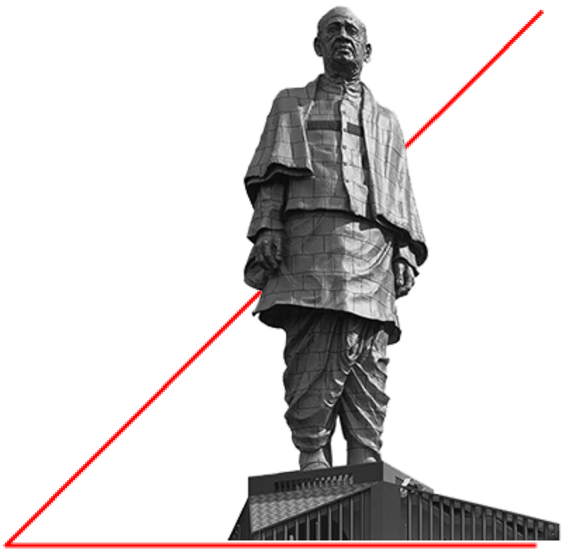
4 Introduction to Trigonometry



120 m

60°

Height = ?

The word trigonometry is derived from the Greek words 'tri' (Meaning three), 'gon' (Meaning sides) and 'metron' (meaning measure). Infact trigonometry is the study of relationships between the sides and angles of a triangle.

Let us take some examples from our surroundings where right triangles can be imagined to be formed.

Suppose the students of a school are visiting Statue of Unity. Now if a student is looking at the top of the tower, a right-angle triangle can be imagined to be made, as shown in fig.

Active Maths 1

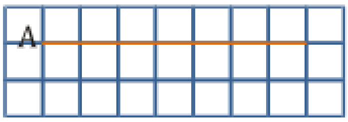
Finding ratios of side lengths in similar right triangles.

Work with a partner. You will need.

- Graph paper
- A protractor

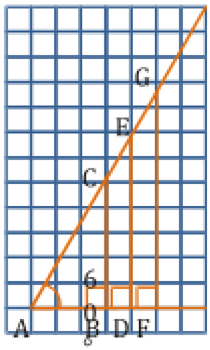
Exploring the concept

1. Draw a segment along a horizontal grid line on your graph paper. Label one end point A.



2. Use a protractor to draw a 60° angle at A.

3. Draw segments along vertical grid lines to form right triangles, such as ΔABC, ΔADE and ΔAFG shown in figure.



Drawing conclusions

1. Why are ΔABC, ΔADE and ΔAFG similar triangles?

2. Cut a strip of graph paper to use as a ruler. Use your ruler to find the value of following ratio for ΔABC and for ΔADE :

$$\frac{\text{Length of leg opposite}\angle A}{\text{Length of hypotenuse}} = \frac{\text{Perpendicular}}{\text{hypotenuse}}$$

What do you notice?

3. Make a conjecture about the value of the ratio for any triangle similar to ΔABC to test your conjecture, find the value of this ratio for ΔAFG [i.e. perpendicular (p), Base(b) and hypotenuse(h)].

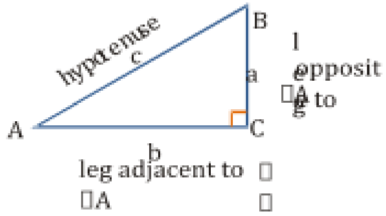
In the Exploration you found that corresponding ratios of side lengths in similar right triangles do not depend on the lengths of the sides. These ratios depend only on the shape of the triangles as determined by the measures of the acute angles. These constant ratios are so important that they are given names as sine, cosine, tangent, secant, cosecant, cotangent.

Here side opposite to angle A is perpendicular; side adjacent to angle A is base & side opposite to right angle is hypotenuse.

Trigonometric ratios

Sine and Cosine of an angle

In right ΔABC, the sine of ∠A, which is written "sin A", is given by



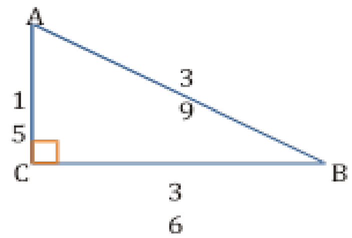
$$\sin A = \frac{\text{Length of leg opposite} \angle A}{\text{Length of hypotenuse}} = \frac{a}{c}$$

and the cosine of ∠A, which is written "cos A", is given by

$$\cos A = \frac{\text{Length of leg adjacent} \angle A}{\text{Length of hypotenuse}} = \frac{b}{c}$$

Numerical Ability 1

In fig., for ΔABC, find sin A and cos A.



Solution

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{36}{39} \approx 0.9231$$

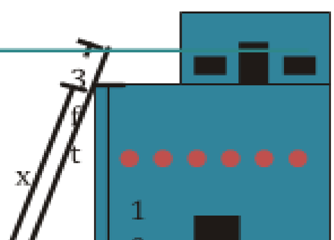
$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{39} \approx 0.3846$$

Building Concepts 1

Between A.D. 1000 and 1300, the Anasazi people lived in cliff dwellings in the southwestern part of the United States. The doors to the cliff dwellings opened onto balconies that were reached by climbing ladders. Suppose the ladder shown rests on the ground and extends 3 ft above the balcony. How long is the ladder?

