Lecture :- 13

Lagrange Multipliers

goal :- is to min/max a function $f(x,y,z)$
where x,y,z are not independent.

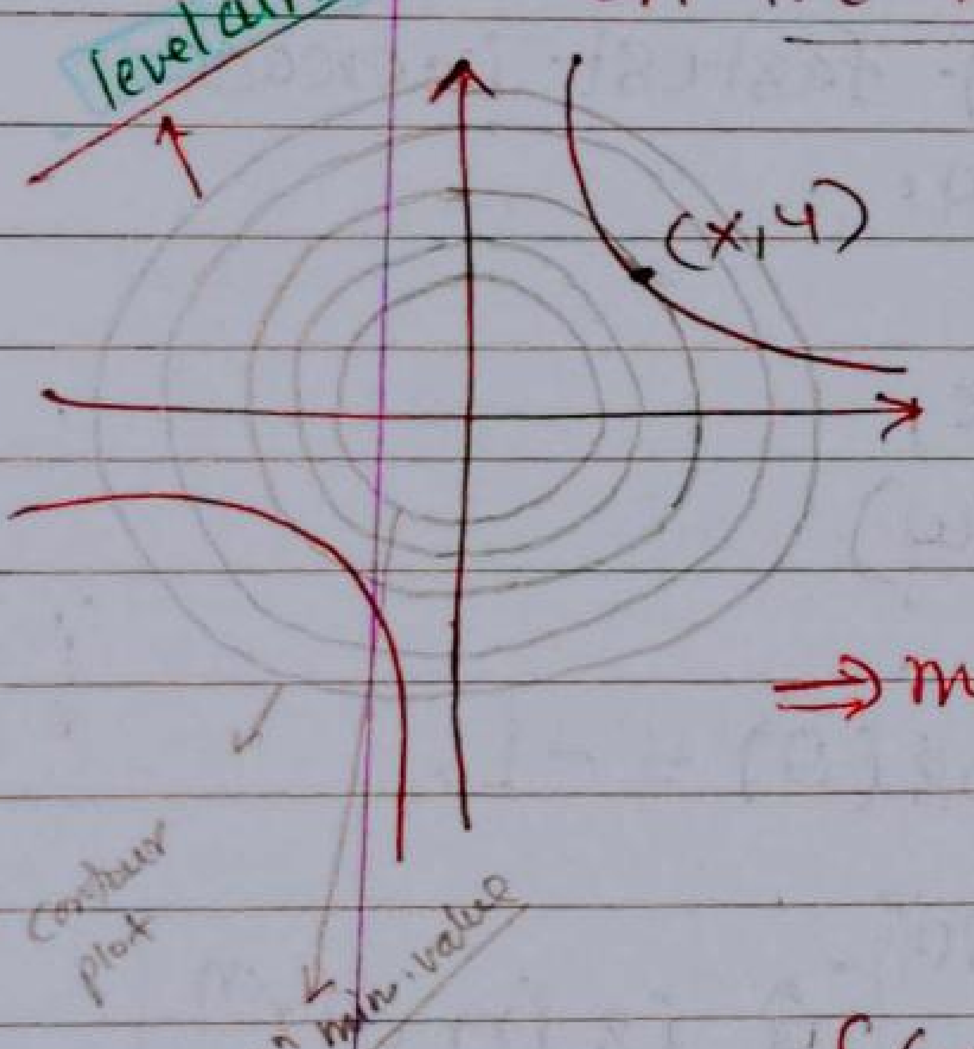
$$g(x,y,z) = c$$

ex: $PV = nRT$ in physics.

observations ① we cannot use our usual method of looking for critical point of f.
Because critical points of f typically will not ~~s~~ satisfy this condition $g(x,y,z) = c$, & so won't be good solⁿ.

②

examples I want to find point closest to the origin on the hyperbola $xy = 3$.



→ we can solve this by elementary geometry.

