C \vec{F} \cdot \vec{dr} = \int_C M dx + N dy.}$

Method to evaluate express x, y in terms of a single variable & substitute &

$$\int_C \vec{F} \cdot \vec{dr} = \int_C -y\, dx + x\, dy = \underline{\quad \text{in terms of 't'} \quad}$$

$x = t \qquad dx = dt$

$y = t^2 \qquad dy = 2t\,dt$

$$\int_C \vec{F} \cdot \vec{dr} = \int_C -t^2 dt + t \cdot 2t\,dt$$

$$= \int_0^1 t^2\, dt = 1/3;$$

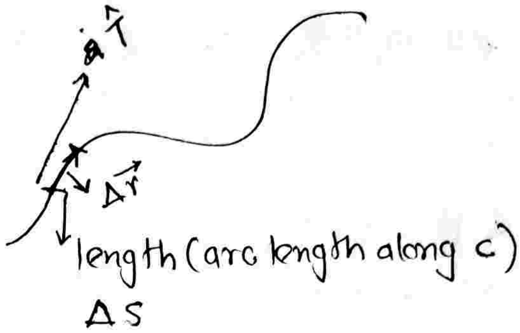
General Method: If you are given a curve then you first have to figure out how do you express x & y in terms of the same thing.

Note: $\int_C \vec{F} \cdot \vec{dr}$ depends on the trajectory C but not on parametrization.

e.g. could be $\begin{cases} x = \sin\theta \\ y = \sin^2\theta \end{cases}$ $0 \leq \theta \leq \frac{\pi}{2}$

(NOT PRACTICAL)

Geometric Approach:



Δr:

If I take a very small piece of trajectory then my vector $\Delta\vec{r}$ will be tangent to the trajectory.

→ It will be going in same direction as the unit tangent ($\hat{T}$). & its length, arc length along trajectory (Δs)

s → distance along trajectory.

$$\boxed{d\vec{r} = \langle dx, dy \rangle = \hat{T} \cdot ds}$$

(<u>Note:</u> $\frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \hat{T} \cdot \frac{ds}{dt}$)

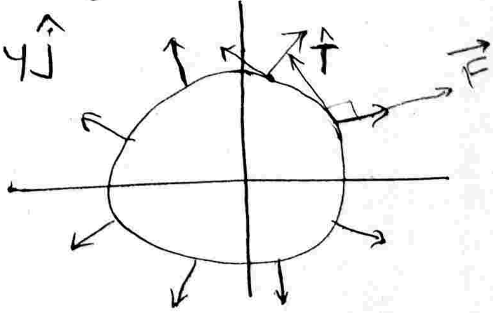
So,

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy = \int_C \underbrace{\vec{F} \cdot \hat{T}}\, ds}$$

↳ scalar quantity
It is the tangent component of force.

<u>Example:</u> C: circle of radius 'a' at origin; counterclockwise, $\vec{F} = x\hat{i} + y\hat{j}$

⇒ $\hat{T}$ ⊥r to radial direction $\vec{F}$



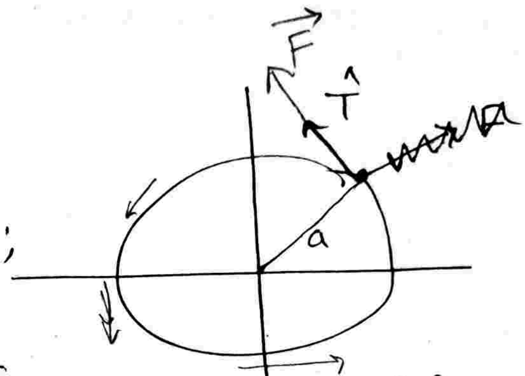
<u>Sol:</u> $\vec{F} \perp \hat{T}$; so $\vec{F} \cdot \hat{T} = 0$

$$\int_C \vec{F} \cdot \hat{T}\, ds = 0.$$

example 6

C: same

$$\vec{F} = -y\hat{i} + x\hat{j}$$



Sol: $\vec{F} \parallel \hat{T}$; $\vec{F}\cdot\hat{T} = |\vec{F}| = a$;

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C a\,ds = a\int_C ds = a\cdot \text{length}(C)$$
$$= 2\pi a^2$$

