IIT JEE

MATHS

FORMULA SHEET

2025-2026

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WARNING

Joint Entrance Examination

INDEX

UNIT-I (SETS, RELATIONS AND FUNCTIONS)

Sets

➡ **Set** :- A set is well defined collections of objects.

(i) Objects, elements and members of set are synonyms terms.

(ii) Sets are usually denoted by capital letters A, B, C, X, Y, Z etc.

(iii) The elements of a set are represented by small letters a, b, c, x, y, z etc.

➡ **Roaster OR tabular form** : Elements are listed, separated by commas and enclosed within curly brackets { } Example : {a, e, i, o, u} set of vowels.

➡ **Set builder form** : All elements possess a single common property. Example : { x : x is a vowel in English alphabet} .

➡ **Cardinal number** : Number of elements of a Set A is called cardinal number and denoted by n(A) .

➡ **Empty Set** : A set which does not contain any element is called the empty set or the null set or the void set.

➡ **Finite Set** : A set which is empty or consists of a definite number of elements is called finite set.

➡ **Infinite Set** : A set which is not empty & consists of a indefinite number of elements is called infinite set.

➡ **Equal Set** : Two sets A and B are said to be equal if they have exactly elements and we write A = B. Otherwise, the sets are said the same to be unequal and we write $A \neq B$

➡ **Equivalent sets** : Two finite sets A and B are equivalent if their number of elements are same. i.e. $\boxed{n(A) = n(B)}$

NOTE Equal sets are always equivalent but equivalent sets may not be equal.

➡ **Subset** : A set A is said to be a subset of a set B if every element of A is also an element of B. $\boxed{A \subseteq B \text{ if } a \in A \Rightarrow a \in B}$

➡ **Proper subset** : If $A \subset B$ and $A \neq B$, then A is called a proper subset of B and B is called superset of A.

NOTE 1 :- Every set is a subset of itself i.e. $A \subseteq A$ for all A

NOTE 2 :- Empty set ϕ is a subset of every set.

NOTE 3 :- Clearly $N \subset W \subset Z \subset Q \subset R \subset C$

NOTE 4 :- The total number of subsets of a finite set containing n elements is 2^n.

➡ **Singleton set** : If a set A has only one element, we call it singleton set.

NOTE Subsets of set of real numbers $\boxed{N \subset Z \subset Q,\ Q \subset R,\ N \not\subset T}$

T = Irrational number

➡ **Intervals as subsets of R** :

- $(a,b) = \{x : a < x < b\}$ is an open interval, does not contain end points a & b.
- $[a,b] = \{x : a \leq x \leq b\}$ is an closed interval, contain end points also.
- $[a,b) = \{x : a \leq x < b\}$ is an open interval from a to b, including a but excluding b.
- $(a,b] = \{x : a < x \leq b\}$ is an open interval from a to b, including b but excluding a.

- **Length of any interval** : The number $(b-a)$ is called the length of any of the intervals (a, b), $[a, b]$, $[a, b)$ or $(a, b]$.

- **Power Set** : The collection of all subsets of a set A is called the power set of A. denoted by $P(A)$.

- **Universal Set** : A set that contains all sets in a given context is called Universal set denoted by U.

- **Union of Sets** : The union of A and B is the set which consists of all the elements of B, the common elements being taken only once.

The symbol '$\cup$' is used to denote the union. $\boxed{A \cup B = \{x : x \in A \text{ or } x \in B\}}$

- **Some Properties of the operation of union**

(i) $\boxed{A \cup B = B \cup A}$ (Commutative law)

(ii) $\boxed{(A \cup B) \cup C = A \cup (B \cup C)}$ (Associative law)

(iii) $\boxed{A \cup \phi = A}$ (Law of identity element, ϕ is the identity of $\cup$)

(iv) $\boxed{A \cup A = A}$ (Idempotent law)

(v) $\boxed{U \cup A = U}$ (Law of U)

- **Intersection of Sets** : The intersection of A and B is the set of all the elements which are common to both A and B. The symbol '$\cap$' is used to denote the intersection. $\boxed{A \cap B = \{x : x \in A \text{ and } x \in B\}}$

- **Some Properties of the operation of intersection**

(i) $\boxed{A \cap B = B \cap A}$ (Commutative law)

(ii) $\boxed{(A \cap B) \cap C = A \cap (B \cap C)}$ (Associative law)

(iii) $\boxed{\phi \cap A = \phi,\ U \cap A = A}$ (Law of ϕ and U)

(iv) $\boxed{A \cap A = A}$ (Idempotent law)

v $\boxed{A - (B \cup C) = (A - B) \cap (A - C)\ ;\ A - (B \cap C) = (A - B) \cup (A - C)}$

vi Distributive laws : $\boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

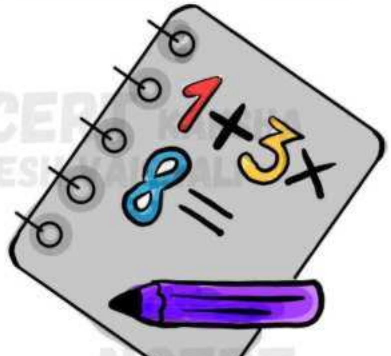
$\boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

vii $\boxed{A \cap B \subseteq A \; ; \; A \cap B \subseteq B}$

viii $\boxed{A \subseteq A \cup B \; ; \; B \subseteq A \cup B}$

ix $\boxed{A \subseteq B \Rightarrow A \cap B = A}$

x $\boxed{A \subseteq B \Rightarrow A \cup B = B}$



➡ **Disjoint Sets** : If $A \cap B = \phi$, then A, B are disjoint.

NOTE $\boxed{A \cap A' = \phi}$ $\therefore$ A, A' are disjoint.

➡ **Difference of Sets** : The difference of the two sets A and B in this order is the set of elements which belong to A but not to B. $\boxed{A - B = \{x : x \in A \text{ and } x \notin B\}}$

NOTE $\boxed{A - B \neq B - A}$

➡ **Complement of a Set** : Let U be the universal set and A a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. denoted by A'

$\boxed{A' = \{x : x \in U \text{ and } x \notin A\} \text{ obviously } A' = U - A}$

➡ **Symmetric difference of Sets** :

$A \Delta B = (A - B) \cup (B - A)$

- $\boxed{(A')' = A}$
- $\boxed{A \subseteq B \Leftrightarrow B' \subseteq A'}$

If there are any two sets, then

(i) $A - B = A \cap B'$

(ii) $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \phi$

(iv) $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \phi$

(vi) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Some properties of Complement Sets

1. **Complement laws** : (i) $\boxed{A \cup A' = U}$ (ii) $\boxed{A \cap A' = \phi}$

2. **De Morgan's law** : (i) $\boxed{(A \cup B)' = A' \cap B'}$ (ii) $\boxed{(A \cap B)' = A' \cup B'}$

3. **Law of double complementation** : $\boxed{(A')' = A}$

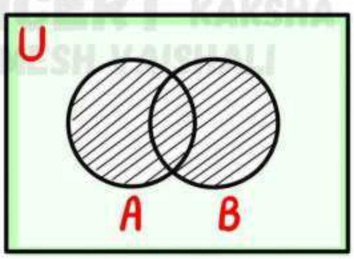
4. **Laws of empty set and universal set** : $\boxed{\phi' = U}$ and $\boxed{U' = \phi}$

NOTE

If A is a subset of the universal set U, then its complement A' is also a subset of U.

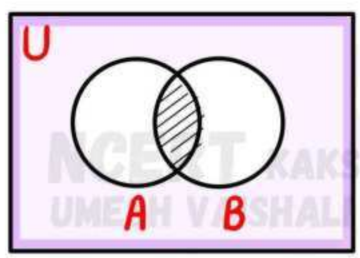
VEN DIAGRAM

(A)



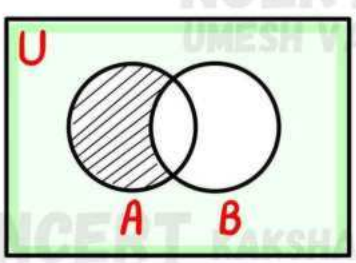
A ∪ B

(B)



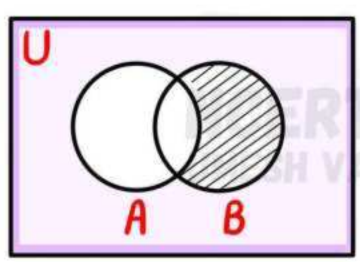
A ∩ B

(C)



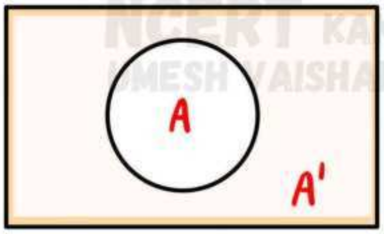
A - B

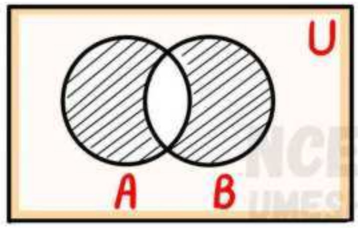
(D)

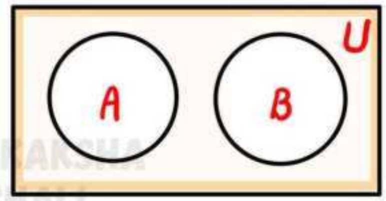


B - A

Clearly $(A-B) \cup (B-A) \cup (A \cup B) = A \cup B$







Disjoints sets

$$A \Delta B = (A-B) \cup (B-A)$$

NOTE $A \cap A' = \phi$, $A \cup A' = U$

Important Results

If A, B and C are finite sets and U be the finite universal set, then

(i) $\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint sets

(iii) $n(A-B) = n(A) - n(A \cap B)$ i.e. $n(A-B) + n(A \cap B) = n(A)$

(iv) $$\boxed{n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)}$$

(v) Number of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(vi) Number of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(vii) $\boxed{n(A' \cup B') = n[(A \cap B)'] = n(U) - n(A \cap B)}$

(viii) $\boxed{n(A' \cap B') = n[(A \cup B)'] = n(U) - n(A \cup B)}$

(ix) If $A_1, A_2 \ldots\ldots A_n$ are finite sets, then

$$n\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} n(A_i) - \sum_{1 \le i \le j \le n} n(A_i \cap A_j)$$

$$+ \sum_{1 \le i \le j \le n} n(A_i \cap A_j \cap A_k) - \ldots\ldots + (-1)^{n-1} n(A_1 \cap A_2 \cap \ldots A_n)$$

NOTE

$=$	equal to	$<$	less then
$\neq$	Not equal to	$\leq$	less then equal to
$\subset$	subset	$>$	greater than
$\not\subset$	not a subset	$\geq$	greater than equal to
$\Rightarrow$	implies	$\supset$	Superset
$\Leftrightarrow$	if and only if	$\not\supset$	Not superset
$\in$	belongs to OR contains in	$\cup$	Union
$\forall$	for all	$\cap$	Intersection
$:$	such that		

SOME IMPORTANT NUMBER SETS

N = Set of all natural numbers = $\{1, 2, 3, 4, \ldots\ldots\}$

W = Set of all whole numbers = $\{0, 1, 2, 3, \ldots\ldots\}$

Z or I set of all integers = $\{\ldots. -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$

Z^+ = Set of all +ve integers = $\{1, 2, 3, \ldots.\} = N$

Z^- = Set of all −ve integers = $\{-1, -2, -3, \ldots.\}$

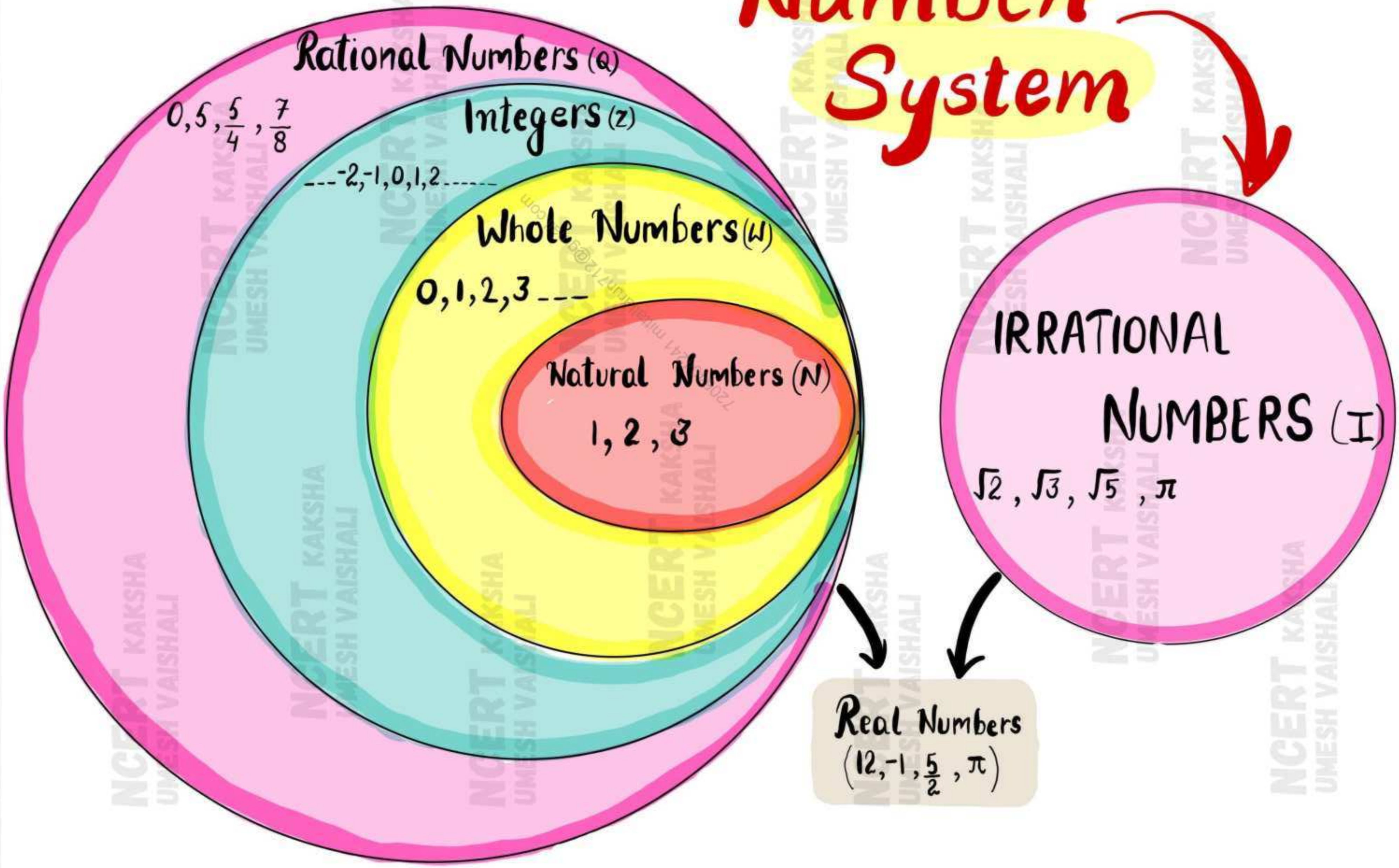
Z_0 = The Set of all non zero integers = $\{+1, +2, +3, \ldots.\}$

Q = The set of all rational numbers = $\left\{\frac{p}{q} : p, q \in 1, q \neq 0\right\}$

R = The set of all real numbers.

$R-Q$ = The set of all real irrational numbers.

Number System



Relations

Relation :- If A and B are two non-empty sets, then any subset R of $A \times B$ is called Relation from set A to B.

i.e. $\boxed{R: A \to B \Leftrightarrow R \subseteq A \times B}$

If $(x,y) \in R$ then we write xRy (read as x is R related to y) and
If $(x,y) \notin R$ then we write $x\not{R}y$ (read as x is not R related to y)

Domain And Range Of a Relation :-

If R is any relation from set A to set B then,

Domain

Domain of R is the set of all first coordinates of elements of R and is denoted by Dom (R).

Range

- Range of R is the set of all second coordinates of R and it is denoted by Range (R). A relation R on set A means, the relation from A to A i.e. $\boxed{R \subseteq A \times A}$

Types Of Relation :-

- **Empty Relation** :- A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e. $\boxed{R = \phi \subset A \times A}$

- **Universal Relation** :-

Universal

A relation R in a set A is called universal relation each of A is related to every element of A, i.e. $\boxed{R = A \times A}$

- **Identity Relation** :-

$\boxed{R = \{(x,y) : x \in A, y \in A, x = y\}}$

OR

$\boxed{R = \{(x,x) ; x \in A\}}$

- **Reflexive Relation** :-

If $(a,a) \in R$, for every $a \in A$

- **Symmetric Relation** :- If $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$ for all $a_1, a_2 \in A$.

Transitive Relation :- If $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies $(a_1, a_3) \in R$ for all $a_1, a_2, a_3 \in A$.

Equivalence Relation :- A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

Antisymmetric Relation :- A relation R in a set A is antisymmetric if $\boxed{(a,b) \in R \ \ (b,a) \in R \Rightarrow a = b \ \forall \ a, b \in R}$

Inverse Relation :- If A and B are two non-empty sets and R be a relation from A to B, such that $R = \{(a,b) : a \in A, b \in B\}$, then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $\boxed{R^{-1} = \{(b,a) : (a,b) \in R\}}$

$\boxed{\text{Domain } (R) = \text{Range } (R^{-1})}$ and $\boxed{\text{Range } (R) = \text{Domain } (R^{-1})}$

Relation on a set :
If R is a relation from set A to A itself then R is called Relation on Set A.

Equivalence Class :- let R be an equivalence relation on a non-empty set A for all $a \in A$, the equivalence class of 'a' is defined as the set of all such elements of A which are related to 'a' under R. It is denoted by [a]

i.e. $\boxed{[a] = \text{equivalence class of 'a'} = \{x \in A : (x, a) \in R\}}$

Total number of Relation : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then A X B consists of mn orders pairs. So total number of subsets of A X B is 2^{mn}.

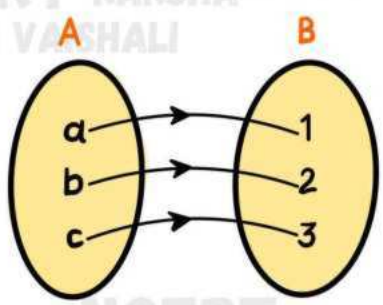
Functions

Function :- let X and Y be two non-empty sets. Then a rule f which associates to each element $x \in X$, a unique element, denoted by $f(x)$ of Y, is called a function from X to Y and written as $f: X \to Y$ where, $f(x)$ is called image of x and x is called the pre-image of $f(x)$ and set Y is called the co-domain of f and $Y = f(x) = \{f(x) : x \in X\}$ is called the range of f. and X is known as domain of F.

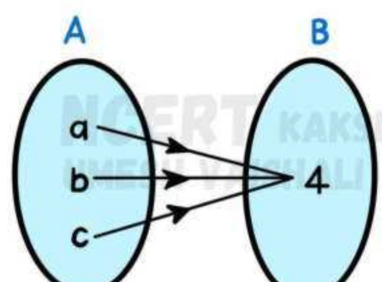
NOTE Range is a subset of co-domain.

TYPES OF FUNCTIONS

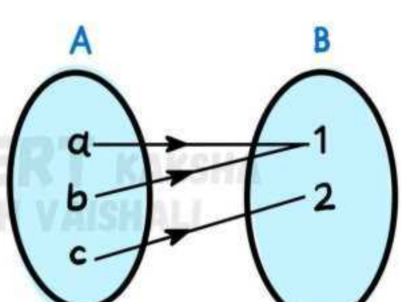
One to One Function



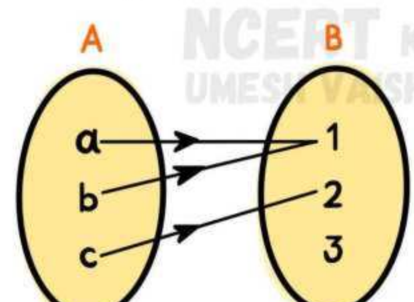
Many to One Function



Onto Function



Into Function



One-One OR Injective Function :- A function $f: X \to Y$ is defined to be one-one if the images of distinct element of X under f are distinct; i.e. $\boxed{x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$ otherwise f is called many-one.

NOTE

- Set always denoted by capital Letters $(A, B, C, D \dots)$
- Elements of a set always denoted by small letters $(a, b, c, d \dots)$

Onto OR Surjective :- A function $f: X \to Y$ is said to be onto if every element of Y is the image of some element of X under f; i.e. for every $y \in Y$, element of Y there exists an element x in X such that $f(x) = y$.

One-One and onto or Bijective :- A function $f: X \to Y$ is said to be one-one and onto, if f is both one-one and onto.

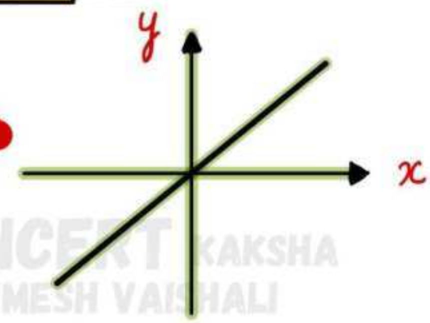
NOTE $f: X \to Y$ is onto if and only if Range of $f = Y$

Composition Of Function :- let $f: A \to B$ and $g: B \to C$ be two function then the composition of f and g denoted by gof and defined as the function $gof: A \to C$

$$\boxed{gof(x) = g[f(x)], \forall x \in A}$$

Identity function

let R be the set of real numbers, a function $I: R \to R$ such that $I(x) = x \ \forall x \in R$ is called identity function.

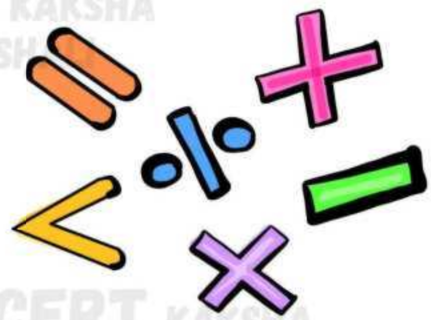


Invertible Function :- A function $f: X \to Y$ is defined to be invertible, if there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}.

Binary Operation :- A binary operation $*$ on a set A is a function $*: A \times A \to A$. We denote $*(a,b)$ by $a * b$.

- A binary operation $*$ on a set A is called commutative, if $a*b = b*a$, for every $a, b \in A$.
- A binary operation $*: A \times A \to A$ is said to be associative if $(a*b)*c = a*(b*c)$, $\forall\, a, b, c \in A$.
- A binary operation $*: A \times A \to A$, an element $e \in A$ (identity element), if it exists, is called identity for the operation $*$, if $a*e = e*a$, $\forall\, a \in A$.
- A binary operation $*: A \times A \to A$ with the identity element e in A, an element $a \in A$ is said to be invertible with respect to the operation $*$, if there exists an element b in A such that $a*b = e = b*a$ and b is called the inverse of a and is denoted by a^{-1}.

No. Of Function :-



let $f: A \to B$ be any mapping and $|A| = n$ and $|B| = m$

where, $|A|$ represent no. of elements in set A

$|B|$ represent no. of elements in set B

Then, Total no. of function from A to $B = m^n$

- **case (i)** If $n = m$; then

  Total no. of mapping $= n^n$
  Total no. of one-one mapping $= n!$
  Total no. of onto mapping $= n!$

- **case (ii)** If $n < m$; then

  Total no. of mapping $= m^n$
  Total no. of one-one mapping $= {}^mC_n \, n!$
  Total no. of onto mapping $= 0$

- **case (iii)** If $n > m$; then

  Total no. of mapping $= m^n$
  Total no. of one-one mapping $= 0$
  Total no. of onto mapping $= \sum_{r=0}^{m-1} (-1)^r \, {}^mC_r \, (m-r)^n$

Important types of function :

(A) Polynomial function :-

If a function 'f' is called by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, \ldots a_n$ are real numbers and $a \neq 0$, then f is called a polynomial function of degree n.

(i) A polynomial of degree one with no constant term is called an odd linear function, i.e. $\boxed{f(x) = ax}$, $a \neq 0$

(ii) There are four polynomial functions, satisfying the relation,

$$\boxed{f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)}$$

They are :

(a) $f(x) = x^n + 1$, $n \in N$
(b) $f(x) = 1 - x^n$, $n \in N$
(c) $f(x) = 0$
(d) $f(x) = 2$

(iii) Domain of a polynomial function is R.

(iv) Range of odd degree polynomial is R whereas range of an even degree polynomial is never R.

(b) Algebric function :

A function 'f' is called an algebric function if it can be constructed using algebric operations (such as addition, subtraction, multiplication, divison and taking radicals) starting with polynomials.

(C) Rational function :

A Rational function is a function of the form

$$\boxed{y = f(x) = \frac{g(x)}{h(x)}}$$

, where $g(x)$ & $h(x)$ are polynomials & $h(x) \neq 0$.

Domain : $R - \{x \mid h(x) = 0\}$

NOTE Any rational function is automatically an algebric function.

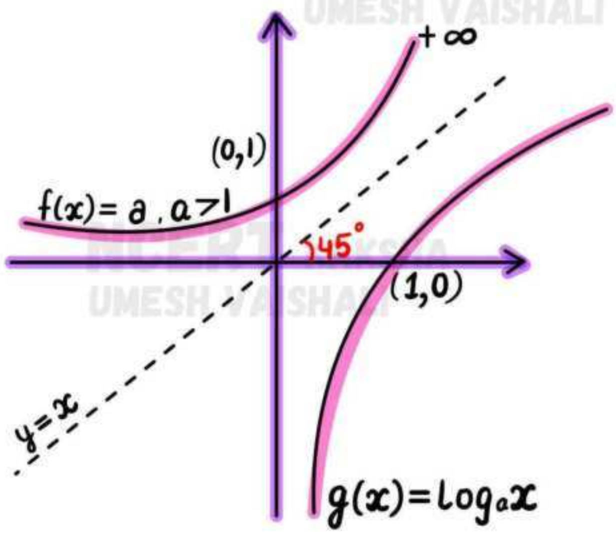
(d) Exponential and logarithmic function :-

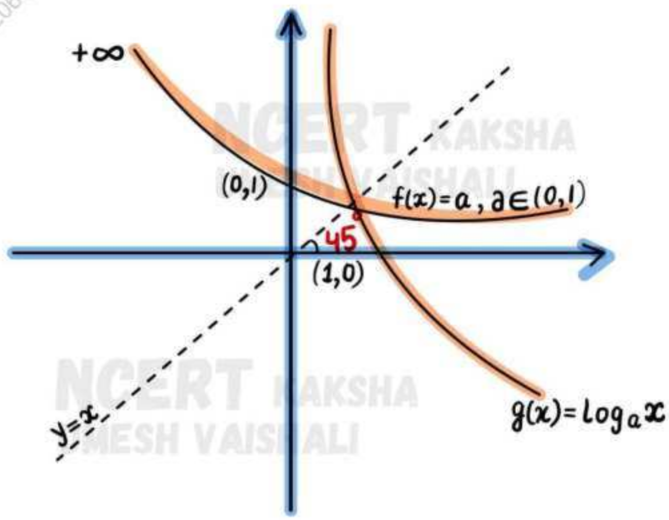
A function $f(x) = a^x$ $(a > 0)$, $a \neq 1$, $x \in R$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $\boxed{g(x) = \log_a x}$

Note that $f(x)$ & $g(x)$ are inverse of each other & their graphs are as shown. (Functions are miror image of each other about the line $y = x$)

Domain of a^x is R. Range R^+

Domain of $\log_a x$ is R^+ Range R





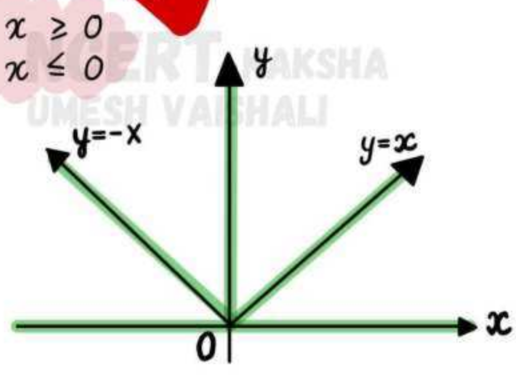
(e) Absolute value function :

It is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$

Also defined as $\max \{x, -x\}$

Domain : R Range : $[0, \infty)$

Domain : $R - \{0\}$ Range : R^+



NOTE $f(x) = \frac{1}{|x|}$

Properties of modulus function :

For any $x, y, a \in R$

(i) $|x| \geq 0$ (ii) $|x| = |-x|$

(iii) $|xy| = |x||y|$

(iv) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$; $y \neq 0$

(v) $|x| = a \Rightarrow x = \pm a$, $a > 0$

(vi) $\sqrt{x^2} = |x|$

(vii) $|x| \geq a \Rightarrow x \geq a, x \leq -a$. where a is positive.

(viii) $|x| \leq a \Rightarrow x \in [-a, a]$. where a is positive.

(ix) $|x| > |y| \Rightarrow x^2 > y^2$

(x) $||x| - |y|| \leq |x + y| \leq |x| + |y|$

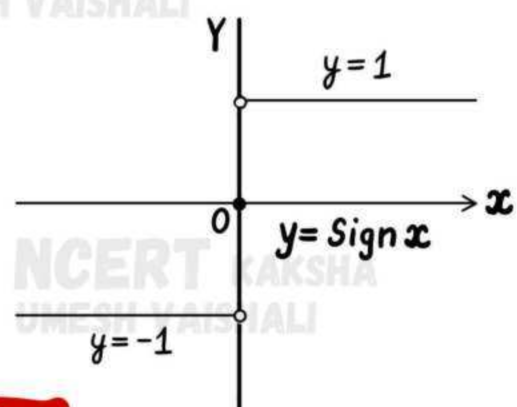
Note that

(a) $|x| + |y| = |x + y| \Rightarrow xy \geq 0$

(b) $|x| + |y| = |x - y| \Rightarrow xy \leq 0$

(f) **Signum function** : Signum function $y = sgn(x)$ is defined as follows

$$y = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$



Domain : R

Range : $\{-1, 0, 1\}$

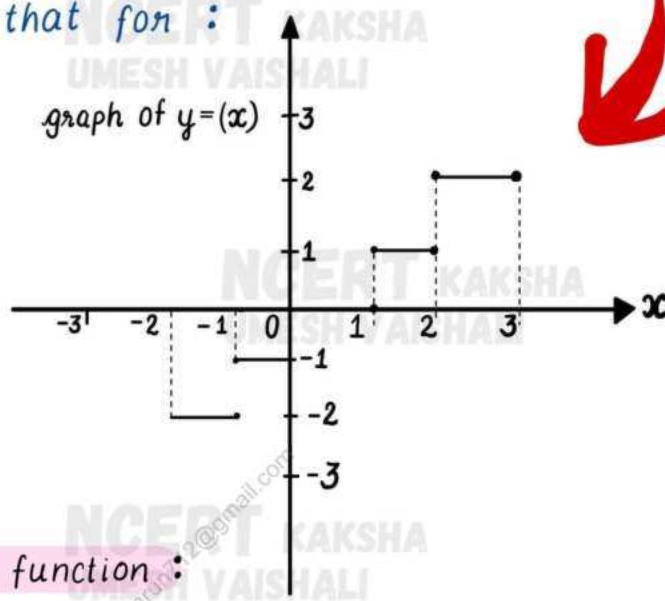
(g) **Greatest integer or step up function :**

The function $y = f(x) = \{x\}$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x. Note that for :

Domain : R

Range : I

x	(x)
(-2,-1)	-2
(-1,0)	-1
(0,1)	0
(1,2)	1



Properties of greatest integer function :

(i) $x - 1 < [x] \leq x < [x] + 1,\ 0 \leq x - \{x\} < 1$

(ii) $[x+y] = \begin{cases} [x]+[y] \\ [x]+[y]+1 \end{cases}$

(iii) $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

(iv) $[x] + [-x] = \begin{cases} 0, & x \in I \\ 1, & x \notin I \end{cases}$

NOTE

$f(x) = \dfrac{1}{[x]}$

Domain : $R - [0, 1)$

Range : $\left\{ x \mid x = \dfrac{1}{n},\ n \in I - \{0\} \right\}$

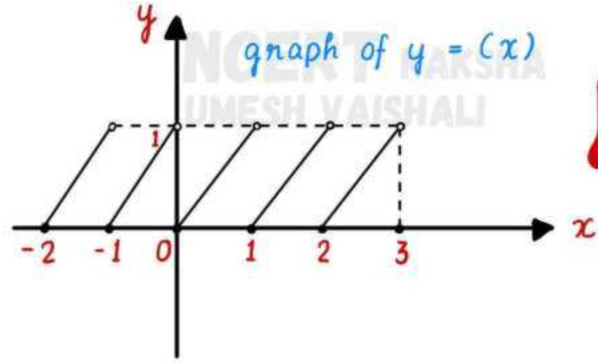
(h) **Fractional part function** : It is defined as : $g(x) = \{x\} = x - [x]$ e.g.

x	(x)
(-2,-1)	$x+2$
(-1,0)	$x+1$
(0,1)	x
(1,2)	$x-1$

Domain : R

Range : $[0, 1)$

Period : 1



NOTE $f(x) = \dfrac{1}{\{x\}}$ Domain : $R - 1$ Range : $(1, \infty)$

(j) Constant function :

$f : A \rightarrow B$ is said to be constant function if every element of A has the same f image in B. Thus $f : A \longrightarrow B$; $f(x) = c$ $\forall$ $x \in A$, $c \in B$ is constant function.