→ but we are going to do lagrange multipliers

⇒ means : minimize distance from origin

$$\cancel{f(x,y) = \sqrt{x^2+y^2}}$$

$$f(x,y) = x^2+y^2$$

subject to constraint $\underbrace{xy = 3}_{g(x,y)}$

⇒ actually we are looking smallest value of f; that be realized on the hyperbola.
↳ ~~when~~ minimum when circle is tangent to hyperbola

↳ So smaller value of fⁿ → no solⁿ
~~larger value of f → solution.~~

Key observations

when we have minimum, level curve of f is actually tangent to our hyperbola.

level curve of f is tangent to hyperbola

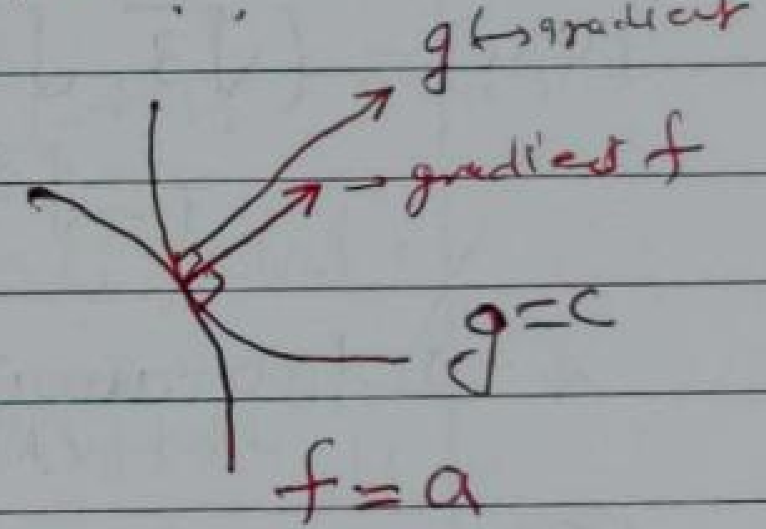
$\boxed{g = 3}$

so we have a level curve of 'f' and a level curve of 'g' that are tangent to each other.
Points where this happens?

→ means same tangent line !!

→ means normal vector should be parallel. !!

→ $\nabla f \parallel \nabla g$



→ parallel when they are proportional to each other.

→ so $\boxed{\nabla f = \lambda \nabla g}$ $\lambda \to$ lambda.

min/max
2 variables x, y
constraint $g(x,y) = c$

system of eq^n

$$\nabla f = \lambda \nabla g \begin{cases} f_x = \lambda g_x & \text{— (i)} \\ f_y = \lambda g_y & \text{— (ii)} \end{cases}$$

→ unknown $\boxed{x, y, \lambda}$
3 unknown, 2 eq^n, 1 eq^n is missing

constraint $g = c$ — (iii)

$f = x^2 + y^2$ $\quad f_x = \lambda g_x$
$g = xy$ $\quad f_y = \lambda g_y$

$$\begin{cases} 2x = \lambda y & \text{— (i)} \\ 2y = \lambda x & \text{— (ii)} \\ xy = 3 & \text{— (iii)} \end{cases}$$

$$\begin{array}{l} 2x - \lambda y = 0 \\ \lambda x - 2y = 0 \\ xy = 3 \end{array} \Leftrightarrow \underbrace{\begin{bmatrix} 2 & -\lambda \\ \lambda & -2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{Y}$$

$\downarrow$

Trivial sol^n (0,0) doesnot solve (iii)

another sol^n: only if determinant of matrix is zero

$$\begin{vmatrix} 2 & -\lambda \\ \lambda & -2 \end{vmatrix} = -4+\lambda^2 = 0; \Leftrightarrow \lambda^2 = 4$$
$$\Leftrightarrow \lambda = \pm 2$$

$\lambda = 2$	$\lambda = -2$
(i) $\rightarrow x = y$	(i) $\rightarrow x = -y$
(iii) $\rightarrow x^2 = 3$	(iii) $\rightarrow -x^2 = 3$.
$(x,y) = (\sqrt{3}, \sqrt{3})$ or $(-\sqrt{3}, -\sqrt{3})$	no sol^n

$\lambda \Rightarrow$ lagrange multiplier.