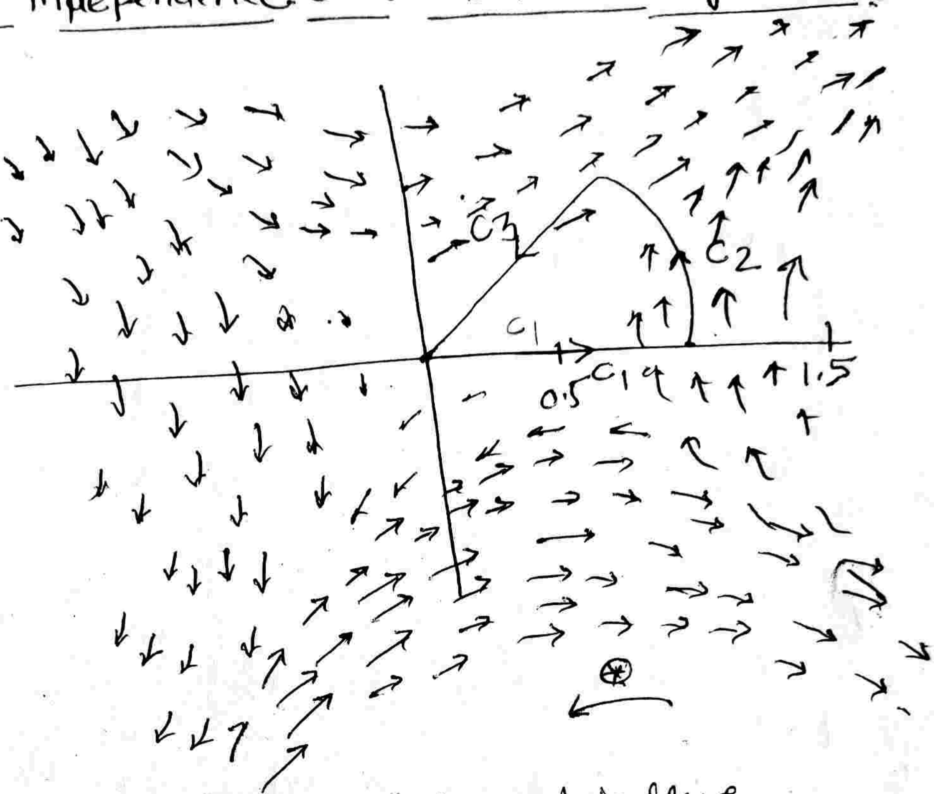
(or: computing : $\int_C -y\,dx + x\,dy = \int_C -(a\sin\theta)(-a\sin\theta)d\theta$
$+(a\cos\theta)(a\cos\theta\,d\theta)$

$x = a\cos\theta$ $\quad 0 \le \theta \le 2\pi$
$y = a\sin\theta$

$$= \int_0^{2\pi} a^2(\underbrace{\sin^2\theta + \cos^2\theta}_{1})\,d\theta$$

Lecture : 20

Path Independence and conservative fields.



example vector field:

$\vec{F} = \langle y, x \rangle$

$\int_C \vec{F} \cdot d\vec{r}$; $C = C_1 + C_2 + C_3$

enclosing sector of unit disk

$\boxed{0 \leq \theta \leq \frac{\pi}{4}}$

Need $\int_{C_i} y\,dx + x\,dy$;

ⓐ x-axis :-

from $(0,0)$ to $(1,0)$:-

$y = 0$; $dy = 0$;

$\int_{C_1} \cancel{y}^{\,0}\,dx + \cancel{x\,dy}^{\,0} = \int_{C_1} 0\,dx + 0 = 0$;

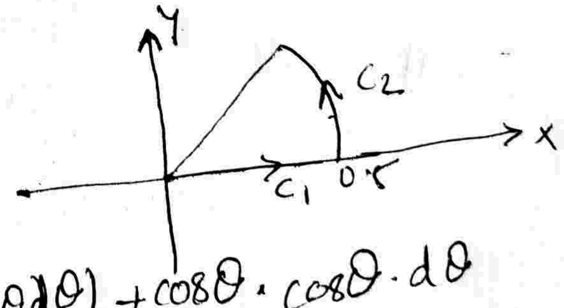
ⓐ Geometrically:

Along the x-axis, vector field is pointing vertically (y-direction)

Work done will be zero

$\boxed{\vec{F} \parallel \hat{j}\ ;\ \vec{F} \cdot \hat{T} = 0.}$

ⓑ C_2 : portion of unit circle:



$x = \cos\theta \qquad dx = -\sin\theta\,d\theta$

$y = \sin\theta \qquad dy = \cos\theta\,d\theta$

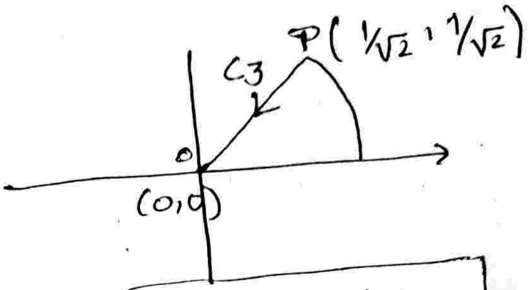
$0 \leq \theta \leq \pi/4$

$$\int_C y\,dx + x\,dy = \int_0^{\pi/4} \sin\theta \cdot (-\sin\theta\,d\theta) + \cos\theta \cdot \cos\theta \cdot d\theta$$

$$= \int_0^{\pi/4} \underbrace{\cos^2\theta - \sin^2\theta}_{\cos(2\theta)}\,d\theta = \left[\frac{1}{2} \sin 2\theta\right]_0^{\pi/4} = 1/2$$

(c)

figure out a way to express x, & y as a same parameter.



$\int_{C_3} y\,dx + x\,dy$: if

Could do : $\boxed{x = \underbrace{\frac{1}{\sqrt{2}}}_{\substack{\text{start} \\ \text{from P} \\ \text{at t0}}} - \underbrace{\frac{1}{\sqrt{2}}}_{\substack{\text{end at} \\ \text{point O} \\ \text{at t1}}} t}$

$\boxed{y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} t}$

$\boxed{0 \le t \le 1}$

Better

$\underbrace{\begin{array}{l} x = t \\ y = t \end{array}}_{\text{paramaterization}}$ t from 0 to $1/\sqrt{2}$ gives us $(-C_3)$ (C_3 backwards)

$dx = dt$; $dy = dt$

$$\int_{-C_3} y\,dx + x\,dy = \int_0^{1/\sqrt{2}} t\,dt + t\,dt = \int_0^{1/\sqrt{2}} 2t\,dt = \left[t^2\right]_0^{1/\sqrt{2}}$$
$$= 1/2$$

$$\boxed{\int_{C_3} y\,dx + x\,dy = -1/2}$$

Total Work $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + \frac{1}{2} - \frac{1}{2} = 0.$

Special Cases

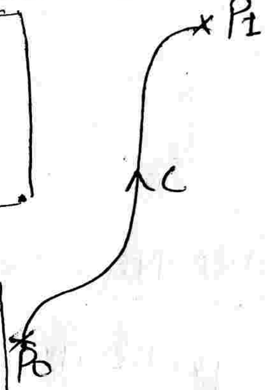
Say that $\vec{F}$ is gradient of some function so its a gradient field $\boxed{\vec{F} = \nabla f}$ ($f(x,y)$ is called potential)

Then we can simplify evaluation of $\int_C \vec{F}\cdot d\vec{r}$.

fundamental theorem of calculus for line integrals

$$\cancel{\int_C \vec{F}\cdot d\vec{r}} \quad \boxed{\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)}$$

