Mechanical Systems and Signal Processing (1991) 5(2), 119-137

A COMPLEX MODAL TESTING THEORY FOR ROTATING MACHINERY

CHONG-WON LEE

Department of Mechanical Engineering, Korea Advanced Institute of Science and Technology, P.O. Box 150, Chongryangri, Seoul, Korea

(*Received* 1 *September* 1989, *accepted* 1 *June* 1990)

A new modal testing method, termed complex modal testing, is proposed for modal parameter identification of rotating machinery and compared with the classical method. The proposed method is essentially based on the use of complex notation which has been known to be a powerful mathematical tool in the analysis of, particularly isotropic, rotor systems. Complex modal testing not only allows clear physical insight into the backward and forward modes, but also enables the separation of those modes in the frequency domain so that effective modal parameter identification is possible. In addition, it does not require additional testing effort to obtain the adjoint mode shapes of isotropic rotors which are characterised by non-self-adjointness.

1. INTRODUCTION

Rotors, unlike other stationary structures, exhibit a peculiar phenomenon in their modal characteristics. As a rotor starts rotating, two different kinds of modes known as the backward and forward modes are observed. Although the presence of backward and forward modes in rotor dynamics has been extensively investigated in the literature, the directivity of the modes is often neglected in the usual (real) formulations in the dynamic analysis of rotors. In particular, the nature of the modes is completely veiled when one relies on the numerical analysis of complicated rotor systems, calculating only the natural frequencies or critical speeds, and ignoring other important modal parameters such as the modal vectors and the adjoint modal vectors.

In order to gain physical understanding of the modal characteristics of rotors, an analytical modal analysis, which takes into account the non-self-adjointness of distributed parameter rotor-bearing systems, has recently been developed [1-3]. This approach adopts the complex notation in the dynamic analysis of rotors. The modal analysis emphasises, based on a rigorous mathematical treatment, the importance of the directivity, the mode shapes and the adjoint of the backward and forward modes in understanding rotor dynamics. The complex modal testing method proposed in this paper stems essentially from the analytical results in [1-3].

The classical modal testing method has been widely and successfully used for modal parameter identification of structures of all kinds, except rotating machinery. Although a few attempts [4, 5] have recently been made to develop a modal testing method for rotating machinery, they are not different, at least in theory, from the classical method which completely ignores the directivity of a mode. Application of classical modal testing to rotating machinery therefore has resulted in the heavy overlapping of the backward and forward modes in the frequency domain. In the worst case, the non-self-adjointness of rotating systems requires a series of modal tests corresponding to one row plus one

0888-3270/91/020119+19 \$03.00

column of the frequency response matrix (FRM) in order to identify the adjoint modal parameters as well. This is, in practice, quite a burden to test engineers.

The complex modal testing theory developed in this paper gives not only the directivity of the backward and forward modes, but also completely separates those modes in the frequency domain so that effective modal parameter identification is possible. In addition, the necessity of additional tests to identify the adjoint modal parameters is relaxed under some practical conditions, requiring modal tests for only one row or column of the FRM as in the case of classical self-adjoint dynamic systems.

2. MODAL ANALYSIS OF MULTI-DEGREE OF FREEDOM ROTOR-BEARING SYSTEMS

2.1. USE OF REAL NOTATION—THE CLASSICAL APPROACH

The equation of motion of a multi-dof rotor-bearing system may be written as [6, 7]

$$\mathbf{M}\ddot{\mathbf{q}}+(\mathbf{G}+\mathbf{D})\dot{\mathbf{q}}+(\mathbf{S}+\mathbf{H})\mathbf{q}=\mathbf{f}(t) \tag{1}$$

where the positive definite, but not necessarily diagonal, matrix $\mathbf{M}$ is known as the mass(inertia) matrix, the skew symmetric matrices $\mathbf{G}$ and $\mathbf{H}$ are referred to as the gyroscopic and the circulatory matrices, respectively, and the indefinite non-symmetric matrices $\mathbf{D}$ and $\mathbf{S}$ are called the damping and the stiffness matrices, respectively. The displacement and force vectors consist of

$$\mathbf{q}(t)=\begin{Bmatrix}\mathbf{y}(t)\\ \mathbf{z}(t)\end{Bmatrix}, \qquad \mathbf{f}(t)=\begin{Bmatrix}\mathbf{f}_y(t)\\ \mathbf{f}_z(t)\end{Bmatrix} \tag{2}$$

where $\mathbf{y}$ and $\mathbf{f}_y$ are the $N\times 1$ y-directional displacement and force vectors, N being the dimension of the directional co-ordinate vectors, and $\mathbf{z}$ and $\mathbf{f}_z$ are the $N\times 1$ z-directional displacement and force vectors. The $2N\times 2N$ matrices $\mathbf{M}$, $\mathbf{G}$, $\mathbf{D}$, $\mathbf{S}$ and $\mathbf{H}$ are in general rotational speed, Ω, dependent [8]. For Ω given, equation (1) can be written as

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{f}(t) \tag{3}$$

where the generalised damping and stiffness matrices, $\mathbf{C}$ and $\mathbf{K}$, are now neither positive (negative) definite nor symmetric. It is assumed here, however, that $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are real matrices, without loss of generality.

The equation of motion of a $2N$ dof system (3) can be written in state space form as [1-3, 9, 10]

$$\mathbf{A}\dot{\mathbf{w}}=\mathbf{B}\mathbf{w}+\mathbf{F} \tag{4}$$

where

$$\mathbf{A}=\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}, \qquad \mathbf{B}=\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & -\mathbf{K}\end{bmatrix}, \qquad \mathbf{w}=\begin{Bmatrix}\dot{\mathbf{q}}\\ \mathbf{q}\end{Bmatrix}, \qquad \mathbf{F}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{f}\end{Bmatrix}.$$

The $4N\times 4N$ matrices $\mathbf{A}$ and $\mathbf{B}$ are again real but indefinite, non-symmetric in general, resulting in a non-self-adjoint eigenvalue problem. The eigenvalue problem associated with equation (4) and its adjoint are then given by

$$\begin{aligned}\lambda_r^i\mathbf{A}\mathbf{r}_r^i&=\mathbf{B}\mathbf{r}_r^i, \qquad r, s=\pm 1, \pm 2, \ldots, \pm N\\ \bar{\lambda}_s^i\mathbf{A}'\mathbf{l}_s^i&=\mathbf{B}'\mathbf{l}_s^i, \qquad i=B, F\end{aligned} \tag{5}$$

where the bar indicates the complex conjugate, the prime denotes the transpose, and the superscripts B and F implicitly refer to the backward and forward modes. The modes associated with positive and negative integer subscripts form complex conjugate pairs.

In equation (5), the (right) eigenvectors, $\mathbf{r}$, and the adjoint (complex conjugate left) eigenvectors, $\mathbf{l}$, are composed as

$$\mathbf{r}_r^i = \begin{Bmatrix} \lambda \mathbf{u} \\ \mathbf{u} \end{Bmatrix}_r^i, \qquad \mathbf{l}_s^i = \begin{Bmatrix} \bar{\lambda} \mathbf{v} \\ \mathbf{v} \end{Bmatrix}_s^i \tag{6}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the modal and the adjoint modal vectors corresponding to the $4N$ eigenvalues λ determined from the characteristic(frequency) polynomial of order $4N$

$$|\lambda \mathbf{A} - \mathbf{B}| = 0 \quad \text{or} \quad |\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}| = 0. \tag{7}$$

The modal and the adjoint vectors may be directly obtained from

$$(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K})\mathbf{u} = \mathbf{0} \quad \text{and} \quad \bar{\mathbf{v}}'(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}) = \mathbf{0}'. \tag{8}$$

The eigenvectors or modal vectors and the adjoint eigenvectors may be biorthonormalised so as to satisfy

$$\begin{aligned} \bar{\mathbf{l}}_s^{k\prime} \mathbf{A} \mathbf{r}_r^i &= \delta_{rs}^{ik} \qquad i, k = B, F \\ \bar{\mathbf{l}}_s^{k\prime} \mathbf{B} \mathbf{r}_r^i &= \lambda_r^i \delta_{rs}^{ik} \qquad r, s = \pm 1, \pm 2, \ldots, \pm N \end{aligned} \tag{9a}$$

or

$$\begin{aligned} (\lambda_r^i + \lambda_s^k) \bar{\mathbf{v}}_s^{k\prime} \mathbf{M} \mathbf{u}_r^i + \bar{\mathbf{v}}_s^{k\prime} \mathbf{C} \mathbf{u}_r^i &= \delta_{rs}^{ik} \\ \lambda_r^i \lambda_s^k \bar{\mathbf{v}}_s^{k\prime} \mathbf{M} \mathbf{u}_r^i - \bar{\mathbf{v}}_s^{k\prime} \mathbf{K} \mathbf{u}_r^i &= \lambda_r^i \delta_{rs}^{ik} \end{aligned} \tag{9b}$$

where the Kronecker delta is defined as

$$\delta_{rs}^{ik} = \begin{cases} 1; & \text{when} \quad i = k \quad \text{and} \quad r = s \\ 0; & \text{otherwise} \end{cases}. \tag{10}$$

The $4N \times 1$ state vector $\mathbf{w}(t)$ can be expanded in terms of the eigenvectors as

$$\mathbf{w}(t) = \sum_{i=B,F} \sum_{r=-N}^{N}{}' \{\mathbf{r}\eta(t)\}_r^i \tag{11}$$

where the prime notation in the summation implies the exclusion of $r = 0$ and $\eta(t)$ are the principal co-ordinates. Substitution of equation (11) into equation (4) and use of the biorthonormality conditions (9) yield the $4N$ sets of modal equations of motion

$$\dot{\eta}_r^i = \lambda_r^i \eta_r^i + \bar{\mathbf{v}}_r^{i\prime} \mathbf{f}, \qquad i = B, F; \; r = \pm 1, \pm 2, \ldots, \pm N. \tag{12}$$

The forced response of the system (3) then becomes

$$\mathbf{q}(t) = \sum_{i=B,F} \sum_{r=-N}^{N}{}' \{\mathbf{u}\eta(t)\}_r^i = \sum_{i=B,F} \sum_{r=1}^{N} \{\mathbf{u}\eta(t) + \bar{\mathbf{u}}\bar{\eta}(t)\}_r^i. \tag{13}$$

Introducing the Laplace transforms $\mathbf{Q}(s)$ and $\mathbf{F}(s)$ of the vectors $\mathbf{q}(t)$ and $\mathbf{f}(t)$, equations (12) and (13) may be transformed into, s being the Laplace variable,

$$\mathbf{Q}(s) = \mathbf{H}(s)\mathbf{F}(s)$$

or (14)

$$\begin{Bmatrix} \mathbf{Y}(s) \\ \mathbf{Z}(s) \end{Bmatrix} = \begin{bmatrix} \mathbf{H}_{yy}(s) & \mathbf{H}_{yz}(s) \\ \mathbf{H}_{zy}(s) & \mathbf{H}_{zz}(s) \end{bmatrix} \begin{Bmatrix} \mathbf{F}_y(s) \\ \mathbf{F}_z(s) \end{Bmatrix}$$

where $\mathbf{Y}(s)$, $\mathbf{Z}(s)$, $\mathbf{F}_y(s)$ and $\mathbf{F}_z(s)$ are the Laplace transforms of $\mathbf{y}(t)$, $\mathbf{z}(t)$, $\mathbf{f}_y(t)$ and $\mathbf{f}_z(t)$, respectively. The FRM can then be written as, by replacing s in the transfer matrix $\mathbf{H}(s)$ by $j\omega$,

$$\mathbf{H}(j\omega) = \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}\bar{\mathbf{v}}'}{j\omega - \lambda} + \frac{\bar{\mathbf{u}}\mathbf{v}'}{j\omega - \bar{\lambda}} \right]_r^i . \tag{15}$$

By introducing the definitions

$$\mathbf{u}_r^i = \begin{Bmatrix} \mathbf{u}_y \\ \mathbf{u}_z \end{Bmatrix}_r^i, \qquad \mathbf{v}_r^i = \begin{Bmatrix} \mathbf{v}_y \\ \mathbf{v}_z \end{Bmatrix}_r^i \tag{16}$$

the partitioned block FRMs can be expressed from equation (15) as

$$\begin{aligned}
\mathbf{H}_{yy}(j\omega) &= \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}_y\bar{\mathbf{v}}_y'}{j\omega - \lambda} + \frac{\bar{\mathbf{u}}_y\mathbf{v}_y'}{j\omega - \bar{\lambda}} \right]_r^i \\
\mathbf{H}_{yz}(j\omega) &= \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}_y\bar{\mathbf{v}}_z'}{j\omega - \lambda} + \frac{\bar{\mathbf{u}}_y\mathbf{v}_z'}{j\omega - \bar{\lambda}} \right]_r^i \\
\mathbf{H}_{zy}(j\omega) &= \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}_z\bar{\mathbf{v}}_y'}{j\omega - \lambda} + \frac{\bar{\mathbf{u}}_z\mathbf{v}_y'}{j\omega - \bar{\lambda}} \right]_r^i \\
\mathbf{H}_{zz}(j\omega) &= \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}_z\bar{\mathbf{v}}_z'}{j\omega - \lambda} + \frac{\bar{\mathbf{u}}_z\mathbf{v}_z'}{j\omega - \bar{\lambda}} \right]_r^i
\end{aligned} \tag{17}$$

In the above expressions, $\mathbf{u}_y$ and $\mathbf{u}_z$ are the modal vectors in the y and z directions and $\mathbf{v}_y$ and $\mathbf{v}_z$ are the corresponding adjoint modal vectors. From equation (15) or (17), one can see that the FRM also contains the information on adjoint modal vector as well as natural frequency, damping ratio and modal vector. Each column of the numerator contains the same modal vector multiplied by a component of the adjoint modal vector whereas each row contains the same adjoint modal vector multiplied by a component of the modal vector. In other words, the modal information is completely identified from a single row as well as a single column arbitrarily chosen from the FRM except those corresponding to components of the modal vector and the adjoint which are zero. Therefore, one row plus one column of the FRM needs to be measured in order to identify all the modal parameters of a rotor. From equation (17), it holds

$$\mathbf{H}_{ik}(-j\omega) = \bar{\mathbf{H}}_{ik}(j\omega), \qquad i, k = y, z \tag{18}$$

which implies that a frequency response function (FRF) over the negative frequency region, when written in real expressions, is merely a duplicate of that over the positive frequency region and vice versa. This is the reason why all modal testing schemes developed so far essentially deal only with one-sided, conventionally over the positive frequency region, FRFs. However, as equation (18) indicates, a $2N \times 2N$ FRM in non-self-adjoint vibratory systems such as rotor systems requires thorough knowledge of four one-sided partitioned block FRMs of dimension $N \times N$. The worst thing in dealing with equation (18) is that, since the forward and backward modes are not distinguishable in the frequency domain, the frequency response characteristics of those two physically different modes are mixed together over the one-sided frequency region, resulting in heavy overlapping of the otherwise physically well separated modes. This fact will be further discussed in detail in the next sections.

2.2. USE OF COMPLEX NOTATION—THE NEW APPROACH

In rotor dynamic analysis, it is often convenient to introduce the complex notation $\mathbf{p}=\mathbf{y}+j\mathbf{z}$ and $\mathbf{g}=\mathbf{f}_y+j\mathbf{f}_z$. Then the equation of motion (3) can be rewritten as [11]

$$\mathbf{M}_f\ddot{\mathbf{p}}+\mathbf{M}_b\ddot{\bar{\mathbf{p}}}+\mathbf{C}_f\dot{\mathbf{p}}+\mathbf{C}_b\dot{\bar{\mathbf{p}}}+\mathbf{K}_f\mathbf{p}+\mathbf{K}_b\bar{\mathbf{p}}=\mathbf{g} \tag{19}$$

where

$$2\mathbf{M}_f=(\mathbf{M}_{yy}+\mathbf{M}_{zz})-j(\mathbf{M}_{yz}-\mathbf{M}_{zy})$$
$$2\mathbf{M}_b=(\mathbf{M}_{yy}-\mathbf{M}_{zz})+j(\mathbf{M}_{yz}+\mathbf{M}_{zy})$$
$$2\mathbf{C}_f=(\mathbf{C}_{yy}+\mathbf{C}_{zz})-j(\mathbf{C}_{yz}-\mathbf{C}_{zy})$$
$$2\mathbf{C}_b=(\mathbf{C}_{yy}-\mathbf{C}_{zz})+j(\mathbf{C}_{yz}+\mathbf{C}_{zy})$$
$$2\mathbf{K}_f=(\mathbf{K}_{yy}+\mathbf{K}_{zz})-j(\mathbf{K}_{yz}-\mathbf{K}_{zy})$$
$$2\mathbf{K}_b=(\mathbf{K}_{yy}-\mathbf{K}_{zz})+j(\mathbf{K}_{yz}+\mathbf{K}_{zy})$$

and

$$\mathbf{M}=\begin{bmatrix}\mathbf{M}_{yy} & \mathbf{M}_{yz}\\ \mathbf{M}_{zy} & \mathbf{M}_{zz}\end{bmatrix},\qquad \mathbf{C}=\begin{bmatrix}\mathbf{C}_{yy} & \mathbf{C}_{yz}\\ \mathbf{C}_{zy} & \mathbf{C}_{zz}\end{bmatrix},\qquad \mathbf{K}=\begin{bmatrix}\mathbf{K}_{yy} & \mathbf{K}_{yz}\\ \mathbf{K}_{zy} & \mathbf{K}_{zz}\end{bmatrix}.$$

The FRM in complex domain can easily be derived by using the relations, $\mathbf{P}(s)$, $\hat{\mathbf{P}}(s)$, $\mathbf{G}(s)$ and $\hat{\mathbf{G}}(s)$ being the Laplace transforms of $\mathbf{p}(t)$, $\bar{\mathbf{p}}(t)$, $\mathbf{g}(t)$ and $\bar{\mathbf{g}}(t)$, respectively,

$$\begin{Bmatrix}\mathbf{Y}(j\omega)\\ \mathbf{Z}(j\omega)\end{Bmatrix}=\frac{1}{2}\begin{bmatrix}1 & 1\\ -j & j\end{bmatrix}\begin{Bmatrix}\mathbf{P}(j\omega)\\ \hat{\mathbf{P}}(j\omega)\end{Bmatrix}$$
$$\begin{Bmatrix}\mathbf{F}_y(j\omega)\\ \mathbf{F}_z(j\omega)\end{Bmatrix}=\frac{1}{2}\begin{bmatrix}1 & 1\\ -j & j\end{bmatrix}\begin{Bmatrix}\mathbf{G}(j\omega)\\ \hat{\mathbf{G}}(j\omega)\end{Bmatrix} \tag{20}$$

as

$$\begin{Bmatrix}\mathbf{P}(j\omega)\\ \hat{\mathbf{P}}(j\omega)\end{Bmatrix}=\begin{bmatrix}\mathbf{H}_{pg}(j\omega) & \mathbf{H}_{p\hat{g}}(j\omega)\\ \mathbf{H}_{\hat{p}g}(j\omega) & \mathbf{H}_{\hat{p}\hat{g}}(j\omega)\end{bmatrix}\begin{Bmatrix}\mathbf{G}(j\omega)\\ \hat{\mathbf{G}}(j\omega)\end{Bmatrix} \tag{21}$$

where

$$2\mathbf{H}_{pg}(j\omega)=\mathbf{H}_{yy}+\mathbf{H}_{zz}-j(\mathbf{H}_{yz}-\mathbf{H}_{zy})$$
$$2\mathbf{H}_{p\hat{g}}(j\omega)=\mathbf{H}_{yy}-\mathbf{H}_{zz}+j(\mathbf{H}_{yz}+\mathbf{H}_{zy})$$
$$2\mathbf{H}_{\hat{p}g}(j\omega)=\mathbf{H}_{yy}-\mathbf{H}_{zz}-j(\mathbf{H}_{yz}+\mathbf{H}_{zy})$$
$$2\mathbf{H}_{\hat{p}\hat{g}}(j\omega)=\mathbf{H}_{yy}+\mathbf{H}_{zz}+j(\mathbf{H}_{yz}-\mathbf{H}_{zy}). \tag{22}$$

From equations (18) and (22), it can be concluded that

$$\mathbf{H}_{\hat{p}g}(j\omega)=\bar{\mathbf{H}}_{p\hat{g}}(-j\omega)$$
$$\mathbf{H}_{\hat{p}\hat{g}}(j\omega)=\bar{\mathbf{H}}_{pg}(-j\omega). \tag{23}$$

Therefore, in order for the modal testing to be complete, it is sufficient to consider two $N\times N$ block FRMs, i.e.

$$\mathbf{P}(j\omega)=[\mathbf{H}_{pg}(j\omega)\ \ \mathbf{H}_{p\hat{g}}(j\omega)]\begin{Bmatrix}\mathbf{G}(j\omega)\\ \hat{\mathbf{G}}(j\omega)\end{Bmatrix}. \tag{24}$$

However $\mathbf{H}_{pg}(j\omega)$ and $\mathbf{H}_{p\hat{g}}(j\omega)$, which do not satisfy a relation such as equation (18), should be considered over the two-sided frequency region. In equation (24), $\mathbf{H}_{pg}(j\omega)$ and $\mathbf{H}_{p\hat{g}}(j\omega)$ are the FRMs associated with the complex response vector, $\mathbf{p}(t)$, and a pair of complex excitation vectors, $\mathbf{g}(t)$ and $\bar{\mathbf{g}}(t)$, which may be represented as

$$\begin{aligned}\mathbf{H}_{pg}(j\omega) &= \frac{1}{2}\sum_{i=B,F}\sum_{r=1}^{N}\left[\frac{(\mathbf{u}_y+j\mathbf{u}_z)(\bar{\mathbf{v}}_y{}'-j\bar{\mathbf{v}}_z{}')}{j\omega-\lambda}+\frac{(\bar{\mathbf{u}}_y+j\bar{\mathbf{u}}_z)(\mathbf{v}_y{}'-j\mathbf{v}_z{}')}{j\omega-\bar{\lambda}}\right]_r^i\\ \mathbf{H}_{p\hat{g}}(j\omega) &= \frac{1}{2}\sum_{i=B,F}\sum_{r=1}^{N}\left[\frac{(\mathbf{u}_y+j\mathbf{u}_z)(\bar{\mathbf{v}}_y{}'+j\bar{\mathbf{v}}_z{}')}{j\omega-\lambda}+\frac{(\bar{\mathbf{u}}_y+j\bar{\mathbf{u}}_z)(\mathbf{v}_y{}'+j\mathbf{v}_z{}')}{j\omega-\bar{\lambda}}\right]_r^i.\end{aligned} \tag{25}$$

The advantages of using equation (25) instead of equation (17) will be discussed in the next sections.

3. MODAL PROPERTIES OF DISCRETE ISOTROPIC ROTORS

3.1. ISOTROPIC ROTORS

Isotropic rotors may be defined as the rotors whose response directions are always coincident with the excitation directions [2, 11]. The equation of motion of an anisotropic rotor (19) becomes that of an isotropic rotor when

$$\mathbf{M}_b = \mathbf{C}_b = \mathbf{K}_b = \mathbf{0} \tag{26}$$

or

$$\begin{aligned}\mathbf{M}_{yy} &= \mathbf{M}_{zz} = \mathbf{M}_1, &\quad \mathbf{M}_{yz} &= -\mathbf{M}_{zy} = \mathbf{M}_2\\ \mathbf{C}_{yy} &= \mathbf{C}_{zz} = \mathbf{C}_1, &\quad \mathbf{C}_{yz} &= -\mathbf{C}_{zy} = \mathbf{C}_2\\ \mathbf{K}_{yy} &= \mathbf{K}_{zz} = \mathbf{K}_1, &\quad \mathbf{K}_{yz} &= -\mathbf{K}_{zy} = \mathbf{K}_2.\end{aligned}$$

Substitution of equation (26) into equation (19) yields

$$\mathbf{M}_c\ddot{\mathbf{p}} + \mathbf{C}_c\dot{\mathbf{p}} + \mathbf{K}_c\mathbf{p} = \mathbf{g} \tag{27a}$$

where

$$\mathbf{M}_c = \mathbf{M}_1 - j\mathbf{M}_2, \qquad \mathbf{C}_c = \mathbf{C}_1 - j\mathbf{C}_2, \qquad \mathbf{K}_c = \mathbf{K}_1 - j\mathbf{K}_2$$

or, in state space form,

$$\mathbf{A}_c\dot{\mathbf{w}}_c = \mathbf{B}_c\mathbf{w}_c + \mathbf{F}_c \tag{27b}$$

where

$$\mathbf{A}_c = \begin{bmatrix}\mathbf{0} & \mathbf{M}_c\\ \mathbf{M}_c & \mathbf{C}_c\end{bmatrix}, \qquad \mathbf{B}_c = \begin{bmatrix}\mathbf{M}_c & \mathbf{0}\\ \mathbf{0} & -\mathbf{K}_c\end{bmatrix}, \qquad \mathbf{w}_c = \begin{Bmatrix}\dot{\mathbf{p}}\\ \mathbf{p}\end{Bmatrix}, \qquad \mathbf{F}_c = \begin{Bmatrix}\mathbf{0}\\ \mathbf{g}\end{Bmatrix}.$$

The $2N \times 2N$ complex matrices $\mathbf{A}_c$ and $\mathbf{B}_c$ are indefinite, non-Hermitian in general, resulting in a non-self-adjoint eigenvalue problem. The eigenvalue problem associated with equation (27b) and the adjoint are given, in a similar way to the procedure in the previous section, by

$$\begin{aligned}\lambda_r^i\mathbf{A}_c\mathbf{r}_{cr}^i &= \mathbf{B}_c\mathbf{r}_{cr}^i, &\qquad r, s &= 1, 2, \ldots, N\\ \bar{\lambda}_s^i\bar{\mathbf{A}}_c{}'\mathbf{l}_{cs}^i &= \bar{\mathbf{B}}_c{}'\mathbf{l}_{cs}^i, &\qquad i &= B, F\end{aligned} \tag{28}$$

where the (right) eigenvectors, $\mathbf{r}_c$, and the adjoint(complex conjugate left) eigenvectors, $\mathbf{l}_c$, are composed as

$$\mathbf{r}_{cr}^i = \begin{Bmatrix}\lambda\mathbf{u}_c\\ \mathbf{u}_c\end{Bmatrix}_r^i, \qquad \mathbf{l}_{cs}^i = \begin{Bmatrix}\bar{\lambda}\mathbf{v}_c\\ \mathbf{v}_c\end{Bmatrix}_s^i \tag{29}$$

where $\mathbf{u}_c$ and $\mathbf{v}_c$ are the complex modal and the adjoint vectors corresponding to the $2N$ eigenvalues λ determined from

$$|\lambda \mathbf{A}_c - \mathbf{B}_c| = 0 \quad \text{or} \quad |\lambda^2 \mathbf{M}_c + \lambda \mathbf{C}_c + \mathbf{K}_c| = 0. \tag{30}$$

The complex modal and the adjoint vectors may be directly obtained from

$$(\lambda^2 \mathbf{M}_c + \lambda \mathbf{C}_c + \mathbf{K}_c)\mathbf{u}_c = \mathbf{0} \quad \text{and} \quad \bar{\mathbf{v}}_c'(\lambda^2 \mathbf{M}_c + \lambda \mathbf{C}_c + \mathbf{K}_c) = \mathbf{0}'. \tag{31}$$

The eigenvectors(modal vectors) and the adjoint vectors(modal vectors) may be biorthonormalised so as to satisfy

$$\begin{aligned} \bar{\mathbf{l}}_{cs}^{k\prime} \mathbf{A}_c \mathbf{r}_{cr}^i &= \delta_{rs}^{ik}, \qquad i, k = B, F \\ \bar{\mathbf{l}}_{cs}^{k\prime} \mathbf{B}_c \mathbf{r}_{cr}^i &= \lambda_r^i \delta_{rs}^{ik}, \qquad r, s = 1, 2, 3, \ldots, N \end{aligned} \tag{32a}$$

or

$$\begin{aligned} (\lambda_r^i + \lambda_s^k)\bar{\mathbf{v}}_{cs}^{k\prime} \mathbf{M}_c \mathbf{u}_{cr}^i + \bar{\mathbf{v}}_{cs}^{k\prime} \mathbf{C}_c \mathbf{u}_{cr}^i &= \delta_{rs}^{ik} \\ \lambda_r^i \lambda_s^k \bar{\mathbf{v}}_{cs}^{k\prime} \mathbf{M}_c \mathbf{u}_{cr}^i - \bar{\mathbf{v}}_{cs}^{k\prime} \mathbf{K}_c \mathbf{u}_{cr}^i &= \lambda_r^i \delta_{rs}^{ik}. \end{aligned} \tag{32b}$$

The complex state vector $\mathbf{w}_c(t)$ can be expanded in terms of the eigenvectors as

$$\mathbf{w}_c(t) = \sum_{i=B,\,F} \sum_{r=1}^{N} \{\mathbf{r}_c \eta_c(t)\}_r^i \tag{33}$$

where $\eta_c(t)$ are the complex principal coordinates. Substitution of equation (33) into equation (27b) and use of the biorthonormality conditions (32) yield the $2N$ sets of complex modal equations of motion

$$\dot{\eta}_{cr}^i = \lambda_r^i \eta_{cr}^i + \bar{\mathbf{v}}_{cr}^{i\prime} \mathbf{g}, \qquad i = B, F;\ r = 1, 2, \ldots, N. \tag{34}$$

The forced response of the system (27) then becomes

$$\mathbf{p}(t) = \sum_{i=B,F} \sum_{r=1}^{N} \{\mathbf{u}_c \eta_c(t)\}_r^i. \tag{35}$$

Therefore the complex FRM can be written as

$$\mathbf{P}(j\omega) = \mathbf{H}_{pg}(j\omega)\mathbf{G}(j\omega) \tag{36}$$

where

$$\mathbf{H}_{pg}(j\omega) = \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}_c \bar{\mathbf{v}}_c'}{j\omega - \lambda}\right]_r^i.$$

Notice here that the FRM excludes the trivial, conjugate, modes [1, 2].

3.2. RELATIONSHIP BETWEEN THE CLASSICAL AND COMPLEX MODAL TESTING METHODS

In equation (31), it is obvious that the complex conjugates of the eigenvalues determined from equation (30) will not be the characteristic roots, unless $\mathbf{M}_2 = \mathbf{C}_2 = \mathbf{K}_2 = \mathbf{0}$. The complex vectors, not the complex modal vectors, associated with the complex conjugates of the eigenvalues should be trivial, zero in a complex sense, vectors. By introducing the following relations, compatible with the biorthonormality conditions (9) and (32),

$$\mathbf{u}_{cr}^i = \frac{1}{\sqrt{2}}(\mathbf{u}_y + j\mathbf{u}_z)_r^i, \qquad \mathbf{v}_{cr}^i = \frac{1}{\sqrt{2}}(\mathbf{v}_y + j\mathbf{v}_z)_r^i \tag{37}$$

the trivial vectors associated with the complex conjugates of the eigenvalues can be written as [2, 3]

$$(\mathbf{u}_y - j\mathbf{u}_z)_r^i = \mathbf{0}, \qquad (\mathbf{v}_y - j\mathbf{v}_z)_r^i = \mathbf{0}. \tag{38}$$

Using the relation (38), the complex modal vectors and the adjoint can be rewritten as

$$\mathbf{u}_{cr}^i = \sqrt{2}\,\mathbf{u}_{yr}^i = \sqrt{2}\,\mathbf{u}_{0r}^i, \qquad \mathbf{v}_{cr}^i = \sqrt{2}\,\mathbf{v}_{yr}^i = \sqrt{2}\,\mathbf{v}_{0r}^i. \tag{39}$$

Substitution of equations (38) and (39) into equation (25) yields

$$\mathbf{H}_{pg}(j\omega) = 2 \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}_0\bar{\mathbf{v}}_0'}{j\omega - \lambda}\right]_r^i \tag{40a}$$

$$\mathbf{H}_{p\hat{g}}(j\omega) = \mathbf{0}. \tag{40b}$$

The relation (40b) should always hold for isotropic systems, which could be another definition of isotropy. It should also be noted from equation (40a) that $\mathbf{H}_{pg}(j\omega)$ is not only a single $N \times N$ FRM containing all modal information required but also it normally provides with well separated modes, excluding the unnecessary trivial modes.

On the other hand, substitution of equations (38) and (39) into equation (17) gives, since $\mathrm{Im}\,(\lambda_r^B) < 0$ and $\mathrm{Im}\,(\lambda_r^F) > 0$ in most cases,

$$\begin{aligned}
\mathbf{H}_{yy}(j\omega) = \mathbf{H}_{zz}(j\omega) = \mathbf{H}_1(j\omega) &= \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}_0\bar{\mathbf{v}}_0'}{j\omega - \lambda} + \frac{\bar{\mathbf{u}}_0\mathbf{v}_0'}{j\omega - \bar{\lambda}}\right]_r^i \\
&\cong \sum_{r=1}^{N} \left\{\left[\frac{\mathbf{u}_0\bar{\mathbf{v}}_0'}{j\omega - \lambda}\right]_r^F + \left[\frac{\bar{\mathbf{u}}_0\mathbf{v}_0'}{j\omega - \bar{\lambda}}\right]_r^B\right\}; \qquad \omega > 0 \\
&\cong \sum_{r=1}^{N} \left\{\left[\frac{\mathbf{u}_0\bar{\mathbf{v}}_0'}{j\omega - \lambda}\right]_r^B + \left[\frac{\bar{\mathbf{u}}_0\mathbf{v}_0'}{j\omega - \bar{\lambda}}\right]_r^F\right\}; \qquad \omega < 0 \\
\mathbf{H}_{yz}(j\omega) = -\mathbf{H}_{zy}(j\omega) = \mathbf{H}_2(j\omega) &= j \sum_{i=B,F} \sum_{r=1}^{N} \left[\frac{\mathbf{u}_0\bar{\mathbf{v}}_0'}{j\omega - \lambda} - \frac{\bar{\mathbf{u}}_0\mathbf{v}_0'}{j\omega - \bar{\lambda}}\right]_r^i \\
&\cong j \sum_{r=1}^{N} \left\{\left[\frac{\mathbf{u}_0\bar{\mathbf{v}}_0'}{j\omega - \lambda}\right]_r^F - \left[\frac{\bar{\mathbf{u}}_0\mathbf{v}_0'}{j\omega - \bar{\lambda}}\right]_r^B\right\}; \qquad \omega > 0 \\
&\cong j \sum_{r=1}^{N} \left\{\left[\frac{\mathbf{u}_0\bar{\mathbf{v}}_0'}{j\omega - \lambda}\right]_r^B - \left[\frac{\bar{\mathbf{u}}_0\mathbf{v}_0'}{j\omega - \bar{\lambda}}\right]_r^F\right\}; \qquad \omega < 0
\end{aligned} \tag{41}$$

From equation (41), it can be concluded that the classical method fails in the complete separation of the backward and forward modes in frequency domain, unlike the new method given in equation (40a). The primary reason is that the classical method always yields conjugate pairs of eigensolutions, losing the important directivity information of the modes. Using equation (41), equation (40) may be rewritten as

$$\begin{aligned}
\mathbf{H}_{pg}(j\omega) &= \mathbf{H}_1(j\omega) - j\mathbf{H}_2(j\omega) \\
&\cong 2 \sum_{r=1}^{N} \left[\frac{\mathbf{u}_0\bar{\mathbf{v}}_0'}{j\omega - \lambda}\right]_r^F; \qquad \omega > 0 \\
&\cong 2 \sum_{r=1}^{N} \left[\frac{\mathbf{u}_0\bar{\mathbf{v}}_0'}{j\omega - \lambda}\right]_r^B; \qquad \omega < 0
\end{aligned} \tag{42a}$$

$$\mathbf{H}_{p\hat{g}}(j\omega) = \mathbf{0} \tag{42b}$$

which confirms the results in equation (40). Comparison of equation (42) with equation (41) suggests that in the classical FRMs $\mathbf{H}_{yy}(j\omega)$ and $\mathbf{H}_{yz}(j\omega)$, the forward and backward modes are likely to overlap, whereas they are completely separated in $\mathbf{H}_{pg}(j\omega)$. In the case when $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{K}_1$ and $\mathbf{K}_2$ are all symmetric, i.e.

$$\mathbf{M}_k = \mathbf{M}_k', \qquad \mathbf{C}_k = \mathbf{C}_k', \qquad \mathbf{K}_k = \mathbf{K}_k', \qquad ; k = 1, 2 \tag{43}$$

the relation between the complex modal vectors and the adjoint simply becomes [1, 2]

$$\bar{\mathbf{v}}^i_{0r} = K^i_r \mathbf{u}^i_{0r} \tag{44}$$

where K^i_r is a complex quantity which is to be determined using the biorthonormality conditions (32). Since the condition (43) are satisfied in most isotropic rotor systems [1, 2], only one row or column of the FRM needs to be measured in order to identify all the modal parameters of an isotropic rotor, in the same way as for stationary structures.

4. FREQUENCY RESPONSE MATRICES OF WEAKLY ANISOTROPIC ROTORS

The equation of motion of a weakly anisotropic rotor system may conveniently be expressed as

$$\mathbf{M}_f\ddot{\mathbf{p}} + \Delta\mathbf{M}_b\ddot{\bar{\mathbf{p}}} + \mathbf{C}_f\dot{\mathbf{p}} + \Delta\mathbf{C}_b\dot{\bar{\mathbf{p}}} + \mathbf{K}_f\mathbf{p} + \Delta\mathbf{K}_b\bar{\mathbf{p}} = \mathbf{g} \tag{45}$$

where the terms preceded by Δ denote the perturbed terms from the isotropic rotor system. The complex conjugates of the eigensolutions of the corresponding isotropic system do not completely vanish as in pure isotropic systems. The relations (38) do not hold any longer. Instead,

$$\begin{aligned} \mathbf{u}^i_{yr} &\cong (\mathbf{u}_0 + \varepsilon\mathbf{u}_{1y})^i_r, \qquad & \mathbf{u}^i_{zr} &\cong -j(\mathbf{u}_0 + \varepsilon\mathbf{u}_{1z})^i_r \\ \mathbf{v}^i_{yr} &\cong (\mathbf{v}_0 + \varepsilon\mathbf{v}_{1y})^i_r, \qquad & \mathbf{v}^i_{zr} &\cong -j(\mathbf{v}_0 + \varepsilon\mathbf{v}_{1z})^i_r \end{aligned} \tag{46}$$

where $\varepsilon\mathbf{u}_1$ and $\varepsilon\mathbf{v}_1$ denote the first order perturbed modal vectors and their adjoints. Substitution of equation (46) into equation (17) gives

$$\begin{aligned}
\mathbf{H}_{yy}(j\omega) &= \sum_{r=1}^{N}\left\{\left[\frac{(\mathbf{u}_0+\varepsilon\mathbf{u}_{1y})(\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1y})'}{j\omega-\lambda}\right]^B_r + \left[\frac{(\bar{\mathbf{u}}_0+\varepsilon\bar{\mathbf{u}}_{1y})(\mathbf{v}_0+\varepsilon\mathbf{v}_{1y})'}{j\omega-\bar{\lambda}}\right]^F_r\right.\\
&\quad\left. + \left[\frac{(\mathbf{u}_0+\varepsilon\mathbf{u}_{1y})(\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1y})'}{j\omega-\lambda}\right]^F_r + \left[\frac{(\bar{\mathbf{u}}_0+\varepsilon\bar{\mathbf{u}}_{1y})(\mathbf{v}_0+\varepsilon\mathbf{v}_{1y})'}{j\omega-\bar{\lambda}}\right]^B_r\right\}\\
\mathbf{H}_{yz}(j\omega) &= j\sum_{r=1}^{N}\left\{\left[\frac{(\mathbf{u}_0+\varepsilon\mathbf{u}_{1y})(\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1z})'}{j\omega-\lambda}\right]^B_r - \left[\frac{(\bar{\mathbf{u}}_0+\varepsilon\bar{\mathbf{u}}_{1y})(\mathbf{v}_0+\varepsilon\mathbf{v}_{1z})'}{j\omega-\bar{\lambda}}\right]^F_r\right.\\
&\quad\left. + \left[\frac{(\mathbf{u}_0+\varepsilon\mathbf{u}_{1y})(\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1z})'}{j\omega-\lambda}\right]^F_r - \left[\frac{(\bar{\mathbf{u}}_0+\varepsilon\bar{\mathbf{u}}_{1y})(\mathbf{v}_0+\varepsilon\mathbf{v}_{1z})'}{j\omega-\bar{\lambda}}\right]^B_r\right\}\\
\mathbf{H}_{zy}(j\omega) &= j\sum_{r=1}^{N}\left\{-\left[\frac{(\mathbf{u}_0+\varepsilon\mathbf{u}_{1z})(\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1y})'}{j\omega-\lambda}\right]^B_r + \left[\frac{(\bar{\mathbf{u}}_0+\varepsilon\bar{\mathbf{u}}_{1z})(\mathbf{v}_0+\varepsilon\mathbf{v}_{1y})'}{j\omega-\bar{\lambda}}\right]^F_r\right.\\
&\quad\left. - \left[\frac{(\mathbf{u}_0+\varepsilon\mathbf{u}_{1z})(\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1y})'}{j\omega-\lambda}\right]^F_r + \left[\frac{(\bar{\mathbf{u}}_0+\varepsilon\bar{\mathbf{u}}_{1z})(\mathbf{v}_0+\varepsilon\mathbf{v}_{1y})'}{j\omega-\bar{\lambda}}\right]^B_r\right\}\\
\mathbf{H}_{zz}(j\omega) &= \sum_{r=1}^{N}\left\{\left[\frac{(\mathbf{u}_0+\varepsilon\mathbf{u}_{1z})(\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1z})'}{j\omega-\lambda}\right]^B_r + \left[\frac{(\bar{\mathbf{u}}_0+\varepsilon\bar{\mathbf{u}}_{1z})(\mathbf{v}_0+\varepsilon\mathbf{v}_{1z})'}{j\omega-\bar{\lambda}}\right]^F_r\right.\\
&\quad\left. + \left[\frac{(\mathbf{u}_0+\varepsilon\mathbf{u}_{1z})(\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1z})'}{j\omega-\lambda}\right]^F_r + \left[\frac{(\bar{\mathbf{u}}_0+\varepsilon\bar{\mathbf{u}}_{1z})(\mathbf{v}_0+\varepsilon\mathbf{v}_{1z})'}{j\omega-\bar{\lambda}}\right]^B_r\right\}.
\end{aligned} \tag{47}$$

In equation (47), the terms in the first and second brackets of the summations normally correspond to the two neighbouring modes over the negative frequency region, and those in the third and fourth brackets are the corresponding mirror images over the positive frequency region. Those two neighbouring modes are heavily overlapped in the usual case where the residue matrices are equally significant. It is not, therefore, a simple task to decouple those modes by conventional circle fit techniques used for modal parameter identification. Substitution of equation (46) into equation (25) gives, on the other hand,

$$\begin{aligned}
\mathbf{H}_{pg}(j\omega) = \frac{1}{2}\sum_{r=1}^{N}\Bigg\{&\left[\frac{(2\mathbf{u}_0+\varepsilon\mathbf{u}_{1y}+\varepsilon\mathbf{u}_{1z})(2\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1y}+\varepsilon\bar{\mathbf{v}}_{1z})'}{j\omega-\lambda}\right]_r^B \\
&+\left[\frac{(\varepsilon\bar{\mathbf{u}}_{1y}-\varepsilon\bar{\mathbf{u}}_{1z})(\varepsilon\mathbf{v}_{1y}-\varepsilon\mathbf{v}_{1z})'}{j\omega-\bar{\lambda}}\right]_r^F+\left[\frac{(2\mathbf{u}_0+\varepsilon\mathbf{u}_{1y}+\varepsilon\mathbf{u}_{1z})(2\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1y}+\varepsilon\bar{\mathbf{v}}_{1z})'}{j\omega-\lambda}\right]_r^F \\
&+\left[\frac{(\varepsilon\bar{\mathbf{u}}_{1y}-\varepsilon\bar{\mathbf{u}}_{1z})(\varepsilon\mathbf{v}_{1y}-\varepsilon\mathbf{v}_{1z})'}{j\omega-\bar{\lambda}}\right]_r^B\Bigg\} \\
\mathbf{H}_{p\hat{g}}(j\omega) = \frac{1}{2}\sum_{r=1}^{N}\Bigg\{&\left[\frac{(2\mathbf{u}_0+\varepsilon\mathbf{u}_{1y}+\varepsilon\mathbf{u}_{1z})(\varepsilon\bar{\mathbf{v}}_{1y}-\varepsilon\bar{\mathbf{v}}_{1z})'}{j\omega-\lambda}\right]_r^B \\
&+\left[\frac{(\varepsilon\mathbf{u}_{1y}-\varepsilon\mathbf{u}_{1z})(2\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1y}+\varepsilon\bar{\mathbf{v}}_{1z})'}{j\omega-\bar{\lambda}}\right]_r^F+\left[\frac{(2\mathbf{u}_0+\varepsilon\mathbf{u}_{1y}+\varepsilon\mathbf{u}_{1z})(\varepsilon\bar{\mathbf{v}}_{1y}-\varepsilon\bar{\mathbf{v}}_{1z})'}{j\omega-\lambda}\right]_r^F \\
&+\left[\frac{(\varepsilon\mathbf{u}_{1y}-\varepsilon\mathbf{u}_{1z})(2\bar{\mathbf{v}}_0+\varepsilon\bar{\mathbf{v}}_{1y}+\varepsilon\bar{\mathbf{v}}_{1z})'}{j\omega-\bar{\lambda}}\right]_r^B\Bigg\}.
\end{aligned} \tag{48}$$

In equation (48), the terms in the first and second brackets of the summations normally correspond to the two neighbouring modes over the negative frequency region, and those in the third and fourth brackets are not the corresponding mirror images but another two neighbouring modes over the positive frequency region. Those two neighbouring modes overlap in the usual cases. However the norms of the corresponding residue matrices are different by the order of ε^2 in $\mathbf{H}_{pg}(j\omega)$ and equally small (order of ε) in $\mathbf{H}_{p\hat{g}}(j\omega)$. Since one of the two modes in $\mathbf{H}_{pg}(j\omega)$, although it is overlapped with another(conjugate mode), is significantly dominant [2, 3], little error due to a single mode circle fit is expected.

Anisotropy of a rotor is best identified in theory when it is at rest. However, the anisotropic nature of rotating components, in particular bearings, is not adequately revealed in practice either at rest or even at a low rotational speed. Moreover the modal properties of a rotor are normally of interest near its operating speeds. Once a rotor rotates at its operating speed, it is often hard to judge from the FRFs obtained by using the classical modal testing if the separation of two neighbouring modal frequencies is due to the inherent system anisotropy or the gyroscopic effects and if those two modes are associated with a pair of backward and forward modes. Complex modal testing, on the other hand, clearly answers these intricate questions. The FRMs, given by equation (48), obtained by using the complex modal testing not only separate the backward modes from the forward, but also indicate the strength of the system anisotropy, if it exists, in terms of the norm of residue matrices. When the anisotropy is found to be insignificant, the rotor may be assumed to be isotropic and great advantages in modal testing will be gained. First, the modal parameter identification requires modal testing for only a single row or column of the FRM given by equation (40a). Secondly, the number of FRFs needed is reduced by one half compared with the classical method. The assumption of isotropy in many applications will remain effective particularly when the experimental errors during modal testing are of the same order of magnitude as the errors involved with such an assumption.

5. ILLUSTRATIVE EXAMPLES

In this section, two illustrative examples are treated to demonstrate the differences between the classical and complex modal testing methods for rotating machinery. The first example deals with a rigid rotor supported at the ends by two identical isotropic bearings. In the first example, the FRFs obtained by the classical and new methods are compared. The second example is a two dof anisotropic rotor bearing system for which analytical eigensolutions are available. Emphasis in the second example is put on the verification of the mathematical treatments given in the previous sections.

5.1. ISOTROPIC RIGID ROTOR-BEARING SYSTEM [11]

Let us consider a rigid rotor supported at the ends by two identical isotropic bearings as shown in Fig. 1. The equation of motion of the system takes the form of equation (27), where [11]

$$\mathbf{M}_c=\begin{bmatrix} ml_2^2+i_t & ml_1l_2-i_t \\ ml_1l_2-i_t & ml_1^2+i_t \end{bmatrix},\qquad \mathbf{C}_c=\begin{bmatrix} c_f-ji_p\Omega & ji_p\Omega \\ ji_p\Omega & c_f-ji_p\Omega \end{bmatrix},\qquad \mathbf{K}_c=\begin{bmatrix} k_f & 0 \\ 0 & k_f \end{bmatrix}$$

$$\mathbf{p}(t)=\mathbf{y}(t)+\mathbf{j}\mathbf{z}(t)=\begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix}+j\begin{Bmatrix} z_1(t) \\ z_2(t) \end{Bmatrix},\qquad \mathbf{g}(t)=\mathbf{f}_y(t)+j\mathbf{f}_z(t)=\begin{Bmatrix} f_{y1}(t) \\ f_{y2}(t) \end{Bmatrix}+j\begin{Bmatrix} f_{z1}(t) \\ f_{z2}(t) \end{Bmatrix}$$

$$i_t=\frac{I_t}{L^2},\qquad i_p=\frac{I_p}{L^2},\qquad l_k=\frac{L_k}{L},\qquad k=1,2$$

$$2c_f=c_{yy}+c_{zz}+j(c_{zy}-c_{yz})\quad \text{with } c_{yy}=c_{zz} \text{ and } c_{zy}=-c_{yz}$$

$$2k_f=k_{yy}+k_{zz}+j(k_{zy}-k_{yz})\quad \text{with}\quad k_{yy}=k_{zz}\quad \text{and}\quad k_{zy}=-k_{yz}.$$

In the above expressions, m is the total mass of the rotor, L is the bearing span, I_t and I_p are the transverse and polar mass moments of inertia about the centre of gravity of the rotor, respectively, L_k, $k=1,2$, is the distance of the kth bearing from the centre of gravity and c_{ik} and k_{ik}, $i, k=y, z$, are the damping and stiffness coefficients of each bearing. It should be noticed that equation (43) holds in this case, requiring the modal testing of only one row or column of FRM $\mathbf{H}_{pg}(j\omega)$ to identify the modal and adjoint parameters.

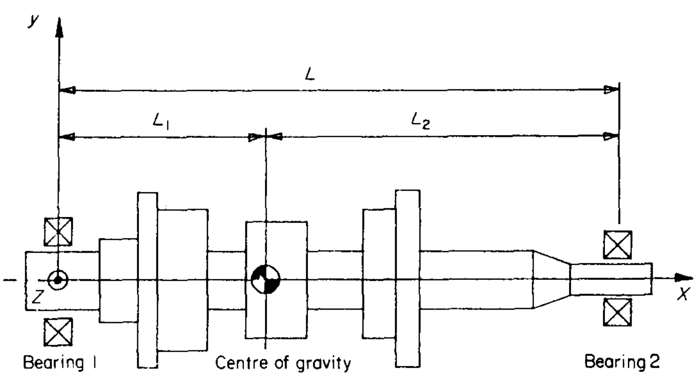

Figure 1. Isotropic rigid rotor-bearing system.

The FRFs obtained by the classical modal testing method are shown in Fig. 2 when $m = 6{\cdot}70$ kg, $l_1 = l_2 = 0{\cdot}5$, $i_t = 0{\cdot}56$ kg, $i_p = 0{\cdot}15$ kg, $\Omega = 10\,000$ rpm, $c_{yy} = 330$ N · s/m, $c_{yz} =$ 17 N · s/m, $k_{yy} = 3\,700\,000$ N/m and $k_{yz} = 71\,000$ N/m. It shows that the first backward and forward modes appearing near at 0·17 kHz are not distinguishable since they are associated with the translatory, non-gyroscopic, motion of the rotor. On the other hand, the second backward and forward modes appearing near to 0·3 kHz are different but heavily overlapped. The reason is that the FRFs are conjugate even with respect to frequency, i.e. the two-sided spectral information is folded into the one-sided spectral domain, losing the directivity of modes. The overlapping or coupling of the second forward and backward modes associated with gyroscopic motion is best seen in the Nyquist plots.

The FRFs obtained by the complex modal testing method are shown in Fig. 3. It shows that the second backward and forward modes are now well separated on the Nyquist

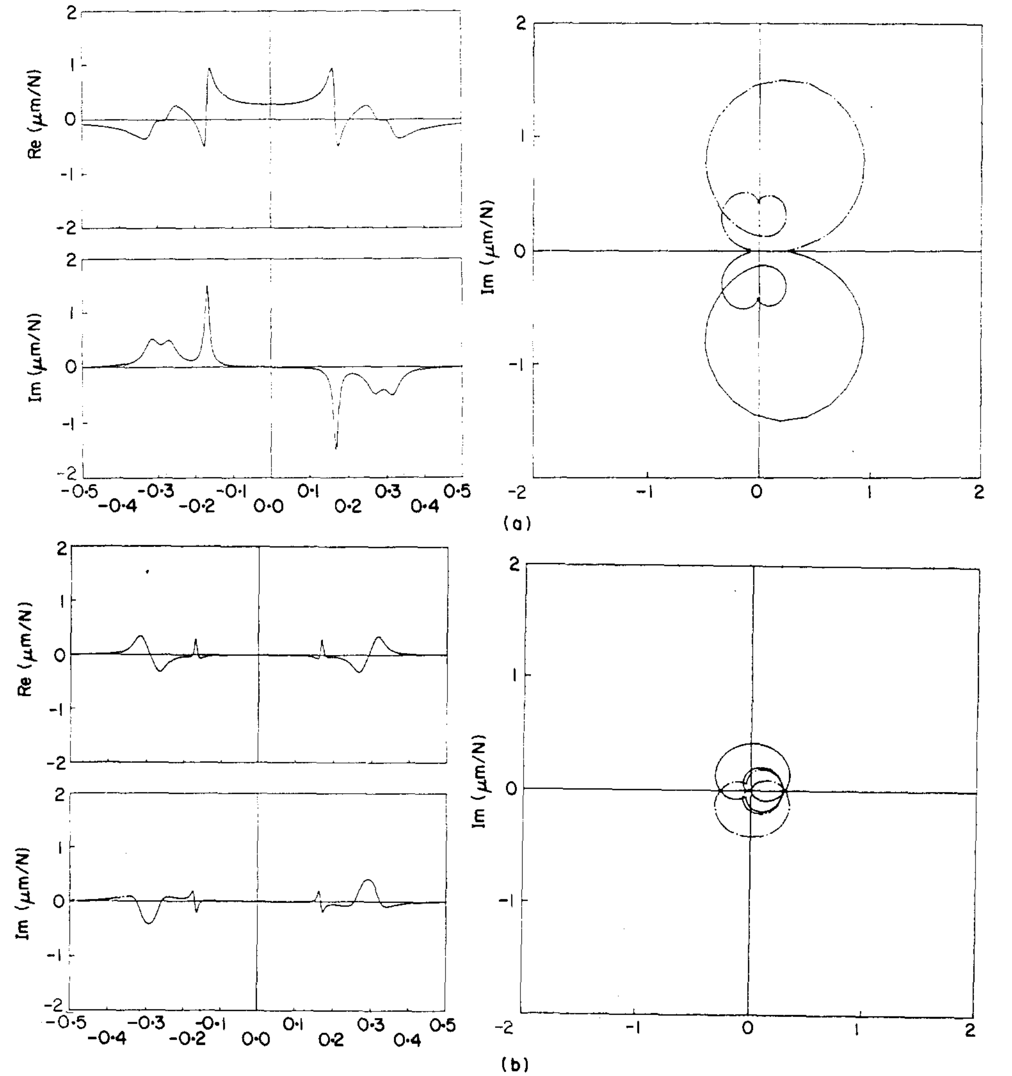

Figure 2. Frequency response functions of isotropic rigid rotor-bearing system obtained by classical modal testing method. (a) Co-quad and Nyquist plots of $H_{y_1y_1}(j\omega)$; (b) co-quad and Nyquist plots of $H_{y_1z_1}(j\omega)$; (c) co-quad and Nyquist plots of $H_{y_2y_1}(j\omega)$; (d) co-quad and Nyquist plots of $H_{y_2z_1}(j\omega)$.

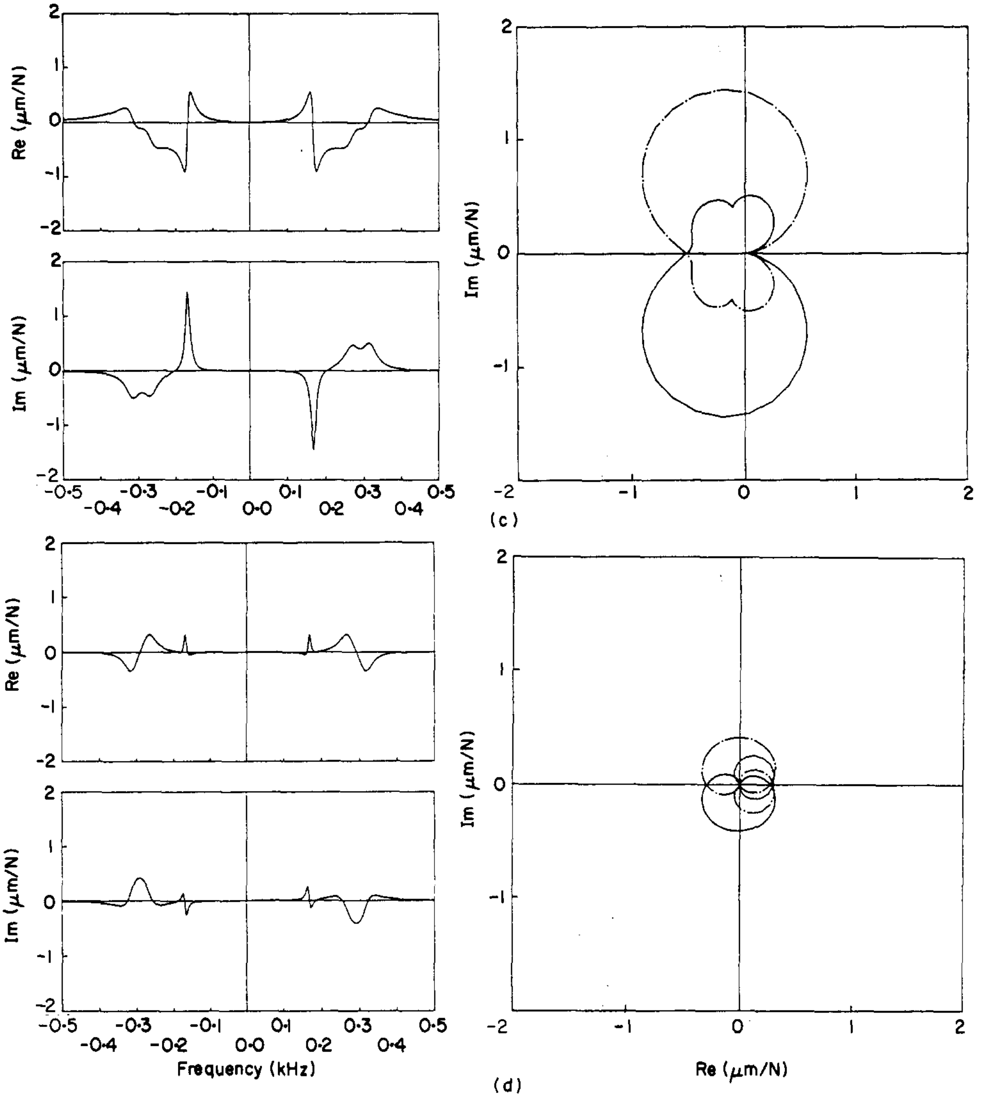

Figure 2. Continued.

plots as well as the Co-quad plots. The two-sided FRFs are no longer conjugate. Therefore, the spectral information preserving the directivity of modes is spread over the two-sided frequency domain, never resulting in the overlapping of the forward and backward modes. As shown in the Nyquist plots, a single mode circle fit can work well with each mode.

5.2. TWO dof ANISOTROPIC ROTOR-BEARING SYSTEM [3]

Let us consider a two dof anisotropic rotor-bearing system with non-dimensionalised equation of motion

$$\begin{Bmatrix}\ddot{y}\\ \ddot{z}\end{Bmatrix}+\begin{bmatrix}0 & \Omega_p\\ -\Omega_p & 0\end{bmatrix}\begin{Bmatrix}\dot{y}\\ \dot{z}\end{Bmatrix}+\begin{bmatrix}1+\Delta & 0\\ 0 & 1-\Delta\end{bmatrix}\begin{Bmatrix}y\\ z\end{Bmatrix}=\begin{Bmatrix}f_y\\ f_z\end{Bmatrix}$$

where $|\Delta|(<1)$ indicates the degree of anisotropy of the undamped support bearing and Ω_p corresponds to the gyroscopic effect. The eigensolutions and the adjoint eigensolutions,

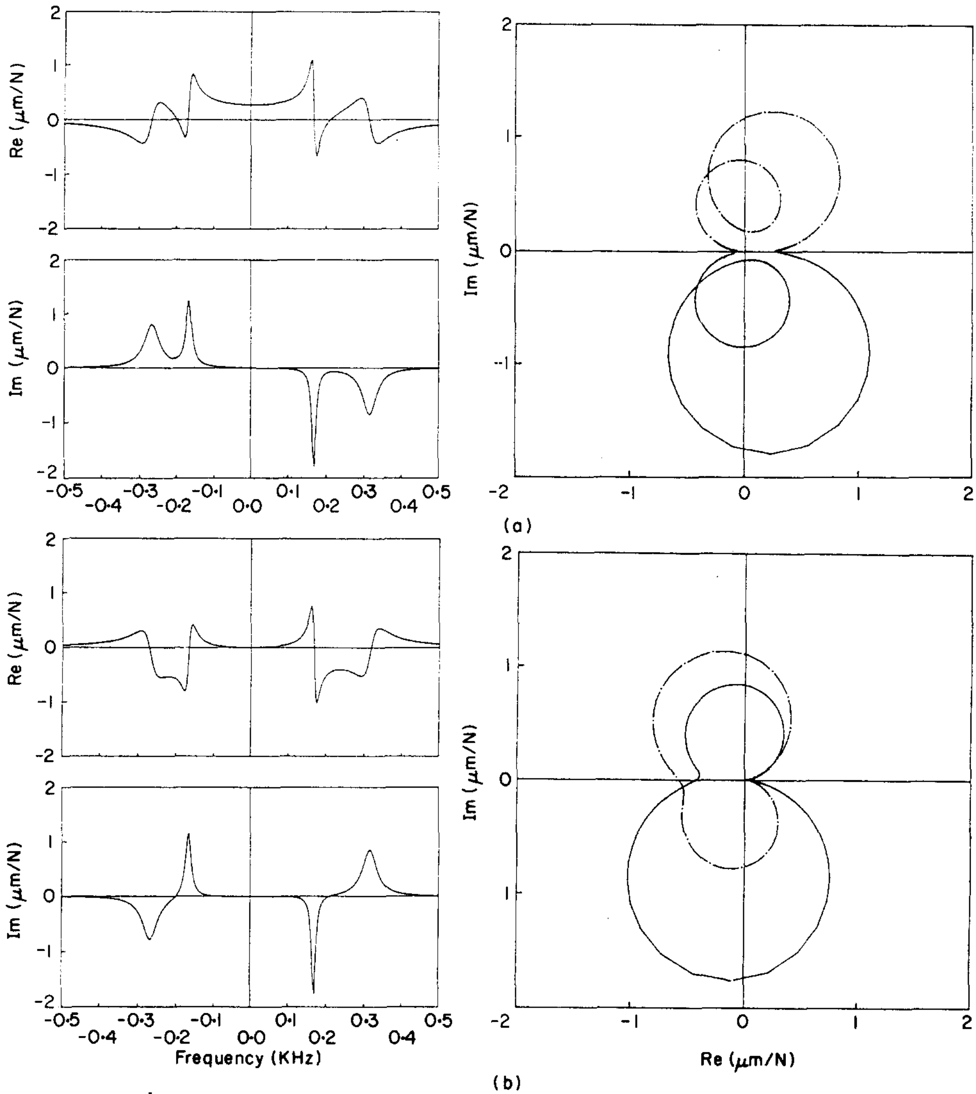

Figure 3. Frequency response functions of isotropic rigid rotor-bearing system obtained by complex modal testing method. (a) Co-quad and Nyquist plots of $H_{p_1g_1}(j\omega)$; (b) co-quad and Nyquist plots of $H_{p_2g_1}(j\omega)$.

after normalising the modal vectors with respect to the y-directional component, are found to be:

$$\lambda_1^i = j\omega^i; \qquad \mathbf{u}_1^i = \{1, ja^i\}', \qquad \mathbf{v}_1^i = K^i\{1, ja^i\}'$$
$$\lambda_{-1}^i = -j\omega^i; \qquad \mathbf{u}_{-1}^i = \{1, -ja^i\}', \qquad \mathbf{v}_{-1}^i = \bar{K}^i\{1, -ja^i\}'$$
$$i = B, F$$

where the backward and forward modal frequencies, $\omega^B < 0$ and $\omega^F > 0$, are given by

$$\omega^B = -\sqrt{1 + \frac{\Omega_p^2}{2} - \sqrt{\left(1 + \frac{\Omega_p^2}{2}\right)^2 - 1 + \Delta^2}}$$
$$\omega^F = \sqrt{1 + \frac{\Omega_p^2}{2} + \sqrt{\left(1 + \frac{\Omega_p^2}{2}\right)^2 - 1 + \Delta^2}}$$

and the components of the modal vectors, a^i, which are now real quantities, and the modal norms, K^i, which are now pure imaginary quantities, become

$$a^i = \frac{\Omega_p \omega^i}{-(\omega^i)^2 + 1 - \Delta}$$

$$K^i = \frac{j}{2[\omega^i\{1+(a^i)^2\} + a^i \Omega_p]} = \frac{-j}{\kappa^i}.$$

The values ω^i and a^i, $i = B, F$, are plotted in Fig. 4 with respect to Ω_p and Δ. This figure shows that a^i approaches -1 as the system anisotropy vanishes irrespective of the strength of the gyroscopic effect Ω_p.

The classical FRM can be constructed as

$$\begin{Bmatrix} Y(j\omega) \\ Z(j\omega) \end{Bmatrix} = \begin{bmatrix} H_{yy}(j\omega) & H_{yz}(j\omega) \\ H_{zy}(j\omega) & H_{zz}(j\omega) \end{bmatrix} \begin{Bmatrix} F_y(j\omega) \\ F_z(j\omega) \end{Bmatrix}$$

where the classical FRFs are represented, using the above results, as

$$H_{yy}(j\omega) = \frac{1}{\kappa^B(\omega - \omega^B)} - \frac{1}{\kappa^B(\omega + \omega^B)} + \frac{1}{\kappa^F(\omega - \omega^F)} - \frac{1}{\kappa^F(\omega + \omega^F)}$$

$$-H_{yz}(j\omega) = \frac{ja^B}{\kappa^B(\omega - \omega^B)} + \frac{ja^B}{\kappa^B(\omega + \omega^B)} + \frac{ja^F}{\kappa^F(\omega - \omega^F)} + \frac{ja^F}{\kappa^F(\omega + \omega^F)}$$

$$H_{zy}(j\omega) = -H_{yz}(j\omega)$$

$$H_{zz}(j\omega) = \frac{(a^B)^2}{\kappa^B(\omega - \omega^B)} - \frac{(a^B)^2}{\kappa^B(\omega + \omega^B)} + \frac{(a^F)^2}{\kappa^F(\omega - \omega^F)} - \frac{(a^F)^2}{\kappa^F(\omega + \omega^F)}.$$

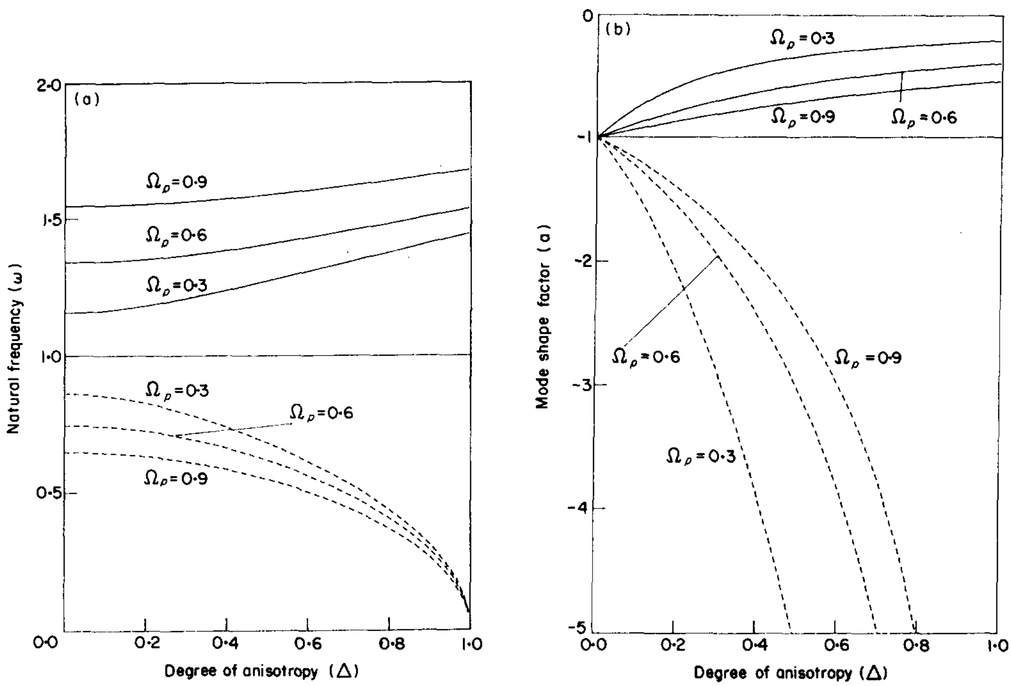

Figure 4. Modal properties of anisotropic rotor-bearing system. (a) Natural frequency vs. degree of anisotropy; ——, ω^F; - - -, $-\omega^B$. (b) mode shape factor vs. degree of anisotropy. ——, a^F; - - -, a^B.

Notice here that modal overlapping in all classical FRFs is inevitable, causing difficulty in parameter identification. On the other hand, the new FRFs are represented as

$$2H_{pg}(j\omega)=\frac{(1-a^B)^2}{\kappa^B(\omega-\omega^B)}-\frac{(1+a^B)^2}{\kappa^B(\omega+\omega^B)}+\frac{(1-a^F)^2}{\kappa^F(\omega-\omega^F)}-\frac{(1+a^F)^2}{\kappa^F(\omega+\omega^F)}$$

$$2H_{p\hat{g}}(j\omega)=\frac{1-(a^B)^2}{\kappa^B(\omega-\omega^B)}-\frac{1-(a^B)^2}{\kappa^B(\omega+\omega^B)}+\frac{1-(a^F)^2}{\kappa^F(\omega-\omega^F)}-\frac{1-(a^F)^2}{\kappa^F(\omega+\omega^F)}.$$

Notice that, as discussed before, $H_{p\hat{g}}(j\omega)$ becomes insignificant and the modal overlapping in $H_{pg}(j\omega)$ becomes less important in the case of weak anisotropy since a^i becomes a negative unity as Δ approaches zero. The FRFs obtained by the classical and new methods are shown in Figs 5 and 6, respectively, when $\Omega_p = 0{\cdot}3$ and $\Delta = 0{\cdot}1$. In Figs 5 and 6, the modal damping ratio of 0·05 is equally applied to all modes for graphical convenience.

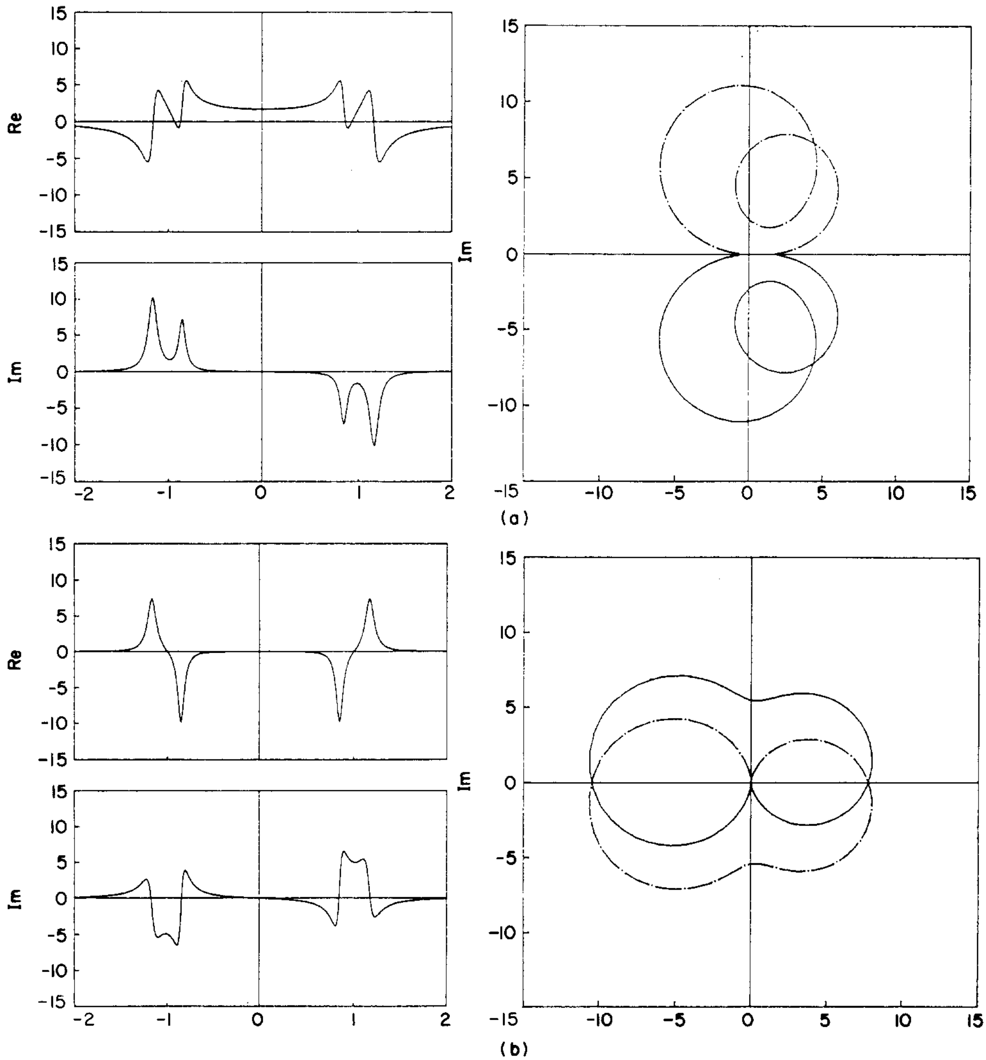

Figure 5. Frequency response functions of weakly anisotropic rotor-bearing system obtained by classical modal testing method. (a) Co-quad and Nyquist plots of $2H_{yy}(j\omega)$; (b) co-quad and Nyquist plots of $2H_{yz}(j\omega)$; (c) co-quad and Nyquist plots of $2H_{zy}(j\omega)$; (d) co-quad and Nyquist plots of $2H_{zz}(j\omega)$.

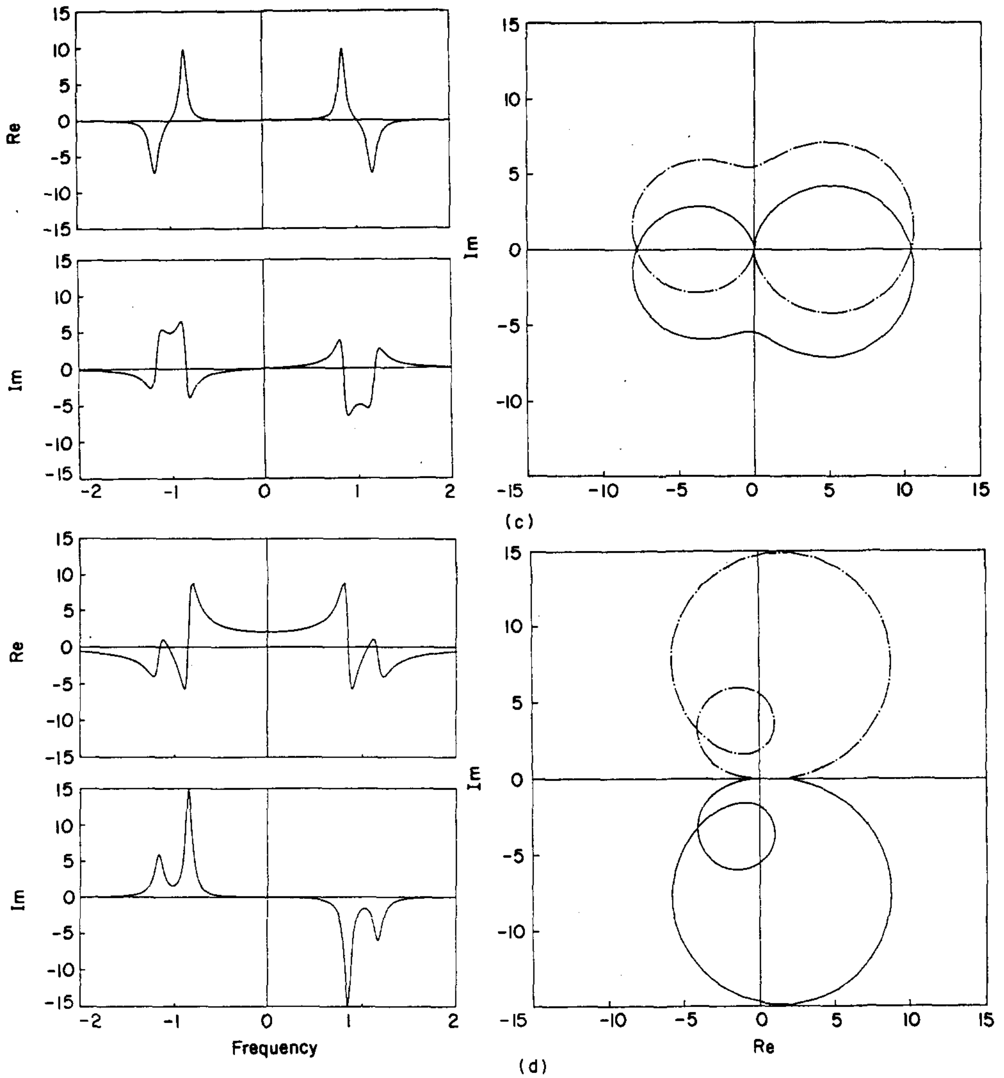

Figure 5. Continued.

As shown in Fig. 5, the modal overlapping is severe and it is difficult to tell if the separation of modes is due to system anisotropy or due to the gyroscopic effect. It is not even certain which one corresponds to the forward or backward mode. On the other hand, the forward and backward modes are clearly separated in Fig. 6, and the degree of anisotropy is indicated by the presence of the little 'kinks' on the Nyquist plot of $H_{pg}(j\omega)$. It can be concluded that the modal separation is mostly due to the gyroscopic effect. Not only are the amplitudes of the 'kinks' small, but the magnitude of $H_{p\hat{g}}(j\omega)$ is far smaller than that of $H_{pg}(j\omega)$. This result suggests that this system can be well approximated as an isotropic system in practice.

6. CONCLUSIONS

A new modal testing method, employing the complex notation, is proposed for the modal and the adjoint modal parameter identification of rotating machinery and compared

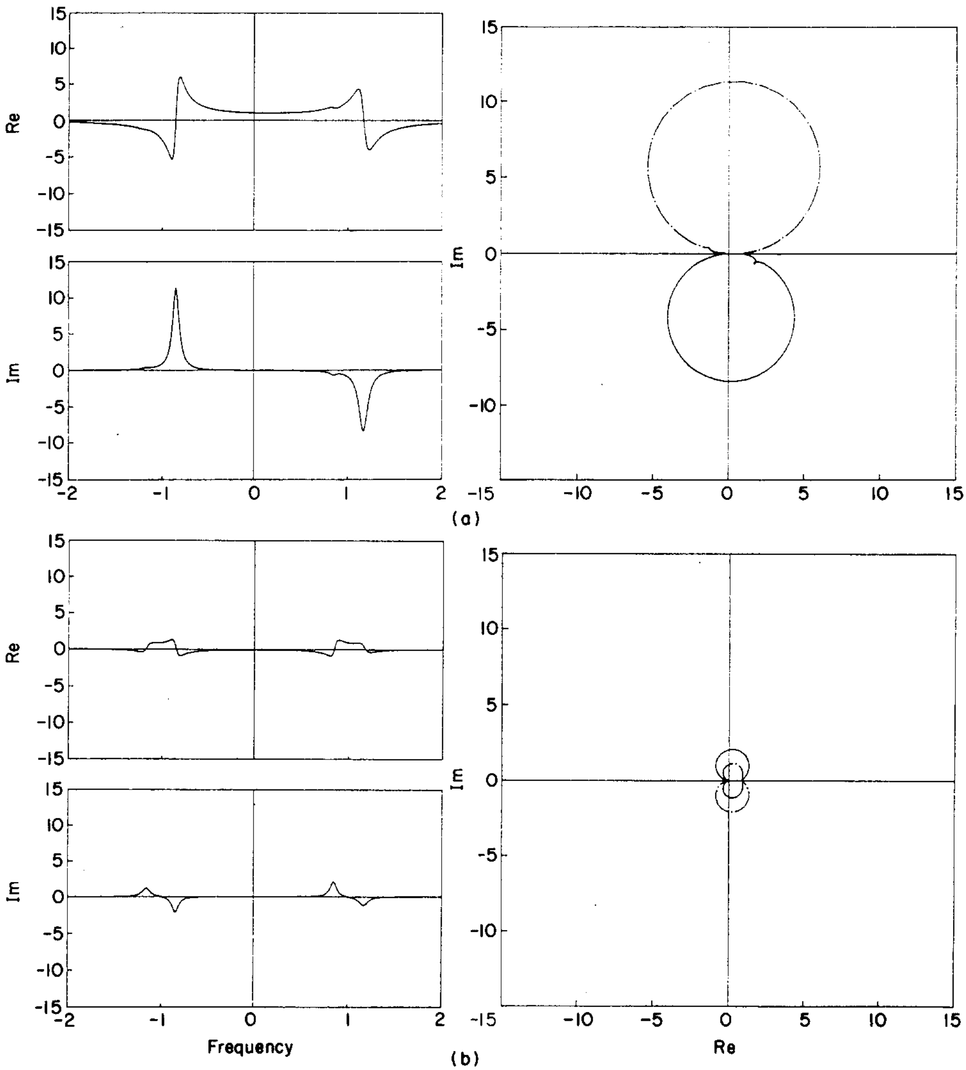

Figure 6. Frequency response functions of weakly anisotropic rotor-bearing system obtained by complex modal testing method. (a) Co-quad and Nyquist plots of $H_{pg}(j\omega)$; (b) co-quad and Nyquist plots of $H_{p\hat{g}}(j\omega)$.

with the classical modal testing method. The proposed complex modal testing method not only allows clear physical insight into the backward and forward modes, but also enables the separation of those modes in the frequency domain. Thus effective modal parameter identification, which could not be accomplished by the classical method, becomes possible. In particular, the complex modal testing method preserves the important directivity information of the modes. The necessity of additional tests to identify the adjoint modal parameters is also relaxed for isotropic and weakly anisotropic rotors, requiring modal testing for only a single row or column of FRM as in the case of classical self-adjoint dynamic systems.

ACKNOWLEDGEMENT

The author would like to thank Dr S. W. Hong for his help with the numerical computation and Dr J. S. Kim for helpful discussions.

REFERENCES

1. C. W. LEE, R. KATZ, A. G. ULSOY and R. A. SCOTT 1988 *Journal of Sound and Vibration* **122**, 119–130. Modal analysis of a distributed parameter rotating shaft.
2. C. W. LEE and Y. G. JEI 1988 *Journal of Sound and Vibration* **126**, 345–361. Modal analysis of continuous rotor-bearing systems.
3. Y. G. JEI and C. W. LEE September 1989 *Twelfth Biennial ASME Conference on Mechanical Vibration and Noise*, Montreal, 89–96. Vibrations of anisotropic rotor-bearing systems.
4. P. J. ROGERS and E. J. EWINS 1989 *Proceedings of the 7th International Modal Analysis Conference, Las Vegas*, 466–473. Modal testing of an operating rotor system using a structural dynamics approach.
5. A. MUSZYNSKA 1986 *International Journal of Analytical and Experimental Modal Analysis*, 15–34. Modal testing of rotor/bearing systems.
6. L. MEIROVITCH 1980 *Computational Methods in Structural Dynamics.* Rockville, Sijthoff & Noordhoff.
7. Y. D. KIM and C. W. LEE 1986 *Journal of Sound and Vibration* **111**, 441–456. Finite element analysis of rotor bearing systems using a modal transformation matrix.
8. C. W. LEE and S. W. HONG 1990 *The International Journal of Analytical and Experimental Modal Analysis*, 51–64. Asynchronous harmonic response analysis of rotor bearing systems.
9. R. NORDMANN 1984 Modal analysis in rotor dynamics. In: *Dynamics of Rotors: Stability and System Identification*, pp. 3–27. Berlin: Springer-Verlag.
10. P. LANCASTER 1966 *Lambda Matrices and Vibrating Systems.* Oxford: Pergamon Press.
11. C. W. LEE and S. W. HONG 1989 *Proceedings of the Institution of Mechanical Engineers* **203**C, 93–101. Identification of bearing dynamic coefficients by unbalance response measurements.