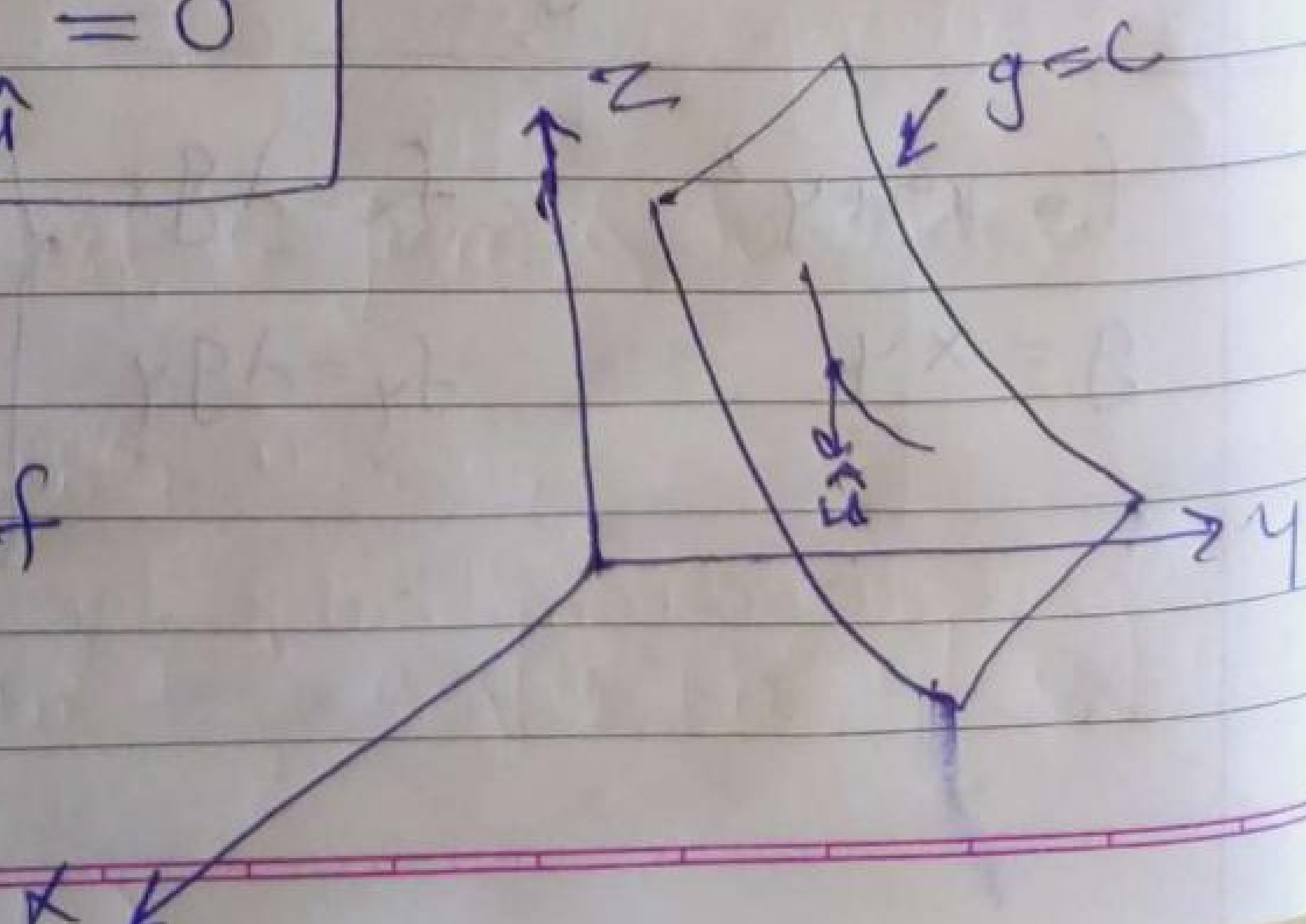
Why is this method valid?

