Lecture: 9

Max-min problems; least squares:

Partial derivatives:

$$f(x,y) \rightsquigarrow \frac{\partial f}{\partial x} = f_x \quad (\text{vary } x; / y = \text{const})$$

$$\frac{\partial f}{\partial y} = f_y \quad (\text{vary } y; / x = \text{const}).$$

Approximation formula:

(when you both x & y)

→ if we do both at the same time then the two effects will addup with each other, because first you change x and then you will change y. Or other way around. It doesn't matter.

→ If we change x by $x \rightsquigarrow x + \Delta x$
$y \rightsquigarrow y + \Delta y$

$z = f(x,y)$: then $\boxed{\Delta z \simeq f_x \Delta x + f_y \Delta y}$ ✓

↳ Justify this formula: tangent plane to

$z = f(x,y)$?

Know: f_x, f_y are slopes of 2 tangent lines.

If $\frac{\partial f}{\partial x}(x_0, y_0) = a \Rightarrow L_1 = \begin{cases} z = z_0 + a(x - x_0) \\ y = y_0 \end{cases}$ ↳ keeping y constant

$\frac{\partial f}{\partial y}(x_0, y_0) = b \Rightarrow L_2 = \begin{cases} z = z_0 + b(y - y_0) \\ x = x_0 \end{cases}$

These two Lines L_1 & L_2 are both going to be in the tangent plane to the surface.

tangent eq^n

Page No.
Date:

L_1, L_2 are both tangent to the graph $z = f(x,y)$.
Together they determine a plane.

plane eq^n : $\boxed{z = z_0 + a(x - x_0) + b(y - y_0)}$

z equals to constant time x + contant time y.

If I hold y constant and vary x, I will get the first line L_1, on holding x ~~holding~~ constant I get L_2

Another way to do it is would ~~eq~~ provide ~~actually~~ parametric eq^n of these lines, get vectors along them and then take the cross-product to get the normal vector to the plane. And then get this eq^n for plane using normal vector.

Now what this approximation formula says?

$$\boxed{\Delta z \approx f_x \Delta x + f_y \Delta y}$$

that graph of a f^n is close to the tangent plane.
If we were moving ~~me~~ on the tangent plane this would be an actual equality.

Δz would be a linear f^n of Δx & Δy. And the graph of the f^n is near the tangent plane, but is not quite the same ~~near the tangent planes~~ so it is only an approximation for small Δx & small Δy.

says:- graph of f is close to its tangent plane.

Application of partial derivatives :-

Page No.
Date:

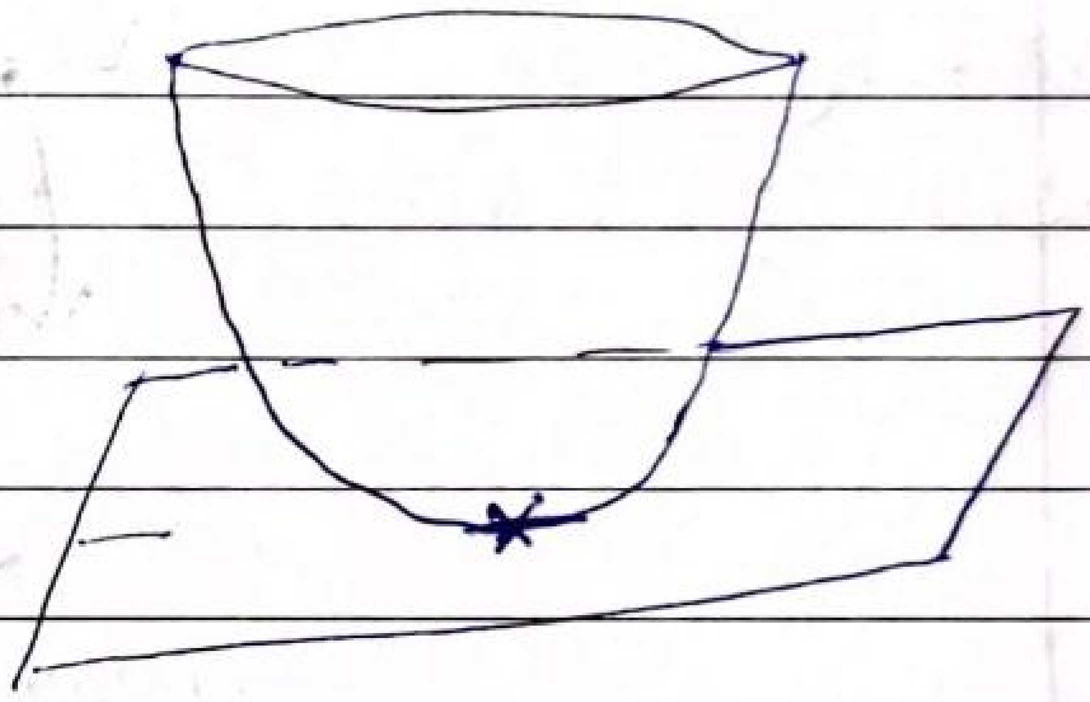
① OPTIMIZATION PROBLEMS :