(use sin60° = $\sqrt{3}/2$)

Explanation



Use the sine ratio to find the unknown side length.

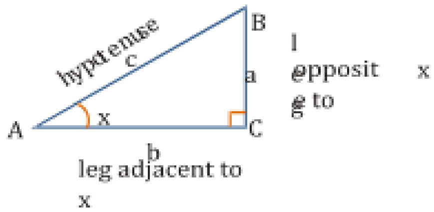
$$\sin 60° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{10}{x}$$

$$x = \frac{10}{\sin 60°} = \frac{10}{\sqrt{3}/2} = \frac{20}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{20\sqrt{3}}{3} = 11.54 \approx 12$$

The ladder is about 12 + 3 = 15 ft. long.

Building Concepts 2

Indicate the perpendicular, the hypotenuse and the base (in that order) with respect to the angle marked x.



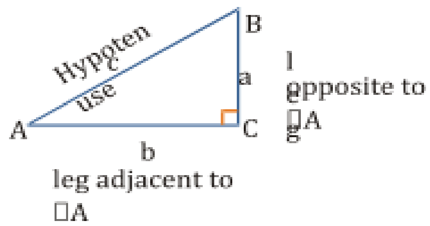
Explanation

Perpendicular = a

Hypotenuse = c

Base = b

Tangent of an angle



In addition to the sine and cosine ratios, you can use the tangent ratio to find the measures of the sides and angles of a right triangle.

In right ΔABC, the tangent of ∠A, which is written "tan A", is given by

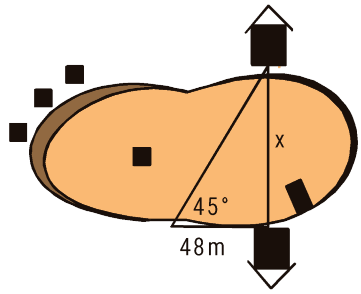
$$\frac{\text{length of leg opposite to } \angle A}{\text{length of leg adjacent to } \angle A} = \frac{a}{b}$$

Building Concepts 3

Surveying surveyors use trigonometry to calculate distances that would otherwise be difficult to find, such as the distance between two houses located across a lake from each other, as shown in the diagram. What is this distance? (Use tan45° = 1)

Explanation

Let x = the distance in meters across the lake. Write an equation involving x and a trigonometric ratio.



$\Rightarrow \quad \dfrac{x}{48} = \tan 45°$

$\Rightarrow \quad x = 48 \tan 45°$

$\Rightarrow \quad x = 48(1) = 48$

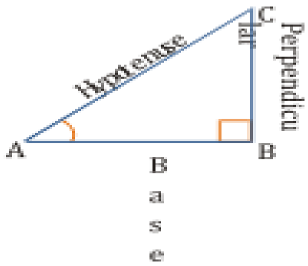
The distance between the two houses is about 48 m.

Cosecant, secant and cotangent of an Angle

In ΔABC, let $\angle B = 90°$ and let $\angle A$ be acute.

For $\angle A$, we have;

Base = AB, Perpendicular = BC and Hypotenuse = AC. Then



(i) Cosecant A = $\dfrac{\text{Hypotenuse (H)}}{\text{Perpendicular (P)}} = \dfrac{AC}{BC}$, written as cosec A.

(ii) Secant A = $\dfrac{\text{Hypotenuse (H)}}{\text{Base (B)}} = \dfrac{AC}{AB}$, written as sec A.

(iii) Cotangent A = $\dfrac{\text{Base (B)}}{\text{Perpendicular (P)}} = \dfrac{AB}{BC}$, written as cot A.

Thus, there are six trigonometric ratios based on the three sides of a right-angled triangle.

Quick Tips

- The sine, cosine, and tangent ratios in a right triangle can be remembered by representing them as strings of letters, as in **SOH-CAH-TOA.**

Sine = Opposite ÷ Hypotenuse

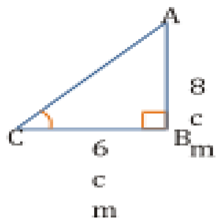
Cosine = Adjacent ÷ Hypotenuse

Tangent = Opposite ÷ Adjacent

The memorization of this mnemonic can be aided by expanding it into a phrase, such as "Some Officers Have Curly Auburn Hair Till Old Age".

Numerical Ability 2

Using the information given in figure. write the values of all trigonometric ratios of angle C.



Solution

AC = $\sqrt{AB^2 + BC^2}$

AC = $\sqrt{8^2 + 6^2}$ = 10

Using the definition of t-ratios,

sin C = $\dfrac{AB}{AC} = \dfrac{8}{10} = \dfrac{4}{5}$; cosC = $\dfrac{BC}{AC} = \dfrac{6}{10} = \dfrac{3}{5}$

tan C = $\frac{AB}{BC} = \frac{8}{6} = \frac{4}{3}$; cotC = $\frac{BC}{AC} = \frac{6}{8} = \frac{3}{4}$

sec C = $\frac{AC}{BC} = \frac{10}{6} = \frac{5}{3}$; and cosec C = $\frac{AC}{AB} = \frac{10}{8} = \frac{5}{4}$

Do You Remember ?