At constrained min/max,
in any direction along level $g = c$;
the rate of change of 'f' must be '0'.

for any $\hat{u}$ tangent to $g = c$,
we must have $\boxed{\left.\frac{df}{ds}\right|_{\hat{u}} = 0}$

$\nabla f \cdot \hat{u}$

so any such $\hat{u}$ is $\perp r$ ∇f



so; $\nabla f \perp$ level set of g

know $\nabla g \perp$ level set of g

so $\boxed{\nabla f \parallel \nabla g}$

Critical point: If we had a constrained min or max, if we move in the level set of g, f doesn't change.

Well, it doesn't change to first order.

It is the same idea as when you are looking for a minimum, you set the derivative equal to zero.

So the derivative in any direction, tangent to g equals c, should be the directional derivation of f, in any such direction should be zero.

WARNING: the method does not tell, whether a sol^n is a minimum or maximum.

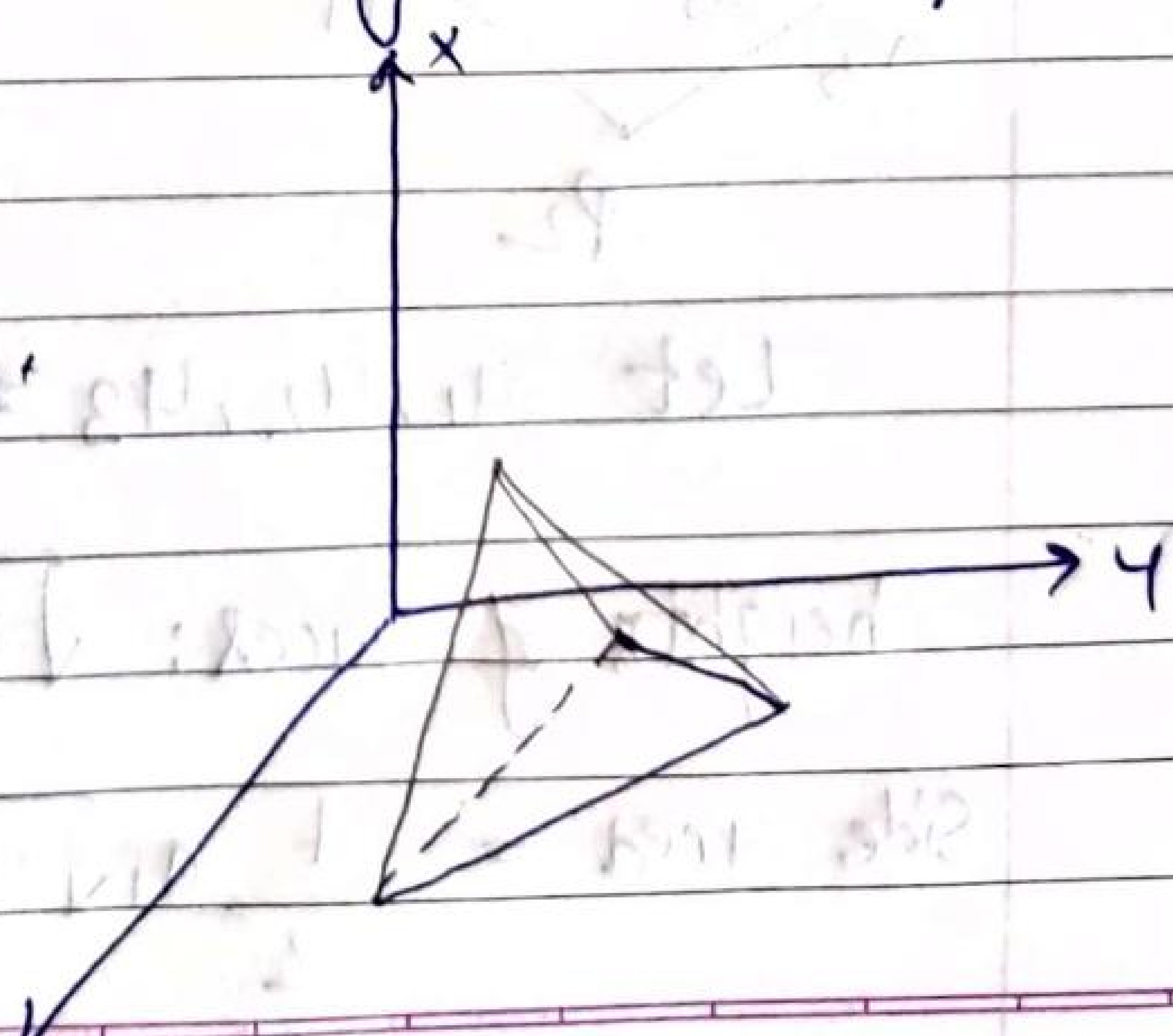
↳ we cannot use the second derivative test.