for gradient



$$\boxed{\int_C f_x\,dx + f_y\,dy = \int_C df = f(P_1) - f(P_0)}$$

Proof: $\int_C \nabla f \cdot d\vec{r} = \int_C f_x\,dx + f_y\,dy$

C: $x = x(t)$
$y = y(t)$
$t_0 \le t \le t_1$

Sol: $x = x(t)$, $dx = x'(t)\,dt$

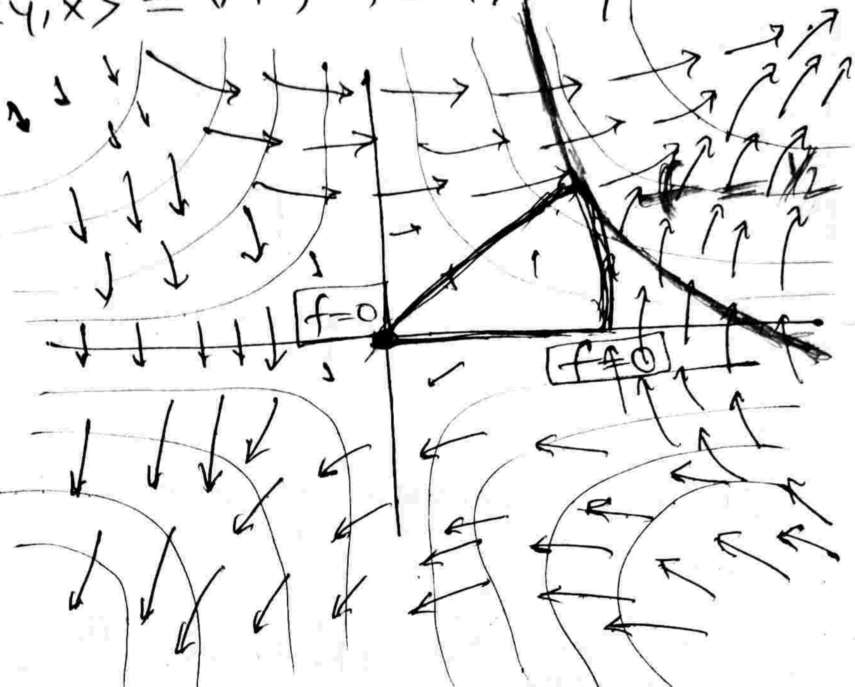
$y = y(t)$; $dy = y'(t)\,dt$

$$\int_C \nabla f\cdot d\vec{r} = \int_C \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt}\right) dt$$

$$= \int_{t_0}^{t_1} \frac{df}{dt}\,dt = \Big[f\big(x(t), y(t)\big)\Big]_{t_0}^{t_1} = f(P_1) - f(P_0)$$

$$\boxed{\int_C \nabla f\cdot d\vec{r} = f(P_1) - f(P_0)}$$

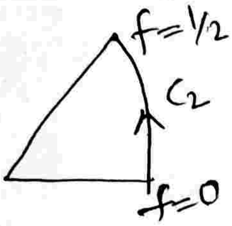
examples: $\vec{F} = \langle y, x \rangle = \nabla f, \quad f(x,y) = xy$



So:

$$\int_{C_2} \vec{F}\cdot d\vec{r} = f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) - f(1, 0)$$

$$= \frac{1}{2} - 0 = 1/2$$



WARNING: Everything today only applies if $\vec{F}$ is a gradient field! Not true otherwise!.

Consequences of fundamental theorem: IF $\vec{F}$ is a gradient field

→ Path-Independence

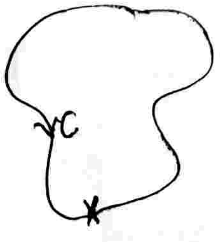
$$\boxed{\int_{C_1} \nabla f\cdot d\vec{r} = \int_{C_2} \nabla f\cdot d\vec{r}}$$

if C_1, C_2 have same start & end point.

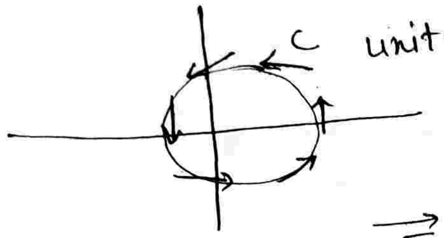
→ $\vec{F} = \nabla f$ is conservative.

⇒ if 'C' is closed curve ⇒ $\int_C \vec{F}\cdot d\vec{r} = 0$

Work = 0.



Remarks: $\vec{F} = \langle -y, x \rangle$;



$\int_C \vec{F} \cdot \hat{T}\, ds = \int_C 1\, ds = 2\pi \neq 0$..

On unit circle, $\vec{F} \parallel \hat{T}$

$\vec{F} \cdot \hat{T} = |\vec{F}| = 1$

NOT CONSERVATIVE.
SO ITS NOT A GRADIENT VECTOR.

NOT PATH INDEPEDENT

Physics: If force $\vec{F}$ is gradient of a potential

$$\boxed{\vec{F} = \nabla f}$$

→ work of $\vec{F}$ = change in value of potential
(E.g: gravitational field vs. gravitational
electrical field   electrical potential.

→ conservativeness means no energy can be.
extracted from the field "for free"!
— total Energy is conserved.

→ EQUIVALENT PROPERTIES:

① $\vec{F}$ is conservative: $\int_C \vec{F} \cdot d\vec{r} = 0$ along all closed curves C.

$\Updownarrow$

② $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

$\Updownarrow$

(3) $\vec{F}$ is a gradient field $\vec{F} = \langle f_x, f_y \rangle$

$\Updownarrow$

(4) $M\,dx + N\,dy$ is an exact differential $= df$.