- **Pythagoras theorem** : In a right angle triangle,

  Hypotenuse^2 = Perpendicular^2 + Base^2

Reciprocal Relations

Clearly, we have :

(i) cosecθ = $\frac{1}{\sin\theta}$ (ii) secθ = $\frac{1}{\cos\theta}$ (iii) cotθ = $\frac{1}{\tan\theta}$

Thus, we have :

(i) sinθ.cosecθ = 1 (ii) cosθ.secθ = 1 (iii) tanθ.cotθ = 1

Quotient Relations

Consider a right angled triangle in which for an acute angle θ, we have :

sinθ = $\frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{P}{H}$

cos θ = $\frac{\text{Base}}{\text{Hypotenuse}} = \frac{B}{H}$

Now, $\frac{\sin\theta}{\cos\theta} = \frac{\frac{P}{H}}{\frac{B}{H}} = \frac{P}{H} \times \frac{H}{B} = \frac{P}{B}$ = tanθ (by def.)

and $\dfrac{\cos\theta}{\sin\theta} = \dfrac{\frac{B}{H}}{\frac{P}{H}} = \dfrac{B}{H}\times\dfrac{H}{P} = \dfrac{B}{P} = \cot\theta$ (by def.)

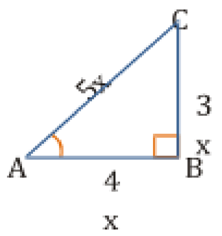
Thus, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Numerical Ability 3

In a right ΔABC, if ∠A is acute and $\tan A = \dfrac{3}{4}$, find the remaining trigonometric ratios of ∠A.

Solution

Consider a ΔABC in which ∠B = 90°.



For ∠A, we have :

Base = AB, perpendicular = BC and Hypotenuse = AC.

$\therefore \quad \tan A = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{3}{4}$

$\Rightarrow \quad \dfrac{BC}{AB} = \dfrac{3}{4}$

Let, BC = 3x units and AB = 4x units.

Then, AC $= \sqrt{AB^2 + BC^2}$

$= \sqrt{(4x)^2 + (3x)^2}$

$= \sqrt{25x^2}$

= 5x units.

$\sin A = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{3}{5}$

$\cos A = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{4}{5}$

$\cot A = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{4}{3}$

$\text{cosec } A = \dfrac{1}{\sin A} = \dfrac{5}{3}$

$\sec A = \dfrac{1}{\cos A} = \dfrac{5}{4}$

Power of T-ratios

We denote :

(i) $(\sin \theta)^2$ by $\sin^2 \theta$

(ii) $(\cos \theta)^2$ by $\cos^2 \theta$

(iii) $(\sin \theta)^3$ by $\sin^3 \theta$

(iv) $(\cos \theta)^3$ by $\cos^3 \theta$ and so on.

Be Alert !

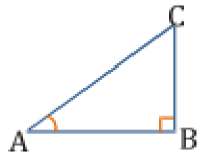
- The symbol sinA is used as an abbreviation for 'the sine of the angle A'. sin A is not the product of 'sin' and A. 'sin' separated from A has no meaning. Similarly, cos A is not the product of 'cos' and A. Similar interpretations follow for other trigonometric ratios also.
- We may write $\sin^2 A$, $\cos^2 A$, etc., in place of $(\sin A)^2$, $(\cos A)^2$, etc., respectively.

  But cosec A = $(\sin A)^{-1} \neq \sin^{-1}A$ (it is called sine inverse A). $\sin^{-1}A$ has a different meaning, which will be discussed in higher classes. Similar conventions hold for the other trigonometric ratios as well.
- Since the hypotenuse is the longest side in a right triangle, the value of sin A or cos A is always less than 1 (or, in particular, equal to 1).

Numerical Ability **4**

If sin A = $\frac{1}{2}$, verify that $2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

Solution



We know that

sin A = $\frac{BC}{AC} = \frac{1}{2}$

Let BC = k and AC = 2k

$\therefore$ AB = $\sqrt{AC^2 - BC^2}$ (Pythagoras theorem)

= $\sqrt{(2k)^2 - k^2}$ = $\sqrt{4k^2 - k^2}$

= $\sqrt{3k^2}$ = $\sqrt{3}k$

Now, cos A = $\frac{AB}{AC}$ = $\frac{\sqrt{3}k}{2k}$ = $\frac{\sqrt{3}}{2}$

and tan A = $\frac{BC}{AB}$ = $\frac{k}{\sqrt{3}k}$ = $\frac{1}{\sqrt{3}}$

Now 2 sin A cos A = 2.$\frac{1}{2}.\frac{\sqrt{3}}{2}$ = $\frac{\sqrt{3}}{2}$ ... (i) and $\frac{2\tan A}{1+\tan^2 A} = \frac{2.\frac{1}{\sqrt{3}}}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{1+\frac{1}{3}} = \frac{\frac{2}{\sqrt{3}}}{\frac{4}{3}}$

