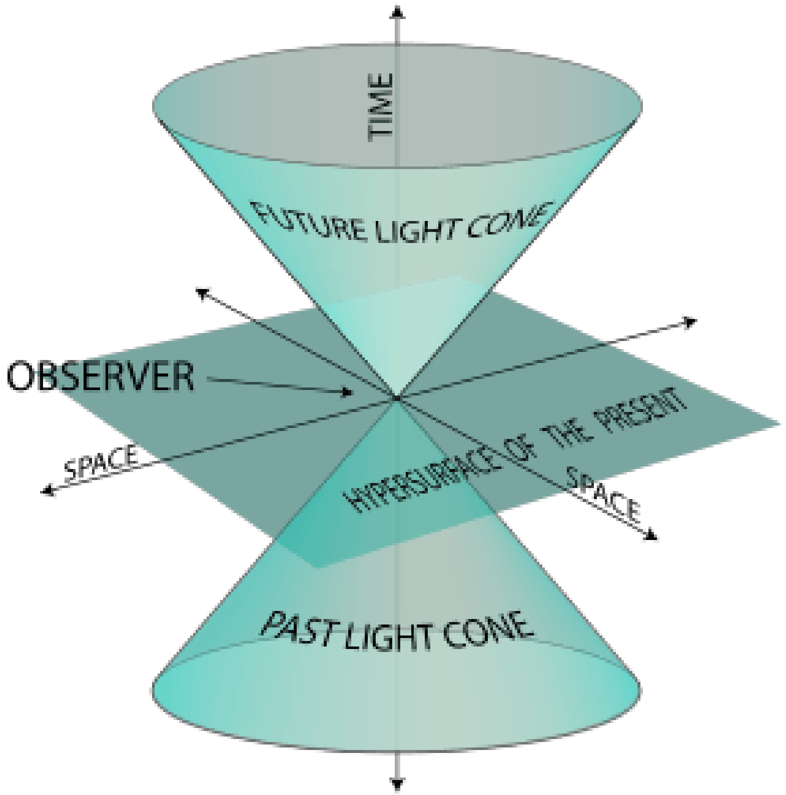
Cross Scholarship Research

Analogs of the Schwarzschild Metric in Flat Space-Time.



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Content:

General Relativity and EFE

General Theory of Relativity: Geometric theory of gravitation.

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Einstein's field equation. The terms denote the Ricci Curvature tensor, the Ricci scalar, the metric tensor, the cosmological constant, Gravitational constant and stress-energy tensor.

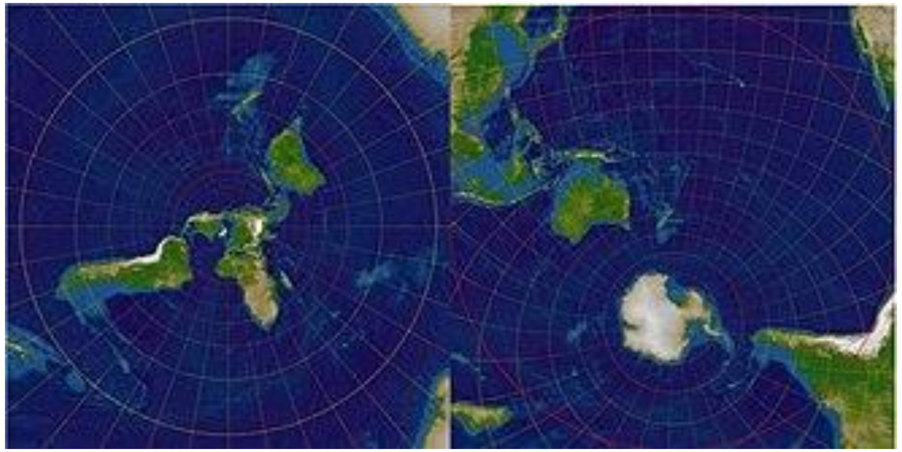
Assuming the cosmological constant to be zero, the field equation in vacuum given by:

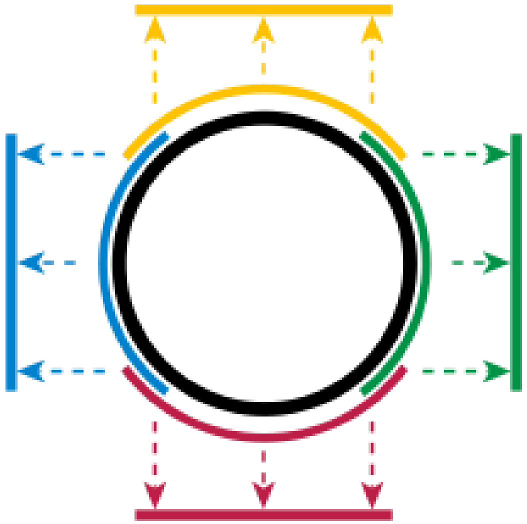
$$R_{\mu\nu} = 0$$

Manifolds and Metrics

Manifolds

- Topological space that locally resembles Euclidean space.