So, To find min or max we compare values of f at the various sol^n to Lagrange equations.

Advanced example:

Want to build a pyramid with given triangular base & given volume

↳ minimize total surface area.



[surface minimizing pyramid]

<u>Sol!</u> ↳ We have to look for the position of that top point.

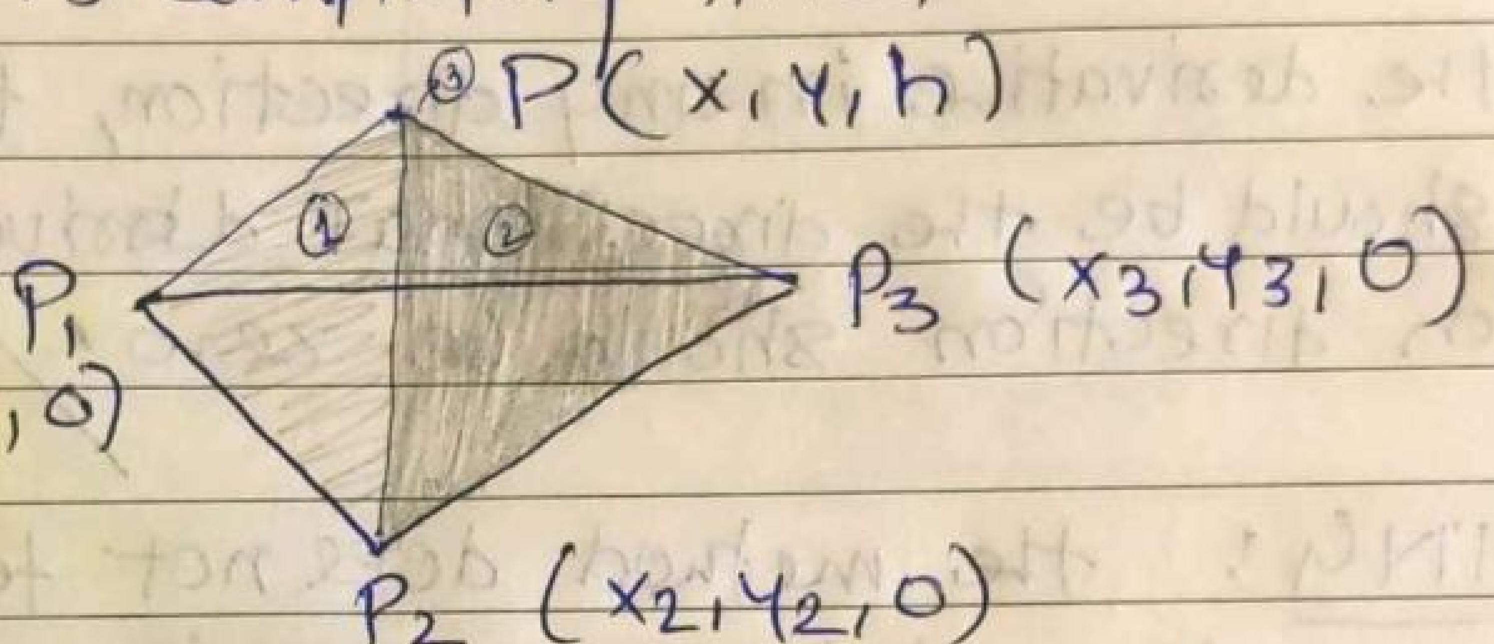
⇒ Volume of a pyramid is $\frac{1}{3}$ Area(base) × height