$= \frac{2}{\sqrt{3}} \times \frac{3}{4} = \frac{\sqrt{3}}{2}$ ... (ii)

Hence from (i) and (ii)

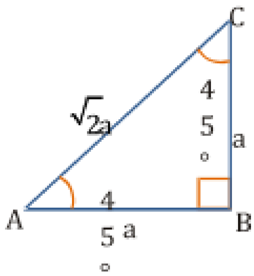
$2 \sin A \cos A = \frac{2\tan A}{1+\tan^2 A}$

Trigonometric ratio of standard angles

T-ratios of 45°

Consider a ΔABC in which $\angle B = 90°$ and $\angle A = 45°$.

Then, clearly, $\angle C = 45°$.



$\therefore$ AB = BC = a (say).

$AC = \sqrt{AB^2 + BC^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} = \sqrt{2}a$.

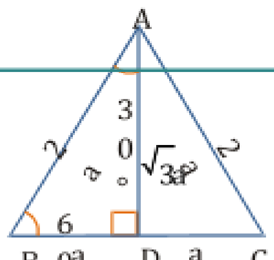
$\sin 45° = \frac{BC}{AC} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$; $\cos 45° = \frac{AB}{AC} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$

$\tan 45° = \frac{BC}{AB} = \frac{a}{a} = 1$

$\text{cosec } 45° = \frac{1}{\sin 45^\circ} = \sqrt{2}$; $\sec 45 = \frac{1}{\cos 45^\circ} = \sqrt{2}$

$\cot 45° = \frac{AB}{BC} = \frac{1}{\tan 45^\circ} = 1$

T-ratios of 60° and 30°



Draw an equilateral ΔABC with each side = 2a.

Then, $\angle A = \angle B = \angle C = 60°$.

From A, draw $AD \perp BC$.

Then, $BD = DC = a$, $\angle BAD = 30°$ and $\angle ADB = 90°$.

Also, $AD = \sqrt{AB^2 - BD^2} = \sqrt{4a^2 - a^2} = \sqrt{3a^2} = \sqrt{3}a$

T-ratios of 60°

In ΔADB we have : $\angle ADB = 90°$ and $\angle ABD = 60°$.

Base = BD = a, Perpendicular = AD = $\sqrt{3}a$ and Hypotenuse AB = 2a.

$\therefore$ $\sin 60° = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$; $\cos 60° = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$

$\tan 60° = \frac{AD}{BD} = \frac{\sqrt{3}a}{a} = \sqrt{3}$

$\therefore$ $\operatorname{cosec} 60° = \frac{1}{\sin 60°} = \frac{2}{\sqrt{3}}$; $\sec 60° = \frac{1}{\cos 60°} = 2$

$\cot 60° = \frac{1}{\tan 60°} = \frac{1}{\sqrt{3}}$

T-ratios of 30°

In ΔADB we have: $\angle ADB = 90°$ and $\angle BAD = 30°$.

$\therefore$ Base = AD = $\sqrt{3}a$, Perpendicular = BD = a and Hypotenuse = AB = 2a.

$\therefore$ $\sin 30° = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$; $\cos 30° = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{BD}{AD} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$

$\operatorname{cosec} 30° = \frac{1}{\sin 30°} = 2$; $\sec 30° = \frac{1}{\cos 30°} = \frac{2}{\sqrt{3}}$;

cot 30° = $\frac{1}{\tan 30°} = \sqrt{3}$

Table for T-ratios of standard angles

Ratio \ Angle θ	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cot θ	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cosec θ	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Quick Tips

(i) As θ increases from 0° to 90°, sin θ increases from 0 to 1.

(ii) As θ increases from 0° to 90°, cos θ decreases from 1 to 0.

(iii) As θ increases from 0° to 90°, tan θ increases from 0 to ∞.

(iv) The maximum value of $\frac{1}{\sec\theta}$, 0° ≤ θ ≤ 90° is one.

(v) As cos θ decreases from 1 to 0, θ increases from 0° to 90°.

(vi) sin θ and cos θ can not be greater than one numerically.

(vii) sec θ and cosec θ can not be less than one numerically.

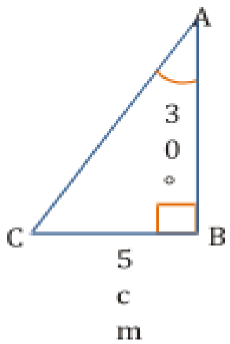
(viii) tan θ and cot θ can have any value.

Numerical Ability 5

In ΔABC, right angled at B, BC = 5 cm, ∠BAC = 30°, find the length of the sides AB and AC.