- Each point has a neighbourhood that is homoeomorphic to the Euclidean space of the same dimension.

Manifolds and Metrics

Pseudo-Riemannian manifold

- A Riemannian manifold is a differentiable manifold with a Riemannian metric.

- A Pseudo-Riemannian manifold is a differentiable manifold with a metric tensor that is nondegenerate throughout it.

Pseudo-Euclidean space

- A Pseudo-Euclidean space is a finite-dimensional real n-space together with a non-degenerate quadratic form q.
- Unlike Euclidean space, there can exist vectors with negative magnitudes. Minkowski space is an example with n=4 and k=3.

Manifolds and Metrics

Metric Tensors

- Tensors are mathematical objects that obey certain transformation rules and contain information on the relationship between objects and their vector spaces.

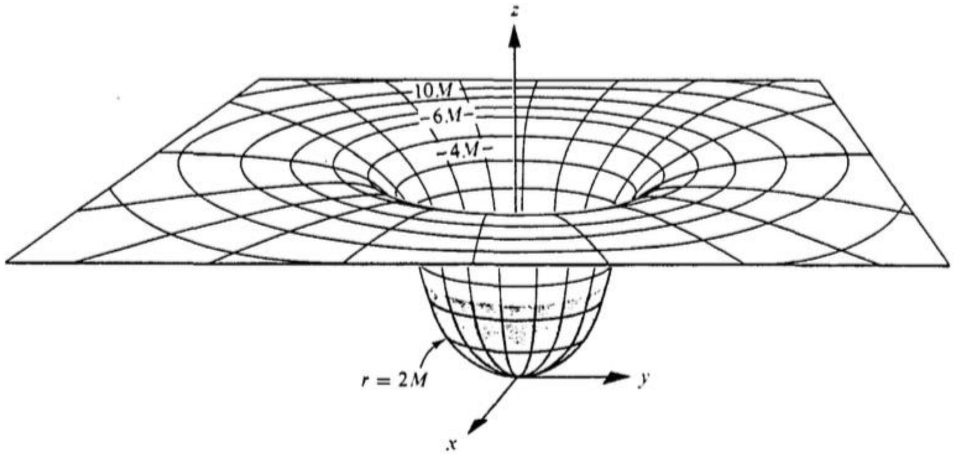
- A metric tensor is a covariant, second-degree, symmetric tensor on a differentiable manifold M that varies from point to point.

- Example of a metric *(t, x, y, x)* in a Lorentzian manifold.

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$\begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Manifolds and Metrics



The Schwarzschild Metric

- The German physicist Karl Schwarzschild was the first to find an exact solution to Einstein’s field equations.
- Solution to the field equations that describe the gravitational field outside a spherical mass given some parameters (charge, angular momentum, and universal cosmological constant =0).
- Useful approximation for slow-rotating objects.

The Schwarzschild Metric

$$g = -c^2\, d\tau^2 = -\left(1 - \frac{r_s}{r}\right) c^2\, dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 g_\Omega$$

With the last term $g_\Omega = \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$ being a metric on a sphere.

- Definition of terms; *c* is the speed of light, *t* is the time coordinate, *r* is the radial coordinate, Φ is the longitude of the point, θ is the colatitude, and r_s is the Schwarzschild radius.

Christoffel symbols

Christoffel symbols are associated with the differentiation of vector fields in a Riemannian or a pseudo Riemannian manifold M, called the Levi-Civita connection, which is is a generalization of the covariant derivative of vector fields in the Euclidean space. Locally the Levi-civita connection is defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

[1]

And the Christoffel symbol is given by

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell} g^{k\ell} \left\{ \frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{\ell i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right\}$$

Reimann Curvature Tensor

Given the Christoffel symbol, the Riemann curvature tensor is locally given by

$$R^{\ell}_{ijk} = \frac{\partial}{\partial x^{j}}\Gamma^{\ell}_{ik} - \frac{\partial}{\partial x^{k}}\Gamma^{\ell}_{ij} + \sum_{p}\left\{\Gamma^{\ell}_{jp}\Gamma^{p}_{ik} - \Gamma^{\ell}_{kp}\Gamma^{p}_{ij}\right\}$$ [1]

The Riemann curvature tensor can then be used to obtain both the Ricci Curvature tensor, the Ricci scalar.

$$\mathrm{Ric}_{p}\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right) = \sum_{k} R^{k}_{ikj}$$

$$\mathrm{Scal}(p) = \sum_{i} g^{ii} R_{ii}$$

Flow-chart for summary

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

[2]

To show that a metric is an exact solution to the EFE.