⇝ find min/max of f^n $f(x,y)$

ⓐ If we have a local minimum or a local maximum then both partial derivative are both zero at the same time.

$$f_x = 0 \text{ and } f_y = 0.$$

⟺ tangent plane to the graph $z = f(x,y)$ is horizontal !!



Def^n: (x_0, y_0) is a critical point of f

if $f_x(x_0, y_0) = 0$ and
$f_y(x_0, y_0) = 0$

Example : $f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$.
max & min ?

sol^n:
$$\begin{cases} f_x = 2x - 2y + 2 = 0 \\ f_y = -2x + 6y - 2 = 0 \end{cases}$$

sum : $4y = 0$; $\rightarrow y = 0$;
$2x + 2 = 0$; $\rightarrow x = -1$ //

$\Rightarrow$ $\boxed{1 \text{ critical point : } (x,y) = (-1, 0)}$

Possibilities:
→ local min
→ local max
→ saddle.

$$f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$$

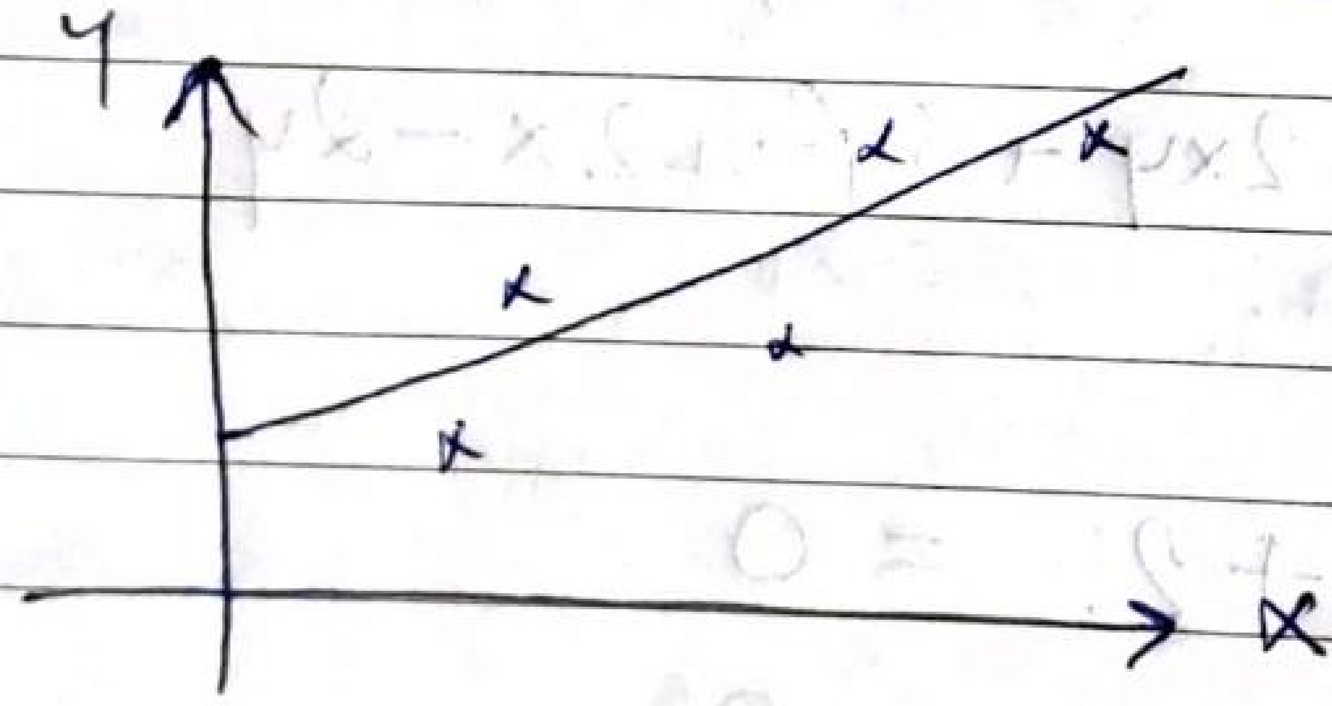
$$f(x,y) = (x-y)^2 + 2y^2 + 2x - 2y$$
(complete square)

$$f(x,y) = \underbrace{((x-y)+1)^2}_{\geq 0} + \underbrace{2y^2}_{\geq 0} - 1 \geq -1 = f(-1,0)$$

always +ve   always +ve   so.. minimum

always get atleast -1

LEAST - SQUARES - INTERPOLATIONS:



GIVEN EXPERIMENTAL DATA

$(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$

find "best fit" line $y = ax + b$
find "best" a & b.   a, b unknown

→ minimizing total square ~~er~~ deviation.

deviation for each data point : $y_i - (ax_i + b)$

Minimize $D(a,b) = \sum_{i=1}^{n} \left[y_i - (ax_i - b) \right]^2$

Want:
$$\begin{cases} \dfrac{\partial D}{\partial a} = \displaystyle\sum_{i=1}^{n} \cancel{2}\left(y_i - (ax_i+b)\right)(-x_i) = 0 \\[2ex] \dfrac{\partial D}{\partial b} = \displaystyle\sum_{i=1}^{n} \cancel{2}\left(y_i - (ax_i+b)\right)(-1) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \displaystyle\sum_{i=1}^{n} \left(x_i^2 a + x_i b - x_i y_i\right) = 0 \\[2ex] \displaystyle\sum_{i=1}^{n} \left(x_i a + b - y_i\right) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(\displaystyle\sum_{i=1}^{n} x_i^2\right) a + \left(\displaystyle\sum_{i=1}^{n} x_i\right) b = \displaystyle\sum_{i=1}^{n} x_i y_i \\[2ex] \left(\displaystyle\sum_{i=1}^{n} x_i\right) a + n b = \displaystyle\sum_{i=1}^{n} y_i \end{cases}$$

$\rightsquigarrow$ 2x2 linear system!

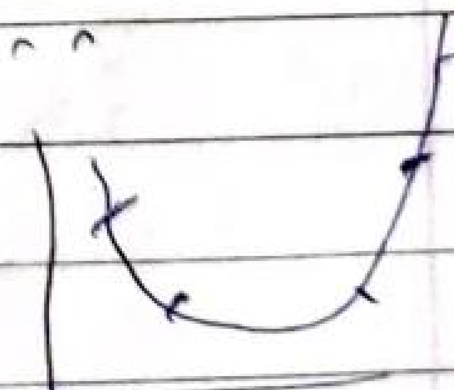
Can show: it's a minimum.

Least squares is more general!

Best exponential fit: $\boxed{y = ce^{ax}}$

$\Leftrightarrow$ $\boxed{\ln(y) = \ln(c) + ax}$ ← leo

→ best line...



- quadratic law: $y = ax^2 + bx + c$

$$\boxed{D(a,b,c) = \sum_{i=1}^{n} \left(y_i - (ax_i^2 + bx_i + c)\right)^2}$$