Solution

We are given



∠BAC = 30°, i.e., ∠A = 30° and BC = 5cm

Now, $\sin A = \frac{BC}{AC}$ or $\sin 30° = \frac{5}{AC}$

or $\frac{5}{AC} = \frac{1}{2}$ $\quad [\sin 30° = \frac{1}{2}]$

or AC = 2 × 5 or 10 cm

To find AB, we have,

$\frac{AB}{AC} = \cos A$

or $\frac{AB}{10} = \cos 30°$

or $\frac{AB}{10} = \frac{\sqrt{3}}{2}$ $\quad [\cos 30° = \frac{\sqrt{3}}{2}]$

∴ $AB = \frac{\sqrt{3}}{2} \times 10$

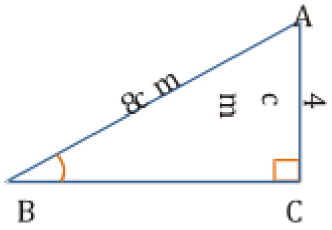
or $5\sqrt{3}$ cm

Hence, AB = $5\sqrt{3}$ cm and AC = 10 cm.

Numerical Ability 6

In ΔABC, right angled at C, if AC = 4 cm and AB = 8 cm. Find ∠A and ∠B.

Solution



We are given, AC = 4 cm and AB = 8 cm

Now $\sin B = \frac{AC}{AB} = \frac{4}{8} = \frac{1}{2}$

But we know that $\sin 30° = \frac{1}{2}$

$\therefore \quad B = 30°$

Now, $\angle A = 90° - \angle B$ $\quad\quad$ $[\angle A + \angle B = 90°]$

$= 90° - 30° = 60°$

Hence, $\angle A = 60°$ and $\angle B = 30°$.

Numerical Ability 7

Find the value of θ in each of the following:

(i) $2 \sin 2\theta = \sqrt{3}$ (ii) $2 \cos 3\theta = 1$ (iii) $\sqrt{3} \tan 2\theta - 3 = 0$

Solution

(i) we have,

$2 \sin 2\theta = \sqrt{3} \Rightarrow \sin 2\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \sin 2\theta = \sin 60°$

$\Rightarrow 2\theta = 60°$

$\Rightarrow \theta = 30°$

(ii) we have,

$2 \cos 3\theta = 1$

$\Rightarrow \quad \cos 3\theta = \frac{1}{2}$

$\Rightarrow \quad \cos 3\theta = \cos 60°$

$\Rightarrow \quad 3\theta = 60°$

$\Rightarrow \quad \theta = 20°$

(iii) we have,

$\sqrt{3}\tan 2\theta - 3 = 0$

$\Rightarrow \quad \sqrt{3}\tan 2\theta = 3$

$\Rightarrow \quad \tan 2\theta = \frac{3}{\sqrt{3}} = \sqrt{3}$

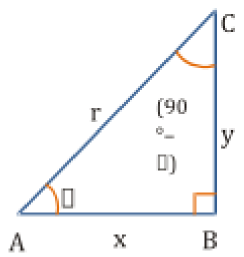
$\Rightarrow \quad \tan 2\theta = \tan 60°$

$\Rightarrow \quad 2\theta = 60°$

$\Rightarrow \quad \theta = 30°$

T-ratios of complementary angles

Complementary angles



Two angles are said to be complementary, if their sum is 90°.

Thus, θ° and (90° – θ) are complementary angles.

T-ratios of complementary angles

Consider ΔABC in which ∠B = 90° and ∠A = θ°.

$\therefore \quad \angle C = (90° - \theta)$.

Let AB = x. BC = y and AC = r.

Then, $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$ and $\tan\theta = \frac{y}{x}$.

When we consider the T-ratios of (90° – θ), then

Base = BC, **Perpendicular** = AB and **Hypotenuse** = AC.

∴ $\sin(90° - \theta) = \frac{AB}{AC} = \frac{x}{r} = \cos\theta.$

$\cos(90° - \theta) = \frac{BC}{AC} = \frac{y}{r} = \sin\theta.$

$\tan(90° - \theta) = \frac{AB}{BC} = \frac{x}{y} = \cot\theta.$

∴ $\text{cosec}(90° - \theta) = \frac{1}{\sin(90° - \theta)} = \frac{1}{\cos\theta} = \sec\theta.$

$\sec(90° - \theta) = \frac{1}{\cos(90° - \theta)} = \frac{1}{\sin\theta} = \text{cosec}\,\theta.$

$\cot(90° - \theta) = \frac{1}{\tan(90° - \theta)} = \frac{1}{\cot\theta} = \tan\theta.$

(i) sin(90° – θ) = cosθ (ii) cos(90° – θ) = sinθ

(iii) tan(90° – θ) = cotθ (iv) cosec(90° – θ) = secθ

(v) sec(90° – θ) = cosecθ (vi) cot(90° – θ) = tanθ

Add co, if co is not there. Remove co, If co is there.

SPOT LIGHT

In other words:
sin (angle) = cos (complement); cos (angle) = sin (complement)
tan (angle) = cot (complement); cot (angle) = tan (complement)
sec (angle) = cosec (complement); cosec (angle) = sec (complement)

(∴ where complement = 90° – angle)

Numerical Ability 8

Without using tables, evaluate:

(i) $\dfrac{\sin 53^\circ}{\cos 37^\circ}$ **(ii)** $\dfrac{\cos 49^\circ}{\sin 41^\circ}$ **(iii)** $\dfrac{\tan 66^\circ}{\cot 24^\circ}$

Solution

(i) $\dfrac{\sin 53^\circ}{\cos 37^\circ} = \dfrac{\sin(90^\circ - 37^\circ)}{\cos 37^\circ} = \dfrac{\cos 37^\circ}{\cos 37^\circ} = 1$ [∴ sin (90° – θ) = cos θ]

(ii) $\dfrac{\cos 49^\circ}{\sin 41^\circ} = \dfrac{\cos(90^\circ - 41^\circ)}{\sin 41^\circ} = \dfrac{\sin 41^\circ}{\sin 41^\circ} = 1$ [∴ cos (90° – θ) = sin θ]

(iii) $\dfrac{\tan 66^\circ}{\cot 24^\circ} = \dfrac{\tan(90^\circ - 24^\circ)}{\cot 24^\circ} = \dfrac{\cot 24^\circ}{\cot 24^\circ} = 1$ [∴ tan (90° – θ) = cot θ]

Quick Tips

- The above example suggests that out of the two t-ratios, we convert one in term of the t-ratio of the complement.
- For uniformity, we usually convert the angle greater than 45° in terms of its complement.

Numerical Ability 9

Without using tables, show that (cos 35° cos 55° – sin 35° sin 55°) = 0.

Solution

LHS = (cos 35° cos 55° – sin 35° sin 55°)

= [(cos 35° cos 55° – sin (90° – 55°) sin (90° – 35°)]

= (cos 35° cos 55° – cos 55° cos 35°) = 0 = RHS.

[∴ sin (90° – θ) = cos θ and cos (90° – θ) = sin θ]

Numerical Ability 10

Express (sin 85° + cosec 85°) in terms of trigonometric ratios of angles between 0° and 45°.

Solution

(sin 85° + cosec 85°) = sin (90° – 5°) + cosec (90° – 5°)

= (cos 5° + sec 5°).

Numerical Ability 11

Evaluate : $\dfrac{\sec 29^{o}}{\operatorname{cosec} 61^{o}} + 2\cot 8^{\circ}\cot 17^{\circ}\cot 45^{\circ}\cot 73^{\circ}\cot 82^{\circ}$

Solution

$\dfrac{\sec 29^{o}}{\operatorname{cosec} 61^{o}} + 2 \cot 8^{\circ} \cot 17^{\circ} \cot 45^{\circ} \cot 73^{\circ} \cot 82^{\circ}.$

$= \dfrac{\sec 29^{o}}{\operatorname{cosec}(90^{o} - 29^{o})} + 2 \cot 8^{\circ} \cot 17^{\circ} (1) \cot (90^{\circ} - 17^{\circ}) \cot (90^{\circ} - 8^{\circ})$

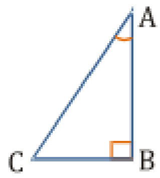
$= \dfrac{\sec 29^{o}}{\sec 29^{o}} + 2 \cot 8^{\circ} \cot 17^{\circ} \tan 17^{\circ} \tan 8^{\circ}.$ [cot (90° – θ) = tan θ]

$= 1 + 2 \cot 8^{\circ} \cot 17^{\circ} \dfrac{1}{\cot 17^{o}} \dfrac{1}{\cot 8^{o}}$ $\left(\because \tan\theta = \dfrac{1}{\cot\theta}\right)$

= 1 + 2 = 3.

Trigonometric identities

We know that an equation is called an identity when it is true for all values of the variables involved. Similarly, "an equation involving trigonometric ratios of an angle is called a trigonometric identity, if it is true for all values of the angle(s) involved."



The three Fundamental Trigonometric Identities are –

(i) $\cos^2 A + \sin^2 A = 1;\ 0^{\circ} \le A \le 90^{\circ}$

(ii) $1 + \tan^2 A = \sec^2 A;\ 0^{\circ} \le A < 90^{\circ}$

(iii) $1 + \cot^2 A = \operatorname{cosec}^2 A;\ 0^{\circ} < A \le 90^{\circ}$

Geometrical proof

Consider a ΔABC, right angled at B. Then we have :

$AB^2 + BC^2 = AC^2$ … (i)

By Pythagoras theorem

(i) $\cos^2 A + \sin^2 A = 1;\ 0° \le A \le 90°$

Dividing each term of (i) by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2} \text{ i.e. } \left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = \left(\frac{AC}{AC}\right)^2$$

i.e., $(\cos A)^2 + (\sin A)^2 = 1$

i.e., $\cos^2 A + \sin^2 A = 1$ … (ii)

This is true for all A such that $0° \le A \le 90°$

So, this is a trigonometric identity.

(ii) $1 + \tan^2 A = \sec^2 A\ ;\ 0° \le A < 90°$

Let us now divide (i) by AB^2. We get

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2} \text{ or, } \left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

i.e., $1 + \tan^2 A = \sec^2 A$ … (iii)

This equation is true for $A = 0°$. Since tan A and sec A are not defined for $A = 90°$, so (iii) is true for all A such that $0° \le A < 90°$

(iii) $1 + \cot^2 A = \operatorname{cosec}^2 A\ ;\ 0° < A \le 90°$

Again, let us divide (i) by BC^2, we get

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

$$\Rightarrow \left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

$\Rightarrow \quad 1 + \cot^2 A = \operatorname{cosec}^2 A \qquad \ldots \text{(iv)}$

Since cosec A and cot A are not defined for A = 0°, therefore (iv) is true for all A such that $0° < A \le 90°$

Using the above trigonometric identities, we can express each trigonometric ratio in terms of the other trigonometric ratios, i.e., if any one of the ratios is known, we can also determine the values of other trigonometric ratios.

Fundamental identities (results)

$\sin^2\theta + \cos^2\theta = 1$	$1 + \tan^2\theta = \sec^2\theta$	$1 + \cot^2\theta = \operatorname{cosec}^2\theta$
$\sin^2\theta = 1 - \cos^2\theta$	$\sec^2\theta - \tan^2\theta = 1$	$\operatorname{cosec}^2\theta - \cot^2\theta = 1$
$\cos^2\theta = 1 - \sin^2\theta$	$\tan^2\theta = \sec^2\theta - 1$	$\cot^2\theta = \operatorname{cosec}^2\theta - 1$

To prove trigonometrical identities

The following methods are to be followed:

Method-I : Take the more complicated side of the identity (L.H.S. or R.H.S. as the case may be) and by using suitable trigonometric and algebraic formulae prove it equal to the other side.

Method-II : When neither side of the identity is in a simple form, simplify the L.H.S. and R.H.S. separately by using suitable formulae (by expressing all the T-ratios occurring in the identity in terms of the sine and cosine and show that the results are equal).

Numerical Ability 12

Prove $\sqrt{sec^2\theta + cosec^2\theta} = tan\,\theta + cot\,\theta$

Solution

$$\text{LHS} = \sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \sqrt{(1+\tan^2\theta)+(1+\cot^2\theta)}$$

$$= \sqrt{\tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta} \qquad (\tan\theta \,.\, \cot\theta = 1)$$

$$= \sqrt{(\tan\theta + \cot\theta)^2}$$

$= \tan\theta + \cot\theta = \text{RHS}$

Hence proved.

Numerical Ability 13

Prove that : $\dfrac{\sec^2\theta\sin^2\theta - \text{cosec}^2\theta + \text{cosec}^2\theta\cos^2\theta}{\sec^2\theta\sin^2\theta - \text{cosec}^2\theta\cos^2\theta} = \sin^2\theta$

Solution

$$\text{LHS} = \frac{\sec^2\theta\sin^2\theta - \text{cosec}^2\theta + \text{cosec}^2\theta\cos^2\theta}{\sec^2\theta\sin^2\theta - \text{cosec}^2\theta\cos^2\theta}$$

$$= \frac{\dfrac{\sin^2\theta}{\cos^2\theta} - \dfrac{1}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta}}{\dfrac{\sin^2\theta}{\cos^2\theta} - \dfrac{\cos^2\theta}{\sin^2\theta}}$$

$$= \frac{\dfrac{\sin^2\theta}{\cos^2\theta} - \left(\dfrac{1-\cos^2\theta}{\sin^2\theta}\right)}{\dfrac{\sin^4\theta - \cos^4\theta}{\sin^2\theta\cos^2\theta}}$$

$$= \frac{\dfrac{\sin^2\theta}{\cos^2\theta} - 1}{\dfrac{(\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)}{\sin^2\theta\cos^2\theta}}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{\cos^2\theta} \times \frac{\sin^2\theta\cos^2\theta}{\sin^2\theta - \cos^2\theta}$$

$= \sin^2\theta = \text{RHS}$

Hence Proved.

Numerical Ability 14

Prove that : $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0$

Solution

LHS $= 2\,(\sin^6\theta + \cos^6\theta) - 3\,(\sin^4\theta + \cos^4\theta) + 1$

$= 2\,[(\sin^2\theta)^3 + (\cos^2\theta)^3] - 3\,[(\sin^2\theta)^2 + (\cos^2\theta)^2] + 1$

$= 2\,[(\sin^2\theta + \cos^2\theta)^3] - 3\sin^2\theta\cos^2\theta\,(\sin^2\theta + \cos^2\theta)] - 3\,[(\sin^2\theta + \cos^2\theta)^2 - 2$

$\sin^2\theta\cos^2\theta\,] + 1$

$= 2\,\{1 - 3\sin^2\theta\cos^2\theta\} - 3\,\{1 - 2\sin^2\theta\cos^2\theta\} + 1$

$= 2 - 6\sin^2\theta\cos^2\theta - 3 + 6\sin^2\theta\cos^2\theta + 1 = 3 - 3 = 0 =$ RHS.

Hence Proved.

Numerical Ability 15

Prove the following by using suitable identities:

$$\frac{1-\sin x}{1+\sec x} - \frac{1+\sin x}{1-\sec x} = 2\cos x(\cot x + \mathrm{cosec}^2 x)$$

Solution

$$\text{LHS} = \frac{1-\sin x}{1+\sec x} - \frac{1+\sin x}{1-\sec x} = \frac{1-\sin x}{1+\frac{1}{\cos x}} - \frac{1+\sin x}{1-\frac{1}{\cos x}}$$

$$= \frac{\cos x(1-\sin x)}{\cos x+1} - \frac{\cos x(1+\sin x)}{\cos x-1} = \cos x\left[\frac{(1-\sin x)}{\cos x+1} - \frac{1+\sin x}{\cos x-1}\right]$$

$$= \cos x\left\{\frac{(1-\sin x)(\cos x-1)-(\cos x+1)(1+\sin x)}{\cos^2 x-1}\right\}$$

$$= \cos x\left\{\frac{(1-\sin x)(1-\cos x)+(1+\cos x)(1+\sin x)}{1-\cos^2 x}\right\}$$

$$= \cos x\left\{\frac{1-\cos x-\sin x+\sin x\cos x+1+\sin x+\cos x+\sin x\cos x}{\sin^2 x}\right\}$$

$$= \cos x\left\{\frac{2+2\sin x\cos x}{\sin^2 x}\right\} = 2\cos x\left(\frac{1+\sin x\cos x}{\sin^2 x}\right)$$

$$\text{LHS} = 2\cos x\left\{\frac{1}{\sin^2 x}+\frac{\cos x}{\sin x}\right\} = 2\cos x\,(\text{cosec}^2 x + \cot x) = \text{RHS}.$$

Hence Proved.

Quick Tips

- In proving question, generally we convert all the trigonometric ratios in terms of $\sin\theta$ and $\cos\theta$ to make it easy and convenient.

Numerical Ability 16

Prove that : $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \sec\theta + \tan\theta = \dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1-\sin\theta} = \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}$

Solution

Consider the numerator of the LHS of the given expression.

i.e. $\tan\theta + \sec\theta - 1$ [Use $1 = \sec^2\theta - \tan^2\theta$]

$= (\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)$ [Use $a^2 - b^2 = (a + b)(a - b)$]

$= (\tan\theta + \sec\theta) - (\sec\theta + \tan\theta)(\sec\theta - \tan\theta)$ [Take $(\sec\theta + \tan\theta)$ as common]

$= (\sec\theta + \tan\theta)(1 - \sec\theta + \tan\theta)$ [Note this step very carefully]

$= (\sec\theta + \tan\theta)(\tan\theta - \sec\theta + 1)$ ... (1) [Rearranging]

LHS $= \dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$

$= \dfrac{(\sec\theta+\tan\theta)(\tan\theta-\sec\theta+1)}{(\tan\theta-\sec\theta+1)}$ [Use (1)]

= sec θ + tan θ = RHS [First form]

$= \dfrac{1}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta} = \dfrac{1+\sin\theta}{\cos\theta} =$ RHS … (2) [second form]

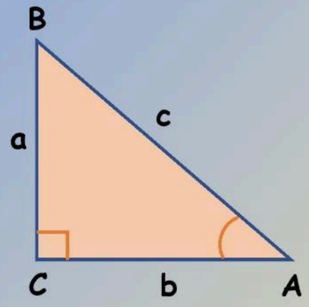
$= \dfrac{(1+\sin\theta)(1-\sin\theta)}{\cos\theta(1-\sin\theta)} = \dfrac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)} = \dfrac{\cos^2\theta}{\cos\theta(1-\sin\theta)}$

$= \dfrac{\cos\theta}{(1-\sin\theta)}$ [Third form]

$= \dfrac{\sqrt{1+\sin\theta}.\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}\sqrt{1-\sin\theta}}$ [use $\cos^2\theta = 1 - \sin^2\theta$]

$= \dfrac{\sqrt{1+\sin\theta}}{\sqrt{1-\sin\theta}}$ = RHS [Fourth form]

Memory map



- $\sin A = \dfrac{a}{c}$
- $\cos A = \dfrac{b}{c}$
- $\tan A = \dfrac{a}{b}$
- $\operatorname{cosec} A = \dfrac{c}{a}$
- $\sec A = \dfrac{c}{b}$
- $\cot A = \dfrac{b}{a}$

Reciprocal Relation

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$

Quotient Relations

$\dfrac{\sin\theta}{\cos\theta} = \tan\theta$

$\dfrac{\cos\theta}{\sin\theta} = \cot\theta$

Power of T-ratios

$(\sin\theta)^2 = \sin^2\theta$

$(\cos\theta)^2 = \cos^2\theta$

$(\sin\theta)^3 = \sin^3\theta$

$(\cos\theta)^3 = \cos^3\theta$

Trigonometrical Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Table of T-Ratios of Standard

Angle θ / Ratio	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\operatorname{cosec}\theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1