Domain : R Range : $\{c\}$

(k) Trigonometric functions :

(i) Sine function : $f(x) = \sin x$

Domain : R

Range : $[-1, 1]$, period 2π

(ii) Cosine function : $f(x) = \cos x$

Domain : R

Range : $[-1, 1]$, period 2π

(iii) Tangent function : $f(x) = \tan x$

Domain : $R - \left\{ x \mid x = \dfrac{(2n+1)\pi}{2},\ n \in I \right\}$

Range : R, period π

(iv) Cosecant function : $f(x) = \operatorname{cosec} x$

Domain : $R - \{ x \mid x = n\pi,\ n \in I \}$

Range : $R - (-1, 1)$, period 2π

(v) Secant function : $f(x) = \sec x$

Domain : $R - \{ x \mid x = (2n+1)\frac{\pi}{2},\ n \in I \}$

Range : $R - (-1, 1)$, period 2π

(vi) Cotangent function : $f(x) = \cot x$

Domain : $R - \{ x \mid x = n\pi,\ n \in I \}$

Range : R, period π

Equal or identical function :

Two function f and g are said to be equal if :

(a) The domain of f = the domain of g

(b) The range of f = range of g and

(c) $f(x) = g(x)$ for every x belonging to their common domain (i.e. should have the same graph)

Algebric operation on functions :

If f and g are real valued functions of x with domain set A, B respectively, $f+g$, $f-g$, $(f \cdot g)$ & $\left(\frac{f}{g}\right)$ as follows :

(a) $(f \pm g)(x) = f(x) \pm g(x)$, domain in each case is $A \cap B$

(b) $(f \cdot g)(x) = f(x) \cdot g(x)$, domain is $A \cap B$

(c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, domain $A \cap B - \{ x \mid g(x) = 0 \}$

Odd and even functions :

If a function is such that whenever 'x' is in it's domain '$-x$' is also in it's domain & its satisfies

$\boxed{f(-x) = f(x)}$ then it is an even function.

NOTE

(i) A function may neither be odd nor even.

(ii) Inverse of an even function is not defined, as it is many-one function.

(iii) Every even function is symmetric about the y-axis and every odd function is symmetric about the origin.

(iv) Every function which has '$-x$' in its domain whenever 'x' is in it's domain, can be expressed as the sum of an even and an odd function.

e.g. $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{Even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{Odd}}$

(v) The only function which is defined on the entire number line and even and odd at the same time is $f(x) = 0$.

(vi) If $f(x)$ and $g(x)$ both are even or both are odd then the function $f(x).g(x)$ will be even but if any one of them is odd and other is even, then $f.g$ will be odd.

Complex Numbers

(real numbers + imaginary number)

➡ Complex Numbers (z) : General form $\boxed{z = a + ib}$

a (real part) Re z

ib (imaginary part) Im z

a, b = real numbers

Every complex number can be regarded As

- **Purely real** if $b = 0$
- **Imaginary** if $b \neq 0$
  - ↓ **Purely imaginary** if $a = 0$

NOTE

(i) $N \subset W \subset I \subset Q \subset R \subset C$

(ii) $z = 0$ both purely real as well as purely imaginary but not imaginary.

(iii) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of a or b non - negative.

NOTE Two complex numbers $z = a + ib$ and $z = c + id$ are equal if $a = c$ and $b = d$

➡ **Algebra of Complex numbers :**

1. Addition of two complex numbers :

(a) The closure law : $\boxed{z_1 + z_2}$ z_1, z_2 = two complex no.

(b) The commutative law : $\boxed{z_1 + z_2 = z_2 + z_1}$

(c) The associative law : $\boxed{(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)}$

(d) The existence of additive identity : $0 + i0$ denoted as 0 (zero complex no.) $\boxed{z + 0 = z}$ additive identity

(e) The existence of additive inverse : $-a + i(-b)$ denoted as $-z$ (negative of z)

$\boxed{z + (-z) = 0}$

additive inverse

2. Diffenence of two complex numbers : $\boxed{z_1 - z_2 = z_1 + (-z_2)}$

3. Multiplication of two complex numbers : Let $z_1 = a + ib$ and $z_2 = c + id$, then, the product $z_1 z_2$ is $z_1 z_2 = (ac - bd) + i(ad + bc)$

(a) The closure law : $\boxed{z_1 z_2}$ z_1, z_2 = two complex no.

(b) The commutative law : $\boxed{z_1 z_2 = z_2 z_1}$

(c) The associative law : $\boxed{(z_1 z_2) z_3 = z_1 (z_2 z_3)}$

(d) The existence of multiplicative identity : $1 + i0$ denoted as 1 $\boxed{z \cdot 1 = z}$ multiplicative identity

(e) The existence of multiplicative inverse : $\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$

denoted as $\frac{1}{z}$ OR z^{-1} $\boxed{z \cdot \frac{1}{z} = 1}$ multiplicative inverse

(f) The distribution law : (a) $\boxed{z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3}$

(b) $\boxed{(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3}$ z_1, z_2, z_3 = three complex no

4. Division of two complex numbers : $\boxed{\frac{z_1}{z_2} = z_1 \frac{1}{z_2}}$ $z_2 \neq 0$

➡ Power of i : $\boxed{i = \sqrt{-1}}$ $\boxed{i^2 = -1}$ $\boxed{i^3 = -i}$ $\boxed{i^4 = 1}$ $\boxed{i^5 = i}$ $\boxed{i^6 = -1}$

NOTE Any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

➡ Identities

$\boxed{(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1 z_2}$

$\boxed{(z_1 - z_2)^2 = z_1^2 + z_2^2 - 2z_1 z_2}$

$\boxed{(z_1 + z_2)^3 = z_1^3 + 3z_1^2 z_2 + 3z_1 z_2^2 + z_2^3}$

$\boxed{(z_1 - z_2)^3 = z_1^3 - 3z_1^2 z_2 + 3z_1 z_2^2 - z_2^3}$

$\boxed{z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)}$

Modulus : Let $z = a + ib$

Modulus of z $\boxed{|z| = \sqrt{a^2 + b^2}}$

Conjugate : Let $z = a + ib$

conjugate of z $\boxed{\bar{z} = a - ib}$

NOTE

- $z + \bar{z} = 2\,Re(z)$
- $z - \bar{z} = 2i\,Im(z)$
- $z\bar{z} = a^2 + b^2$ which is real
- If z is purely real then $z - \bar{z} = 0$
- If z is purely imaginary then $z + \bar{z} = 0$

NOTE

(a) $\boxed{|z_1 z_2| = |z_1||z_2|}$

(b) $\boxed{\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}}$

(c) $\boxed{\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}}$ $z_2 \neq 0$

(d) $\boxed{\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}}$

(e) $\boxed{\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}}$

(f) $\boxed{z\bar{z} = |z|^2}$

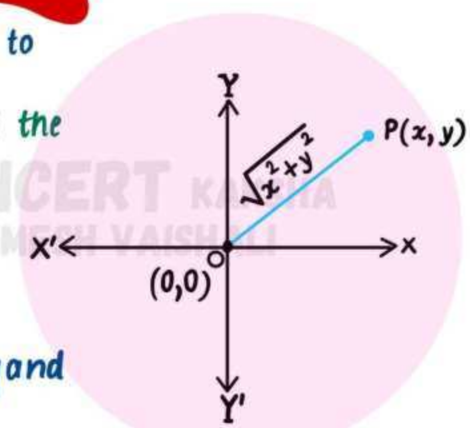
(g) $\boxed{\overline{(\bar{z})} = z}$

(h) If f is a polynomial with real cofficient such that

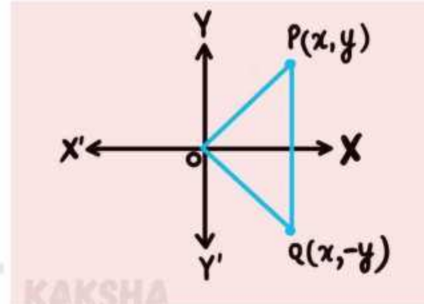
then $\boxed{f(\alpha + i\beta) = x + iy}$, $\boxed{f(\alpha - i\beta) = x - iy}$

Argand Plane (Cartesian Form) :-

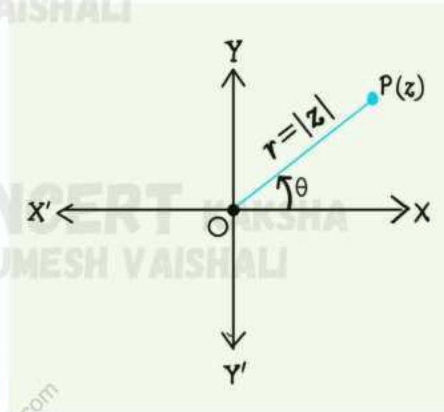
The Plane having a complex number assigned to each of its point is called the complex plane on the argand plane $x + iy = \sqrt{x^2 + y^2}$ is the distance between the point $P(x, y)$ and the origin $O\ (0,0)$ The x-axis and y-axis in the argand plane, respectivelly, the real axis and



the imaginary axis. The point $(x, -y)$ is the mirror image of the point (x, y) on the real axis.



Polar form of the complex no. :



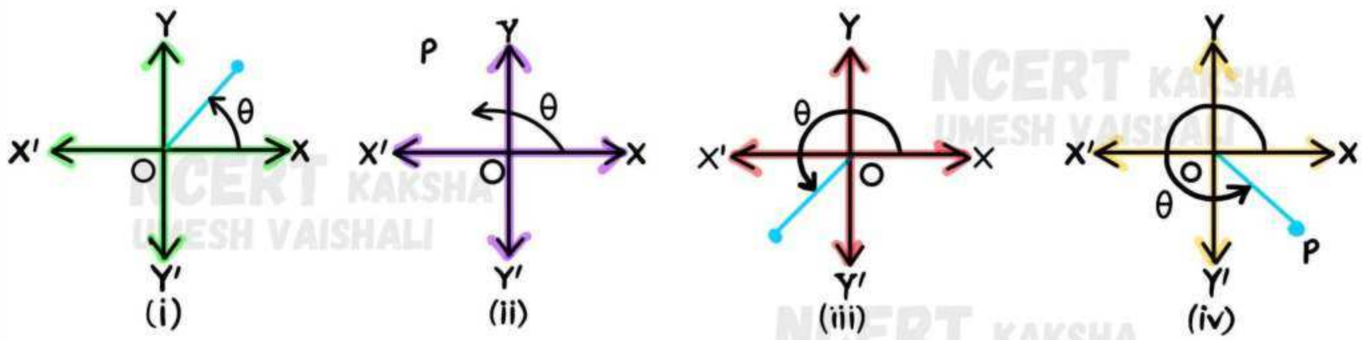
Let the point P represent the non-zero complex no. $z = x + iy$ $\quad z = r(\cos\theta + i\sin\theta)$

where $x = r\cos\theta$, $y = r\sin\theta$

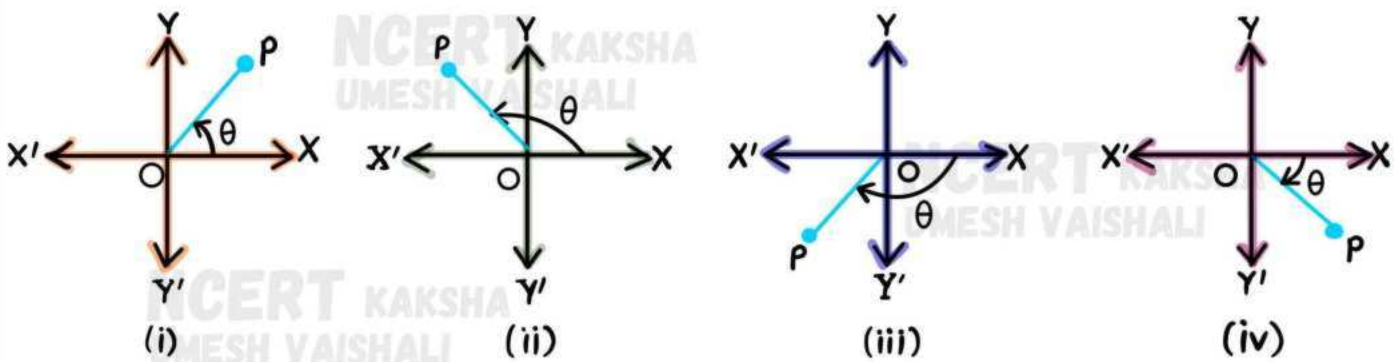
$r = \sqrt{x^2 + y^2} = |z|$ (modulus of z)

θ = argument of z $(\arg z) = \tan^{-1}\left(\frac{y}{x}\right)$

For any complex no. $z \neq 0$, there corresponds only one value of θ in $0 \leq \theta < 2\pi$. The value of θ, such that $-\pi < \theta \leq \pi$ is called the principal argument of z.



$(0 \leq \theta < 2\pi)$



$(-\pi < \theta \leq \pi)$

Quadratic Equations $ax^2 + bx + c = 0$

where $a, b, c \in R$, $a \neq 0$, $b^2 - 4ac < 0$

then, the solution of the quadratic equation is, $\boxed{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{4ac - b^2}\, i}{2a}}$

NOTE

A polynomial equation has at least one root.

A polynomial equation of degree n has n roots.

Trigonometric / Polar representation :

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

where $|z| = r$

$\arg z = \theta$

$\bar{z} = r(\cos\theta - i\sin\theta)$

NOTE

$\cos\theta + i\sin\theta$ is also written as $\text{Cis}\,\theta$.

Euler's formula : The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula. Also $\boxed{\cos x = \dfrac{e^{ix} + e^{-ix}}{2}}$ & $\boxed{\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}}$

are known as Euler's identities.

Exponential Representation : Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $\boxed{z = r.e^{i\theta}}$

IMPORTANT PROPERTIES OF MODULUS :

(a) $|z| \geq 0$

(b) $|z| \geq \text{Re}(z)$

(c) $|z| \geq \text{Im}(z)$

(d) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(e) $z\bar{z} = |z|^2$

(f) $|z^n| = |z|^n$

(g) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$

OR $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(k) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

(l) $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$ [Triangular Inequality]

(m) $||z_1| - |z_2|| \le |z_1 - z_2| \le |z_1| + |z_2|$ [Triangular Inequality]

(n) If $\left|z + \frac{1}{z}\right| = a \ (a > 0)$, then $\max|z| = \frac{a + \sqrt{a^2+4}}{2}$

& $\min|z| = \frac{1}{2}\left(\sqrt{a^2+4} - a\right)$

➡ **De'moiver's Theorem** : The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ if 'n' is integer & it is one of the values of $(\cos\theta + i\sin\theta)^n$ if n is a rational number of the form $\frac{p}{q}$

➡ **Cube root of unity** :

(a) The cube roots of unity are 1, $\omega = \frac{-1 + i\sqrt{3}}{2} = e^{i2\pi/3}$

& $\omega^2 = \frac{-1 - i\sqrt{3}}{2} = e^{i4\pi/3}$

(b) $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$, in general

$$1 + \omega^r + \omega^{2r} = \begin{cases} 0, & r \text{ is not integral multiple of } 3 \\ 3, & r \text{ is multiple of } 3 \end{cases}$$

(c) $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

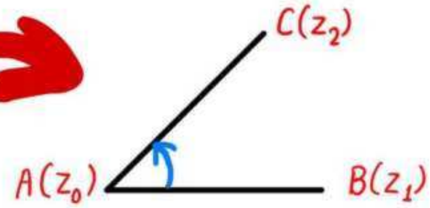
$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$

$x^2 + x + 1 = (x - \omega)(x - \omega^2)$

Rotation :

$$\boxed{\frac{z_2 - z_0}{z_2 - z_0} = \frac{z_1 - z_0}{z_1 - z_0}\, e^{i\theta}}$$



Take θ in anticlockwise direction.

Geometry in complex number :

(A) **Distance formula** : $|z_1 - z_2|$ = distance between the points z_1 & z_2 on the Argand plane.

(B) **Section formula** : If z_1 & z_2 are two complex numbers then the complex number $z = \dfrac{nz_1 + mz_2}{m+n}$ divides the join of z_1 & z_2 in the ratio $m : n$.

(C) If the vertices A, B, C of a triangle represent the complex numbers z_1, z_2, z_3 respectively, then :

- Centroid of the $\Delta ABC = \dfrac{z_1 + z_2 + z_3}{3}$

- Orthocentre of the $\Delta ABC = \dfrac{(a \sec A) z_1 + (b \sec B) z_2 + (c \sec C) z_3}{a \sec A + b \sec B + c \sec C}$

  OR $\dfrac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$

- Incentre of the $\Delta ABC = \dfrac{az_1 + bz_2 + cz_3}{a+b+c}$

- Circumcentre of the $\Delta ABC = \dfrac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

Equation of circle :

(a) Circle whose centre is z_0 & radius = r

$$\boxed{|z - z_0| = r}$$

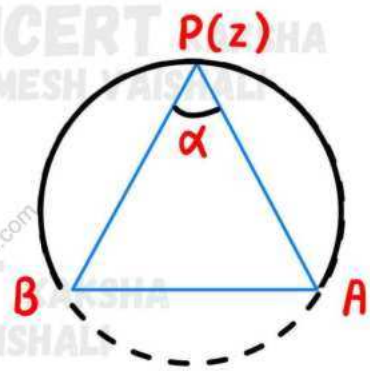
(b) General equation of circle is

$$\boxed{z\bar{z} + a\bar{z} + \bar{a}z + b = 0}$$

centre '$-a$' & radius $\sqrt{|a|^2 - b}$

(c) Diameter form $\boxed{(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0}$

(d) Equation $\boxed{\left|\dfrac{z - z_1}{z - z_2}\right| = k}$ represent a circle if $k \neq 1$ and a straight line if $k = 1$.



(e) Equation $\boxed{|z - z_1|^2 + |z - z_1|^2 = k}$ represent circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$

(f) $\boxed{\arg\left[\dfrac{z - z_1}{z - z_2}\right] = \alpha}$ $0 < \alpha < \pi$, $\alpha \neq \dfrac{\pi}{2}$

represent a segment of circle passing through $A(z_1)$ & $B(z_2)$

➡ **Standard Loci :**

(a) $|z - z_1| + |z - z_2| = 2k$ (a constant) represent

(i) If $2k > |z_1 - z_2|$ An ellipse

(ii) If $2k = |z_1 - z_2|$ A line segment

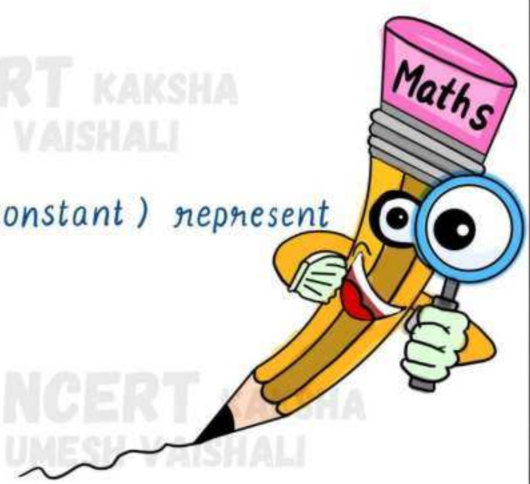
(iii) If $2k < |z_1 - z_2|$ No solution

(b) Equation $||z - z_1| + |z - z_2|| = 2k$ (a constant) represent

(i) If $2k > |z_1 - z_2|$ A hyperbola

(ii) If $2k = |z_1 - z_2|$ A line ray

(iii) If $2k < |z_1 - z_2|$ No solution



Quadratic Equations

➡ Solution Of Quadratic Equation And Relation between Roots and Co-Efficients :

[a] The solutions of the quadratic equations, $ax^2+bx+c=0$ is given by $\boxed{x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}}$

[b] The expression $\boxed{b^2-4ac\equiv D}$ is called the discriminant of the quadratic equation.

[c] If α & β are the roots of the quadratic equation $ax^2+bx+C=0$, then ;

1. $\boxed{\alpha+\beta=-b/a}$ 2. $\boxed{\alpha\beta=C/a}$ 3. $\boxed{|\alpha-\beta|=\sqrt{D}/|a|}$

[d] Quadratic equation whose roots are α & β is $(x-\alpha)(x-\beta)=0$ i.e. $x^2-(\alpha+\beta)x+\alpha\beta=0$ i.e. $\boxed{x^2-(\text{sum of roots})\,x+\text{product of roots}=0}$

[e] If α,β are roots of equation $ax^2+bx+c=0$, we have identity in x as $\boxed{ax^2+bx+c=a(x-\alpha)(x-\beta)}$

➡ Nature Of Roots :-

(a) Consider the quadratic equation $ax^2+bx+c=0$ where $a,b,C\in R$ and $a\neq 0$ then;

- $D>0 \Leftrightarrow$ roots are real and distinct (unequal).
- $D=0 \Leftrightarrow$ roots are real and coincident (equal).
- $D<0 \Leftrightarrow$ roots are imaginary.
- If $p+iq$ is one root of a quadratic equation, then the other root

must be the conjugate $p-iq$ and vice versa. ($p, q \in R$ & $i=\sqrt{-1}$).

(b) Consider the quadratic equation $ax^2+bx+c=0$ where $a, b, c \in Q$ and $a \neq 0$ then,

- If D is a perfect square, then roots are rational.
- If $\alpha = p+\sqrt{q}$ is one root in this case, (where p is rational and $\sqrt{q}$ is a surd) then other root will be $p-\sqrt{q}$.

Common Roots Of Two Quadratic Equations :-

(a) At least one common root.

Let α be the common root of $ax^2+bx+C=0$ and $a'x^2+b'x+c'=0$ then $a\alpha^2+b\alpha+c=0$ and $a'\alpha^2+b'\alpha+c'=0$. By Cramer's Rule

$$\frac{\alpha^2}{bc'-b'c} = \frac{\alpha}{a'c-ac'} = \frac{1}{ab'-a'b}$$

Therefore $$\alpha = \frac{ca'-ca'}{ab'-a'b} = \frac{bc'-b'c}{a'c-ac'}$$

So the condition for a common root is

$$\boxed{(ca'-c'a)^2 = (ab'-a'b)(bc'-b'c)}$$

(b) If both roots are same then $\boxed{\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}}$

Roots under particular cases :-

Let the quadratic equation $ax^2+bx+c=0$ has real roots and

(a) If $b=0 \Rightarrow$ roots are equal magnitude but of opposite sign

(b) If $c=0 \Rightarrow$ one roots is zero other is $-\frac{b}{a}$

(c) If $a=c \Rightarrow$ roots are reciprocal to each other

(d) If $ac<0 \Rightarrow$ roots are of opposite sign

(e) If $\left.\begin{matrix} a>0, b>0, c>0 \\ a<0, b<0, c<0 \end{matrix}\right\} \Rightarrow$ both roots are negative.

(f) If $\left.\begin{array}{l} a>0,\ b<0,\ c>0 \\ a<0,\ b>0,\ c<0 \end{array}\right\} \Rightarrow$ both roots are positive.

(g) If sign of a = sign of b $\neq$ sign of c $\Rightarrow$ Greater root in magnitude is negative.

(h) If sign of b = sign of c $\neq$ sign of a $\Rightarrow$ Greater root in magnitude is positive.

(i) If $a+b+c=0 \Rightarrow$ one root is 1 and second root is $\frac{c}{a}$.

➡ Maximum & Minimum values of quadratic expression :

Maximum or Minimum values of expression $y = ax^2+bx+c$ is $\frac{-D}{4a}$ which occurs at $x = -\frac{b}{2a}$ according as $a<0$ or $a>0$. $y \in \left[\frac{-D}{4a}, \infty\right)$ if $a>0$ & $y \in \left(-\infty, -\frac{D}{4a}\right]$ if $a<0$.

➡ General Quadratic expression in Two variables :

$f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if ;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

➡ Theory of Equation :-

If $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_n$ are the roots of the equation ;

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n = 0$

where $a_0, a_1, \ldots\ldots a_n$ are constant $a_0 \neq 0$ then,

$$\sum \alpha_1 = -\frac{a_1}{a_0}, \quad \sum \alpha_1\alpha_2 = -\frac{a_2}{a_0}, \quad \sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, \ldots\ldots, \alpha_1\alpha_2\alpha_3\ldots\alpha_n = (-1)^n\frac{a_n}{a_0}$$

NOTE

(i) Every odd degree equation has at least one real root whose sign is opposite to that of its constant term, when coefficient of highest degree term is (+ve) {If not then make it (+) ve}.

Ex : $x^3 - x^2 + x - 1 = 0$

(ii) Even degree polynomial whose constant term is (-)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+) ve & one (-) ve.

(iii) If equation contains only even power of x & all coefficient are (+) ve, then all roots are imaginary.

(iv) Rational root theorem : If a rational number $\frac{p}{q}$ $(p, q \in Z_0)$ is a root of polynomial equation with integral coefficient $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$, then p divides a_0 and q divides a_n.

Matrices

Matrix :- A matrix is a rectangular arrangement of numbers or functions arranged into a fixed number of rows and columns. A matrix is written inside brackets [ ]. Each entry in a matrix is called an element of the matrix.

$$A = \begin{array}{c} \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] \begin{array}{l} \rightarrow \text{First row } (R_1) \\ \rightarrow \text{Second row } (R_2) \\ \\ \end{array} \\ \begin{array}{cc} \downarrow & \downarrow \\ (C_1)\ \text{First column} & \text{Second column}\ (C_2) \end{array} \end{array}$$

$m \times n \rightarrow$ order of Matrix (No. of rows X No. of Columns)

We shall write $A = [a_{ij}]_{m \times n}$ → order of Matrix

i-th row, j-th column

Types Of Matrices~

★ **Column Matrix** :-

A matrix is said to be a column matrix if it has only one column. Ex:- $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4 \times 1}$ column

★ **Row Matrix** :-

A matrix is said to be a row matrix if it has only one row.

Ex- $[abcd]_{1 \times 4}$ row

★ **Square Matrix** : No. of rows (m) = No. of columns (n)

$$\text{Ex:-} \begin{array}{cc} \begin{bmatrix} 3 & 5 & 9 \\ 4 & 7 & 4 \\ 1 & 0 & 2 \end{bmatrix} & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \\ \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} & \end{array} \quad (3 \times 3) \quad \begin{array}{l} \rightarrow m \\ \rightarrow n \end{array}$$

NOTE

1. The pair of elements a_{ij} & a_{ji} are called Conjugate Elements.
2. The Elements a_{11}, a_{22}, a_{33} .... a_{nn} are called Diagonal Elements. The line along which the diagonal elements lie is called "Principal or Leading diagonal." The quantity Σa_{ii} = trace of the matrix written as, $t_r(A)$.

★ **Horizontal Matrix** :- A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

★ **Vertical Matrix** :- A matrix of order $m \times n$ is a vertical matrix if $m > n$.

★ **Diagonal Matrix** :- A square matrix is said to be a diagonal matrix if all its non diagonal elements are zero.

Ex $\begin{bmatrix} -1 & \boxed{0} \\ \boxed{0} & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ non-diagonal elements zero.

★ **Scalar Matrix** :- A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal.

Ex- $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$ diagonal elements equal

★ **Identity Matrix** :- A square matrix in which all diagonal elements are 1 and rest are all zero.

Ex- $[1]$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ diagonal elements (1)

★ **Null OR Zero Matrix** :- If all its elements are zero. We donate zero matrix by O. Ex- $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

➡ **Equal Matrix** :- Two matrices

$A = [a_{ij}]$ and $B = [b_{ij}]$

are said to be equal if-

(i) They are of the same order.

(ii) Each elements of A is equal to the corresponding element of B i.e. $a_{ij} = b_{ij}$ for all i and j.

Upper triangular Matrix :- An upper triangular matrix, if $a_{ij} = 0 \ \forall \ i > j$, i.e. all entries below principal diagonal are zero.

Example – $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ upper triangular

lower triangular Matrix :- A lower triangular matrix, if $a_{ij} = 0 \ \forall \ i < j$, i.e. all entries above principal diagonal are zero.

Example – $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$ lower triangular

NOTE

1. Minimum number of zeros in triangular matrix of order $n = n(n-1)/2$.
2. Minimum number of zeros in a diagonal matrix of order $n = n(n-1)$
3. Null square matrix is also a diagonal matrix.

Transpose OF A Matrix :- Matrix obtained by interchanging rows and columns of A and denoted by A' OR A^T.

example :- if $A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & \frac{-1}{5} \end{bmatrix}_{3\times 2}$, then $A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & \frac{-1}{5} \end{bmatrix}_{2\times 3}$

Properties OF Transport of the matrices :-

(i) $\boxed{(A^T)^T = A}$ (ii) $\boxed{(kA)^T = kA^T}$ k is any constant

(iii) $\boxed{(A+B)^T = A^T + B^T}$ (iv) $\boxed{(AB)^T = B^T A^T}$ (V) $\boxed{(ABC)^T = C^T B^T A^T}$

Symmetric Matrices :- $\boxed{A' = A}$ For example $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

is a symmetric matrix as $A' = A$

NOTE Maximum number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.

➡ **Skew-Symmetric Matrices :-** $\boxed{B' = -B}$ For example $B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$

is a skew symmetric matrix as $B' = -B$

Important Points :-

▲ Diagonal Elements Of a skew symmetric matrice are zero.

▲ For any square matrix A with real entries than $(A+A^T)$ is a symmetric and $(A-A^T)$ is skew symmetric.

▲ Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix. $\boxed{A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')}$

▲ The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.

▲ If A & B are symmetric matrices then,

(1) $AB + BA$ is a symmetric matrix.

(2) $AB - BA$ is a skew symmetric matrix.