$$\text{Vol} = \frac{1}{3}\,\text{Area(base)} \cdot \underbrace{\text{height}}_{\text{fixed } = h}$$

↳ fixing the volume, knowing that we have fixed the area of a base, means that we are fixing the height of the pyramid.

↳ so height is completely fixed.

could do:-

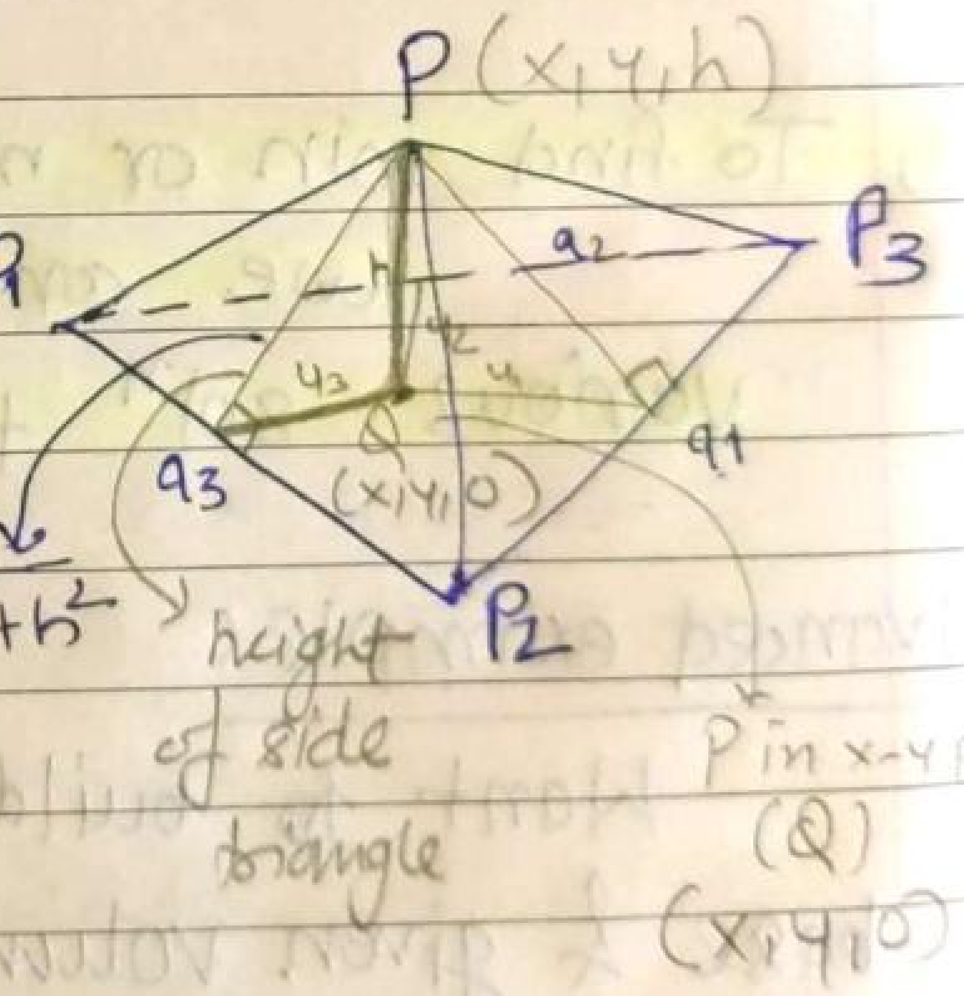


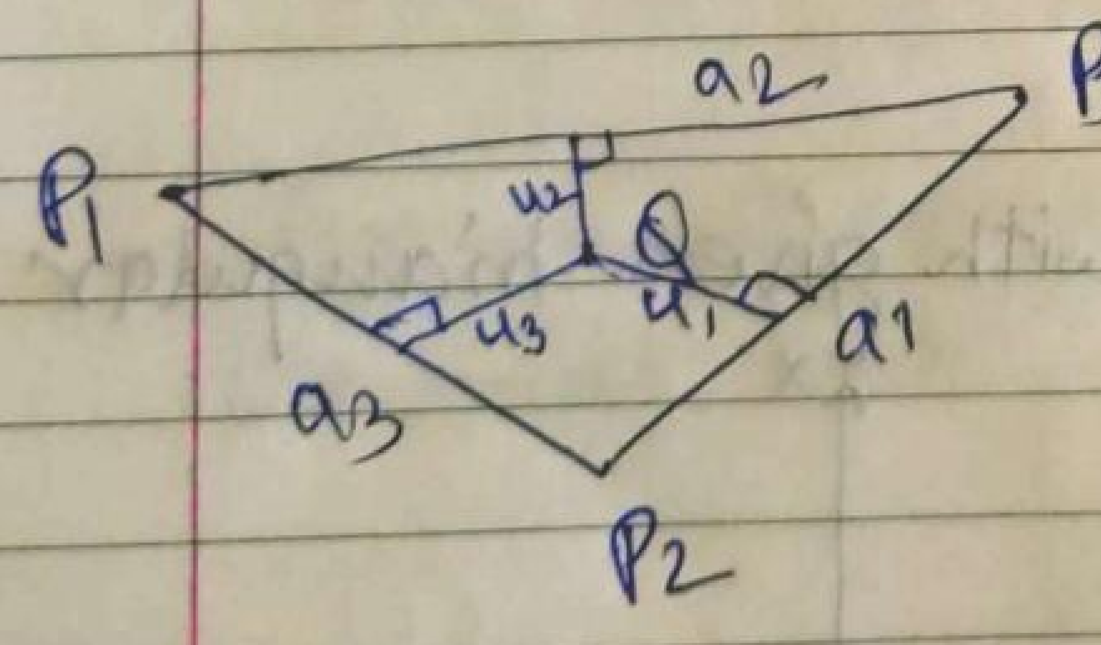
<u>goal</u>: minimize the area of triangles in pyramid

⇝ Area(x,y) can't solve min.

Area = (sum of 3 bases & 3 height)

$Q = (x,y,0)$





Let u_1, u_2, u_3 = dist from Q to sides.

heights of faces: $\sqrt{u_1^2+h^2}$ + . . . , – – –

$$\text{Side area} = \frac{1}{2}a_1\sqrt{u_1^2+h^2} + \frac{1}{2}a_2\sqrt{u_2^2+h^2} + \frac{1}{2}a_3\sqrt{u_3^2+h^2}$$
$$= f(u_1,u_2,u_3)$$

Cut the base into 3 $\Rightarrow$ Area (base) $\longrightarrow$

$$\frac{1}{2} a_1 u_1 + \frac{1}{2} a_2 u_2 + \frac{1}{2} a_3 u_3$$

$$\boxed{\nabla f = \lambda \nabla g}$$

$$\frac{\partial f}{\partial u_1} = \cancel{\frac{1}{2}} a_1 \frac{u_1}{\sqrt{u_1^2 + h^2}} = \lambda \cancel{\frac{1}{2}} a_1$$

$$\frac{\partial f}{\partial u_2} = \cancel{\frac{1}{2}} a_2 \frac{u_2}{\sqrt{u_2^2 + h^2}} = \lambda \cancel{\frac{1}{2}} a_2$$

$$\frac{\partial f}{\partial u_3} = \frac{u_3}{\sqrt{u^2 + h^2}} = \lambda$$

$\boxed{u_1 = u_2 = u_3}$ Q = incenter ✓

means: