Lectures 5

Parametric equations for lines & curves:-

EQUATIONS OF LINES:

We have seen line = intersection of 2 planes.

Another representation of line:

(line is the trajectory of a point as time varies.)

line = trajectory of moving point = "parametric equation"

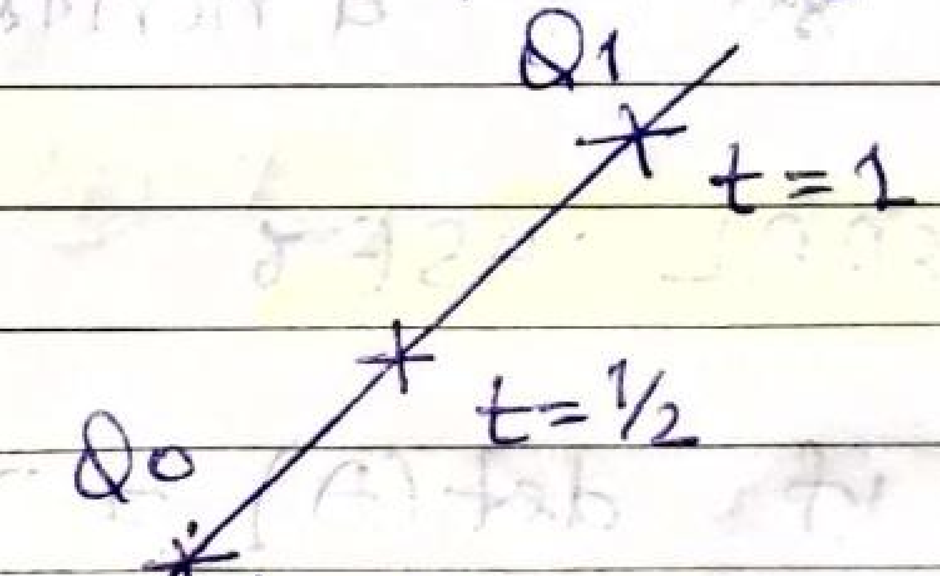
example:-

Line through

$Q_0 = (-1, 2, 2)$; $Q_1 = (1, 3, -1)$

Now, how to find all the other points?

Solution:

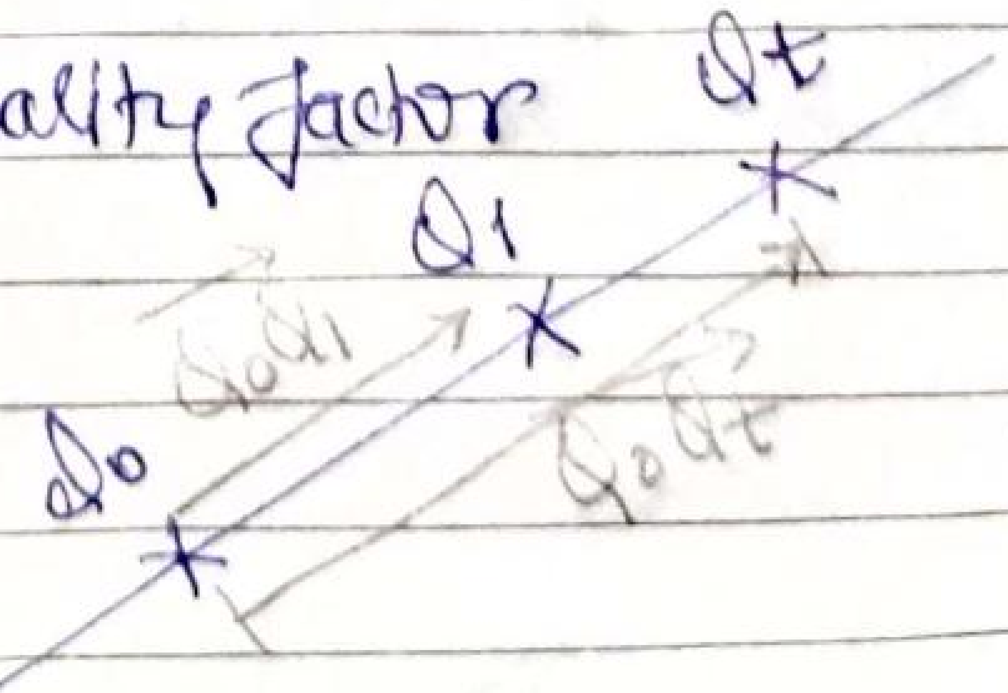


Let's say $Q(t)$ is moving point; and at $t=0$; it is Q_0 & moves at constant speed on the line

What is the position at time $t=0$ · $Q(t)$?

$$\boxed{\overrightarrow{Q_0Q(t)} = t\,\overrightarrow{Q_0Q_1}}$$

↳ proportionality factor



$\overrightarrow{Q_0Q(t)} = t\,\langle 2, 1, -3 \rangle$

$Q(t) = (x(t), y(t), z(t))$

↳ co-ordinate of moving point

$$\begin{cases} x(t)+1 = t\cdot 2 \\ y(t)-2 = t \quad (3-2) \\ z(t)-2 = -3t \quad (-1-2) \end{cases}$$

$$\overrightarrow{Q_0Q(t)} = t\,\overrightarrow{Q_0Q_1}$$

$(x(t)-(-1) = (1+1$

$$\begin{cases} x(t) = -1+2t \\ y(t) = 2+t \\ z(t) = 2-3t \end{cases}$$

→ where we are at t_0

→ $\overrightarrow{Q_0Q_1}$

$$\boxed{Q(t) = Q_0 + t\,\overrightarrow{Q_0Q_1}}$$

Applications: Intersection with a plane?

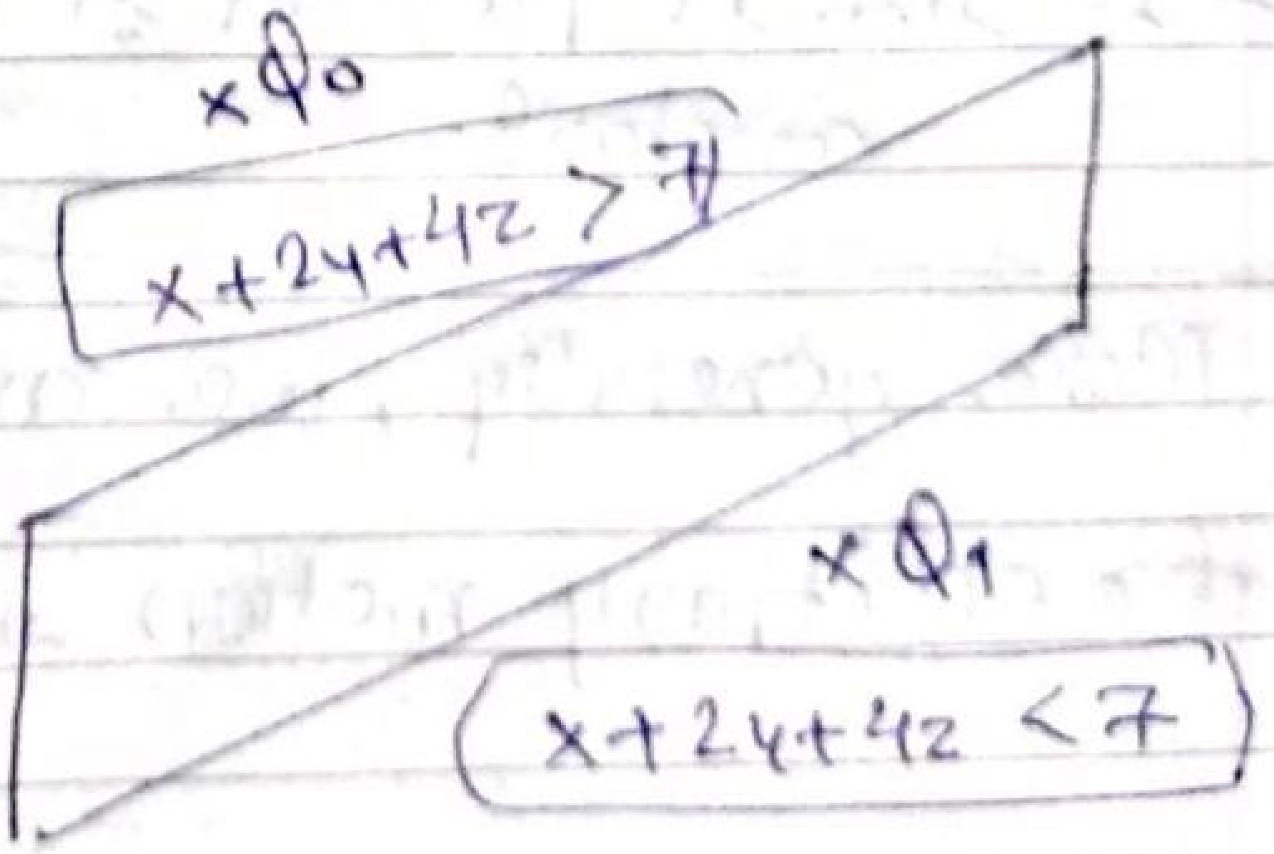
Consider the plane $x+2y+4z=7$
where does the line Q_0Q_1 intersect the plane?

Q: Relative to $x+2y+4z=7$;
$Q_0=(-1,2,2)$ & $Q_1=(1,3,-1)$ are on

Sol: $Q_0=(-1,2,2)$; $x+2y+4z = (-1)+2\times 2+4\cdot 2 = 11 > 7$
(so Q_0 not in the plane)

$Q_1=(1,3,-1)$; $x+2y+4z = 1+2\times 3-4 = 3<7$.
(not in plane)

$\boxed{Q_0 \text{ \& } Q_1 \text{ are opposite sides of the plane}}$



↳ What about moving point $Q(t)$?

$$x(t) + 2y(t) + 4z(t)$$

$$= (-1+2t) + 2(2+t) + 4(2-3t)$$

$$= -8t + 11 \overset{?}{=} 7$$

↳ $Q(t)$ is the plane when

$$-8t + 11 = 7$$

$$\Leftrightarrow t = 1/2$$

Then $Q(1/2) = (0, 5/2, 1/2)$ (Plug the values into $(-1+2t) + (2+t) + (2-3t)$ → $\boxed{Q_0 + t\overrightarrow{Q_0Q_1}}$ ✓

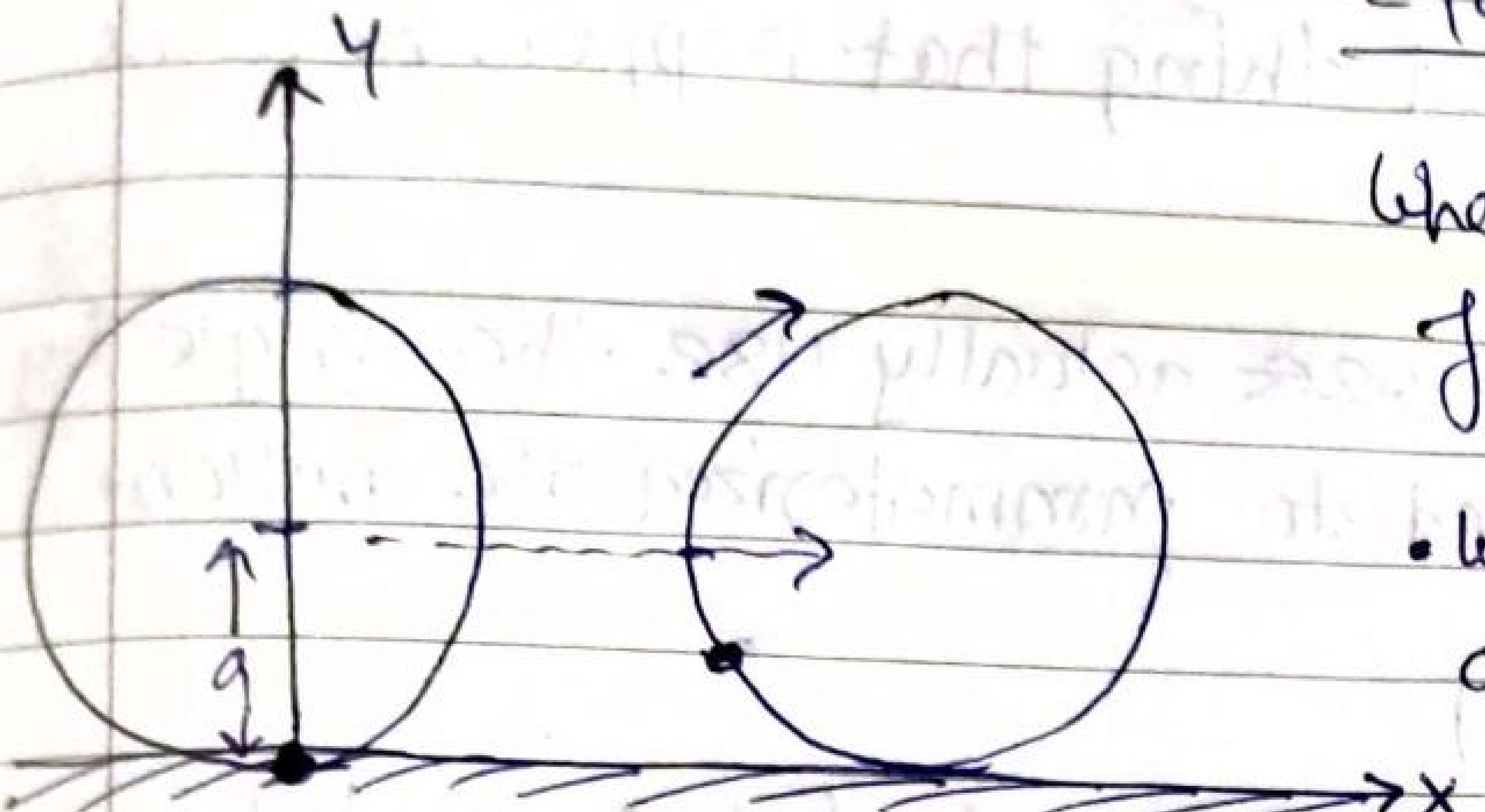
→ this is the line where the plane intersects. $(0, 5/2, 1/2)$ ✓

Notes If a line in the plan then plugging $(x(t), y(t), z(t))$ into eq^n always gives 7!

& If line is parallel to plane, we get another constant.

↳ More generally, we can use parametric eq^n for arbitrary motion in the plane or in space!

example cycloid



cycloid:

Wheel of radius a rolling on floor = x-axis

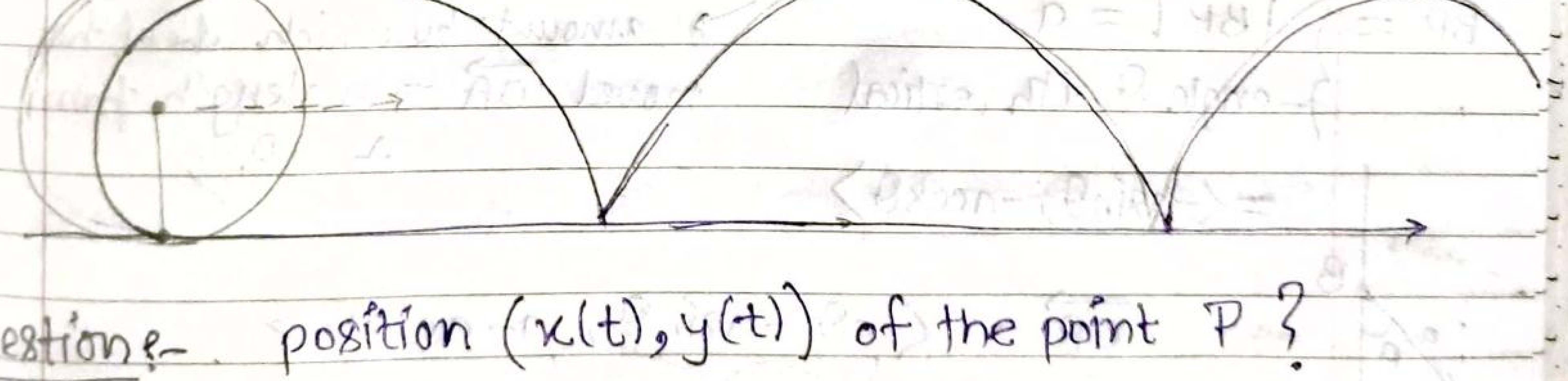
- We have point 'P', painted on the wheel.
- Initially, it's at the origin

But as time goes by, it moves on the wheel.

⇒ P is the point on the rim of wheel. starts at origin

In particular, can we find the position of this point. x(t), y(t), as a function of time?

Trajectory of cycloid-



Question:- position $(x(t), y(t))$ of the point P?

Sol: I am expressing the position in terms of time. Let's see, is time going to be a good thing to do? Well, suddenly, the position changes over time. But doesn't actually matter how fast the wheel is rolling? No; because trajectory ~~to~~ ~~is~~ will be same.

So, in fact, time is not most relevant thing here. Matter, is how far wheel has gone. So we can use

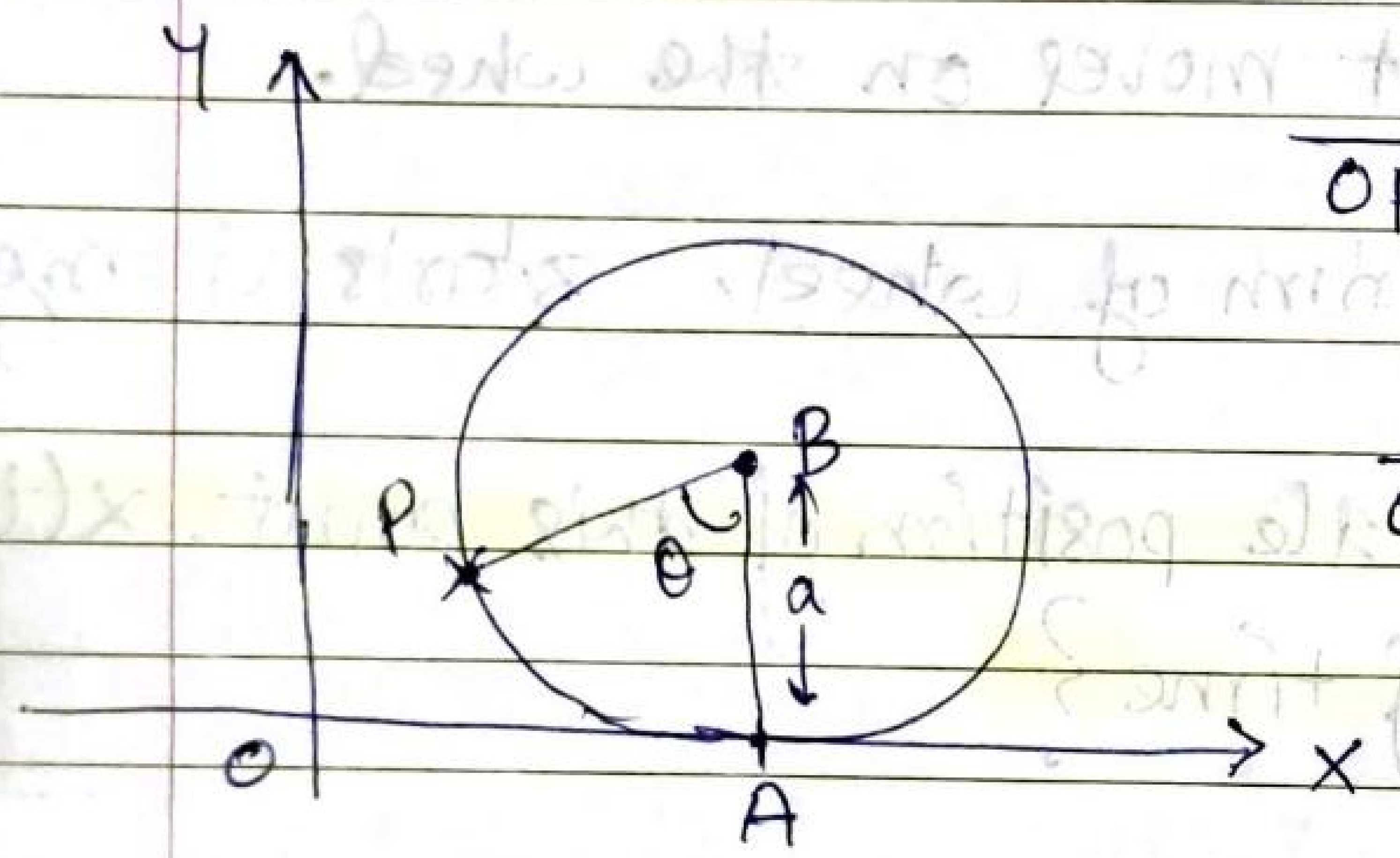
→ parameter, for example, the distance by which the wheel has moved.

We can ~~even~~ do even better because we see that, really, the most complicated thing that happen here is really the rotation.

So, maybe we can ~~use~~ actually use the angle by which the wheel has turned to parameterized the motion.

$$(x(\theta), y(\theta))$$

as a f^n of the angle by which wheel has rotated.



$$\overrightarrow{OP} = \underbrace{\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}}$$

$\overrightarrow{OA} = \langle a\theta, 0 \rangle$

amount

two components

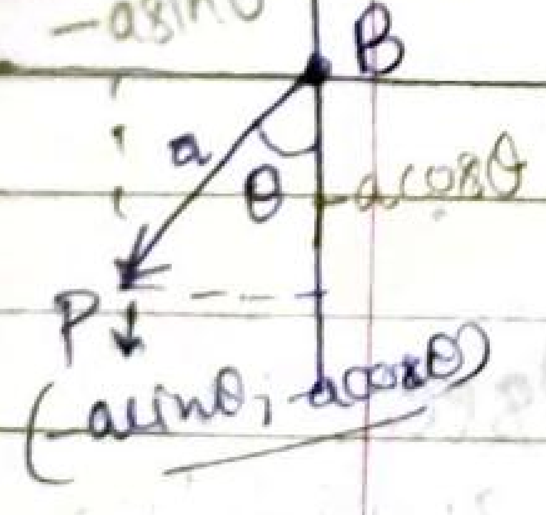
↳ y will be 0, because its directed along x-axis

amount by which wheel has moved $\overrightarrow{OA}$ = arclength from A to P.

$\overrightarrow{AB} = \langle 0, a \rangle$

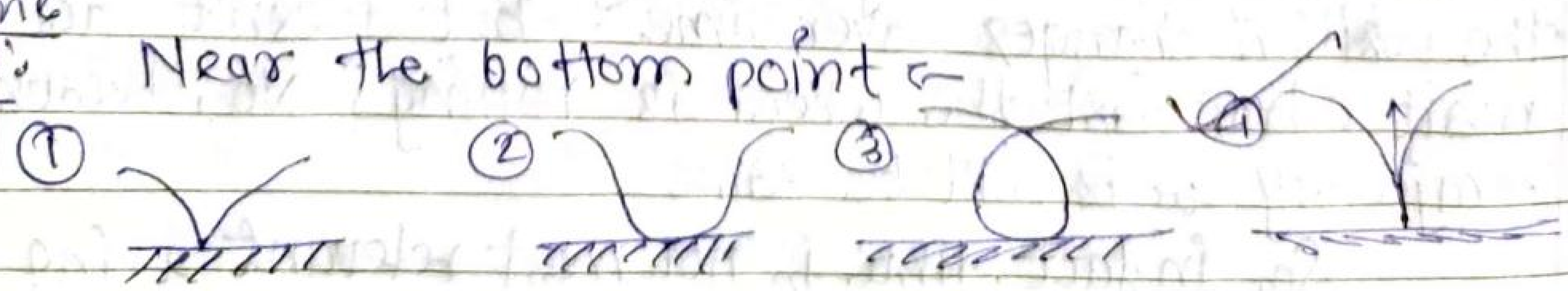
$\overrightarrow{BP} = \begin{cases} |\overrightarrow{BP}| = a \\ \text{angle } \theta \text{ with vertical} \end{cases}$

$= \langle -a\sin\theta, -a\cos\theta \rangle$



$$\overrightarrow{OP} = \langle \underbrace{a\theta - a\sin\theta}_{x(\theta)}, \underbrace{a - a\cos\theta}_{y(\theta)} \rangle$$

Questions
Note: Near the bottom point –



let's try to figure out from formula

$$\vec{OP} = \langle (a\theta - a\sin\theta),\ (a - a\cos\theta) \rangle$$

$x(\theta)$ $\qquad\qquad$ $y(\theta)$.

Q: what happens near the bottom point?

A:

take length unit = radius $\therefore a = 1$;

$$\begin{cases} x(\theta) = \theta - \sin\theta \\ y(\theta) = 1 - \cos\theta \end{cases}$$

$\sin(\theta) \sim \theta$ ← very small angle.

$\cos(\theta) \sim 1$ ← for very small

Better approximation?

Taylor's expansion:-

~~for~~ if t is small, then value of f^n

$$\boxed{f(t) \simeq f(0)} + t f'(0) + \frac{t^2}{2} f''(0) + \frac{t^3}{6} f'''(0)$$

⟶ to be more precise.

$\sin(\theta) \simeq \theta - \frac{\theta^3}{6}$ $\qquad$ $\cos(\theta) \simeq 1 - \frac{\theta^2}{2}$

$$x(\theta) \simeq \theta - \left(\theta - \frac{\theta^3}{6}\right) \simeq \theta^3/6$$

$$y(\theta) \simeq 1 - \left(1 - \frac{\theta^2}{2}\right) \simeq \theta^2/2 \quad \checkmark \text{ is large}$$

$$\boxed{|x| << |y|}$$

$$\frac{y}{x} \simeq \frac{\theta^2/2}{\theta^3/6} = \frac{3}{\theta} \to \infty \quad \text{when } \theta \to 0$$

Slope at origin is ∞.