➡ **Operation On Matrices :-**

1. **Addition Of Matrices :-** If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ two matrices of the same order $m \times n$, then their sum $A+B$ is $m \times n$ matrix such that, $(A+B)_{ij} = a_{ij} + b_{ij}$ $\forall\, i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots n$

★ **Properties Of Matrix addition :-**

(i) Commutativity $\boxed{A+B = B+A}$ (ii) Associativity $\boxed{(A+B)+C = A+(B+C)}$

(iii) Existence of identity $\boxed{A+O = A+O = A}$

zero matrix

(iv) Existence of inverse $\boxed{A+(-A) = O = (-A)+A}$

(v) Cancellation laws $\boxed{A+B = A+C \Rightarrow B=C \text{ and } B+A = C+A \Rightarrow B=C}$

2. **Scalar Multiplication Of a matrix** :- let $A=[a_{ij}]_{m\times n}$ be a matrix and k is a scalar. Then the matrix obtained by multiplying each element of matrix A by k and is denoted by kA or Ak

Multiplication of a matrix by a Scalar :

If $A = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then $KA = \begin{vmatrix} Ka & Kb & Kc \\ Kb & Kc & Ka \\ Kc & Ka & Kb \end{vmatrix}$

★ Properties Of Scalar Multiplication Of a Matrix :-

If $A=[a_{ij}]$ and $B=[b_{ij}]$ be two matrices of the same order, say $m\times n$, and k and l are scalars, then (i) $\boxed{k(A+B) = kA+kB}$ (ii) $\boxed{(k+l)A = kA+lA}$

3. **Multiplication Of Matrices** :- Two Matrices A and B are said to be defined for multiplication, if the number of columns of A (pre-multiplier) is equal to the number of rows of B (post multiplier)

$$\boxed{A_{m\times n} \times B_{n\times p} = AB_{m\times p}}$$

equal

★ Properties Of Multiplication Of Matrices :-

1. Associative law $\boxed{(AB)C = A(BC)}$
2. Distributive law (a) $\boxed{A(B+C) = AB+AC}$ (b) $\boxed{(A+B)C = AC+BC}$
3. Existence Of Multiplicative identity $\boxed{IA = AI = A}$ Identity matrix
4. $AB = O \not\Rightarrow A=O$ or $B=O$ (in general)

NOTE

If A and B are two non-zero matrices such that $AB = 0$, then A and B are called the divisors of zero. If A and B are two matrices such that

(i) $AB = BA$ then A and B are said to commute.

(ii) $AB = -BA$ then A and B are said to anticommute

5. **Positive Integral Powers of A square matrix:**

(a) $A^m A^n = A^{m+n}$

(b) $(A^m)^n = A^{mn} = (A^n)^m$

(c) $I^m = I, \; m, n \in N$

Elementary Operation (transformation) of a Matrix :-

There are six operations (transformation) on a matrix 3 of which due to row and 3 of due to column, called Elementary Transformation.

- The interchange of any two rows or two column.
- The multiplication of the elements of any row or column by a non-zero number.
- The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non-zero number.

Invertible Matrix :- If A is square matrix of order $m \times n$ and if there exist another square matrix B of the same order such that $AB = BA = I_m$

The A is invertible and B is called inverse of A.

Inverse of a square matrix, if it exists, is unique.

If A and B are invertible matrices of the same order, then

$(AB)^{-1} = B^{-1}A^{-1}$

Properties Of Invertible Matrix :-

If A is an invertible Matrix, then

1. $\boxed{(A^{-1})^{-1} = A}$
2. $\boxed{A^{-1} = \frac{1}{|A|}(\text{adj} A)}$
3. $\boxed{(A^T)^{-1} = (A^{-1})^T}$
4. $\boxed{|A^{-1}| = \frac{1}{|A|}}$
5. Square Matrix is invertible; If it is non-singular.
6. The inverse of an invertible symmetric matrix is a symmetric matrix.

Theorem : If A & B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$

Characteristic Equation :-

Let A be a square matrix. Then the polynomial in x, $|A - xI|$ is called as characteristic polynomial of A and the equation $|A - xI| = 0$ is called characteristic equation of A.

Cayley - Hamilton Theorem :-

Every square matrix A satisfy its characteristic equation i.e. $a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$ is the characteristic eq. of matrix A, then $a_0 A^n + a_1 A^{n-1} + \dots + a_{n-1} A + a_n I = 0$.

Orthogonal Matrix :-

A square matrix is said to be orthogonal matrix if $AA^T = I$

NOTE

(i) The determinant value of orthogonal matrix is either 1 or −1. Hence orthogonal matrix is always invertible.

(ii) $AA^T = I = A^T A$ Hence $A^{-1} = A^T$.

SOME SPECIAL SQUARE MATRICES :-

(a) **Idempotent Matrix** – A square matrix is idempotent provided $A^2 = A$. For idempotent matrix note the following :

- $A^n = A \; \forall n \in N$.
- determinant value of idempotent matrix is either 0 or 1.
- If idempotent matrix is invertible then it will be an identity matrix i.e. I.

(b) **Periodic Matrix** :- A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.

(c) **Nilpotent matrix** :- A square matrix is said to be nilpotent matrix of order m, $m \in N$, if $A^m = 0$, $A^{m-1} \neq 0$. Note that a nilpotent matrix will not be invertible.

(d) **Involutary Matrix** :- If $A^2 = 1$, the matrix is said to be an involutary matrix.

Note that $A = A^{-1}$ for an involutary matrix.

(e) If A and B are square matrices of same order and AB = BA then

$$\boxed{(A+B)^n = {}^nC_0 A^n + {}^nC_1 A^{n-1} B + {}^nC_2 {}^{n-2} B^2 + \ldots + {}^nC_n B^n}$$

ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}. \text{ Then } (\text{adj } A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \text{Transpose of cofactor matrix}$$

NOTE

If A be a square matrix.

- $A(\text{adj } A) = |A| I_n = (\text{adj } A) A$
- $|\text{adj } A| = |A|^{n-1}, \quad n \geq 2$
- $\text{adj}(\text{adj } A) = |A|^{n-2} A, \quad |A| \neq 0$.
- $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
- $\text{adj}(KA) = K^{n-1}(\text{adj } A)$ where K is a scalar.

SYSTEM OF EQUATION & CRITERIA FOR CONSISTENCY

Gauss - Jordan method :

Example :-
$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

$$\Rightarrow \begin{bmatrix} a_1 x + b_1 y + c_1 z \\ a_2 x + b_2 y + c_2 z \\ a_3 x + b_3 y + c_3 z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow AX = B \Rightarrow A^{-1}AX = A^{-1}B, \text{ if } |A| \neq 0 \Rightarrow X = A^{-1}B = \frac{\text{Adj } A}{|A|} \cdot B$$

NOTE

(i) If $|A| \neq 0$, system is consistent having unique solution.

(ii) If $|A| \neq 0$ & $(\text{adj } A) \cdot B \neq 0$ (Null matrix) system is consistent having unique non-trivial solution.

(iii) If $|A| \neq 0$ & $(\text{adj } A) \cdot B = 0$ (Null matrix) system is consistent having unique trivial solution.

(iv) If $|A| = 0$, then **matrix method fails**

If $(\text{adj } A).B = O$ (null matrix) → Infinite solutions OR no solution

If $(\text{adj } A).B \neq O$ → Inconsistent (no solution)

SUCCESS is a journey not a Destination.....

Determinants

Determinant :- To every square matrix $A=[a_{ij}]$ of order n, we can associate a no. (real or complex) called determinant of the square matrix A, where $a_{ij} = (i,j)^{th}$ element of A. Donated as: $\det A$ or $|A|$.

Determinant of Matrix of order one :-

let $A = [a]$ be the matrix of order 1, then determinant of A is defined to be equal to a.

Determinant of Matrix of order 2X2 :- let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \Rightarrow \det(A) = |A| = \Delta$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2\times 2} = a_{11}a_{22} - a_{21}a_{22}$$

Determinant of matrix of order 3X3 :- let
$$A = \begin{array}{c} \begin{array}{ccc} \oplus & \ominus & \oplus \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

Expansion along First row (R_1)

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

ऐसे ही 2^{nd}, 3^{rd} row and 1^{st}, 2^{nd}, 3^{rd} column के through भी expansion कर सकते हैं। और answer हर बार same आयेगा चाहे expansion कैसे भी करो।

note:- • For matrix A, $|A|$ is read as determinant of A and not modulus of A.

▪ Only square matrices have determinants.

➡ Properties Of Determinants :-

▲ The value of the determinant remains unchanged if its rows and columns interchanged.

▲ If any two rows (or columns) of a determinant are interchanged, then value of determinant is zero.

▲ If any two rows (or columns) of a determinant are identical, then value of determinant is zero.

▲ If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

▲ If some or all elements of a row or column of a determinant, is expressed sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

▲ If, to each element of any row or column of a determinant, the equimultiple of corresponding elements of other row (or column) are added, then the value of determinant remains the same i.e. the value of determinant remains same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

▲ If $D(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$, where f_r, g_r, h_r ; $r = 1, 2, 3$ are three differentiable functions.

then
$$\frac{d}{dx} D(x) = \begin{vmatrix} f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1' & g_2' & g_3' \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1' & h_2' & h_3' \end{vmatrix}$$

➡ Area Of Triangle :-

Area of triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\boxed{\Delta = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]}$$

NOTE

(i) Area is a positive quantity, we always take the absolute value of Δ.

(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero.

Minors :- Minor of an element a_{ij} of the $|A|$ is determinant obtained by deleting i^{th} row and j^{th} column and is denoted by M_{ij}.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ and the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have "9 minors".

Cofactors :- Cofactor of an element a_{ij}, denoted by A_{ij} is defined by

$A_{ij} = (-1)^{i+j}\, M_{ij}$ mirror of a_{ij}

Adjoint of a matrix :- The adjoint of a square matrix $A = [a_{ij}]$ is defined as the transpose of the matrix $[A_{ij}]_{m \times n}$. Adjoint of the matrix A denoted by adj A.

(the cofactor of the element a_{ij})

Singular Matrix :- A square matrix A is said to be singular if $|A| = 0$

Non-Singular Matrix :-

A square matrix A is said to be non-singular if $|A| \neq 0$.

Then the system of equations can be written as,

$$AX = B \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

case I :- If A is a non-singular matrix, then its inverse exists. Now

$X = A^{-1}B$ (consistent & unique sol^n)

Case II :- If A is a singular matrix,

then $|A| = 0$ $(Adj\,A)\,B \neq 0$ sol^n does not exist (incosistent)

$(Adj\,A)\,B = 0$ infinitely many sol^n or no sol^n (consistent or inconsistent)

➡ **Theorem 1** ~ If A be any given square matrix of order n, then

$$A(adj\,A) = (adj\,A)\,A = |A|\,I.$$

➡ **Theorem 2** ~ If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

➡ **Theorem 3** ~ $|AB| = |A|\,|B|$ where A and B are square matrices of same order.

➡ **Theorem 4** ~ A square matrix A is invertible if and only if A is non-singular matrix.

➡ **Consistent System** :- If system of equation have solution (one or more) exists.

➡ **Inconsistent System** :- If system has no solution or solution does not exist.

➡ **System of linear equation using inverse of a matrix** :-

consider the equations,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Here, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B \Rightarrow \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then the system of equations can be written as,

$$AX = B \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case I :- If A is a non-singular matrix, then its inverse exists. Now

$X = A^{-1}B$ (consistent & unique solⁿ)

Case II :- If A is a singular matrix,

then $|A| = 0$, $(\text{Adj}\,A)\,B \neq 0$ solⁿ does not exist (incosistent)

$(\text{Adj}\,A)\,B = 0$ infinitely many solⁿ or no solⁿ (consistent or inconsistent)

➡ **System Of Equation** : $\boxed{AX = B}$
- if $\boxed{B = 0}$ (Homogenous system of Equation)
- if $\boxed{B \neq 0}$ (Non-Homogenous system of Equation)

➡ **Homogenous System Of Equation** :- $\boxed{AX = 0}$
- if $\boxed{|A| = 0}$ then unique solution (Trivial Solution)
- if $\boxed{|A| \neq 0}$ then infinitely solutions.

- If A is skew symmetric matrix of odd order, then $|A| = 0$
- The determinant of a skey-symmetric Matrix of even order is a perfect square.

➡ SPECIAL Determinants :

(a) **Symmetric Determinant** :- Elements of a determinant are such that $a_{ij} = a_{ji}$.

e.g. $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

(b) **Other Important Determinants** :-

(i) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(ii) $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

Permutations and Combinations

Fundamental Principle of counting (counting without actually counting) :—

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of different ways of :

(a) Simultaneous occurence of both events in a definite order is $m \times n$. This can be extented to any number of events (known as multiplication principle).

(b) Happening of exactly one of the events is $m + n$ (known as additional principle).

Permutations : The number of permutations of n different things taken r at a time, where repetition is not allowed, is denoted ^nP_r.

$$\boxed{n! = 1 \times 2 \times 3 \times \ldots \times n}$$

$$\boxed{n! = n \times (n-1)!}$$

$$\boxed{^nP_r = \frac{n!}{(n-r)!}}$$

where $0 \le r \le n$

Factorial :-

A useful Notation : $n! = n(n-1)(n-2)\ldots\ldots 3.2.1$;

$$\boxed{n! = n(n-1)!}$$

$$\boxed{0! = 1! = 1}$$

$$\boxed{(2n)! = 2^n \cdot n! \, [1.3.5.7 \ldots\ldots (2n-1)]}$$

NOTE THAT

(i) Factorial of negative integers is not defined.

(ii) Let p be a prime number and n be a positive integer, then exponent of p in $n!$ is denoted by $E_p(n!)$ and is given by

$$\boxed{E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots\ldots}$$

★ **Theorem 1 :** The number of permutations of n different things, taken r at a time. where $0 < r \leq n$ and the objects do not repeat is $n(n-1)(n-2)\ldots\ (n-r+1)$ which is denoted by ${}^nP_r$.

★ **Theorem 2 :** The number of permutations of n different things, taken r at a time, where repetition is allowed, is n^r.

★ **Theorem 3 :** The number of permutations of n objects, where p objects are of the same kind and rest are all different $= \dfrac{n!}{p!}$

★ **Theorem 4 :** The number of permutations of n objects taken all at a time, where p_1 objects are of first kind, p_2 objects are of the second kind, ..., p_k objects are of the k^{th} kind and rest, if any, are all different is

$$\boxed{\frac{n!}{p_1!\,p_2!\ldots p_k!}}$$

NOTE

- The number of circular permutations of n different things taken all at a time is ; $(n-1)! = \dfrac{{}^nP_n}{n}$.

  If clockwise & anti-clockwise circular permutations are considered to be same, then it is $\dfrac{(n-1)!}{2}$.

- The number of circular permutations of n different things taking

r at a time distinguishing clockwise & anticlockwise arrangement is $\frac{^nP_r}{r}$.

Combinations : The number of combinations of n different things taken r at a time, denoted by ^nC_r

$$\boxed{^nC_r = \frac{n!}{r!(n-r)!} \; ; \; 0 \le r \le n}$$

★ **Theorem 5** : $\boxed{^nP_r = {^nC_r}\, r! \; ; \; 0 < r \le n}$

★ **Theorem 6** : $\boxed{^nC_r + {^nC_{r-1}} = {^{n+1}C_r}}$

NOTE

1. From above $\frac{n!}{(n-r)!} = {^nC_r} \times r!$, i.e. $\boxed{^nC_r = \frac{n!}{r!(n-r)!}}$

In particular, if $r = n$, $\boxed{^nC_n = \frac{n!}{n!\,0!} = 1}$

2. We define $^nC_0 = 1$, i.e., the number of combinations of n different things taken nothing at all is considered to be 1. Counting combinations is merely counting the no. of ways in which some or all objects at a time are selected. Selecting nothing at all is the same as leaving behind all the objects and we know that there is only one way of doing so. This way we define $^nC_0 = 1$.

3. As $\frac{n!}{0!(n-0)!} = 1 = {^nC_0}$, the formula $^nC_r = \frac{n!}{r!(n-r)!}$ is applicable for $r = 0$ also.

Hence $\boxed{^nC_r = \frac{n!}{r!(n-r)!}}$; $0 \le r \le n$.

4. $\boxed{^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!} = {^nC_r}}$ i.e., selecting r objects out of n objects is same as rejecting $(n-r)$ objects.

5. $\boxed{^{n}C_{a} \Rightarrow {}^{n}C_{a} \Rightarrow a = b}$ or $a = n - b$, i.e., $n = a + b$

NOTE

- The number of combination of n different things taking r at a time.

→ when p particular things are always to be included = $^{n-p}C_{r-p}$

→ when p particular things are always to be excluded = $^{n-p}C_{r}$

→ when p particular things are always to be included and q particular things are to be excluded = $^{n-p-q}C_{r-p}$

- Given n different objects, the number of ways of selecting atleast one of them is, $\boxed{^{n}C_{1} + {}^{n}C_{2} + {}^{n}C_{3} + \dots\dots + {}^{n}C_{n} = 2^{n} - 1}$

  This can also be stated as the total number of non - empty combinations of n distinct things.

Believe in yourself and ANYTHING is possible.

Umesh Bhaiya ❤
Always with you

Vaishali Didi ❤
Always with you

Binomial Theorem And Its Simple Applications

NOTE

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^2 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = (a+b)^3(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

➡ **Pascal's Triangle** : The cofficients of the expansion are arranged in an array. This array is called Pascal's Triangle.

➡ **The expansion of a binomial for any positive integral n**

$$\boxed{(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots + {}^nC_{n-1} a . b^{n-1} + {}^nC_n b^n}$$

➡ **Observations** :

1. The notation $\sum_{k=0}^{n} {}^nC_k a^{n-k} b^k$ stands for ${}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots\ldots + {}^nC_n a^{n-n} b^n$, where $b^0 = 1 = a^{n-n}$ Hence the theorem can also be stated as $\boxed{(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^{n-k} b^k}$.

2. The cofficients ${}^nC_r$ occuring in the binomial theorem are known as binomial cofficients.

3. There are $(n+1)$ terms in the expansion of $(a+b)^n$, i.e. one more than the index.

4. In the successive terms of the expansion the index of a goes on decreasing by unity. It is n in the first term, $(n-1)$ in the second term, and so on ending with zero in the last term. At the same time the index of b increases by unity, starting with zero in the first term, 1 in the second and so on ending with n in the last term.

5. In the expansion of $(a+b)^n$, the sum of the indices of a and b is $n+0 = n$ in the first term, $(n-1)+1 = n$ in the second term and so on $0+n = n$ in the last term. Thus it can be seen that the sum of the indices of a and b in every term of the expansion.

Some special cases

$a = x$ and $b = -y$ $\quad \boxed{(x-y)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 - \ldots + (-1)^n\, {}^nC_n y^n}$

$a = 1$ and $b = x$ $\quad \boxed{(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots + {}^nC_n x^n}$

$a = 1$ and $b = -x$ $\quad \boxed{(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - \ldots + (-1)^n\, {}^nC_n x^n}$

General term

: In General term of an expansion $(a+b)^n$ is

$$\boxed{T_{r+1} = {}^nC_r\, a^{n-r} . b^r} \quad [(r+1)^{th} \text{ term}]$$

Middle terms

(i) If n is even, then the number of terms in the expansion will be $n+1$. Since n is even so $(n+1)$ is odd. Therefore, the middle term is $\left[\frac{n+1+1}{2}\right]^{th}$, i.e. $\left[\frac{n}{2}+1\right]^{th}$ term.

(ii) If n is odd, then $(n+1)$ is even, so there will be two middle terms in the expansion, namely, $\left[\frac{n+1}{2}\right]^{th}$ term and $\left[\frac{n+1}{2}+1\right]^{th}$ term.

(iii) In the expansion of $\left[x+\frac{1}{x}\right]^{2n}$, where $x \neq 0$, the middle term is $\left[\frac{2n+1+1}{2}\right]^{th}$ i.e. $(n+1)^{th}$ term, $2n$ is even. It is given by $\boxed{{}^{2n}C_n\, x^n \left[\frac{1}{x}\right]^n = {}^{2n}C_n}$ (constant).

SOME RESULTS ON BINOMIAL COEFFICIENTS :-

(a) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(b) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

(c) $C_0 + C_1 + C_2 + \ldots\ldots = C_n = 2^n$, $C_r = {}^nC_r$

(d) $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n+1}$, $C_r = {}^nC_r$

(e) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!\,n!}$, $C_r = {}^nC_r$

Greatest coefficient & greatest term in expansion of $(x+a)^n$:

(a) If n is even, greatest binomial cofficient is ${}^nC_{n/2}$

If n is even, greatest binomial cofficient is ${}^nC_{\frac{n-1}{2}}$ OR ${}^nC_{\frac{n+1}{2}}$

(b) For greatest term :

$$\text{Greatest term} \begin{cases} T_p \ \& \ T_{p+1} \text{ if } \dfrac{n+1}{\left|\frac{x}{a}\right|+1} \text{ is an integer equal to } p. \\[2ex] T_{q+1} \text{ if } \dfrac{n+1}{\left|\frac{x}{a}\right|+1} \text{ is non integer and } \in (q, q+1), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q \in I \end{cases}$$

Binomial Theorem for negative or fractional indices :

If $n \in R$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \infty \text{ provided } |x| < 1.$$

NOTE

(i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots\dots\dots \infty$

(ii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots\dots\dots \infty$

(iii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots\dots\dots \infty$

(iv) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots\dots\dots \infty$

Exponential Series :

(a) $\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots \infty}$; where x may be any real or complex & $e = \lim\limits_{n \to \infty} \left[1 + \frac{1}{n}\right]^n$

(b) $\boxed{a^x = 1 + \frac{x}{1!}\ln a + \frac{x^2}{2!}\ln^2 a + \frac{x^3}{3!}\ln^3 a + \dots\dots \infty}$, where $a > 0$

Logarithmic series :

(a) $\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \infty}$, where $-1 < x \leq 1$

(b) $\boxed{\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots \infty}$, where $-1 \leq x < 1$

(c) $\boxed{\ln\frac{(1+x)}{(1-x)} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots \infty\right]}$, $|x| < 1$

Sequence and Series

➡ **Sequence** : A sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it of the type $\{1, 2, 3 \ldots k\}$. Sometimes, we use the functional notation $a(n)$ for a_n.

➡ **Series** : Let $a_1, a_2, a_3 \ldots\ldots, a_n$ be a given sequence. Then, the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called the series associated with the given sequence.

$$\boxed{a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k}$$

➡ **Arithmetic progression** : $a,\ a+d,\ a+2d \ldots\ldots a+(n-1)d, \ldots$

n^{th} term (General term) $\boxed{T_n = a_n = a+(n-1)d}$ $\boxed{l = a+(n-1)d}$

The sum of n terms $\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$

Sum of A.P. when first and last term is given, $\boxed{S_n = \frac{n}{2}[a+l]}$

a = first term
l = last term
d = common difference
n = the no. of terms
S_n = the sum of n terms

➡ **Arithmetic Mean** (A.M.) $\boxed{A = \frac{a+b}{2}}$

a and b = two numbers
A = Arithmetic Mean

Also n^{th} term $\boxed{T_n = S_n - S_{n-1}}$

NOTE

- Three numbers in AP can be taken as $a-d,\ a,\ a+d$; four numbers in AP can be taken as $a-3d,\ a-d,\ a+d,\ a+3d$ five numbers in AP are $a-2d,\ a-d,\ a,\ a+d,\ a+2d$ & six terms in AP are $a-5d,\ a-3d,\ a-d,\ a+d,\ a+3d,\ a+5d$ etc.

- If a, b, c are in A.P., then $\boxed{b = \frac{a+c}{2}}$

- If $a_1, a_2, a_3 \ldots\ldots$ and $b_1, b_2, b_3 \ldots\ldots$ are two A.P.s, then $a_1 \pm b_1,\ a_2 \pm b_2,\ a_3 \pm b_3 \ldots\ldots$ are also in A.P.

(a) If each term of an A.P. is increased or decreased by the same number, then the resulting sequence is also an A.P. having the same condition difference.

(b) If each term of an A.P. is multiplied or divided by the same non zero number (k), then the resulting sequence is also an A.P. whose common difference is kd & $\frac{d}{k}$ respectively, where d is common difference of original A.P.

- Any term of an AP (expect the first & last) is equal to half the sum of terms which are equidistant from it.

$$T_n = \frac{T_{n-k} + T_{n+k}}{2}, \quad k < n$$

➡ Geometric Progression (G.P.) $a, ar, ar^2, ar^3, \ldots$

- General term of G.P. $\boxed{T_n = a_n = ar^{n-1}}$ r = common ratio
- Sum of n terms of G.P. $\boxed{S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}}$

Case I If $r = 1$ $\boxed{S_n = na}$

Case II If $r \neq 1$ $\boxed{S_n = \frac{a(1-r^n)}{1-r}}$ OR $\boxed{S_n = \frac{a(r^n-1)}{r-1}}$

$r < 1$ $\quad r > 1$

➡ Relationship between A.M and G.M. $\boxed{A - G = \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0}$; $a, b > 0$

➡ Sum of first n natural numbers $S_n = 1 + 2 + 3 + \ldots + n$; $\boxed{S_n = \frac{n(n+1)}{2}}$

➡ Sum of squares of the first n natural numbers

$S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$; $\boxed{S_n = \frac{n(n+1)(2n+1)}{6}}$

➡ Sum of cubes of the first n natural numbers

$S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3$; $\boxed{S_n = \frac{[n(n+1)]^2}{4}}$

➡ Geometric Mean (G.M.)

$\boxed{G = \sqrt{ab}}$; $a, b > 0$

➡ Sum of infinite GP when $|r| < 1$ $(n \to \infty, r^n \to 0)$

$$\boxed{S_\infty = \frac{a}{1-r} \; ; |r| < 1}$$

➡ If a, b, c are in GP $\Rightarrow \boxed{b^2 = ac} \Rightarrow$ loga, logb, logc are in A.P.

NOTE

- In a G.P. product of k^{th} term from begining and k^{th} term from the last is always constant which equal to product of first term and last term.

- Three numbers in G.P. : $\frac{a}{r}, a, ar$

  Five numbers in G.P. : $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

  Four numbers in G.P. : $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

  Six numbers in G.P. : $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$

- If each term of a G.P. be raised to the same power, then resulting series is also a G.P.

- If each term of a G.P. be multiplied or divided by the same non zero quantity, then the resulting sequence is also a G.P.

- If $a_1, a_2, a_3 \ldots$ and $b_1, b_2, b_3 \ldots\ldots$ be two G.P.'s of common ratio r_1 and r_2 respectively, then $a_1 b_1$, $a_2 b_2 \ldots\ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3} \ldots\ldots$ will also form a G.P. common ratio will be r_1, r_2 and $\frac{r_1}{r_2}$ respectively.

- In a positive G.P. every term (except first) is equal to square root of product of its two terms which are equidistant from it. i.e. $\boxed{T_r = \sqrt{T_{r-k}\, T_{r+k}}}$, $k < r$

- If $a_1, a_2, a_3 \ldots\ldots a_n$ is a G.P. of non zero, non negative terms, then $\log a_1, \log a_2 \ldots\ldots \log a_n$ is an A.P. and vice-versa.

Harmonic Progression (HP) :

A sequence is said to HP if the reciprocals of its terms are in AP. If the sequence $a_1, a_2, a_3, \ldots\ldots, a_n$ is an HP then $\frac{1}{a_1}, \frac{1}{a_2}, \ldots\ldots, \frac{1}{a_n}$ is an AP & vice versa. Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is

$$\boxed{\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots\ldots, \frac{1}{a+(n-1)d}}$$

NOTE

- No term of any H.P. can be zero.
- If a, b, c are in HP $\Rightarrow$ $\boxed{b = \frac{2ac}{a+c}}$ OR $\boxed{\frac{a}{c} = \frac{a-b}{b-c}}$

MEANS

(a) Arithmetic mean (AM) :

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c

n - arithmetic means between two numbers :

If a, b are any two given numbers & $A_1, A_2, \ldots\ldots\ldots, A_n$, b are in AP then $A_1, A_2, \ldots\ldots, A_n$ are the n AM's between a & b, then $\boxed{A_1 = a+d,\ A_2 = a+2d, \ldots\ldots, A_n = a+nd}$

where $d = \frac{b-a}{n+1}$

NOTE

Sum of n AM's inserted between a & b is equal to n times the single AM between a & b

i.e. $\boxed{\sum_{r=1}^{n} A_r = nA}$ where A is the single AM between a & b i.e. $\frac{a+b}{2}$.

(b) **Geometric mean (GM)** : If a, b, c are in GP, then b is the GM between a & c i.e. $b^2 = ac$, therefore $b = \sqrt{ac}$.

n-geometric means between two numbers :

If a, b are two given positive numbers & $a, G_1, G_2, \ldots\ldots, G_n, b$ are in GP then $G_1, G_2, G_3, \ldots\ldots G_n$ are n GMs between a & b.

$$\boxed{G_1 = ar, \quad G_2 = ar^2, \ldots\ldots G_n = ar^n}, \quad \text{where } r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

NOTE

The product of n GMs between a & b is equal to n^{th} power of the single GM between a & b i.e.

$$\boxed{\prod_{r=1}^{n} G_r = (G)^n}$$

where G is the single GM between a & b i.e. $\sqrt{ab}$.

Limits

➡ **limit** : If the right and left hand limits equal, then that common value is called the limit of $f(x)$ at $x = a$ and denote it by $\lim\limits_{x \to a} f(x)$.

$\lim\limits_{x \to a^-} f(x)$ left hand limit of f at a.

$\lim\limits_{x \to a^-} f(x)$ right hand limit of $f(x)$ at a.

➡ **LEFT HAND LIMIT & RIGHT HAND LIMIT OF A FUNCTION :**

$$\boxed{\text{Left hand limit (LHL)} = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h),\ h > 0.}$$

$$\boxed{\text{Right hand limit (RHL)} = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h),\ h > 0.}$$

Limit of a function $f(x)$ is said to exist as $x \to a$ when $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) =$ Finite and fixed quantity.

NOTE

In $\lim\limits_{x \to a} f(x)$, $x \to a$ necessarily implies $x \neq a$. That is while evaluating limit at $x = a$, we are not concerned with the value of the function at $x = a$, In fact the function may or may not be defined at $x = a$.

Also it is necessary to note that if $f(x)$ is defined only on one side of '$x = a$', one sided limits are good enough to establish the existence of limits, & if $f(x)$ is defined on either side of 'a' both sided limits are to be considered.

➡ **Theorem 1** : Let f and g be two functions such that both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist, then

(i) $\boxed{\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)}$

(ii) $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

➡ **Theorem 2 :** For any positive integer n, $\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$

➡ **Theorem 3 :** Let f and g be any two real valued functions with the same domain such that $f(x) \le g(x)$ for all x in the domain of definition, for some a, if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

➡ **Theorem 4 : (Sandwich theorem) :** Let f, g and h be real functions such that $f(x) \le g(x) \le h(x)$ for all x in the common domain of definition. For some real number a, if

$$\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x), \text{ then } \lim_{x\to a} g(x) = l$$

➡ **Theorem 5 :** The following are two important limits

(i) $\lim_{x\to 0}\frac{\sin x}{x} = 1$ (ii) $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

➡ **Power rule :**

$$\lim_{x\to a}[f(x)]^{g(x)} = \lim_{x\to a}[f(x)]^{\lim_{x\to a} g(x)} \text{ ; provided } \lim_{x\to a} f(x) > 0$$

$$\lim_{x\to a} f[g(x)] = f\left[\lim_{x\to a} g(x)\right] = f(m) \text{ ; provided } f(x) \text{ is continuous at } x = m.$$

➡ **INTERMINATE FORMS :**

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 1^{∞}, 0^0, ∞^0

➡ **GENERAL METHODS TO BE USED TO EVALUATE LIMITS :**

(a) Factorization : Important factors –

(i) $x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + \ldots\ldots + a^{n-1})$, $n \in N$

(ii) $x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + \ldots\ldots + a^{n-1})$, n is an odd natural number.

NOTE $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

➡ LIMIT OF TRIGONOMETRIC FUNCTIONS :

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\tan^{-1} x}{x} = \lim_{x \to 0} \frac{\sin^{-1} x}{x}$$

where x is measured in radians

Further if $\lim\limits_{x \to a} f(x) = 0$, then $\lim\limits_{x \to a} \dfrac{\sin f(x)}{f(x)} = 1$

➡ LIMIT OF EXPONENTIAL FUNCTIONS :

(a) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a \quad (a > 0)$ In particular $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$.

In general if $\lim\limits_{x \to a} f(x) = 0$, then $\boxed{\lim\limits_{x \to a} \dfrac{a^{f(x)} - 1}{f(x)} = \log_e a}$, $a > 0$

(b) $\boxed{\lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = 1}$

(c) $\boxed{\lim\limits_{x \to 0} (1+x)^{1/x} = e = \lim\limits_{x \to \infty} \left[1 + \dfrac{1}{x}\right]^x}$

NOTE The base and exponent depends on the same variable.

In general, if $\lim\limits_{x \to a} f(x) = 0$, then $\lim\limits_{x \to a} [1 + f(x)]^{1/f(x)} = e$

(d) If $\lim\limits_{x \to a} f(x) = 1$ and $\lim\limits_{x \to a} \phi(x) = \infty$

then $\boxed{\lim\limits_{x \to a} [f(x)]^{\phi(x)} = e^k}$ where $k = \lim\limits_{x \to a} \phi(x)[f(x) - 1]$

(e) If $\lim\limits_{x \to a} f(x) = A > 0$ and $\lim\limits_{x \to a} \phi(x) = B$ (a finite quantity),

then $\boxed{\lim\limits_{x \to a} [f(x)]^{\phi(x)} = e^{B \log A} = A^B}$

Derivative: The derivative of a function f at a is defined by

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Derivative of a function f at a point x is defined by

$$\boxed{f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

first principle of derivative

NOTE $\lim\limits_{x \to a}[(\lambda \cdot f)(x)] = \lambda \lim\limits_{x \to a} f(x)$

Limits of polynomials and rational functions: A function f is said to be a polynomial function if $f(x)$ is zero function or if $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$. where a_is are real numbers such that $a_n \neq 0$ for some natural number n

Theorem 6: For functions u and v the following holds:

(i) $\boxed{(u \pm v)' = u' \pm v'}$ (ii) $\boxed{(uv)' = u'v + uv'}$ (Leibnitz rule)

(iii) $\boxed{\left[\frac{u}{v}\right]' = \frac{u'v - uv'}{v^2}}$ provided all are defined

Theorem 7: $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ for any positive integer n.

Theorem 8: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0$ be a polynomial function where a_is are all real numbers and $a_n \neq 0$. Then the derivative function is given by

$$\boxed{\frac{df(x)}{dx} = n a_n x^{n-1} + (n-1) a_{n-1} x^{x-2} + \ldots\ldots + 2a_2 x + a_1}$$

- $\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$
- $\boxed{\frac{d}{dx}(\sin x) = \cos x}$
- $\boxed{\frac{d}{dx} = -\sin x}$

Continuity

Continuity :- suppose F is real function on a subset of the real numbers and let c be a point in the domain F. Then F is continuous at c if $\boxed{\lim_{x \to c} f(x) = f(c)}$

If $\lim f(c-h) = \lim f(c+h) = f(c)$ = finite and fixed quantity $(h > 0)$

i.e. $LHL|_{x=c} = RHL|_{x=c}$ = value of $f(x)|_{x=c}$ = finite and fixed quantity.

NOTE

At isolated points functions are considered to be continuous.

Discontinuity :- A function said to be discontinuous at point $x = a$, it is not continuous at this point. This point $x = a$ where the function is not continuous is called the point of discontinuity.

Theorem :- 1 Suppose f and g be two real functions continuous at a real no. then,

1. $f + g$ is continuous at $x = c$
2. $f - g$ is continuous at $x = c$
3. $f \cdot g$ is continuous at $x = c$
4. $\frac{f}{g}$ is continuous at $x = c$, {provided $g(c) \neq 0$}
5. $a \cdot f$ is continuous at $x = c$, where a is constant.
6. $|f|$ is continuous at $x = c$

↗ Every constant function is continuous function.

↗ Every Polynomial is continuous function.

↗ Identity function is continuous function.

↗ Every Exponential & logarithmic function is continuous function

➡ Theorem :- 2

Suppose f and g are real valued functions such that $(f \circ g)$ is defined at c. If g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c.

➡ CONTINUITY OF THE FUNCTION IN AN INTERVAL :

(a) A function is said to be continuous in (a, b) if f is continuous at each & every point belonging to (a, b).

(b) A function is said to be continuous in a closed interval $[a, b]$ if :

- f is continuous in the open interval (a, b).
- f is right continuous at 'a' i.e. $\text{Lim } f(x) = f(a) =$ a finite quantity.
- f is left continuous at 'b' i.e. $\text{Lim } f(x) = f(b) =$ a finite quantity.
- If f and g are continuous at $x = c$, then $f+g$, $f-g$, $f \cdot g$ may still be continuous.
- Sum or difference of a continuous and a discontinuous function is always discontinuous.

➡ REASONS OF CONTINUITY :

(a) Limit does not exist i.e. $\text{Lim } f(x) \neq \text{Lim } f(x)$

(b) $\text{Lim } f(x) \neq f(a)$.

Differentiability

Differentiability :- Suppose f is a real function and c is a point in its domain. The derivative of f at c defined by $\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ provided this limit exists. Derivative of at c is denoted by $f'(c)$ or $\dfrac{d}{dx}[f(x)]_c$.

The function defined by $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ wherever the limit exists is defined to be the derivative of f. The derivative denoted by $f'(x)$ or $\dfrac{d}{dx}[f(x)]$ or if $y=f(x)$ by $\dfrac{dy}{dx}$ or y'.

RIGHT HAND & LEFT HAND DERIVATIVE :

(a) **Right hand derivative** :-

$f'(a) = \lim\limits_{h\to 0^+} f(a+h) - f(a)$, provided the limit exists & is finite.

(b) **Left hand derivative** :-

$f'(a) = \lim\limits_{h\to 0^+} f(a-h) - f(a)$, provided the limit exists & is finite.

(c) **Derivative of function at a point** :-

If $f'_+(a) = f'_-(a) =$ finite quantity, then $f(x)$ is said to be derivable or differentiable at $x = a$. In such case $f'_+(a) = f'_-(a) = f'(a)$ & it is called derivative or differential coefficient of $f(x)$ at $x = a$

NOTE

(i) All polynomial, trigonometric, inverse trigonometric, logarithmic and exponential function are continuous and differentiable in their domains, except at end points.

(ii) If $f(x)$ & $g(x)$ are derivable at $x = a$ then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$ will also be derivable at $x = a$ & if $g(a) \neq 0$ then the function will also be derivable at $x = a$.

DERIVABILITY OVER AN INTERVAL :

(a) $f(x)$ is said to be derivable over an open interval (a, b) if it is derivable at each & every point of the open interval (a, b).

(b) $f(x)$ is said to be derivable over the closed interval $[a, b]$ if :

(i) $f(x)$ is derivable in (a, b) &

(ii) for the points a and b $f'(a)$ & $f'(b)$ exist.

NOTE

If $f(x)$ is differentiable at $x = a$ & $g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x) \cdot g(x)$ can still be differentiable at x.

If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function ; $F(x) = f(x) \cdot g(x)$ can still be differentiable at $x = a$.

If $f(x)$ & $g(x)$ both are non - derivable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function.

If $f(x)$ is derivable at $x = a \nRightarrow f'(x)$ is continuous at $x = a$.

Sum or difference of a differentiable and a non - differentiable function is always is non - differentiable

Algebra Of Derivaties :-

(Quotient Rule)

3. $\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, \text{ whenever } v \neq 0}$

ALGEBRA OF DERIVATIES

1. $\boxed{(u \pm v)' = u' \pm v'}$

2. $\boxed{(uv)' = u'v + uv'}$

(Leibnitz or product rule)

Theorem :- 3

If a function f is differentiable at a point c, then it is also continuous at that point.

NOTE Every differentiable function is continuous.

Chain Rule :-

let f be a real valued function which is a composite of two functions u and v i.e. $f = vou$; suppose $t = u(x)$ and if $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, we have $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$ Suppose f is real valued function which is a composite of three functions u, v and w ; i.e. $f = (wou) ov$ and if $t = v(x)$ and $S = u(t)$ then

$$\frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Some Properties Of logorithmic Function :-

$\boxed{\log_a p = \frac{\log_b p}{\log_b a}}$

$\boxed{\log_b pq = \log_b p + \log_b q}$

$$\log_b P^2 = \log_b P + \log_b P = 2\log_b P$$

$$\log_b P^n = n\log_b P$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b x = \frac{1}{\log_x b}$$

➡ **Learn it Important for limit Questions:-**

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$

$$\log|1-x| = -x - \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} - \cdots$$

$$e^{-x} = 1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \cdots$$

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$$

$$a^x = 1 + \frac{x\log_e a}{\lfloor 1} + \frac{x^2(\log_e a)^2}{\lfloor 2} + \cdots$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$$

$$\log|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^5 + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)x^3}{\lfloor 3} + \cdots$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots, \quad x \in R$$

$$\sin^{-1}x = x + \frac{1^2}{3}x^3 + \frac{1^2.3^2}{5}x^5 + \frac{1^2.3^2.5^2}{7}x^7 + \ldots\ldots, \quad x \in [-1,1]$$

➡ **Rolle's Theorem:-**

If $f:[a,b]\to R$ is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b)$ then there exists some C in (a,b) such that $f'(c) = 0$.

➡ **Langrage Theorem Or Mean Value Theorem:**

If $F:[a,b]\to R$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists some c in (a,b) such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

Some Standard derivative :-

$\frac{d}{dx}(c) = 0$ c = constant

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\log_e x) = \frac{1}{x}$

$\frac{d}{dx}(a^x) = a^x \log_e a$

$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

Derivative of functions in parametric forms :-

$x = f(t) = g(t)$ parametric form with t as a parameter.

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{whenever } \frac{dx}{dt} \neq 0$$

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \quad \left[\text{as } \frac{dy}{dt} = g'(t) \text{ and } \frac{dx}{dt} = f'(t)\right] \left\{\text{provided } f'(t) \neq 0\right\}$$

NOTE

Exponential form	logarithim form
$2^3 = 8$	$\log_2 8 = 3$
$b^1 = b$	$\log_b b = 1$
$b^0 = 1$	$\log_b 1 = 0$

NOTE

Higher order derivative may be defined similarly.

Logarithmic Differentiation :-

$$y = f(x) = [u(x)]^{v(x)}$$

$\log y = v(x) \log [u(x)]$ taking log both sides,

$\frac{1}{y}\frac{dy}{dx} = v(x) \cdot \frac{1}{u(x)} \cdot u'(x) + v'(x) \cdot \log[u(x)]$ using chain rule to differentiate

$$\boxed{\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)} \cdot u'(x) + v'(x) \cdot \log[u(x)]\right]}$$

Second order derivative :-

Let $y = f(x) \Rightarrow \frac{dy}{dx} = f'(x)$ ——— (i)

differenciate (i) again w.r.t to x,

$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}[f'(x)] \Rightarrow \boxed{\frac{d^2y}{dx^2} = f''(x)}$ denoted by D^2y or D''

Differentiation Of Determinants :-

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where $f, g, h, l, m, n, u, v, w$ are differentiable functions of x, then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Similarly one can also proceed columnwise.

Application Of Derivatives

➡ Rate Of Change :-

If a quantity y varies with another quantity x, satisfying some rule $y=f(x)$, then $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x=x_0$.

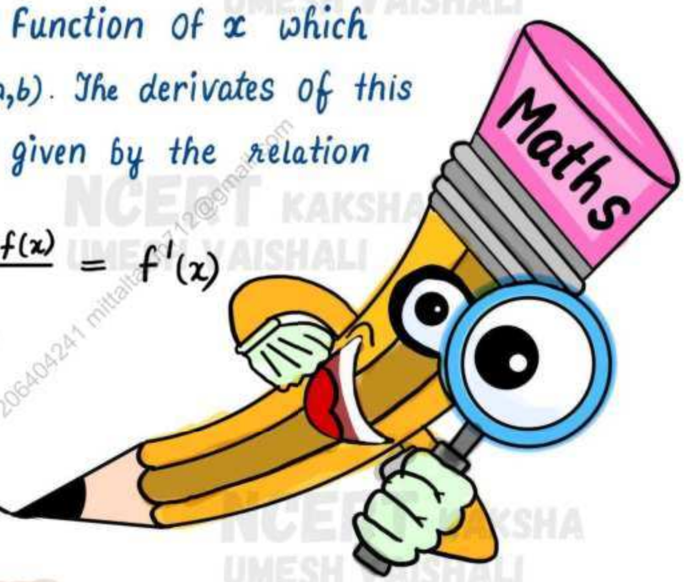
➡ Differentials :-

Let $y=f(x)$ be any function of x which is differentiable in (a,b). The derivates of this function at some point x of (a,b) is given by the relation

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)$$

$$\Rightarrow \frac{dy}{dx}=f'(x)\Rightarrow dy=f'(x)\,dx$$

differential of the function



➡ Increasing And decreasing Functions :-

Increasing	Decreasing
(a) Increasing on an interval (a,b) if $x_1<x_2$ in $(a,b)\Rightarrow f(x_1)\le f(x_2)$ for all $x_1, x_2 \in (a,b)$.	(b) decreasing on an interval (a,b) if $x_1<x_2$ in $(a,b)\Rightarrow f(x_1)\ge f(x_2)$ for all $x_1, x_2\in(a,b)$

➡ Theorem 1 ~

(a) f is increasing in $[a,b]$ if $f'(x)>0$ for each $x\in(a,b)$.

(b) f is decreasing in $[a,b]$ if $f'(x)<0$ for each $x\in(a,b)$

(c) f is a constant function in $[a,b]$ if $f'(x)=0$ for each $x\in(a,b)$

Tangent to a curve :- The equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by-

$\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$ OR $f'(x_0) = m$ = slope of tangent at (x_0, y_0)

$$\boxed{y - y_0 = \left.\frac{dy}{dx}\right]_{(x_0, y_0)} (x - x_0)}$$

⇒ If $\frac{dy}{dx}$ does not exist at the point (x_0, y_0), then the tangent at this point is parallel to the Y-axis and its equation is $x = x_0$.

⇒ If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to x-axis, then $\left.\frac{dy}{dx}\right]_{x = x_0} = 0$

Normal to the curve :-

Equation of the normal to the curve $y = f(x)$ at a point (x_0, y_0) is given by,

$$y - y_0 = \frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0, y_0)}} (x - x_0)$$

$\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$ OR $f'(x_0) = m$ = slope of tangent at (x_0, y_0)

- If $\frac{dy}{dx}$ at the point (x_0, y_0) is zero, then equation of the normal is $x = x_0$
- If $\frac{dy}{dx}$ at the point (x_0, y_0) does not exist, then the normal is parallel to x-axis and its eq. $y = y_0$

$$\boxed{\text{Slope of the Normal} = \frac{-1}{\text{slope of the tangent}}}$$

Approximation :- Let $y = f(x)$, Δx be a small increament in x and Δy be the increment in y corresponding to the increment in x, i.e. $\Delta y = f(x + \Delta x) - f(x)$. Then approximate value of $\Delta y = \left(\frac{dy}{dx}\right) \Delta x$.

Maximum or Minimum value of a Function :-

- A function f is said to attain maximum value at a point $a \in D_f$, if $f(a) \geq f(x)$ $\forall$ $x \in D_f$ then $f(a)$ is called absolute maximum value of f.

★ A function f is said to attain minimum value at a point $b \in D_f$, if $f(b) \leq f(x)$ $\forall\ x \in D_f$ then $f(b)$ is called absolute minimum value of f.

➡ Local Maxima And Local Minima (Relative Extrema) :-

Local Maxima ~ A function $f(x)$ is said to attain a local maxima at $x = a$, if there exists a neighbourhood $(a-\delta, a+\delta)$ of 'a' such that $f(x) < f(a)\ \forall\ x \in (a-\delta, a+\delta)$, $x \neq a$, then $f(a)$ is the local maximum value of $f(x)$ at $x = a$.

Local Minima :- A function $f(x)$ is said to attain a local minima at $x = a$, if there exists a neighbourhood $(a-\delta, a+\delta)$ of 'a' such that $f(x) > f(a)\ \forall\ x \in (a-\delta, a+\delta)$, $x \neq a$, then $f(a)$ is the local minimum value of $f(x)$ at $x = a$.

(a) **First derivative test** :-

- If $f'(x)$ changes sign from positive to negative as x increases through c, then 'c' is a point of local maxima and $f(c)$ is local maximum value.
- If $f'(x)$ changes sign from negative to positive as x increases through c, then 'c' is a point of local minima and $f(c)$ is local minimum value.
- If $f'(x)$ does not changes sign as x increases through c, then c is neither a point of local minima nor a point of local maxima. Such a point is called point of inflection.

(b) **Second Derivative Test** :- Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then,

- $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. In this case $f(c)$ is called local maximum value.
- $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. In this case $f(c)$ is called local minimum value.

- The test fails if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to first derivative test.

Working Rule for Finding absolute maximum or absolute minimum values :-

Step I :- Find all the critical points of f in the given interval, i.e, find points x where either $f'(x) = 0$ or f is not differentiable.

Step II :- Take the end points of the interval.

Step III :- At all these points, Calculate the value of f.

Step IV :- Identify the maximum and minimum value of f out the values calculated in step III. The maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.

Critical Point :-

A point C in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a critical point of f.

Useful for Questions :-

Area of Square $= x^2$ & Perimeter $= 4x$

Side of Square

Area of Rectangle $= x \cdot y$ & Perimeter $= 2(x + y)$

length

Breadth

Area of Circle $= \pi r^2$

circumference $= 2\pi r$

radius

Area Of Trapezium $= \frac{1}{2}$ Sum of Parallel Side $\times$ Perpendicular distance between them

CUBE = Surface Area $= 6x^2$ Side of Cube
Volume $= x^3$

Right Circular Cylinder
Total surface Area $= 2\pi r h + 2\pi r^2$ Height
Curved surface Area $= 2\pi r h$
Volume $= \pi r^2 h$

Area Of Equilateral Triangle $= \frac{\sqrt{3}}{4} (\text{Side})^2$

Right circular cone
Total surface Area $= \pi r^2 + \pi r l$ Slant height
Curved surface Area $= \pi r l$
Volume $= \frac{1}{3} \pi r^2 h$

Sphere
Volume of Sphere $= \frac{4}{3}\pi r^3$
Surface Area $= 4\pi r^2$

Integrals Calculas

➡ **Integration** (Anti differentiation) :- Integration is the inverse process of differentiation. Instead of differentiating a function. We are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called integration or anti differentiation.

Example :- $\boxed{y = \int f(x)\,dx}$

Derivatives	Integrals (Antideratives)
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
$\frac{d}{dx}(x) = 1$	$\int dx = x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$	$\int \sec x \tan x\,dx = \sec x + c$
$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$	$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{dx}{1-x^2} = -\cos^{-1}x + C$
$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \tan^{-1}x + c$
$\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$	$\int \frac{dx}{1+x^2} = -\cot^{-1}x + c$
$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + c$
$\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = -\text{cosec}^{-1}x + c$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}(\log x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log\|x\| + c$
$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$	$\int a^x dx = \frac{a^x}{\log a} + C$

Integration By Substitution Method :-

$\int \tan x \, dx = \log|\sec x| + c$

$\int \cot x \, dx = \log|\sin x| + c$

$\int \sec x \, dx = \log|\sec x + \tan x| + c$

$\int \text{cosec} x \, dx = \log|\text{cosec} x - \cot x|$

Integrals Of Some Particular functions :-

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + c$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

$$\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$$

➡ To find the integral $\int \frac{dx}{ax^2+bx+c}$

we write, $ax^2+bx+c = a\left[x^2+\frac{bx}{a}+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$

Now; put $x+\frac{b}{2a} = t \Rightarrow dx = dt$ and $\frac{c}{a}-\frac{b^2}{4a^2} = \pm K^2$

The integral becomes $\frac{1}{a}\int \frac{dt}{t^2 \pm K^2}$

➡ To find the integrated of the type:

$\int \frac{px+q}{ax^2+bx+c}dx \Rightarrow$ where p, q, a, b, c are constants.

To find the real numbers A, B such that,

$$px+q = A\frac{d}{dx}(ax^2+bx+c) + B = A(2ax+b) + B$$

NOTE

Integral of the form $\int \sin^m x \cdot \cos^n x \, dx$

(i) If (m) exponent of $\sin x$ is an odd positive integer, then put $\cos x = t$

(ii) If (n) exponent of $\cos x$ is an odd integer, then put $\sin x = t$

(iii) If $m+n$ is negative even integer then put $\tan x = t$.

(iv) If m and n both even positive integar then use

$\sin^2 x = \dfrac{1-\cos 2x}{2}$, $\cos^2 x = \dfrac{1+\cos 2x}{2}$

Integration by Partial fraction :-

Form of Rational function	Form of partial function
$\dfrac{px+q}{(x-a)(x-b)}$; $a \neq b$	$\dfrac{A}{x-a} + \dfrac{B}{x-b}$
$\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{x-a} + \dfrac{B}{(x-a)^2}$
$\dfrac{px+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$
$\dfrac{px+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-b)}$
$\dfrac{px+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{(x-a)} + \dfrac{Bx+C}{(x^2+bx+c)}$

where x^2+bx+c cannot be factorised further.

Integration By Parts :-

$$\int f(x)\, g(x)\, dx = f(x) \int g(x)\, dx - \int \left[f'(x) \int g(x)\, dx \right] dx$$

- **Integral of the type :-**

$$\int e^x[f(x)+f'(x)]\,dx = \int e^x f(x)\,dx$$

$$\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\left|\log\left(x+\sqrt{x^2-a^2}\right)\right| + C$$

$$\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\left|\log\left(x+\sqrt{x^2+a^2}\right)\right| + C$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

- **Fundamental Theorem Of Calculus :-**

- **Area Function :** $A(x) = \int_a^x f(x)\,dx$

- **First Fundamental Theorem Of Integral calculus :**

Theorem 1 ~ Let f be a continuous function on the closed interval $[a,b]$ and let $A(x)$ be the area function. then $\boxed{A'(x)=f(x)}$, for all $x\in[a,b]$

- **Second fundamental theorem of integral calculus :**

Theorem 2 ~ f be continuous function defined on the closed interval $[a,b]$ and F be an antiderivative of f.

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b)-F(a)$$

- **Definite Integral :-** If $F(x)$ is the integral of $f(x)$ over the interval $[a,b]$, i.e. $\int f(x)\,dx = F(x)$ then the definite integral of $f(x)$ over the interval $[a,b]$ is denoted by $\int_a^b f(x)$ is defined as

upper Limit, Lower Limit: $\int_a^b f(x)\,dx = F(b) - F(a)$

➡ Define integral as the limit of the sum :-

$$\int_a^b f(x)\,dx = \lim_{h\to\infty} h\left[f(a) + f(a+h) + \dots f(a+(n-1)h)\right]$$

$$\int_a^b f(x)\,dx = (b-a) \lim_{n\to\infty} \frac{1}{n}\left[f(a) + f(a+h) + \dots f(a+(n-1)h)\right]$$

where $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

➡ Some Properties Of Define Integrals :-

P_0 : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P_1 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P_2 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $(a<c<b)$

P_3 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P_4 : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

P_5 : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

P_6 : $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & : \text{if } f(2a-x) = f(x) \\ 0 & : f(2a-x) = -f(x) \end{cases}$

P_7 : (i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$: $f(x)$ is even function, i.e. $f(-x) = f(x)$

(ii) $\int_{-a}^a f(x)\,dx = 0$ if $f(x)$ is odd function. i.e. $f(-x) = -f(x)$

P_8 : $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$; if $f(x) = f(a+x)$

! NOTE $\int_x^{T+x} f(t)\,dt$ will be independent of x and equal to $\int_0^T f(t)\,dt$

NOTE

- $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$ where $f(x)$ is periodic with period T & $n \in I$
- $\int_{ma}^{na} f(x)\,dx = (n-m)\int_0^a f(x)\,dx$, $(n, m \in I)$ if $f(x)$ is periodic with period 'a'.

WALLI's FORMULA :

(a) $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)\ldots\ldots(1 \text{ or } 2)}{n(n-2)\ldots\ldots(1 \text{ or } 2)} K$$

where $K = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

(b) $$\int_0^{\pi/2} \sin^n x \cdot \cos^m x\,dx = \frac{[(n-1)(n-3)(n-5)\ldots\ldots 1 \text{ or } 2][(m-1)(m-3)\ldots\ldots 1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)\ldots\ldots 1 \text{ or } 2}$$

where $K = \begin{cases} \frac{\pi}{2} & \text{if both } m \text{ and } n \text{ are even } (m, n \in N) \\ 1 & \text{otherwise} \end{cases}$

DERIVATIVE OF ANTIDERIVATIVE FUNCTION :

(Newton- Leibnitz Formula)

If $h(x)$ & $g(x)$ are differentiable functions of x then,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f[h(x)] \cdot h'(x) - f[g(x)] \cdot g'(x)$$

SOME STANDARD RESULTS :

(a) $$\int_0^{\pi/2} \log \sin x\,dx = -\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log \cos x\,dx$$

(b) $$\int_a^b \{x\}\,dx = \frac{b-a}{2}\ ;\ a, b \in I$$

(c) $$\int_a^b \frac{|x|}{x}\,dx = |b| - |a|.$$

Applications Of Integrals

- The area of region bounded by the curve $y=f(x)$, x-axis and the lines $x=a$ and $x=b$ $(b>a)$ is given by $\boxed{\text{Area} = \int_a^b f(x)\,dx = \int_a^b y\,dx}$
- The area of region bounded by the curve $x=\phi(y)$, y-axis and the lines $y=c$ and $y=d$ is $\boxed{\text{Area} = \int_c^d x\,dy = \int_c^d \phi(y)\,dy}$
- Area enclosed between; $y=f(x)$ and $y=g(x)$ and the lines; $x=a$, $x=b$

$$\boxed{\text{Area} = \int_a^b [f(x)-g(x)]\,dx \;;\; f(x) \geq g(x) \text{ in } [a,b]}$$

- If $f(x) \geq g(x)$ in $[a,c]$ and $f(x) \leq g(x)$ in $[c,b]$, $a<c<b$ then,

$$\boxed{\text{Area} = \int_a^c [f(x)-g(x)]\,dx + \int_c^b [g(x)-f(x)]\,dx}$$

Important For Questions :-

$$\Sigma(n-1) = 1+2+3+\ldots+(n-1) = \frac{n(n-1)}{2}$$

$$\Sigma(n-1)^2 = 1^2+2^2+3^2+\ldots+(n-1)^2 = \frac{n(n-1)(2n-1)}{6}$$

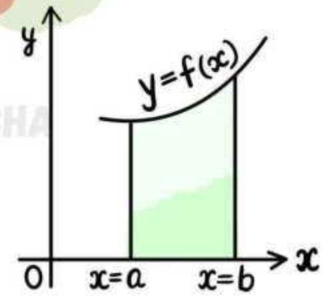
$$\Sigma(n-1)^3 = 1^3+2^3+3^3+\ldots+(n-1)^3 = \left[\frac{n(n-1)}{2}\right]^2$$

$$a + ar + ar^2 + \ldots + ar^{n-1} = \begin{cases} a\left(\dfrac{r^n-1}{r-1}\right) & \text{if } r>1 \\ a\left(\dfrac{1-r^n}{1-r}\right) & \text{if } r<1 \end{cases}$$

Area Under The Curve

1. The area bounded by the curve $y=f(x)$, the x-axis and the ordinates $x=a$ & $x=b$ is given by,

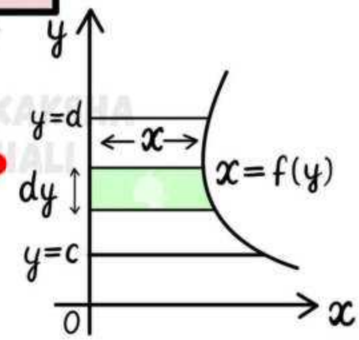
$$\boxed{A=\int_a^b f(x)\,dx=\int_a^b y\,dx,\ f(x)\geq 0.}$$



2. If the area is below the x-axis then A is negative. The convention is to consider the magnitude only i.e. $\boxed{A=\left|\int_a^b y\,dx\right|}$ in this case.

3. The area bounded by the curve $x=f(y)$, y-axis and abscissa $y=c$, $y=d$ is given by,

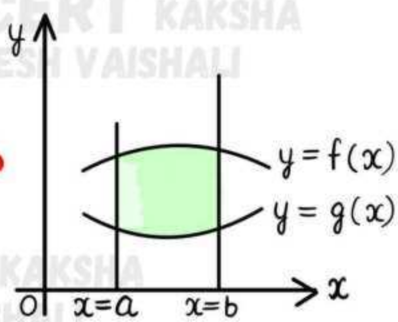
$$\boxed{Area=\int_c^d x\,dy=\int_c^d f(y)\,dy,\ f(y)\geq 0}$$



4. Area between the curves $y=f(x)$ and $y=g(x)$ between the ordinates $x=a$ and $x=b$ is given by,

$$\boxed{A=\int_a^b f(x)\,dx-\int_a^b g(x)\,dx}$$

$$\boxed{A=\int_a^b [f(x)-g(x)]\,dx,\ f(x)\geq g(x)\ \forall\, x\in(a,b).}$$



5. Average value of a function $y=f(x)$ w.r.t. x over an interval $a\leq x\leq b$ is defined as : $\boxed{y(av)=\frac{1}{b-a}\int_a^b f(x)\,dx.}$

➡ CURVE TRACING :

The following outline procedure is to be applied in sketching the graph of a function $y=f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) **Symmetry** : The symmetry of the curve is judged as follows:

(i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

(ii) If all the powers of x are even, the curve is symmetrical about the axis of y.

(iii) If powers of x and y both are even, the curve is symmetrical about the axis of x as well as y.

(iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

(v) If on replacing 'x' by '$-x$' and 'y' by '$-y$', the equation of the curve is unaltered then there is symmetry in opposite quadrants, i.e. symmetric about the origin.

(b) Find dy/dx and equate it to zero to find the points on the curve where you have horizontal tangents.

(c) Find the points where the curve crosses the x-axis and also the y-axis.

(d) Examine if possible the intervals when $f(x)$ is increasing or decreasing Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

➡ USEFUL RESULTS :

(a) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab.

(b) Area enclosed between the parabolas $y^2 = 4ax$ and $x^2 = 4by$ is $16ab/3$.

(c) Area included between the parabola $y^2 = 4ax$ and the line $y = mx$ is $8a^2/3m^3$.

(d) Area enclosed by the parabola and its double ordinate P, Q is two-third of area of rectangle PQRS, where R, S lie on tangent at the vertex.

UNIT-9 DIFFERENTIAL EQUATIONS

Differential Equations

Differentiation :-

An equation involving the independent variable x say, dependent variable y say and the differential coefficients of dependent variable with respect to independent variable i.e. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots$, etc.

Example :- $\frac{dy}{dx} + 8y = 3x$, $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 7y = x^2$ are differential equations.

Order And Degree Of a differential equation :-

The order of a differential equation is the highest order derivative occurring in the differential equation.

The degree of a differential equation is the degree of the highest order derivative occurring in the equation, when the differential coefficients are made free from radicals, fractions and it is written as a polynomial in differential co-efficient.

Example ~ $\boxed{\frac{d^3y}{dx^3}} + \frac{8d^2y}{dx^2} - \frac{3dy}{dx} + y = 0$

Highest order derivative = 3 ⇒ order :- 3

The degree of the highest order derivative occurring in the equation ⇒ degree = 1

$\left(\frac{d^2y}{dx^2}\right)^3 + \sin\frac{dy}{dx} = 0$

order :- 2

degree = not defined (because this differential eq^n cannot be written in the form of polynomial in diff^n co-efficient.)

note :-

order and degree of a differential eq^n are always positive integers

Classification Of Differential Equation :-

A According to D.E. Order
- → 1st order D.E. (in which only 1^{st} order derivative of the dependent variable occurs)
- → Higher order D.E. (in which two or more order derivative of the dependent variable occurs)

B According to D.E. linearity
- → linear D.E.
- → Non-linear D.E.

linear And Non-linear D.E. :-

A D.E. is which the dependent variable and its derivative occurs only in the 1^{st} degree & are not multiplied together, is called a linear D.E. other it is Non-linear D.E.

note:-

Every linear D.E. is always of the 1^{st} degree but D.E. of 1^{st} degree need not to be the linear D.E.

note:-

If the homogenous differential equation is in the form $\frac{dx}{dy} = f(x,y)$ then we substitute $x = vy$ and so $\frac{dx}{dy} = v + y\frac{dv}{dy}$ and proceed as above.

General Solution :-

The solution which contains arbitrary constants is called the general solution (primitive) of the differential equation.

Particular Solution : The solution obtained from the general solution by giving particular values to the arbitrary constants is called a particular solution of the differential equation.

Equations in Variable separable form :-

Consider the Equation $\frac{dy}{dx} = X \cdot Y$

where X is a function of x only and Y is a function of y only.

[i] Put the equation in the form $\frac{1}{Y}\,dy = X\cdot dx$

[ii] Integrating both the sides, we get $\int \frac{dy}{Y} = \int X dx + C$ where c is an arbitrary constant.
Thus the required solⁿ is obtained.

➡ Equations Reducible to variables seperable form :-

1. Write the given equation in form $\frac{dy}{dx} = f(ax+by+c)$

2. Put $ax+by+c = z$, so that $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$

3. Putting this $\frac{dy}{dx}$ in the given equation, we get $\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$. This eqⁿ is reduced in the form : $\frac{dz}{a+bf(z)} = dx$. After integrating, we get the required result.

➡ Homogenous differential Equation :-

A differential equation of the form $\frac{dy}{dx} = F(x,y)$ is said to be homogenous differential equation if $F(x,y)$ is a homogenous function of degree zero.

1. Suppose $y = vx$ and so $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

2. The value $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ is substituted in given eqⁿ. The eqⁿ reduces to variable separable form, which can be solved by integrating both sides.

4. Finally v is replaced by $\frac{y}{x}$ to get the required solution.

➡ First order linear differential equation :-

$\frac{dy}{dx} + Py = Q$ —(i)

where P and Q are constants or function of x only

$I.F. = e^{\int P dx}$ (I.F.= Integrating factor)

solution of (i) is,

$\boxed{y.(I.F.) = \int Q \times (I.F.)\, dx + c}$

$\frac{dx}{dy} + Px = Q$ —(i)

where P and Q are constants or function of y only

$I.F. = e^{\int P dy}$ (I.F.= Integrating factor)

solution of (i) is,

$\boxed{x.(I.F.) = \int Q \times (I.F.)\, dy + c}$

TRAJECTORIES :-

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

Orthogonal trajectories :

A curve making at each of its points a right angle with the curve of the family passing through that point is called an orthogonal trajectory of that family.

We set up the differential equation of the given family of curves. Let it be of the form $F(x, y, y') = 0$

The differential equation of the orthogonal trajectories is of the form $F\left[x, y, \frac{-1}{y'}\right] = 0$

The general integral of this equation $\phi(x, y, C) = 0$ gives the family of orthogonal trajectories.

NOTE

Following exact differentials must be remembered :

(i) $dx + dy = d(x+y)$

(ii) $dx - dy = d(x-y)$

(iii) $xdy + ydx = d(xy)$

(iv) $\frac{xdy - ydx}{x^2} = d\left[\frac{y}{x}\right]$

(v) $\frac{ydx - xdy}{y^2} = d\left[\frac{x}{y}\right]$

(vi) $2(xdx + ydy) = d(x^2 + y^2)$

It should be observed that :

(i) $\frac{xdy + ydx}{xy} = d\left(\log_e(xy)\right)$

(ii) $\frac{dx + dy}{x+y} = d\left(\log_e(x+y)\right)$

(iii) $\frac{xdy - ydx}{xy} = d\left[\log_e\left(\frac{y}{x}\right)\right]$

(iv) $\frac{ydx - xdy}{xy} = d\left[\log_e\left(\frac{x}{y}\right)\right]$

(v) $\dfrac{x\,dy - y\,dx}{x^2+y^2} = d\left[\tan^{-1}\dfrac{y}{x}\right]$

(vi) $\dfrac{y\,dx - x\,dy}{x^2+y^2} = d\left[\tan^{-1}\dfrac{x}{y}\right]$

(vii) $\dfrac{x\,dx + y\,dy}{x^2+y^2} = d\left[\log_e\sqrt{x^2+y^2}\right]$

(viii) $d\left[-\dfrac{1}{xy}\right] = \dfrac{x\,dy + y\,dx}{x^2y^2}$

(ix) $d\left[\dfrac{e^x}{y}\right] = \dfrac{ye^x\,dx - e^x\,dy}{y^2}$

(x) $d\left[\dfrac{e^y}{x}\right] = \dfrac{xe^y\,dy - e^y\,dx}{x^2}$

Straight Lines

➡ Distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\boxed{PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$

Coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m:n$ are

$$\left[\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right]$$

In particular, If $m = n$, the coordinates of the mid point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are $\left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right]$

➡ Area of triangle $\boxed{\frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]}$

vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3)

If the area of the triangle ABC is zero, then three points A, B and C lie on a line, i.e., they are collinear.

➡ Slope of a line $\boxed{m = \tan\theta}$ $(\theta \neq 90^\circ)$

The slope of x-axis is zero and slope of y-axis is not defined.

➡ Slope of the line through the points (x_1, y_1) and (x_2, y_2) $\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$

➡ If the line l_1 is parallel to l_2 $\boxed{m_1 = m_2}$

$\boxed{\tan\alpha = \tan\beta}$

➡ If the line l_1 and l_2 are perpendicular $\boxed{m_2 = -\frac{1}{m_1}}$ OR $\boxed{m_1 m_2 = -1}$

$$\boxed{\begin{aligned}\tan\beta &= \tan(\alpha + 90^\circ)\\ &= -\cot\alpha = -\frac{1}{\tan\alpha}\end{aligned}}$$

- **Acute angle θ between two lines with slopes m_1 and m_2**

$$\boxed{\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, \quad 1 + m_1 m_2 \neq 0}$$

- **Collinearity of three points** Three points are collinear if and only if

$$\boxed{\text{slope of } AB = \text{slope of } BC}$$

- **Point - slope form** $\boxed{y - y_1 = m(x - x_1)}$

- **Two - point form** $\boxed{y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$ OR $\boxed{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0}$

- **Slope - intercept form**

Case I $\boxed{y = mx + c}$ slope m and y - intercept c

Case II $\boxed{y = m(x - d)}$ slope m and x - intercept d

- **Intercept form** $\boxed{\dfrac{x}{a} + \dfrac{y}{b} = 1}$ x - intercept a and y - intercept b

- **Normal form** $\boxed{x\cos\omega + y\sin\omega = p}$ Normal distance from the origin.

- **Distance of a point from a line** $\boxed{d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}}$

$Ax + By + C = 0$ from a point (x_1, y_1)

- **Distance between two parallel lines** $\boxed{d = \dfrac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}}$

two parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$

- **CONCURRENCY OF LINES :**

Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$

are concurrent, if $\boxed{\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0}$

NOTE

If lines are concurrent then $\Delta = 0$ but if $\Delta = 0$ then lines may or may not be concurrent {lines may be parallel}.

REFLECTION OF A POINT :

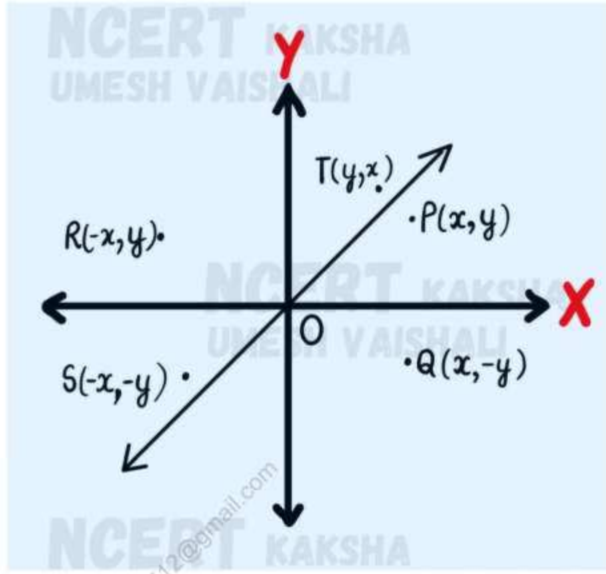
Let $P(x,y)$ be any point, then its image with respect to

(a) x-axis is $Q(x,-y)$

(b) y-axis is $R(-x,y)$

(c) origin is $S(-x,-y)$

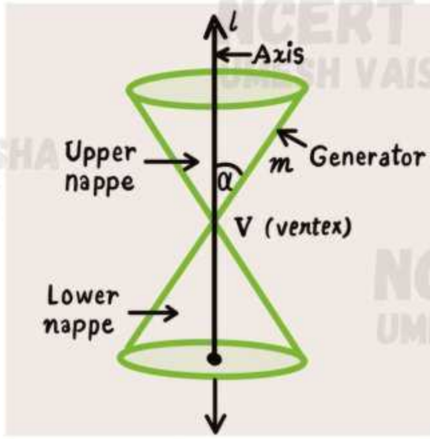
(d) line $y=x$ is $T(y,x)$

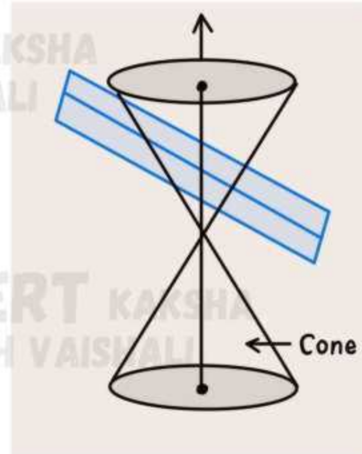


Conic Sections

➡ Sections of a cone :

The intersection of a plane with a cone, the section so obtained is called a conic section.





➡ Circle, ellipse, parabola and hyperbola :

When the plane cuts nappe of the cone, we have the following situations :

(a) When $\beta = 90^\circ$, the section is a circle.

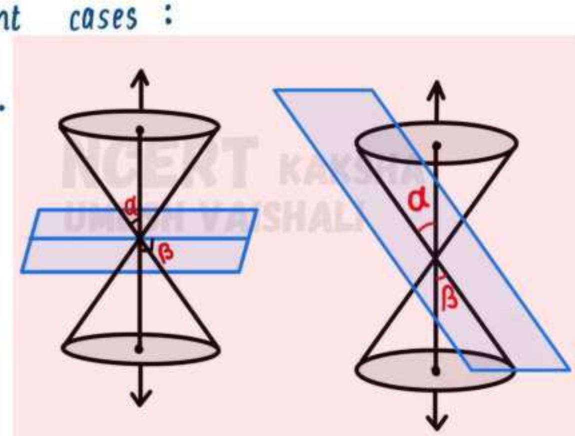
(b) When $\alpha < \beta < 90^\circ$, the section is an ellipse.

(c) When $\beta = \alpha$; the section is a parabola.

(d) When $0 \leq \beta < \alpha$; the plane cuts through both the nappes and the curves of intersection is a hyperbola.

➡ Degenerated conic sections :

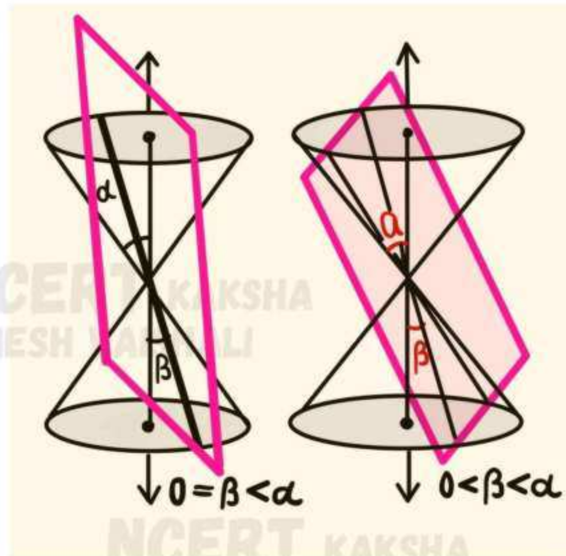
When the plane cuts at the vertex of the cone, we have the following different cases :



(a) When $\alpha < \beta \leq 90^\circ$, then the section is a point.

(b) When $\beta = \alpha$, the plane contains a generator of the cone and the section is a straight line. It is the degenerated case of a parabola.

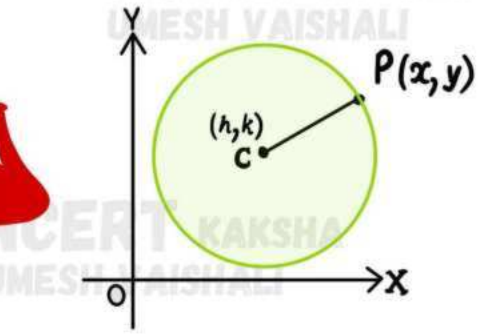
(c) when $0 \le \beta < \alpha$, the section is a pair of intersecting straight lines. It is degenerated case of a hyperbola.



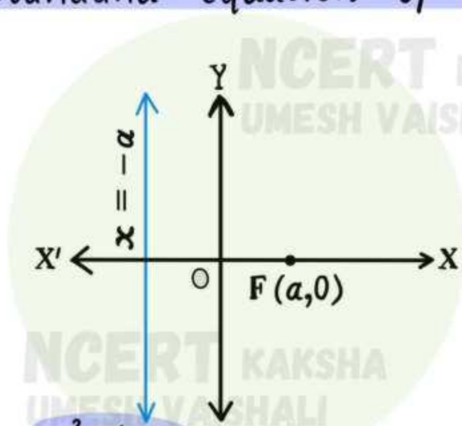
➡ **Circle equation**

centre at (h, k)
circle radius $= r$

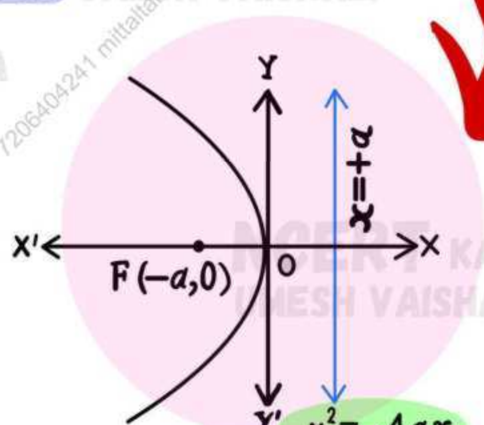
$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$



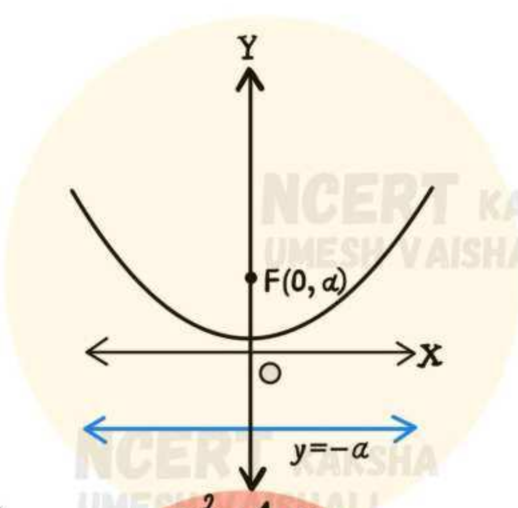
➡ **Standard equation of parabola**



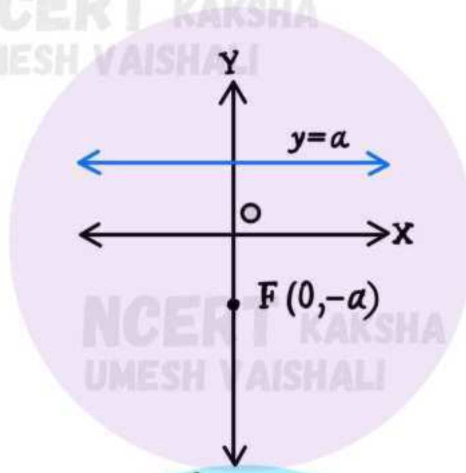
(a) $y^2 = 4ax$



(b) $y^2 = -4ax$



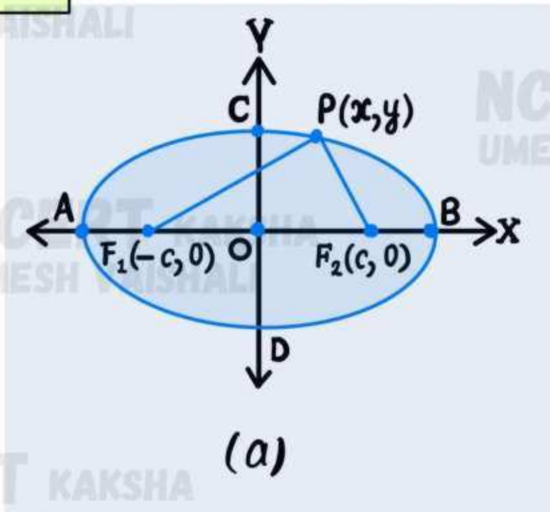
(c) $x^2 = 4ay$



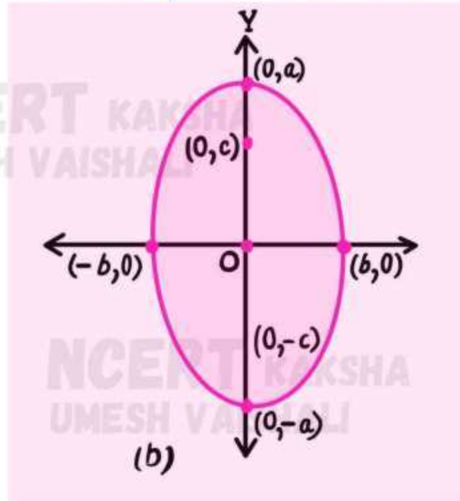
(d) $x^2 = -4ay$

Standard equations of an ellipse

$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ centre of the ellipse is at the origin and foci are



(a)



(b)

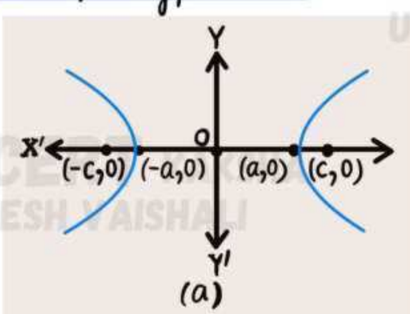
- Latus rectum of parabola $\boxed{4a}$
- Latus rectum of ellipse $\boxed{\frac{2b^2}{a}}$
- The eccentricity of an ellipse $\boxed{e=\frac{c}{a}}$ distance from the centre
- Relationship between semi-major axis, semi minor axis and the distance of the focus from the centre of the ellipse.

$\boxed{a^2=b^2+c^2}$ OR $\boxed{c=\sqrt{a^2-b^2}}$

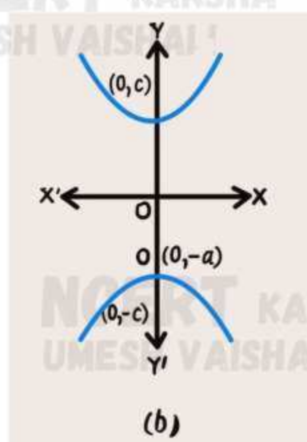
NOTE
A hyperbola in which $a=b$ is called an equilateral hyperbola.

- Standard equation of hyperbola

$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$



(a)



(b)

- Latus rectum of hyperbola $\boxed{\frac{2b^2}{a}}$
- The eccentricity of an hyperbola $\boxed{e=\frac{c}{a}}$ distance from the centre

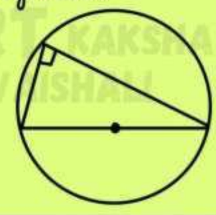
Circles

➡ Circle : Important Geometrical Properties

The angle at the centre is twice the angle at the circumference.

Angles in the same segment are equal.

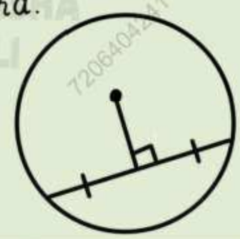
The angle in a semicircle is 90 degrees.



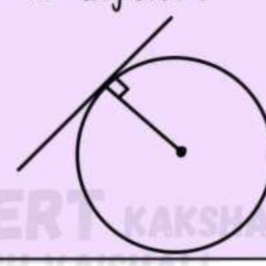
The angle in a semicircle is 90 degrees

$a+b=180^\circ$
$c+d=180^\circ$

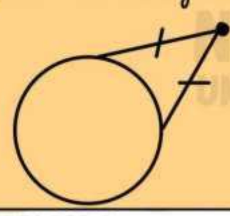
The perpendicular from the center to the chord bisects the chord.



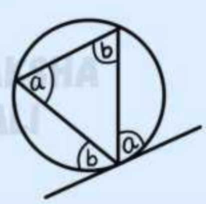
The angle between a tangent and a radius is 90 degrees.



Tangents from a point outside a circle are equal in length.



Alternate segment theorem.



➡ Definition and Standard Equations of a circle :-

1. <u>Central form of the Equation of a Circle</u>

From the definition of a circle, we find the equation of its locus.

$$\boxed{\sqrt{(x-x_1)^2+(y-y_1)^2} = r \Rightarrow (x-x_1)^2+(y-y_1)^2 = r^2}$$

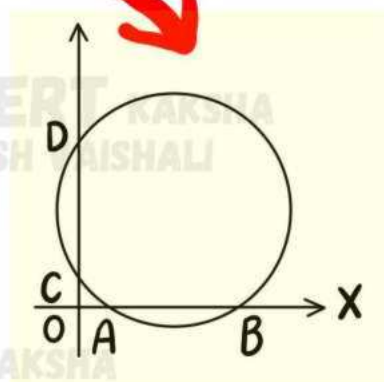
This is called central form of circle with centre (x_1, y_1) and radius 'r'.

General Form of the Equation of a circle :-

$$\boxed{x^2 + y^2 + 2gx + 2fy + c = 0}$$

where $(x_1, y_1) = (-g, -f)$ and $r = \sqrt{g^2 + f^2 - c}$

Intercepts made by a circle on the Axes :-

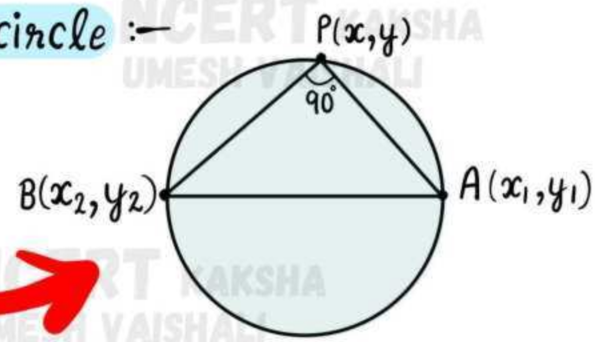


1. The length of the intercept made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

   (a) x-axis $= AB = 2\sqrt{g^2 - c}$ (b) y-axis $= CD = 2\sqrt{f^2 - c}$

2. Intercepts are always positive.
3. If the circle touches x-axis then $AB = 0$ $\therefore c = g^2$
4. If the circle touches y-axis then $CD = 0$ $\therefore c = f^2$
5. If the circle touches both the axes, then $AB = 0 = CD$ $\therefore c = g^2 = f^2$

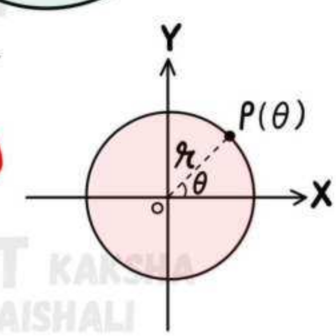
Diametric form of the Equation of a circle :-



The circle whose diametric endpoints are $A(x_1, y_1)$ and $B(x_2, y_2)$ has the Equation

$$\boxed{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.}$$

Parametric form of the Equation Of a Circle :-

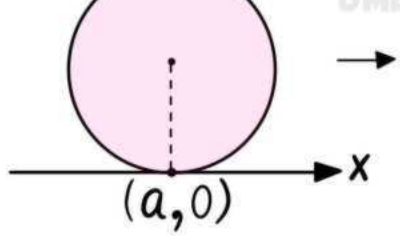


(a) $x^2 + y^2 = r^2 \Rightarrow x = r\cos\theta,\ y = r\sin\theta$

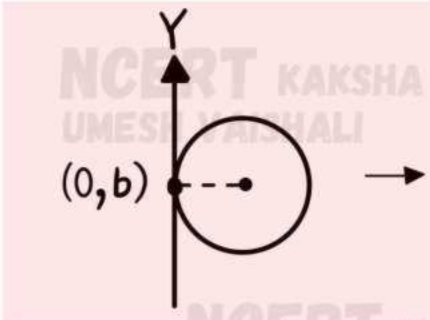
(b) $(x - x_1)^2 + (y - y_1)^2 = r^2 \Rightarrow x = x_1 + r\cos\theta,\ y = y_1 + r\sin\theta$

Some Special Circles :-

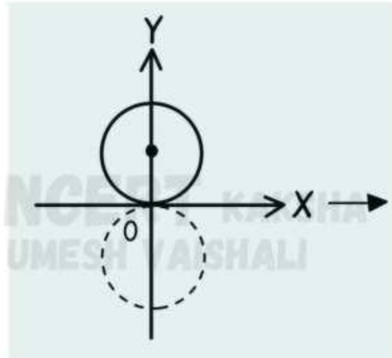
1. Circle touching x-axis



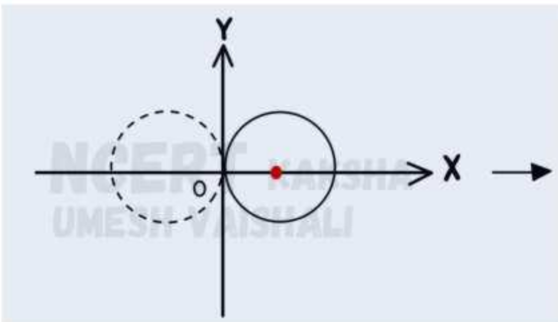
2. Circle touching Y-axis



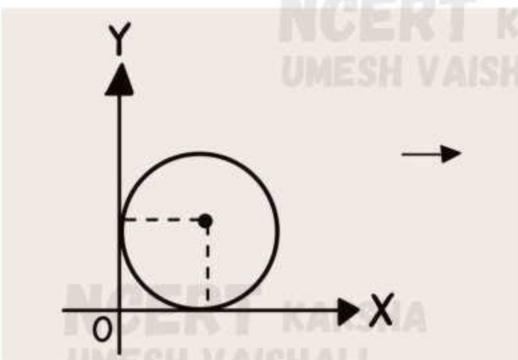
3. Circle touching X-axis at origin

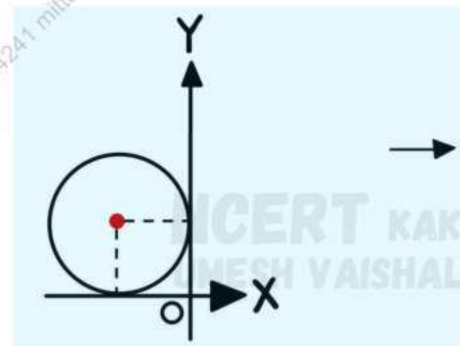


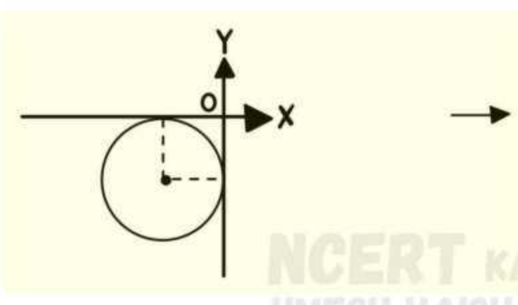
4. Circle touching Y-axis at origin

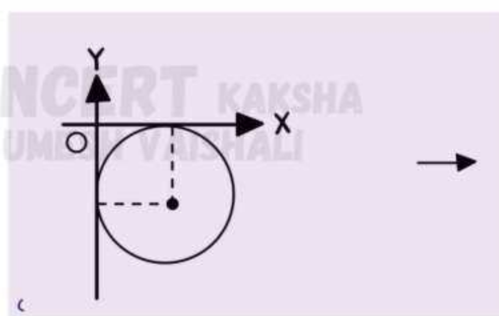


5. Circle touching both axes









Some Standard Notations :-

Any second degree equation in two variables, that is,

$\boxed{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$ will be represented as $S = 0$

1. $S \equiv x^2 + y^2 + 2gx + 2fy + c$

2. Consider a point (x_1, y_1). Value of S at (x_1, y_1) is represented by S_1

that is, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

3. Consider a point (x_1, y_1).

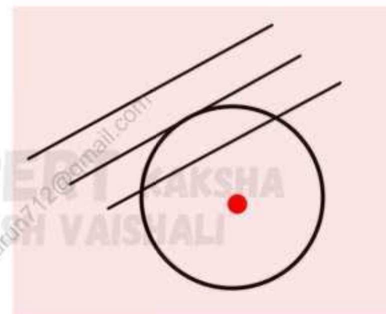
If we replace $x^2 \to xx_1$, $y^2 \to yy_1$, $x \to \frac{x+x_1}{2}$, $y \to \frac{y+y_1}{2}$, $xy \to \frac{xy_1 + x_1y}{2}$,

in S, then we get T, that is

$$T = xx_1 + yy_1 + 2g\left(\frac{x+x_1}{2}\right) + 2f\left(\frac{y+y_1}{2}\right) + c$$

Position Of a Line with respect to a circle and Equations with Tangents

For a given line and a circle, either



(a) line cuts the circle, or

(b) line touches the circle, or

(c) line does not meet the circle

Condition Of Tangency :-

The straight line $y = mx + c$ will be a tangent to the circle $x^2 + y^2 = a^2$ if

$c = \pm a\sqrt{1+m^2}$

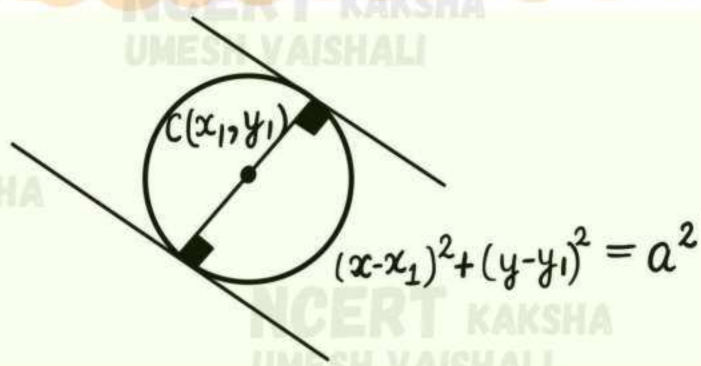
NOTE A line will touch a circle if and only if the length of the $\perp$ from the centre of the circle to the line is equal to the radius of the circle.

Similarly, For this circle

$(x-x_1)^2 + (y-y_1)^2 = r^2$

We have,

$(y-y_1) = m(x-x_1) \pm a\sqrt{1+m^2}$



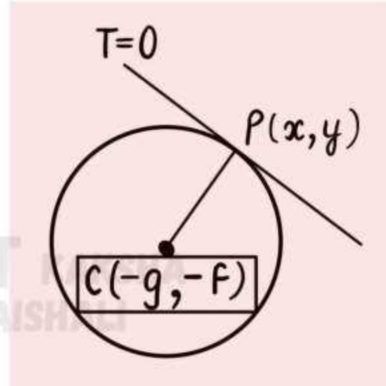
Tangent : (Point Form)

1. To Circle : $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) :

$\boxed{T=0}$: $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$

2. To Circle : $(x^2 + y^2 = a^2)$ at (x_1, y_1) :

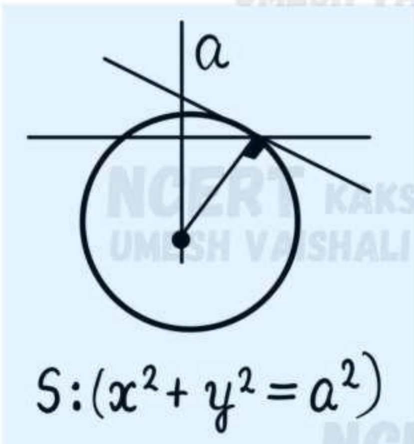
$\boxed{T=0}$: $xx_1 + yy_1 = a^2$

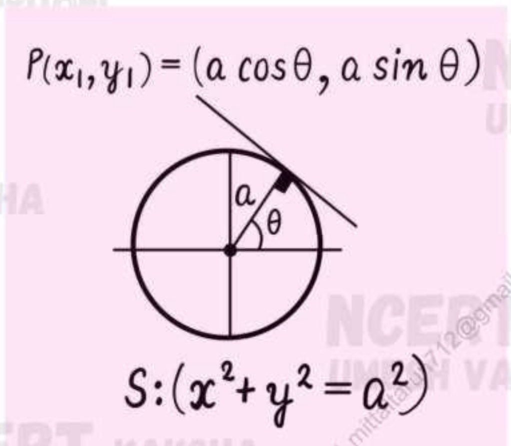


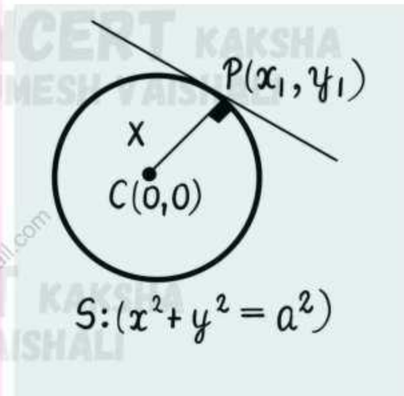
3. To Circle : $(x^2 + y^2 = a^2)$ at $(a\cos\theta, a\sin\theta)$:

$x(\cos\theta) + y(\sin\theta) = a$

$P(x_1, y_1) \equiv (a\cos\theta, a\sin\theta)$







Various Equations Of Tangents Of a Circle :-

Slope Form

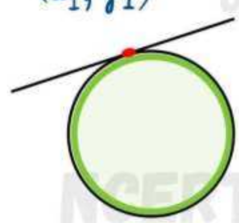
slope = m

$x^2 + y^2 = r^2$

$y = mx \pm r\sqrt{1+m^2}$

Tangent at a Point on a Circle

(x_1, y_1)

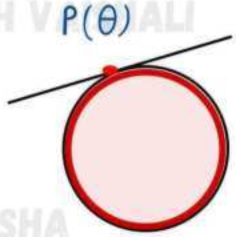


T=0

$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + C = 0$

Parametric form

$P(\theta)$

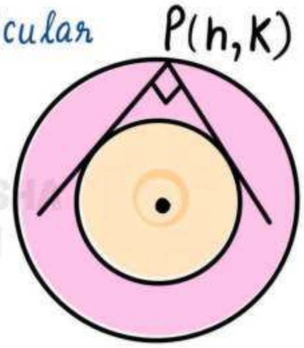


T=0

$x(\cos\theta) + y(\sin\theta) = a$

Director Circle :-

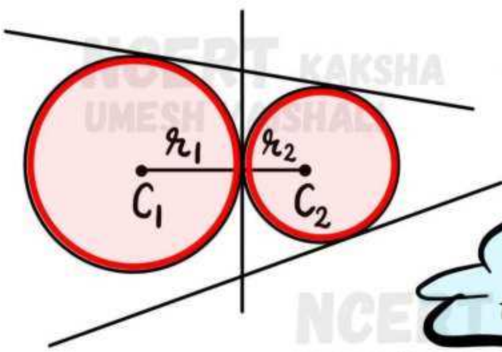
1. The locus of the point of intersection of two perpendicular tangents to a circle is called the Director circle.



2. Director Circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the given circle.

COMMON TANGENTS

1. Circles touching each other externally :-

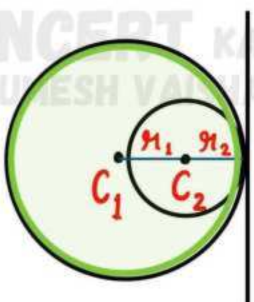


Direct common tangents has both circles on same sides

Transverse common tangents has two circles on different sides.

$t(C_1C_2) = r_1 + r_2$ Three common tangents

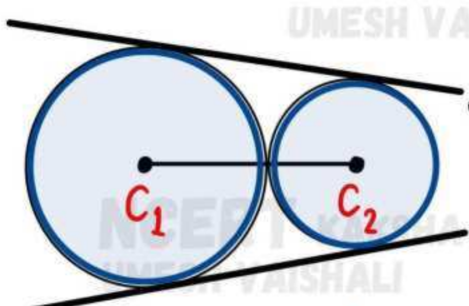
2. Circles touching each other internally :-



Direct common tangents has both circles on same side.

$t(C_1C_2) = |r_1 - r_2|$ only one common tangents

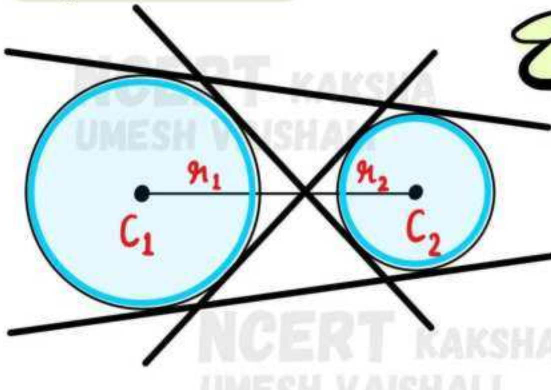
3. Circles intersecting each other :-



Direct common tangents has both circles on same side.

$|r_1 - r_2| < I(C_1C_2) < r_1 + r_2$ Two common tangent

4. **Disjoint circles** :-



Direct common tangents has both circles on same side.

Transverse common tangents has two circles on different sides

$I(C_1C_2) > r_1 + r_2$ Four common tangent

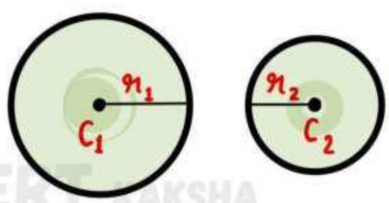
5. **One circle inside the other** :-

$I(C_1C_2) < r_1 - r_2$

No common tangent

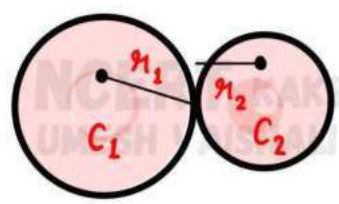
Try to observe how we can comment upon the positions of two circles depending on their radii and the distance between their centres.

(1)



$|C_1C_2| > r_1 + r_2$

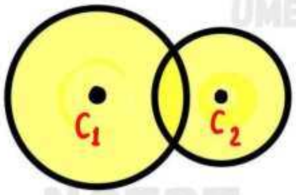
(2)



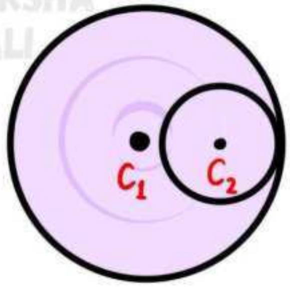
$|C_1C_2| = r_1 + r_2$

Try to observe how we can comment upon the positions of two circles depending on their radii and the distance between their centres.

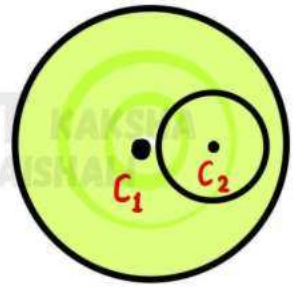
(3)



$|r_1 - r_2| < |C_1C_2| < r_1 + r_2$



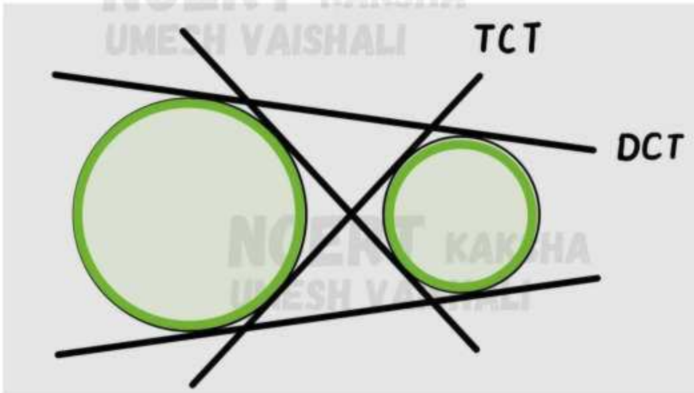
$|C_1C_2| = r_1 - r_2$



$|C_1C_2| < r_1 - r_2$

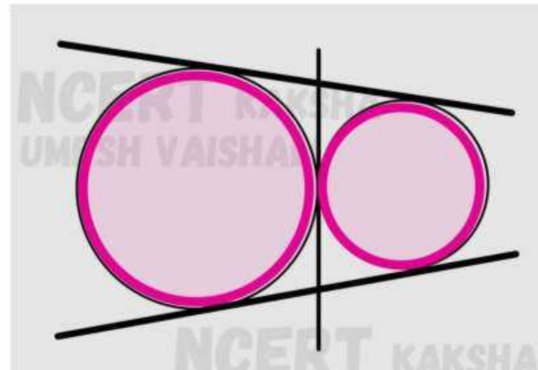
NUMBER OF COMMON TANGENTS :

(1) $|C_1C_2| > r_1 + r_2$



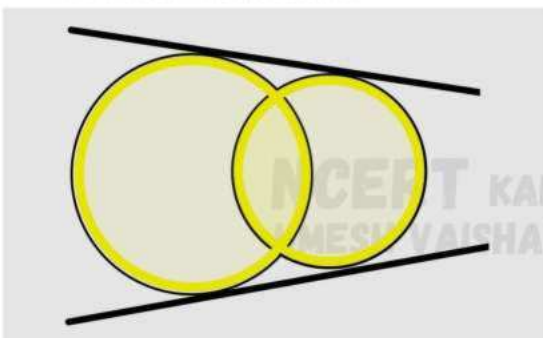
4 common tangents

(2) $|C_1C_2| = r_1 + r_2$



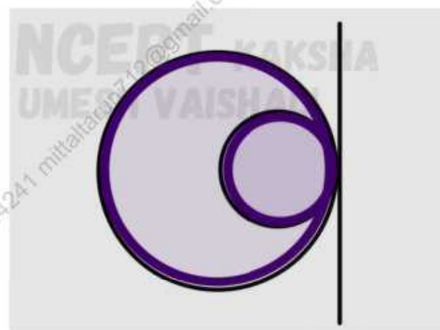
3 common tangents

(3) $|r_1 - r_2| < |C_1C_2| < r_1 + r_2$



2 common tangents

(4) $|C_1C_2| = r_1 - r_2$



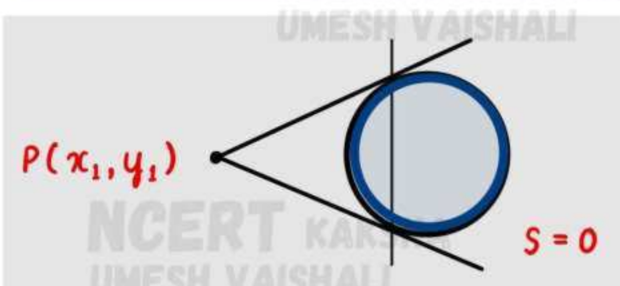
1 common tangents

(5) $|C_1C_2| < r_1 - r_2$

0 common tangents

EQUATION OF CHORDS

(1) Equation of CoC (chord of contact) with respect to $P(x_1, y_1)$

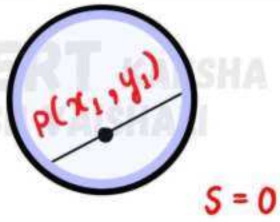


Length of Chord of contact

$$= \frac{2LR}{\sqrt{L^2 + R^2}}$$

Its equation is given by $T = 0$

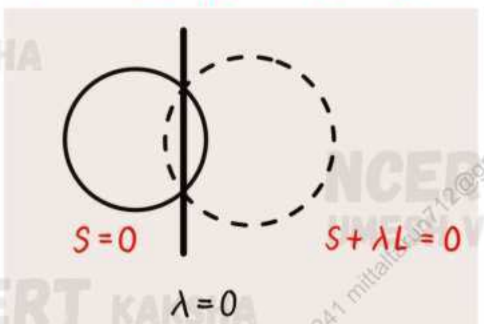
(2) Equation of chord with given midpoint $P(x_1, y_1)$



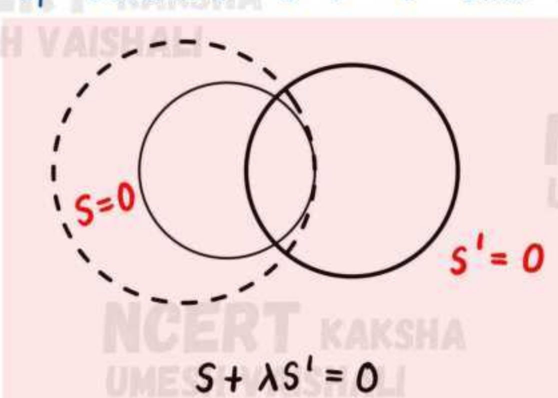
Its equation given by $T = S_1$

➡ FAMILY OF CIRCLES :

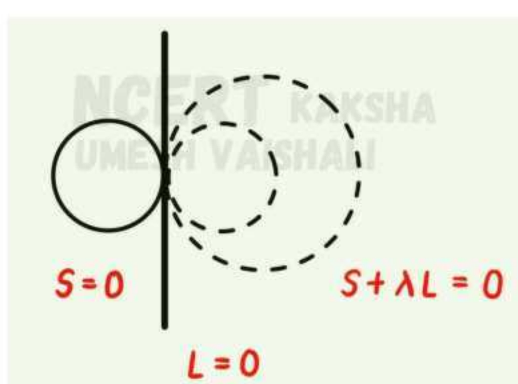
(1) The equation of the family of circles passing through the point of intersection of circle $S = 0$ and a line $L = 0$ is given as $S + \lambda L = 0$, (where λ is a parameter)



(2) The equation of the family of circles passing through the point of intersection of circle $S = 0$ and $S' = 0$ is given as $S + \lambda S' = 0$ (where λ is a parameter, $\lambda \neq -1$). But it is better to find first equation of common $S - S' = 0$ and then use $S + \lambda(S - S') = 0$



3. The equation of the family of circles touching the circles $S = 0$ and the Line $L = 0$ at their point of contact P is $S + \lambda L = 0$, (where λ is a parameter)



4. The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is

$$\boxed{(x-x_1)^2 + (y-y_1)^2 + K[y - y_1 - m(x-x_1)] = 0}$$

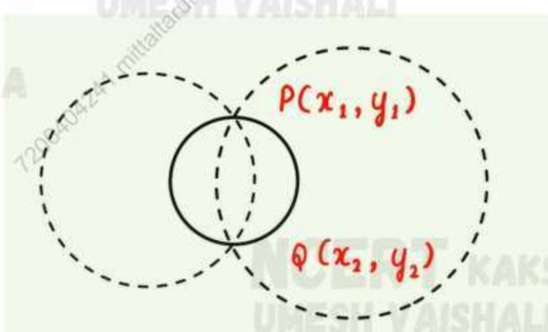
where K is a parameter.

5. The equation of a family of circles passing through two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be written in the form

$$\boxed{(x-x_1)(x_1-x_2) + (y-y_1)(y_1-y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0}$$

$$\boxed{S + \lambda L = 0}$$

(where λ is a parameter)



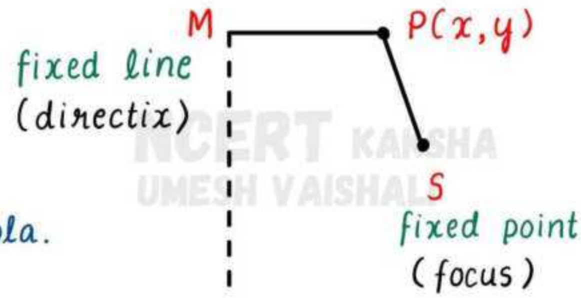
UNIT-10 (COORDINATE GEOMETRY)

Parabola

➡ DEFINITION :

The locus of a point in a plane having distance from the fixed point and the fixed straight line (Directrix) should always be the same, is a parabola.

$$\boxed{\frac{PS}{PM} = e \text{ (eccentricity)}}$$



If $e = 1$, the conic is called as parabola.

Standard form	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Coordinates of vertex	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
Coordinates of focus	$(a,0)$	$(-a,0)$	$(0,a)$	$(0,-a)$
Equation of the directix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of the latusrectum	$4a$	$4a$	$4a$	$4a$
Focal distance of a point $P(x,y)$	$x + a$	$a - x$	$y + a$	$a - y$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Parametric equations	$x = at^2$ $y = 2at$	$x = -at^2$ $y = 2at$	$x = 2at$ $y = at^2$	$x = 2at$ $y = -at^2$

➡ EQUATION OF TANGENT IN DIFFERENT FORMS :

(1) Point form -

The equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is $\boxed{yy_1 = 2a(x + x_1).}$

(2) Parametric form -

The equation of tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is $\boxed{ty = x + at^2}$

(3) Slope form -

The equation of the tangent to parabola $y^2 = 4ax$ in terms of slope 'm' is $\boxed{y = mx + \frac{a}{m}}$

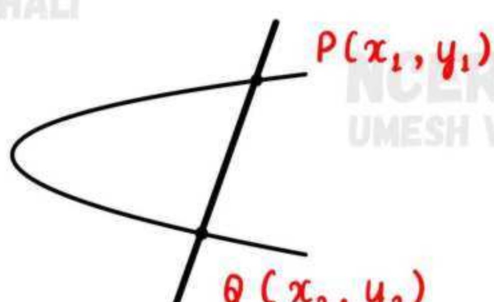
The coordinates of the point contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Parabola	Line	Point of contact	Condition of tangency
$y^2 = 4ax$	$y = mx + c$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$c = \frac{a}{m}$
$y^2 = -4ax$	$y = mx + c$	$\left(\frac{-a}{m^2}, \frac{-2a}{m}\right)$	$c = -\frac{a}{m}$
$x^2 = 4ay$	$x = my + c$	$\left(\frac{2a}{m}, \frac{a}{m^2}\right)$	$c = \frac{a}{m}$
$x^2 = -4ay$	$x = my + c$	$\left(\frac{-2a}{m}, \frac{-a}{m^2}\right)$	$c = -\frac{a}{m}$

EQUATION OF A CHORD

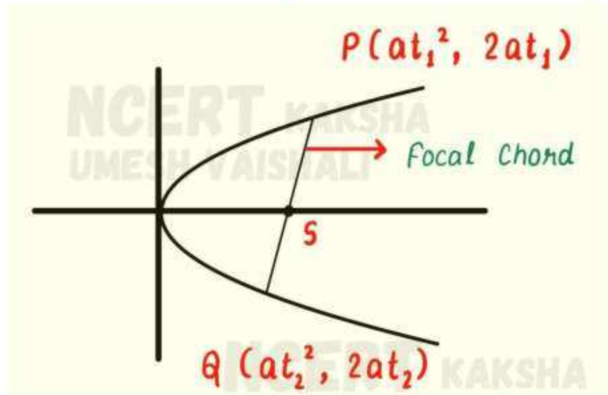
1. The equation of chord joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the parabola $y^2 = 4ax$ is

2. The equation of chord joining $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is $y(t_1 + t_2) = 2(x + t_1 t_2)$



➡ CONDITION FOR THE CHORD TO BE A FOCAL CHORD

The chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ passes through focus if $t_1 t_2 = -1$



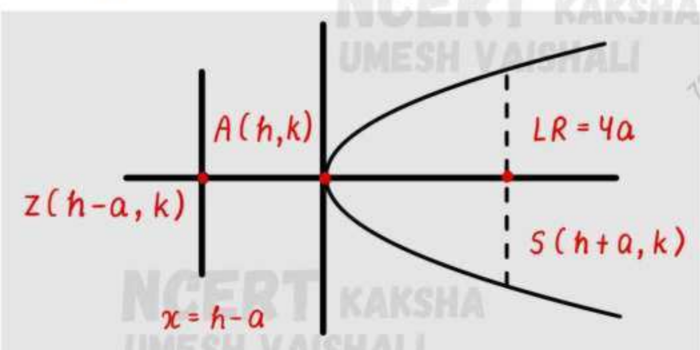
➡ LENGTH OF FOCAL CHORD

The length of focal length chord joining P $(at_1^2, 2at_1)$ and Q $(at_2^2, 2at_2)$ is PQ $a(t_2 - t_1)^2$.

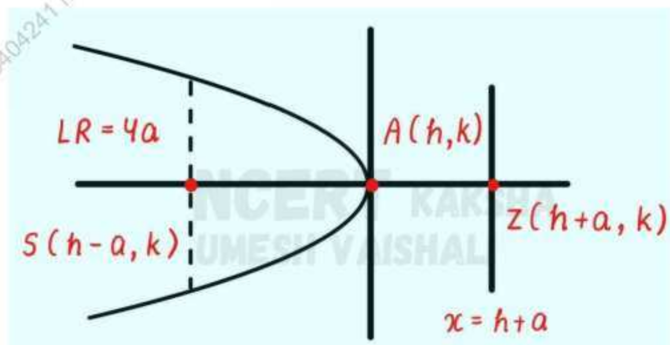
➡ STANDARD PARABOLAS HAVING VERTEX AT ANY POINT :

Consider the following equations for $a > 0$ and remember their graphs.

1. $(y - k)^2 = 4a(x - h)$

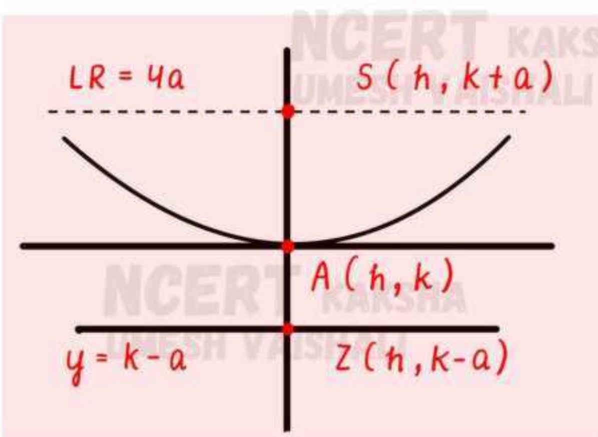


2. $(y - k)^2 = -4a(x - h)$

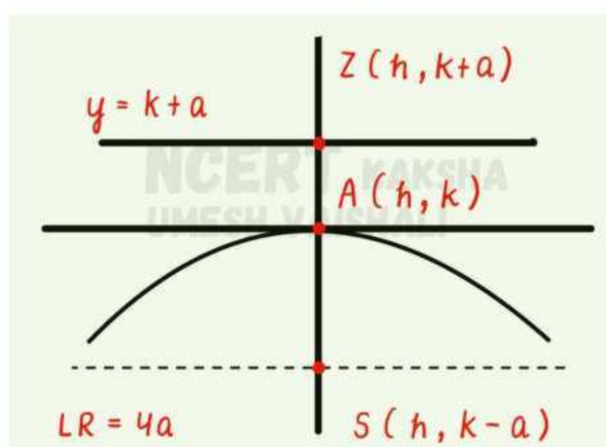


Consider the following equations for $a > 0$ and remember their graphs.

1. $(x - h)^2 = 4a(y - k)$



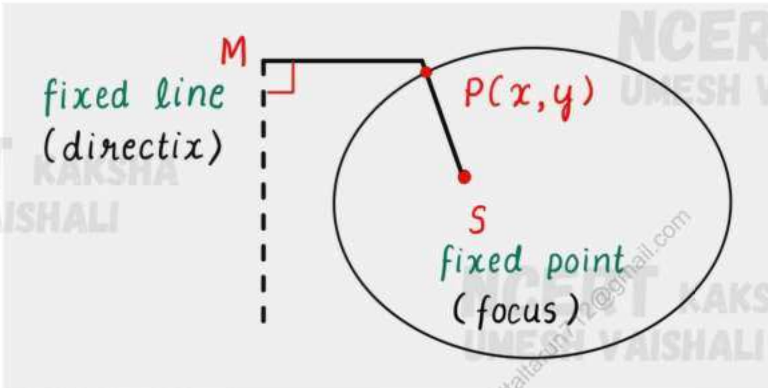
2. $(x - h)^2 = -4a(y - k)$



Ellipse

DEFINITION :

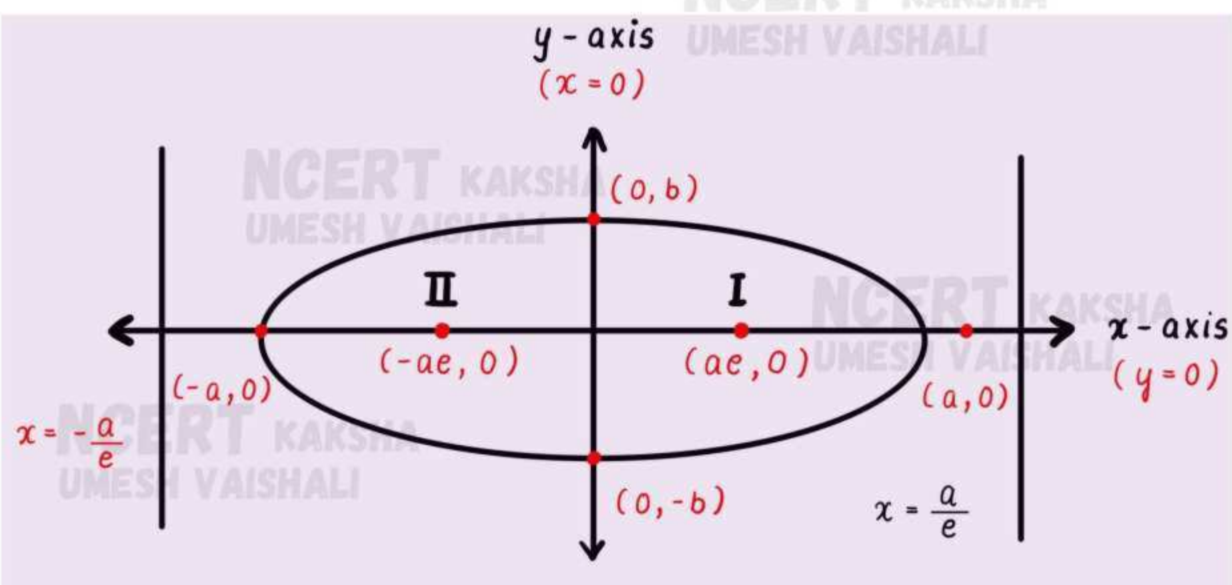
It is locus of a point which moves such that ratio of its distance from a fixed point to its distance from a fixed line is always a constant less than 1.



$\boxed{\frac{PS}{PM} = \text{constant} = e}$ where e is eccentricity $0 < e < 1$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1}$$

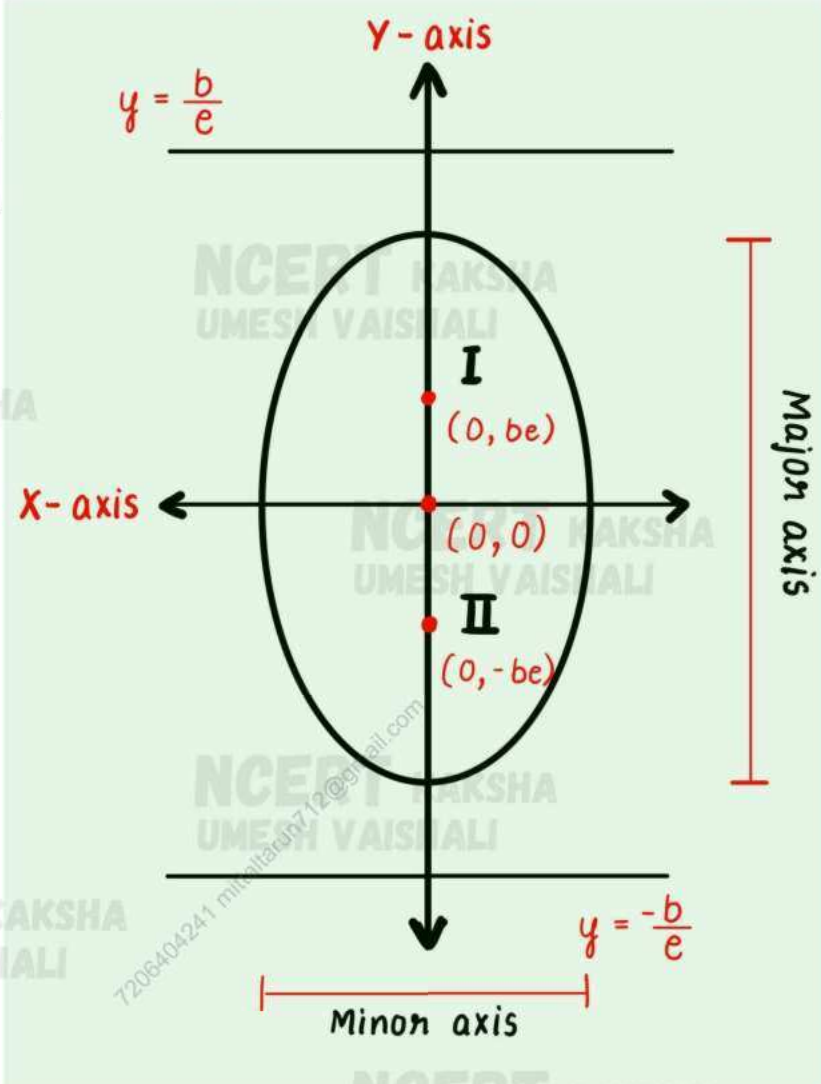
This is standard equation of ellipse with focus $(ae, 0)$ and directix $x = \frac{a}{e}$.



Standard Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ when } b > a$$

$$e^2 = 1 - \frac{a^2}{b^2}$$



	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a < b$
coordinates of the centre	$(0,0)$	$(0,0)$
coordinates of the vertices	$(a,0)$ and $(-a,0)$	$(0,b)$ and $(0,-b)$
coordinates of foci	$(\pm ae, 0)$	$(0, \pm be)$
Length of the minor axis	$2b$	$2a$
Length of the major axis	$2a$	$2b$
Equations of the minor axis	$x = 0$	$y = 0$
Equation of the major axis	$y = 0$	$x = 0$
Equations of the directrices	$x = \frac{a}{e}$ and $x = -\frac{a}{e}$	$x = \frac{b}{e}$ and $x = -\frac{b}{e}$

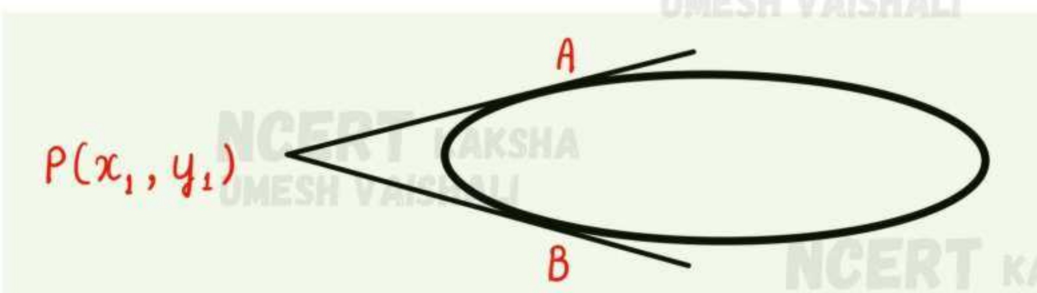
	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a<b$
Electricity	$e=\sqrt{1-\frac{b^2}{a^2}}$	$e=\sqrt{1-\frac{a^2}{b^2}}$
length of the latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
focal distance of a point (x,y)	$a \pm ex$	$b \pm ex$

➡ EQUATION OF TANGENT IN DIFFERENT FORMS :

1. **Point form** : The equation of the tangent to the ellipse $\boxed{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}$

2. **Slopeform** : If the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then $c^2 = a^2m^2 + b^2$. Hence, the straight line $\boxed{y = mx \pm \sqrt{a^2m^2+b^2}}$ always represents the tangent to the ellipse.

3. **Parametric form** : The equation of the tangent to any point $(a\cos\theta, b\sin\theta)$ is $\boxed{\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1}$



➡ EQUATION OF TANGENT IN DIFFERENT FORMS :

1. **Point form** : The equation of the normal at (x_1, y_1) to the

ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 - b^2$

2. Parametric form : The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$.

3. Slopeform : If m is the slope of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the equation of normal is

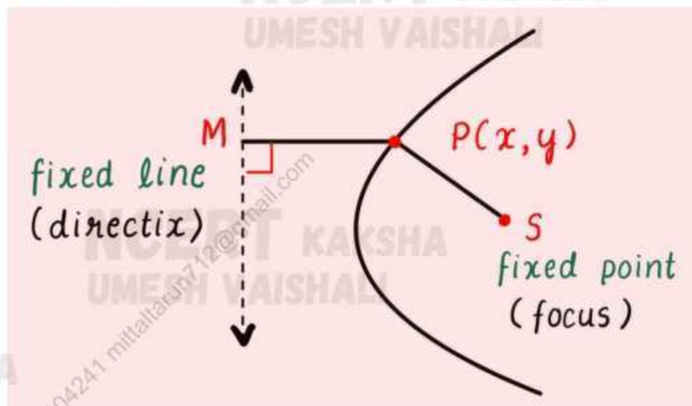
$$\boxed{y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}}.$$

Hyperbola

DEFINITION :

It is locus of a point which moves such that ratio of its distance from a fixed point to its distance from a fixed line is always a constant greater than 1.

$$\boxed{\frac{PS}{PM} = \text{constant} = e}$$



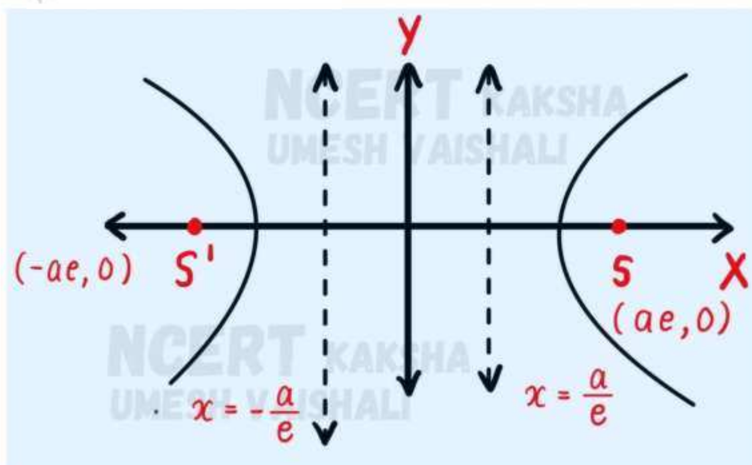
where e is eccentricity : $e > 1$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1}$$

Here, $e > 1$ So, $a^2(1-e^2)$ is negative.

We put $a^2(1-e^2) = -b^2$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$



This is standard equation of hyperbola.

Standard hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

Choice of transverse axis and conjugate axis depends on whose sign is negative in equation of hyperbola.

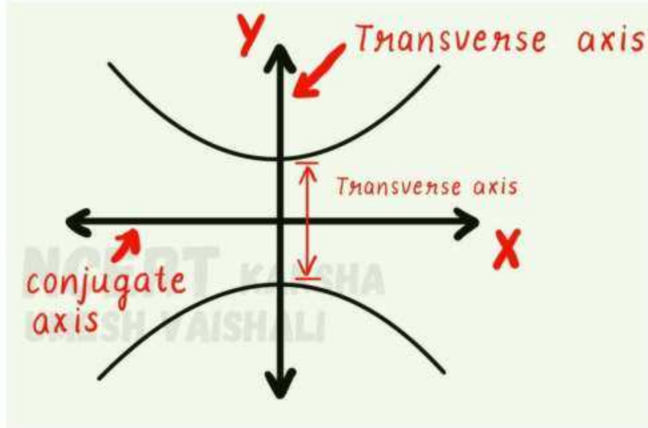
We have studied hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

where, transverse axis is along x-axis.

- For hyperbola

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

Transverse axis is along y-axis.



	Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Conjugate $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
• Coordinates of the centre	$(0,0)$	$(0,0)$
• Coordinates of the vertices	$(a,0)$ and $(-a,0)$	$(0,b)$ and $(0,-b)$
• Coordinates of foci	$(\pm ae, 0)$	$(0, \pm be)$
• Length of the transverse axis	$2a$	$2b$
• Length of the conjugate axis	$2b$	$2a$
• Equations of the directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
• Eccentricity	$e = \sqrt{\frac{a^2+b^2}{a^2}}$	$e = \sqrt{\frac{b^2+a^2}{b^2}}$
• Length of the latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
• Equation of the transverse axis	$y=0$	$x=0$
• Equation of the conjugate axis	$x=0$	$y=0$

Equation of Tangent in different forms :-

(i) Slope Form : $\boxed{y = mx \pm \sqrt{a^2m^2 - b^2}}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(ii) **Point Form** : Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(x_1 y_1)$ is $\boxed{\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1}$

(iii) **Parametric Form** : Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec\theta, b \tan\theta)$ $\boxed{\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1}$

➡ Equation of Tangent in different forms :-

(i) Point form :- The equation of normal to the phyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_2) is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$

The equation of normal at the point (x_1, y_1) to the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}$

(ii) **Parametric Form** : The equation of normal at $(a \sec\theta, b \tan\theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(iii) **Slope Form** : The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of the slop m of normal is $y = mx \pm \frac{m(a^2+b^2)}{\sqrt{a^2 - b^2 m^2}}$

(iv) **Condition for normality** : If $y = 3x + c$ is the normal of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c = \pm \frac{m(a^2+b^2)}{\sqrt{a^2 - m^2 b^2}}$ or $c^2 = \frac{m^2(a^2+b^2)}{(a^2 - m^2 b^2)}$, which is condition of normality.

Three Dimensional Geometry

- In three dimensions, the coordinate axes of a rectangular Cartesian coordinate system are three mutually perpendicular lines. The axes are called the x, y and z axes.
- The three planes determined by the pair of axes are the coordinate planes, called XY, YZ and ZX- planes
- The three coordinate planes divide the space into eight parts known as octants.
- The coordinates of a point P in three dimensional geometry is always written in the form of triplet like (x, y, z). Here x, y and z are the distances from the XY, YZ and ZX - planes.

(i) Any point on x-axis is of the form $(x, 0, 0)$
(ii) Any point on y-axis is of the form $(0, y, 0)$
(iii) Any point on z-axis is of the form $(0, 0, z)$

The coordinates of the origin O are $(0,0,0)$

- Signs of the coordinates in eight octant :

Octants → / Coordinates ↓	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

- Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

$$\boxed{PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

➡ The coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally and externally in the ratio $m : n$ is given by

$$\left[\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right] \text{ and } \left[\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right]$$

Case I: The coordinates of the mid-point of the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right]$$

Case II: The coordinates of the point R which divides PQ in the ratio $k:1$ are obtained by taking $k = \frac{m}{n}$ which are as given below

$$\left[\frac{kx_2 + x_1}{1+k}, \frac{ky_2 + y_1}{1+k}, \frac{kz_2 + z_1}{1+k}\right]$$

The coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are

$$\left[\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right]$$

➡ Relation between the direction cosines of a line

$$\boxed{l^2 + m^2 + n^2 = 1}$$ direction cosines

➡ Direction cosines of a line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\frac{x_2 - x_1}{PQ} = \frac{y_2 - y_1}{PQ} = \frac{z_2 - z_1}{PQ}$$ where $PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

➡ If l, m, n are the direction cosines and a, b, c are the direction ratios of a line $\boxed{l = \frac{a}{\sqrt{a^2+b^2+c^2}}}$; $\boxed{m = \frac{b}{\sqrt{a^2+b^2+c^2}}}$; $\boxed{n = \frac{c}{\sqrt{a^2+b^2+c^2}}}$

➡ Vector Equation of a line that passes through the given point whose position vectors $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\boxed{\vec{r} = \vec{a} + \lambda\vec{b}}$ (vector form)

Cartesian Equation,

$$\boxed{\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}} \quad \text{OR} \quad \boxed{\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}}$$

direction Ratios

➡ Vector Equation of a line that passes through two points

$\boxed{\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})}$, $\lambda \in R$

Position vectors

Cartesian Equation points :- (x_1, y_1, z_1) (x_2, y_2, z_2) $\boxed{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}}$

➡ If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines and θ is the acute angle between two lines then;

$$\boxed{\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|}$$

Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are

(i) Perpendicular $\theta = 90°$ $\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$

(ii) parallel $\theta = 0°$ $\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$

If θ is the acute angle between the line $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda\vec{b_2}$; then θ is given by : $\boxed{\cos\theta = \left|\frac{|\vec{b_1}\cdot\vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}\right|}$ OR $\boxed{\theta = \cos^{-1}\frac{|\vec{b_1}\cdot\vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}}$

cartesian form,

lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is

$$\boxed{\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}}$$

- The shortest distance between the lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ is $\boxed{\left|\frac{(\vec{b_1}\times\vec{b_2})\cdot(\vec{a_2}-\vec{a_1})}{|\vec{b_1}\times\vec{b_2}|}\right|}$ (vector form)

- Shortest Distance between Parallel lines $\vec{r}=\vec{a_1}+\lambda\vec{b}$ and $\vec{r}=\vec{a_2}+\mu\vec{b}$ is $\boxed{\left|\frac{\vec{b}\times(\vec{a_2}-\vec{a_1})}{|\vec{b}|}\right|}$

- Equation of a plane in a normal form $\boxed{\vec{r}\cdot\hat{n}=d}$ (Vector Form)

Position vector — unit normal vector — distance from origin

cartesian form, $\boxed{lx+my+nz=d}$

- The equation of a plane through a point whose Position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $\boxed{(\vec{r}-\vec{a})\cdot\vec{N}=0}$ (Vector Form)

cartesian form, $\boxed{A(x-x_1)+B(y-y_1)+C(z-z_1)=0}$

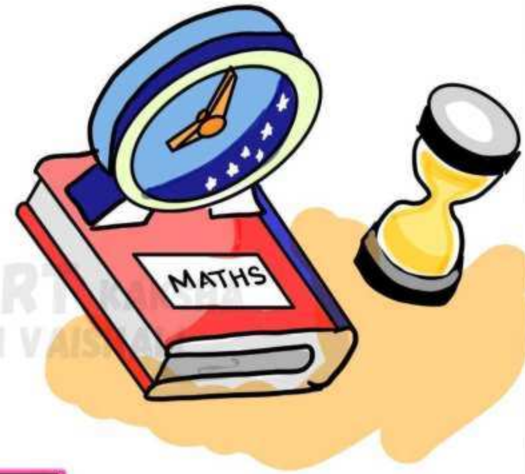
- Equation of a Plane Passing through three non collinear points

$\boxed{(\vec{r}-\vec{a})[\vec{b}-\vec{a}]\times(\vec{c}-\vec{a})]=0}$ (Vector Form)

cartesian form, $$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

- Plane Passing through the intersection of Two given Planes $\boxed{\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2})=d_1+\lambda d_2}$ (Vector Form) cartesian form,

$$\boxed{(A_1x+B_1y+C_1z-d_1)+\lambda(A_2x+B_2y+C_2z-d_2)=0}$$



➡ Angle between Two Planes

$$\boxed{\cos\theta=\left|\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right|}$$

cartesian form, $\boxed{\cos\theta=\left|\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}\right|}$

➡ Angle between a line and a plane $\boxed{\cos\theta=\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\cdot|\vec{n}|}\right|}$ the angle ϕ between the line and the plane is given by $90^\circ-\theta$, i.e.

$\sin(90^\circ-\theta)=\cos\theta$ $\boxed{\sin\phi=\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\cdot|\vec{n}|}\right|}$ **OR** $\boxed{\phi=\sin^{-1}\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\cdot|\vec{n}|}\right|}$

NOTE

If $\vec{b}\cdot\vec{n}=0 \Rightarrow \sin\phi=0 \Rightarrow \phi=0^\circ \Rightarrow$ line is parallel to the plane

➡ Equation Of a Plane in intercept form: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ where a,b,c are intercepts made by the Plane on the x-axis, y-axis & z-axis respectively.

NOTE

(i) If $\vec{n_1}\cdot\vec{n_2}=0 \Rightarrow$ planes are perpendicular

(ii) If $\vec{n_1}=\lambda\vec{n_2} \Rightarrow$ both planes are parallel

(iii) angle between two planes is always taken as acute angle.

Vector Algebra

➡ **Vector** :- A quantity that has magnitude as well as direction is called a vector denoted by $\overrightarrow{AB}$ or $\vec{a}$. $A \xrightarrow[\text{vector}]{\vec{a}} B$

➡ **Initial Point**

The point A where from the vector $\overrightarrow{AB}$ starts is known as initial point.

➡ **Terminal Point**

The point B, where it ends is said to be the terminal point.

➡ **Magnitude** :-

The distance between initial and terminal points of a vector is called the magnitude (or length) of the vector.

➡ **Scalar** :-

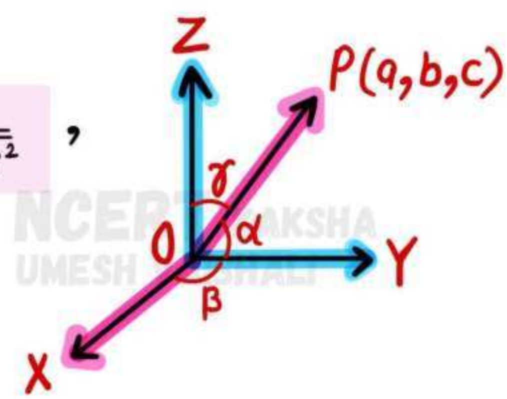
Those physical quantities which have only magnitude are called scalar.

Example - area, volume, mass etc.

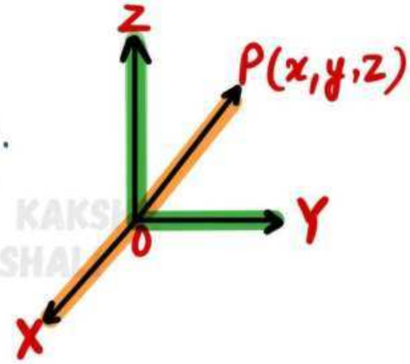
➡ **Direction Cosines** :- If $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ makes angle α, β, γ with +ve direction of x-axis, y-axis and z-axis respectively, than $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of $\vec{r}$ and are denoted by l, m and n where,

$$l = \cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$n = \cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$$



➡ **Direction Ratios** :- If numbers a, b, c are proportional to direction cosines l, m and n respectively of $\vec{r}$, then a, b, c are called direction ratios of $\vec{r}$.



➡ **Position Vector** :- Consider a point (x, y, z) in space. The vector $\overrightarrow{OP}$ with initial point, origin O and terminal point P, is called the position vector of P.

➡ **Zero Vector** :- A vector whose initial and terminal points coincide is known as zero vector.

➡ **Unit Vector** :- A vector whose magnitude is unity is said to be unit vector denoted by $\hat{a}$. $\boxed{\hat{a} = \frac{1}{|\vec{a}|}\vec{a}}$

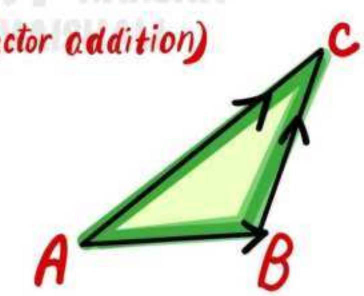
➡ **Co-initial Vectors** :- Two or more vectors having the same initial point are called coinitial vectors.

➡ **Collinear Vectors** :- Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

➡ **Equal Vectors** :- Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points, and written as $\vec{a} = \vec{b}$.

➡ **Negative of a vector** :- A vector whose magnitude is the same as that of a given vector, but direction is opposite to that of it, is called negative of the given vector. $\overrightarrow{BA} = -\overrightarrow{AB}$

➡ **Addition of Vectors** :- $\boxed{\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}}$ (Triangle law of Vector addition)



➡ **Properties of A Vector addition** :-

(i) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (ii) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

(iii) $\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$

➡ **Multiplication Of A Vector by a scalar :-**

$\boxed{|\lambda\vec{a}|=|\lambda||\vec{a}|}$ Scalar

➡ **Midpoint :-** $\boxed{\vec{r}=\dfrac{\vec{a}+\vec{b}}{2}}$

NOTE

For any scalar k,

$\boxed{k\vec{0}=\vec{0}}$

➡ **Component Form :-**

$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ vector component

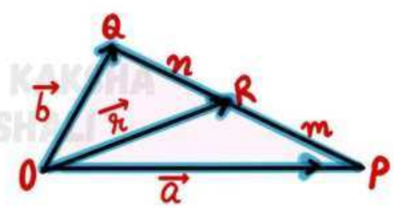
$|\vec{r}|=\sqrt{x^2+y^2+z^2}$

x, y, z = scalar components of $\vec{r}$

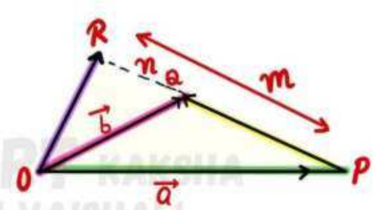
➡ **Vector Joining Two Points :** $|\overrightarrow{P_1P_2}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

➡ **Section Formula :-**

case I When R divides PQ internally $\boxed{\vec{r}=\dfrac{m\vec{b}+n\vec{a}}{m+n}}$



Case II When R divides PQ externally $\boxed{\vec{r}=\dfrac{m\vec{b}-n\vec{a}}{m-n}}$



➡ **Scalar (or dot) product Of Two vectors :-**

$\boxed{\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta}$

θ is the angle between $\vec{a}$ and $\vec{b}$, $0\le\theta\le\pi$

$\boxed{\cos\theta=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}$

$\boxed{\theta=\cos^{-1}\left[\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right]}$

Properties :-

(i) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

(2) $\vec{a}.(\vec{b}+c) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$

(3) $(\lambda\vec{a}).\vec{b} = \lambda(\vec{a}.\vec{b}) = \vec{a}.(\lambda\vec{b})$

(4) $\vec{a}.\vec{b}=0 \Leftrightarrow \vec{a}=0, \vec{b}=0$ OR $\vec{a} \perp \vec{b}$

5. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

6. Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$ and Projection vector of $\vec{a}$ on $\vec{b} = \left|\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right|.\vec{b}$

7. Projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$ and Projection vector of $\vec{b}$ on $\vec{a} = \left|\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right|.\vec{a}$

NOTE

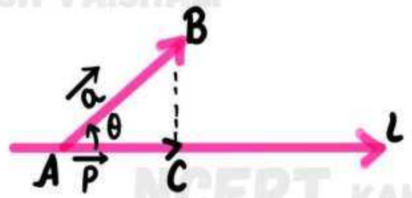
- If two vectors $\vec{a}$ and $\vec{b}$ are given in component form as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ $\quad \vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B, then $\vec{AB} = \vec{b} - \vec{a}$
- Two vectors are said to be arthogonal if they are perpendicular to each other.

Observations :-

- $\vec{a}.\vec{b}$ is a real number.
- Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors, then $\vec{a}.\vec{b}=0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other i.e. $\vec{a}.\vec{b}=0 \Leftrightarrow \vec{a} \perp \vec{b}$
- If $\theta=0$ then $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$. In Particular $\vec{a}.\vec{a} = |\vec{a}|^2$, as θ in this case is 0.

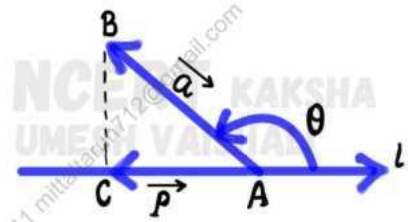
- If $\theta = \pi$ then $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$. In Particular $\vec{a}\cdot(-\vec{a}) = -|\vec{a}|^2$, as θ in this case is π.
- In view of the observations 2 and 3, for mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ $\quad$ $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$
- The scalar product is commutative i.e. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

Projection of a vector on a line :- The $\vec{P}$ is called the projection vector and its magnitude $|\vec{P}|$ is simply called as the projection of the vector $\vec{AB}$ on the directed line l.



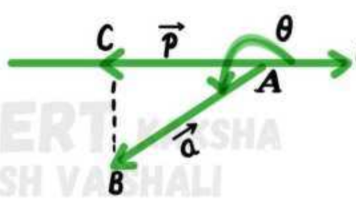
$(0^\circ < \theta < 90^\circ)$

(i)



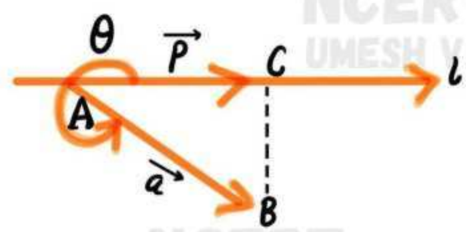
$(90^\circ < \theta < 180^\circ)$

(ii)



$(180^\circ < \theta < 270^\circ)$

(iii)



$(270^\circ < \theta < 360^\circ)$ (iv)

Observations :-

- If $\vec{P}$ is the unit vector along a line l, then the projection of a vector $\vec{a}$ on the line l is given by $\vec{a}\cdot\hat{p}$.
- Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by

$$\vec{a}\cdot\hat{b} \text{ OR } \vec{a}\cdot\left[\frac{\vec{b}}{|\vec{b}|}\right] \text{ OR } \frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})$$

* If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.

* If $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

NOTE

If α, β and γ are the direction angles of vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then its direction cosines may be given as -

$$\cos\alpha = \frac{\vec{a}.\hat{i}}{|\vec{a}||\hat{i}|} = \frac{a_1}{|\vec{a}|}, \quad \cos\beta = \frac{a_2}{|\vec{a}|} \quad \text{and} \quad \cos\gamma = \frac{a_3}{|\vec{a}|}$$

$$\vec{a} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

$\vec{a}$ = unit vector

a_1, a_2, a_3 = scalar components

➡ Vector (or cross) Product of Two Vectors :-

$$\boxed{\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\,\hat{n}} \quad \text{OR} \quad \boxed{\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}}$$

θ is the angle between $\vec{a}$ & $\vec{b}$, $0 \le \theta \le \pi$

n = unit vector $\perp$ to the plane $\vec{a}$ and $\vec{b}$

➡ Properties :-

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

2. If $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a} = 0$, $\vec{b} = 0$ or $\vec{a} \parallel \vec{b}$

3. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

➡ Observations :-

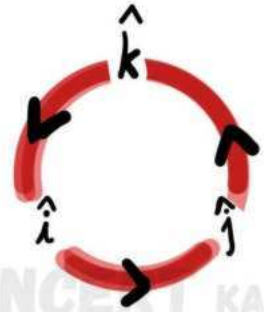
[1] $\vec{a} \times \vec{b}$ is a vector.

[2] If $\theta = \frac{\pi}{2}$ then $\boxed{\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|}$

[3] Angle between two vectors $\vec{a}$ and $\vec{b}$ $\boxed{\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}}$

[4] $\boxed{\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}}$

$\boxed{\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}}$



(5) $\boxed{\hat{j}\times\hat{i} = -\hat{k}}$, $\boxed{\hat{k}\times\hat{j} = -\hat{i}}$ and $\boxed{\hat{i}\times\hat{k} = -\hat{j}}$

- Area of triangle ABC = $\frac{1}{2}|\vec{b}||\vec{a}|\sin\theta = \boxed{\frac{1}{2}|\vec{a}\times\vec{b}|}$

- Area of Parallelogram ABCD $= |\vec{b}||\vec{a}|\sin\theta = \boxed{|\vec{a}\times\vec{b}|}$

- Projection Formulae :-

  - $a = b\cos C + c\cos B$
  - $b = c\cos A + a\cos C$
  - $c = a\cos B + b\cos A$

- Lagrange's Identity :-

$$\boxed{|\vec{a}\times\vec{b}|^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}}$$

OR

$$\boxed{(\vec{a}\cdot\vec{b})^2 + (\vec{a}\times\vec{b})^2 = |\vec{a}|^2 + |\vec{b}|^2}$$

NOTE

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$$

$$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2\left[|\vec{a}|^2 + |\vec{b}|^2\right]$$

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$$

- Scalar Triple Product :- $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

NOTE

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \Leftrightarrow \vec{a}.\vec{b}.\vec{c} \text{ are coplanar}$$

Remember that :

(i) $[\vec{a}.\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$

(ii) $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2\,[\vec{a}\ \vec{b}\ \vec{c}]$

(iii) $$[\vec{a}\,\vec{b}\,\vec{c}]^2 = [\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$$

Vector Triple Product :

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then that expression $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector and is called a vector triple product.

(a) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

(b) $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(c) $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

Reciprocal System Of Vectors :

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$ then the two systems are called Reciprocal System of vectors.

NOTE

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}\ ;\quad \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}\ ;\quad \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

UNIT-13 STATISTICS AND PROBABILITY

Statistics

Mean	$\bar{x} = \frac{\Sigma x}{n}$
Median	If n is odd, then $M = \left(\frac{n+1}{2}\right)^{th}$ term; If n is even, then $M = \frac{\left(\frac{n}{2}\right)^{th} \text{term} + \left(\frac{n+1}{2}\right)^{th} \text{term}}{2}$
Mode	The value which occurs most frequently
Variance	$\sigma^2 = \frac{\Sigma (x-\bar{x})^2}{n}$
Standard Deviation	$S = \sigma = \sqrt{\frac{\Sigma (x-\bar{x})^2}{n}}$

x = observations given

n = Total no. of observations

$\bar{x}$ = Mean

➡ $\boxed{\text{Range} = \text{Maximum value} - \text{Minimum value}}$

➡ Mean Deviation

$$\text{M.D}(a) = \frac{\text{Sum of absolute values of deviations from 'a'}}{\text{No. of observations}}$$

Mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

➡ Mean deviation for ungrouped data

Let n observations be $x_1, x_2, x_3, \ldots\ldots x_n$.

$$\text{M.D.}(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}| \quad \& \quad \text{M.D.}(M) = \frac{1}{n}\sum_{i=1}^{n}|x_i - M|$$

$\bar{x}$ = Mean

M = Median

➡ Mean deviation for grouped data

$$(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i \qquad \text{M.D.}(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i|x_i - \bar{x}| \qquad \text{M.D.}(M) = \frac{1}{N}\sum_{i=1}^{n} f_i|x_i - M|$$

, where $N = \sum_{i=1}^{n} f_i$

➡ Shortcut method for calculating mean deviation about mean

$$\bar{x} = a + \frac{\sum_{i=1}^{n} f_i d_i}{N} \times h$$

assumed mean

common factor

$$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times h$$

Variance and standard deviation

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \qquad \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Variance (σ^2)
Standard deviation (σ)

Cofficient of variation (C.V.) $\frac{\sigma}{\bar{x}} \times 100$, $\bar{x} \neq 0$

Variance and standard deviation of a discrete frequency distribution

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2 \qquad \sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}$$

Variance and standard deviation of a continuous frequency distribution

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2 \qquad \sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2} \quad \text{OR} \quad \sigma = \frac{1}{N}\sqrt{N\sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i\right)^2}$$

Shortcut method to find variance and standard deviation

$$\sigma^2 = \frac{h^2}{N^2}\left[N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2\right] \qquad \sigma = \frac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2}$$

where $y_i = \frac{x_i - A}{h}$

MEDIAN :

The median of a series is the value of middle term of the series when the values are written in ascending order. Therefore median, divides an arranged series into two equal parts.

Formulae Of Median :

(i) For ungrouped distribution : Let n be the number of variate in a series then,

$$\text{Median} = \begin{cases} \left[\frac{n+1}{2}\right]^{th} \text{ term, (when } n \text{ is odd)} \\ \text{Mean of } \left(\frac{n}{2}\right)^{th} \text{ and } \left(\frac{n}{2}+1\right)^{th} \text{ terms, (when } n \text{ is even)} \end{cases}$$

(ii) For ungrouped freq. dist. : First we prepare the cumulative frequency (c.f.) column and find value of N then,

$$\text{Median} = \begin{cases} \left[\dfrac{N+1}{2}\right]^{th} \text{ term, (when N is odd)} \\ \text{Mean of } \left(\dfrac{N}{2}\right)^{th} \text{ and } \left(\dfrac{N}{2}+1\right)^{th} \text{ terms, (when N is even)} \end{cases}$$

(iii) For grouped freq. dist : Prepare c.f. column and find value of $\frac{N}{2}$ then find the class which contain value of c.f. is equal or just greater to N/2, this is median class.

$$\therefore \boxed{\text{Median} = l_{oge} + \frac{\left[\frac{N}{2}-F\right]}{f} \times h}$$

where l (l_{oge}) − lower limit of median class

f − freq. of median class

F − c.f. of the class preceding median class

h − Class interval of median class

MODE :

In a frequency distribution the mode is the value of that variate which have the maximum frequency.

Method for determining mode :

(i) For ungrouped dist − The value of that variate which is repeated maximum number of times.

(ii) For ungrouped freq. dist. : The value of that variate which have maximum frequency.

(iii) For grouped freq. dist. : First we find the class which have maximum frequency, this is model class.

$\therefore$ Mode = l – lower limit of model class

f_0 – freq. of the model class

f_1 – freq. of the class preceding model class.

f_2 – freq. of the class succeeding model class

h – class interval of model class.

➡ Relation Between mean, median and mode :

In a moderately asymmetric distribution following is the relation between mean, median and mode of a distribution. It is known as impirical formula.

$$\boxed{\text{Mode} = 3\ \text{Median} - 2\ \text{Mean}}$$

NOTE

(i) Median always lies between mean and mode

(ii) For a symmetric distribution the mean, median and mode coincide.

UNIT - 13 STATISTICS AND PROBABILITY

Probability

- **Probability formula** : $P(E) = \dfrac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$

- **Outcomes space** : A possible result of a random experiment is called its outcome.

- **Sample space** : The set of outcomes is called the sample space of the experiment.

- **Sample point** : Each element of the sample space is called a sample point.

- **Event** : Any subset E of a sample space S is called an event.

- **Types of events**

1. **Impossible Events and Sure Events** : The empty set ϕ Impossible Events.
   The whole sample space S Sure Events.
2. **Simple Event** : If any event E has only one sample point of a sample space, it is called a simple event. (Elementary Event)
3. **Compound Event** : If an event has more than one sample point, it is called a compound event.

- **Algebra of events**

1. **Complementary Event of A** : The set A' or $S - A$
2. **The Event 'A' or 'B'** : The set $A \cup B$
3. **The Event 'A' and 'B'** : The set $A \cap B$
4. **The Event 'A' but not 'B'** : The set $A - B$

- **Mutually exclusive events** : A and B are mutually exclusive if $A \cap B = \phi$

NOTE Simple events of a sample space are always mutually exclusive.

➡ **Exhaustive Events**: If $E_1, E_2, \ldots, E_n$ are n events of a sample space S and if $\boxed{E_1 \cup E_2 \cup \ldots \cup E_n = \bigcup_{i=1}^{n} E_i = S}$ then $E_1, E_2, \ldots, E_n$ are called exhaustive events.

➡ **Probability**: Number $P(\omega_i)$ associated with sample point ω_i such that.

(i) $0 \le P(\omega_i) \le 1$

(ii) $\Sigma P(\omega_i)$ for all $\omega_i \in S = 1$

(iii) $P(A) = \Sigma P(\omega_i)$ for all $\omega_i \in A$

The no. of $P(\omega_i)$ is called probability of the outcome ω_i.

➡ **Equally likes outcomes**: All outcomes with equal probability

➡ **Probability of an event**: For a finite sample space with equally likely outcomes. $\boxed{P(A) = \frac{n(A)}{n(S)}}$

$n(A)$ → number of elements in the set A

$n(S)$ → number of elements in the set S

➡ If A and B are any two events, then $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

➡ If A and B are mutually exclusive, then $\boxed{P(A \cup B) = P(A) + P(B)}$

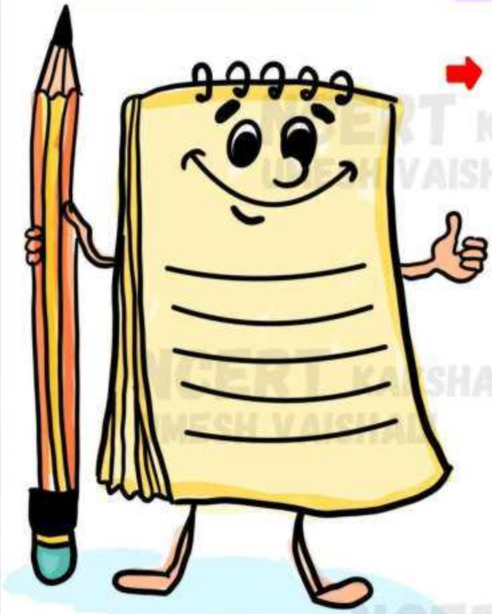
➡ If A is any event then $\boxed{P(\text{not } A) = 1 - P(A)}$

➡ **Conditional Probability**:- If E and F are two events associated with the same sample space of a random experiment, then the conditional probability of the event E under the condition that the event F has occurred written as $P\left(\frac{E}{F}\right)$ is given by, $\boxed{P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)}}$; $P(F) \neq 0$

NOTE $\boxed{0 \le P\left(\frac{E}{F}\right) \le 1}$

Properties of Conditional Probability :- let E and F be events associated with the sample space S of an experiment. Then,

- $P\left(\frac{S}{F}\right) = P\left(\frac{F}{F}\right) = 1$
- $P\left[\frac{(A \cup B)}{F}\right] = P\left(\frac{A}{F}\right) + P\left(\frac{B}{F}\right) - P\left[\frac{(A \cap B)}{F}\right]$
- $P\left(\frac{E'}{F}\right) = 1 - P\left(\frac{E}{F}\right)$



Multiplication Theorem on Probability :-

let E and F be two events associated with a sample space of an experiment. Then,

$$P(E \cap F) = P(E)\,P\left(\frac{F}{E}\right);\ P(E) \neq 0$$
$$= P(F)\,P\left(\frac{E}{F}\right);\ P(F) \neq 0$$

If E, F and G are 3 events associated with a sample space then, $P(E \cap F \cap G) = P(E)\ P\left(\frac{F}{E}\right)\ P\left[\frac{G}{E \cap F}\right]$

Independent Events :- Let E and F be two events associated with the same random experiment, then E and F are said to be independent if,

$$P(E \cap F) = P(E).\,P(F)$$

NOTE $P\left(\frac{E}{F}\right) = P(E),\ P(F) \neq 0$ $\qquad$ $P\left(\frac{F}{E}\right) = P(F),\ P(E) \neq 0$

Dependent Events : Two events E and F are said to be dependent if they are not independent, i.e. if $P(E \cap F) \neq P(E).\,P(F)$

Three events A, B and C are said to be independent of all the following conditions hold :

$$P(A \cap B) = P(A)\ P(B)$$
$$P(A \cap C) = P(A)\ P(C)$$
$$P(B \cap C) = P(B)\ P(C)$$
$$\text{and } P(A \cap B \cap C) = P(A)\ P(B)\ P(C)$$

Bayes' Theorem :- If $E_1, E_2 \ldots E_n$ are mutually exclusive and exhaustive events associated with a sample space and A as any event of non-zero probability then;

$$\boxed{P\left(\frac{E_i}{A}\right) = \frac{P(E_i)\,P\left(\frac{A}{E_i}\right)}{\sum_{j=1}^{n} P(E_j)\,P\left(\frac{A}{E_j}\right)}}$$

Theorem Of Total Probability :-

let $\{E_1, E_2, \ldots E_n\}$ be a partition of the sample space S. Let A be any event associated with S, then:

$$\boxed{P(A) = \sum_{j=1}^{n} P(E_j)\,P\left(\frac{A}{E_j}\right)}$$

Random variable and its probability Distribution :-

A random variable is a real valued whose domain is the sample space of a random experiment. The Probability distribution of a random variable X is the system of numbers.

$$\begin{cases} X : & x_1 \quad x_2 \ldots\ldots x_n \\ P(X) : & P_1 \quad P_2 \ldots\ldots P_n \end{cases}$$ where $P_i > 0; \sum_{i=1}^{n} P_i = 1 \quad i = 1, 2 \ldots n$

Mean Of a Random variable :- Let X be a random variable assume $x_1, x_2, \ldots x_n$ [The expectation of X or E(X)]

with probabilities $P_1, P_2, \ldots P_n$ respectively. Mean of X, denoted by μ is the number $\sum_{i=1}^{n} x_i P_i$

$$\boxed{E(X) = \mu = \sum_{i=1}^{n} x_i P_i = x_1 P_1 + x_2 P_2 + \ldots + x_n P_n}$$

Variance Of A Random Variable :- $\boxed{\sigma^2 = \sum_{i=1}^{n} (x_i - \mu)^2 P_i = \sum_{i=1}^{n} x_i^2 P_i - \mu^2}$

or equivalently $\boxed{\sigma^2 = E(X-\mu)^2}$

Standard deviation of the random variable X is defined as :

$$\boxed{\sigma = \sqrt{\text{Variance}(X)} = \sqrt{\sum_{i=1}^{n}(x_i-\mu)^2 P_i}}$$

Bernoulli Trials :- Trials of a random experiment are called Bernoulli trivals, if they satisfy the following conditions :-

1. There should be fine no. of trials.
2. The trials should be independent.
3. Each trial has exactly two outcomes: success or failure.
4. The Probability of success (or failure) remains the same in each trial.

Binomial Distribution :- A random variable X taking values 0, 1, 2 ....., n is said to have binomial distribution with parameters n and P of its probability distribution is given by :

$\boxed{P(X=r) = {}^nC_r\, p^r q^{n-r}}$ where $q = 1-p$ and $r = 0, 1, 2 \ldots\ldots n$

NOTE $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

In the case of three events ~

$$\boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)}$$

$\boxed{P(A \cup B) = P(A) + P(B)}$, If two events A and B are mutually exclusive.

$\boxed{P(\bar{A} \cap B) = P(B) - P(A \cap B)}$, where $\bar{A}$ and B are independent events.

$\boxed{P(A\cap\bar{B}) = P(A) - P(A\cap B)}$, where A and $\bar{B}$ are independent events.

$\boxed{P(\bar{A}\cap\bar{B}) = P(\overline{A\cup B}) = 1 - P(A\cup B) = P(\bar{A}) \times P(\bar{B})}$, where A and B are mutually exclusive events

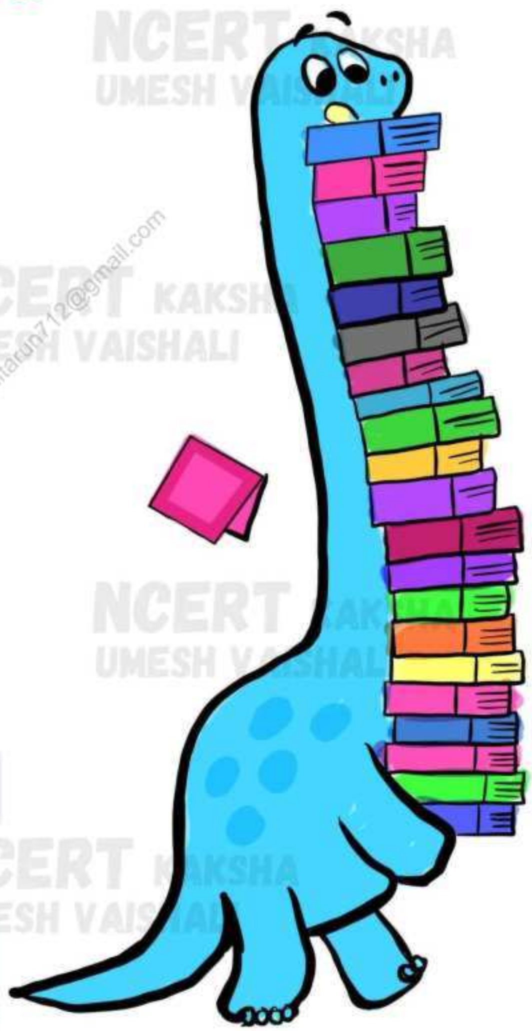
$\boxed{P\left(\frac{\bar{A}}{B}\right) = \frac{P(\bar{A}\cap B)}{P(B)} = \frac{P(B) - P(A\cap B)}{P(B)}}$, where $\bar{A}$ and B are independent events and $P(B) \neq 0$.

$\boxed{P\left(\frac{\bar{B}}{\bar{A}}\right) = \frac{P(\bar{A}\cap\bar{B})}{P(\bar{A})} = \frac{1 - P(A\cap B)}{1 - P(A)}}$, where $\bar{A}$ and $\bar{B}$ are independent events and $P(A) \neq 0$.

Some Important Results :

(a) Let A and B be two events, then

- $P(A) + P(\bar{A}) = 1$
- $P(A+B) = 1 - P(\bar{A}\,\bar{B})$
- $P(A/B) = \frac{P(AB)}{P(B)}$
- $P(A+B) = P(AB) + P(\bar{A}B) + P(A\bar{B})$
- $A \subset B \Rightarrow P(A) \leq P(B)$
- $P(\bar{A}B) = P(B) - P(AB)$
- $P(AB) \leq P(A)\ P(B) \leq P(A+B) \leq P(A) + P(B)$
- $P(AB) = P(A) + P(B) - P(A+B)$
- $P(\text{Exactly one event}) = P(A\bar{B}) + P(\bar{A}B)$
  $= P(A) + P(B) - 2P(AB) = P(A+B) - P(AB)$
- $P(\text{neither } A \text{ nor } B) = P(\bar{A}\,\bar{B}) = 1 - P(A+B)$
- $P(\bar{A}+\bar{B}) = 1 - P(AB)$



(b) Number of exhaustive cases of tossing n coins simultaneously (or of tossing a coin n times) $= 2^n$

(c) Number of exhaustive cases of throwing n dice simultaneously (or throwing one dice n times) $= 6^n$

(d) **Playing Cards :**

- Total Cards : 52 (26 red, 26 black)
- Four suits : Heart, Diamond, Spade, Club - 13 cards each.
- Court Cards : 12 (4 Kings, 4 queens, 4 jacks)
- Honour Cards : 16 (4 aces, 4 Kings, 4 queens, 4 jacks)

(e) **Probability regarding n letters and their envelopes :**

If n letters are placed into n directed envelopes at random, then

- Probability that all letters are in right envelopes $= \boxed{\frac{1}{n!}}$
- Probability that all letters are not in right envelopes $= \boxed{1 - \frac{1}{n!}}$
- Probability that no letters is in right envelopes

$$= \boxed{\frac{1}{2!} - \frac{1}{3!} - \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}}$$

- Probability that exactly r letters are in right envelopes

$$= \boxed{\frac{1}{r!}\left[\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^{n-r} \frac{1}{(n-r)!}\right]}$$

Trignometry

$$\sin A = \frac{\text{Height}}{\text{Hypotenuse}} = \frac{BC}{AC}$$

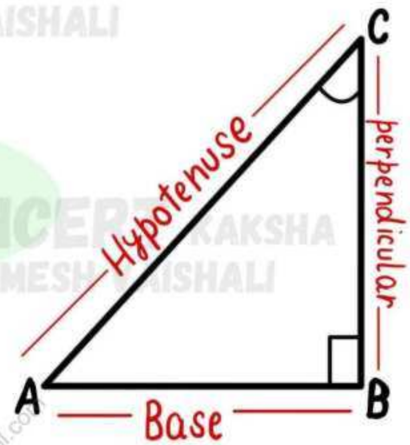
$$\cos A = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{AB}{AC}$$

$$\tan A = \frac{\text{Height}}{\text{Base}} = \frac{BC}{AB}$$

$$\operatorname{cosec} A = \frac{\text{Hypotenuse}}{\text{Height}} = \frac{AC}{BC}$$

$$\sec A = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{AC}{AB}$$

$$\cot A = \frac{\text{Base}}{\text{Height}} = \frac{AB}{BC}$$



Pythagoras theorem

$$(\text{Hypotenuse})^2 = (\text{perpendicular})^2 + (\text{Base})^2$$

$\sin A = \frac{1}{\operatorname{cosec} A}$	$\cos A = \frac{1}{\sec A}$	$\tan A = \frac{1}{\cot A}$
$\operatorname{cosec} A = \frac{1}{\sin A}$	$\sec A = \frac{1}{\cos A}$	$\cot A = \frac{1}{\tan A}$

☑ TRIGNOMETRIC RATIOS OF COMPLEMENTARY ANGLES :-

$\sin(90^\circ - A) = \cos A$, $\cos(90^\circ - A) = \sin A$,

$\tan(90^\circ - A) = \cot A$, $\cot(90^\circ - A) = \tan A$,

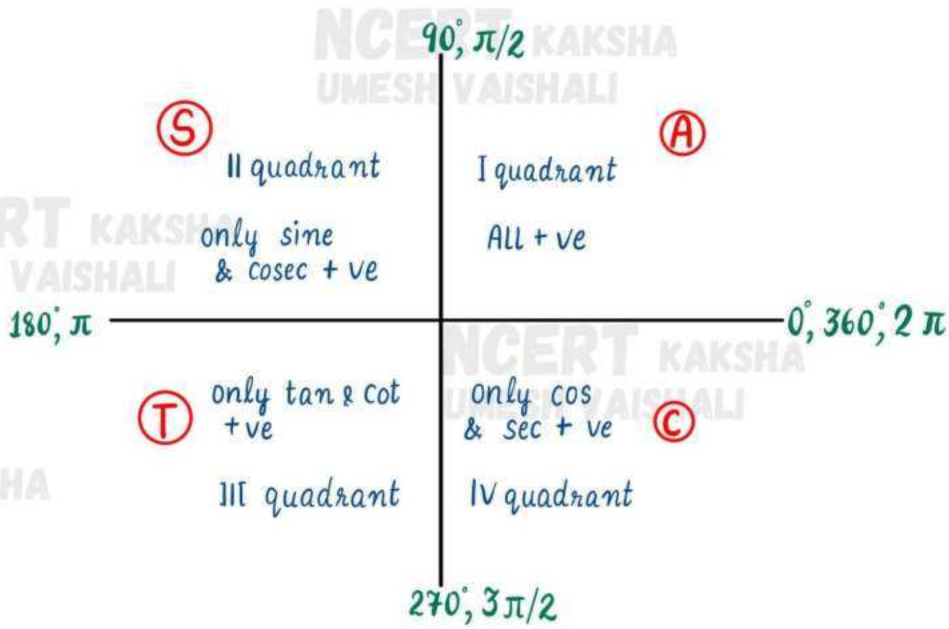
$\sec(90^\circ - A) = \operatorname{cosec} A$, $\operatorname{cosec}(90^\circ - A) = \sec A$,

TABLE :

$\angle A$	$0°$	$30°$	$45°$	$60°$	$90°$
Sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
Cosec A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
Sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
Cot A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

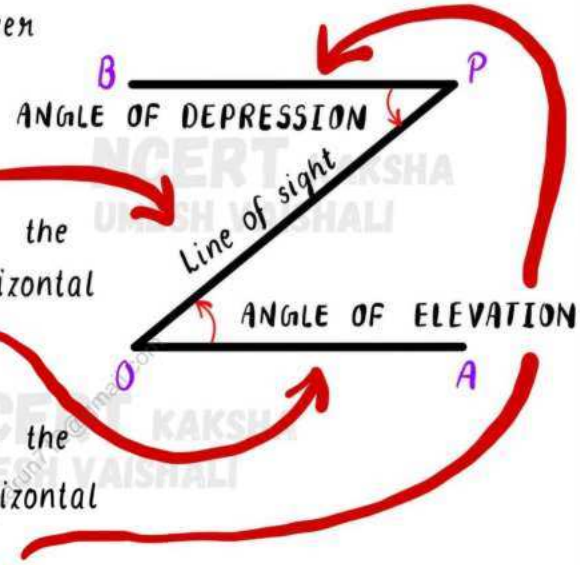
☑ TRIGNOMETRIC SIGN Functions :

- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = -\cos\theta$
- $\tan(-\theta) = -\tan\theta$
- $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$
- $\sec(-\theta) = -\sec\theta$
- $\cot(-\theta) = -\cot\theta$



Heights And Distances

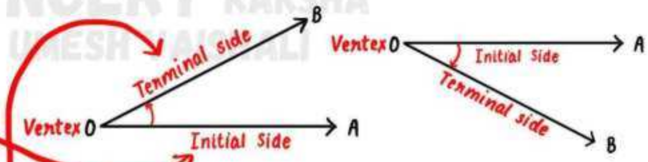
- The line drawn from the eye of an observer to a point in the object where the person is viewing is called **the line of sight.**
- The angle formed by the line of sight with the horizontal when the object is above the horizontal level is called the **angle of elevation.**
- The angle formed by the line of sight with the horizontal when the object is below the horizontal level is called the **angle of depression.**



- The height of an object on the distance between distant objects can be determined with the help of trigonometric ratios.

Trigonometric Functions

- Angle is a measure of rotation of a given ray about its initial point.
- The original ray is called the initial side. The final position of the ray after rotation is called terminal side.



(Positive angle) Rotation is anticlockwise

(Negative angle) Rotation is clockwise

- $1^\circ = 60'$
- $1' = 60''$

1° (one degree) / $1'$ (one minute) / $1''$ (one second)

$$\boxed{\theta = \frac{l}{r}}$$

θ = angle
l = an arc of length
r = radius of a circle

- Value of $\pi = \frac{22}{7} = 3.14$
- 1 radian $= \frac{180^\circ}{\pi}$
- 2π radian $= 360^\circ$
- π radian $= 180^\circ$
- $1^\circ = \frac{\pi}{180}$ radian

Radian measure $= \frac{\pi}{180} \times$ Degree measure

Degree measure $= \frac{180}{\pi} \times$ Radian measure

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$
$\sin(x-y) = \sin x \cos y - \cos x \sin y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$
$\cos(x-y) = \cos x \cos y + \sin x \sin y$

$\cos\left(\frac{\pi}{2}-x\right) = \sin x$

$\cos\left(\frac{\pi}{2}+x\right) = -\sin x$

$\sin\left(\frac{\pi}{2}-x\right) = \cos x$

$\sin\left(\frac{\pi}{2}+x\right) = \cos x$

$\cos(\pi-x) = -\cos x$

$\sin(\pi-x) = \sin x$

$\cos(\pi+x) = -\cos x$

$\sin(\pi+x) = -\sin x$

$\cos(2\pi-x) = \cos x$

$\sin(2\pi-x) = -\sin x$

$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

$\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot y + \cot x}$

$\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \operatorname{cosec}^2 x$

$\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$

$\cos 2x = \cos^2 x - \sin^2 x$ OR $\dfrac{1-\tan^2 x}{1+\tan^2 x}$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$

$\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

$\sin 3x = 3\sin x - 4\sin^3 x$
$\cos 3x = 4\cos^3 x - 3\cos x$
$\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

(xxvi) $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$.

(xxvii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$.

(xxviii) $\sin(A+B+C)$

$= \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B$

$- \sin A\sin B\sin C$

$= \Sigma \sin A\cos B\cos C - \Pi \sin A$

$= \cos A\cos B\cos C\,[\tan A + \tan B + \tan C - \tan A\tan B\tan C]$

(xxix) $\cos(A+B+C)$

$= \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C$

$- \cos A\sin B\sin C$

$= \Pi\cos A - \Sigma \sin A\sin B\cos C$

$= \cos A\cos B\cos C\,[1 - \tan A\tan B - \tan B\tan C - \tan C\tan A]$

(xxx) $\tan(A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A} = \frac{S_1 - S_3}{1 - S_2}$$

(xxxi) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin(\alpha+\overline{n-1}\beta)$

$$= \frac{\sin\left\{\alpha + \left[\frac{n-1}{2}\right]\beta\right\}\sin\left[\frac{n\beta}{2}\right]}{\sin\left[\frac{\beta}{2}\right]}$$

(xxxii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+\overline{n-1}\beta)$

$$= \frac{\cos\left\{\alpha + \left[\frac{n-1}{2}\right]\beta\right\}\sin\left[\frac{n\beta}{2}\right]}{\sin\left[\frac{\beta}{2}\right]}$$

GENERAL SOLUTIONS OF SOME TRIGNOMETRIC EQUATIONS :-

- $\sin x = 0$ implies $\boxed{x = n\pi}$, where n is any integer.
- $\cos x = 0$ implies $\boxed{x = (2n+1)\frac{\pi}{2}}$, where n is any integer.
- $\tan x = 0$ implies $\boxed{x = n\pi}$, $n \in I$
- $\sin x = \sin y$ implies $\boxed{x = n\pi + (-1)^n y}$, where $n \in Z$
- $\cos x = \cos y$ implies $x = 2n\pi \pm y$, where $n \in Z$
- $\tan x = \tan y$ implies $x = n\pi + y$, where $n \in Z$
- If $\sin x = 1$, then $\boxed{x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}}$, $n \in I$
- If $\cos x = 1$ then $\boxed{x = 2n\pi}$, $n \in I$
- If $\sin^2 x = \sin^2 y$ or $\cos^2 x = \cos^2 y$ or $\tan^2 x = \tan^2 y$, then $\boxed{x = n\pi \pm y}$, $n \in I$
- $\boxed{\sin(n\pi + x) = (-1)^n \sin x}$, $n \in I$
- $\boxed{\cos(n\pi + x) = (-1)^n \cos x}$, $n \in I$

$\operatorname{cosec} x = \frac{1}{\sin x}$, $x \neq n\pi$, where n is any integer.

$\sec x = \frac{1}{\cos x}$, $x \neq (2n+1)\frac{\pi}{2}$, where n is any integer.

$\tan x = \frac{\sin x}{\cos x}$, $x \neq (2n+1)\frac{\pi}{2}$, where n is any integer.

$\cot x = \frac{\cos x}{\sin x}$, $x \neq n\pi$, where n is any integer.

Trigonometric Equations : Equations involving trigonometric functions of a variable are called trigonometric Equations.

Principal solutions : The solutions of a trigometric equation for which $0 \leq x \leq 2\pi$ are called principal solutions.

➡ General solution : The expression involving integer 'n' which give all solutions of a trigonometric equation is called the general function.

➡ **VALUES OF SOME T-RATIOS FOR ANGLES 18°, 36°, 15°, 22.5°, 67.5° etc.**

(a) $\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ = \sin \frac{\pi}{10}$

(b) $\cos 36^\circ = \frac{\sqrt{5}+1}{4} = \sin 54^\circ = \cos \frac{\pi}{5}$

(c) $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ = \sin \frac{\pi}{12}$

(d) $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ = \cos \frac{\pi}{12}$

(e) $\tan \frac{\pi}{12} = 2-\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot \frac{5\pi}{12}$

(f) $\tan \frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot \frac{5\pi}{12}$

(g) $\tan (22.5^\circ) = \sqrt{2}-1 = \cot (67.5^\circ) = \cot \frac{3\pi}{8} = \tan \frac{\pi}{8}$

(h) $\tan (67.5^\circ) = \sqrt{2}+1 = \cot (22.5^\circ)$

➡ **CONDITIONAL IDENTITIES :**

If $A+B+C = 180^\circ$, then

(a) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(b) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(c) $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$

(d) $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

(e) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(f) $\cos 2A + \cos 2B + \cos 2C = -1-4 \cos A \cos B \cos C$

(g) $\sin A + \sin B + \sin C = 4\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

(h) $\cos A + \cos B + \cos C = 1+4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

DOMAINS, RANGES AND PERIODICITY OF TRIGNOMETRIC FUNCTION

T-Ratio	Domain	Range	Period
$\sin x$	R	$[-1,1]$	2π
$\cos x$	R	$[-1,1]$	2π
$\tan x$	$R-\{(2n+1)\pi/2; n\in I\}$	R	π
$\cot x$	$R-[n\pi : n \in I]$	R	π
$\sec x$	$R-\{(2n+1)\pi/2 : n\in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π
$\mathrm{cosec}\, x$	$R-\{n\pi : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π

IMPORTANT NOTE :

(a) The sum of interior angles of a polygon of n-sides

$= (n-2) \times 180^\circ = (n-2)\pi$.

(b) Each interior angle of a regular polygon of n sides

$= \frac{(n-2)}{n} \times 180^\circ = \frac{(n-2)}{n}\pi$

(c) Sum of exterior angles of a polygon of any number of sides

$= 360^\circ = 2\pi$.

Inverse Trigonometric Functions

Functions	Domain (x)	Range
$y=\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$y=\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$y=\tan^{-1}x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$y=\operatorname{cosec}^{-1}x$	$R-(-1,1)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$
$y=\sec^{-1}x$	$R-(-1,1)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$
$y=\cot^{-1}x$	R	$(0,\pi)$

note:- $\in \rightarrow$ belongs to
$\forall \rightarrow$ for all

$R=$ Real Numbers

	Principal Value	General Value
$\sin\theta=\sin\alpha$	$\theta=\alpha$ if $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$	$\theta=n\pi+(-1)^n\alpha$; $n\in Z$
$\cos\theta=\cos\alpha$	$\theta=\alpha$ if $0<\alpha<\pi$	$\theta=2n\pi\pm\alpha$; $n\in Z$
$\tan\theta=\tan\alpha$	$\theta=\alpha$ if $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$	$\theta=n\pi+\alpha$; $n\in Z$

➡ **Properties of Inverse trigonometric Functions** :-

1. $\sin^{-1}(\sin x)=x$, $\forall x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
2. $\cos^{-1}(\cos x)=x$, $\forall x\in[0,\pi]$
3. $\tan^{-1}(\tan x)=x$, $\forall x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
4. $\cot^{-1}(\cot x)=x$, $\forall x\in(0,\pi)$
5. $\sec^{-1}(\sec x)=x$, $\forall x\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$
6. $\operatorname{cosec}^{-1}(\operatorname{cosec} x)=x$, $\forall x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

$$\sin(\sin^{-1}x) = x \quad \forall \; x \in [-1,1]$$
$$\cos(\cos^{-1}x) = x \quad \forall \; x \in [-1,1]$$
$$\tan(\tan^{-1}x) = x \quad \forall \; x \in R$$
$$\cot(\cot^{-1}x) = x \quad \forall \; x \in R$$
$$\sec(\sec^{-1}x) = x \quad \forall \; x \in R-(-1,1)$$
$$\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x \quad \forall \; x \in R-(-1,1)$$

$$\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x \quad \forall \; x \in R-(-1,1)$$
$$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x \quad \forall \; x \in R-(-1,1)$$
$$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x \quad \forall \; x > 0$$
$$\tan^{-1}\left(\frac{1}{x}\right) = -\pi + \cot^{-1}x \quad \forall \; x < 0$$

$$\sin^{-1}(-x) = -\sin^{-1}x \quad \forall \; x \in [-1,1]$$
$$\cos^{-1}(-x) = \pi - \cos^{-1}x \quad \forall \; x \in [-1,1]$$
$$\tan^{-1}(-x) = -\tan^{-1}x \quad \forall \; x \in R$$
$$\cot^{-1}(-x) = \pi - \cot^{-1}x \quad \forall \; x \in R$$
$$\sec^{-1}(-x) = \pi - \sec^{-1}x \quad \forall \; x \in R-(-1,1)$$
$$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x \quad \forall \; x \in R-(-1,1)$$

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \quad \forall \; x \in [-1,1]$$
$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \quad \forall \; x \in R$$
$$\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2} \quad \forall \; x \in R-(-1,1)$$

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\} \text{ if } -1 \le x, y \le 1 \;\&\; x^2+y^2 \le 1 \text{ OR } xy < 0 \;\&\; x^2+y^2 > 1$$
$$\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\} \text{ if } -1 \le x, y \le 1 \;\&\; x^2+y^2 \le 1 \text{ OR } xy > 0 \;\&\; x^2+y^2 > 1$$
$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} \text{ if } -1 \le x, y \le 1 \;\&\; x+y \ge 0$$
$$\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\} \text{ if } -1 \le x, y \le 1 \;\&\; x \le y$$
$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy < 1$$
$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) \text{ if } xy > -1$$

$$3\sin^{-1}x = \sin^{-1}(3x-4x^3) \quad \text{if } -\frac{1}{2} \le x \le \frac{1}{2}$$

$$3\cos^{-1}x = \cos^{-1}(4x^3-3x) \quad \text{if } \frac{1}{2} \le x \le 1$$

$$3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x}\right) \quad \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

$$\sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

$$\cos^{-1}x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

$$= \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right) = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

$$2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \quad \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$$

$$2\cos^{-1}x = \cos^{-1}(2x^2-1) \quad \text{if } 0 \le x \le 1$$

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad \text{if } -1 < x < 1$$

$$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \quad \text{if } -1 \le x \le 1$$

$$2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \quad \text{if } 0 \le x < \infty$$

$$2\tan^{-1}x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad \text{if } x > 1$$

Expression	Substitution
a^2+x^2 OR $\sqrt{a^2+x^2}$	$x=a\tan\theta$ OR $x=a\cot\theta$
a^2-x^2 OR $\sqrt{a^2-x^2}$	$x=a\sin\theta$ OR $x=a\cos\theta$
x^2-a^2 OR $\sqrt{x^2-a^2}$	$x=a\sec\theta$ OR $x=a\,\mathrm{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ OR $\sqrt{\frac{a+x}{a-x}}$	$x=a\cos 2\theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ OR $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2=a^2\cos 2\theta$
$\sqrt{\frac{x}{a-x}}$ OR $\sqrt{\frac{a-x}{x}}$	$x=a\sin^2\theta$ OR $x=a\cos^2\theta$
$\sqrt{\frac{x}{a+x}}$ $\sqrt{\frac{a+x}{x}}$	$x=a\tan^2\theta$ OR $x=a\cot^2\theta$

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IMPORTANT LOGARITHMIC FORMULAS

(a) $\log_a N = x$, read as log of N to the base $a \Leftrightarrow a^x = N$

If $a = 10$ then we write $\log N$ or $\log_{10} N$ and if $a = e$ we write $\ln N$ or $\log_e N$ (Natural log)

(b) Necessary conditions : $N > 0$; $a > 0$; $a \neq 1$

(c) $\log_a 1 = 0$ (d) $\log_a a = 1$ (e) $\log_{1/a} a = -1$

(f) $\log_a (x.y) = \log_a x + \log_a y$; $x, y > 0$

(g) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$; $x, y > 0$

(h) $\log_a x^p = p \log_a x$; $x > 0$

(i) $\log_{a^q} x = \frac{1}{q} \log_a x$; $x > 0$

(j) $\log_a x = \frac{1}{\log_x a}$; $x > 0$, $x \neq 1$

(k) $\log_a x = \log_b x / \log_b a$; $x > 0$, $a, b > 0$, $b \neq 1$, $a \neq 1$

(l) $\log_a b . \log_b c . \log_c d = \log_a d$; $a, b, c, d > 0, \neq 1$

(m) $a^{\log_a x} = x$; $a > 0$, $a \neq 1$ (n) $a^{\log_b c} = c^{\log_b a}$; $a, b, c > 0$; $b \neq 1$

(o) $\log_a x < \log_a y \Leftrightarrow \left[\begin{array}{ll} x < y & \text{if } a > 1 \\ x > y & \text{if } 0 < a < 1 \end{array}\right.$

(p) $\log_a x = \log_a y \Rightarrow x = y$; $x, y > 0$; $a > 0$, $a \neq 1$ (q) $e^{\ln a^x} = a^x$

(r) $\log_{10} 2 = 0.3010$; $\log_{10} 3 = 0.4771$; $\ln 2 = 0.693$, $\ln 10 = 2.303$

(s) If $a > 1$ then $\log_a x < p \Rightarrow 0 < x < a^p$

(t) If $a > 1$ then $\log_a x > p \Rightarrow x > a^p$

(u) If $0 < a < 1$ then $\log_a x < p \Rightarrow x > a^p$

(v) If $0 < a < 1$ then $\log_a x > p \Rightarrow 0 < x < a^p$