1) Find the Christoffel symbols.
2) Use that to find the Reimann Curvature tensor.
3) Use that to find the Ricci Curvature tensor and scalar.
4) Establish values for the stress-energy tensor and solve the EFE.

Euclidean Space-time

Assuming that the speed of light and the gravitational constant = 1, the Schwarzschild metric in Minkowski 4-spacetime is given by

$$-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2d\Omega^2$$

Where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the unit sphere S^2

This metric is a spherically symmetric asymptotically flat solution of the vacuum Einstein equation.

Euclidean Space-time

Using Wick's rotation for $t \rightarrow it$ we get a metric in Euclidean 4-spacetime.

$$\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

We prove that this metric too is a spherically symmetric asymptotically flat solution of the vacuum Einstein equation in a Euclidean 4-spacetime.

Pseudo-Euclidean Space-time

We denote $\mathbb{R}^4$ as an pseudo-Euclidean 4-spacetime with coordinates *(t,x,y,z)* and the indefinite metric

$$-dt^2 - dx^2 + dy^2 + dz^2$$

This is a 4-dimensional flat space-time with two time coordinates. It is obvious that our space-time does not admit spherical symmetry, we seek to show that it does however admit hyperbolic symmetry.

Pseudo-Euclidean Space-time

We use the following transformations;

$$x = \rho \cosh \tau$$
$$y = \rho \sinh \tau \cos \theta$$
$$z = \rho \sinh \tau \sin \theta$$

To derive our metric on the hyperbola as;

$$-d\rho^2 + \rho^2 d\tau^2 + \rho^2 \sinh^2 \tau d\theta^2$$

Pseudo-Euclidean Space-time

Given the following metric, we are currently trying to see if our space-time admits an analog of the Schwarzschild metric, i.e. a static asymptotically flat solution of the vacuum Einstein with hyperbolic symmetry.

$$ds^2 = -A(\rho)dt^2 - B(\rho)d\tau^2 + \rho^2[d\tau^2 + \sinh^2\tau d\theta^2]$$

Where, $\lim_{\rho\to\infty} A(\rho) = \lim_{\rho\to\infty} B(\rho) = 1$

Previous Project

- Schwarzschild Black Hole in De Sitter Space-time

$$R_{1,1} = -\frac{A'(r)B'(r)}{4e^{2ct}B(r)^2} + \frac{A''(r)}{2e^{2ct}B(r)} - \frac{(A'(r))^2}{4e^{2ct}A(r)B(r)} - 3c^2 + \frac{A'(r)}{re^{2ct}B(r)}$$

$$R_{1,2} = \frac{cA'(r)}{A(r)}$$

$$R_{2,1} = \frac{cA'(r)}{A(r)}$$

$$R_{2,2} = \frac{3c^2e^{2ct}B(r)}{A(r)} + \frac{(A'(r))^2}{4A(r)^2} - \frac{A''(r)}{2A(r)} + \frac{A'(r)B'(r)}{4A(r)B(r)} + \frac{B'(r)}{rB(r)}$$

$$R_{3,3} = \frac{3c^2e^{2ct}r^2}{A(r)} - \frac{(A(r)r}{2A(r)B(r)} + \frac{rB'(r)}{B(r)^2} + 1 - \frac{1}{B(r)}$$

$$R_{4,4} = \frac{3c^2e^{2ct}r^2\sin(\theta)^2}{A(r)} - \frac{A'(r)r\sin(\theta)^2}{2A(r)B(r)} + \frac{r\sin(\theta)^2B'(r)}{2B(r)^2} + \sin(\theta)^2 - \frac{\sin(\theta)^2}{B(r)}$$

Citations

- [1] The Curvature, the Einstein Equations, and the Black Hole https://www.math.usm.edu/lee/curvature/index.html
- [2] Images Used https://www.markushanke.net/schwarzschild-spacetime-and-black-holes/
- [3] Derivation of the metric https://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf

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Questions?