Численный анализ математических моделей механики сплошной среды

(tentative version of the lecture course)

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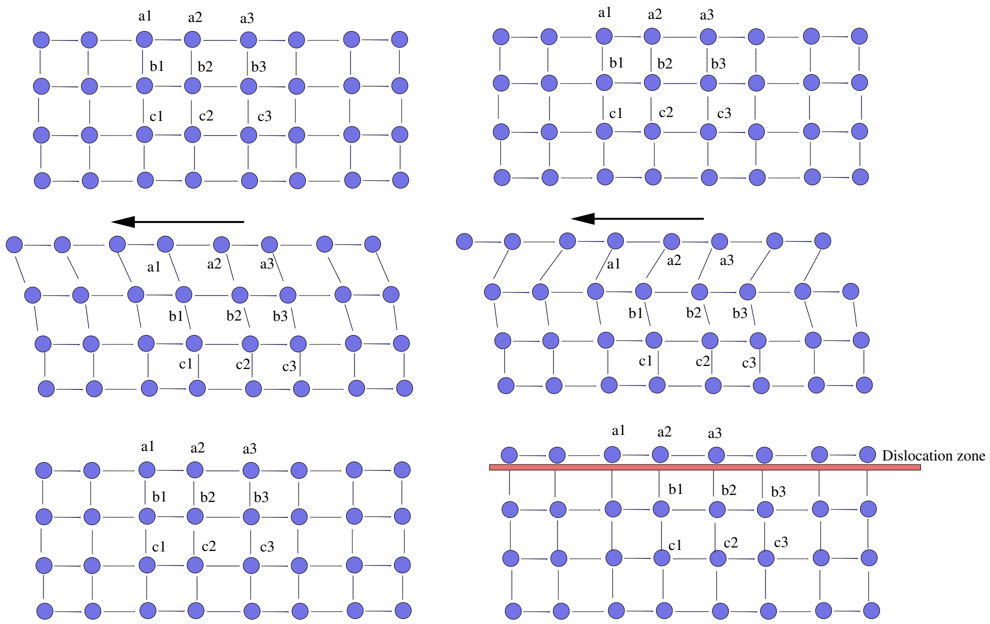
Содержание

Глава I

МАТЕМАТИЧЕСКАЯ ТЕОРИЯ ПЛАСТИЧНОСТИ

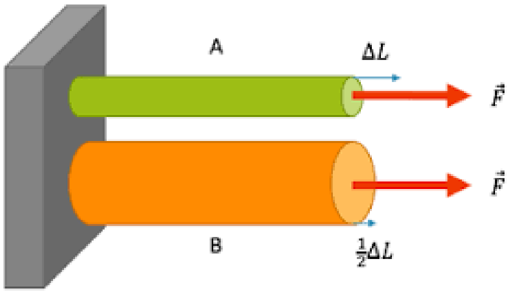
I.1 Физические основы пластичности

Основным отличием моделей пластичности от различных моделей упругого тела является возникновение необратимых деформаций.



Математические модели пластичности [1] следуют из физических экспериментов. Рассмотрим простейшую модель: растяжение стержня

[1] see Ilushin 48, Ishlinski 03, Kachanov 69, Nadai 54



Здесь S -площадь поперечного сечения, F–сила, $\sigma = \frac{F}{S}$ – напряжение, $\varepsilon = \frac{\triangle l}{l}$ – деформация.

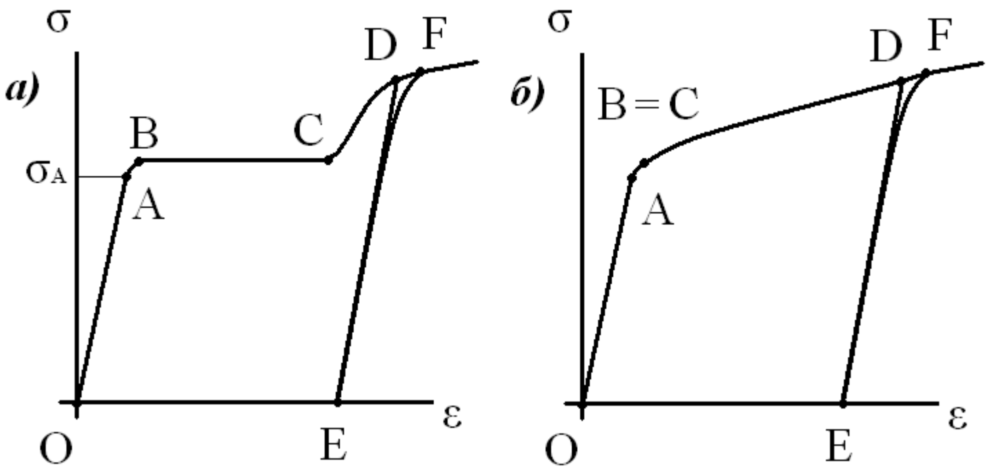


Figure I.1: Физическая диаграмма "напряжения–деформация"

Диаграмма "напряжения–деформация"[2] (Stress-strain relations):
(a) мягкая сталь;
(b) высокопрочная сталь
Здесь: OA это линейный участок, где деформации обратимы.
σ_A, в точке A называется пределом пропорциональности, далее AB деформации остаются обратимыми, но связь с напряжениями уже становится нелинейной.
В точке B возникают пластические (необратимые) деформации. Напряжение σ_B называется пределом текучести (yield limit).
Эта величина является характеристикой материала. Ряд металлов имеет длинное плато после B, где деформации растут при практически постоянном напряжении, другие материалы могут вести себя по другому и деформации растут только при росте напряжений.

[2]Данная диаграмма не включает явление разрушения, которое имеет место всегда, если нагрузки достаточно велики.

CD отражает так называемую закалку. Если сейчас мы уменьшим нагрузки, тогда кривая «разгрузки» будет DE, а OE показывает значение остаточных пластических деформаций, которые не исчезают. Если бы сейчас мы снова начали нагружать, то кривая растяжения-деформации пойдет по DE и далее по основной кривой к точке F. Стоит отметить, что значения предела эластичности увеличились. до значения σ_D и планка стала «жестче» по отношению к напряжению (от этого происходит термин «закаливание»).

Конечно, реальное поведение различных упругопластических материалов может быть гораздо сложнее, чем то, что мы видим на рис. I.1. Например, некоторые материалы подвержены так называемому эффекту Баушингера. который наблюдается для большинства поликристаллических металлов (уменьшении предела упругой деформации после предварительной малой пластической деформации противоположного знака).

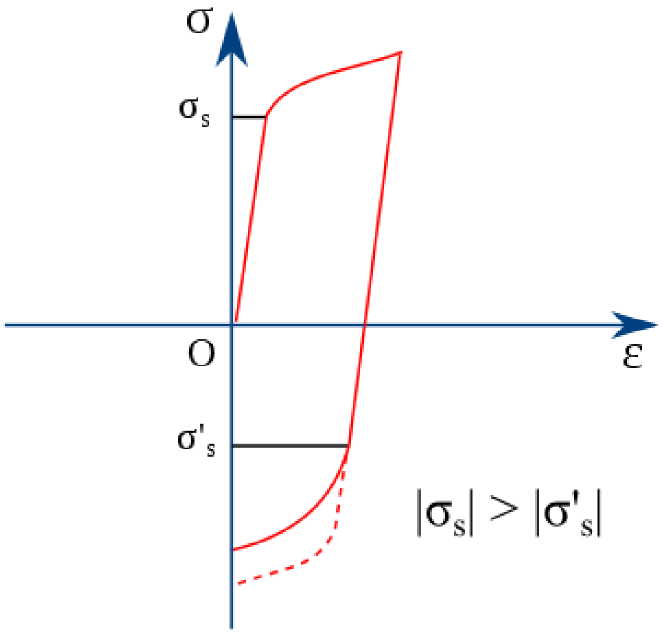


Figure I.2: Эффект Баушингера

Существует несколько математических моделей, которые аппроксимируют физические наблюдаемые зависимости между напряжениями и деформациями. К ним относятся:

- жестко–пластическая модель;
- идеально упруго–пластическая;
- упруго-пластичность с линейным упрочнением;
- модели с нелинейным упрочнением.

Различные модели часто ассоциируют с пиктограммами, которые отражают

особенности физической модели.

В большинстве моделей пластичности предполагается, что полная деформация ε разлагается в сумму упругих ε^e и пластических ε^p деформаций.

В *жесткопластической модели* зона упругости не учитывается. Предполагается что упругие деформации, возникающие на этом первом этапе, пренебрежимо малы относительно пластических.

В этой модели

$$\varepsilon = \varepsilon^p = 0, \quad \text{if} \quad \sigma < k_*;$$
$$\varepsilon = \varepsilon^p > 0, \quad \text{if} \quad \sigma = k_*.$$

Здесь все допустимые состояния соответствуют точкам горизонтальной линии и напряжениями, равными пределу текучести, т. е. $\sigma = k_*, \varepsilon \geq 0$. Эта грубая модель в основном используется при расчетах так называемой *предельной нагрузки*, которая характеризует предельную способность тела сопротивляться внешним силам.

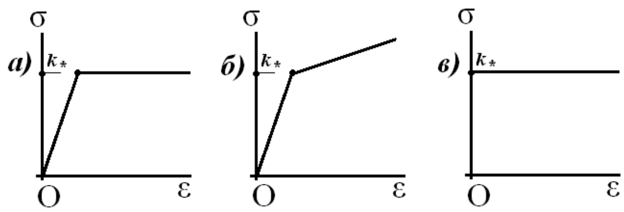


Figure I.3: Диаграмма напряжения–деформации
(a) идеально упруго–пластическая среда;
(b) упруго-пластичность с линейным упрочнением;
(c) жестко–пластическая среда

Рассмотрим случай *идеального упруго-пластического* материала. Пусть на стержень действует монотонно возрастающее растягивающая нагрузка. В определенный момент пластическая деформация появляется и увеличивается пропорционально параметру времени t. Соответствующий коэффициент пропорциональности равен λ.

Если напряжение σ меньше k_*, то приращение деформации $d\varepsilon$ пропорционально приращению напряжения $d\sigma$, i.e,

$$d\varepsilon = d\varepsilon^e = \frac{1}{E} d\sigma, \quad d\varepsilon^p = 0, \quad \text{если} \quad \sigma < k_*,$$

где E это модуль Юнга. Если $\sigma = k_*$ то общая деформация возрастает исключительно за счет пластической составляющей. Таким образом,

$$d\varepsilon = d\varepsilon^p = \lambda dt, \quad d\varepsilon^e = 0, \quad d\sigma = 0, \quad \text{если} \quad \sigma = k_*.$$

Объединив все соотношения, получаем

$$d\varepsilon = \frac{1}{E}d\sigma + \lambda dt,$$

где

$$\lambda = 0, \quad d\sigma > 0 \quad \text{и} \quad \sigma < k_*;$$
$$\lambda \geq 0, \quad d\sigma = 0 \quad \text{и} \quad \sigma = k_*.$$

Поделим на dt и устремим dt к нулю.

$$\boxed{\dot{\varepsilon} = \tfrac{1}{E}\dot{\sigma} + \lambda = \dot{\varepsilon}^e + \dot{\varepsilon}^p,}$$

где $\dot{\varepsilon}$, $\dot{\varepsilon}^e$ и λ можно назвать скоростями общей, упругой, и пластической деформации, соответственно. Соотношения, которые определяют пластическую деформацию, можно записать в виде

$$\boxed{\lambda(\tau - \sigma) \leq 0, \quad \forall \tau \leq k_*.} \tag{I.1.1}$$

Нетрудно видеть, что если $\sigma < k_*$, то $\lambda = 0$, а если $\sigma = k_*$, то $\lambda > 0$.

Мы рассмотрели пример который описывает простую (одномерную) задачу напряжения-деформации. Проблемы реальной жизни связаны с гораздо более сложными случаями, которые требуют соотношений напряжение-деформация в терминах тензоров. В этом случае соответствующие соотношения пластичности можно рассматривать как определенное обобщение этого отношения.

I.2 Определяющие соотношения упруго–пластической среды

Важной особенностью моделей теории пластичности является наличие ограничений на допустимые напряжения

I.2.1 Поверхность текучести

Общая теория описывает эффекты пластичности с помощью специальной функции текучести $\mathcal{F} : \mathbb{M}^{d \times d} \to \mathbb{R}$, которая определяется через тензор напряжений.

В теории идеальной пластичности напряжения должны удовлетворять соотношению

$$\boxed{\mathcal{F}(\sigma) \leq 0.} \tag{I.2.1}$$

Напряжения, удовлетворяющие условию $\mathcal{F}(\sigma) = 0$ образуют так называемую **поверхность текучести (yield surface)**

Любое напряжение внутри поверхности текучести может достигается за счет чисто упругих деформаций.

Пластические деформации возникают только на поверхности текучести.

В моделях упрочняющихся материалов поверхность текучести может изменить свою форму в процессе загрузки.

Модели идеальной пластичности обычно используют фиксированные поверхности текучести.

И более сложных моделях конфигурация поверхности текучести зависит от различных параметров и может меняться.

Points of Ω where $\mathcal{F}(\sigma) < 0$ belong to the **elastic** part (zone). Another part, where $\mathcal{F}(\sigma) = 0$ is called the **plastic** zone.

Configurations of elastic and plastic zones are unknown a priori. Therefore problems of plasticity theory belong to the class of **free boundary problems**.

$\mathcal{F}(\sigma)$ depends on media properties, but it must be independent of coordinate system.

For this reason, yield function is a function of three invariants of σ, which are generated by the coefficients of the corresponding characteristic polynomial. Usually, they are given by the relations

$$\begin{aligned}
I_1(\sigma) &= \mathrm{tr}\sigma = \sigma_1 + \sigma_2 + \sigma_3; \\
I_2(\sigma) &= \frac{1}{2}\left((\mathrm{tr}\sigma)^2 - \mathrm{tr}\sigma^2\right) = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1; \\
I_3(\sigma) &= \det(\sigma) = \sigma_1\sigma_2\sigma_3,
\end{aligned}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the eigenvalues of, which are also called *main stresses*.

Thus, the yield function must be represented in the form

$$\boxed{\boldsymbol{\mathcal{F}(\sigma) = f\left(I_1(\sigma), I_2(\sigma), I_3(\sigma)\right) - k_*,}}$$

where k_* is the material constant (yield limit).

In certain cases (e.g., for some metals), the influence of $\mathrm{tr}\sigma$ on the form of yield surface is insignificant. Then, we use orthogonal decomposition of σ into the spherical and shear parts $\sigma = \sigma^D + \frac{1}{n}\mathrm{tr}\sigma\mathbb{I}$ and obtain

$$\mathcal{F} = \mathcal{F}\left(I_2(\sigma^D), I_3(\sigma^D)\right). \tag{I.2.2}$$

Treska-Saint-Venant condition

$$
\begin{aligned}
2\,|\tau_1| &= |\sigma_2 - \sigma_3| \le \sqrt{3}\,k_*,\\
2\,|\tau_2| &= |\sigma_3 - \sigma_1| \le \sqrt{3}\,k_*,\\
2\,|\tau_3| &= |\sigma_2 - \sigma_1| \le \sqrt{3}\,k_*,
\end{aligned}
$$

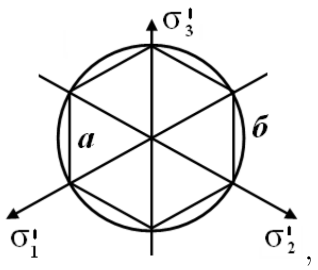
where τ_1, τ_2, τ_3 are the main tangential (shear) stresses. Elastic zone corresponds to strict inequalities in while in the plastic zone at least one of them holds as equality (obviously all the equalities cannot be simultaneously satisfied).

von Mises condition

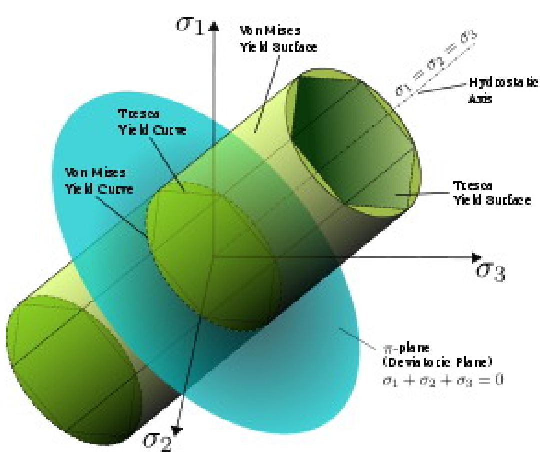
$$\left|\sigma^D\right| \le \sqrt{2}\,k_*, \quad \text{or} \quad \sigma^D : \sigma^D \le 2k_*^2.$$

In terms of the main stresses von Mises condition has the form

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \le 6k_*^2.$$



(a) Treska-Saint-Venant yield surface, (b) Mises yield surface



Schlieher-Mohr condition

Schlieher-Mohr condition reflects the fact that in many cases (e.g., for porous media, sand, clay) yield limit depends on compression, what implies

$$\mathcal{F}(\sigma) = \sigma^D - C(\mathrm{tr}\sigma).$$

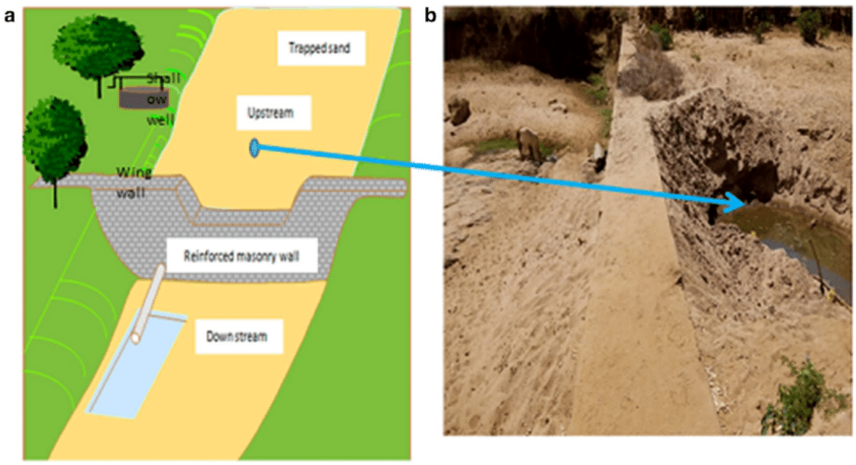


Figure I.4: Земляная дамба

Usually it is presented in the form:

$$\left|\sigma^D\right| + g(\mathrm{tr}\sigma) \le \sqrt{2}\, k_*.$$

where g is a nonnegative monotone function.

Particular cases: **Coulomb-Mohr** (or **Drucker-Prager** condition)

$$g(\mathrm{tr}\sigma) = c_0\, \mathrm{tr}\sigma, \quad \text{where} \ \ c_0 \ge 0.$$

The corresponding yield surface is a cone which has no intersection with the negative part of the horizontal axis. This fact reflects experimental observations showing that in the situation of all-sided compression plastic deformations usually do not arise.

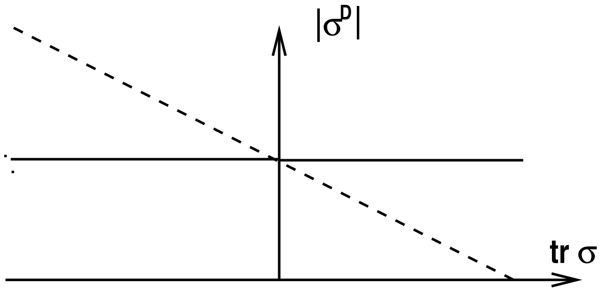


Figure I.5: Coulomb-Mohr and Mises yield surfaces in coordinates $|\sigma^D|$ and trσ.

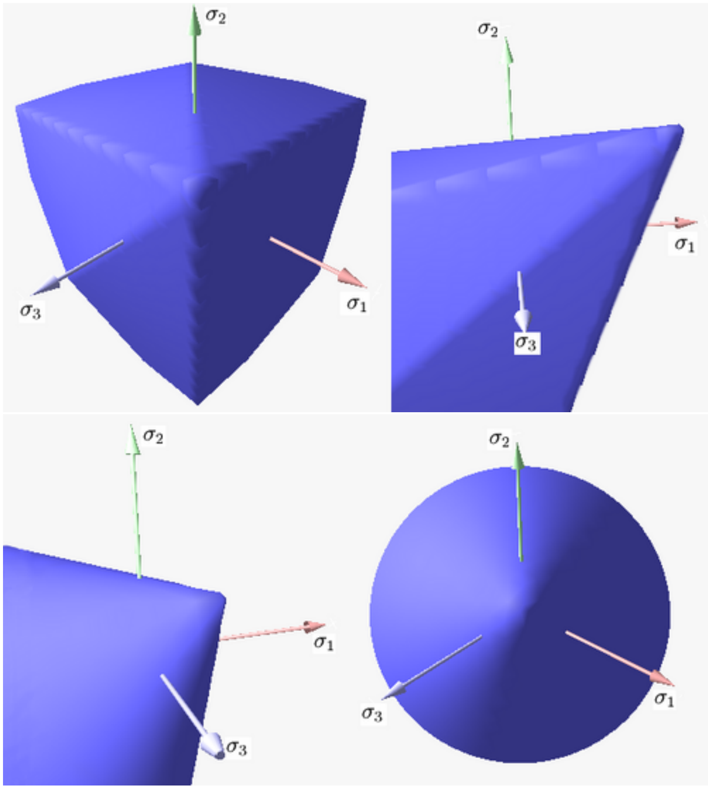


Figure I.6: Coulomb-Mohr and Mises yield surfaces (main stresses)

I.2.2 Аддитивность деформаций и постулат Друкера

Постулат I: полная деформация представима в виде суммы упругой и пластической деформации (аддитивность)

$$\boxed{\dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}) = \underbrace{\mathbb{L}^{-1}\dot{\boldsymbol{\sigma}}}_{\text{elastic strain rate}} + \overbrace{\boldsymbol{\lambda}}^{\text{plastic strain rate}}} \tag{I.2.3}$$

Here
u is the displacement vector,
σ is the stress tensor,
and $\mathbb{L}$ is the tensor of elasticity constants.

From this relation, we see that $\boldsymbol{\lambda}$ is the difference between the velocity of total and elastic deformations.

Постулат II (Drucker): поверхность текучести выпукла и пластическая деформация ортогональна этой поверхности.

$$\boxed{\boldsymbol{\lambda} : (\boldsymbol{\tau} - \boldsymbol{\sigma}) \leq 0, \quad \forall \boldsymbol{\tau} \in \mathbb{M}^{n \times n}, \quad \mathcal{F}(\boldsymbol{\tau}) \leq 0.} \tag{I.2.4}$$

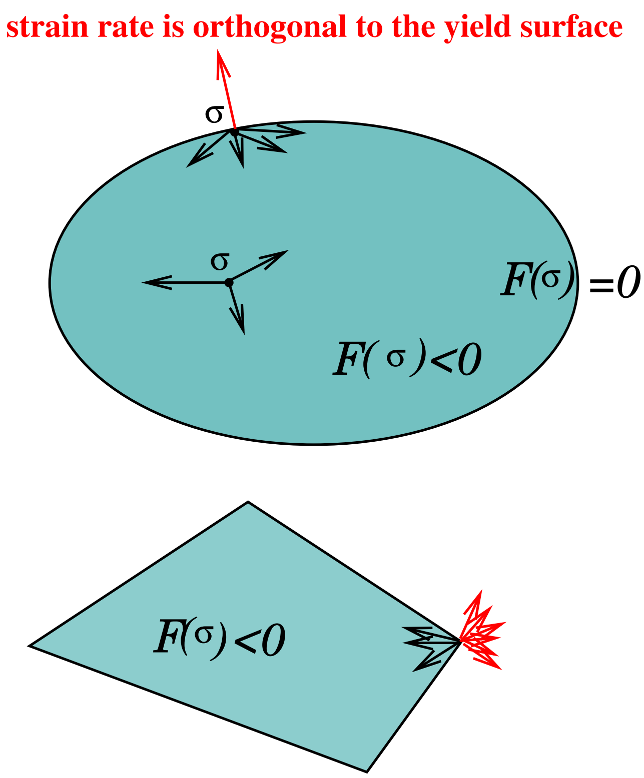


Figure I.7: Yield surface and plastic strain

I.2.3 Полная система уравнений

Уравнение движения (equation of motion)

$$\rho \frac{\partial^2 u}{\partial t^2} = \mathrm{div}\sigma + f \quad \text{in} \quad \Omega. \tag{I.2.5}$$

Малые деформации $$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right). \tag{I.2.6}$$

Определяющие соотношения среды

$$\dot{\varepsilon}(u) = \mathbb{L}^{-1}\dot{\sigma} + \boldsymbol{\lambda}. \qquad \mathcal{F}(\sigma) \leq 0. \tag{I.2.7}$$

Закон Друкера $$\boldsymbol{\lambda} : (\tau - \sigma) \leq 0, \qquad \forall \tau \in \mathbb{M}^{n\times n}, \quad \mathcal{F}(\tau) \leq 0. \tag{I.2.8}$$

Краевые условия $$u = u_0 \quad \text{on} \quad \Gamma_D, \tag{I.2.9}$$

$$\sigma\nu = F \quad \text{on} \quad \Gamma_N. \tag{I.2.10}$$

Начальные условия $$\sigma|_{t=0} = \hat{\sigma}, \tag{I.2.11}$$

$$u\,|_{t=0} = \hat{u}, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \hat{\dot{u}}, \tag{I.2.12}$$

Здесь ρ это плотность среды, f – объемная сила, u_0 – заданное перемещение на части границы Γ_D, F – поверхностная сила на части границы Γ_N, ν – вектор единичной внешней нормали к $\partial\Omega$, $\hat{\sigma}$, $\hat{u}$ и $\hat{\dot{u}}$ – начальные данные.

Prandtl-Reuss model of perfect plasticity

Эта модель описывает медленную (квазистатическую) эволюцию упруго–пластической среды. При этом считается, что

$$\boxed{\frac{\partial^2 u}{\partial^2 t} \approx 0}$$

и этот член в уравнении можно исключить.

В области $\Omega \in \mathbb{R}^d$ надо найти функции $u(x,t)$, $\varepsilon(x,t)$ и $\sigma(x,t)$ удовлетворяющие системе соотношений

$$\mathrm{Div}\sigma + f(x,t) = 0 \quad \text{in} \quad \Omega, \text{ for a.a. } t \in [0,T],$$

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right),$$

$$\dot{\varepsilon}(u) = \mathbb{L}^{-1}\dot{\sigma} + \boldsymbol{\lambda}, \qquad \mathcal{F}(\sigma(x,t)) \leq 0,$$

$$\boldsymbol{\lambda} : (\tau - \sigma) \leq 0, \qquad \forall \tau \in \mathbb{M}^{n\times n}, \quad \mathcal{F}(\tau) \leq 0,$$

$$u = u_0 \quad \text{on} \quad \Gamma_D,$$

$$\sigma\nu = F \quad \text{on} \quad \Gamma_N.$$

В математическом плане эта модель описывается эволюционным вариационным неравенством.

Дальнейшее упрощение модели приводит к стационарным формулировкам, которые используются в тех случаях, когда нагружение было монотонным (без разгрузки) и нас интересует только финальное напряженно-деформированное состояние.

I.3 Деформационная теория пластичности

Эту модель можно также рассматривать как вариант физически нелинейной упругости при условии малости деформаций

$$
\begin{aligned}
&\mathrm{div}\sigma + f = 0 \quad \text{in} \quad \Omega;\\
&\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right);\\
&\varepsilon(u) = \mathbb{L}^{-1}\sigma + \boldsymbol{\lambda}, \qquad \mathcal{F}(\sigma) \leq 0;\\
&\boldsymbol{\lambda} : (\tau - \sigma) \leq 0, \qquad \forall \tau \in \mathbb{M}^{n\times n}, \quad \mathcal{F}(\tau) \leq 0;\\
&u = u_0 \quad \text{on} \quad \Gamma_D;\\
&\sigma\nu = F \quad \text{on} \quad \Gamma_N.
\end{aligned}
$$

Mathematical statement of the problem

Определим множество допустимых напряжений

$$K := \{\tau \in \Sigma \ \mid \ \mathcal{F}(\tau) \leq 0 \text{ a.e. in}\, \Omega\}, \text{ where } \Sigma = L^2(\Omega, \mathbb{M}^{n\times n}),$$

и перемещений

$$V_0 + u_0 := \left\{w \in H^1(\Omega, \mathbb{R}^d) \ \mid \ w = w_0 + u_0, \ \ w_0 \ = \ 0 \ \text{ on } \ \Gamma_D, \ \right\}.$$

Пусть

$$f_i \in L^2(\Omega), \quad F_i \in L^2(\Gamma_N), \quad i = 1, ..n.$$

Множество

$$Q_f = \left\{\tau \in \Sigma \mid \int\limits_\Omega \varepsilon(w) : \tau\, dx = \int\limits_\Omega f \cdot w\, dx + \int_{\Gamma_N} F \cdot w d\Gamma, \quad \forall w \ \in V_0, \right\}.$$

содержит тензор функции, которые удовлетворяют условиям равновесия и краевым условиям Неймана. Действительно, $\tau \in Q_f$ означает, что

$$\mathrm{div}\tau + f = 0 \text{ в } \Omega, \text{ и } \ \tau\nu = F \text{ на } \Gamma_N$$

Определим билинейную форму

$$a(\tau,\sigma) = \int\limits_{\Omega} \mathbb{L}^{-1}\tau : \sigma \, d\,\Omega$$

где $\mathbb{L}^{-1}$ удовлетворяет соотношениям

$$\alpha_1 \xi : \xi \leq \mathbb{L}^{-1}\xi : \xi \leq \alpha_2 \xi : \xi, \quad 0 < \alpha_1 \leq \alpha_2, \quad \forall \xi \in \mathbb{M}^{n\times n}.$$

Теорема I.3.1 (Вариационный принцип Хаара–Кармана)**.** *Если $u \in V_0 + u_0$ и $\sigma \in K \cap Q_f$ являются решением задачи $\mathcal{P}$, то эта тензор функция минимизирует функционал*

$$I^*(\sigma) = \frac{1}{2}\, a(\sigma,\sigma) - \int_{\Gamma_D} \sigma\nu \cdot u_0 \, d\Gamma.$$

на множестве $K \cap Q_f$.

Proof. Take

$$\varepsilon(u) = \mathbb{L}^{-1}\sigma + \boldsymbol{\lambda}$$

and multiply by $\tau - \sigma$, where τ belongs to $K \cap Q_f$.

Integrate the relation over Ω.

$$\int\limits_{\Omega} \varepsilon(u) : (\tau - \sigma)\, dx = \int\limits_{\Omega} (\mathbb{L}^{-1}\sigma + \boldsymbol{\lambda}) : (\tau - \sigma)\, dx.$$

Note that

$$\boldsymbol{\lambda} : (\tau - \sigma) \leq 0.$$

Hence,

$$\int\limits_{\Omega} \varepsilon(u) : (\tau - \sigma)\, dx \leq \int\limits_{\Omega} \mathbb{L}^{-1}\sigma : (\tau - \sigma)\, dx = a(\sigma, \tau - \sigma). \qquad \text{(I.3.1)}$$

Now we transform the left hand side of (I.3.1)

$$\int\limits_{\Omega} \varepsilon(u) : (\tau - \sigma)\, dx = \int\limits_{\Omega} \varepsilon(w + u_0) : (\tau - \sigma)\, dx, \quad \text{where} \quad w \in V_0.$$

Since $\tau \in Q_f$ and $\sigma \in Q_f$, we have

$$\int\limits_{\Omega} \varepsilon(w) : (\tau - \sigma)\, dx = 0,$$

which shows that

$$\int\limits_{\Omega} \varepsilon(u) : (\tau - \sigma)\, dx = \int\limits_{\Omega} \varepsilon(u_0) : (\tau - \sigma)\, dx = \int_{\Gamma_D} (\tau - \sigma)\nu \cdot u_0\, d\Gamma.$$

We arrive at the conclusion that σ satisfies the *variational inequality*

$$a(\sigma, \tau - \sigma) \geq \int_{\Gamma_D} (\tau - \sigma)\nu \cdot u_0\, d\Gamma, \qquad \forall \tau \in K \cap Q_f.$$

In view of LS lemma, σ minimizes $I^*(\sigma)$ over the set $K \cap Q_f$.

Замечание I.3.1. $-I^*(\sigma)$ is a strictly convex functional, defined on on a convex closed set $K \cap Q_f$. Thus, the problem

$$\boxed{\min_{K \cap Q_f} \frac{1}{2} a(\sigma, \sigma) - \int_{\Gamma_D} \sigma\nu \cdot u_0\, d\Gamma} \tag{I.3.2}$$

is uniquely solvable provided that $K \cap Q_f \neq \emptyset$. In the literature this problem is known as the Haar-Karman variational principle.

Замечание I.3.2. Haar-Karman variational principle can serve a basis for numerical methods. However, on this way we are faced with serious technical difficulties because the condition $\sigma \in Q_f$ must be either exactly satisfied (which can be done by so-called equilibrated finite element approximations) or taken into account with the help of penalty type methods. Besides, we must satisfy the pointwise conditions stipulated by the yield law. In general, numerical schemes of such a type are rather cumbersome.

I.4 Минимаксные формулировки задач пластичности

На множестве $K \times V_0 + u_0$ определим Лагранжиан

$$\boxed{L(\sigma, u) = \int\limits_{\Omega} \Big(\varepsilon(u) : \sigma - \frac{1}{2}\mathbb{L}^{-1}\sigma : \sigma - f \cdot u\Big)\, dx - \int_{\Gamma_N} F \cdot u\, d\Gamma.}$$

Нетрудно показать, что вариационный принцип Хаара–Кармана эквивалентен минимаксной формулировке

$$\sup_{\sigma \in K} \inf_{u \in V_0 + u_0} L(\sigma, u). \tag{I.4.1}$$

Действительно, $L(\tau, v)$ является аффинной функцией относительно v. Поэтому либо

$$\inf_{u \in V_0 + u_0} L(\sigma, u) = -\infty,$$

либо

$$L(\sigma, u) \; = \; -\frac{1}{2} a(\sigma, \sigma) + \text{const.}$$

Представим функцию u в виде $u = w + u_0$,

$$\begin{aligned}
&\inf_{u \in V_0 + u_0} L(\sigma, u) \\
&= \inf_{w \in V_0} \left(\int\limits_{\Omega} \varepsilon(w + u_0) : \sigma \; - \frac{1}{2} \mathbb{L}^{-1} \sigma : \sigma - f \cdot (w + u_0) \, dx - \int_{\Gamma_N} F \cdot (w + u_0) \, d\Gamma \right) = \\
&\qquad \inf_{w \in V_0} \left(\int\limits_{\Omega} (\varepsilon(w) : \sigma \; - f \cdot w) \, dx - \int_{\Gamma_N} F \cdot w d\Gamma \right) \\
&\qquad \quad + \int\limits_{\Omega} (\varepsilon(u_0) : \sigma \; - \frac{1}{2} \mathbb{L}^{-1} \sigma : \sigma - f \cdot u_0) \, dx - \int_{\Gamma_N} F \cdot u_0 \, d\Gamma
\end{aligned}$$

Если $\sigma \in Q_f$ то инфимум конечен

$$\begin{aligned}
&\int\limits_{\Omega} (\varepsilon(u_0) : \sigma \; - \frac{1}{2} \mathbb{L}^{-1} \sigma : \sigma - f \cdot u_0) \, dx - \int_{\Gamma_N} F \cdot u_0 \, d\Gamma = \\
&\qquad = \int_{\Omega} (-\text{div}\sigma \cdot u_0 - f \cdot u_0) \, dx + \int_{\Gamma_N} (\sigma \nu - F) \cdot u_0 \, d\Gamma - \\
&\qquad \qquad - \int_{\Omega} \frac{1}{2} \mathbb{L}^{-1} \sigma : \sigma \, dx + \int_{\Gamma_D} \sigma \nu \cdot u_0 \, d\Gamma = -I^*(\sigma).
\end{aligned}$$

Следовательно

$$\inf_{u \in V_0 + u_0} L(\sigma, u) = \begin{cases} -\infty, & \sigma \notin Q_f; \\ -I^*(\sigma), & \sigma \in Q_f. \end{cases}$$

Таким образом,

$$\sup_{\sigma\in K}\inf_{u\in V_0+u_0} L(\sigma,u)=\sup_{\sigma\in K\cap Q_f} -I^*(\sigma).$$

Мы пришли к вариационной формулировке (I.3.2).

Variational problem in terms of displacements

In accordance with the saddle point theory the problem I.4.1 has a dual counterpart

$$\inf_{u\in V_0+u_0}\sup_{\sigma\in K} L(\sigma,u).$$

Let us find explicit form of this problem for the isotropic material and $n=3$.

We have

$$\mathbb{L}^{-1}\sigma=\frac{1}{2\mu}\sigma^D+\frac{1}{9K_0}\mathrm{tr}\sigma\mathbb{I},$$

where μ and K_0 are the shear and volume constants, respectively. We recall that

$$\sigma^D:\mathbb{I}=\left(\sigma-\frac{1}{3}\mathrm{tr}\sigma\,\mathbb{I}\right):\mathbb{I}=\mathrm{tr}\sigma-\mathrm{tr}\sigma=0,$$

i.e., spherical and deviatorical parts are orthogonal. In view of this fact,

$$\mathbb{L}^{-1}\sigma:\sigma=\mathbb{L}^{-1}_{ijkl}\sigma_{ij}\sigma_{kl}=\frac{1}{2\mu}|\sigma^D|^2+\frac{1}{9K_0}(\mathrm{tr}\sigma)^2.$$

Then, components of L depending on σ can be represented as follows have

$$\sup_{\sigma\in K}\int\limits_\Omega\left(\varepsilon(u):\sigma-\frac{1}{2}\mathbb{L}^{-1}\sigma:\sigma\right)dx=$$
$$=\sup_{\sigma\in K}\int\limits_\Omega\left(\varepsilon^D(u):\sigma^D+\frac{1}{3}\mathrm{tr}\varepsilon(u)\mathrm{tr}\sigma-\frac{1}{4\mu}|\sigma^D|^2-\frac{1}{18K_0}(\mathrm{tr}\sigma)^2\right)dx. \quad \text{(I.4.2)}$$

Consider von Mises yield condition. In this case,

$$K=\left\{\sigma\in\Sigma\ \mid\ |\sigma^D|\le\sqrt{2}k_*\ \text{a.e. in}\ \Omega\right\}.$$

and the problem is reduced to separate maximization with respect to $\sigma^D\in K^D$ and $\mathrm{tr}\sigma\in L(\Omega)$, where

$$K^D=\left\{\sigma^D\in\Sigma\ \mid\ |\sigma^D|\le\sqrt{2}k_*\text{a.e. in}\ \ \Omega\right\}.$$

In other words, we must solve two problems:

$$\sup_{\sigma^D \in K^D} \int_\Omega \left(\varepsilon^D(u) : \sigma^D - \frac{1}{4\mu} \left|\sigma^D\right|^2 \right) dx \tag{I.4.3}$$

and

$$\sup_{\mathrm{tr}\sigma} \int_\Omega \left(\frac{1}{3}\mathrm{tr}\sigma \;\; \mathrm{tr}\,\varepsilon(u) - \frac{1}{18K_0}(\mathrm{tr}\sigma)^2 \right) dx. \tag{I.4.4}$$

Both problems are simple. In (I.4.4) we need to maximize the integrand function. The corresponding condition is that for almost all $x \in \Omega$

$$\mathrm{tr}\sigma(x) = 3K_0 \,\mathrm{tr}\,\varepsilon(u)(x) = 3K_0 \,\mathrm{div}u(x).$$

Thus, the term (I.4.4) is equal to

$$\frac{K_0}{2} \int_\Omega (\mathrm{div}u)^2 \, dx. \tag{I.4.5}$$

For the term (I.4.3), two options are possible.
1. If for the maximizer the condition $|\sigma^D(x)| \leq \sqrt{2}k_*$ at $x \in \Omega$ is not active, then the maximum value of the integrand is defined by the relation $\sigma^D = 2\mu\,\varepsilon^D$. In this case, $\left|\varepsilon^D\right| < k_*/(\sqrt{2}\,\mu)$ and supremum equals to

$$\mu \left|\varepsilon^D\right|^2. \tag{I.4.6}$$

2. If $\left|\varepsilon^D\right| \geq k_*/(\sqrt{2}\,\mu)$ then maximal value is attained on the boundary of K^D, i.e.,

$$\left|\sigma^D\right| = \sqrt{2}\,k_*.$$

Obviously, scalar product attains maximum if the components are collinear. For this reason, $\sigma^D : \varepsilon^D$ is maximal if

$$\sigma^D = \sqrt{2}\,k_* \frac{\varepsilon^D}{|\varepsilon^D|}.$$

Then supremum in (I.4.3) is equal to

$$\int_\Omega \left(\sqrt{2}\,k_* \left|\varepsilon^D\right| - \frac{k_*^2}{2\,\mu} \right) dx. \tag{I.4.7}$$

We combine (I.4.6) and (I.4.7) and find that

$$\sup_{\sigma^D \in K^D} \int\limits_{\Omega} \left(\varepsilon^D : \sigma^D - \frac{1}{4\mu} \left|\sigma^D\right|^2 \right) dx = \int\limits_{\Omega} \Phi\left(\left|\varepsilon^D\right|\right) dx,$$

where

$$\Phi(\gamma) = \begin{cases} \mu\gamma^2 & \text{if} \quad 0 \le \gamma < \frac{k_*}{\sqrt{2}\,\mu}; \\ \\ \sqrt{2}k_*\gamma - k_*^2/(2\mu) & \text{if} \quad \gamma \ge \frac{k_*}{\sqrt{2}\,\mu}. \end{cases} \tag{I.4.8}$$

The corresponding variational problem is formulated in terms of displacements and has the form

$$\inf_{u \in V_0 + u_0} J(u), \tag{I.4.9}$$

where

$$\boxed{\boldsymbol{J(u) := \int\limits_{\Omega} \left(\frac{K_0}{2} (\mathrm{div} u)^2 + \Phi\left(\left|\varepsilon^D(u)\right|\right) \right) dx - \int\limits_{\Omega} f \cdot u dx - \int\limits_{\Gamma_N} F \cdot u d\Gamma.}}$$

(I.4.10)

Замечание I.4.1. Functional (I.4.10) is coercive only on a nonreflexive space. Therefore problem (I.4.9) may have no solution. Nevertheless, it can be used for constructing a minimizing sequence.

I.5 Вариационные формулировки задач нелинейной механики

Modern theories describing stationary models of continuum media are usually based on two main variables: tensor-function of stresses σ and vector-function of displacements u. These functions must satisfy boundary conditions, equilibrium equations, and *constitutive relations* that postulate a certain correlation between deformations $\varepsilon(u)(x)$ and stresses $\sigma(x)$. In simple models, constitutive relations are defined by explicit algebraic relations (e.g., by Hooke's law). A unified form of constitutive relations applicable for a wide spectrum of real life media models is presented by the conception of *superpotential*.

Superpotential is defined by a convex, l.s.c. functional ψ with values in $\mathbb{R} \cup +\infty$. With the help of this functional, we define constitutive relations as follows:

$$\boxed{\boldsymbol{\sigma(x) \in \partial\psi(\varepsilon(u)(x))}} \tag{I.5.1}$$

where $\partial\psi$ denotes subdifferential of ψ. In view of (**??**)–(**??**), this relation is equivalent to

$$\boxed{\boldsymbol{\varepsilon(u)(x) \in \partial\psi^*(\sigma(x)),}} \tag{I.5.2}$$

where ψ^* is the functional conjugate to ψ, i. e.,

$$\psi^*(\xi^*) = \sup_{\xi \in \mathbb{M}^{n\times n}} \{\xi : \xi^* - \psi(\xi)\}.$$

It is clear that

$$\psi(\xi) + \psi^*(\xi^*) - \xi : \xi^* \geq 0, \quad \forall \xi, \xi^* \in \mathbb{M}^{n\times n}$$

and the equality is possible if and only if ξ and ξ^* belong to the sets $\partial\psi^*(\xi^*)$ and $\partial\psi(\xi)$, respectively. Thus, we have the relation

$$\psi(\varepsilon(u)(x)) + \psi^*(\sigma(x)) = \varepsilon(u)(x) : \sigma(x).$$

We recall that $\partial\psi(\xi)$ may contain more than one element, so that the relations (I.5.1)–(I.5.2) should be understood in terms of sets. In general, they may assign a particular σ to different strain tensors.

If ψ and ψ^* are Gateaux differentiable, then (I.5.1) has the form

$$\sigma(x) = \psi'(\varepsilon(u)(x)). \tag{I.5.3}$$

Relation (I.5.2) also comes in the form of equality

$$\varepsilon(u)(x) = \psi^{*\prime}(\sigma(x)). \tag{I.5.4}$$

Our goal is to show that (I.5.1)–(I.5.2) generate variational problems for displacements and and stresses.

I.5.1 Вариационная формулировка задачи в перемещениях

First, we demonstrate this with the paradigm of the primal problem (associated with displacements). Assume that, that $u \in H^1(\Omega, \mathbb{R}^n)$. Then, for almost all $x \in \Omega$ the strain tensor $\varepsilon(u)(x)$ is defined. Since ψ is a convex function, we have

$$\psi(\xi) \geq \psi(\varepsilon(u)(x)) + \sigma(x) : (\xi - \varepsilon(u)(x)), \qquad \forall \xi \in \mathbb{M}^{n \times n}.$$

In particular, if set $\xi = \varepsilon(v)(x)$, then we obtain the relation

$$\psi(\varepsilon(v)(x)) \geq \psi(\varepsilon(u)(x)) + \sigma(x) : (\varepsilon(v)(x) - \varepsilon(u)(x)), \tag{I.5.5}$$

which is valid for any function $v \in H^1(\Omega, \mathbb{R}^n)$. Denote

$$\Psi(p) = \int_\Omega \psi(p(x))\, dx$$

and integrate (I.5.5) over Ω. We have

$$\Psi(\varepsilon(v)) \geq \Psi(\varepsilon(u)) + \int_\Omega \sigma(x) : (\varepsilon(v)(x) - \varepsilon(u)(x))\, dx.$$

Stress tensor σ must satisfy the equilibrium equations and boundary conditions on Γ_N. This means that

$$\int_\Omega \sigma : \varepsilon(v-u)\, dx = \int_\Omega f \cdot (v-u)\, dx + \int_{\Gamma_N} F \cdot (v-u)\, d\Gamma, \tag{I.5.6}$$

where $v \in V_0 + u_0$. Hence,

$$\Psi(\varepsilon(v)) \geq \Psi(\varepsilon(u)) + \int_\Omega f \cdot (v-u)\, dx + \int_{\Gamma_N} F \cdot (v-u)\, d\Gamma.$$

This inequality shows that u is a minimizer of the problem

$$\boxed{\inf_{\boldsymbol{v} \in \boldsymbol{V_0} + \boldsymbol{u_0}} \boldsymbol{J(v)}, \qquad \boldsymbol{J(v) := \Psi(\varepsilon(v)) - \int_\Omega f \cdot v dx - \int_{\Gamma_N} F \cdot v d\Gamma.}} \tag{I.5.7}$$

is called *energy variational functional in terms of displacements.*

I.5.2 Вариационная формулировка задачи в напряжениях

Now we deduce the variational problem that is minimized by the stress tensor σ. Assume that, that $\sigma \in \Sigma$. Then for almost all $x \in \Omega$ we have (using (I.5.2) and convexity of ψ^*)

$$\psi^*(\xi^*) \geq \psi^*(\sigma(x)) + \varepsilon(u)(x) : (\xi^* - \sigma(x)), \quad \forall \xi^* \in \mathbb{M}^{n\times n}.$$

Let $\tau \in \Sigma$. For almost all $x \in \Omega$, we have

$$\psi^*(\tau(x)) \geq \psi^*(\sigma(x)) + \varepsilon(u)(x) : (\tau(x) - \sigma(x)). \tag{I.5.8}$$

Similarly to the previous case, we denote

$$\Psi^*(\tau) = \int_\Omega \psi^*(\tau(x))\, dx$$

and integrate (I.5.8) over Ω. Then we arrive at the relation

$$\Psi^*(\tau) \geq \Psi^*(\sigma) + \int_\Omega \varepsilon(u)(x) : (\tau(x) - \sigma(x))\, dx.$$

Assume that τ (as well as σ) is an equilibrated field and satisfies (in a generalized sense) Neumann boundary conditions. Then, by (I.3.1), we obtain

$$\int_\Omega \varepsilon(u) : (\tau - \sigma)\, dx = -\int_\Omega \operatorname{div}(\tau - \sigma) \cdot u\, dx + \\ + \int_{\partial\Omega} (\tau - \sigma)\nu \cdot u\, d\Gamma = \int_{\Gamma_D} (\tau - \sigma)\nu \cdot u_0\, d\Gamma.$$

Thus σ is a solution of the problem

$$\boxed{\inf_{\boldsymbol{\tau} \in \boldsymbol{Q_f}} \boldsymbol{I}^*(\boldsymbol{\tau}), \qquad \boldsymbol{I}^*(\boldsymbol{\tau}) = -\boldsymbol{\Psi}^*(\boldsymbol{\tau}) + \int\limits_{\boldsymbol{\Gamma_D}} \boldsymbol{\tau\nu} \cdot \boldsymbol{u_0}\, \boldsymbol{d\Gamma}.} \tag{I.5.9}$$

is the *energy functional in terms of stresses.*

Problems (I.5.7) and (I.5.9) form a pair of dual variational problems.

I.6 Физические модели твердых тел

I.6.1 Изотропная линейно упругая среда

In this case

$$\psi_e(\xi) = \frac{K_0}{2}(\mathrm{tr}\xi)^2 + \mu|\xi^D|^2, \quad \forall \xi \in \mathbb{M}^{n\times n},$$

where K_0 and μ are elasticity constants. It is easy to see that

$$\psi_e'(\xi) = K_0 \mathrm{tr}\xi \mathbb{I} + 2\mu\xi^D,$$

and (I.5.3) reads (Hooke's law)

$$\sigma = K_0 \mathrm{tr}\varepsilon \mathbb{I} + 2\mu\varepsilon^D.$$

By the definition, we have

$$\psi_e^*(\xi^*) = \sup_{\xi\in\mathbb{M}^{n\times n}} \left\{ \xi^* : \xi - \frac{K_0}{2}(\mathrm{tr}\xi)^2 - \mu|\xi^D|^2 \right\}.$$

Since supremum is attained if

$$\xi^* = K_0 \mathrm{tr}\xi_0 \mathbb{I} + 2\mu\xi_0^D,$$

we conclude that

$$\mathrm{tr}\xi^* = \xi^* : \mathbb{I} = 3K_0 \mathrm{tr}\xi_0, \quad \xi^{*D} = 2\mu\xi_0^D.$$

Thus,

$$\psi_e^*(\xi^*) = \xi^* : \xi_0 - \left(\frac{K_0}{2}(\mathrm{tr}\xi_0)^2 + \mu|\xi_0^D|^2 \right) = \\ = \frac{K_0}{2}(\mathrm{tr}\xi_0)^2 + \mu|\xi_0^D|^2 = \frac{1}{18K_0}(\mathrm{tr}\xi^*)^2 + \frac{1}{4\mu}|\xi^{*D}|^2. \quad \text{(I.6.1)}$$

If the media is not isotropic, then ψ_e is defined by the relation

$$\psi_e(\xi) = \frac{1}{2}\mathbb{L}\,\xi : \xi,$$

where $\mathbb{L}$ is the elasticity tensor.

Then the conjugate energy functional has the form

$$\psi_e^*(\xi^*) = \frac{1}{2}\mathbb{L}^{-1}\xi^* : \xi^*.$$

I.6.2 Деформационная теория пластичности

Различные модели деформационной теории пластичности задаются с помощью двойственного потенциала ψ^*. Если аргумент ψ^* принадлежит множеству K, то ψ^* совпадает с соответствующим упругим потенциалом ψ_e^*. В противном случае потенциал бесконечен (напряжения вне K являются недопустимыми). Таким образом,

$$\psi^*(\xi^*) = \begin{cases} \psi_e^*(\xi^*), & \text{If} \quad \xi^* \in K; \\ +\infty, & \text{If} \quad \xi^* \notin K, \end{cases}$$

Различные модели определяются заданием выпуклого множества K. Например в разделе (I.4) мы рассмотрели случай, когда K определяется условием Мизеса (von Mises condition). Такие модели также называют моделями пластичности Генки.

$$K = \{\xi^* \in \mathbb{M}^{n\times n} \mid |\xi^{*D}| \leq \sqrt{2}k_*\}.$$

Для построения вариационной задачи в перемещениях надо построить функцию $\psi(\xi)$.

Напомним, что для выпуклых функций ψ совпадает со второй сопряженной ψ^{**}, т.е. $(\psi^*)^*(\xi) = \psi(\xi)$.

Таким образом,

$$\boxed{\boldsymbol{\psi(\xi) = \sup_{\xi^* \in \mathbb{M}^{n\times n}} \{\xi : \xi^* - \psi^*(\xi^*)\} = \sup_{\xi^* \in K} \{\xi : \xi^* - \psi_e^*(\xi^*)\}.}} \tag{I.6.2}$$

Для случая изотропной упругости $\psi_e^*(\xi^*)$ (see (I.6.1) и модели пластичности Генки мы произвели соответствующие вычисления в разделе (I.4.2)) и установили, что

$$\psi(\xi) = \frac{K_0}{2}(\text{tr}\xi)^2 + \Phi\left(\left|\xi^D\right|\right), \tag{I.6.3}$$

где Φ определено в соответствии с (I.4.8).

Ясно, что (I.6.3) приводит (I.5.7) к виду (I.4.10).

I.6.3 Эласто-вякопластическая среда

Вязкопластичность

В вязких средах напряжения зависят от скорости приложения нагрузок В моделях вязкопластичности считается, что

$$\boxed{\boldsymbol{\sigma = \nu \dot{\varepsilon}_{vpl}.}}$$

В вязкопластическом теле пластические деформации связанные с вязкостью возникают только после того, как напряжения достигают некоторого предела.

ν (viscosity), is a nonlinear function $\nu(|\sigma|, p)$ and physical parameters p.
Example: Norton-Hoff viscosity model:

$$\nu = \mu \left(\frac{\sigma}{\xi(|\sigma|)} \right)^{\alpha}.$$

Example: 1D elastic perfectly viscoplastic material:

$$\dot{\varepsilon} = \begin{cases} \frac{1}{E}\dot{\sigma} & \text{if } |\sigma| < k_*, \\ \dot{\varepsilon}_{el} + \dot{\varepsilon}_{vpl} = \frac{1}{E}\dot{\sigma} + \frac{\dot{\sigma}}{\nu}\left(1 - \frac{k_*}{|\sigma|}\right) & \text{if } |\sigma| \geq k_* \end{cases}$$

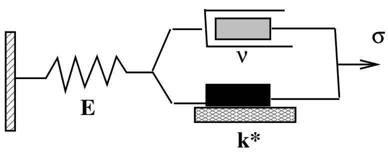


Figure I.8: Yield surface and plastic strain

3D аналогом этих соотношений в случае условия пластичности Мизеса являются

$$\dot{\varepsilon} = \begin{cases} \mathbb{L}^{-1}\dot{\sigma} & \text{if } |\sigma^D| < k_*, \\ \dot{\varepsilon}_{el} + \dot{\varepsilon}_{vpl} = \mathbb{L}^{-1}\dot{\sigma} + \frac{\dot{\sigma}}{\nu}\left(1 - \frac{k_*}{|\sigma^D|}\right) & \text{if } |\sigma^D| \geq k_* \end{cases}$$

A wide collection of models arises if different viscosity relations are combined with various yield laws.

Аналоги этих соотношений в рамках деформационной модели возникают если положить

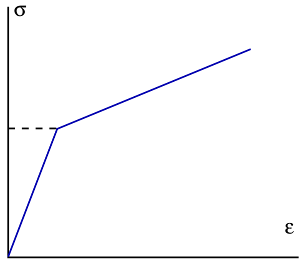
$$\psi^*(\tau) = \frac{1}{2}\mathbb{L}\tau : \tau + \frac{1}{2\theta}(\tau - \pi_K \tau)^2,$$

где π_K это ортогональный проектор на множество K. Нетрудно видеть, что производная Гато от $\frac{1}{2}(\tau - \pi_K\tau)^2$ равна $\tau - \pi_K\tau$, так что соотношение

$$\varepsilon(u) \in \partial\psi^*(\sigma).$$

приводит к

$$\epsilon(u) = \mathbb{L}^{-1}\sigma + \frac{1}{2\theta}(\sigma - \pi_K\sigma)$$



Если $\mathbb{K}$ определяется условием Мизеса, то

$$\pi_K(\tau) = \begin{cases} \tau & \text{if } |\tau^D| \le \sqrt{2}k_*, \\ \sqrt{2}k_* \frac{\tau^D}{|\tau^D|} & \text{if } |\tau^D| > \sqrt{2}k_*. \end{cases}$$

Поэтому

$$\inf_{\eta\in K} |\eta - \tau|^2 = \inf_{\eta\in K} \left(|\tau^D - \eta^D|^2 + \frac{1}{3}\mathrm{tr}(\eta - \tau)^2 \right) =$$

$$= \begin{cases} 0 & \text{if } |\tau^D| \le \sqrt{2}k_*, \\ \left|\tau^D - \sqrt{2}k_* \frac{\tau^D}{|\tau^D|}\right|^2 = \left||\tau^D| - \sqrt{2}k_*\right|^2 & \text{if } |\tau^D| > \sqrt{2}k_*. \end{cases}$$

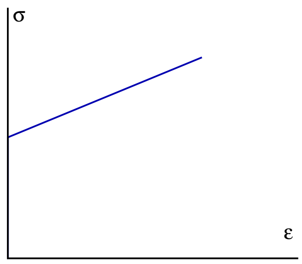
В этом случае

$$\psi^*(\tau) = \mathbb{L}^{-1}\tau + (|\tau^D| - \sqrt{2}k_*)_\oplus.$$

Эта модель может считаться регуляризацией модели Генки.

Жестко-вязкопластическая среда

$$\psi^*(\tau) = \frac{1}{2\mu}(\tau - \pi_K\tau)^2$$



Here $\epsilon(u) = \frac{1}{2\mu}\frac{\sigma^D}{|\sigma^D|}(|\sigma^D| - \sqrt{2}k_*)_+$ and $\varepsilon = 0$ if $\sigma \in K$.

I.6.4 Модель Друкера–Прагера

Здесь

$$\boxed{\mathbb{K} := \left\{\boldsymbol{\tau} \in \boldsymbol{L_2}(\boldsymbol{\Omega}, \mathbb{M}^{n\times n}) \;\mid\; \left|\boldsymbol{\sigma}^{\boldsymbol{D}}\right| + \boldsymbol{\alpha}(\mathrm{tr}\boldsymbol{\sigma}) \leq \sqrt{\mathbf{2}}\,\boldsymbol{k_*}\right\}, \qquad \boldsymbol{\alpha} > \mathbf{0}.}$$

В соответствии с (I.6.2), для того, чтобы построить вариационную задачу в перемещениях надо вычислить

$$\psi(\varepsilon) = \sup_{\substack{\tau\in\mathbb{M}_s^{n\times n},\\ \mathcal{F}(\tau)\leq 0}} \{\varepsilon : \tau - \frac{1}{2}a(\tau,\tau)\}$$

В общем случае $a(\tau,\tau) = \mathbb{L}^{-1}\tau : \tau$ и найти явное выражение ψ трудно. Однако для случая изотропного тела это сделать можно. Приведем соответствующий результат.

Определим

$$\phi(\varepsilon) = \begin{cases} \frac{K_0}{2}\left(\mathrm{tr}\varepsilon + \frac{\sqrt{2}k_*}{\alpha n K_0}\right)^2 + \mu|\varepsilon^D|^2 & \text{if } 2\mu|\varepsilon^D| + \alpha K_0 n \mathrm{tr}\varepsilon \leq 0,\\ \frac{k_*^2}{\alpha^2 K_0 n^2} + \frac{\sqrt{2}k_*}{\alpha n}\mathrm{tr}\varepsilon + D\left[(|\varepsilon^D| - \frac{1}{\alpha n}\mathrm{tr}\varepsilon)_{\oplus}\right]^2 & \text{if } 2\mu|\varepsilon^D| + \alpha K_0 n \mathrm{tr}\varepsilon > 0 \end{cases}$$

где

$$D = \frac{1}{2}\left(\frac{1}{2\mu} + \frac{1}{\alpha^2 K_0 n^2}\right)^{-1}.$$

Тогда

$$\psi(\varepsilon) = \phi\left(\varepsilon - \frac{\sqrt{2}k_*}{\alpha K_0 n^2}\mathbb{I}\right)$$

и мы приходим к следующей вариационной задаче: [3] найти $u \in V_0 + u_0$ такую, что

$$J(u) \leq J(v) \quad \forall v \in V_0 + u_0,$$

где

$$J(v) = \int\limits_{\Omega} \psi(\varepsilon)dx - L(v).$$

[3]Корректность этой вариационной проблемы доказана в работе S. Repin, G. Seregin Existence of a weak solution of the minimax problem arising in Coulomb-Mohr plasticity, Nonlinear evolution equations, 189220 - Amer. Math. Soc. Transl.(2), 1995

Нетрудно установить, что при $|\varepsilon^D| \leq \frac{1}{\alpha n}\mathrm{tr}\varepsilon$ функция ϕ (которая контролирует поведение внутренней энергии) удовлетворяет двойному неравенству

$$\beta\frac{\sqrt{2}k_*}{\alpha\sqrt{n}} \leq \frac{\Phi(\varepsilon)}{|\varepsilon|} - \frac{k_*^2}{\alpha^2 n^2 K_0|\varepsilon|} \leq \beta\frac{\sqrt{2}k_*}{\alpha\sqrt{n}}$$

Иначе говоря, эта функция имеет область, где ее рост (при возрастании $|\varepsilon|$) является линейным. Здесь, также как и для модели Генки невозможно доказать коэрцитивнисть функционала энергии в терминах стандартных рефлексивных пространств.

I.6.5 Модели с потенциалами степенного роста

Модель Рамберга–Осгуда (Ramberg-Osgood model) (используется для различных сплавов, в частности алюминиевых сплавов)

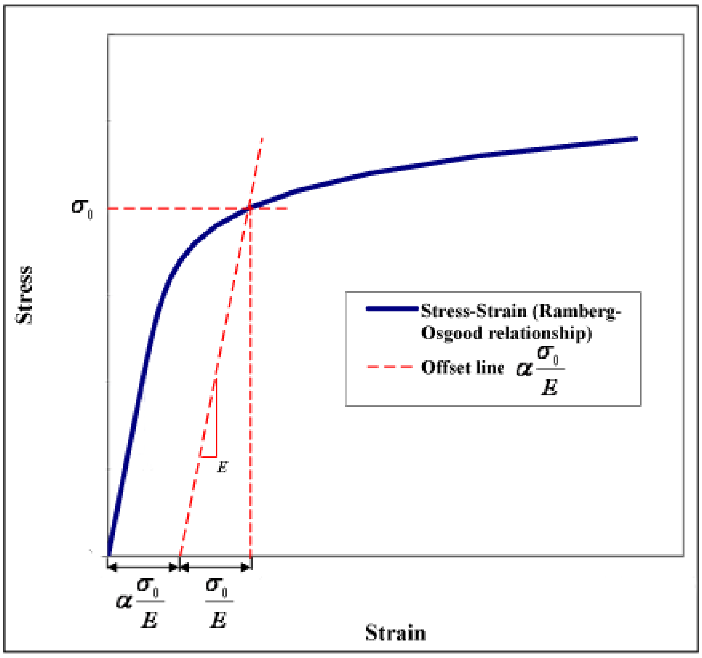
$$\psi^*(\tau) = \frac{1}{2}\mathbb{L}^{-1}\tau : \tau + \theta|\sigma^D|^q,$$

Реологический закон

$$\varepsilon(u) = \mathbb{L}^{-1}\sigma + \theta|\sigma^D|^{q-2}\sigma^D,\ q > 1.$$

Для простейшей модели (E – модуль Юнга)

$$\varepsilon = \sigma E + K\left(\frac{\sigma}{E}\right)^n$$



I.6.6 Упрочняющиеся среды

Поверхность текучести может существенно изменяться в процессе нагружения.

Введем так называемые параметры упрочнения $\zeta \in \mathbb{R}^m$ (hardening parameters), которые регулируют форму поверхности текучести.

Now $\mathcal{F} : \mathbb{M}^{d\times d} \times \mathbb{R}^m \to \mathbb{R}$ and admissible stress fields satisfy

$$\mathcal{F}_\delta(\tau, \zeta) \le 0.$$

Two main cases:
1. Isotropic hardening.

$$\mathbb{K}^I_\delta := \left\{(\tau, \zeta) \in \mathbb{M}^{d\times d} \times \mathbb{R} \mid |\tau^D| \le \sqrt{2}k_* + \delta\theta\right\}.$$

This is the simplest model with only one parameter ζ usually defined throughout the value of plastic rate.

2. Kinematic hardening. Let $\mathbb{M}^{d\times d}_D$ denote the space of $d \times d$ tensors with zero trace.

$$\mathbb{K}^K_\delta := \left\{(\tau, \zeta) \in \mathbb{M}^{d\times d} \times \mathbb{M}^{d\times d}_D \mid |\tau^D - \delta\zeta| \le \sqrt{2}k_*\right\}$$

These models can be presented by relations:

$$(\sigma, \zeta) \subset \partial\psi_\delta(\epsilon(u), 0),$$
$$(\epsilon(u), 0) \subset \partial\psi^*_\delta(\sigma, \zeta).$$

A consequent discussion can be found in:

B. Halpern, N. Q. Son. Sur les matériaux standard généralises. J. Mechanique, 14(1975), 39-63.
C. Johnson. On plasticity with hardening. J. Math. Anal. Appl., 62(1978), 325-336.

Numerical methods has been deeply studied in the works of I. Hlaváček, W. Han an D. Reddy.

I.7 Кручение упругопластического стержня

Finally, we consider one of the classical problems in solid mechanics: torsion of elasto-plastic beam.

We consider a long cylinder with constant cross section. Let us consider a part of it far from ends having length h. It occupies the cylindrical domain

Ω. Upper and lower faces of Ω we denote by Γ_0 and Γ_D. The lateral surface is denoted by and Γ_N (see Fig. (I.9).

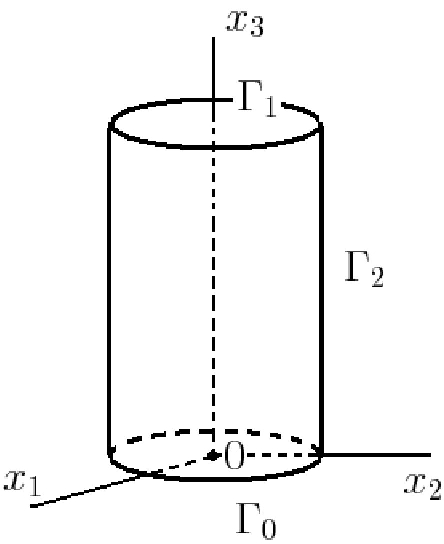


Figure I.9: Цилиндрический стержень

We assume that Γ_N is free from loads, Γ_0 and Γ_D are free from tension loads, and volume forces are negligibly small. Deformations are generated by torsion around the axis x_3 and α is the torsion angle per length unit. Formally, the above conditions are written as follows:

$$f = 0 \qquad \text{in } \Omega; \tag{I.7.1}$$
$$\sigma\nu = 0 \qquad \text{on } \Gamma_N; \tag{I.7.2}$$
$$\sigma_{33} = 0 \qquad \text{on } \Gamma_0 \text{ and } \Gamma_D; \tag{I.7.3}$$

Let

$$u_1 = u_2 = 0 \qquad \text{on } \Gamma_0; \tag{I.7.4}$$
$$u_1 = -\alpha h x_2 \qquad \text{on } \Gamma_D; \tag{I.7.5}$$
$$u_2 = +\alpha h x_1 \qquad \text{on } \Gamma_D. \tag{I.7.6}$$

It is easy to see that

$$\sigma\nu \cdot u_0 = 0 \qquad \text{on } \Gamma_0 \text{ and } \Gamma_N;$$

$$\sigma\nu \cdot u_0 = \sigma_{ij}\nu_j u_{0i} = \sigma_{i3} u_{0i} \qquad \text{on } \Gamma_D.$$

Therefore σ minimizes the energy functional

$$\frac{1}{2}a(\sigma,\sigma) - \alpha h \int_{\Gamma_D} (x_1\sigma_{23} - x_2\sigma_{13})\, d\Gamma$$

on the set on $K \cap Q_f$. In our case,

$$Q_f = \{\sigma \in \Sigma \mid \ \mathrm{div}\sigma = 0 \text{ in } \Omega,\ \sigma\nu = 0 \text{ on } \ \Gamma_N,\ \sigma_{33} = 0 \text{ on } \ \Gamma_0 \text{ and } \Gamma_D\}\,.$$

It can be shown, that minimizer has zero components except σ_{13} and σ_{23}. Then only one equilibrium equation is nontrivial:

$$\sigma_{13,1} + \sigma_{23,2} = 0. \tag{I.7.7}$$

We satisfy the equilibrium equation with the help of stream function $\theta = \theta(x_1, x_2)$ that satisfies the relations

$$\sigma_{13} = \sigma_{31} = \frac{\partial\theta}{\partial x_2};$$

$$\sigma_{32} = \sigma_{23} = -\frac{\partial\theta}{\partial x_1}.$$

Evidently, (I.7.7) holds. Now, the equation (I.7.2) comes in the form

$$\nu_1\sigma_{13} + \nu_2\sigma_{23} = \nu_1\frac{\partial\theta}{\partial x_2} - \nu_2\frac{\partial\theta}{\partial x_1} = 0.$$

Let $\tau = (\tau_1, \tau_2)$ (see Fig. I.10) be the tangential unit vector. Since $\tau_1 = -\nu_2$, $\tau_2 = \nu_1$,

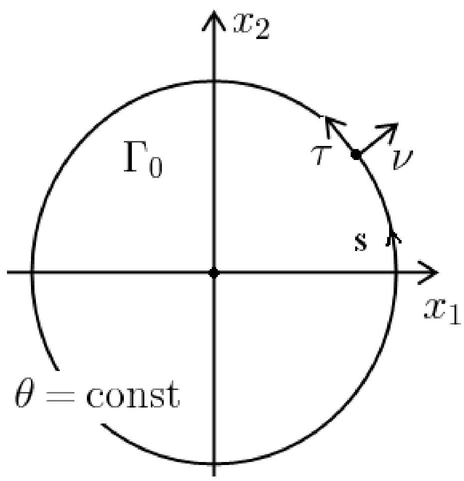


Figure I.10: Поперечное сечение цилиндра we find that

$$\nu_1 \frac{\partial \theta}{\partial x_2} - \nu_2 \frac{\partial \theta}{\partial x_1} = \tau \cdot \nabla \theta = \frac{\partial \theta}{\partial \tau} = \frac{\partial \theta}{\partial s},$$

where s is the tangential coordinate on Γ_0. Thus,

$$\frac{\partial \theta}{\partial s} = 0,$$

and we conclude that θ is constant on the boundary of Γ_0.

However, the function θ is defined up to a constant, so that without a loss of generality we assume that is zero on the boundary. We recall that $\sigma_{i3} \in L^2(\Omega)$, $i = 1, 2$, which means that we must consider θ as a function in $\overset{\circ}{H}(\Gamma_0)$.

Also, we must satisfy is the von Mises yield condition. Since

$$\sigma^D = \begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 0 \end{bmatrix}, \quad \left|\sigma^D\right|^2 = 2(\sigma_{13}^2 + \sigma_{23}^2),$$

this condition has the form

$$\sigma_{13}^2 + \sigma_{23}^2 \leq k_*^2 \quad \text{almost everywhere on } \Gamma_0.$$

In terms of the function θ, we have

$$|\nabla \theta|^2 \leq k_*^2 \quad \text{almost everywhere on } \Gamma_0.$$

For the isotropic media,

$$\mathbb{L}^{-1}\sigma : \sigma = \frac{1}{2\mu}\left|\sigma^D\right|^2 + \frac{1}{9K_0}(\mathrm{tr}\sigma)^2,$$

and (note that $\mathrm{tr}\sigma = 0$), we have

$$\frac{1}{2}a(\sigma, \sigma) = \int_0^h \left(\int_{\Gamma_0} \frac{1}{4\mu}\left|\sigma^D\right|^2 dx\right) dx_3 = \frac{h}{2\mu}\int_{\Gamma_0} |\nabla \theta|^2 \, dx,$$

where $dx = dx_1 dx_2$. Without a loss of generality, we assume that $h = 1$.

In this model, all cross sections in the central part of the beam are assumed to have identical stress fields (which does not depend on x_3). Hence, the variational problem has the form (after integration over x_3 form 0 to h): find $\theta^* \in K_\theta$ that minimizes the functional

$$\frac{1}{2}\int_{\Gamma_0} |\nabla \theta|^2 \, dx + \mu\alpha \int_{\Gamma_0} \left(x_1 \frac{\partial \theta}{\partial x_1} + x_2 \frac{\partial \theta}{\partial x_2}\right) dx,$$

where

$$K_\theta = \left\{ \theta \in \overset{\circ}{H}(\Gamma_0) \,\middle|\, |\nabla\theta| \le k_* \quad \text{a.e. on } \Gamma_0 \right\}.$$

In view of the relations

$$\int_{\Gamma_0} x_1 \frac{\partial\theta}{\partial x_1}\, dx = x_1\theta\Big|_{\partial\Gamma_0} - \int_{\Gamma_0} \theta\, dx = -\int_{\Gamma_0} \theta\, dx;$$
$$\int_{\Gamma_0} x_2 \frac{\partial\theta}{\partial x_2}\, dx = x_2\theta\Big|_{\partial\Gamma_0} - \int_{\Gamma_0} \theta\, dx = -\int_{\Gamma_0} \theta\, dx,$$

and finally we have the problem

$$\inf_{K_\theta} I^*(\theta), \tag{I.7.8}$$

where

$$I^*(\theta) = \frac{1}{2}\int_{\Gamma_0} |\nabla\theta|^2 \, dx - 2\mu\alpha \int_{\Gamma_0} \theta\, dx. \tag{I.7.9}$$

Define

$$\tilde{a}(\theta,\varphi) = \int_{\Gamma_0} \nabla\theta \cdot \nabla\varphi\, dx;$$
$$c = 2\mu\alpha;$$
$$l(\theta) = \int_{\Gamma_0} c\theta\, dx,$$

Represent the functional I.7.9 in the form

$$I^*(\theta) = \frac{1}{2}\tilde{a}(\theta,\theta) - l(\theta).$$

Then by above proved theorems,we conclude that θ^* satisfies the variational inequality

$$\tilde{a}(\theta^*, \varphi - \theta^*) \ge l(\varphi - \theta^*), \quad \forall \varphi \in K_\theta. \tag{I.7.10}$$

Evidently, θ^* exists and is unique.

Частный случай: стержень круглого поперечного сечения

We consider elasto-plastic beam which cross section is a circle of radius R, i.e.,

$$\Gamma_0 = \left\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < R^2\right\}$$

It is convenient to consider the problem in polar coordinate system (r, φ) and find an axisymmetric solution $\theta = \theta(r)$. Problem I.7.8 has the form

$$\inf_{\theta \in K} \int_0^R G(r, \theta, \theta_r) dr,$$

where

$$G(r, \theta, \theta_r) = \frac{1}{2}\,\theta_r^2\, r - c\,\theta\, r, \quad \theta_r = \frac{d\theta}{dr};$$
$$K = \left\{\theta \in H^1([0, R]),\, \theta|_R = 0 \mid |\theta_r| \le k_*\right\}.$$

If at a certain point $\theta < k_*$, then the corresponding Euler equation must be satisfied, i.e.,

$$\frac{d}{dr} G_{\theta_r} - G_\theta = 0.$$

Thus, we have

$$\frac{d}{dr}\,(r\theta_r) + cr = 0;$$
$$r\theta_r = -\frac{cr^2}{2} + D, \quad D = \text{const};$$
$$\theta_r = -\frac{cr}{2} + \frac{D}{r}.$$

Since the derivative θ_r must be bounded, we arrive at the conclusion that $D = 0$. Consequently,

$$\theta = E - \frac{c\,r^2}{4}, \quad E = \text{const}, \quad |\theta_r| = \frac{c\,r}{2}.$$

Inside the domain, we have the inequality

$$|\theta_r| \le k_*, \quad \text{если} \quad c\,r < 2k_*.$$

If c is small, then this solution is valid for all $r \in [0, R]$. In this case, we define E by the boundary condition $\theta(R) = 0$, i.e.,

$$E = \frac{c}{4} R^2.$$

We find that

$$\theta = \frac{c}{4}(R^2 - r^2) \quad \text{as} \quad c \le 2\frac{k_*}{R}.$$

If $c > 2\frac{k_*}{R}$, then for all $r \in [0, R']$, where $c = 2\frac{k_*}{R'}$, we have

$$\theta = E - \frac{c}{4}r^2, \tag{I.7.11}$$

and for $r \in [R', R]$ (since $\theta = 0$ at $r = R$), we have

$$\theta = k_*(R - r), \quad |\theta_r| = k_*. \tag{I.7.12}$$

Now E is defined by the continuity condition of $\theta(r)$ at $r = R'$

$$E - \frac{c}{4}R'^2 = k_*(R - R'), \quad R' = \frac{2k_*}{c};$$

$$E = \frac{c}{4}\frac{4k_*^2}{c^2} + k_*R - \frac{2k_*^2}{c} = k_*\left(R - \frac{k_*}{c}\right).$$

As a result, we find that (see Fig. I.11)

$$\theta(r) = \begin{cases} k_*(R - k_*/c) - cr^2/4 & \text{if } \ 0 \le r < R'; \\ k_*(R - r) & \text{if } \ R' \le r \le R. \end{cases}$$

We note that the function θ that we have constructed is continuous and has continuous derivative.

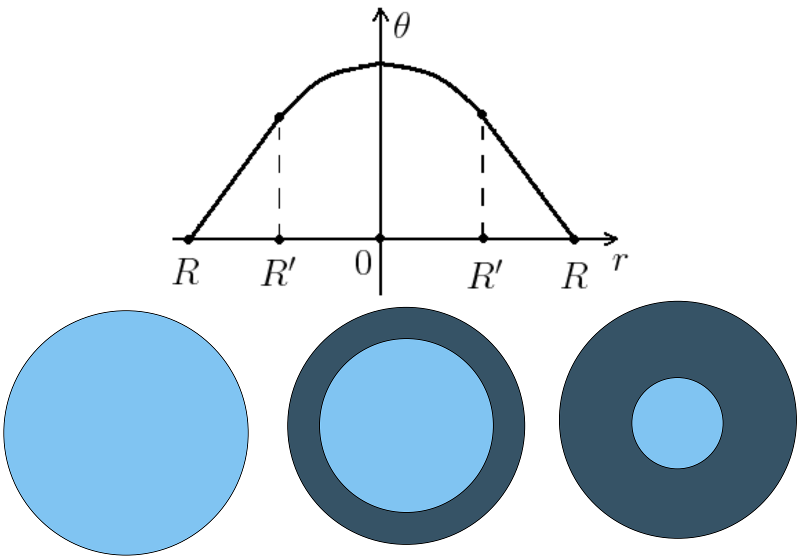


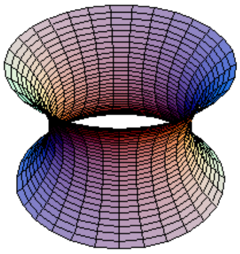
Figure I.11: Solution profile

I.8 Задачи с вариационными функционалами линейного роста

Существует важный класс вариационных задач в которых функционал имеет линейный рост на бесконечности относительно нормы соответствующего дифференциального оператора. Эти задачи сильно отличаются от других вариационных задач. В общем виде такой функционал представим в виде

$$\boldsymbol{J(v,\Omega) = \int\limits_{\Omega} g(v, \Lambda v)\, dx - \int\limits_{\Omega} f v\, dx.}$$

Здесь $\Lambda : V \to Y$ is a bounded linear operator,
$g : V \times Y \to \mathbb{R}$ это выпуклый функционал, имеющий линейный рост если $\|\Lambda v\|_Y \to +\infty$.

I.8.1 Important representatives for the case $\Lambda = \nabla$

$$\int\limits_{\Omega} g(\nabla v)\, dx \quad \text{(models of surfaces)},$$

$$\|v - g\|^2_{L_2} + \int\limits_{\Omega} g(\nabla v)\, dx \quad \text{(image recovery models)}.$$

$$J(v,\Omega) = \int\limits_{\Omega} \sqrt{1 + |\nabla v|^2}\, dx + \int\limits_{\Omega} j(v) dx \text{ minimal/capillary surface}$$

I.8.2 Case $\Lambda = \varepsilon(v)$: mechanical models with saturation

$$\boxed{\int_\Omega \boldsymbol{g(\varepsilon(v))\, dx} \quad \textbf{Internal energy : } \boldsymbol{g(\varepsilon) = \sigma : \varepsilon,}}$$

We have a certain constitutive law

$$\boxed{\boldsymbol{\sigma = C(\varepsilon)} \qquad [= \boldsymbol{\partial\psi(\varepsilon)}]}$$

- If $C(\varepsilon) = L\varepsilon$ (linear), then $\sigma : \varepsilon = L\varepsilon : \varepsilon$ (quadratic growth),
- If $C(\varepsilon) \geq c|\varepsilon|^\alpha$, $\alpha > 0$, then the energy $\sigma : \varepsilon$ has power growth.
- Assume that admissible stresses are bounded, i.e. $\sigma \in K$
  Then,

$$\boxed{\int_\Omega \boldsymbol{\sigma : \varepsilon(v) dx} \quad \text{has zones with linear growth}}$$

Example: $K = \{|\sigma^D| \leq k_*\}$.
Then $\int\limits_\Omega \sigma^D : \varepsilon^D dx$ cannot grow faster than $k_* \int\limits_\Omega |\varepsilon^D| dx$.

I.8.3 Specific difficulties of problems with linear growth

Problems with linear growth functionals lead to nonuniformly elliptic equations, e.g.,

$$-\mathrm{div}\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} - ku = f.$$

Usually they are incorrect in the initial form.

For scalar valued problems existence theory dates back to 60' and brings origin from the works of E. de Giorgi (1960-65). Later existence and regularity has been studied by many authors (Giaquinta, Anzellotti, Giusti, Modica, Finn...)

Mathematical correctness

$$\inf_V J(v), \quad J(v) := \int\limits_{\Omega} g(\nabla v)dx + < \ell, v >$$
$$\beta|\eta| - c_1 \leq g(\eta) \leq \beta|\eta| + c_2, \quad c_1, c_2 > 0.$$

What is suitable $\boldsymbol{V}$?

It is easy to see that J is coercive on $W^{1,1}$. Let us try to use this space as V. Consider a minimizing sequence $\{v_k\}$. It is clear that $\|v_k\|_{W^{1,1}} \leq C$

However, $W^{1,1}$ is not a reflexive space, therefore this boundedness does not mean that v_k tends to some $v \in V$ weakly (in terms of a subsequence).

Simplest example: $\int_a^b g(v')dx$, $a < 0$, $b > 0$.

$$v_\epsilon = \begin{cases} 0 & x \in [a,0), \\ \frac{1}{\epsilon}x, & x \in [0,\epsilon] \\ 1, & x\,[\epsilon, b] \end{cases}, \qquad v'_\epsilon = \begin{cases} 0 & x \in [a,0), \\ \frac{1}{\epsilon}, & x \in [0,\epsilon] \\ 0, & x\,[\epsilon, b] \end{cases},$$

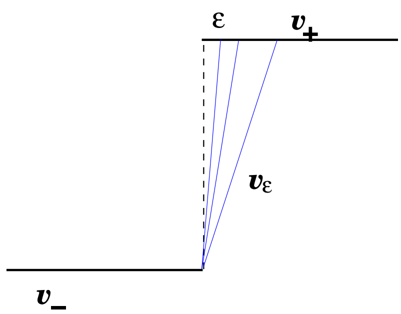
$$\int_a^b g(v'_\epsilon)dx \leq \int_0^\epsilon \beta|v'_\epsilon|dx + const = \int_0^\epsilon \beta\,\frac{1}{\epsilon}dx + const.$$

$J(v_\epsilon)$ is bounded! We cannot exclude such type sequence from those that might converge to infimum!

At almost all points of $[a,b]$ v_ϵ tends to the "step function",

$$\chi = \begin{cases} 0 & x \in [a,0), \\ 1, & x \in [0,b] \end{cases}$$

which does not belong to $W^{1,1}$.



Indeed, generalized derivative Dv is defined by

$$\int_a^b v\phi' dx = -\int_a^b Dv\phi dx \quad \forall \phi \in C_0^1(a,b),$$

However, whatever ϕ we take

$$\lim_{\epsilon\to 0}\int_a^b v_\epsilon \phi' dx = \int_{0-\epsilon}^{0+\epsilon} \phi' dx = \phi(0+\epsilon) - \phi(0-\epsilon) = \phi(0)$$

Hence Dv is the δ_0-function and the limit does not belong to $W^{1,1}$.

I.8.4 Nonparametric Plateau problem.

Ω is the ring $r \in (a,b)$. Axisymmetric minimizer of the nonparametric Plateau's problem minimises the functional

$$J(v) = \int_a^b \sqrt{1+(v')^2}\, r dr, \quad v(a) = U,\, v(b) = 0.$$

Equation

$$\frac{d}{dr}\left(\frac{rv'}{\sqrt{1+(v')^2}}\right) = 0 \;\Rightarrow\; \frac{rv'}{\sqrt{1+(v')^2}} = C.$$

$$\boxed{\frac{r^2(v')^2}{1+(v')^2} = C^2 \,\forall r \in [a,b]}.$$

It is clear that $C \leq a$.

$$\begin{aligned} v' &= -\frac{C}{\sqrt{r^2-C^2}}, \\ v &= -C\ln\left(r+\sqrt{r^2-C^2}\right) + D. \end{aligned}$$

$$v(b) = 0 \;\Rightarrow\quad D = C\ln\left(r + \sqrt{b^2 - C^2}\right),$$
$$v(a) = U \;\Rightarrow\quad \boxed{U = C\left[\ln\left(b + \sqrt{b^2 - C^2}\right) - \ln\left(a + \sqrt{a^2 - C^2}\right)\right]}$$

Since C is bounded, the second condition cannot be always satisfied, i.e., if U is sufficiently large, then the constant C cannot be defined.

For some boundary conditions solution of the problem may not exist in the set of continuous functions.

I.9 Bounded measures, BV, BD

By $\overset{\circ}{C}{}^{\infty}(\Omega)$ we denote the space of all continuous functions with compact supports in Ω that have continuous (classical) derivatives of any order. The sequence $\{\varphi_i\} \in \overset{\circ}{C}{}^{\infty}(\Omega)$ is said to be convergent to zero if

1. there exists a set $\Omega_1 \subset \Omega$ such that $\mathrm{supp}\varphi_i \subset \Omega_1$ for all $i \in \mathbb{N}$,
2. all derivatives of φ_i tend to zero uniformly as $i \to \infty$.

Definition. Linear functionals defined on the functions of the space $\overset{\circ}{C}{}^{\infty}(\Omega)$ are called distributions.

Traditionally, the space of distributions is denoted by $\mathcal{D}'(\Omega)$. The value of a distribution g on a function $\varphi \in \overset{\circ}{C}{}^{\infty}(\Omega)$ is denoted by $\langle g, \varphi\rangle$. We say that the distributions g_1 and g_2 are equal in Ω if

$$\langle g_1, \varphi\rangle = \langle g_2, \varphi\rangle, \quad \forall\varphi \in \overset{\circ}{C}{}^{\infty}(\Omega).$$

If a distribution can be identified with a locally integrable function, then it is called *regular*. In this case, the action of g is given by the Lebesgue integral

$$\langle g, \varphi\rangle = \int_\Omega g\varphi\, dx.$$

Other distributions are called *singular* (e.g., δ-function, $\langle \delta, \varphi\rangle = \varphi(0)$).

Distributions have derivatives of any order if differentiation is understood in a special (generalized) sense.

Definition. Let g be a distribution. Its generalized derivative $D^\alpha g$ is a linear functional defined for any $\varphi \in \overset{\circ}{C}{}^\infty(\Omega)$ by the following rule:

$$\langle D^\alpha g, \varphi\rangle := (-1)^{|\alpha|}\langle g, D^\alpha \varphi\rangle. \tag{I.9.1}$$

I.9.1 Spaces of bounded measures

Consider distributions of a special kind: let Ω be a bounded domain. Distributions such that

$$\sup_{\phi\in \overset{\circ}{C}{}^\infty(\Omega),\ |\varphi|\le 1} < \mu, \varphi > \ := \int\limits_\Omega |\mu| < +\infty$$

form the space $M(\Omega)$ of bounded measures on Ω.

It is a Banach space with respect to the above norm.

Example 1: $L_1(\Omega) \subset M(\Omega)$. Indeed, let $\mu = hdx$, where $h \in L_1(\Omega)$. Then,

$$< \mu, \varphi >= \int\limits_\Omega \phi\, hdx < +\infty.$$

Example 2: δ function belongs $M(\Omega)$.

$$\sup_{\phi\in \overset{\circ}{C}(\Omega),\, |\varphi|\le 1} < \mu, \varphi >:= \phi(0) \le 1$$

Remark. We can take sup over $\overset{\circ}{C}\ (\Omega)$ and, therefore, $M(\Omega)$ can be considered as topologically conjugate to $\overset{\circ}{C}\ (\Omega)$.

Analogous definitions can be introduced for spaces of vector functions.

$M(\Omega, \mathbb{R}^d)$ is a Banach space with respect to the norm:

$$\sup_{\phi\in \overset{\circ}{C}{}^\infty(\Omega,\mathbb{R}^d),\ |\varphi|_{\mathbb{R}^d}\le 1} < \mu, \varphi >:= \int\limits_\Omega |\mu| < +\infty$$

I.9.2 Space $BV(\Omega)$

L^1 functions with generalized derivatives in $M(\Omega)$ form the space BV of functions of bounded variation BV.

$$BV(\Omega) := \left\{ v \in L^1(\Omega)\ v_i \in M(\Omega) \right\}$$

$$\|v\|_{BV} := \|v\|_{L^1} + \sum_i \|v_{,i}\|_M$$

Compare with the space $W^{1,1}(\Omega)$. We can define the norm of $W^{1,1}$ as

$$\|v\|_{1,1} := \|v\|_{L^1} + \sum_i \|v_{,i}\|_{L^1}.$$

Another form

$$\begin{aligned} &BV(\Omega) := \left\{ v \in L^1(\Omega) \ \mid \ \nabla v \in M(\Omega, \mathbb{R}^d) \right\}, \\ &\|v\|_{BV} := \|v\|_{L^1} + \int_\Omega |Dv| \le +\infty, \end{aligned}$$

Here

$$\int_\Omega |Dv| := \sup_{\phi \in \overset{\circ}{C}^\infty(\Omega, \mathbb{R}^d),\ |\varphi|_{\mathbb{R}^d} \le 1} \int_\Omega v \operatorname{div} \phi dx. \qquad Hint:\ ... = \int_\Omega \nabla u \cdot \phi dx$$

By some properties BV is similar to $W^{1,1}$:

- $$W^{1,1} \subset BV(\Omega), \quad \text{but } W^{1,1} \neq BV(\Omega)\,!$$
- For domains with smooth boundaries $v \in BV(\Omega)$ has a trace on $\partial\Omega$, which is in $L^1(\partial\Omega)$. Hence, we can define boundary conditions.
- BV is continuously embedded in $L^{d/(d-1)}$, $d \ge 2$, for bounded domains and $p \in [1, d/(d-1))$ embedding is compact.
- BV is not separable and regular functions does not create a dense subset (in terms of the original norm) of BV. However, BV functions can be approximated by regular functions in weaker norms.

I.9.3 BD(Ω)

This space was introduced by P. Suquet and studied in works of G. Anzelotti, M. Giaquinta, R. Kohn, G. Streng, R. Temam,...

> L^1 vector functions having that have ε_{ij} in $M(\Omega)$ form the space BD of **functions of bounded deformation**.

$$BD(\Omega) := \left\{v \in L^1(\Omega, \mathbb{R}^d)\ \varepsilon_{ij}(v) \in M(\Omega)\right\}$$

$$\|v\|_{BD} := \|v\|_{L^1} + \sum_i \|\varepsilon_{ij}(v)\|_M$$

Another definition

$$\|v\|_{BD} := \|v\|_{L^1} + \int\limits_\Omega |\varepsilon(v)| \leq +\infty,$$

$$\int\limits_\Omega |\varepsilon(v)| = \sup_{\tau \in \overset{\circ}{C}^\infty(\Omega, \mathbb{M}^{d\times d}),\ |\tau|_{\mathbb{M}^{d\times d}} \leq 1} \int\limits_\Omega u \cdot \mathrm{Div}\tau dx.$$

By properties BD is quite analogous to BV:

- $$W^{1,1}(\Omega, \mathbb{R}^d) \in BD(\Omega), \quad W^{1,1}(\Omega, \mathbb{R}^d) \neq BD(\Omega)!$$
- For domains with smooth boundaries (even for C^1) BV functions have L^1 traces that for continuous functions coincide with the classical trace.

  on $\partial\Omega$. Hence, we can define boundary conditions.
- BD is continuously embedded in $L^{d/(d-1)}$, $d \geq 2$, for bounded domains and $p \in [1, d/(d-1))$ embedding is compact.
- Regular functions does not create a dense subset (in terms of the original norm) of BD. However, BV functions can be approximated by regular functions in weaker norms.

I.9.4 Convex functionals of bounded measures.

Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a convex function having linear growth and such that $\psi^*(\xi^*) := \sup_{\xi\in\mathbb{R}^d}(\xi^*\xi - \psi(\xi))$ is uniformly bounded on $\operatorname{dom}\psi^*$.

Let μ be a bounded (Radon) measure. Define

$$\Psi(\mu) = \int\limits_\Omega \psi(\mu),$$

where $\psi(\mu)$ is a new Radon measure generated by ψ. We can define it as follows:

$$\Psi(\mu) = \sup_{v\in \overset{\circ}{C}{}^\infty(\Omega)} \left\{ \int\limits_\Omega v\mu - \int\limits_\Omega \psi^*(v)dx \right\} = \sup_{v\in \overset{\circ}{C}{}^\infty(\Omega)} \left\{ <\mu, v> - \int\limits_\Omega \psi^*(v)dx \right\}$$

A measure can be decomposed into regular and singular parts: $\boxed{\mu = hdx + \hat{\mu}.}$

In the theory of measures, it is proved: $\boxed{\Psi(\mu) = \int\limits_\Omega \psi(h(x))dx + \int\limits_\Omega \psi_\infty(\hat{\mu})}$

Actually, the second term allows to compute singular components of measures which in perfect plasticity generate "penalties for discontinuity".

Variational problems with linear growth functionals are defined on spaces of functions that may contain singular components.

I.9.5 Relaxation of the minimal surface problem

Classical minimal surface problem:

$$\inf_{w\in \overset{\circ}{W}{}^{1,1}+u_0(\Omega)} \int\limits_\Omega \sqrt{1+|\nabla w|^2}.$$

Relaxation of the minimal surface problem:

$$\inf_{w\in BV(\Omega)+u_0} \int\limits_\Omega \sqrt{1+|\nabla w|^2}.$$

Here $\sqrt{1+|\nabla w|^2}$ is a bounded measure in $BV(\Omega)$.

De Giorgi problem:

Let $\partial\Omega \in C^2$ and u_0 be a smooth function in Ω. The minimal surface problem can be reformulated as follows:

$$\inf_{w\in W^{1,1}(\Omega)} \int\limits_\Omega \sqrt{1+|\nabla w|^2} + \int\limits_{\partial\Omega} |w-u_0|ds.$$

One can prove that this problem has the same lower bound as the classical minimal surface problem

Moreover, if the classical problem possesses a minimizer, then it coincides with the solution of relaxed formulation.

Now above considered axisymmetric problem is reduced to minimization of the functional

$$\int_a^b \sqrt{1+(v')^2}rdr + a|v(a) - U|, \quad v(b) = 0$$

For $U \leq U^*$ we can define the solution that minimizes the functional as it has been done. It satisfies the corresponding differential equation inside Ω and satisfies the condition $v(0) = U$.

Let $U > U^*$. Necessary conditions:

$$\frac{rv'}{\sqrt{1+(v')^2}} = C, \quad v(b) = 0,$$

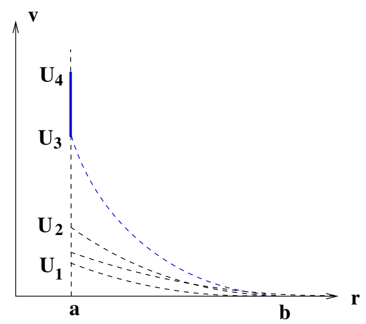
lead to

$$v = C\left(\ln\left(r + \sqrt{b^2 - C^2}\right) - \ln\left(r + \sqrt{r^2 - C^2}\right)\right).$$

Now instead of $v(a) = U$ we have a weaker condition: $\boxed{|v(a) - U| \to \ min}$ which leads to

$$v(a) = C\left(\ln\left(r + \sqrt{b^2 - C^2}\right) - \ln\left(a + \sqrt{a^2 - C^2}\right)\right) \to \ max,$$

i.e., the constant C must be taken as large as it is possible and the generalized solution looks as



I.9.6 Hencky problem

We recall the Lagrangian

$$L(\sigma, u) = \int_{\Omega} \left(\varepsilon(u) : \sigma - \frac{1}{2}\mathbb{L}^{-1}\sigma : \sigma - f \cdot u \right) dx - \int_{\Gamma_N} F \cdot u \, d\Gamma.$$

and the energy functional

$$J(u) := \int_{\Omega} \left(\frac{K_0}{2}(\mathrm{div} u)^2 + \Phi\left(\left|\varepsilon^D(u)\right|\right) \right) dx - \int_{\Omega} f \cdot u dx - \int_{\Gamma_N} F \cdot u d\Gamma.$$

Model problem

$\Omega = (a, b) \times [-\pi, \pi)$. Polar coordinates $r \in (a, b)$, $\theta \in [-\pi, \pi)$. Axisymmetric solution of the elasto-plasic torsion problem in a ring type domain, where the inner boundar $(r = a)$ is clamped: $\mathbf{v}(a) = 0$.

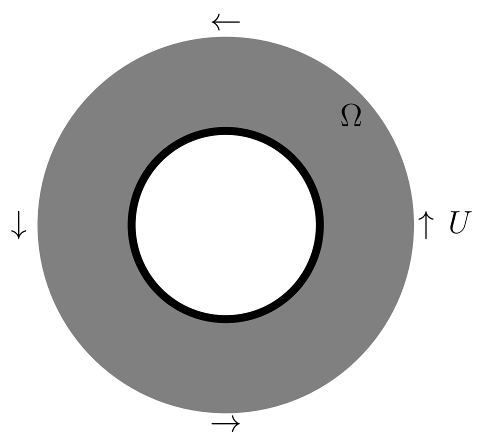


Figure I.12: Domain Ω.

Let $v = (v_r, v_\theta)$, $v_r(b) = 0$ and $v_\theta(b) = U$. In this case, we will find the solution such that $v_r = 0$ and $v_\theta = v_\theta(r)$.

We have

$$\varepsilon_{rr} = \tfrac{\partial v_r}{\partial r} = 0, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} = 0,$$

$$\varepsilon_{r\theta} = \tfrac{1}{2}\left(\tfrac{1}{r}\tfrac{\partial v_r}{\partial \theta} + \tfrac{\partial v_\theta}{\partial r} - \tfrac{v_r}{r}\right) = \tfrac{1}{2}\tfrac{\partial v_r}{\partial \theta}, \quad |\varepsilon^D| = \frac{1}{2}\left|\frac{\partial v_\theta}{\partial r}\right|$$

Henceforth, for simplicity we write v instead of v_θ.

The set $V_0 + u_0 := \{v \in H^1((a,b)),\ v(a) = 0,\ v(b) = U\}$ contains u_0 such that $u_0(b) = U$.

Stress tensor σ contains only one component: shear stress

$$\sigma_{rr} = 0, \quad \sigma_{r\theta} = \tau, \quad \sigma_{\theta\theta} = 0; \quad |\sigma^D| = \sqrt{2}|\tau|.$$

Then $\sigma : \varepsilon = \tau \, \frac{dv_r}{dr}$ and the condition $|\sigma^D| \le \sqrt{2}k_*$ reads

$$|\tau| \le k_*.$$

For simplicity, we henceforth assume that $k_* = 1$. Then

$$L(v,\tau) := \int_a^b \left(\frac{dv_r}{dr}\tau - \frac{1}{2}|\tau|^2 \right) r dr$$

and

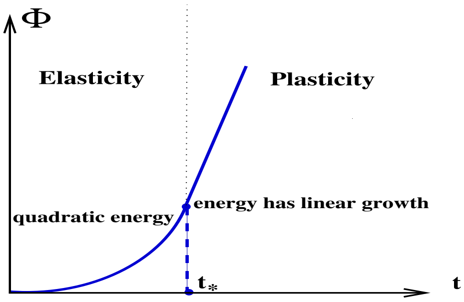
$$\tau \in K := \Big\{ L^2(a,b),\ |\tau| \le 1 \Big\}.$$

Primal problem:

$$J(u) = \inf_{v \in V_0+u_0} J(v); \qquad J(v) = \sup_{\tau \in K} L(v,\tau) = \int_a^b \Phi\left(\frac{dv}{dr}\right) r dr, \tag{I.9.2}$$

where

$$\Phi(t) = \begin{cases} \frac{1}{2}t^2 & |t| \le 1, \\ t - \frac{1}{2} & |t| > 1. \end{cases}$$



Dual problem:

$$\sup_{\tau \in K} \inf_{v \in V_0+u_0} L(v,\tau). \tag{I.9.3}$$

$$\inf_{v \in V_0+u_0} L(v,\tau) = \inf_{v \in V_0+u_0} \int_a^b \left((\frac{du_0}{dr} + \frac{dw}{dr})\tau - \frac{1}{2}|\tau|^2 \right) r dr,$$

where $w(a) = w(b) = 0$. Hence, we arrive at the following simple problem

$$\sup_{\tau \in Q_f \cap K} I(\tau),$$

where

$$I(\tau) = \int_a^b \left((\frac{du_0}{dr}\tau - \frac{1}{2}|\tau|^2 \right) r dr,$$

$$Q_f := \left\{ \tau \in L_2(a,b) : \int_a^b \frac{dw}{dr}\tau r dr = 0 \quad \forall w \in V_0 \right\}$$

All the functions in Q_f satisfy

$$\frac{d}{dr}(\tau\, r) = 0,$$

i.e., $\boxed{\tau = \frac{C}{r}.}$ Therefore,

$$I(\tau) = \int_a^b \left(\frac{du_0}{dr} C - \frac{1}{2}\frac{C^2}{r} \right) dr = CU - C^2 \frac{1}{2} \ln \frac{b}{a}.$$

Recall that

$$|\tau| \le 1 \;\Rightarrow\; \left| \frac{C}{r} \right| \le 1 \;\Rightarrow\; |C| \le\; a.$$

In fact, we arrive at the problem

$$\inf_{|C| \le a} \frac{1}{2} C^2 \ln \frac{b}{a} - CU.$$

Unconditional minimum is attained if

$$C = \frac{U}{\ln(\frac{b}{a})} \le a,$$

what is possible if $U \le U^* := a \ln(\frac{b}{a})$.

If $U > U^*$, then such a solution is impossible and instead $C = a$. Hence, $\bar{\tau}$ (solution of the dual problem) exists and has the form:

$$\bar{\tau} = \begin{cases} \frac{1}{r}\frac{U}{\ln(\frac{b}{a})} & U \le U^*, \\ \frac{a}{r} & U > U^*. \end{cases}$$

Let $U > U^*$.
Then $\sigma = \frac{a}{r}$ and for all $r \in (a, b]$, we have $|\bar{\tau}| < 1$. In other words, the stress is in "elastic zone".

Now, we need to find the corresponding displacement. For this purpose we use the stress-strain relations written in the form of the relation $\boxed{\sigma \in \partial\Phi(\varepsilon)}$
In our case, it reads

$$\bar{\tau} = \partial\Phi(u_{,r}) \quad \Rightarrow \quad \bar{\tau} = \begin{cases} u_{,r} & |u_{,r}| \leq 1, \\ 1 & |u_{,r}| > 1. \end{cases}$$

Thus, the solution u satisfies the equation

$$\frac{du}{dr} = \frac{a}{r} \quad \Rightarrow \quad \boxed{u = a \ln r + D}$$

$$a \ln b + D = u(b) = U \;\; \Rightarrow \;\; D = U - a \ln b,$$

We find that $\boxed{u = U - a \ln \frac{b}{r}}$

If $U = U > U^* = U^* = a \ln(\frac{b}{a})$, then the boundary condition at $r = a$ cannot be satisfied and the problem (I.9.2) **cannot have a continuous solution.**

We need to modify (I.9.2) and replace it by the *relaxed* problem

$$\inf_{v \in V^+} J^+(v); \qquad J^+(u) = \int_a^b \Phi\left(\frac{dv}{dr}\right) r dr + a\,|v - U| \tag{I.9.4}$$

where

$$V^+ := \Big\{ v \in H^1(a, b), \; v(a) = 0 \Big\}.$$

Relaxation of the Hencky problem

A relaxation (lower semicontinuous extension) should be carried out in order to obtain a well–posed variational posing. Relaxation means that we introduce ab extended set V^+ such that $V \subset V^+$ and extend the functional J such that $J(v) = J^+(v)$ for all $v \in V$.
Key requirement: lower bound must be the same:

$$\boxed{\inf \boldsymbol{\mathcal{P}} = \inf \boldsymbol{\mathcal{P}}^+}$$

If the problem $\mathcal{P}^+$ is correct (has a solution), then we obtain a complete relaxation of an incorrect variational problem. Otherwise, the relaxation is *partial*.

Relaxation of the Hencky problem

$$J(v) = \int\limits_\Omega \left(\frac{K_0}{2} (\mathrm{div} v)^2 + \Phi\left(\left|\varepsilon^D(v)\right|\right) \right) dx - \int\limits_\Omega f \cdot w dx - \int_{\Gamma_2} F \cdot v ds.$$

$$\Phi(\gamma) = \begin{cases} \mu\gamma^2 & \text{if} \quad 0 \leq \gamma < \frac{k_*}{\sqrt{2}\mu}; \\ \sqrt{2}k_*\gamma - k_*^2/(2\mu) & \text{if} \quad \gamma \geq \frac{k_*}{\sqrt{2}\mu}. \end{cases}$$

We introduce the

$$V^+ := \left\{ v \in BD(\Omega) \mid \mathrm{div} \in L^2(\Omega) \right\}$$

and define

$$J^+(v) = \int\limits_\Omega \frac{K_0}{2} (\mathrm{div} v)^2 dx + \int\limits_\Omega \Phi\left(\left|\varepsilon^D(v)\right|\right) - \int\limits_\Omega f \cdot w dx - \int_{\Gamma_2} F \cdot v ds.$$

where the second term is understood in the sense of measures.

It has been proved that

$$\inf \mathcal{P} = \inf \mathcal{P}^+.$$

Problem $\mathcal{P}^+$ is too abstract to be directly applied for creating numerical methods. We should better use Problem $\widehat{\mathcal{P}}$, in which the extended functional is defined on a set $\widehat{V}$ such that $\boxed{V_0 + u_0 \subset \widehat{V} \subset V^+}$ and $\widehat{V}$ contains a selection of certain discontinuous displacement fields.

Problem $\widehat{\mathcal{P}}$ (partially relaxed) Find $\widehat{u} \in \widehat{V}$ such that

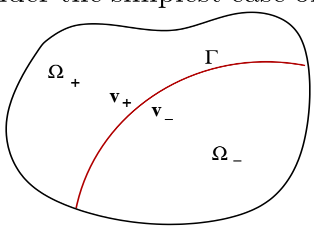
$$\widehat{I}(\widehat{u}) = \inf\{\widehat{I}(v) \mid v \in \widehat{V}\}$$

Properties of the above problems are as follows:

$$\left.\begin{array}{l}\text{Problem}\,\mathcal{P}\\ \text{Problem}\,\widehat{\mathcal{P}}\\ \text{Problem}\,\mathcal{P}^+\end{array}\right\} \Leftrightarrow \quad \text{have the same dual problem}\,\mathcal{P}^*$$

$$\inf \mathcal{P} = \inf \widehat{\mathcal{P}} = \inf \mathcal{P}^+ = \sup \mathcal{P}^*$$

Practical case: we admit discontinuities (jumps) on line(s) Γ, i.e. $\widehat{V}$ consists of piecewise continuous functions. Consider the simplest case of only one



discontinuity line
Here

$$\widehat{V} = \quad \{v = v^- \,\mathrm{in}\, \Omega^-,\, v = v^+ \,\mathrm{in}\, \Omega^+,\; v_i \in V_0 + u_0,\, i = 1, 2\}$$

and ν is the unit vector normal to the line γ which separates the domains Ω^- and Ω^+ (ν is the outer normal for Ω^-). Then

$$\widehat{I}(v) = \int\limits_{\Omega} \psi(\varepsilon(v))dx - \ell(v) + \Upsilon_\Gamma(v^+ - v^-),$$

where

$$\Upsilon_\Gamma(v^+ - v^-) = \sup_{\tau \in K} \int_\Gamma \tau : S(\nu, v^+ - v^-)\, dl,$$

$$S(a,b) = \tfrac{1}{2}(a \otimes b + b \otimes a) \ \ \forall a, b \in \mathbb{R}^n$$

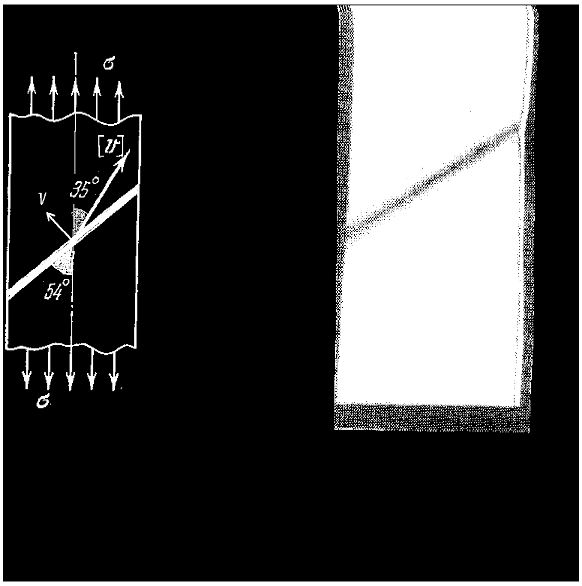
and $\otimes$ denotes the tensor product in $\mathbb{R}^n$. Let

$$[v(x)] = v^+(x) - v^-(x) \ \ x \in \gamma$$

denote the "jump" of v at the point x. The functional $\Upsilon_\Gamma([v(x)])$ depends on the type of plasticity model. It may admit only slip type discontinuities (for Mises condition) or also normal jumps.

Example.

Plane stress elasto-plastic problem with von Mises yield condition. Here, ν and the "jump"–vector v which are nontrivial solutions of the above discussed system are shown at the left hand side
and the physical experiment is presented at the right (picture is taken from `L. M. Kachanov, Plasticity theory, M. Nauka, 1969`).



Замечание I.9.1. The problem $\widehat{\mathcal{P}}$ also can be formulated as a minimax problem for the Lagrangian

$$L_\Gamma(v,\tau) = L(v,\tau) + \Upsilon_\Gamma([v])$$

The minimax formulation

$$L_\Gamma(\sigma, \widehat{v}^*) \leq L_\Gamma(\sigma^*, \widehat{v}^*) \leq L_\Gamma(\sigma^*, \widehat{v}), \quad \hat{v} \in \hat{V} \tag{I.9.5}$$

implies numerical strategies that have been earlier discussed.

Глава II

НЕЛИНЕЙНЫЕ КРАЕВЫЕ УСЛОВИЯ

Как правило в "академических" формулировках краевых задач используются условия Дирихле и Неймана. Однако при изучении моделей механики сплошной среды приходится использовать существенно более сложные краевые условия, такик как условия контакта, трения, мягкого основания и т.п. Большую часть этих условий можно рассматривать в рамках единого подхода, основанного на понятии граничного потенциала

II.1 Граничный потенциал

Let Γ be a part of Lipschitz booundary $\partial\Omega$. We introduce normal and tangential components of u and σ on Γ:

$$\begin{aligned}
&u_n = u \cdot n = u_i n_i \ (scalar), \quad u_t = u - (u \cdot n)n \ (vector),\\
&\sigma_n = \sigma n \ (vector),\\
&\sigma_{nn} = \sigma_n \cdot n = \sigma_{ij} n_i n_j \ (scalar), \quad \sigma_t = \sigma_n - \sigma_{nn} n \ (vector)
\end{aligned}$$

On Γ:

$$\boxed{\boldsymbol{u} = \boldsymbol{u_n n} + \boldsymbol{u_t}, \qquad \text{and} \quad \boldsymbol{\sigma_n} = \boldsymbol{\sigma_{nn} n} + \boldsymbol{\sigma_t}}$$

We present boundary conditions in a unified form:

$$\boxed{-\boldsymbol{\sigma_n}(\boldsymbol{x}) \in \boldsymbol{\partial j}(\boldsymbol{u}(\boldsymbol{x})) \qquad \boldsymbol{x} \in \boldsymbol{\Gamma}\,,} \tag{II.1.1}$$

Here $j : \mathbb{R}^d \to \mathbb{R}$ is a convex lower semicontinuous functional (boundary dissipative potential)

We will see that various boundary conditions can be taken into account by adding

$$\int_\Gamma j(u)\,ds$$

to the energy functional.

II.1.1 Examples

Neumann condition.

Set $j(u) \equiv Fu$, which means that $\sigma_n = F$ on Γ.

Dirishlet condition condition.

If

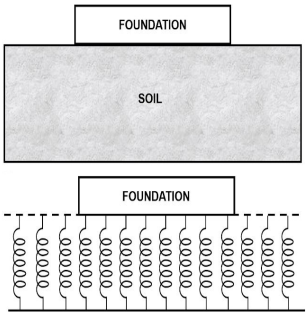
$$j(u) = \begin{cases} 0 & \text{if } u = u_0, \\ +\infty & \text{if } u \neq u_0, \end{cases}$$

then the functional is finite (and equals zero) if $u = u_0$ and infinite in all other cases.

Winkler law

This boundary condition

$$j(u_n) = \frac{1}{2} k u_n^2, \qquad \boxed{-\sigma_n = k u_n}$$



Contact with absolutely rigid obstacle ψ

Here

$$j(u_n) = \begin{cases} 0, & u_n - \psi \le 0 \\ +\infty & u_n - \psi > 0 \end{cases}$$

$$-\sigma_n = \partial j(u_n) = \begin{cases} 0, & u_n - \psi \le 0 \\ \big[0, +\infty), & u_n - \psi = 0 \end{cases}$$

These are Signorini type conditions[1]

II.2 Traces of functions.

In mechanics, boundary conditions are often understood in the classical (pointwise) sense. However, in a formal mathematical statement, they must be associated with properties of the corresponding energy space containing a weak solution.

Let $V = H^1(\Omega)$.

By $\gamma \in \mathcal{L}\big(H^1(\Omega), H^{1/2}(\Gamma)\big)$ we denote the trace operator of H^1–functions on the boundary Γ.

It is proved that the space of traces $H^{1/2}(\Gamma)$ is Banach space with the norm

$$\|\phi\|^2_{H^{1/2}} := \|\phi\|^2_{L^2} + \int_\Gamma \int_\Gamma \frac{|\phi(x_1) - \phi(x_2)|^2}{|x_1 - x_2|^d} dx_1 dx_2$$

[1]Signorini problem: what will be the equilibrium configuration of the orange spherically shaped elastic body resting on the blue rigid frictionless plane?.

Another definition of the norm is:

$$\|\phi\|_{H^{1/2}} = \inf_{\substack{v\in H^1(\Omega)\\ \gamma v=\phi}} \|v\|_{H^1(\Omega)}$$

$H^{1/2}(\Gamma)$ continuously embedded in $L_2(\Gamma)$ and is dense in it.
The space $H^1_0(\Omega)$ is the kernel of γ.

For any function $\phi \in H^{1/2}(\Gamma)$, one can define the continuation (lifting) operator $\mu \in \mathcal{L}(H^{1/2}(\Gamma), H^1(\Omega))$ such that $\mu\phi = w$, $\gamma w = \phi$ on Γ and

$$\|\phi\|_{1/2,\Gamma} \leq c_\gamma \|w\|_{1,\Omega}, \qquad \|w\|_{1,\Omega} \leq c_\mu \|\phi\|_{1/2,\Gamma}, \tag{II.2.1}$$

where $\|\cdot\|_{1,\Omega}$ and $\|\cdot\|_{1/2,\Gamma}$ are norms in H^1 and $H^{1/2}$, respectively.

Traces of vector valued functions

Ω – Lipschitz domain, Γ_0 is a measurable part of $\partial\Omega$. Let $v \in V := H^1(\Omega, \mathbb{R}^d)$.

For any $v \in V$, the operator

$$\gamma^d v := (\gamma v_1, \gamma v_2, ...\gamma v_d)$$

defines the trace of a vector–valued function. The range of the operator γ^d is $H^{1/2}(\Gamma, \mathbb{R}^d)$.

Analogously, one can define the continuation operator μ^d.

The operators γ^d and μ^d inherits the continuity property of γ and μ and meet inequalities analogous to (II.2.1).

If at almost all points n is uniquely defined, then we define the normal component of $\gamma^d v$ as

$$\gamma^d_n v := (\gamma^d v) \cdot n.$$

It is easy to check that for a smooth function v the quantity $\gamma^d_n v$ coincides with v_n.

Next, by the operator γ^d we define the subspace

$$V_0 := \{v \in V \mid \gamma^d v = 0 \text{ a.e. on } \Gamma_0\},$$

The question is how to undrstand $\boxed{\sigma n = F \qquad \text{on } \Gamma}$.

Assume that

$$\sigma \in L^2(\mathbb{M}^{d\times d}, \Omega), \qquad \mathrm{Div}\sigma \in L_2(\mathbb{R}^d, \Omega), \Rightarrow \sigma \in H(\Omega, \mathrm{Div}).$$

Can we define normal traces for such tensor fields?

Trace σ_n

Really, for any smooth τ^* and any $v \in V_0$, we have the relation

$$\int_\Gamma \tau_n^* \cdot \gamma^d v\, dx \;=\; \underbrace{\int_\Omega (\tau^* : \epsilon(v) + \mathrm{div}\tau^* \cdot v)\, dx}_{=\Lambda_{\tau^*}(v)} . \tag{II.2.2}$$

For any $\tau^* \in H(\Omega, \mathrm{Div})$, the right–hand side of this identity is a linear continuous functional $\boxed{\Lambda_{\tau^*} : V \to \mathbb{R}}$.

Лемма II.2.1. *The functional Λ_{τ^*} satisfies the following relations:*

$$\Lambda_{\tau^*}(v) \;=\; 0 \qquad \forall v \in H_0^1(\Omega, \mathbb{R}^d)\,, \tag{II.2.3}$$

$$|\Lambda_{\tau^*}(v)| \le c_\mu \, \|\tau^*\|_{H(\Omega,\mathrm{Div})} \, \|\gamma^d v\|_{1/2,\Gamma} \,. \tag{II.2.4}$$

Доказательство. Let $v \in H_0^1(\Omega, \mathbb{R}^d)$. Smooth functions with compact supports are dense in H_0^1. Therefore, there is a sequence of smooth functions $\{w_k\}$ such that $w_k \to v$ in H^1. For any w_k, we have

$$\int_\Omega \tau^* : \epsilon(w_k)\, dx \;+\; \int_\Omega \mathrm{div}\tau^* \cdot w_k\, dx \;=\; \int_\Gamma \tau^* \cdot n\, w_k ds = 0\,.$$

Passing to the limit, we arrive at (II.2.3).

The inequality (II.2.4 follows directly from the definition of Λ_{τ^*}:

$$|\Lambda_{\tau^*}(v)| \;\le\; \|\tau^*\|_{H(\Omega,\mathrm{Div})} \, \|v\|_{H^1(\Omega,\mathbb{R}^d)} \le c_\mu \, \|\tau^*\|_{H(\Omega,\mathrm{Div})} \, \|\gamma^d v\|_{1/2,\Gamma} \,.$$

□

In essence, Lemma II.2.1 shows that Λ_{τ^*}, is a linear continuous mapping defined on a factorspace of V_0. Indeed,

$$\Lambda_{\tau^*}(v_1) = \Lambda_{\tau^*}(v_2) \qquad \text{if} \;\; v_1, v_2 \in V_0 \text{ and } \gamma^d v_1 = \gamma^d v_2\,.$$

Thus, in this factorspace (denote it Z) two functions belong to one class if they have the same trace on Γ.

This means that Λ_{τ^*} is a bounded linear mapping from Z to $\mathbb{R}$. Hence it can be identified with a certain element in Z^*, which we denote $\delta_n^d \tau^*$ and call the trace of τ^* on Γ.

We follow the usual convention and denote the value of the functional $\xi^* \in Z^*$ on $\xi \in Z$ by means of duality pairing $< \xi^*, \xi >_\Gamma$.

Then,

$$\Lambda_{\tau^*}(\gamma^d v) :=< \delta_n^d \tau^*, \gamma^d v >_\Gamma = \int_\Omega (\tau^* : \epsilon(v) + \mathrm{div}\tau^* \cdot v)\, dx \, .$$

The norm of such a functional is given by standard relation

$$\|\delta_n^d \tau^*\|_{Z^*} \; = \sup_{v \in V_0} \frac{\int_\Omega (\tau^* : \epsilon(v) + \mathrm{div}\tau^* \cdot v)\, dx}{\|\gamma^d v\|_Z}$$

This norm is bounded:

$$\|\delta_n^d \tau^*\|_{Z^*} \; \leq \; c_\mu \, \|\tau^*\|_{H(\Omega, \mathrm{Div})} \; . \tag{II.2.5}$$

II.3 Boundary conditions in a variational form

For any $\xi \in Z$ (i.e., for any boundary trace) we define the functional

$$\Upsilon(\xi) := \int_\Gamma \, j(\xi)\, d\Gamma \, ,$$

where

- $j : \mathbb{R}^d \to \mathbb{R}^d$ is a nonnegative, convex, and l.s.c. functional;
- $j(0) = 0$
- $\mathrm{dom}\, j := \{p \in \mathbb{R}^d \,\|\, j(p) < +\infty\} \; \neq \emptyset \, ,$

In this case, the functional $\Upsilon(\xi)$ is nonnegative, convex and l.s.c. on Z. Since γ^d is a bounded linear operator, the functional $\Upsilon(\gamma^d v)$ also possesses the above properties as the functional on V_0.

Let us introduce a new functional

$$\Upsilon^*(\xi^*) := \sup_{\xi \in Z} \{ < \xi^*, \xi >_\Gamma - \Upsilon(\xi) \} \; , \tag{II.3.1}$$

which we call *conjugate* (in the sense of Young–Fenchel) to the functional Υ.

Under the above assumptions,

$$\Upsilon(\xi) = \sup_{\xi^* \in Z^*} \{ < \xi^*, \xi >_\Gamma - \Upsilon^*(\xi^*) \} \tag{II.3.2}$$

In other words, we have the following duality in the definitions of traces $\gamma^d v$ and $\delta_n^d \tau^*$

$$\Upsilon(\gamma^d v) = \sup_{\tau^* \in H(\Omega, \mathrm{Div})} \int_\Omega (\tau^* : \epsilon(v) + \mathrm{Div}\tau^* \cdot v)\; dx - \Upsilon^*(\delta_n^d \tau^*), \tag{II.3.3}$$

$$\Upsilon^*(\delta_n^d \tau^*) = \sup_{w \in V} \int_\Omega (\tau^* : \epsilon(w) + \mathrm{Div}\tau^* \cdot w)\; dx - \Upsilon(\gamma^d w). \tag{II.3.4}$$

Now we see how the condition (II.1.1) should be understood from the mathematical point of view

$$\boxed{-\delta_n^d \boldsymbol{\sigma_n} \in \boldsymbol{\partial \Upsilon}(\gamma^d \boldsymbol{v})}$$

This is equivalent to

$$0 = D_\Gamma(\gamma^d v, -\delta_n^d \sigma_n) := \Upsilon(\gamma^d v) + \Upsilon^*(-\delta_n^d \sigma_n) + < \gamma^d v, \delta_n^d \sigma_n >_\Gamma . \tag{II.3.5}$$

Here $D_\Gamma(\gamma^d v, -\delta_n^d \sigma_n)$ is the *compound* functional defined on the boundary. We recall that compound functionals are nonnegative and vanish if and only if their arguments satisfy the above differential condition.

Lemma below shows a way to compute $\Upsilon^*(-\delta_n^d \sigma_n)$.

Лемма II.3.1. *If* $\delta_n^d \sigma_n \in L_2(\Gamma, \mathbb{R}^d)$, *then*

$$\Upsilon^*(-\delta_n^d \sigma_n) = \int_\Gamma j^*(-\delta_n^d \sigma_n)\, dx\,,$$

where $j^* : \mathbb{R}^d \to \mathbb{R}$ *is the function conjugate to* j, *i.e.*

$$j^*(q^*) = \sup_{q \in \mathbb{R}^d} \{ q^* \cdot q - j(q) \}\ .$$

Замечание II.3.1. Now we omit γ^d and δ_n^d and write (II.3.5) in the form

$$\Upsilon(v) + \Upsilon^*(-\sigma_n) + \int_\Omega (\sigma : \epsilon(v) + \mathrm{Div}\sigma \cdot v)\, dx = 0 \tag{II.3.6}$$

or

$$\Upsilon(v) + \sup_{w \in V} \Big(\int_\Omega (\sigma_n : \epsilon(v-w) + \mathrm{Div}\sigma_n \cdot (v-w))\; dx - \Upsilon(w) \Big) = 0 \tag{II.3.7}$$

Linear elasticity with nonlinear boundary conditions

How to implement these type nonlinear conditions in the form most convenient for numerical analysis?

On $V \times V$ we define the bilinear form

$$a(u, v) := \int_\Omega L\epsilon(u) : \epsilon(v)\, dx$$

that defines internal energy of an elastic body occupying Ω. The action of external forces is described by the linear functional

$$l(v) := \int_\Omega f \cdot v\, dx \qquad f \in L_2(\Omega, \mathbb{R}^d). \tag{II.3.8}$$

Henceforth, we assume that Γ consists of two nonintersecting parts Γ_D and Γ_1. On Γ_D we have the Dirichlet condition

$$u = u_0 \text{ on } \Gamma_D, \qquad u_0 \in V(\Omega)\,. \tag{II.3.9}$$

Consider the problem

$$\boxed{\boldsymbol{J(u) = \inf_{w \in V_0 + u_0} J(w), \quad J(w) = \frac{1}{2} a(w, w) + \Upsilon(w) + (f, w)}}$$

where

$$\Upsilon(\xi) := \int_{\Gamma_1} j(\xi)\, d\Gamma\,,$$

Let u be the minimizer. Then

$$J(u) \leq J(u + w) \qquad \forall w \in V_0.$$

Hence

$$\int_\Omega \sigma : \varepsilon(u) dx + \Upsilon(u) + (f, u) \leq$$
$$\int_\Omega \sigma : \varepsilon(u + w) dx + \Upsilon(u + w) + (f, u + w)$$

and

$$\int_\Omega \sigma : \varepsilon(w)dx + \Upsilon(u+w) - \Upsilon(u) + (f,w) \geq 0$$

$$\int_\Omega (f - \mathrm{Div}\sigma)\cdot w dx + \int_{\Gamma_1} (\sigma n)\cdot w ds + \Upsilon(u+w) - \Upsilon(u) \geq 0$$

From here, we find that

$$\Upsilon(u+w) - \Upsilon(u) \geq -\int_\Gamma (\sigma n)\cdot w ds,$$

what amounts

$$\boxed{-\boldsymbol{\sigma n} \in \boldsymbol{\partial j(u)} \qquad \text{on } \boldsymbol{\Gamma_1}.}$$

Similar arguments can be used for problems where Ω is occupied by a nonlinear media. In this case, the internal energy is defied by the functional

$$J_0(w) = \int_\Omega \psi(\varepsilon)dx = \int_\Omega \sigma : \varepsilon dx.$$

The variational problem

$$\boxed{\boldsymbol{J(u) = \inf_{w\in V_0+u_0} J(w), \quad J(w) = J_0(w) + \Upsilon(w) + (f,w)}}$$

defines u, which satisfies the nonlinear boundary condition on Γ_1.

Minimax setting

We can set the problem in a saddle point form. Let

$$L(v,\zeta^*) = \frac{1}{2}a(v,v) + (f,w) + \langle\zeta^*, v\rangle_{\Gamma_1} - \int_{\Gamma_1} j^*(\zeta^*)ds$$

It is easy to see that

$$\sup_{\zeta^*\in Z^*} L(v,\zeta^*) = J(v).$$

Hence

$$\inf_{v\in V_0+u_0} \sup_{\zeta^*\in Z^*} L(v,\zeta^*)$$

is equivalent to the above variational problem.

In some cases, the conditions for v_n and v_τ can be splitted and the functional setting has the form

$$J(u) = \inf_{w \in V_0 + u_0} J(w), \quad J(w) = J_0(w) + \Upsilon_n(w_n) + \Upsilon_\tau(w_\tau) + (f, w),$$

where Υ_n and Υ_τ are convex functionals associated with the corresponding part of boundary conditions.

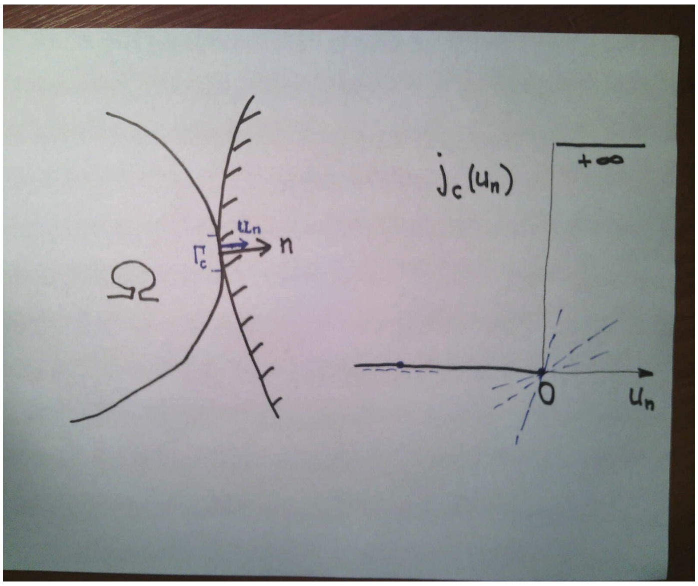
Theis problem generates the variational inequality

$$a(u, w-u) - (f, w-u) + \int\limits_{\Gamma_1} j_n(w_n)ds - \int\limits_{\Gamma_1} j_n(u_n)ds + \int\limits_{\Gamma_1} j_\tau(w_\tau)ds - \int\limits_{\Gamma_1} j_\tau(u_\tau)ds \geq 0$$

By similar arguments, we arrive at the conclusion that minimizer of this variational problem satisfies

$$\begin{aligned}
&\mathrm{Div} L\varepsilon(u) - f = 0, \\
&j_n(w_n) - j_n(u_n) \geq \int\limits_{\Gamma_1} (-\sigma_{nn}(w_n - u_n)ds \\
&j_\tau(w_\tau) - j_\tau(u_\tau) \geq \int\limits_{\Gamma_1} (-\sigma_\tau \cdot (w_\tau - u_\tau)ds
\end{aligned}$$

Contact problem



The function

$$j_c(v) = \begin{cases} 0 & v_n \le 0 \\ +\infty & v_n > 0 \end{cases}$$

generates the simplest contact condition

$$\begin{aligned} v_n < 0 \quad &\Rightarrow \ \sigma_{nn} = 0, \\ v_n = 0 \quad &\Rightarrow \ \sigma_{nn} \le 0 \end{aligned}$$

We can write this condition in the functional form

$$j_c(v_n) + j_c^*(-\sigma_n n) + u_n \sigma_{nn} = 0. \tag{II.3.10}$$

Here

$$j_c^*(v^*) = \sup_v \{v^* v - j_c(v)\} = \sup_{v \le 0} v^* v = \begin{cases} 0 & v^* \le 0 \\ +\infty & v^* > 0 \end{cases}$$

Therefore, (II.3.10) means that $\boxed{u_n \le 0}$ and $\boxed{-\sigma_{nn} \ge 0}$.

We have two options:

1. $u_n < 0$. Then $u_n \sigma_{nn} = 0$ and we conclude that $\sigma_{nn} = 0$.
2. $u_n = 0$. Then $u_n \sigma_{nn} = 0$ holds for any $\sigma_{nn} < 0$.